Q1 p.d.f. X

$$f(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

c.d.f. F_1 $\rightarrow P(X \le x)$

$$F_1(x) = \begin{cases} \int_a^x f(x)dx = \dfrac{x-a}{b-a} & a \le x \le b \\ 0 & x < a \\ 1 & x > b \end{cases}$$

x

Q2.

$$g_2(y|x) = \frac{f(x,y)}{f_1(x)}$$

$$\int_0^1 c(x+y^2)dy = cxy + \frac{1}{3}cy^3\Big|_0^1 = c\left(x+\frac{1}{3}\right)$$

$$f_1(x) = \begin{cases} \int_0^1 f(x,y)dy & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

thus, for $0 \le x \le 1$,

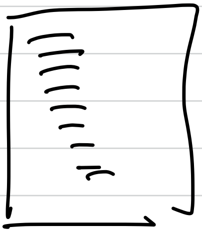
$$g_2(y|x) \begin{cases} \dfrac{\cancel{c}(x+y^2)}{\cancel{c}(x+\frac{1}{3})} & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Chap 4_ Expectations

r.v. X → distribution of X

→ contains all the information for X.

↳ too huge to understand



X →

→ need some "summarization" of X.

↳ measure.

↳ average, expectation, mean

Def X = discrete r.v., bounded

(i.e. X can have only finite possibilities)

f = p.f. of X

then expectation (mean) of X, $E(X)$,

is defined as

$$\boxed{E(X) = \sum_{\forall x} x \cdot f(x)}$$

Ex) Bernoulli r.v. X. (with p.f. f)

$$f(x) = \begin{cases} p & x = 1 \\ 1-p & x = 0 \end{cases}$$

$$E(X) = \sum_{\forall x} x \cdot f(x) = 0 \cdot (1-p) + 1 \cdot p = p.$$

*unbounded case.

$\Rightarrow$ The expectation of X is defined iff at least one of $\sum_{x>0} x f(x)$ or $\sum_{x<0} x f(x)$ is bounded (otherwise $E(X)$ is undefined)

Ex) X = r.v. with p.f. f

$$f(x) = \begin{cases} \dfrac{1}{2|x|(|x|+1)} & \text{when } x = \pm 1, \pm 2 \cdots \\ 0 & \end{cases}$$

↳ non-zero integers

$$\sum_{\forall x} f(x) = 1$$

$$\sum_{x>0} x f(x) = \sum_{x>0} \frac{x}{2x(x+1)} = \sum_{x>0} \frac{1}{x+1} \to \infty$$

harmonic series

$$\sum_{x>0} \frac{1}{x^2} \to \text{converge}$$

$$\sum_{x<0} x f(x) = \sum_{-x<0} \frac{x}{-2x(-x+1)} = \sum_{x<0} \frac{1}{2(x-1)} \to -\infty$$

$\Rightarrow E(X)$ is undefined.

<u>Def</u> X = continuous r.v., bounded

$\quad$ $\langle$ interval.

f = p.d.f of X

then $E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$.

Ex) X = r.v. with p.d.f. f

$$f(x) = \begin{cases} 2x & 0<x<1 \\ 0 & \text{otherwise.} \end{cases}$$

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x) = \int_0^1 2x^2\,dx = \frac{2}{3}x^3 \Big|_0^1$$

$$= \frac{2}{3}$$

- unbounded case.

$\rightarrow$ In this case $E(X)$ is defined iff

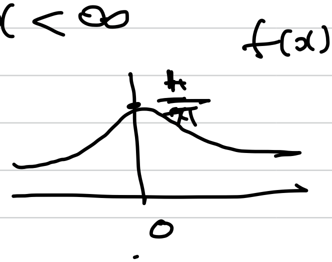
$\int_0^{\infty} x f(x)\,dx$ or $\int_{-\infty}^{0} x f(x)\,dx$ is finite.

Ex) Cauchy distribution. p.d.f f.

→ derivation $\tan^{-1}x$

$$f(x)=\frac{1}{\pi(1+x^2)}, \quad -\infty<x<\infty$$

$$\int_{-\infty}^{\infty} f(x)=1 \quad \rightarrow$$

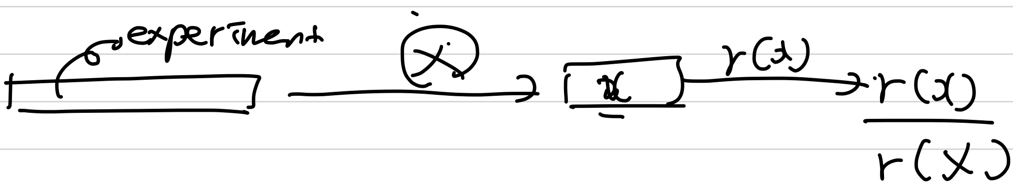


$$E(X) \Rightarrow \int_{-\infty}^{0} x f(x)\,dx \rightarrow -\infty$$

$$\int_{0}^{\infty} x f(x)\,dx \rightarrow \infty$$

→ Expectation is undefined. //

* Expectation of a function.



Expectation of r(X)?

Thm 4.1.1 (Law of Unconscious Statistician) → LOTUS

↳ unknown

X : r.v.

r = real-valued function $\mathbb{R} \to \mathbb{R}$

then i) X is discrete.

p.f. of X

$$E(r(X)) = \sum_{\forall x} r(x) \cdot f(x)$$

ii) X is continuous

p.d.f of X.

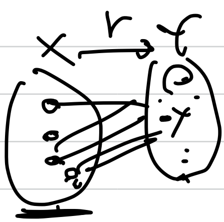
$$E(r(X)) = \int_{-\infty}^{\infty} r(x) \cdot f(x)\, dx.$$

pf) case when both X and r are discrete

r.v. $Y = r(X) \Rightarrow Y$ is discrete r.v.

$$E(Y) = \sum_{\forall y} y \cdot g(y) = \sum_{\forall y} y \cdot P(Y=y)$$

↳ p.f. of Y.



$$= \sum_{\forall y} y \cdot P(r(X)=y)$$

$$= \sum_{\forall y} y \cdot \sum_{\forall x \cdot r(x)=y} f(x)$$

$$= \sum_{\forall y} \sum_{\forall x,\, r(x)=y} y \cdot f(x)$$

$$= \sum_{\forall y} \sum_{\forall x,\, r(x)=y} r(x) \cdot f(x)$$

$$= \sum_{\forall x} r(x) \cdot f(x)$$

Ex) X = r.v. with p.d.f. f

$$f(x) = \begin{cases} 3x^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$E\left(\frac{1}{X}\right)$?

$\hookrightarrow r(x) = \frac{1}{x}$

$$E\left(\frac{1}{X}\right) = \int_{-\infty}^{\infty} \frac{1}{x} \cdot f(x)\, dx$$

LOTUS

$$= \int_0^1 \frac{1}{x} \cdot 3x^2\, dx = \frac{3}{2} x^2 \Big|_0^1 = \frac{3}{2}$$

→ Without LOTUS

r.v. $Y = \frac{1}{X}$ p.d.f. Y ?

c.d.f. $F_Y(y)$ of Y

$$F_Y(y) = P(Y \le y) = P\left(\frac{1}{X} \le y\right)$$

$$= P\left(\frac{1}{y} \le X\right)$$

$$= \int_{\frac{1}{y}}^{\infty} f(x)\,dx = \begin{cases} x^3 \Big|_{\frac{1}{y}}^{1} = 1 - \frac{1}{y^3} & y > 1 \\ 0 & \text{otherwise} \end{cases}$$

$\underline{3x^2}$

p.d.f $g(y)$ of Y

$$g(y) = \begin{cases} 3y^{-4} & y > 1 \\ 0 & \text{otherwise} \end{cases} \qquad -y^{-3}$$

$$E(Y) = \int_{-\infty}^{\infty} y \cdot g(y)\,dy = \int_{1}^{\infty} y^{-3} \cdot 3\,dy$$

$3y^{-3}$

$$= -\frac{3}{2} y^{-2} \Big|_{1}^{\infty}$$

$$= \frac{3}{2}$$

4-2 Properties of expectations (X = r.v.)

Thm (linear function): If r.v. $Y = aX + b$
4-2-1 (a, b = constant) then

$$E(Y) = aE(X) + b$$

pf) Suppose X is conti. r.v. with p.d.f f

$$E(Y) = E(aX+b)$$

$$= \int_{-\infty}^{\infty} (ax+b) \cdot f(x)\,dx$$

LOTUS

$$= a \underbrace{\int_{-\infty}^{\infty} x f(x)\,dx}_{E(X)} + b \underbrace{\int_{-\infty}^{\infty} f(x)\,dx}_{1}$$

$$= aE(X) + b$$

Ex) $E(X) = 5$

$$E(3X-5) = 3E(X) - 5 = 10$$

$$E(-3X+15) = 0$$

Cor if $X = c$ (c = constant), $E(X) = c$.

Thm 4.2.2 If $\exists$ constant a s.t.

i) $P(X \geq a) = 1 \rightarrow E(X) \geq a$

ii) $P(X \leq a) = 1 \rightarrow E(X) \leq a$,

pf) i) X = conti r.v. with p.d.f f.

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{a}^{\infty} x f(x)\,dx$$

$$\geq \int_{a}^{\infty} a \cdot f(x)\,dx = a \int_{a}^{\infty} f(x)\,dx$$

$$= a \cdot \underbrace{P(X \geq a)}_{1}$$

$$= a$$

Thm 4.2.4 $X_1 \cdots X_n$ = r.v. where $E(X_1) \cdots E(X_n)$ are finite.

$$E(X_1 + X_2 \cdots + X_n)$$

$$= E(X_1) + E(X_2) \cdots + E(X_n)$$

pf) the case $n=2$

suppose X_1 and X_2 are cont. r-vs with joint p.d.f f.

$$E(\underbrace{X_1+X_2}_{r(X_1,X_2)}) \overset{\text{LOTUS}}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x_1+x_2)\,f(x_1,x_2)\,dx_1\,dx_2$$

→ marginal p.d.f. X_1

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1\,f(x_1,x_2)\,dx_2\,dx_1 + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_2\,f(x_1,x_2)\,dx_1\,dx_2$$

$$= \underbrace{\int_{-\infty}^{\infty} x_1\cdot f_1(x_1)\,dx_1}_{E(X_1)} + \underbrace{\int_{-\infty}^{\infty} x_2\cdot f_2(x_2)\,dx_2}_{E(X_2)}$$

$$= E(X_1)+E(X_2)$$

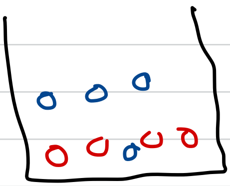
→ NO condition for independence.

Cor. $E(X_i)$ is finite for all $i = 1 \cdots n$

$a_1 \cdots a_n, b =$ constant

$$E(a_1X_1 + a_2X_2 + \cdots a_nX_n + b)$$
$$= a_1E(X_1) + a_2E(X_2) \cdots + a_nE(X_n) + b.$$

EX) selecting balls



\# total balls : N

\# red balls : $N \cdot p$ $(0 \le p \le 1)$

- select n balls one-by-one, without replacement

r.v. X = # red balls in n selected balls

$E(X)$?

$\Rightarrow$ Let n random variables $X_1 \cdots X_n$ as

$$\forall i \quad X_i = \begin{cases} 1 & \rightarrow i\text{-th selected ball is red} \\ 0 & \rightarrow \text{otherwise.} \end{cases}$$

$$E(X) = E(X_1 + X_2 + \cdots X_n)$$

$$= E(X_1) + E(X_2) \cdots + E(X_n)$$

$$P(X_1 = 1) = p$$
$$P(X_1 = 0) = 1-p$$

$$E(X_1) = 1 \cdot p + (1-p)0$$
$$= p.$$

$$E(X_2)? = p.$$

$$P(X_2 = 1) = p$$
$$= P(X_2 = 1 \mid X_1 = 0)$$
$$+ P(X_2 = 1 \mid X_1 = 1)$$

marginal distribution of X_2

$$\cdots \forall n, \; E(X_n) = p$$

$$E(X) = n \cdot p.$$

— For general function $g(x)$

$$E(g(x)) \neq g(E(x))$$

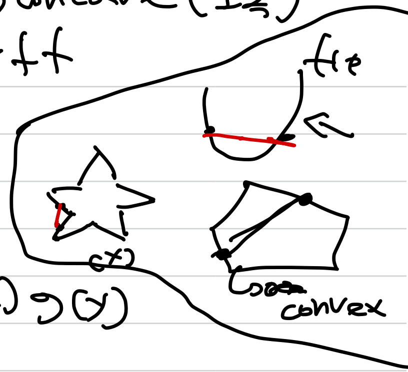
(= when g is linear)

Def Function g is **convex** 볼록 iff

→ concave (오목)

$\forall x, y,\ \lambda \in (0,1)$

$$g(\lambda x + (1-\lambda) y) \geq \lambda g(x) + (1-\lambda) g(y)$$



Thm 4.2.5 (Jensen's inequality)

g = convex function

X = r.v. with finite mean

$$\Rightarrow \boxed{E(g(X)) \geq g(E(X))}$$

Ex n letters, n envelopes. → place each letters in an envelope in a random manner

→ $\exists$ unique correct envelope for each letter.

r.v. X = # letters placed correctly

$E(X)$?

⇒ let $X_1 \cdots X_n$ be n r.v.s where

$\forall i \quad X_i = \begin{cases} 1 & \text{if } i\text{th letter placed correctly.} \\ 0 & \text{otherwise.} \end{cases}$

$$E(X) = E(X_1 + X_2 - \quad + X_n)$$
$$= E(X_1) + E(X_2) \cdots \quad + E(X_n)$$

$$E(X_1) = 1 \cdot \frac{1}{n} + 0 \cdot \frac{n-1}{n} = \frac{1}{n}$$

$\forall i, E(X_i) = \frac{1}{n}$, $E(X) = 1$

\# **Thm 4.2.6** $X_1 \cdots X_n$: n r.v.s, mutually independent.

$E(X_1) \cdots, E(X_n)$ are finite

$$E\left(\prod_{i=1}^{n} X_i\right) = \prod_{i=1}^{n} E(X_i)$$

product.

pf) $n=2$, X_1 and X_2 are conti r.v.s

with joint p.d.f f

$$f(x_1, x_2) \longrightarrow f_1(x_1) \cdot f_2(x_2)$$

X_1 and X_2 are independent

$$E(X_1 X_2) \xrightarrow[\text{LOTUS}]{} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 \cdot f(x_1, x_2)\, dx_1\, dx_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 \cdot f_1(x_1) \cdot x_2 \cdot f_2(x_2)\, dx_1\, dx_2$$

($\because$ X_1 and X_2 are independent)

$$= \underbrace{\int_{-\infty}^{\infty} x_2 \cdot f_2(x_2)}_{E(X_2)} \cdot \underbrace{\int_{-\infty}^{\infty} x_1 \cdot f_1(x_1)\, dx_1\, dx_2}_{E(X_1)}$$

$$= E(X_1) \cdot E(X_2) \text{ //}$$

ex) $E(X_1) = E(X_2) = E(X_3) = 0$

$E(X_1^2) = E(X_2^2) = E(X_3^2) = 1$

X_1, X_2, X_3 = independent

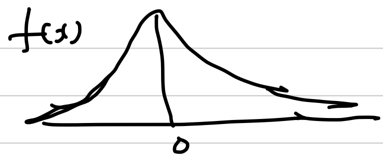
$$E(X_1^2 \cdot (X_2 - 4X_3)^2) = E(X_1^2) \cdot E((X_2 - 4X_3)^2)$$

$$= E(X_2^2 - 4X_2X_3 + 16X_3^2)$$

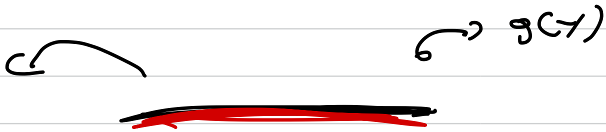
$E(X_2) \cdot E(X_3)$

$$= E(X_2^2) - 4E(X_2X_3) + 16 \cdot E(X_3^2) = 17.$$

4.2 Variance



$E(X)=0$



$E(Y)=0$

$\Rightarrow$ need a measure how spread out the distribution

$\rightarrow$ Variance.

<u>Def</u> X = r.v with finite mean $E(X)=\mu$

constant

then the variance of X, $Var(X)$, is

$$Var(X)=E((X-\mu)^2)$$

$\hookrightarrow \sigma_X^2$ or σ^2

standard derivation of X is $\sqrt{Var(X)}$

$\rightarrow \sigma_X$, σ.

Ex) X = discrete r.v., X can have only 5 values $-2, 0, 1, 3, 4$.

$$P(X=-2) = P(X=0) = P(X=1) = P(X=3) = P(X=4) = \frac{1}{5}.$$

$Var(X)$?

$$\mu = E(X) = \frac{1}{5}(-2+0+1+3+4) = \frac{6}{5}.$$

1.2

$$Var(X) = E\left(\underline{(X-\mu)^2}\right)$$

$-2 - 1.2$

	-2	0	1	3	4
$(X-\mu)^2$	$(-3.2)^2$	$(-1.2)^2$	$(-0.2)^2$	$(1.8)^2$	$(2.8)^2$
$f((X-\mu)^2)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

$$Var(X) = \frac{1}{5}\left((-3.2)^2 + (-1.2)^2 + (-0.2)^2 + (1.8)^2 + (2.8)^2\right) = 4.56$$

Thm 4.3.1 X : r.v.

$$Var(X) = \underline{E(X^2)} - (\underline{E(X)})^2$$

<u>pf)</u> $Var(X) = E((X-\mu)^2)$

$$= E(\underline{X}^2 - \underline{2X\mu} + \underline{\mu^2})$$

$$= E(X^2) - 2\underbrace{E(X)}_{\mu}\mu + \mu^2$$

$$= E(X^2) - \underbrace{\mu^2}_{E(X)}$$

$\Rightarrow$ try to solve the former ex using this Thm.