Quiz 3.

1. $\underline{E(X(X-1))} = \underline{\mu(\mu-1)+\sigma^2}$

$E(X^2) = ?$

$\underbrace{Var(X)}_{\sigma^2} = E(X^2) - (\underbrace{E(X)}_{\mu})^2$

$\therefore E(X^2) = \sigma^2 + \mu^2$

$\therefore E(X(X-1)) = E(X^2) - E(X)$

$= \sigma^2 + \mu^2 - \mu$

$= \mu(\mu-1) + \sigma^2$

2. $Cov(aX, bY) = ab \cdot Cov(X, Y)$

$\underline{a} \cdot E(X) \cdot \underline{b} \cdot E(Y)$

$Cov(aX, bY) = E(\underline{ab}XY) - \underbrace{E(aX)E(bY)}$

$= ab(E\underbrace{(XY) - E(X)E(Y)}_{Cov(X,Y)})$

$= ab\, Cov(X, Y)$,

Chap 5. Special distributions.

5-2 The Benoulli distribution & Binomial distribution

<u>Def)</u> r.v. X has the Benoulli distribution with parameter p ($0 \le p \le 1$)

$\Longleftrightarrow$ X can only have either 0 or 1. with probability

$$P(X=0) = 1-p.$$

$$P(X=1) = p$$

, in this cas, the p.f. of X

$$f(x \mid p) = \begin{cases} p^x \cdot (1-p)^{1-x} & x = 0 \text{ or } 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$E(X) = 0 \cdot \boxed{(1-p)} + 1 \cdot \boxed{p} = p$$

$$Var(X) = \underbrace{E(X^2)}_{} - (E(X))^2 = p - p^2 = p(1-p)$$

$\hookrightarrow 0^2(1-p) + 1^2 \cdot p = p$

m.g.f $\psi(t)$ of X

$$\psi(t) = E(e^{t \cdot x}) = 1(1-p) + e^t \cdot p = \underline{e^t \cdot p + (1-p)}$$

$\hookrightarrow$ 1, e^t or

finite   infinite

Def (Bernoulli trials / process)

: $X_1, X_2, \cdots$ be a sequence of i.i.d. r-v.s.

→ (identically, independently, distributed)

$X_1, X_2, \cdots$ have the same distribution

mutually independent

, where each of X_i has the Bernoulli distribution with parameter p. Then the sequence is called Bernoulli trials (if the sequence is finite) or process (otherwise)

ex) Tossing a coin (10 times)

- tossed one-by-one, each trial is fair (i.i.d.)

r-v. $X_i = \begin{cases} 1 & \text{head is obtained on the } i\text{-th toss} \\ 0 & \text{tail} \quad \text{"} \end{cases}$

X_i has the Bernoulli distribution with parameter $1/2$.

$X_1, X_2, \cdots X_{10}$: Bernoulli trials with parameter $1/2$.

r.v. $X = X_1 + X_2 \cdots + X_{10}$

Q: What is the distribution of X?

- Binomial distribution

<u>Def</u> (Binomial distribution)

r.v. X has the Binomial distribution with parameters p, n. ($n > 0$, integer, $0 \le p \le 1$),
denoted as $X \sim B(n, p)$

$\Longleftrightarrow$ X is a discrete r.v. with the following p.f. f.

$$f(x \mid n, p) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x = 0 \ldots n \\ 0 & \text{otherwise} \end{cases}$$

<u>Thm</u> 5-2-1 $X_1 \cdots X_n$: Bernoulli trials with parameter p.

then $X_1 + X_2 \cdots + X_n \sim B(n, p)$

pf) $X = X_1 + X_2 \cdots + X_n$ $\boxed{\text{x of n r-vs should be 1.}}$

$$P(X=x) \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x=0 \cdots n. \\ 0 & \text{otherwise} \end{cases}$$

$\therefore X \sim B(n,p)$

$E(X) = E(X_1 + X_2 \cdots + X_n) = pn.$

$Var(X) = Var(X_1) + Var(X_2) \cdots + Var(X_n)$

$X_1 \cdots X_n$ are independent $= np(1-p)$

m.g.f. $\psi(t) = E(e^{tX}) = \prod_{i=1}^{n} (e^{tX_i}) = [e^t p + (1-p)]^n.$

Thm 5-2-2 $X_1 \cdots X_k$: independent

$\forall i, \; X_i \sim B(n_i, p).$

then $X_1 + X_2 + \cdots X_k \sim B(\sum_{i=1}^{k} n_i, p)$

pf

$$\left\{ \begin{array}{l} X_1 \\ \vdots \\ X_i \to \text{distribution of summation of } n_i \text{ Bernoulli r.v. (i.i.d) with parameter } p. \\ \vdots \\ X_k \end{array} \right.$$

$+$

$X_1 + X_2 \cdots + X_k = \sum_{i=1}^{k} n_i \quad // \quad \sim B(\sum_{i=1}^{k} n_i, p)$

french

5.4 The Poisson distribution

- store : there are 4.5 costomers per hour. (in avg)

- want to calculate P(x costomers come in hour)

⇒ assume that during each second, 0 or 1 costomer come.

r.v X_i = # costomers in i-th second

r.v $X = \sum_{i=1}^{3600} X_i$

$\sim B(3600, 0.00125)$ → $P(X=x)$

large; very small

⇒ approximate (n is large, p is small)

f = p.f. of X. $f(x) = f(x|n,p)$

$$\frac{f(x+1)}{f(x)} = \frac{\binom{n}{x+1} \cdot \boxed{p^{\cancel{x+1}} \cdot (1-p)^{n-(\cancel{x+1})}}}{\binom{n}{x} \cdot \boxed{\cancel{p^{x}} \cdot (1-p)^{\cancel{n-x}}}} = \frac{n-x}{x+1} \frac{p}{1-p}$$

$$\frac{\cancel{n\cdot(n-1)\cdots}(n-(x+1)+1)}{\cancel{(x+1)!}\ x+1} \qquad n-x$$

$$\frac{\cancel{n\cdot(n-1)\cdots}(n-x+1)}{\cancel{x!}}$$

$$\frac{n-x}{x+1}\cdot\frac{p}{1-p} \approx \frac{n}{x+1}\cdot\frac{p}{1} = \frac{n.p}{x+1}$$

$n.p \to \lambda$

$$f(0) = f(0)\,\frac{\lambda^0}{0!}$$

$$f(1) = f(0)\cdot\lambda \qquad \frac{\lambda}{1}$$

$$f(2) = f(1)\cdot\frac{\lambda}{2} = f(0)\cdot\lambda\cdot\frac{\lambda}{2}$$

$$= f(0)\cdot\frac{\lambda^2}{2!}$$

$$f(3) = f(2)\cdot\frac{\lambda}{3}$$

$$= f(0)\cdot\frac{\lambda^2\cdot\lambda}{2!\cdot 3} = f(0)\cdot\frac{\lambda^3}{3!}$$

$\vdots$

$+$

$$1 = f(0)\sum_{x=0}^{n}\frac{\lambda^x}{x!}$$

$$f(0) = \frac{1}{\boxed{\sum_{x=0}^{n}\frac{\lambda^x}{x!}}} \longrightarrow e^{-\lambda}$$

as $n\to\infty$ e^{λ}

$$f(x) = f(0)\cdot\frac{\lambda^x}{x!} = \begin{cases} e^{-\lambda}\cdot\dfrac{\lambda^x}{x!} & x=0,1\cdots \\ 0 & \text{otherwise.}\end{cases}$$

$\hookrightarrow$ p.f of Poisson distribution

Def (Poisson distribution)

for $\lambda > 0$ r.v. X has the Poisson distribution with parameter λ, denoted as
(mean)

$X \sim Pois(\lambda)$, $\longleftrightarrow$

the p.f. f of X is defined as

$$f(x \mid \lambda) = \begin{cases} e^{-\lambda} \cdot \dfrac{\lambda^x}{x!} & x = 0, 1, 2 \cdots \\ 0 & \text{otherwise} \end{cases}$$

Thm $X \sim Pois(\lambda)$

5-4-1 +5-4-2

i) $E(X) = \lambda$ ii) $Var(X) = \lambda$.

pf) i). $E(X) \underset{\text{LOTUS}}{=} \displaystyle\sum_{x=0}^{\infty} \cancel{x} \cdot e^{-\lambda} \cdot \frac{\lambda^x}{\cancel{x!}}$ $\lambda^x \leftarrow \lambda \cdot \lambda^{x-1}$, $(x-1)!$

$$= \lambda \sum_{x=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^{x-1}}{(x-1)!} \qquad Y = x-1$$

$$= \lambda \underbrace{\sum_{Y=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^Y}{Y!}}_{\Rightarrow 1} = \lambda.$$

ii) $E(X(X-1))$

$$\overset{\text{LOTUS}}{=} \sum_{x=0}^{\infty} \cancel{x(x-1)} \cdot e^{-\lambda} \frac{\lambda^{x}}{\cancel{x!}} \qquad \lambda^{x} \to \lambda^{2}\cdot\lambda^{x-2}, \quad (x-2)!$$

$$= \lambda^{2} \sum_{x=2}^{\infty} e^{-\lambda} \frac{\lambda^{x-2}}{(x-2)!} \qquad y = x-2$$

$$= \lambda^{2} \cdot \underbrace{\sum_{y=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^{y}}{y!}}_{\Rightarrow 1} = \lambda^{2}$$

$$E(X^{2}) - \underset{\to\lambda}{E(X)} = \lambda^{2}$$

$$E(X^{2}) = \lambda^{2} + \lambda$$

$$Var(X) = E(X^{2}) - [E(X)]^{2} = \lambda^{2} + \lambda - \lambda^{2}$$

$$= \lambda$$

Thm 5-4-3 $X \sim Poi(\lambda)$. Then

m.g.f. $\psi(t)$ of X is

$$\psi(t) = e^{\lambda(e^{t}-1)}$$

pf) $\psi(t) = E(e^{tX})$

$$\overset{\text{LOTUS}}{=} \sum_{x=0}^{\infty} e^{tx} \cdot e^{-\lambda} \cdot \frac{\lambda^{x}}{x!}$$

ce

$e^{tx} \cdot \lambda^{x}$

$= (e^{t}\cdot\lambda)^{x}$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} = e^{-\lambda} \cdot e^{\lambda \cdot e^t}$$

$$\underbrace{\sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}}_{e^{\lambda \cdot e^t}} \qquad = e^{\lambda(e^t - 1)}$$

Thm 5.4.4 r.v. $X_1 \cdots X_k$ = independent

$\forall i$, $X_i \sim Poi(\lambda_i)$

$$X_1 + X_2 + \cdots + X_k \sim Poi(\lambda_1 + \lambda_2 \cdots + \lambda_k)$$

pf) $\psi_i(t)$ = m.g.f of X_i

since $X_1 \cdots X_k$ are independent,

m.g.f. $\psi(t)$ of X ($X = \sum_{i=1}^{k} X_i$)

$$\psi(t) = \prod_{i=1}^{k} \psi_i(t)$$

$$= \prod_{i=1}^{k} e^{\lambda_i (e^t - 1)}$$

$$= e^{\sum_{i=1}^{k} \lambda_i (e^t - 1)}$$

$$= e^{(\lambda_1 + \lambda_2 \cdots \lambda_k)(e^t - 1)}$$

$\Downarrow$

m.g.f. $Poi(\lambda_1 + \lambda_2 \cdots + \lambda_k)$

$\therefore X_1 + X_2 \cdots + X_k \sim Poi(\lambda_1 + \lambda_2 \cdots + \lambda_k)$

- The Poisson approximation to Binomial distribution

when n is quite large, an p is quite small,

$X \sim B(n, p)$ can be approximated to $Poi(np)$

$\lim_{n\to\infty} n \cdot p_n$ is converge.

<u>Thm</u> 5-4-5. : n = positive integer $0 < p < 1$.

$f(x \mid n, p)$ = p.f of $B(n, p)$

$f(x \mid \lambda)$ = p.f of $Poi(\lambda)$

Let $p_1, p_2 \dots$ be a sequence of numbers between 0 and 1. s.t.

$\lim_{n\to\infty} n \cdot p_n = \lambda$ then

$$\lim_{n\to\infty} f(x \mid n, p_n) = f(x \mid \lambda).$$

$$\binom{n}{x} \cdot p_n^x \cdot (1-p_n)^{n-x}$$

pf) $$f(x \mid n, p_n) = \frac{n\cdot(n-1)\cdots(n-x+1)}{x!} \cdot p_n^x \cdot (1-p_n)^{n-x}$$

$\frac{\lambda_n}{n}$

$\lambda_n = n \cdot p_n \qquad \lim_{n\to\infty} \lambda_n = \lambda.$

$\because n-(n-x+1)+1$

$\Rightarrow x$ terms

$$= \boxed{\frac{n(n-1)\cdots(n-x+1)}{x!}} \cdot \left(\frac{\lambda_n}{n}\right)^x \cdot \left(1-\frac{\lambda_n}{n}\right)^{n-x}$$

$$= \frac{\lambda_n^x}{x!} \cdot \frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{n-x+1}{n} \cdot \frac{\left(1-\frac{\lambda_n}{n}\right)^n}{\left(1-\frac{\lambda_n}{n}\right)^x}$$

$$\underset{\to 1}{\frac{n}{n}} \cdot \underset{\to 1}{\frac{n-1}{n}} \cdots \underset{\to 1}{\frac{n-x+1}{n}} \to 1 \quad \text{as } n\to\infty$$

$$\left(1 - \frac{\lambda_n}{n}\right)^{-x} \to 1 \quad \text{as } n \to \infty$$

$\to 0$

$$\left(1 - \frac{\lambda_n}{n}\right)^n \to e^{-\lambda}$$

$$\lim_{n\to\infty} f(x \mid n, p_n) = \frac{\lambda^x}{x!} \cdot e^{-\lambda} = \frac{\lambda^x \cdot e^{-\lambda}}{x!}$$

$$= f(x \mid \lambda)$$

★★

5.6. The normal distribution

(Gaussian distribution)

<u>Def</u> r.v. X has the normal distribution with mean μ and variance σ^2, denote as

$$X \sim N(\mu, \sigma^2), \quad (-\infty < \mu < \infty \quad \sigma^2 \geq 0.)$$

iff X has a continuous distribution with p.d.f. f as

$$e^{\boxed{}}$$

$$f(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot \exp\left(-\frac{1}{2} \cdot \left(\frac{x-\mu}{\sigma}\right)^2\right)$$

$$-\infty < x < \infty$$

<u>Thm 5.6.1</u> $f(x \mid \mu, \sigma^2)$ is a p.d.f.

pf) $f(x \mid \mu, \sigma^2)$ non-negative ? trivial.

enough to prove

$$\int_{-\infty}^{\infty} f(x \mid \mu, \sigma^2) = 1$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) dx$$

$$y = \frac{x-\mu}{\sigma}$$

$$dy = \frac{1}{\sigma} dx$$

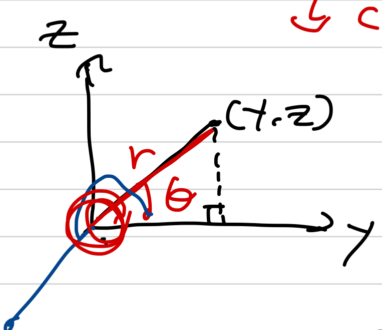
$$= \frac{1}{\sqrt{2\pi}} \cdot \underline{\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}y^2\right) dy} = \text{let } I$$

Let $I = \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}y^2\right) dy$

$$I^2 = \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}y^2\right) dy \cdot \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}z^2\right) dz$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}(y^2+z^2)\right) dy\, dz.$$

change to polar coordinate



$$y = r \cdot \cos\theta$$

$$z = r \sin\theta$$

$$r^2 = y^2 + z^2$$

$$J = \begin{bmatrix} \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$dxdy = \det(J)\, dr\, d\theta$$

$$\det(J) = r\cos^2\theta + r\sin^2\theta = r.$$

$$dxdy = r\, dr\, d\theta.$$

$$I^2 = \int_0^{2\pi}\int_0^{\infty} \exp\left(-\frac{1}{2}r^2\right) \underline{r\,dr}\, d\theta$$

$$\frac{1}{2}r^2 = v$$

$$r\,dr = dv.$$

$$= \int_0^{2\pi}\int_0^{\infty} e^{-v}\, dv\, d\theta = 2\pi.$$

$$-e^{-v}\Big|_0^{\infty}$$

$$\theta$$

$$I = \sqrt{2\pi}$$

$$\therefore \int_{-\infty}^{\infty} f(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}} \cdot \sqrt{2\pi} = 1.$$