p.d.f $\rightarrow$ c.d.f.
p.f $\leftarrow$

Thm 3.3.5 X : conti. r.v.
Let f and F be the p.d.f and c.d.f of X resp.

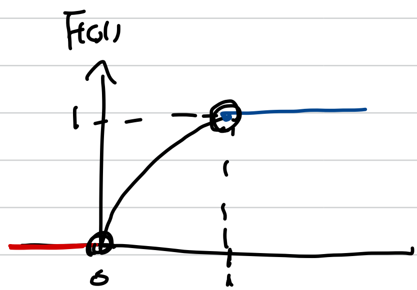
i) F is continuous for $\forall x \in \mathbb{R}$.

ii) $\frac{d}{dx}F(x) = f(x)$ $\boxed{\forall x \text{ where } f \text{ is continuous at } x.}$

(Fundamental thm of Calculus)
↳ FTC.

ex) c.d.f F of a r.v. X

$$F(x) = \begin{cases} 0 & x < 0 \\ x^{2/3} & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$



- F is differentiable $\forall x \in \mathbb{R} \setminus \{0, 1\}$

$$\therefore\ f(x) = \begin{cases} \frac{2}{3}x^{-\frac{1}{3}} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

0

(when $x = 0$ or 1 you can define $f(x)$ arbitrary)

Bivariate (multivariate)

3.4. Joint distribution

→ distribution of multiple r.v.s

Def X, Y = r.v.

The joint distribution of X and Y is a collection $P((X,Y) \in C)$ where $C \subseteq \mathbb{R} \times \mathbb{R}$.

ex) S = select 10 people from 20.

X = # selected people whose age are over 60 → 0 --- 10

Y = " who are healthy. → 0, 1, ---- 10

$P((X,Y) = (x,y))$ = # selected people where x are over 60 and y are healthy among them.

$(x, y \in \{0, 1, \dots, 10\})$

Def) X, Y = r.v.

X and Y have discrete joint distribution $\longleftrightarrow$ $\exists$ countable # possible values for (X, Y).

Thm 3.4.1 X, Y: discrete r.v. then X and Y have a discrete joint distribution.

Def Joint probability function (joint p.f.)

$\Rightarrow$ Joint p.f. of X and Y is a function s.t. $\forall (x,y) \in \underset{R \times R}{\underline{R^2}}$ $f(x,y) = P(X = x$ and $Y = y)$

Thm 3.4.2 X, Y: discrete r.v.

i) if (X, Y) cannot have a ordered pair (x, y) then $f(x,y) = 0$.

ii) $\displaystyle\sum_{\substack{\text{All possible} \\ (x,y)}} f(x,y) = 1$

iii) $P((x,y) \in \underline{C}) = \displaystyle\sum_{(x,y) \in C} f(x,y)$

set of ordered pairs $\subset R \times R$.

ex) Table of joint p.f. $f(x,y)$

$X=x$ \ $Y=y$	1	2	3	4
1	0.1	0	0.1	0
2	0.3	0	0.1	0.2
3	0	0.2	0	0.

$$P(X \geq 2, Y \geq 2) = 0.1 + 0.2 + 0.2 = 0.5.$$

- Continuous joint distribution.

Def X and Y have a continuous joint distribution if $\exists$ non-negative function f s.t.

$$\underline{P((X,Y) \in C \subset R^2)} = \iint_C f(x,y)\,dx\,dy$$

plain

f = joint probability density function (joint p.d.f.)

Thm Any joint p.d.f f satisfies the following

i) Every individual point in x-y plain has prob. 0. ($C = \{(x,y)\}$)

ii) when $C = \{(x,y) \mid y = f(x)\}$ or

(x,y) on the single line.

$C = \{(x,y) \mid x = f(y)\}$

then $\iint_C f(x,y)\,dx\,dy = 0$.

iii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

iv) $f(x,y) \geq 0 \quad \forall (x,y) \in R \times R$.

ex) X, Y = r.v.

suppose X is conti r.v. $X = Y$

$P((x,y) \mid x = y) = 1$

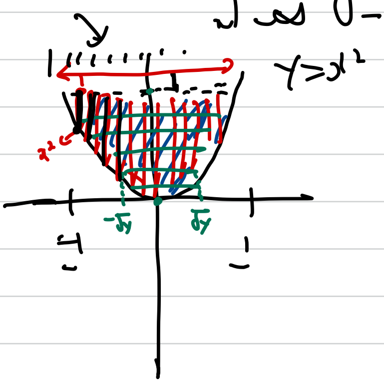
$\Rightarrow$ X and Y cannot have conti. joint. distribution.

ex) $(X, Y) \Rightarrow$ conti joint distribution with joint p.d.f f.

$$f(x,y) = \begin{cases} c x^2 y & \text{for } \underline{x^2 \le y \le 1} \\ \underline{0} & \text{otherwise.} \end{cases}$$

(i) c?

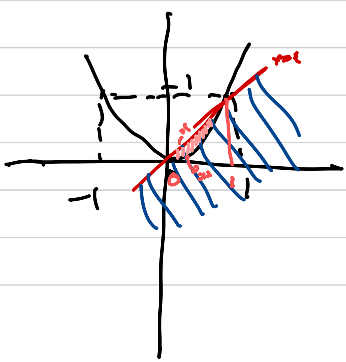
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$



$$\underline{\int_{-1}^{1}\int_{x^2}^{1} c x^2 y \, dy\, dx} = 1 \rightarrow c = \frac{21}{4}$$

$$\int_{0}^{1}\int_{-\sqrt{y}}^{\sqrt{y}} c x^2 y \, dx\, dy = 1$$

(ii) $P(\underline{X \ge Y})$



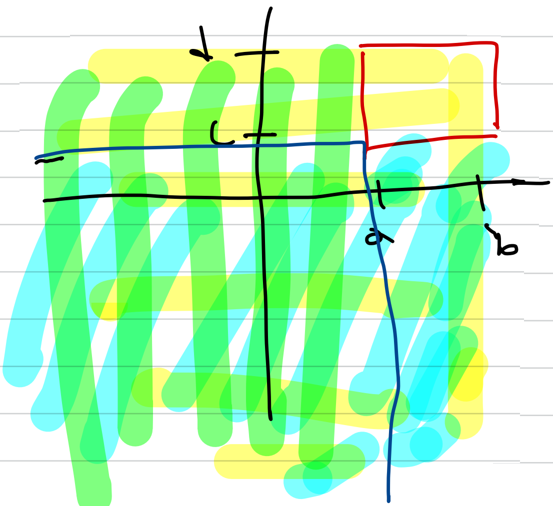
$$\int_{0}^{1}\int_{x^2}^{x} \frac{21}{4} x^2 y \, dy\, dx = \frac{3}{20}$$

* Joint cumulative distribution function
(joint c.d.f.)

<u>Def</u> The joint c.d.f. of two r.v.s X and Y is a function F s.t.

$\forall x, y \in \mathbb{R} \times \mathbb{R}$, $F(x, y) = P\{X \le x \text{ and } Y \le y\}$

<u>Cor</u> $P(a < X \le b \text{ and } c < Y \le d)$



$= F(b, d) - F(a, d) - F(b, c) + F(a, c)$

Thm 3.4.5 F = joint c.d.f. of r.v. X and Y.

$$F_1 \text{ (c.d.f of } X) = \lim_{y\to\infty} F(x,y)$$

↳ marginal c.d.f. of x

↳ 주변

$$F_2 \text{ (c.d.f. of } Y) = \lim_{x\to\infty} F(x,y)$$

Thm X, Y = conti r.v. with joint p.d.f. f.

then joint c.d.f $F(x,y)$ as

$$F(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(r,s)\, dr\, ds$$

↓

$X \le x$ $Y \le y$

Also if F is second order differentiable on (x,y)

$$f(x,y) = \frac{\partial F(x,y)}{\partial x\, \partial y}$$

ex) X, Y = conti r-v.

- X and Y only have values from 0 to 2

$0 \le X \le 2 \quad 0 \le Y \le 2$

- joint c-d-f F of X and Y is

$\frac{1}{16} x y^2$

$$F(x,y) \begin{cases} \frac{1}{16} xy(x+y) & 0 \le X, Y \le 2 \\ 0 & X < 0 \text{ or } Y < 0 \\ 1 & X, Y > 2 \end{cases}$$

i) $F_1(x) \begin{cases} \lim_{y \to \infty} F(x,y) = \lim_{y \to 2} F(x,y) = F(x,2) = \frac{1}{8} x(2+x) & 0 \le x \le 2 \\ 1 & x > 2 \\ 0 & x < 0. \end{cases}$

ii) $F_2(y) \begin{cases} \lim_{x \to \infty} F(x,y) = \frac{1}{8} y(2+y) & 0 \le y \le 2 \\ 1 & y > 2 \\ 0. & y < 0 \end{cases}$

(ii) joint p.d.f of X and Y

$$f(x,y) = \begin{cases} \frac{\partial}{\partial x \partial y} F(x,y) & 0 < x, y < 2 \\ = \frac{\partial}{\partial x}\left(\frac{1}{16}x^2 + \frac{1}{8}xy\right) & \\ = \frac{1}{8}x + \frac{1}{8}y & \\ 0 & \text{otherwise} \end{cases}$$

3.5 Marginal distribution

Def Suppose X and Y have a joint distribution with joint c.d.f / p.f / p.d.f)

Then marginal c.d.f / p.f / p.d.f is a c.d.f / p.f / p.d.f of X (or Y) derived from joint c.d.f / p.f. / p.d.f.

Thm 3.5-1 X, Y : discrete r.v. with joint p.f. f. then marginal p.f. f_1 of X

$$f_1(x) = \sum_{\forall y} f(x,y)$$

similarly, $f_2(y) = \sum_{\forall x} f(x,y)$

marginal p.f. f_2 of Y

ex) Table of $f(x,y)$

$X=x$ \ $Y=y$	1	2	3	4
1	0.1	0	0.1	0
2	0.3	0	0.1	0.2
3	0	0.2	0	0

$f_1(1) = 0.2$

$f_2(2) = 0.2$

Thm 3.5-2 X, Y : conti r.v with joint p.d.f f. then marginal p.d.f f_1 of X

$$f_1(x) = \int_{-\infty}^{\infty} f(x,y)\, dy \qquad -\infty < x < \infty$$

, the marginal p.d.f. f_2 of Y.

$$f_2(y) = \int_{-\infty}^{\infty} f(x,y)\,dx \qquad \forall -\infty < y < \infty$$

ex) Joint p.d.f. f of X and Y.

$$f(x,y) = \begin{cases} \frac{21}{4}x^2 y & x^2 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(i) marginal p.d.f. f_1 of X

$$f_1(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{x^2}^{1} \frac{21}{4}x^2 y\,dy$$

$$= \frac{21}{8}x^2 y^2 \Big|_{x^2}^{1} = \frac{21}{8}x^2(1 - x^4)$$

(ii) 〃 f_2 of Y

$$f_2(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{21}{4}x^2 y\,dx \qquad \frac{1}{3}x^3$$

$$= \frac{7}{4}x^3 y \Big|_{-\sqrt{y}}^{\sqrt{y}} = \frac{7}{4}y^{5/2} + \frac{7}{4}y^{\frac{5}{2}} \qquad y^{\frac{1}{2}}$$

$$= \frac{7}{2}y^{5/2}$$

- Independent random variables

<u>Def</u> Two r.v.s X and Y are independent if for any two sets $A, B \subset \mathbb{R}$:

$$P(X \in A \text{ and } Y \in B) = P(X \in A) \cdot P(Y \in B)$$

$F(x,y)$

ex) $P(X \leq x \text{ and } Y \leq y) = P(X \leq x) \cdot P(Y \leq y)$

X and Y are independent $\quad F_1(x) \quad F_2(y)$

<u>Thm 3.5-4</u> F = joint c.d.f of r.v.s X and Y

F_1 = marginal " " X

F_2 " " Y.

X and Y are independent $\longleftrightarrow$ $F(x,y) = F_1(x) \cdot F_2(y)$ $\forall x, y \in \mathbb{R}$.

Thm 3.5.5. f = joint p.f. / p.d.f of r.v. X and Y

f_1 = marginal " X

f_2 = " Y

X and Y are independent $\Longleftrightarrow$ $\forall x, y \in \mathbb{R}$

$$f(x,y) = f_1(x) \cdot f_2(y)$$

ex) X, Y = conti r.v. with p.d.f. g.

$$g(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and, X and Y are independent.

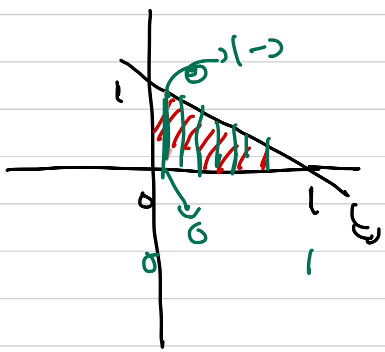
$P(X + Y \le 1)$?

$\Rightarrow$ joint p.d.f f of X and Y

$$f(x,y) = \begin{cases} 4xy & 0 \le x, y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

$P(X+Y\le 1)$

$Y \le 1-x$



$$\int_0^1\int_0^{1-x} 4xy\;dy\,dx$$

$$=\int_0^1 2xy^2\Big|_0^{1-x}\,dx$$

$$=\int_0^1 2x(1-x)^2\,dx$$

$$2x(1-2x+x^2)$$
$$=2x-4x^2+2x^3$$

$$=\int_0^1 2x-4x^2+2x^3\,dx$$

$$=x^2-\frac{4}{3}x^3+\frac{1}{2}x^4\Big|_0^1$$

$$=\underbrace{1-\frac{4}{3}}_{-\frac{1}{3}}+\frac{1}{2}=\frac{1}{6}.$$

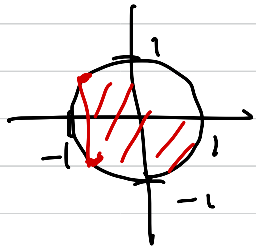
joint p.d.f f of X and Y

Ex) $$f(x,y) = \begin{cases} kx^2y^2 \ (k \neq 0) & x^2+y^2 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

X and Y are independent ?

$$f_1(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} kx^2y^2\,dy$$



$$= \frac{1}{3}kx^2y^3 \Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \quad {}^{3/2}$$

$$= \frac{2}{3}kx^2(1-x^2)^{3/2}$$

$$f_2(y) = \frac{2}{3}k^2y^2(1-y^2)^{3/2}$$

X and Y are not independent

$$\underbrace{f(x,y)}_{0} \neq \underbrace{f_1(x)}_{\neq 0} \cdot \underbrace{f_2(y)}_{\neq 0}$$

$(x,y) = (0.9, 0.9)$

$0.9^2 + 0.9^2 = 1.xx$

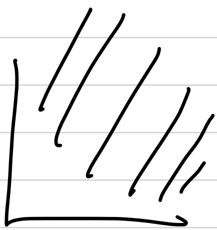
* Suppose $\{(x,y) \mid f(x,y)>0\}$ is a (unbounded) rectangular range, and

$$f(x,y) = \underline{h_1(x)} \cdot \underline{h_2(y)} \quad \text{"} \quad Y$$

↳ function only depends on x

then X and Y are independent.

ex) $$f(x,y) = \begin{cases} ke^{-(x+2y)} & x \ge 0,\ y \ge 0 \\ (k \neq 0) & \\ 0 & \end{cases}$$



k and $ke^{-(x+2y)}$

$$= \underbrace{ke^{-x}}_{h_1(x)} \cdot \underbrace{e^{-2y}}_{h_2(y)}$$

∴ X and Y are independent.

3.6. Conditional distributions

Def X, Y : discrete / conti r.v. with joint p.f / p.d.f. f.

f_2 : marginal p.f / p.d.f of Y.

then for $\forall y$ with $f_2(y) > 0$, $g_1(x|y) = \dfrac{f(x,y)}{f_2(y)}$ is conditional p.f. / p.d.f. of X given $Y=y$.

Similarly, $g_2(y|x) = \dfrac{f(x,y)}{f_1(x)}$ $\forall x$ $f_1(x) > 0$

conditional p.f / p.d.f of Y.

for a conditional p.f. $g_1(x|y)$

$$\sum_{\forall x} g_1(x|y) = 1 = \frac{\boxed{\sum_{\forall x} f(x,y)}}{f_2(y)} = \frac{f_2(y)}{f_2(y)} = 1$$

// p.d.f $g_1(x|y)$

$$\int_{-\infty}^{\infty} g_1(x|y)\,dx = 1.$$

ex)

$X=$ \ $Y=$	1	2	3	4
1	0.1	0	0.1	0
2	0.3	0	0.1	0.2
3	0	0.2	0	0

$$g_1(x|2) = \frac{f(x,2)}{f_2(2)} = \frac{f(x,2)}{0.2}$$

$$g_1(1|2) = \frac{0}{0.2} = 0$$

$$g_1(3|2) = \frac{0.2}{0.2} = 1$$

ex) X, Y = conti r.v. with joint p.d.f f

$$f(x,y) = \begin{cases} e^{-x} & 0 \le y \le x < \infty \\ 0 & \text{otherwise.} \end{cases}$$

for $y>0$, $g_1(x|y)$

$$g_1(x|y) = \frac{f(x,y)}{f_2(y)} \quad \forall y \text{ s.t. } f_2(y) > 0$$

$$f_2(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_y^{\infty} e^{-x}\,dx$$

$$= -e^{-x}\Big|_y^{\infty}$$

$$= e^{-y}.$$

$$f_2(y) \begin{cases} e^{-y} & \underline{y \geq 0} \\ \underline{0} & \text{otherwise} \end{cases}$$

thus, for each $y \geq 0$,

$$g_1(x|y) = \frac{e^{-x}}{e^{-y}} = \begin{cases} e^{y-x} & y \leq x \\ 0 & \text{otherwise} \end{cases}$$

<u>ex)</u> X = r.v. from the uniform distribution on the interval $[0, 1]$

- after the value $X = x$ $(0 < x < 1)$ is chosen, Let Y = r.v. from the uniform distribution on the intervals $[x, 1]$

marginal p.d.f f_2 of Y?

$\Rightarrow$ marginal p.d.f f_1 of X

$$f_1(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

when $0<x<1$, $\underline{g_2(y|x)}$ $= \begin{cases} \dfrac{1}{1-x} & \underline{0 < x < y < 1} \\ 0 & \text{otherwise} \end{cases}$

$$g_2(y|x) = \frac{f(x,y)}{f_1(x)}$$

$$f(x,y) = g_2(y|x) \cdot f_1(x)$$

$$= \begin{cases} \underline{\dfrac{1}{1-x}} & 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_2(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^y \frac{1}{1-x}\,dx$$

$$= -\ln(1-x)\Big|_0^y = \underline{-\ln(1-y)}$$

$$f_2(y) = \begin{cases} -\ln(1-y) & 0<y<1 \\ 0 & \text{otherwise.} \end{cases}$$

Thm $X, Y =$ r.v.

X and Y are independent

$\longleftrightarrow$ $\forall y$ s.t. $f_2(y) > 0$ and $\forall x$

$$g_1(x|y) = f_1(x)$$

$$\hookrightarrow = \frac{f(x,y)}{f_2(y)} = \frac{f_1(x)\cdot\cancel{f_2(y)}}{\cancel{f_2(y)}} = f_1(x)$$

Similarly, $\forall x$ s.t. $f_1(x) > 0$, $\forall y$

$$g_2(y|x) = f_2(y)$$

//