Chap 1. Introduction to probability

* Experiment : the possible outputs $\boxed{\text{can be identified}}$ after the experiment (real or hypothesis)
→ 가정

progress

ex) <u>throw a dice</u>

1
2
3
4
5
6

→ output of the experiment.

Def (Sample space) : A set of all possible outputs of an experiment.

$$S = \{1, 2, 3, 4, 5, 6\}$$

Def (Event) : $\boxed{\text{Well-defined}}$ set of possible outputs of an experiment

→ subset of S.

$2n \quad \forall n \in \mathbb{Z}$

E : even number is obtained.

odd

$$\Rightarrow \{2, 4, 6\}.$$

Cor. Sample space is an event.

1.4 set theory. (operations & relations)

1. · subset ($\subset$)

$A \subset B \rightarrow$ A is a subset of B.

$\rightarrow$ B contains A

$\forall a \in A$, $a \in B$

Thm 1-4-1, $\forall$ set (event) A, B, C. , S = sample space.

i) $A \subset S$ ii) if $A \subset B$. and $B \subset C$, $A \subset C$. (transitivity)

iii) $A \subset B$ and $B \subset A$ then $A = B$. $\rightleftarrows$

ex) A : even number is obtained

B : a number greater than 1 is obtained

$A = \{2, 4, 6\}$, $B = \{2, 3, 4, 5, 6\}$

$\Rightarrow A \subset B$.

2. Empty set ($\emptyset$)

$E = \emptyset \longrightarrow$ event E cannot occur

ex) C : a number > 7 is obtained $\rightarrow C = \emptyset$.

여

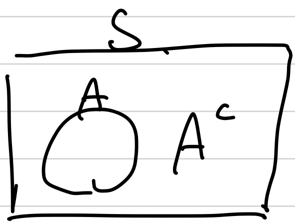
i) Complement (A^c)

A = event A^c = the event that A does not occur.

ex) A = even number is obtained ⇒ $A = \{2, 4, 6\}$

A^c = " not " ⇒ $A^c = \{1, 3, 5\}$

Venn diagram



Thm 1-4-3   A = event $(A^c)^c = A$ $\phi^c = S$

$S^c = \phi$

ii) Union

A, B = event

$A \cup B$: Event either A or B or both A and B occur.

* union of n sets $A_1, \cdots, A_n$

$\rightarrow \bigcup_{i=1}^{n} A_i$ $\left(\bigcup_{i=1}^{\infty} A_i \right)$

cup.

Thm 1-4-4. A, B = event

i) $A \cup B = B \cup A$ ii) $A \cup A = A$ iii) $A \cup A^c = S$

iv) $A \cup \emptyset = A$ v) $A \cup S = S$ vi) if $A \subset B$ then $A \cup B = B$.

Thm associative property. ; A, B, C = event

$$A \cup B \cup C = (A \cup B) \cup C = A \cup (B \cup C)$$

iii) Intersection : A, B : event

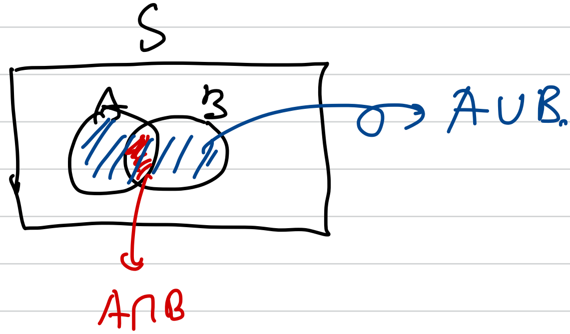
$A \cap B$ = Event that both A and B are occured.

ex) A = even is obtained → $A = \{2, 4, 6\}$

B : < 4 is obtained → $B = \{1, 2, 3\}$

$A \cup B = \{1, 2, 3, 4, 6\}$

$A \cap B = \{2\}$.



* Intersection of sets $A_1 \cdots, A_n$.

$\cap$ ⇒ $\bigcap_{i=1}^{n} A_i$ $\left(\bigcap_{i=1}^{\infty} A_i\right)$

cap

<u>Thm</u> A, B : event

i) $A \cap B = B \cap A$ ii) $A \cap A = A$. iii) $A \cap A^c = \emptyset$.

iv) $A \cap \emptyset = \emptyset$ v) $A \cap S = A$

<u>Def</u> A and B are disjoint $\leftrightarrow A \cap B = \emptyset$.

$A_1 \cdots, A_n$ are mutually disjoint

$\leftrightarrow \forall i, j, \ A_i \cap A_j = \emptyset$.

ex) Tossing a coin : Coin is tossed 3 times

H : head T = tail

$S = \{ \overset{s_1}{HHH}, \overset{s_2}{THH}, \overset{s_3}{HTH}, \overset{s_4}{HHT}, \overset{s_5}{HTT}, \overset{s_6}{THT},$
$\overset{s_7}{TTH}, \overset{s_8}{TTT} \}$

A : at least one head is obtained

$= \{ s_1, s_2, s_3, s_4, s_5, s_6, s_7 \}$

B : a head is obtained on the second toss

$= \{ s_1, s_2, s_4, s_6 \}$..

C = tail is obtained on the 3rd toss

$\Rightarrow \{S_4, S_5, S_6, S_8\}$

i) $B \subset A$ ii) $A^c = \{S_8\}$ iii) $B \cap C = \{S_4, S_6\}$

iv) $B \cup C = \{S_1, S_2, S_4, S_5, S_6, S_8\}$

v) $A \cap (B \cup C) = \{S_1, S_2, S_4, S_5, S_6\}$

* additional properties of sets.

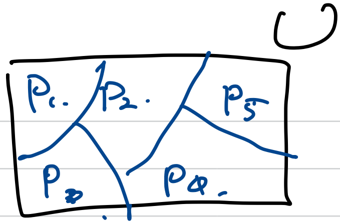
i) De Morgan's law.

- $(A \cap B)^c = A^c \cup B^c$
- $(A \cup B)^c = A^c \cap B^c$

ii) Distributive property.

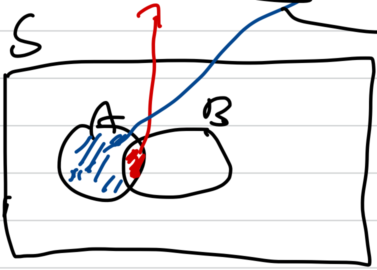
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

※ iii) Partitioning a set



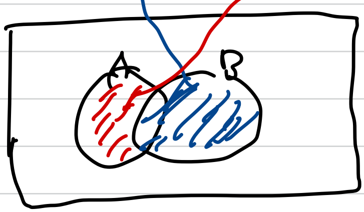
$\bigcup P_i = U$
$\forall i, j,\ P_i \cap P_j = \phi$

- $A = (A \cap B) \cup (A \cap B^c)$



partition of A.

- $A \cup B = B \cup (A \cap B^c)$



partition of $A \cup B$.

1.5. The definition of probability

length, S: sample space E = event.

$P(E) \Rightarrow$ a value (that must satisfy

measure. the axioms as follows:

i) $\forall A \subset S,\ P(A) \geq 0.$

ii) $P(S) = 1$

$\sigma \to$ 1-1 cor to $\mathbb{N}$. — $\mathbb{N}$, $\mathbb{Z}$, Rational, X irrational, X Real.

iii) for every countable infinite sequence of mutually disjoint events $A_1, A_2, \cdots$,

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

X sigma.

* iii) is also satisfied for any finite collection of events $A_1 \cdots A_n$ if they are mutually disjoint

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

Def) S = sample space

Probability (probability measure) on S

$\Rightarrow P(A)$ for $\forall$ events A, satisfies axiom i), ii), and iii)

Thm 1-5-1 $P(\emptyset) = 0$

pf) consider countable infinite collection of sets $A_1, A_2 \cdots$ s.t. $\forall i$, $A_i = \emptyset$.

$\forall i, j$, $A_i \cap A_j = \emptyset$. $\Rightarrow$ mutually disjoint.

Thus, by axiom iii),

$$P(\emptyset) = P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) = \sum_{i=1}^{\infty} P(\emptyset)$$

* the only number where the infinite sum of that is simply the number. $\Rightarrow 0$.

identity of '+'

* other properties (try yourself!)

i) $P(A^c) = 1 - P(A)$

ii) if $A \subset B$ $P(A) \leq P(B)$.

iii) $0 \leq P(A) \leq 1$

iv) $P(A \cap B^c) = P(A) - P(A \cap B)$ $\longrightarrow$

Thm 1.5.1. A, B : event

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

pf). $A \cup B = \underbrace{B \cup (A \cap B^c)}_{\text{disjoint}}$

by axiom iii), $P(A \cup B) = P(B) + P(A \cap B^c)$

$= P(B) + P(A) - P(A \cap B)$.

ex) A student who likes candy or chocolate.

S

$P(\{\text{candy}\}) = 0.3$

$P(\{\text{chocolate}\}) = 0.8$

$P(\{\text{candy \& chocolate}\}) = ?$ $\boxed{0.1}$

$$\underbrace{P(A \cup B)}_{\substack{= P(S) \\ = 1 \text{ by axiom ii)} \\ \therefore}} = \underbrace{P(A)}_{0.3} + \underbrace{P(B)}_{0.8} - P(A \cap B) \Rightarrow \boxed{0.1}$$

<u>Thm</u> 1.5.8 (Bonferroni inequality)

$\forall$ event $A_1 \cdots A_n$ ($\times$ condition on disjointness)

$$P\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{i=1}^{n} P(A_i)$$

<u>pf)</u> induction on n.

* Probability is 0 $\xleftarrow{O}$ $\xrightarrow[X]{}$ impossible