Q1. $0 \le P(A) \le 1$

i) $0 \le P(A)$ → by axiom i)

ii) $P(A) \le P(S)$

$S = \underbrace{A \cup A^c}$ → partition of S

by axiom iii)

$1 = P(S) = P(A) + P(A^c)$

↳ by axiom ii)

since $P(A^c) \ge 0$ (by axiom i))

$1 = P(S) \ge P(A)$

Q2. $\binom{n}{r} + \binom{n-1}{r-1} + \binom{n-1}{r}$

① .. $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$ ③ .. $\binom{n-1}{r} = \dfrac{(n-1)!}{r!(n-r-1)!}$

② .. $\binom{n-1}{r-1} = \dfrac{(n-1)!}{(r-1)!(n-r)!}$

② + ③ $= \dfrac{r(n-1)! + (n-r)(n-1)!}{r!(n-r)!}$

$= \dfrac{(n-1)!(\cancel{r} + n - \cancel{r})}{r!(n-r)!} = \dfrac{n!}{r!(n-r)!} = ①$.

ex) Tossing coins - toss one at a time.

- outcomes are mutually independent. ←

- toss until a tail appears

event A = the game ends after n-th toss

$P(A) = (1 \text{ --- } (n-1)\text{th tail}) \times (n\text{-th head})$

$= \left(\frac{1}{2}\right)^{n-1} \cdot \left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^{n}$

event B = a tail appears sooner or later

$P(B) = P(\text{T at the 1st toss}) + P(\text{T at the 2nd toss})$

$+ \text{ - - - - } +$

$= \sum_{i=1}^{\infty} P(\underline{\text{T appears at the } i\text{-th toss}})$

$= \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^{i} = 1$ ←

2-3 Bayes' theorem.

Event B_1 -- B_k = partition of S s.t.

$\forall i, \; P(B_i) > 0$

Suppose for an event A, $P(\underline{A} | B_1)$ --- $P(A|B_k)$

are given. $(P(A) > 0)$

Goal = $P(B_1|A), P(B_2|A) \cdots, P(B_K|A)$

<u>Thm</u> (Bayes' theorem)

$B_1 \cdots B_K$ = partition of S s.t. $\forall i$ $P(B_i) > 0$

A = an event s.t. $P(A) > 0$

then $\forall i \in \{1 \cdots K\}$

$$P(B_i|A) = \frac{P(B_i \cap A)}{P(A)} = \frac{P(A|B_i) \cdot P(B_i)}{\underline{P(A)}}$$

$$= \frac{P(A|B_i) \cdot P(B_i)}{P\left(\bigcup_{l=1}^{K} (A \cap B_l)\right)} = \frac{P(A|B_i) \cdot P(B_i)}{\sum_{l=1}^{K} P(A \cap B_l)}$$

$$= \boxed{\frac{P(A|B_i) \cdot P(B_i)}{\sum_{l=1}^{K} P(A|B_l) \cdot P(B_l)}}$$

<u>Ex</u> an item I.

one of 3 diff. machines (m_1, m_2, m_3)

produces I.

i) prob. I is produced by $m_1 = 0.2$
ii) " $m_2 = 0.3$
iii) " $m_3 = 0.5$

iv) prob. I produced by m_1 is defective $= 0.01$
v) " m_2 " $= 0.02$
vi) " m_3 " $= 0.03$

Q = prob. if I is defective, it is produced by m_2.

* Define events properly! *

M_1 = I is produced by m_1
M_2 = " m_2
M_3 = " m_3

A = I is defective.

$P(M_1) = 0.2 \quad P(M_2) = 0.3 \quad P(M_3) = 0.5$

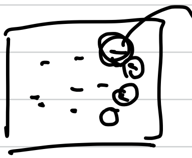
$P(A \mid M_1) = 0.01 \quad P(A \mid M_2) = 0.02$

$P(A \mid M_3) = 0.03$

$$P(M_2 \mid A) = \frac{P(M_2 \cap A)}{P(A)} = \frac{P(A \mid M_2) \cdot P(M_2)}{\sum_{i=1}^{3} P(A \mid M_i) \cdot P(M_i)}$$

$$= \frac{0.3 \times 0.02}{0.2 \times 0.01 + 0.3 \times 0.02 + 0.03 \times 0.5} = 0.26.$$

Chap3. Random variables and distributions



S

$\underline{X} \longrightarrow$ value. (real-value)

function

Def) A random variable (r.v.) is a real-valued function defined on S

ex) $\underline{X}$ = # heads after tossing 10 coins (experiment)

$$|S| = 2^{10}$$

$$X(\underline{HHHTTHTTHH}) = 6.$$

$$\underline{X=6}$$

* X = r.v.

$C \subset \mathbb{R}.$

$$P(X \in C) = P(\underbrace{\{s : X(s) \in C\}}_{\text{event}})$$

Def distribution :

X = r.v.

Distribution of X = Collection of all probabilities

of the form $P(X \in C)$

(where $\{s : X(s) \in C\}$ is an event)

ex) Tossing a coin 10 times

X = # heads after tossing

distribution of X?

$P(X \in C) = P(X = x)$

↳ integers from 0 to 10.

$$P(X = x) = \begin{cases} \dfrac{\binom{10}{x}}{2^{10}} & \text{when } x = \text{integer } 0 \ldots 10 \\ 0 & \text{otherwise} \end{cases}$$

Def (discrete distribution)

⇒ X has a discrete distribution

if X can have countable values

$x_1, x_2, \ldots$

질량

Def (probability (mass) functions) → p.f.

X = discrete r.v.

p.f. of X = a function f. s.t.

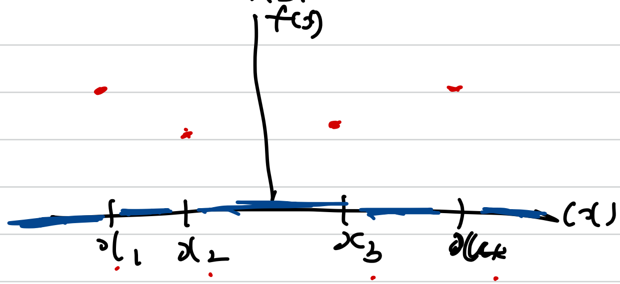
$$\underline{\forall x \in \mathbb{R}} \quad f(x) = P(X = x)$$

Thm. 3.1-1 X = discrete r.v. with p.f. f.

i) if x is not possible value of X

$$f(x) = 0.$$

ii) If $x_1, x_2, \ldots$ include all possible values of X, $\displaystyle\sum_{i=1}^{\infty} f(x_i) = 1$



Thm 3.1-2

$C \subseteq \mathbb{R}$.

X = r.v. with p.f. f.

$$P(X \in C) = \sum_{x_i \in C} f(x_i) \text{ where } f(x_i) > 0.$$

* Some random variables have distributions appear so frequently that the distributions have given names

* Uniform distribution (on integer)

ex) Daily number.

i) pick 3 digits i_1, i_2, and i_3 from 0 to 9. (Chosen at random.)

randomly chosen. ↓.

↪ Simple sample space.

ii) Daily number $= 100i_1 + 10i_2 + i_3$

ex) $i_1 = 0 \quad i_2 = 5 \quad i_3 = 3 \rightarrow 53$

X = daily number.

$P(X = x) = 0.001$ for all integer $x = 0 \cdots 999$.

Def (uniform distribution on integers)

a, b: integer $(a \leq b)$

suppose r.v. X is equally likely to be each of integers a ... b, then

X has an uniform distribution on the integers from a to b

Thm X = uniform distribution on the integers a ... b.

then the p.f. f of X is

$$f(x) = \begin{cases} \dfrac{1}{b-a+1} & x = a \dots b, \text{ integer} \\ 0 & \text{otherwise} \end{cases}$$

$b-a+1$

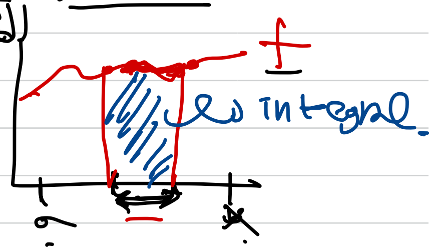
→ choose 1 position randomly (uniformly random)

⇒ prob. position i is picked

→ $\frac{1}{n}$

3.2 continuous distribution

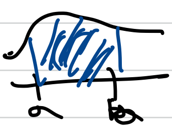
↳ r.v. can have any real value in an interval $[a,b]$
(a, b can unbounded)



Def r.v. X has a continuous distribution if $\exists$ non-negative function f s.t.

$\forall x \in \mathbb{R},\ f(x) \geq 0$

$\forall$ interval $[a,b]$



$$P(a \leq X \leq b) = \int_a^b f(x)\,dx$$

$$P(X \leq a) = \int_{-\infty}^a f(x)\,dx$$

$$P(b \leq X) = \int_b^{\infty} f(x)\,dx$$

In this case, f is called probability density function (p.d.f.)

↳ 밀도

※ Any p.d.f. f satisfies the following:

i) $\forall x \in \mathbb{R}, \quad f(x) \geq 0.$

ii) $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$

ex) X = conti. r.v.

f = p.d.f. of X

$$P(X=a) = \int_a^a f(x)\,dx = 0.$$

ex) X = conti. r.v. with p.d.f. f as

$$f(x) \begin{cases} cx & 0 < x < 4 \\ 0 & \text{otherwise.} \end{cases}$$

i) c? $\Rightarrow \displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$= \underbrace{\int_{-\infty}^{0} 0\,dx}_{0} + \int_0^4 cx\,dx + \underbrace{\int_4^{\infty} 0\,dx}_{0}$$

$$= \frac{1}{2}cx^2 \Big|_0^4$$

$8c = 1$

$c = \frac{1}{8}.$

ii) $P(X \geq 2) = \int_2^\infty f(x)\,dx$

$= \int_2^4 \frac{1}{8} x\,dx + \underbrace{\int_4^\infty 0\,dx}_{0}$

$= \frac{1}{16} x^2 \Big|_2^4 = 1 - \frac{1}{4} = \frac{3}{4}.$

* Uniform distribution on intervals.

ex) Time for watching Youtube per week in the range $[0, 80]$ 8.0

- assume that the prob. of watching time is in the subinterval $[a, b]$ is proportional to the length of the subinterval

$[10, 30]$

$[10, 50]$

Def a, b = two real values s.t $a < b$

X = conti. r.v. defined on $a \leq X \leq b$.

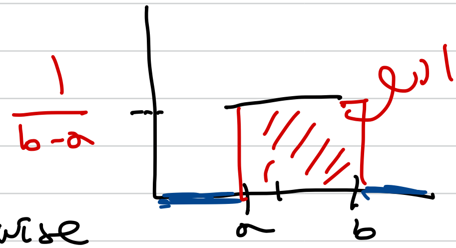
For any subinterval of $[a, b]$, if the prob. of X is in the subinterval is proportional to the length of the subinterval.

then X has an uniform distribution on the interval $[a,b]$.

Thm 3.2-1 X = uniform distribution on the interval $[a,b]$.

then p.d.f. f of X is

$$f(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$



ex) prob. that watching Youtube $[20, 50]$ minutes per week

$$\int_{20}^{50} \frac{1}{80}\,dx = \frac{1}{80}x \Big|_{20}^{50} = \frac{3}{8}.$$

★ Cumulative distribution function.

Def (cumulative distribution function) → c.d.f.
↳ 누적.

The c.d.f. F of a random variable X is a function s.t.

$$F(x) = P(X \le x) \quad \forall\, -\infty < x < \infty$$

ex) Bernoulli distribution

X has a Bernoulli distribution with parameter p.

$\Leftrightarrow P(X=0) = 1-p$

$P(X=1) = p$.

c.d.f. $F(x)$ of X.

$$F(x) = \begin{cases} 0 & x<0 \\ 1-p & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

$\boxed{P(X \le x)}$

$P(X \le 1)$

ex) X = conti. r.v. with p.d.f. f as

$$f(x) = \begin{cases} 0 & x \le 0 \\ \dfrac{1}{(1+x)^2} & x > 0. \end{cases}$$

c.d.f. $F(x) = P(X \le x)$

$$= \int_{-\infty}^{x} f(x)\,dx$$

$$= \begin{cases} 0 & x \le 0 \\ \int_0^x \frac{1}{(1+x)^2}\,dx & x > 0 \end{cases}$$

$(1+x)^{-2}$

$$= -1(1+x)^{-1} \Big|_0^x$$

$$= -\frac{1}{1+x} - (-1) = 1 - \frac{1}{1+x}$$

$P(X \le 3) = F(3) = 1 - \frac{1}{4} = \frac{3}{4}$

* properties of c.d.f F.

i) $P(X \le x) \le P(X \le x + \varepsilon)$

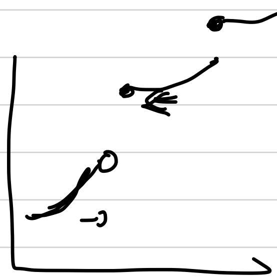
$\Rightarrow$ F is non-decreasing function

i.e. $\forall x_1, x_2$ if $x_1 \le x_2$ $F(x_1) \le F(x_2)$

ii) $\lim\limits_{x\to-\infty} F(x) = 0$ $\quad \lim\limits_{x\to\infty} F(x) = 1$

iii) F is always right continuous

F



$$F(x) = \lim_{\substack{y\to x \\ y>x}} F(y)$$

ex) r.v. X whose c.d.f. F is

$$F(x) = \begin{cases} 0 & x<0 \\ 1-\dfrac{1}{1+x^2} & x\ge 0. \end{cases}$$

i) $P(\underline{2<X\le 4})$

$\overline{15}$

$A = 2<X\le 4$

$B = X\le 2$

$C = \underline{X\le 4}.$

$\underline{P(A)}+P(B) = P(C)$

$P(A) = P(C) - P(B)$

$= F(4) - F(2)$

$= \frac{4}{5} - \frac{2}{3} = \frac{2}{15}$

Thm $\forall x \; P(X > x) = 1 - F(x)$

Thm $\forall x_1 < x_2, \; P(x_1 < X \leq x_2) = F(x_2) - F(x_1)$