1.6 Finite sample space

$\Leftrightarrow |S| = n \quad S = \{s_1 \cdots s_n\}$

Def) simple sample space S

$\Leftrightarrow$ S is finite &.

$P(s_1 \text{ occurs}) = P(s_2 \text{ occurs}) \cdots = P(s_n \text{ occurs})$

$= \frac{1}{n}$

A = an event contains exactly m outcome.

$$P(A) = \frac{m}{n}$$

ex) tossing 3 coins. (H, T)

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

If S is a simple sample space

$P(HHH) = P(HHT) \cdots \cdots = P(TTT) = \frac{1}{8}$ //

P (obtain H on the 1st toss)

$$= \frac{4}{8} = \frac{1}{2}$$

1.7 counting methods

Thm 1.7.1 multiplication rule

Suppose an experiment consists of k steps

$$\underbrace{\square \to \square \to \square \to \square}_{k}$$

1st step $\to n_1$
2nd " $\to n_2$
$\vdots$
kth step $\to n_k$

} possible outcomes.

$S = \{ (u_1 \cdots u_k) \mid u_i \text{ is an outcome of } i\text{th step} \}$

$|S| = n_1 \cdot n_2 \cdot \cdots \cdot n_k$ //

ex) Rolling two dice.

$\underset{n_1=6}{\square} \to \underset{n_2=6}{\square}$ $\Rightarrow 6 \times 6 = 36$ possible outcomes.

ex) Combination lock.

password: 4 digits, 0 – 9.

$\underset{n_1=10}{\square} \to \underset{n_2=10}{\square} \to \underset{n_3=10}{\square} \to \underset{n_4=10}{\square}$

$\Rightarrow 10^4$.

Permutations

1 2 3

1 2
2 1

Def) Permutation of k from n.

$\Rightarrow$ $\exists$ set consist of n elements

\# possible ways to choose k elements from the set,

and order them (replacement is not allowed)

$\square \to \square \to \square \;--\; \to \square$

$n_1 = n \quad n_2 = n-1 \quad n_3 = n-2 \quad \cdots \quad n_k = n-k+1$

$\Rightarrow n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)$

if $k=n \Rightarrow n \cdot (n-1) \cdot (n-2) \cdots 1 = n!$

ex) 25 students, select 1 president of the class,

" 1 vice-president "

\# possible cases to select

- ordered O
- replacement. X

$25 \cdot 24 = 600$.

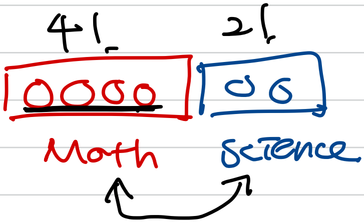
ex) 6 diff. books, # cases to arrange them on a shelf.

- ordering O
- replacement X

$6! = 720.$

ex) 4 diff math books and 2 diff science books.

(each of math and science books should be arranged consecutively)



$\Rightarrow 4! \cdot 2! \cdot 2! = 96.$

* Combinations

→ Suppose ∃ set S of n elements

\# possible subsets of S of size k.

$$= \frac{n \cdot (n-1) \cdots (n-k+1)}{k!} = \frac{n!}{(n-k)!\,k!}$$

$$= \binom{n}{k}$$

↳ n choose k.

ex) 25 students, select two representatives

- ordering X
- replacement X

$\binom{25}{2} = 300$

ex) Tossing 10 coins, the probability of 3 of them are head.

$\hookrightarrow A$

$|S| = 2^{10}$ $|A| = \binom{10}{3}$

- ordering X
- replacement X

$$P(A) = \frac{\binom{10}{3}}{2^{10}}$$

ex) 10 Men and 20 Women.

selecting 10 people from these 30 people.

The probability at most 3 men are selected.

$\frown A$

S = selecting 10 people from 30 $\Rightarrow |S| = \binom{30}{10}$

- A_0 = among 10 selected people, $\exists$ no man
- A_1 = " $\exists$ 1 man
- A_2 = " $\exists$ 2 men
- A_3 = " $\exists$ 3 men.

$$P(A) = P(A_0) + P(A_1) + P(A_2) + P(A_3)$$

$$|A_0| = \binom{20}{10}$$

$$|A_3| = \binom{20}{7} \cdot \binom{10}{3}$$

$$|A_1| = \binom{20}{9} \cdot \binom{10}{1}$$

$$|A_2| = \binom{20}{8} \cdot \binom{10}{2}$$

$$P(A_i) = \frac{|A_i|}{|S|}$$

Section 2. Conditional probability.

ex) Lotto. $\rightarrow$ select 6 numbers from 1-30

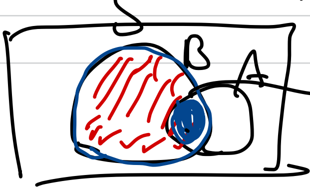
- ordering X
- replacement X

event A : 1, 14, 15, 20, 23, 27 are selected

" B : 15 is selected

problem : If we know B has occured,

prob. A occurs. ← condition



$$\frac{P(A \cap B)}{P(B)}$$

Def. (conditional probability)

⇒ conditional probability of the event A, given that the event B has occured.

$$P(A|B) = \frac{P(A\cap B)}{P(B)} \quad (\underline{\text{if } P(B)>0})$$

<u>ex)</u> $P(B) = \dfrac{|B|}{\binom{30}{6}} = \dfrac{\binom{29}{5}}{\binom{30}{6}} = \dfrac{1}{5}$

$$P(A\cap B) = P(A) = \frac{1}{\binom{30}{6}} = 1.68\times 10^{-6}.$$

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = 8.4\times 10^{-6}.$$

ex) Rolling two dice

$S = \{(a,b) \mid a = \text{1st outcome}, b = \text{2nd outcome}\}$

$A = \{(a,b) \mid a+b < 8\}$

$B = \{(a,b) \mid a+b \text{ is odd}\}$

$P\{A|B\}.$

$P(B)$?

- 3 → (1,2), (2,1) 2.
- 5 → (1,4), (2,3), (3,2), (4,1) 4.
- 7 → (1,6), (2,5) - - - (6,1) 6.
- 9 → (3,6), (4,5), (5,4), (6,3) 4
- 11 → (5,6), (6,5) 2

$P(B) = \frac{1}{2}$ $\quad P(A \cap B) = \frac{12}{36} = \frac{1}{3}$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}.$$

* Multiplication rule for conditional probability.

Thm 2.1.1 A, B : event

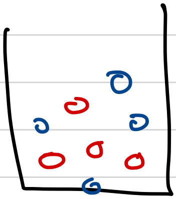
i) If $\underline{P(B) > 0}$ $\Rightarrow P(A \cap B) = P(A|B) \cdot P(B)$

$$P(A|B) = \frac{P(A|B)}{P(B)}$$

ii) If $\underline{P(A) > 0}$ $\Rightarrow P(\underline{A} \cap B) = P(B|A) \cdot P(A)$

ex) Selecting two balls from the box

(box contains r red balls and b blue balls)



E = 1st selected ball is red. and the 2nd " is blue. (without replacement)

① → A

② → B

$$P(E) = \underline{P(A \cap B)}$$

$$P(A) = \frac{r}{r+b}$$

$$\underline{P(B)} = ? \qquad P(B \mid \underline{A}) = \frac{b}{r+b-1}$$

↳ affected by A.

$$P(A \cap B) = P(A) \cdot P(B \mid A) = \frac{r}{r+b} \cdot \frac{b}{r+b-1}$$

Thm 2.1.2 (Generalization form)

$A_1 \, - - \, A_n$: events s.t $\forall i \; P(A_i) > 0$.

$$P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1) \cdot P(A_2 \mid A_1) \cdot P(\underline{A_3} \mid A_1 \cap A_2)$$

$$\cdots \; P(A_n \mid \underline{A_1 \cap A_2 \, - - \cap A_{n-1}}) \; - - ①$$

pf) ① $= P(\cancel{A_1}) \cdot \dfrac{P(A_1 \cap \cancel{A_2})}{P(\cancel{A_1})} \cdot \dfrac{P(A_1 \cap \cancel{A_2} \cap A_3)}{P(\cancel{A_1 \cap A_2})}$

$$\cdots \cdot \boxed{\dfrac{P(A_1 \cap A_2 \cdots \cap A_n)}{P(\cancel{A_1 \cap A_2 \cdots \cap A_{n-1}})}}$$

ex) box (r red balls & b blue balls)

E : selecting 4 balls where the sequence of outcomes is R B R B.

P(E)? let event R_j = select red at j-th try.

〃 B_j = 〃 blue 〃

$$E = R_1 \cap B_2 \cap R_3 \cap B_4.$$

$$P(E) = P(R_1) \cdot P(B_2 | R_1) \cdot P(R_3 | R_1 \cap B_2) \cdot P(B_4 | R_1 \cap B_2 \cap R_3)$$

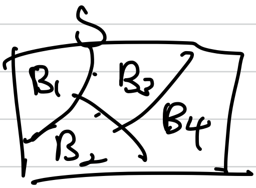
$$= \frac{r}{r+b} \cdot \frac{b}{r+b-1} \cdot \frac{r-1}{r+b-2} \cdot \frac{b-1}{r+b-3}$$

* Conditional probability and partitions

Def (partition) S = sample space.

events $B_1, B_2 \cdots B_k$ form a partition of S

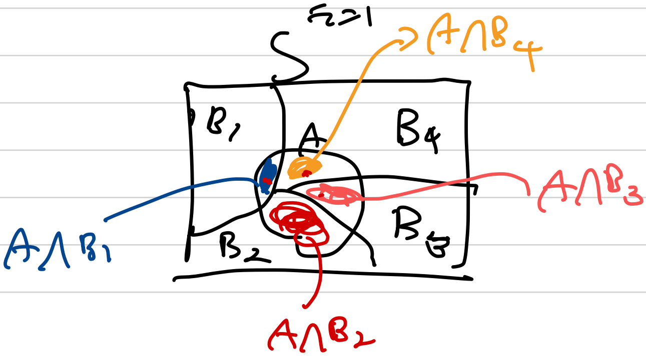
$\longleftrightarrow$ i) $\bigcup_{i=1}^{k} B_i = S$ ii) $B_1 \cdots B_k$ are mutually disjoint.



Thm (Law of total probability)
2.1.4

Event $B_1 \cdots B_k$ = partition of S, and $\forall i$, $P(B_i) > 0$. then for any event $A \subset S$,

$$P(A) = \sum_{i=1}^{k} P(B_i) \cdot P(A \mid B_i)$$



pf) Since $B_1 \cdots B_K$ form a partition of S.

$$A = \underline{(B_1 \cap A)} \cup \underline{(B_2 \cap A)} \cdots \cup \underline{(B_K \cap A)}$$

Since $B_1 \cap A, B_2 \cap A \cdots B_K \cap A$ are mutually disjoint, by axiom iii)

$$P(A) = \sum_{i=1}^{k} P(B_i \cap A), \quad \text{since } P(B_i) > 0$$

for $\forall i$, $P(A) = \sum_{i=1}^{k} P(A \mid B_i) P(B_i)$ //.

12

11

ex) Game.

i) select a number from 1-50. Let the number be X.

ii) select repeatedly until selecting a number $X \leq Y$. (replacement is allowed)

prob. $Y = 50$.

$A : Y = 50$ $\quad P(A \mid B_i) = \dfrac{1}{50 - i + 1}$

$B_i : X = i$

30

30 --- 50 $\quad$ i --- 50

$\hookrightarrow$ $\underline{B_1 \cdots B_{50}}$ are mutually disjoint!

$$P(A) = \sum_{i=1}^{50} \underline{P(B_i)} \cdot P(A \mid B_i)$$

$$= \sum_{i=1}^{50} \frac{1}{50} \cdot \frac{1}{51-i} \quad \sim 0.09.$$

2.2. Independent events.

ex) Tossing two coins

$S = \{\underline{HH}, HT, \underline{TH}, TT\}$

$A = \{\text{2nd outcome is } H\} = \{HH, TH\}$

$B = \{\text{1st outcome is } T\} = \{TH, TT\}$

$$P(A \mid \underline{B}) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

$$P(A) = \frac{1}{2}$$

$\Rightarrow$ Event B does not affect to the event A.

$\Rightarrow$ A and B are independent.

Def (independent) : Two events A and B are independent $\iff$ $P(A|B) = P(A)$

$$\iff \frac{P(A\cap B)}{P(B)} = P(A)$$

$$\iff P(A\cap B) = P(A)\cdot P(B)$$

※ Check independent condition first in the problem!

ex) 2 Machines ($M1$ and $M2$) in the factory.
two machines are operated independently each other

event A = $M1$ become inoperative in 8 hours
" B = $M2$ "

$P(A) = \frac{1}{3}$, $P(B) = \frac{1}{4}$.

$\Rightarrow$ Prob. that either A or B become inoperative in 8 hours. $\Rightarrow P(A\cup B)$

$$P(A \cup B) = \underset{\frac{1}{3}}{P(A)} + \underset{\frac{1}{4}}{P(B)} - \underset{P(A)\cdot P(B) = \frac{1}{12}}{\underline{P(A \cap B)}}$$

$(\because A$ and B are independent$)$

$$= \frac{1}{2}$$

Thm (Independence of complements)
2.2.1

if A, B are two independent events, then $\underline{A \text{ and } B^c}$ are also independent

pf) $P(\underline{A \cap B^c}) = P(A) - \underline{P(A \cap B)}$
$A \setminus B$

$$= P(A) - P(A) \cdot P(B)$$

$(\because A$ and B are independent$)$

$$= P(A)(\underline{1 - P(B)})$$

$$= P(A) \cdot P(B^c)$$

2^k

Def (Mutually independent events)

k events $\underline{A_1 \dots A_k}$ are mutually independent

$\Leftrightarrow \forall$ subset $\{A_{i_1}, \dots A_{i_j}\}$ for $2 \leq \underline{j} \leq k$

$$P(A_{i_1} \cap A_{i_2} \cap \dots A_{i_j}) = P(A_{i_1}) \cdot P(A_{i_2}) \cdots P(A_{i_j})$$

ex) i) A machine produces 6 items 1. -- 6.

ii) The result (defective or nondefective)
for these items are $\boxed{\text{mutually independent}}$

prob. exactly two items are defective.

(each item can be defective with prob. p)

D_i = item i is defective

$P(D_1) = P(D_2) \cdots = P(D_6) = p.$

ex) item 1, 2 → defective & 3, 4, 5, 6 → non-defective

$$P(\underline{D_1} \cap \underline{D_2} \cap \underline{D_3}^c \cap D_4^c \cap D_5^c \cap D_6^c)$$

$$= p \cdot p \cdot (1-p) \cdot (1-p) \cdot (1-p) \cdot (1-p)$$

$$= \underline{p^2 \cdot (1-p)^4}.$$

$$\Rightarrow \underline{\binom{6}{2} \cdot p^2 \cdot (1-p)^4}.$$

//