Unit 2

Boolean Algebra

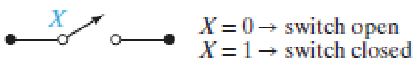
Logic Circuits (Spring 2022)

Introduction

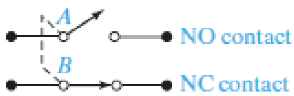
- Two-state variables
  - All switching devices are two-state devices
  - All variables assume only one of two values
  - *Boolean variable* X or Y: input or output of switching circuit
- Symbols “0” and “1”
  - Represent states in a logic circuit (not a numeric value)
  - 0 usually represents range of low voltages and 1 represents range of high voltages in a logic gate circuit
  - 0 represents open switch and 1 represents closed switch in a switching circuit
  - 0 and 1 can represent the two states in any binary valued system
- *Boolean algebra*
  - All variables assume only one of two states
  - Some limited operations are defined on those variables

Basic Operations

- Basic operations of Boolean (switching) algebra
  - AND
  - OR
  - Complement (or inverse)
- Switch contact



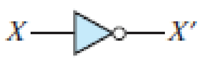
  - Labeled with a variable
  - NC(normally closed) and NO(normally open) contacts are always in opposite states



  - Variable X is assigned to NO contact $\Rightarrow$ X' will be assigned for NC

Basic Operations: Complement

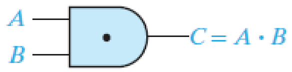
- Complementation
  - Prime (’) denotes complementation
  - 0’=1 and 1’=0
  - X'=1 if X=0 and X'=0 if X=1
- Complementation is also called inversion
  - Circle at the output denotes inversion

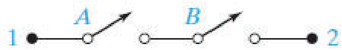


Basic Operations: AND

- Series switching circuits

A B	$C = A \cdot B$
0 0	0
0 1	0
1 0	0
1 1	1





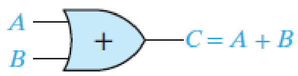
$C = 0 \rightarrow$ open circuit between terminals 1 and 2
$C = 1 \rightarrow$ closed circuit between terminals 1 and 2

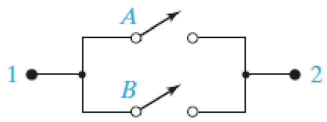
- AND operation
  - Written algebraically as $C=A\cdot B$
  - We will usually write AB instead of $A\cdot B$
  - Also referred to as logical (or Boolean) multiplication

Basic Operations: OR

- Parallel switching circuits

A B	$C = A + B$
0 0	0
0 1	1
1 0	1
1 1	1





a closed circuit if either A or B, or both, are closed
an open circuit only if A and B are both open

- OR operation
  - Written algebraically as $C=A+B$
  - Also referred to as logical (or Boolean) addition

Exclusive-OR

- Definition: $X \oplus Y = 1$ iff $X = 1$ or $Y = 1$, but not both

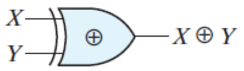
  $0 \oplus 0 = 0 \qquad 0 \oplus 1 = 1$

  $1 \oplus 0 = 1 \qquad 1 \oplus 1 = 0$

- Truth table

X	Y	X ⊕ Y
0	0	0
0	1	1
1	0	1
1	1	0

- Logic symbol



Exclusive-OR

- Some Theorems

$$X \oplus 0 = X$$
$$X \oplus 1 = X'$$
$$X \oplus X = 0$$
$$X \oplus X' = 1$$
$$X \oplus Y = Y \oplus X \text{ (commutative law)}$$
$$(X \oplus Y) \oplus Z = X \oplus (Y \oplus Z) = X \oplus Y \oplus Z \text{ (associative law)}$$
$$X(Y \oplus Z) = XY \oplus XZ \text{ (distributive law)}$$
$$(X \oplus Y)' = X \oplus Y' = X' \oplus Y = XY + X'Y'$$

Equivalence Operations

- Definition: $X \equiv Y = 1$ iff X and Y have the same value

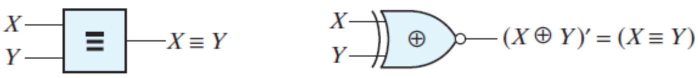
  $(0 \equiv 0) = 1 \qquad (0 \equiv 1) = 0$

  $(1 \equiv 0) = 0 \qquad (1 \equiv 1) = 1$

- Truth table

X	Y	$X \equiv Y$
0	0	1
0	1	0
1	0	0
1	1	1

- Logic symbol



Exclusive OR vs. Equivalence

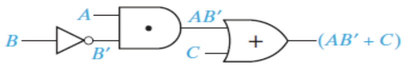
- Logic expressions
  - $X \oplus Y = X Y' + X' Y$
  - $X \equiv Y = X Y + X' Y'$
- Equivalence is the complement of exclusive-OR and vice versa
  - $(X Y' + X' Y)' = X Y + X' Y'$
  - $(X Y + X' Y')' = X Y' + X' Y$

Boolean Expression and Logic Diagram

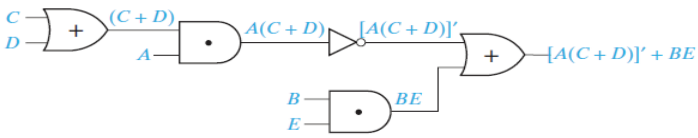
- Example Boolean expression

$$AB' + C \qquad (2\text{-}1)$$
$$[A(C + D)]' + BE \qquad (2\text{-}2)$$

- Order of operations: parentheses – inversion – AND – OR
- Logic diagram



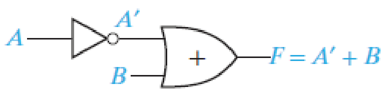
(a)



(b)

Logic Diagram and Truth Table

- Logic diagram



- Truth table
  - Specifies the values of a Boolean expression for every possible combination of values of the variables in the expression
  - n-variable expression $\Rightarrow 2^n$ rows

A	B	A'	F = A' + B
0	0	1	1
0	1	1	1
1	0	0	0
1	1	0	1

Boolean Expressions

- Two Boolean expressions are *equal*
  - If they have the same value for *every* possible combination of the variables

A B C	B'	AB'	$AB' + C$	$A + C$	$B' + C$	$(A + C)(B' + C)$
0 0 0	1	0	0	0	1	0
0 0 1	1	0	1	1	1	1
0 1 0	0	0	0	0	0	0
0 1 1	0	0	1	1	1	1
1 0 0	1	1	1	1	1	1
1 0 1	1	1	1	1	1	1
1 1 0	0	0	0	1	0	0
1 1 1	0	0	1	1	1	1

$$AB' + C = (A + C)(B' + C) \qquad (2\text{-}3)$$

Basic Theorems

Operations with 0 and 1:

1. $X + 0 = X$ 1D. $X \cdot 1 = X$
2. $X + 1 = 1$ 2D. $X \cdot 0 = 0$

Idempotent laws:

3. $X + X = X$ 3D. $X \cdot X = X$

Involution law:

4. $(X')' = X$

Laws of complementarity:

5. $X + X' = 1$ 5D. $X \cdot X' = 0$

Basic Theorems

Commutative laws:
6. $X + Y = Y + X$ — 6D. $XY = YX$

Associative laws:
7. $(X + Y) + Z = X + (Y + Z) = X + Y + Z$ — 7D. $(XY)Z = X(YZ) = XYZ$

Distributive laws:
8. $X(Y + Z) = XY + XZ$ — 8D. $X + YZ = (X + Y)(X + Z)$

DeMorgan's laws:
9. $(X + Y)' = X'Y'$ — 9D. $(XY)' = X' + Y'$

Basic Theorems

- Commutative law
  - Order in which variables are written does not affect result of applying AND and OR operations
  - $XY = YX$
  - $X+Y = Y+X$
- Associative law
  - Result of AND and OR operations is independent of which variables we associate together first
  - $(XY)Z=X(YZ)=XYZ$
  - $(X+Y)+Z= X+(Y+Z)=X+Y+Z$

Basic Theorems

- Distributive laws
  - $X(Y+Z) = XY + XZ$
  - $X+YZ = (X+Y)(X+Z) \Leftarrow$ valid for Boolean algebra only
- DeMorgan's laws
  - $(X+Y)' = X'Y'$
  - $(XY)' = X' + Y'$

X	Y	X'	Y'	X + Y	(X + Y)'	X'Y'	XY	(XY)'	X' + Y'
0	0	1	1	0	1	1	0	1	1
0	1	1	0	1	0	0	0	1	1
1	0	0	1	1	0	0	0	1	1
1	1	0	0	1	0	0	1	0	0

Simplification Theorems

Uniting theorems:
1. $XY + XY' = X$ — 1D. $(X + Y)(X + Y') = X$

Absorption theorems:
2. $X + XY = X$ — 2D. $X(X + Y) = X$

Elimination theorems:
3. $X + X'Y = X + Y$ — 3D. $X(X' + Y) = XY$

Duality:
4. $(X + Y + Z + \cdots)^D = XYZ\ldots$ — 4D. $(XYZ\ldots)^D = X + Y + Z + \cdots$

Theorems for multiplying out and factoring:
5. $(X + Y)(X' + Z) = XZ + X'Y$ — 5D. $XY + X'Z = (X + Z)(X' + Y)$

Consensus theorems:
6. $XY + YZ + X'Z = XY + X'Z$ — 6D. $(X + Y)(Y + Z)(X' + Z) = (X + Y)(X' + Z)$

Simplification Theorems

- How to prove the simplification theorems
  - Using a truth table
  - Using the basic theorems

$$XY + XY' = X(Y + Y') = X(1) = X$$
$$X + XY = X \cdot 1 + XY = X(1 + Y) = X \cdot 1 = X$$
$$X + X'Y = (X + X')(X + Y) = 1(X + Y) = X + Y$$

$$XY + X'Z + YZ = XY + X'Z + (1)YZ =$$
$$XY + X'Z + (X + X')YZ = XY + XYZ + X'Z + X'YZ =$$
$$XY + X'Z \text{ (using absorption twice)}$$

Consensus Theorem

- Consensus theorem
  - $XY + X'Z + YZ = XY + X'Z$
  - $(X + Y)(X' + Z)(Y + Z) = (X + Y)(X' + Z)$
- Consensus term
  - The eliminated term
  - YZ
  - $Y + Z$
- Cases of utilizing Consensus theorem

$$a'b' + ac + bc' + b'c + ab = a'b' + ac + bc'$$

$$(a + b + c')(a + b + d')(b + c + d') = (a + b + c')(b + c + d')$$

Simplifying Boolean Expression

Example 2

Simplify $Z = [\underbrace{A + B'C}_{X} + \underbrace{D + EF}_{Y}]\,[\underbrace{A + B'C}_{X} + \underbrace{(D + EF)'}_{Y'}]$

Substituting: $Z = [\;X \;+\; Y\;]\,[\;X \;+\; Y'\;]$

Then, by the uniting theorem (2-15D), the expression reduces to

$$Z = X = A + B'C$$

Simplifying Boolean Expression

Example 3

Simplify $Z = \underbrace{(AB + C)}_{X'}\,\underbrace{(B'D + C'E')}_{Y} + \underbrace{(AB + C)'}_{X}$

Substituting: $Z = X'\,Y + X$

By the elimination theorem (2-17): $Z = X + Y = B'D + C'E' + (AB + C)'$

Note that in this example we let $X = (AB + C)'$ rather than $(AB + C)$ in order to match the form of the elimination theorem (2-17).

Complementing Boolean Expression

- Successive application of DeMorgan's laws
  - The complement or inverse of any Boolean expression can be found using DeMorgan's Laws

$$(X_1 + X_2 + X_3 + \cdots + X_n)' = X_1'X_2'X_3' \ldots X_n'$$
$$(X_1X_2X_3 \ldots X_n)' = X_1' + X_2' + X_3' + \cdots + X_n'$$

For example, for $n = 3$,

$$(X_1 + X_2 + X_3)' = (X_1 + X_2)'X'_3 = X'_1X'_2X'_3$$

  - The complement of the product is the sum of the complements
  - The complement of the sum is the product of the complements

Complementing Boolean Expression

Example 1

To find the complement of $(A' + B)C'$, first apply (2-13) and then (2-12).

$$[(A' + B)C']' = (A' + B)' + (C')' = AB' + C$$

Complementing Boolean Expression

Example 2

$$
\begin{aligned}
[(AB' + C)D' + E]' &= [(AB' + C)D']'E' && \text{(by (2-12))} \\
&= [(AB' + C)' + D]E' && \text{(by (2-13))} \\
&= [(AB')'C' + D]E' && \text{(by (2-12))} \\
&= [(A' + B)C' + D]E' && \text{(by (2-13))} \quad \text{(2-27)}
\end{aligned}
$$

Note that in the final expressions, the complement operation is applied only to single variables.