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EVOLUTION OF ASYMMETRIC SLAB MAGNETOHYDRODYNAMIC WAVES

Matthew Allcock[1] and Robert Erdélyi[1]

[1] *Solar Physics and Space Plasma Research Centre, School of Mathematics and Statistics, University of Sheffield, Hicks Building, Hounsfield Road, Sheffield, S3 7RH, UK*

ABSTRACT

Abstract (250 word limit for ApJ)

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Corresponding author: Robert Erdélyi
robertus@sheffield.ac.uk

1. INTRODUCTION

Numerical results: Terradas et al. (2006)

Physicality of the principle leaky kink mode: Cally (2003) solved the initial value problem of transverse waves in a cold magnetic flux tube. Ruderman & Roberts (2006a) repeated it showing PLK modes are on an unphysical banch of the complex plane (?). Commented on by Cally (2006) and in return by Ruderman & Roberts (2006b). Settled (?) by considering numerical solution by Terradas et al. (2007) and analytically by Andries & Goossens (2007). So PLK modes are not physical.

2. INITIAL VALUE PROBLEM - INCOMPRESSIBLE MAGNETIC SLAB

Consider an equilibrium plasma with magnetic field $B_0(x)\hat{\mathbf{z}}$, density $\rho_0(x)$, and pressure $p_0(x)$, without gravity. In the absence of structuring in the z-direction and considering perturbations in the (x, z)-plane only, we can take Fourier components for velocity and other parameters like $\mathbf{v}(\mathbf{x}, t) = \hat{\mathbf{v}}(x)e^{i(kz+ly-\omega t)}$. The velocity perturbation amplitude in the inhomogeneous direction of an ideal plasma is given by

$$\frac{d}{dx}\left(\frac{\epsilon(x)}{l^2 + m_0^2(x)}\frac{d\hat{v}_x}{dx}\right) - \epsilon(x)\hat{v}_x = 0, \tag{1}$$

where

$$\epsilon(x) = \rho_0(x)[k^2 v_A^2(x) - \omega^2], \quad m_0^2(x) = \frac{(k^2 c_0^2(x) - \omega^2)(k^2 v_A^2(x) - \omega^2)}{(c_0^2(x) + v_A^2(x))(k^2 c_T^2(x) - \omega^2)}, \tag{2}$$

and

$$c_0(x) = \sqrt{\frac{\gamma p_0(x)}{\rho_0(x)}}, \quad v_A(x) = \frac{B_0(x)}{\sqrt{\mu\rho_0(x)}}, \quad c_T(x) = \frac{c_0(x)v_A(x)}{\sqrt{c_0^2(x) + v_A^2(x)}} \tag{3}$$

are the sound, Alfvén, and tube speeds, respectively.

When the plasma is incompressible, so that $c_0 \to \infty$, we have $c_T^2 \to v_A^2$ and $m_0^2 \to k^2$. After restricting the analysis to propagation only parallel to the magnetic field ($l = 0$), Equation (1) reduces to

$$\frac{d}{dx}\left(\epsilon(x)\frac{d\hat{v}_x}{dx}\right) - k^2\epsilon(x)\hat{v}_x = 0, \tag{4}$$

which is commonly used for solving eigenmode problems in MHD wave physics. In the present work, the temporal evolution of linear asymmetric MHD waves is considered so we must dial back our assumptions about how the wave behaviour through time.

Above we used a Fourier decomposition in time, which is valid when the solutions are homogeneous in time, such as normal mode solutions (rewrite this). To investigate the temporal evolution of solutions, we take only Fourier components in the z-direction, that is $\mathbf{v}(\mathbf{x}, t) = \hat{\mathbf{v}}(x, t)e^{ikz}$, and we take the Laplace transform with respect to time, such that

$$\tilde{\mathbf{v}}(x) = \int_0^\infty \hat{\mathbf{v}}(x, t)e^{i\omega t}dt. \tag{5}$$

This gives us the initial-value form of Equation (4) to be

$$\frac{d}{dx}\left(\epsilon(x)\frac{d\tilde{v}_x}{dx}\right) - k^2\epsilon(x)\tilde{v}_x = f(x), \tag{6}$$

where

$$f(x) = ik\left\{\rho_0\left[\frac{\partial\hat{\Omega}}{\partial t}(x, 0) - i\omega\hat{\Omega}(x, 0)\right] - \left[\frac{\partial\hat{v}_z}{\partial t}(x, 0) - i\omega\hat{v}_z(x, 0)\right]\frac{d\rho_0}{dx}\right\}, \tag{7}$$

where the vorticity, $\Omega(x, t)\hat{\mathbf{y}} = \hat{\Omega}(x, t)e^{ikz}\hat{\mathbf{y}} = \nabla \times \mathbf{v}(\mathbf{x}, t)$, is given by

$$\hat{\Omega}(\mathbf{x}, t) = -\frac{i}{k}\left(\frac{\partial^2\hat{v}_x}{\partial x^2} - k^2\hat{v}_x\right). \tag{8}$$

(this differs from Rae & Roberts (1981) by a factor of -1 due to taking Fourier forms like e^{ikz} rather than e^{-ikz})

Consider equilibrium magnetic field and density profiles given by

$$B(x)=\begin{cases}B_1, & \text{if } x<-x_0,\\ B_0, & \text{if } |x|\le x_0,\\ B_2, & \text{if } x>x_0,\end{cases} \quad \text{and} \quad \rho(x)=\begin{cases}\rho_1, & \text{if } x<-x_0,\\ \rho_0, & \text{if } |x|\le x_0,\\ \rho_2, & \text{if } x>x_0,\end{cases} \tag{9}$$

where B_i and ρ_i are uniform for $i=0,1,2$. This establishes the plasma as magnetic slab embedded in an asymmetric magnetic background.

In this equilibrium, transverse velocity perturbations are related to initial perturbations in the following way:

$$\frac{d^2\tilde{v}_x}{dx^2}-k^2\tilde{v}_x=\begin{cases}f(x)/\epsilon_1, & \text{if } x<-x_0,\\ f(x)/\epsilon_0, & \text{if } |x|<x_0,\\ f(x)/\epsilon_2, & \text{if } x>x_0.\end{cases} \tag{10}$$

2.1. *Attempt 1*

Sturm-Liouville Theory tells us that with the aid of a Green's function, $G(x;s)$, Equation (10) can be solved to give

$$\tilde{v}_x(x)=\begin{cases}\tilde{A}(\cosh kx+\sinh kx)-\frac{1}{\epsilon_1}\int_{-\infty}^{-x_0}G(x;s)f(s)ds, & \text{if } x<-x_0,\\ \tilde{B}\cosh kx+\tilde{C}\sinh kx-\frac{1}{\epsilon_0}\int_{-x_0}^{x_0}G(x;s)f(s)ds, & \text{if } |x|<x_0,\\ \tilde{D}(\cosh kx-\sinh kx)-\frac{1}{\epsilon_2}\int_{x_0}^{\infty}G(x;s)f(s)ds, & \text{if } x>x_0,\end{cases} \tag{11}$$

where

$$G(x;s)=\frac{1}{2k}[e^{ks}e^{-kx}H(x-s)+e^{-ks}e^{kx}H(s-x)] \tag{12}$$

and H is the Heaviside step function. Ensuring continuity of both transverse velocity and total pressure across the boundaries at $x=\pm x_0$ gives us the following system of linear algebraic equations for the constants A, B, C, and D:

$$\begin{pmatrix} c_0-s_0 & -c_0 & s_0 & 0\\ 0 & c_0 & s_0 & s_0-c_0\\ \epsilon_1(c_0-s_0) & \epsilon_0 s_0 & -\epsilon_0 c_0 & 0\\ 0 & \epsilon_0 s_0 & \epsilon_0 c_0 & -\epsilon_2(s_0-c_0)\end{pmatrix}\begin{pmatrix}\tilde{A}\\ \tilde{B}\\ \tilde{C}\\ \tilde{D}\end{pmatrix}=\frac{1}{2k}\begin{pmatrix} e^{kx_0}/\epsilon_1 & -e^{-kx_0}/\epsilon_0 & 0 & 0\\ 0 & 0 & e^{-kx_0}/\epsilon_0 & -e^{kx_0}/\epsilon_2\\ -e^{kx_0} & -e^{-kx_0} & 0 & 0\\ 0 & 0 & -e^{-kx_0} & -e^{kx_0}\end{pmatrix}\begin{pmatrix}I_1\\ I_0^-\\ I_0^+\\ I_2\end{pmatrix}, \tag{13}$$

where $c_0=\cosh kx_0$, $s_0=\sinh kx_0$, and the functionals I_1, I_0^-, I_0^+, and I_2 are given by

$$I_1=\int_{-\infty}^{-x_0}e^{ks}f(s)ds,\quad I_0^-=\int_{-x_0}^{x_0}e^{-ks}f(s)ds,\quad I_0^+=\int_{-x_0}^{x_0}e^{ks}f(s)ds,\quad I_2=\int_{x_0}^{\infty}e^{-ks}f(s)ds. \tag{14}$$

Solving this system of equations gives

$$\tilde{A}=\frac{e^{2kx_0}T_1}{2k\epsilon_1 D_R},\quad \tilde{B}=\frac{e^{2kx_0}T_0^-}{2k\epsilon_0 D_R},\quad \tilde{C}=\frac{e^{2kx_0}T_0^+}{2k\epsilon_0 D_R},\quad \tilde{D}=\frac{e^{2kx_0}T_2}{2k\epsilon_2 D_R}, \tag{15}$$

where

$$T_1(\omega)=I_1[e^{4kx_0}(\epsilon_0-\epsilon_1)(\epsilon_0+\epsilon_2)-(\epsilon_0+\epsilon_1)(\epsilon_0-\epsilon_2)]-4I_2e^{2kx_0}\epsilon_0\epsilon_1-2\epsilon_1\left[I_0^-e^{2kx_0}(\epsilon_0+\epsilon_2)+I_0^+(\epsilon_0-\epsilon_2)\right], \tag{16}$$

$$\begin{aligned}T_0^-(\omega)=&[(I_0^+-I_0^-)(\epsilon_2-\epsilon_1)\epsilon_0-(I_0^++I_0^-)(\epsilon_0^2-\epsilon_1\epsilon_2)]e^{2kx_0}-(I_0^++I_0^-)(\epsilon_0-\epsilon_1)(\epsilon_0-\epsilon_2)\\ &-2\epsilon_0e^{2kx_0}\left[((I_1+I_2)\epsilon_0+I_1\epsilon_2+I_2\epsilon_1)e^{2kx_0}+(I_1+I_2)\epsilon_0-I_1\epsilon_2-I_2\epsilon_1\right],\end{aligned} \tag{17}$$

$$\begin{aligned}T_0^+(\omega)=&[(I_0^++I_0^-)(\epsilon_2-\epsilon_1)\epsilon_0-(I_0^+-I_0^-)(\epsilon_0^2-\epsilon_1\epsilon_2)]e^{2kx_0}-(I_0^+-I_0^-)(\epsilon_0-\epsilon_1)(\epsilon_0-\epsilon_2)\\ &-2\epsilon_0e^{2kx_0}\left[((I_2-I_1)\epsilon_0-I_1\epsilon_2+I_2\epsilon_1)e^{2kx_0}+(I_1-I_2)\epsilon_0-I_1\epsilon_2+I_2\epsilon_1\right],\end{aligned} \tag{18}$$

$$T_2(\omega)=I_2[e^{4kx_0}(\epsilon_0+\epsilon_1)(\epsilon_0-\epsilon_2)-(\epsilon_0-\epsilon_1)(\epsilon_0+\epsilon_2)]-4I_1e^{2kx_0}\epsilon_0\epsilon_2-2\epsilon_2\left[I_0^+e^{2kx_0}(\epsilon_0+\epsilon_1)+I_0^-(\epsilon_0-\epsilon_1)\right], \tag{19}$$

$$D_R(\omega)=(\epsilon_0+\epsilon_1)(\epsilon_0+\epsilon_2)e^{4kx_0}-(\epsilon_0-\epsilon_1)(\epsilon_0-\epsilon_2). \tag{20}$$

The solutions of equation $D_R(\omega) = 0$ are precisely the solutions of the dispersion relation for an incompressible asymmetric magnetic slab. To confirm this, we follow notion from Zsámberger et al. (2018) (with an added superscript z) by observing that for an incompressible plasma, the sound speed, $c_s \to \infty$. Therefore,

$$m_j^2 = \frac{(k^2c_j^2 - \omega^2)(k^2v_{Aj}^2 - \omega^2)}{(c_j^2 + v_{Aj}^2)(k^2c_Tj^2 - \omega^2)} \to k^2, \tag{21}$$

$$\Lambda_j^z = \frac{i\rho_j}{\omega m_j}(k^2v_{Aj}^2 - \omega^2) \to \frac{i\rho_j}{\omega k}(k^2v_{Aj}^2 - \omega^2), \quad \text{for } j = 0, 1, 2. \tag{22}$$

Therefore, after multiplying Equation (20) by $\omega k/i(e^{4kx_0} + 1)$, we recover the dispersion relation in Zsámberger et al. (2018) for an incompressible plasma, namely

$$2(\Lambda_0^2 + \Lambda_1\Lambda_2) + \Lambda_0(\Lambda_1 + \Lambda_2)[\tanh kx_0 + \coth kx_0] = 0. \tag{23}$$

The solution given by Equation (11), with constants (with respect to x) $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$, given by Equation (15) corroborates with those describing the initial value problem of surface waves on an interface between two incompressible plasmas. When the slab width, $2x_0$, approaches zero, it is expected that the solutions here approach those describing the interface. As $x_0 \to 0$, the dispersion function behaves like

$$D_R \to (\epsilon_0 + \epsilon_1)(\epsilon_0 + \epsilon_2) - (\epsilon_0 - \epsilon_1)(\epsilon_0 - \epsilon_2) = 2\epsilon_0(\epsilon_1 + \epsilon_2). \tag{24}$$

Therefore, in this limit, $\tilde{A}$ and $\tilde{D}$ behave like

$$\tilde{A} \to \frac{1}{4k\epsilon_0\epsilon_1(\epsilon_1 + \epsilon_2)}\left\{2I_1\epsilon_0(\epsilon_2 - \epsilon_1) - 4I_2\epsilon_0\epsilon_1\right\} = \frac{1}{k(\epsilon_1 + \epsilon_2)}\left\{-I_2 + \frac{1}{2\epsilon_1}(\epsilon_2 - \epsilon_1)I_1\right\}, \tag{25}$$

$$\tilde{D} \to \frac{1}{4k\epsilon_0\epsilon_2(\epsilon_1 + \epsilon_2)}\left\{2I_2\epsilon_0(\epsilon_1 - \epsilon_2) - 4I_1\epsilon_0\epsilon_2\right\} = \frac{1}{k(\epsilon_1 + \epsilon_2)}\left\{-I_1 + \frac{1}{2\epsilon_2}(\epsilon_1 - \epsilon_2)I_2\right\}, \tag{26}$$

$$\tag{27}$$

which are equal to (the corrected versions of) A_- and A_+, respectively, from Rae & Roberts (1981).

2.1.1. *Solution in time*

To recover the transverse velocity, $v_x(x,t)$, we employ the inverse Laplace transform, such that

$$v_x(x,t) = \frac{1}{2\pi}\lim_{L\to\infty}\int_{i\gamma-L}^{i\gamma+L}\tilde{v}_x(x,\omega)e^{-i\omega t}d\omega, \tag{28}$$

where γ is a real number such that all the singularities of the integrand is below the contour of integration. The integral is evaluated along an infinite horizontal line in the upper half of the complex plane. Since the problem is now reduced to solving a complex integral, it is dependent on the singularities (with respect to ω) of $\tilde{v}_x$, whose residues determine the value of the contour integral.

Focusing firstly on the region $x < -x_0$, the solution is

$$v_x = \frac{1}{2\pi}\lim_{L\to\infty}\int_{i\gamma-L}^{i\gamma+L}\left[\tilde{A}e^{k(x+x_0)} + \frac{1}{\epsilon_1}\int_{-\infty}^{-x_0}G(x;s)f(s)ds\right]e^{-i\omega t}d\omega, \tag{29}$$

$$= \frac{e^{k(x+x_0)}}{2\pi}\left(\lim_{L\to\infty}\int_{i\gamma-L}^{i\gamma+L}\tilde{A}e^{-i\omega t}ds\right) + \left(\int_{-\infty}^{-x_0}G(x;s)f(s)ds\right)\left(\lim_{L\to\infty}\int_{i\gamma-L}^{i\gamma+L}\frac{e^{-i\omega t}}{\epsilon_1}d\omega\right), \tag{30}$$

The first integral in the above solution is calculated as follows: The functions $\epsilon_{0,1,2}$ are polynomial in ω, and are therefore entire. The integrals $I_{1,2}$ and $I_0^{\pm}$ are not functions of ω so also contribute no singularities in ω. Therefore, $T_{1,2}$, and $T_0^{\pm}$ are entire functions.

The final integral in Equation (50) is calculated as follows:

the singularities of $\tilde{v}_x$ are precisely the singularities of A and $1/\epsilon_1$

Considering the functional form of $\tilde{v}_x(x,\omega)$, we see that, in each region, it is made up of a term involving

2.1.2. *Specific initial conditions*

The above approach using arbitrary initial conditions is forced to cease here due to mathematical intractability of the general inverse Laplace transform calculation (however, general asymptotic results are, just barely, tractable). Instead, we progress with specific initial conditions.

Choosing specific initial conditions requires a delicate balance between mathematical tractability and physical applicability.

2.2. *Attempt 2*

The idea behind this second attempt is to choose the Green's function in such a way as to make the constant coefficients of the particular solution part are directly related to the boundary values of the transverse velocity.

We are trying to solve the equation

$$\frac{d^2\tilde{v}_x}{dx^2} - k^2\tilde{v}_x = \begin{cases} f(x)/\epsilon_1, & \text{if } x < -x_0, \\ f(x)/\epsilon_0, & \text{if } |x| < x_0, \\ f(x)/\epsilon_2, & \text{if } x > x_0, \end{cases} \tag{31}$$

under the boundary conditions $\tilde{v}_x(-x_0) = \tilde{A}_1$ and $\tilde{v}_x(x_0) = \tilde{A}_2$.

The Green's function, $G(x;s)$, corresponding to Equation (31) must satisfy

$$\frac{\partial^2 G}{\partial x^2} - k^2 G = \delta(x-s), \quad G(-x_0;s) = G(x_0;s) = 0, \tag{32}$$

where δ denotes the Dirac Delta function. It is instructive to piecewise define the Green's function as

$$G(x;s) = \begin{cases} G_1(x;s), & \text{if } x < -x_0, \\ G_0(x;s), & \text{if } |x| < x_0, \\ G_2(x;s), & \text{if } x_0 < x. \end{cases} \tag{33}$$

The general solution, for $|x| < x_0$, of the equation for G_0 is

$$G_0(x;s) = c_1 \sinh(k(x-x_0)) + c_2 \sinh(k(x+x_0)), \tag{34}$$

where $c_1 = 0$ for $x < s$ and $c_2 = 0$ for $x > s$. Ensuring G_0 and $\partial G_0/\partial x$ have jumps of 0 and 1 at $x = s$, respectively, determines c_1 and c_2, so that $G_0(x;s)$ is

$$G_0(x;s) = \frac{1}{k\sinh(2kx_0)} \begin{cases} \sinh(k(s-x_0))\sinh(k(x+x_0)), & \text{if } -x_0 < x < s, \\ \sinh(k(x-x_0))\sinh(k(s+x_0)), & \text{if } s < x < x_0. \end{cases} \tag{35}$$

Because the boundary conditions are inhomogeneous, we must add to the standard Green's function solution a term that is a solution to the homogeneous equation and the inhomogeneous boundary conditions. In this manner, we find that the solution within the slab is

$$\tilde{v}_x(x) = \frac{1}{\sinh 2kx_0}\left[\tilde{A}_1 \sinh(k(x_0-x)) + \tilde{A}_2 \sinh(k(x_0+x))\right] + \frac{1}{\epsilon_0}\int_{-x_0}^{x_0} G_0(x;s) f(s) ds. \tag{36}$$

Similarly, we find that the Green's function for the plasma outside the slab is

$$G_1(x;s) = \frac{1}{k} \begin{cases} e^{k(x+x_0)}\sinh(k(s+x_0)), & \text{if } x < s, \\ e^{k(s+x_0)}\sinh(k(x+x_0)), & \text{if } s < x < -x_0, \end{cases} \tag{37}$$

for $x < -x_0$, and

$$G_2(x;s) = -\frac{1}{k} \begin{cases} e^{-k(s-x_0)}\sinh(k(x-x_0)), & \text{if } x_0 < x < s, \\ e^{-k(x-x_0)}\sinh(k(s-x_0)), & \text{if } s < x, \end{cases} \tag{38}$$

for $x > x_0$. Therefore the solution is

$$\tilde{v}_x(x) = \tilde{A}_1 e^{k(x_0+x)} + \frac{1}{\epsilon_1}\int_{-\infty}^{-x_0} G_1(x;s)f(s)ds, \tag{39}$$

for $x < -x_0$, and

$$\tilde{v}_x(x) = \tilde{A}_2 e^{k(x_0-x)} + \frac{1}{\epsilon_2}\int_{x_0}^{\infty} G_2(x;s)f(s)ds, \tag{40}$$

for $x > x_0$.

2.2.1. *Matching solutions*

To establish physically relevant solutions, we require that the transverse velocity and the total pressure are continuous over each interface. Given the above solutions, the transverse velocity is automatically continuous over the boundaries. The perturbation in the total pressure, P_T, for a compressible plasma is given (for example, in Allcock & Erdélyi (2017)) by

$$\tilde{P}_T(x) = \frac{\Lambda}{m}\frac{d\tilde{v}_x}{dx}, \tag{41}$$

where

$$\Lambda = \frac{i\rho(\omega^2 - k^2 v_A^2)}{m\omega}, \quad m^2 = \frac{(k^2 v_A^2 - \omega^2)(k^2 c_0^2 - \omega^2)}{(c_0^2 + v_A^2)(k^2 c_T^2 - \omega^2)}, \quad \text{and} \quad c_T^2 = \frac{c_0^2 v_A^2}{c_0^2 + v_A^2}. \tag{42}$$

When the plasma is incompressible, $m^2 \to k^2$, therefore continuity in total pressure is equivalent to continuity in $\epsilon(x)\tilde{v}_x'(x)$ for an incompressible plasma.

Applying this boundary condition gives

$$\tilde{A}_1(\omega) = \frac{T_1(\omega)}{kD(\omega)}, \quad \tilde{A}_2(\omega) = \frac{T_2(\omega)}{kD(\omega)}, \tag{43}$$

where

$$T_1(\omega) = (I_0^- - I_1)[\epsilon_0\cosh(2kx_0) + \epsilon_2\sinh(2kx_0)] - \epsilon_0(I_0^+ + I_2), \tag{44}$$

$$T_2(\omega) = \epsilon_0(I_0^- - I_1) - (I_0^+ + I_2)[\epsilon_0\cosh(2kx_0) + \epsilon_1\sinh(2kx_0)], \tag{45}$$

$$D(\omega) = \epsilon_0(\epsilon_1 + \epsilon_2)\cosh(2kx_0) + (\epsilon_0^2 + \epsilon_1\epsilon_2)\sinh(2kx_0), \tag{46}$$

and

$$I_0^{\pm} = \int_{-x_0}^{x_0} \frac{\sinh(k(s \pm x_0))}{\sinh(2kx_0)} f(s)ds, \quad I_1 = \int_{-\infty}^{-x_0} e^{k(s+x_0)} f(s)ds, \quad I_2 = \int_{x_0}^{\infty} e^{k(x_0-s)} f(s)ds. \tag{47}$$

2.2.2. *Solution in time*

To recover the transverse velocity, $v_x(x,t)$, we employ the inverse Laplace transform (non-standard, discussed in Appendix A), such that

$$v_x(x,t) = \frac{1}{2\pi}\lim_{L\to\infty}\int_{i\gamma-L}^{i\gamma+L} \tilde{v}_x(x,\omega)e^{-i\omega t}d\omega, \tag{48}$$

where γ is a real number such that all the singularities of the integrand is below the contour of integration. The integral is evaluated along an infinite horizontal line in the upper half of the complex plane. Since the problem is now reduced to solving a complex integral, it is dependent on the singularities (with respect to ω) of $\tilde{v}_x$, whose residues determine the value of the contour integral.

Focusing firstly on the region $x < -x_0$, the solution is

$$v_x = \frac{1}{2\pi}\lim_{L\to\infty}\int_{i\gamma-L}^{i\gamma+L}\left[\tilde{A}_1 e^{k(x+x_0)} + \frac{1}{\epsilon_1}\int_{-\infty}^{-x_0} G_1(x;s)f(s)ds\right]e^{-i\omega t}d\omega, \tag{49}$$

$$= \frac{e^{k(x+x_0)}}{2\pi}\left(\lim_{L\to\infty}\int_{i\gamma-L}^{i\gamma+L}\tilde{A}_1 e^{-i\omega t}ds\right) + \left(\int_{-\infty}^{-x_0} G_1(x;s)f(s)ds\right)\left(\lim_{L\to\infty}\int_{i\gamma-L}^{i\gamma+L}\frac{e^{-i\omega t}}{\epsilon_1}d\omega\right), \tag{50}$$

The first integral in the above solution is calculated as follows. The functions $\epsilon_{0,1,2}$ are polynomial in ω, and are therefore entire. The integrals $I_{1,2}$ and $I_0^{\pm}$ are not functions of ω so also contribute no singularities in ω. Therefore, T_1 and T_2 are entire functions.

The zeros of $D(\omega)$ are determined by firstly noting that $D = 0$ is the dispersion relation of the corresponding eigenvalue problem solved by Zsámberger et al. (2018). They show that the dispersion relation governing transverse wave propagation parallel to the magnetic field in an asymmetric slab of compressible plasma is given by

$$2(\Lambda_0^2 + \Lambda_1\Lambda_2) + \Lambda_0(\Lambda_1 + \Lambda_2)[\tanh(m_0x_0) + \coth(m_0x_0)] = 0, \tag{51}$$

where

$$\Lambda_j = -\frac{i\rho_j(k^2v_{Aj}^2 - \omega^2)}{\omega m_j}, \quad m_j^2 = \frac{(k^2c_j^2 - \omega^2)(k^2v_{Aj}^2 - \omega^2)}{(c_j^2 + v_{Aj}^2)(k^2c_{Tj}^2 - \omega^2)}, \quad \text{and} \quad c_{Tj}^2 = \frac{c_j^2v_{Aj}^2}{c_j^2 + v_{Aj}^2}, \tag{52}$$

for $j = 0, 1, 2$. When compressibility is neglected, such that the sound speeds, c_j, approach infinity, we have $c_{Tj}^2 \to v_{Aj}^2$, $m_j^2 \to k^2$, and therefore $\Lambda_j = -i\rho_j(k^2v_{Aj}^2 - \omega^2)/\omega k = -i\epsilon_j/\omega k$, for $j = 0, 1, 2$. Therefore, Equation (51) can be reduced to the dispersion relation for an incompressible magnetic slab, which is

$$2(\epsilon_0^2 + \epsilon_1\epsilon_2) + \epsilon_0(\epsilon_1 + \epsilon_2)[\tanh(m_0x_0) + \coth(m_0x_0)] = 0, \tag{53}$$

which can easily to shown to be equivalent to $D(\omega) = 0$, where $D(\omega)$ is given by Equation (46). It follows that the zeros of $D(\omega)$ are precisely the eigenvalues of the asymmetric incompressible magnetic slab.

The zeros of $D(\omega) = 0$ are found by writing this equation as

$$\epsilon_0(\epsilon_1 + \epsilon_2) + (\epsilon_0^2 + \epsilon_1\epsilon_2)\tanh(2kx_0) = 0 \tag{54}$$

and substituting expressions for $\epsilon(x)$, which gives

$$\rho_0(k^2v_{A0}^2 - \omega^2)[\rho_1(k^2v_{A1}^2 - \omega^2) + \rho_2(k^2v_{A2}^2 - \omega^2)] + [\rho_0^2(k^2v_{A0}^2 - \omega^2)^2 + \rho_1\rho_2(k^2v_{A1}^2 - \omega^2)(k^2v_{A2}^2 - \omega^2)]\tanh(2kx_0) = 0. \tag{55}$$

The above equation can be rewritten as a quadratic in $(\omega/k)^2$, namely

$$\left(\frac{\omega}{k}\right)^4[(\rho_0^2 + \rho_1\rho_2)\tanh(2kx_0) + \rho_0(\rho_1 + \rho_2)] \tag{56}$$

$$-\left(\frac{\omega}{k}\right)^2[(2\rho_0^2v_{A0}^2 + \rho_1\rho_2(v_{A1}^2 + v_{A2}^2))\tanh(2kx_0) + \rho_0v_{A0}^2(\rho_1 + \rho_2) + \rho_0(\rho_1v_{A1}^2 + \rho_2v_{A2}^2)] \tag{57}$$

$$+[(\rho_0^2v_{A0}^4 + \rho_1\rho_2v_{A1}^2v_{A2}^2)\tanh(2kx_0) + \rho_0v_{A0}^2(\rho_1v_{A1}^2 + \rho_2v_{A2}^2)] = 0, \tag{58}$$

which has solutions

$$\left(\frac{\omega_0}{k}\right)^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{59}$$

where

$$a = (\rho_0^2 + \rho_1\rho_2)\tanh(2kx_0) + \rho_0(\rho_1 + \rho_2), \tag{60}$$

$$b = (2\rho_0^2v_{A0}^2 + \rho_1\rho_2(v_{A1}^2 + v_{A2}^2))\tanh(2kx_0) + \rho_0v_{A0}^2(\rho_1 + \rho_2) + \rho_0(\rho_1v_{A1}^2 + \rho_2v_{A2}^2), \tag{61}$$

$$c = (\rho_0^2v_{A0}^4 + \rho_1\rho_2v_{A1}^2v_{A2}^2)\tanh(2kx_0) + \rho_0v_{A0}^2(\rho_1v_{A1}^2 + \rho_2v_{A2}^2). \tag{62}$$

The solutions, ω_0 corroborate with the corresponding incompressible eigenfrequencies for an interface and a symmetric slab, as shown in Appendices B.1 and B.2, respectively.

Given that the function $T_1(\omega)$ is entire, we can construct a closed Bromwich contour, $C = C_0 + C_1$, where C_0 is a straight line from $(-L, \gamma)$ to (L, γ), and C_1 connects $(-L, \gamma)$ and (L, γ) via a semi-circle to ensure that C encloses the zeros of D at $\pm\omega_0$, as shown by Figure 2.2.2 (WHY IS THIS HAPPENING?!).

Firstly, we see that the integrands in question behave like $T_1(\omega)/D(\omega) = \mathcal{O}(|\omega|^{-2})$, as $|\omega| \to \infty$. Therefore,

$$\lim_{L\to\infty}\int_{C_1}\frac{T_1(\omega)}{D(\omega)}e^{-i\omega t}d\omega = 0. \tag{63}$$

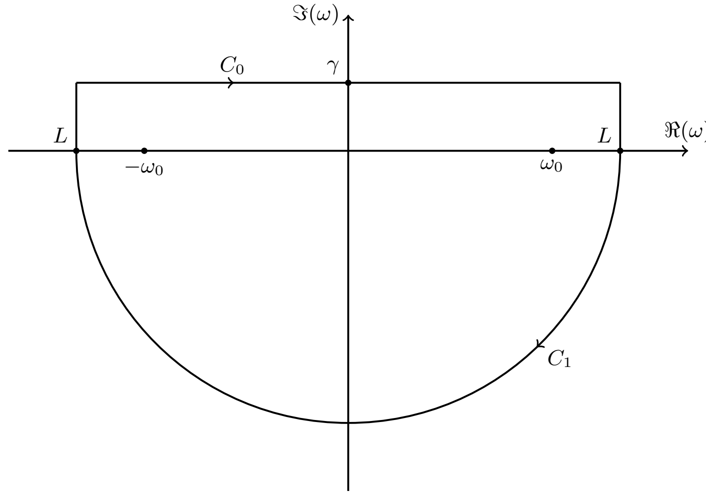

Figure 1. Bromwich contour for the complex integration of $\tilde{A}_{1,2}$.

Secondly, the contour C is integrated in the clockwise direction and therefore is equal to $-2\pi i$ multiplied by the sum of the residues of the integrand at $\omega = \pm\omega_0$. THERE IS A PROBLEM WITH THE BELOW BECAUSE THERE COULD BE 4 ROOTS (BACKWARDS AND FORWARD PROPAGATING SAUS AND KINK). WE ALSO MUST CHECK L'H'S RULE IS VALID HERE, AND THAT THE ROOTS OF D ARE NOT ALSO THE ROOTS OF T OR D'. The residue at $\omega = \omega_0$ can be evaluated using L'Hopital's Rule, to within choice of initial condition, as

$$\operatorname{Res}\left[\frac{T_1(\omega)}{D(\omega)}e^{-i\omega t}, \omega = \omega_0\right] = \lim_{\omega\to\omega_0}\frac{(\omega-\omega_0)T_1(\omega)}{D(\omega)}e^{-i\omega t} \tag{64}$$

$$= \lim_{\omega\to\omega_0}\frac{1}{D'(\omega)}[T_1(\omega)e^{-i\omega t} + (\omega-\omega_0)T_1'(\omega)e^{-i\omega t} - it(\omega-\omega_0)T_1(\omega)e^{-i\omega t}] \tag{65}$$

$$= \chi_1 e^{-i\omega_0 t}, \tag{66}$$

where $\chi_1 = T_1(\omega_0)/D'(\omega_0)$. Similarly, the residue at $\omega = -\omega_0$ is

$$\operatorname{Res}\left[\frac{T_1(\omega)}{D(\omega)}e^{-i\omega t}, \omega = -\omega_0\right] = \lim_{\omega\to-\omega_0}\frac{(\omega+\omega_0)T_1(\omega)}{D(\omega)}e^{-i\omega t} \tag{67}$$

$$= -\chi_1 e^{i\omega_0 t}. \tag{68}$$

where we have used the fact that D and T_1 are even functions and D' is an odd function of ω.

Putting all of the above results together, we find that solution of the first integral of Equation (50), or equivalently the boundary velocity is

$$A_1 = \frac{1}{2\pi k}\lim_{L\to\infty}\int_{C_0}\frac{T_1(\omega)}{D(\omega)}e^{-i\omega t}d\omega \tag{69}$$

$$= \frac{1}{2\pi k}\lim_{L\to\infty}\int_{C}\frac{T_1(\omega)}{D(\omega)}e^{-i\omega t}d\omega \tag{70}$$

$$= \frac{-i}{k}\sum \operatorname{Res}\left[\frac{T_1(\omega)}{D(\omega)}e^{-i\omega t}, \omega = \pm\omega_0\right] \tag{71}$$

$$= \frac{-i}{k}\chi_1(e^{-i\omega_0 t} - e^{i\omega_0 t}) \tag{72}$$

$$= -\frac{2}{k}\chi_1\sin(\omega_0 t). \tag{73}$$

TO DO NEXT: Redo the above with the 4 poles, not just two. (maybe see what these solutions are like in symmetric slab/interface). Evaluate final integral of equation (50). Repeat the above for other two regions of the slab. This gives us the full solution. Then maybe try to work out $\chi_{1,2}$. Test various initial conditions. Publish

The final integral in Equation (50) is calculated as follows:
the singularities of $\tilde{v}_x$ are precisely the singularities of A and $1/\epsilon_1$
Considering the functional form of $\tilde{v}_x(x, \omega)$, we see that, in each region, it is made up of a term involving

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Software: Mayavi

APPENDIX

A. NON-STANDARD LAPLACE TRANSFORM

Consider a function $f(t)$, whose standard Laplace transform, $F_1(\omega)$, and non-standard Laplace transform, $F_2(\omega)$, are

$$F_1(\omega) = \int_0^\infty f(t)e^{-\omega t}dt, \quad \text{and} \quad F_2(\omega) = \int_0^\infty f(t)e^{i\omega t}dt. \tag{A1}$$

Trivially, $F_1(-i\omega) = F_2(\omega)$. Using the standard inverse Laplace transform, and letting γ be real and greater than the real part of all the singularities of $F_1(\omega)$, the original function $f(t)$ can be written

$$f(t) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT} F_1(\omega)e^{\omega t}d\omega, \tag{A2}$$

$$= \frac{1}{2\pi i}\lim_{T\to\infty}\int_{i\gamma-T}^{i\gamma+T} F_1(-i\omega)e^{-i\omega t}(-id\omega), \tag{A3}$$

$$= \frac{1}{2\pi}\lim_{T\to\infty}\int_{i\gamma-T}^{i\gamma+T} F_2(\omega)e^{-i\omega t}d\omega. \tag{A4}$$

Therefore, the inverse transform of the non-standard Laplace transform is

$$f(t) = \frac{1}{2\pi}\lim_{T\to\infty}\int_{i\gamma-T}^{i\gamma+T} F_2(\omega)e^{-i\omega t}d\omega. \tag{A5}$$

B. CORROBORATION OF INCOMPRESSIBLE SOLUTIONS WITH PREVIOUS RESULTS

B.1. *Interface*

When we let the width of an asymmetric slab vanish, we recover the traditional interface geometry. Letting $x_0 \to 0$, the parameters a, b, and c, from Equations (60), (61), and (62), reduce to

$$a = \rho_0(\rho_1 + \rho_2), \tag{B6}$$

$$b = \rho_0 v_{A0}^2(\rho_1 + \rho_2) + \rho_0(\rho_1 v_{A1}^2 + \rho_2 v_{A2}^2), \tag{B7}$$

$$c = \rho_0 v_{A0}^2(\rho_1 v_{A1}^2 + \rho_2 v_{A2}^2). \tag{B8}$$

Therefore, when the slab width vanishes, the eigenmodes given by Equation (59) reduce to

$$\left(\frac{\omega_0}{k}\right)^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \begin{cases} v_{A0}^2, \\ \frac{\rho_1 v_{A1}^2 + \rho_2 v_{A2}^2}{\rho_1 + \rho_2}. \end{cases} \tag{B9}$$

The first solution above is degenerate because, while the parameter v_{A0} makes sense in the limit as the slab width vanishes, it is meaningless in an interface system constructed without an inner region at all. The second solution corroborates with the surface eigenmodes of an interface, as expected (Roberts 1981a).

B.2. *Symmetric slab*

By letting the parameters on each external plasma region be equal (*i.e.* $\rho_1 = \rho_2 = \rho_e$, and similar for the magnetic field and Aflvén speed) the asymmetric slab is reduced to a symmetric slab. In this limit, the parameters a, b, and c, from Equations (60), (61), and (62), can be reduced to

$$a = \frac{2}{\tau_0 + c_0}\left[\rho_0^2 + \rho_e^2 + \rho_0\rho_e(\tau_0 + c_0)\right], \tag{B10}$$

$$b = \frac{2}{\tau_0 + c_0}\left[2(\rho_0^2 v_{A0}^2 + \rho_e^2 v_{Ae}^2) + \rho_0\rho_e(v_{A0}^2 + v_{Ae}^2)(\tau_0 + c_0)\right], \tag{B11}$$

$$c = \frac{2}{\tau_0 + c_0}\left[\rho_0^2 v_{A0}^4 + \rho_e^2 v_{Ae}^4 + \rho_0\rho_e v_{A0}^2 v_{Ae}^2(\tau_0 + c_0)\right], \tag{B12}$$

where $\tau_0 = \tanh kx_0$ and $c_0 = \coth kx_0$. The discriminant in the solution, Equation (59), reduces to

$$b^2 - 4ac = 4\rho_0^2\rho_e^2(v_{A0}^2 - v_{Ae}^2)^2\left(\frac{\tau_0 - c_0}{\tau_0 + c_0}\right)^2. \tag{B13}$$

Therefore, the eigenfrequencies reduce to

$$\left(\frac{\omega_0}{k}\right)^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \begin{cases} \frac{\rho_0 v_{A0}^2 + \rho_e v_{Ae}^2 c_0}{\rho_0 + \rho_e c_0}, \\ \frac{\rho_0 v_{A0}^2 + \rho_e v_{Ae}^2 \tau_0}{\rho_0 + \rho_e \tau_0}, \end{cases} \tag{B14}$$

which corroborates with Equation (12) in Roberts (1981b).

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