Draft version October 16, 2018
Typeset using LaTeX default style in AASTeX61

EVOLUTION OF ASYMMETRIC SLAB MAGNETOHYDRODYNAMIC WAVES

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ABSTRACT

Abstract (250 word limit for ApJ)

Keywords: magnetohydrodynamics — plasmas — Sun: atmosphere — Sun: oscillations — waves

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1. INTRODUCTION

Numerical results: Terradas et al. (2006)

Physicality of the principle leaky kink mode: Cally (2003) solved the initial value problem of transverse waves in a cold magnetic flux tube. Ruderman & Roberts (2006a) repeated it showing PLK modes are on an unphysical banch of the complex plane (?). Commented on by Cally (2006) and in return by Ruderman & Roberts (2006b). Settled (?) by considering numerical solution by Terradas et al. (2007) and analytically by Andries & Goossens (2007). So PLK modes are not physical.

2. INITIAL VALUE PROBLEM - INCOMPRESSIBLE MAGNETIC SLAB

Consider an equilibrium plasma with magnetic field $B_0(x)\hat{\mathbf{z}}$, density $\rho_0(x)$, and pressure $p_0(x)$, without gravity. In the absence of structuring in the z-direction and considering perturbations in the (x, z)-plane only, we can take Fourier components for velocity and other parameters like $\mathbf{v}(\mathbf{x}, t) = \hat{\mathbf{v}}(x)e^{i(kz+ly-\omega t)}$. The velocity perturbation amplitude in the inhomogeneous direction of an ideal plasma is given by

$$\frac{d}{dx}\left(\frac{\epsilon(x)}{l^2+m_0^2(x)}\frac{d\hat{v}_x}{dx}\right) - \epsilon(x)\hat{v}_x = 0, \tag{1}$$

where

$$\epsilon(x) = \rho_0(x)[k^2 v_A^2(x) - \omega^2], \quad m_0^2(x) = \frac{(k^2c_0^2(x) - \omega^2)(k^2v_A^2(x) - \omega^2)}{(c_0^2(x) + v_A^2(x))(k^2c_T^2(x) - \omega^2)}, \tag{2}$$

and

$$c_0(x) = \sqrt{\frac{\gamma p_0(x)}{\rho_0(x)}}, \quad v_A(x) = \frac{B_0(x)}{\sqrt{\mu\rho_0(x)}}, \quad c_T(x) = \frac{c_0(x)v_A(x)}{\sqrt{c_0^2(x) + v_A^2(x)}} \tag{3}$$

are the sound, Alfvén, and tube speeds, respectively.

When the plasma is incompressible, so that $c_0 \to \infty$, we have $c_T^2 \to v_A^2$ and $m_0^2 \to k^2$. After restricting the analysis to propagation only parallel to the magnetic field ($l = 0$), Equation (1) reduces to

$$\frac{d}{dx}\left(\epsilon(x)\frac{d\hat{v}_x}{dx}\right) - k^2\epsilon(x)\hat{v}_x = 0, \tag{4}$$

which is commonly used for solving eigenmode problems in MHD wave physics. In the present work, the temporal evolution of linear asymmetric MHD waves is considered so we must dial back our assumptions about how the wave behaviour changes through time.

Above we have used a Fourier decomposition in both space and time. To investigate the temporal evolution of solutions, we take only Fourier components in the z-direction, that is $\mathbf{v}(\mathbf{x}, t) = \hat{\mathbf{v}}(x, t)e^{ikz}$, and we take the Laplace transform with respect to time, such that

$$\tilde{\mathbf{v}}(x) = \mathcal{L}\{\hat{\mathbf{v}}(x, t)\} = \int_0^\infty \hat{\mathbf{v}}(x, t)e^{i\omega t}dt. \tag{5}$$

This gives us the initial-value form of Equation (4) to be

$$\frac{d}{dx}\left(\epsilon(x)\frac{d\tilde{v}_x}{dx}\right) - k^2\epsilon(x)\tilde{v}_x = f(x), \tag{6}$$

where

$$f(x) = ik\left\{\rho_0\left[\frac{\partial\hat{\Omega}}{\partial t}(x, 0) - i\omega\hat{\Omega}(x, 0)\right] - \left[\frac{\partial\hat{v}_z}{\partial t}(x, 0) - i\omega\hat{v}_z(x, 0)\right]\frac{d\rho_0}{dx}\right\}, \tag{7}$$

where the vorticity, $\Omega(x, t)\hat{\mathbf{y}} = \hat{\Omega}(x, t)e^{ikz}\hat{\mathbf{y}} = \nabla \times \mathbf{v}(\mathbf{x}, t)$, is given by

$$\hat{\Omega}(\mathbf{x}, t) = -\frac{i}{k}\left(\frac{\partial^2\hat{v}_x}{\partial x^2} - k^2\hat{v}_x\right). \tag{8}$$

(this differs from Rae & Roberts (1981) by a factor of -1 due to taking Fourier forms like e^{ikz} rather than e^{-ikz}). For utility later on, we define $\Psi_0 = \Psi(x, 0)$ by function $\Psi(x, t) = k(\rho_0\hat{\Omega}(x, t) - \rho_0'\hat{v}_z(x, t))$ so that $f(x) = \omega\Psi_0 + i\frac{\partial\Psi_0}{\partial t}$.

Consider equilibrium magnetic field and density profiles given by

$$B(x) = \begin{cases} B_1, & \text{if } x < -x_0, \\ B_0, & \text{if } |x| \le x_0, \\ B_2, & \text{if } x > x_0, \end{cases} \quad \text{and} \quad \rho(x) = \begin{cases} \rho_1, & \text{if } x < -x_0, \\ \rho_0, & \text{if } |x| \le x_0, \\ \rho_2, & \text{if } x > x_0, \end{cases} \tag{9}$$

where B_i and ρ_i are uniform for $i = 0, 1, 2$. This establishes the plasma as magnetic slab embedded in an asymmetric magnetic background. So that the structure is in equilibrium, we require that the total pressure in each of the three regions be equal, that is

$$p_1 + \frac{B_1^2}{2\mu} = p_0 + \frac{B_0^2}{2\mu} = p_2 + \frac{B_2^2}{2\mu}, \tag{10}$$

where p_i is the gas pressure in region i and μ is the permeability of free space. The sound speed in each region is $c_i = \sqrt{\gamma p_i/\rho_i}$, for $i = 0, 1, 2$, where γ is the adiabatic index. The Alfvn speed is denoted by $v_{Ai} = B_i/\sqrt{\rho_i \mu}$, for $i = 0, 1, 2$. Equation 10 describing equilibrium pressure balance yields the following relationship between the density ratios and characteristic speeds for any two regions,

$$\frac{\rho_i}{\rho_j} = \frac{c_j^2 + \frac{1}{2}\gamma v_{Aj}^2}{c_i^2 + \frac{1}{2}\gamma v_{Ai}^2}, \quad \text{where} \quad i = 0, 1, 2; \quad j = 0, 1, 2; \quad i \neq j. \tag{11}$$

In this equilibrium, transverse velocity perturbations are related to initial perturbations in the following way:

$$\frac{d^2 \tilde{v}_x}{dx^2} - k^2 \tilde{v}_x = \begin{cases} f(x)/\epsilon_1, & \text{if } x < -x_0, \\ f(x)/\epsilon_0, & \text{if } |x| < x_0, \\ f(x)/\epsilon_2, & \text{if } x > x_0, \end{cases} \tag{12}$$

under the boundary conditions $\tilde{v}_x(-x_0) = \tilde{A}_1$ and $\tilde{v}_x(x_0) = \tilde{A}_2$.

Sturm-Liouville Theory tells us that the Green's function, $G(x; s)$, corresponding to Equation (12) must satisfy

$$\frac{\partial^2 G}{\partial x^2} - k^2 G = \delta(x - s), \quad G(-x_0; s) = G(x_0; s) = 0, \tag{13}$$

where δ denotes the Dirac Delta function. It is instructive to piecewise define the Green's function as

$$G(x; s) = \begin{cases} G_1(x; s), & \text{if } x < -x_0, \\ G_0(x; s), & \text{if } |x| < x_0, \\ G_2(x; s), & \text{if } x_0 < x. \end{cases} \tag{14}$$

The general solution, for $|x| < x_0$, of the equation for G_0 is

$$G_0(x; s) = c_1 \sinh(k(x - x_0)) + c_2 \sinh(k(x + x_0)), \tag{15}$$

where $c_1 = 0$ for $x < s$ and $c_2 = 0$ for $x > s$. In accordance with standard Green's Functions procedure (Boyce & DiPrima 2012), ensuring G_0 and $\partial G_0/\partial x$ have jumps of 0 and 1 at $x = s$, respectively, determines c_1 and c_2, so that $G_0(x; s)$ is

$$G_0(x; s) = \frac{1}{k \sinh(2kx_0)} \begin{cases} \sinh(k(s - x_0)) \sinh(k(x + x_0)), & \text{if } -x_0 < x < s, \\ \sinh(k(x - x_0)) \sinh(k(s + x_0)), & \text{if } s < x < x_0. \end{cases} \tag{16}$$

Because the boundary conditions are inhomogeneous, we must add to the standard Green's function solution a term that is a solution to the homogeneous equation and the inhomogeneous boundary conditions. In this manner, we find that the solution within the slab is

$$\tilde{v}_x(x) = \frac{1}{\sinh 2kx_0} \left[\tilde{A}_1 \sinh(k(x_0 - x)) + \tilde{A}_2 \sinh(k(x_0 + x)) \right] + \frac{1}{\epsilon_0} \int_{-x_0}^{x_0} G_0(x; s) f(s) ds. \tag{17}$$

Similarly, we find that the Green's function for the plasma outside the slab is

$$G_1(x;s) = \frac{1}{k}\begin{cases} e^{k(x+x_0)}\sinh(k(s+x_0)), & \text{if } x < s, \\ e^{k(s+x_0)}\sinh(k(x+x_0)), & \text{if } s < x < -x_0, \end{cases} \tag{18}$$

for $x < -x_0$, and

$$G_2(x;s) = -\frac{1}{k}\begin{cases} e^{-k(s-x_0)}\sinh(k(x-x_0)), & \text{if } x_0 < x < s, \\ e^{-k(x-x_0)}\sinh(k(s-x_0)), & \text{if } s < x, \end{cases} \tag{19}$$

for $x > x_0$. Therefore the solution is

$$\tilde{v}_x(x) = \tilde{A}_1 e^{k(x_0+x)} + \frac{1}{\epsilon_1}\int_{-\infty}^{-x_0} G_1(x;s)f(s)ds, \tag{20}$$

for $x < -x_0$, and

$$\tilde{v}_x(x) = \tilde{A}_2 e^{k(x_0-x)} + \frac{1}{\epsilon_2}\int_{x_0}^{\infty} G_2(x;s)f(s)ds, \tag{21}$$

for $x > x_0$.

2.1. *Matching solutions*

To establish physically relevant solutions, we require that the transverse velocity and the total pressure are continuous over each interface. Given the above solutions, the transverse velocity is automatically continuous over the boundaries. The perturbation in the total pressure, P_T, for a compressible plasma is given (for example, in Allcock & Erdélyi (2017)) by

$$\tilde{P}_T(x) = \frac{\Lambda}{m}\frac{d\tilde{v}_x}{dx}, \tag{22}$$

where

$$\Lambda = \frac{i\rho(\omega^2 - k^2 v_A^2)}{m\omega}, \quad m^2 = \frac{(k^2 v_A^2 - \omega^2)(k^2 c_0^2 - \omega^2)}{(c_0^2 + v_A^2)(k^2 c_T^2 - \omega^2)}, \quad \text{and} \quad c_T^2 = \frac{c_0^2 v_A^2}{c_0^2 + v_A^2}. \tag{23}$$

When the plasma is incompressible, $m^2 \to k^2$, therefore continuity in total pressure is equivalent to continuity in $\epsilon(x)\tilde{v}_x'(x)$ for an incompressible plasma.

Applying this boundary condition gives

$$\tilde{A}_1(\omega) = \frac{T_1(\omega)}{kD(\omega)}, \quad \tilde{A}_2(\omega) = \frac{T_2(\omega)}{kD(\omega)}, \tag{24}$$

where

$$T_1(\omega) = (I_0^- - I_1)[\epsilon_0\cosh(2kx_0) + \epsilon_2\sinh(2kx_0)] - \epsilon_0(I_0^+ + I_2), \tag{25}$$
$$T_2(\omega) = \epsilon_0(I_0^- - I_1) - (I_0^+ + I_2)[\epsilon_0\cosh(2kx_0) + \epsilon_1\sinh(2kx_0)], \tag{26}$$
$$D(\omega) = \epsilon_0(\epsilon_1 + \epsilon_2)\cosh(2kx_0) + (\epsilon_0^2 + \epsilon_1\epsilon_2)\sinh(2kx_0), \tag{27}$$

and

$$I_0^{\pm} = \int_{-x_0}^{x_0} \frac{\sinh(k(s \pm x_0))}{\sinh(2kx_0)} f(s)ds, \quad I_1 = \int_{-\infty}^{-x_0} e^{k(s+x_0)} f(s)ds, \quad I_2 = \int_{x_0}^{\infty} e^{k(x_0-s)} f(s)ds. \tag{28}$$

2.2. *Solution in time*

To recover the transverse velocity, $v_x(x,t)$, we employ the inverse Laplace transform (non-standard, discussed in Appendix A), such that

$$v_x(x,t) = \mathcal{L}^{-1}\{\tilde{v}_x(x,\omega)\} = \frac{1}{2\pi}\lim_{L\to\infty}\int_{i\gamma-L}^{i\gamma+L} \tilde{v}_x(x,\omega)e^{-i\omega t}d\omega, \tag{29}$$

where γ is a real number such that all the singularities of the integrand are below the contour of integration to ensure that all singularities contribute to the integral. The integral is evaluated along an infinite horizontal line in the upper half of the complex plane and is dependent on the singularities (with respect to ω) of $\tilde{v}_x$, whose residues determine the value of the contour integral.

Focusing firstly on the region $x < -x_0$, the solution is

$$v_x = \mathcal{L}^{-1}\left\{\tilde{A}_1 e^{k(x+x_0)} + \frac{1}{\epsilon_1}\int_{-\infty}^{-x_0} G_1(x;s)f(s)ds\right\}, \tag{30}$$

$$= e^{k(x+x_0)}\mathcal{L}^{-1}\left\{\tilde{A}_1\right\} + \int_{-\infty}^{-x_0} G_1(x;s)\mathcal{L}^{-1}\left\{\frac{f(s)}{\epsilon_1}\right\}ds, \tag{31}$$

$$= e^{k(x+x_0)}\mathcal{L}^{-1}\left\{\tilde{A}_1\right\} + \int_{-\infty}^{-x_0} G_1(x;s)\left[\Psi_0\mathcal{L}^{-1}\left\{\frac{\omega}{\epsilon_1}\right\} + i\frac{\partial\Psi_0}{\partial t}\mathcal{L}^{-1}\left\{\frac{1}{\epsilon_1}\right\}\right]ds, \tag{32}$$

CONSIDER ADDING AN OVERVIEW OF THE FOLLOWING METHOD TO EVALUATE THE INTEGRALS IN THE ABOVE.

The first inverse Laplace transform (ILT), $\mathcal{L}^{-1}\left\{\tilde{A}_1\right\}$, in the above solution is calculated as follows. The functions $\epsilon_{0,1,2}$ are polynomial in ω, and are therefore entire. The integrals $I_{1,2}$ and $I_0^{\pm}$ are not functions of ω so also contribute no singularities in ω. Therefore, T_1 and T_2 are entire functions. Hence,

The zeros of $D(\omega)$ are determined by firstly noting that $D = 0$ is the dispersion relation of the corresponding eigenvalue problem solved by Zsámberger et al. (2018). They show that the dispersion relation governing transverse wave propagation parallel to the magnetic field in an asymmetric slab of compressible plasma is given by

$$2(\Lambda_0^2 + \Lambda_1\Lambda_2) + \Lambda_0(\Lambda_1 + \Lambda_2)[\tanh(m_0x_0) + \coth(m_0x_0)] = 0, \tag{33}$$

where

$$\Lambda_j = -\frac{i\rho_j(k^2v_{Aj}^2 - \omega^2)}{\omega m_j}, \quad m_j^2 = \frac{(k^2c_j^2 - \omega^2)(k^2v_{Aj}^2 - \omega^2)}{(c_j^2 + v_{Aj}^2)(k^2c_{Tj}^2 - \omega^2)}, \quad \text{and} \quad c_{Tj}^2 = \frac{c_j^2v_{Aj}^2}{c_j^2 + v_{Aj}^2}, \tag{34}$$

for $j = 0, 1, 2$. When compressibility is neglected, such that the sound speeds, c_j, approach infinity, we have $c_{Tj}^2 \to v_{Aj}^2$, $m_j^2 \to k^2$, and therefore $\Lambda_j = -i\rho_j(k^2v_{Aj}^2 - \omega^2)/\omega k = -i\epsilon_j/\omega k$, for $j = 0, 1, 2$. Therefore, Equation (33) can be reduced to the dispersion relation for an incompressible magnetic slab, which is

$$2(\epsilon_0^2 + \epsilon_1\epsilon_2) + \epsilon_0(\epsilon_1 + \epsilon_2)[\tanh(m_0x_0) + \coth(m_0x_0)] = 0, \tag{35}$$

which can easily to shown to be equivalent to $D(\omega) = 0$, where $D(\omega)$ is given by Equation (27). It follows that the zeros of $D(\omega)$ are precisely the eigenvalues of the asymmetric incompressible magnetic slab. This is a specific case of a powerful general result for initial value problems and their corresponding eigenvalue problems which is explored in the MHD setting by Goedbloed & Poedts (2004), Chapter 10.2.

The zeros of D are found by writing the equation $D(\omega) = 0$ as

$$\epsilon_0(\epsilon_1 + \epsilon_2) + (\epsilon_0^2 + \epsilon_1\epsilon_2)\tanh(2kx_0) = 0 \tag{36}$$

and substituting expressions for $\epsilon(x)$, which gives

$$\rho_0(k^2v_{A0}^2 - \omega^2)[\rho_1(k^2v_{A1}^2 - \omega^2) + \rho_2(k^2v_{A2}^2 - \omega^2)] + [\rho_0^2(k^2v_{A0}^2 - \omega^2)^2 + \rho_1\rho_2(k^2v_{A1}^2 - \omega^2)(k^2v_{A2}^2 - \omega^2)]\tanh(2kx_0) = 0. \tag{37}$$

The above equation can be rewritten as a quadratic in $(\omega/k)^2$, namely

$$\left(\frac{\omega}{k}\right)^4[(\rho_0^2 + \rho_1\rho_2)\tanh(2kx_0) + \rho_0(\rho_1 + \rho_2)] \tag{38}$$

$$-\left(\frac{\omega}{k}\right)^2[(2\rho_0^2v_{A0}^2 + \rho_1\rho_2(v_{A1}^2 + v_{A2}^2))\tanh(2kx_0) + \rho_0\{\rho_1(v_{A0}^2 + v_{A1}^2) + \rho_2(v_{A0}^2 + v_{A2}^2)\}] \tag{39}$$

$$+[(\rho_0^2v_{A0}^4 + \rho_1\rho_2v_{A1}^2v_{A2}^2)\tanh(2kx_0) + \rho_0v_{A0}^2(\rho_1v_{A1}^2 + \rho_2v_{A2}^2)] = 0, \tag{40}$$

which has solutions

$$\left(\frac{\omega_{0+}}{k}\right)^2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \text{and} \quad \left(\frac{\omega_{0-}}{k}\right)^2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \tag{41}$$

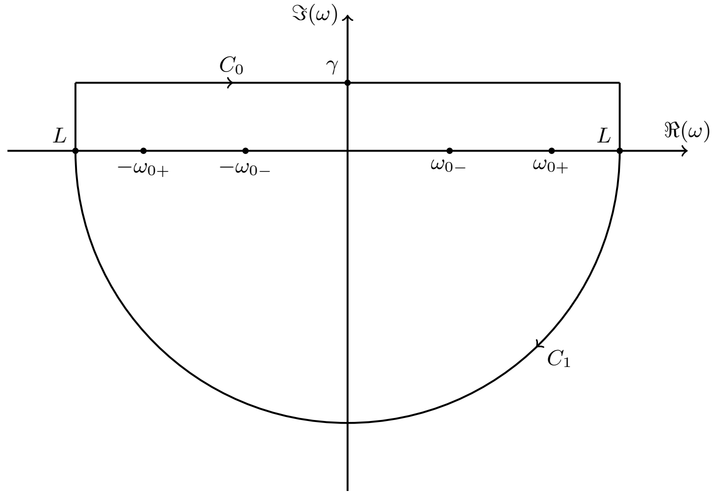

Figure 1. Bromwich contour for the complex integration of $\tilde{A}_{1,2}$.

where

$$a = (\rho_0^2 + \rho_1\rho_2)\tanh(2kx_0) + \rho_0(\rho_1 + \rho_2), \tag{42}$$

$$b = -(2\rho_0^2 v_{A0}^2 + \rho_1\rho_2(v_{A1}^2 + v_{A2}^2))\tanh(2kx_0) - \rho_0\{\rho_1(v_{A0}^2 + v_{A1}^2) + \rho_2(v_{A0}^2 + v_{A2}^2)\}, \tag{43}$$

$$c = (\rho_0^2 v_{A0}^4 + \rho_1\rho_2 v_{A1}^2 v_{A2}^2)\tanh(2kx_0) + \rho_0 v_{A0}^2(\rho_1 v_{A1}^2 + \rho_2 v_{A2}^2). \tag{44}$$

We know from analysis of the dispersion relation (Zsámberger et al. 2018) that the above solutions $\pm\omega_{0\pm}$ must be real. Additionally, the solutions corroborate with the corresponding incompressible eigenfrequencies for an interface and a symmetric slab, as shown in Appendices B.1 and B.2, respectively.

With the location of the singularities of the integrand in hand, we can return to evaluating the first integral in Equation (32) by making use of the famous Residue Theorem of complex analysis. For this theorem to apply, we must integrate around a closed contour instead of the infinite line in Equation (32). To remedy this, we can choose a series of contours such that the limit of the integrals over these contours is equal to the limiting integral in Equation (32). We using the fact that the function $T_1(\omega)$ is entire to construct a Bromwich contour, $C = C_0 + C_1$, where C_0 is a straight line from $(-L, \gamma)$ to (L, γ), and C_1 connects $(-L, \gamma)$ and (L, γ) via a semi-circle to ensure that C encloses the zeros of D at $\pm\omega_{0\pm}$, which is general case is shown by Figure 1. In the limit $L \to \infty$, we recover the desired integral.

Considering first the integral along C_1, we see that the integrand in question behave like $T_1(\omega)/kD(\omega) = \mathcal{O}(|\omega|^{-2})$, as $|\omega| \to \infty$. Therefore, the integral around the semi-circle vanishes, *i.e.*

$$\lim_{L\to\infty} \int_{C_1} \frac{T_1(\omega)}{kD(\omega)} e^{-i\omega t} d\omega = 0. \tag{45}$$

So we have successfully chosen a sequence of contours which limit to the same limit as that of Equation (32). Thankfully the integral around each of these new contours is much easier to calculate than the original, as will be exploited in the following paragraph.

Next, considering the integral along contour C, since it is integrated in the clockwise direction it is therefore equal to $-2\pi i$ multiplied by the sum of the residues of the singularities at $\omega = \pm\omega_{0\pm}$. The residue at $\omega = \omega_0$ can be evaluated using L'Hopital's Rule (the necessary requirements that ensure that this rule can be used are verified in Appendix C, to within a free choice of initial condition, $f(x)$, as

$$\operatorname{Res}\left[\frac{T_1(\omega)}{kD(\omega)} e^{-i\omega t}, \omega = \omega_{0+}\right] = \lim_{\omega\to\omega_{0+}} \frac{(\omega - \omega_{0+})T_1(\omega)}{kD(\omega)} e^{-i\omega t} \tag{46}$$

$$= \lim_{\omega\to\omega_{0+}} \frac{1}{kD'(\omega)} [T_1(\omega) + (\omega - \omega_{0+})T_1'(\omega) - it(\omega - \omega_{0+})T_1(\omega)] e^{-i\omega t} \tag{47}$$

$$= \chi_{1+} e^{-i\omega_{0+} t}, \tag{48}$$

where $\chi_{1+} = T_1(\omega_{0+})/kD'(\omega_{0+})$. In the following, we define $\chi_{1-} = T_1(\omega_{0-})/kD'(\omega_{0-})$, and similar for subscripts 0, and 2. Similarly, the residues at $\omega = -\omega_{0+}$ and $\omega = \pm\omega_{0-}$ are

$$\mathrm{Res}\left[\frac{T_1(\omega)}{kD(\omega)}e^{-i\omega t}, \omega = -\omega_{0+}\right] = -\chi_{1+}e^{i\omega_{0+}t}, \tag{49}$$

$$\mathrm{Res}\left[\frac{T_1(\omega)}{kD(\omega)}e^{-i\omega t}, \omega = \omega_{0-}\right] = \chi_{1-}e^{-i\omega_{0-}t}, \tag{50}$$

$$\mathrm{Res}\left[\frac{T_1(\omega)}{kD(\omega)}e^{-i\omega t}, \omega = -\omega_{0-}\right] = -\chi_{1-}e^{i\omega_{0-}t}, \tag{51}$$

$$\tag{52}$$

respectively. In the above derivation, we have used the fact that D and T_1 are even functions and D' is an odd function of ω.

Putting all of the above results together, we find that solution of the first ILT of Equation (32), or equivalently the boundary velocity, is

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\tilde{A}_1\right\} &= \frac{1}{2\pi}\lim_{L\to\infty}\int_{C_0}\frac{T_1(\omega)}{kD(\omega)}e^{-i\omega t}d\omega && (53)\\
&= \frac{1}{2\pi}\lim_{L\to\infty}\int_{C}\frac{T_1(\omega)}{kD(\omega)}e^{-i\omega t}d\omega && (54)\\
&= -i\sum\mathrm{Res}\left[\frac{T_1(\omega)}{kD(\omega)}e^{-i\omega t}, \omega = \pm\omega_{0\pm}\right] && (55)\\
&= -i[\chi_{1+}(e^{-i\omega_{0+}t} - e^{i\omega_{0+}t}) + \chi_{1-}(e^{-i\omega_{0-}t} - e^{i\omega_{0-}t})] && (56)\\
&= -2[\chi_{1+}\sin(\omega_{0+}t) + \chi_{1-}\sin(\omega_{0-}t)]. && (57)
\end{aligned}$$

The second ILT in Equation (32), $\mathcal{L}^{-1}\{\omega/\epsilon_1\}$, is calculated as follows.

$$\mathcal{L}^{-1}\left\{\frac{\omega}{\epsilon_1}\right\} = \frac{1}{2\pi}\lim_{L\to\infty}\int_{i\gamma-L}^{i\gamma+L}\frac{\omega e^{-i\omega t}}{\epsilon_1}d\omega, = \frac{1}{2\pi\rho_1}\lim_{L\to\infty}\int_{i\gamma-L}^{i\gamma+L}\frac{\omega e^{-i\omega t}}{(kv_{A1}+\omega)(kv_{A1}-\omega)}d\omega, \tag{58}$$

whose integrand has 2 simple poles at $\omega = \pm kv_{A1}$. By noting that the integrand approaches zero as $\omega \to \infty$, we can construct a Bromwich contour as shown in Figure 2. The residues of the integrand at the simple poles are

$$\begin{aligned}
\mathrm{Res}\left[\frac{\omega e^{-i\omega t}}{k^2v_{A1}^2-\omega^2}, \omega = kv_{A1}\right] &= \lim_{\omega\to kv_{A1}}\frac{(\omega - kv_{A1})\omega e^{-i\omega t}}{k^2v_{A1}^2-\omega^2} && (59)\\
&= -\frac{e^{-ikv_{A1}t}}{2}, && (60)\\
\mathrm{Res}\left[\frac{\omega e^{-i\omega t}}{k^2v_{A1}^2-\omega^2}, \omega = -kv_{A1}\right] &= \lim_{\omega\to -kv_{A1}}\frac{(\omega + kv_{A1})\omega e^{-i\omega t}}{k^2v_{A1}^2-\omega^2} && (61)\\
&= -\frac{e^{ikv_{A1}t}}{2}. && (62)
\end{aligned}$$

Therefore, the final integral in Equation (32) is

$$\mathcal{L}^{-1}\left\{\frac{\omega}{\epsilon_1}\right\} = -\frac{i}{\rho_1}\sum\mathrm{Res}\left[\frac{\omega e^{-i\omega t}}{k^2v_{A1}^2-\omega^2}, \omega = \pm kv_{A1}\right] = \frac{1}{\rho_1}\cos kv_{A1}t. \tag{63}$$

Similarly, the third and final ILT in Equation (32), $\mathcal{L}^{-1}\{1/\epsilon_1\}$, is calculated as follows.

$$\mathcal{L}^{-1}\left\{\frac{1}{\epsilon_1}\right\} = \frac{1}{2\pi}\lim_{L\to\infty}\int_{i\gamma-L}^{i\gamma+L}\frac{\omega e^{-i\omega t}}{\epsilon_1}d\omega, = \frac{1}{2\pi\rho_1}\lim_{L\to\infty}\int_{i\gamma-L}^{i\gamma+L}\frac{e^{-i\omega t}}{(kv_{A1}+\omega)(kv_{A1}-\omega)}d\omega, \tag{64}$$

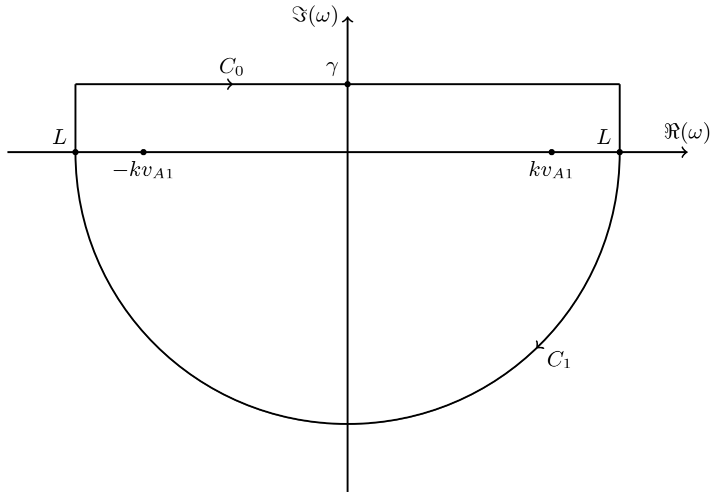

Figure 2. Bromwich contour for the complex integration of the integrand of J_1.

whose integrand has 2 simple poles at $\omega = \pm kv_{A1}$. By noting that the integrand approaches zero as $\omega \to \infty$, we can construct a Bromwich contour as shown in Figure 2. The residues of the integrand at the simple poles are

$$\mathrm{Res}\left[\frac{e^{-i\omega t}}{k^2v_{A1}^2-\omega^2}, \omega = kv_{A1}\right] = \lim_{\omega\to kv_{A1}} \frac{(\omega - kv_{A1})e^{-i\omega t}}{k^2v_{A1}^2-\omega^2} \tag{65}$$

$$= -\frac{e^{-ikv_{A1}t}}{2kv_{A1}}, \tag{66}$$

$$\mathrm{Res}\left[\frac{\omega e^{-i\omega t}}{k^2v_{A1}^2-\omega^2}, \omega = -kv_{A1}\right] = \lim_{\omega\to -kv_{A1}} \frac{(\omega + kv_{A1})e^{-i\omega t}}{k^2v_{A1}^2-\omega^2} \tag{67}$$

$$= \frac{e^{ikv_{A1}t}}{2kv_{A1}}. \tag{68}$$

Therefore, the final integral in Equation (32) is

$$\mathcal{L}^{-1}\left\{\frac{1}{\epsilon_1}\right\} = -\frac{i}{\rho_1}\sum \mathrm{Res}\left[\frac{e^{-i\omega t}}{k^2v_{A1}^2-\omega^2}, \omega = \pm kv_{A1}\right] = \frac{\sin kv_{A1}t}{\rho_1 kv_{A1}}. \tag{69}$$

Combining the above expressions we find that the transverse velocity solution for $x < -x_0$ is

$$v_x = -2e^{k(x+x_0)}[\chi_{1+}\sin(\omega_{0+}t) + \chi_{1-}\sin(\omega_{0-}t)] + \frac{1}{\rho_1}\int_{-\infty}^{-x_0} G_1(x;s)\left[\Psi_0\cos kv_{A1}t + i\frac{\partial\Psi_0}{\partial t}\frac{\sin kv_{A1}t}{kv_{A_1}}\right] ds. \tag{70}$$

Similarly, the transverse velocity for the region $x > x_0$ is

$$v_x = \mathcal{L}^{-1}\left\{\tilde{A}_2 e^{k(x_0-x)} + \frac{1}{\epsilon_2}\int_{x_0}^{\infty} G_2(x;s)f(s)ds\right\}, \tag{71}$$

$$= e^{k(x_0-x)}\mathcal{L}^{-1}\left\{\tilde{A}_2\right\} + \int_{x_0}^{\infty} G_2(x;s)\mathcal{L}^{-1}\left\{\frac{f(s)}{\epsilon_2}\right\} ds, \tag{72}$$

$$= -2e^{k(x_0-x)}[\chi_{2+}\sin(\omega_{0+}t) + \chi_{2-}\sin(\omega_{0-}t)] + \frac{1}{\rho_2}\int_{x_0}^{\infty} G_2(x;s)\left[\Psi_0\cos kv_{A2}t + i\frac{\partial\Psi_0}{\partial t}\frac{\sin kv_{A2}t}{kv_{A_2}}\right] ds. \tag{73}$$

$$\tag{74}$$

Finally, for the region $|x| < x_0$, it is

$$v_x = \mathcal{L}^{-1}\left\{\frac{1}{\sinh 2kx_0}\left[\tilde{A}_1 \sinh(k(x_0 - x)) + \tilde{A}_2 \sinh(k(x_0 + x))\right] + \frac{1}{\epsilon_0}\int_{-x_0}^{x_0} G_0(x;s)f(s)ds\right\}, \tag{75}$$

$$= \frac{1}{\sinh 2kx_0}\left[\mathcal{L}^{-1}\{\tilde{A}_1\}\sinh(k(x_0 - x)) + \mathcal{L}^{-1}\{\tilde{A}_2\}\sinh(k(x_0 + x))\right] + \int_{-x_0}^{x_0} G_0(x;s)\mathcal{L}^{-1}\left\{\frac{f(s)}{\epsilon_0}\right\}ds, \tag{76}$$

$$= -\frac{2}{\sinh 2kx_0}\{[\chi_{1+}\sin(\omega_{0+}t) + \chi_{1-}\sin(\omega_{0-}t)]\sinh(k(x_0 - x)) + [\chi_{2+}\sin(\omega_{0+}t) + \chi_{2-}\sin(\omega_{0-}t)]\sinh(k(x_0 + x))\}$$
$$+ \frac{1}{\rho_2}\int_{-x_0}^{x_0} G_0(x;s)\left[\Psi_0 \cos kv_{A0}t + i\frac{\partial \Psi_0}{\partial t}\frac{\sin kv_{A0}t}{kv_{A_0}}\right]ds. \tag{77}$$

2.3. *Specific initial conditions*

2.3.1. *Uniform initial vorticity*

Let $\Omega(x, 0) = \Omega_0$ be constant. Therefore, $\Psi_0 = k\rho_0\Omega_0$ and $\partial\Psi_0/\partial t = 0$.

M. Allcock acknowledges the support from the University Prize Scholarship at the University of Sheffield. R. Erdélyi acknowledges the support from the UK Science and Technology Facilities Council (STFC grant number: ST/M000826/1) and the Royal Society.

APPENDIX

A. NON-STANDARD LAPLACE TRANSFORM

Consider a function $f(t)$, whose standard Laplace transform, $F_1(\omega)$, and non-standard Laplace transform, $F_2(\omega)$, are

$$F_1(\omega) = \int_0^\infty f(t)e^{-\omega t}dt, \quad \text{and} \quad F_2(\omega) = \int_0^\infty f(t)e^{i\omega t}dt. \tag{A1}$$

Trivially, $F_1(-i\omega) = F_2(\omega)$. Using the standard inverse Laplace transform, and letting γ be real and greater than the real part of all the singularities of $F_1(\omega)$, the original function $f(t)$ can be written

$$f(t) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma - iT}^{\gamma + iT} F_1(\omega)e^{\omega t}d\omega, \tag{A2}$$

$$= \frac{1}{2\pi i}\lim_{T\to\infty}\int_{i\gamma - T}^{i\gamma + T} F_1(-i\omega)e^{-i\omega t}(-id\omega), \tag{A3}$$

$$= \frac{1}{2\pi}\lim_{T\to\infty}\int_{i\gamma - T}^{i\gamma + T} F_2(\omega)e^{-i\omega t}d\omega. \tag{A4}$$

Therefore, the inverse transform of the non-standard Laplace transform is

$$f(t) = \frac{1}{2\pi}\lim_{T\to\infty}\int_{i\gamma - T}^{i\gamma + T} F_2(\omega)e^{-i\omega t}d\omega. \tag{A5}$$

B. CORROBORATION OF INCOMPRESSIBLE SOLUTIONS WITH PREVIOUS RESULTS

B.1. *Interface*

When we let the width of an asymmetric slab vanish, we recover the traditional interface geometry. Letting $x_0 \to 0$, the parameters a, b, and c, from Equations (42), (43), and (44), reduce to

$$a = \rho_0(\rho_1 + \rho_2), \tag{B6}$$

$$b = -\rho_0 v_{A0}^2(\rho_1 + \rho_2) - \rho_0(\rho_1 v_{A1}^2 + \rho_2 v_{A2}^2), \tag{B7}$$

$$c = \rho_0 v_{A0}^2(\rho_1 v_{A1}^2 + \rho_2 v_{A2}^2). \tag{B8}$$

Therefore, when the slab width vanishes, the eigenmodes given by Equation (41) reduce to

$$\left(\frac{\omega_0}{k}\right)^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \begin{cases} v_{A0}^2, \\ \frac{\rho_1 v_{A1}^2 + \rho_2 v_{A2}^2}{\rho_1 + \rho_2}. \end{cases} \tag{B9}$$

The first solution above is degenerate because, while the parameter v_{A0} makes sense in the limit as the slab width vanishes, it is meaningless in an interface system constructed without an inner region. The second solution corroborates with the surface eigenfrequencies of an interface, as expected (Roberts 1981a).

B.2. *Symmetric slab*

By letting the parameters on each external plasma region be equal (*i.e.* $\rho_1 = \rho_2 = \rho_e$, and similar for the magnetic field and Aflvén speed) the asymmetric slab is reduced to a symmetric slab. In this limit, the parameters a, b, and c, from Equations (42), (43), and (44), can be reduced to

$$a = \frac{2}{\tau_0 + c_0}\left[\rho_0^2 + \rho_e^2 + \rho_0\rho_e(\tau_0 + c_0)\right], \tag{B10}$$

$$b = \frac{-2}{\tau_0 + c_0}\left[2(\rho_0^2 v_{A0}^2 + \rho_e^2 v_{Ae}^2) + \rho_0\rho_e(v_{A0}^2 + v_{Ae}^2)(\tau_0 + c_0)\right], \tag{B11}$$

$$c = \frac{2}{\tau_0 + c_0}\left[\rho_0^2 v_{A0}^4 + \rho_e^2 v_{Ae}^4 + \rho_0\rho_e v_{A0}^2 v_{Ae}^2(\tau_0 + c_0)\right], \tag{B12}$$

where $\tau_0 = \tanh kx_0$ and $c_0 = \coth kx_0$. The discriminant in the solution, Equation (41), reduces to

$$b^2 - 4ac = 4\rho_0^2\rho_e^2(v_{A0}^2 - v_{Ae}^2)^2\left(\frac{\tau_0 - c_0}{\tau_0 + c_0}\right)^2. \tag{B13}$$

Therefore, the eigenfrequencies reduce to

$$\left(\frac{\omega_0}{k}\right)^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \begin{cases} \frac{\rho_0 v_{A0}^2 + \rho_e v_{Ae}^2 c_0}{\rho_0 + \rho_e c_0}, \\ \frac{\rho_0 v_{A0}^2 + \rho_e v_{Ae}^2 \tau_0}{\rho_0 + \rho_e \tau_0}, \end{cases} \tag{B14}$$

which corroborates with Equation (12) in Roberts (1981b).

C. VALIDATION OF L'HOPITAL'S RULE

L'Hopital's Rule is a powerful tool for evaluating limits of quotients of functions, provided that these functions satisfy certain necessary criteria. L'Hopital's Rule (for function of complex variables) states that, for functions f and g which are analytic at a point z_0, if $f(x_0) = g(z_0) = 0$, $g'(z_0) \neq 0$ then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z)}{g'(z)}. \tag{C15}$$

There are stronger formulations of the complex L'Hopital's rule which are superfluous for our present needs CITATION?.

Applied to the present problem, the requirements for L'Hopital's rule to hold are:

1. The functions $(\omega - \omega_{0+})T_1(\omega)e^{-i\omega t}$ and $kD(\omega)$ are analytic at ω_{0+},
2. $[(\omega - \omega_{0+})T_1(\omega)e^{-i\omega t}]|_{\omega=\omega_{0+}} = kD(\omega_{0+}) = 0$,
3. $kD'(\omega_{0+}) \neq 0$.

Below, we validate that each of these conditions holds:

1. Functions $T_1(\omega)$ and $D(\omega)$ are polynomials and hence are analytic. Since products of analytic functions are also analytic, $(\omega - \omega_{0+})T_1(\omega)e^{-i\omega t}$ and $kD(\omega)$ are analytic. In particular, they are analytic at ω_{0+}.

2. The point ω_{0+} is a zero of $D(\omega)$ (by definition of ω_{0+}) and $T_1(\omega)$ is regular at ω_{0+}, therefore $[(\omega - \omega_{0+})T_1(\omega)e^{-i\omega t}]|_{\omega=\omega_{0+}} = kD(\omega_{0+}) = 0$.

3. The function $D'(\omega)$ can be rewritten as

$$
\begin{aligned}
D'(\omega) = 2k^2\omega \Bigg[& 2\{c_0\rho_0(\rho_1+\rho_2) + s_0(\rho_0^2+\rho_1\rho_2)\}\frac{\omega^2}{k^2} \\
& - \{c_0\rho_0[\rho_1 v_{A1}^2 + \rho_2 v_{A2}^2 + (\rho_1+\rho_2)v_{A0}^2] + s_0[2\rho_0^2 v_{A0}^2 + \rho_1\rho_2(v_{A1}^2 + v_{A2}^2)]\}\Bigg] \qquad \text{(C16)} \\
= 2k^2\omega \left[2a\frac{\omega^2}{k^2} + b\right], & \qquad \text{(C17)}
\end{aligned}
$$

where a and b are given by Equations (42) and (43). The above equation has zeros at $\omega = 0$ and $\omega = \pm\omega_0$, where

$$\frac{\omega_0^2}{k^2} = -\frac{b}{2a}. \qquad \text{(C18)}$$

Therefore, the zeros of the function D are always at least a factor of i away from the zeros of D' (and are a factor of exactly i away if and only if $d = b^2 - 4ac = 0$). The result follows.

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