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TEMPORAL EVOLUTION OF ASYMMETRIC SLAB MAGNETOHYDRODYNAMIC WAVES

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ABSTRACT

Abstract (250 word limit)

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1. INTRODUCTION

Numerical results: Terradas et al. (2006)

Physicality of the principle leaky kink mode: Cally (2003) solved the initial value problem of transverse waves in a cold magnetic flux tube. Ruderman & Roberts (2006a) repeated it showing PLK modes are on an unphysical banch of the complex plane (?). Commented on by Cally (2006) and in return by Ruderman & Roberts (2006b). Settled (?) by considering numerical solution by Terradas et al. (2007) and analytically by Andries & Goossens (2007). So PLK modes are not physical.

2. INITIAL VALUE PROBLEM - INCOMPRESSIBLE MAGNETIC SLAB

Consider an equilibrium plasma with magnetic field $B_0(x)\hat{\mathbf{z}}$, density $\rho_0(x)$, and pressure $p_0(x)$, without gravity. In the absence of structuring in the z-direction and considering perturbations in the (x,z)-plane only, we can take Fourier components for velocity and other parameters like $\mathbf{v}(\mathbf{x},t) = \hat{\mathbf{v}}(x)e^{i(kz+ly-\omega t)}$. The velocity perturbation amplitude in the inhomogeneous direction of an ideal plasma is given by

$$\frac{d}{dx}\left(\frac{\epsilon(x)}{l^2 - m_0^2(x)}\frac{d\hat{v}_x}{dx}\right) - \epsilon(x)\hat{v}_x = 0, \tag{1}$$

where

$$\epsilon(x) = \rho_0(x)[k^2 v_A^2(x) - \omega^2], \quad m_0^2(x) = \frac{(k^2c_0^2(x) - \omega^2)(k^2v_A^2(x) - \omega^2)}{(c_0^2(x) + v_A^2(x))(k^2c_T^2(x) - \omega^2)}, \tag{2}$$

and

$$c_0(x) = \sqrt{\frac{\gamma p_0}{\rho_0(x)}}, \quad v_A(x) = \frac{B_0(x)}{\sqrt{\mu\rho_0(x)}}, \quad c_T(x) = \frac{c_0(x)v_A(x)}{\sqrt{c_0^2(x) + v_A^2(x)}} \tag{3}$$

are the sound, Alfvén, and tube speeds.

When the plasma is incompressible, so that $c_0 \to \infty$, this reduces to

$$\frac{d}{dx}\left(\epsilon(x)\frac{d\hat{v}_x}{dx}\right) - k^2\epsilon(x)\hat{v}_x = 0, \tag{4}$$

which is commonly used for solving eigenmode problems in MHD wave physics. In the present work, the temporal evolution of linear asymmetric MHD waves is considered so we must dial back our assumptions about how the wave behaviour through time.

Above we used a Fourier decomposition in time, which is valid when the solutions are homogeneous in time, such as normal mode solutions (rewrite this). To investigate the temporal evolution of solutions, we take only Fourier components in the z-direction, that is $\mathbf{v}(\mathbf{x},t) = \hat{\mathbf{v}}(x,t)e^{ikz}$, and we take the Laplace transform with respect to time, such that

$$\tilde{\mathbf{v}}(x) = \int_0^\infty \hat{\mathbf{v}}(x,t)e^{i\omega t}dt. \tag{5}$$

This gives us the initial-value form of Equation (4) to be

$$\frac{d}{dx}\left(\epsilon(x)\frac{d\tilde{v}_x}{dx}\right) - k^2\epsilon(x)\tilde{v}_x = f(x), \tag{6}$$

where

$$f(x) = -ik\left\{\rho_0\left[\frac{\partial\hat{\Omega}}{\partial t}(x,0) - i\omega\hat{\Omega}(x,0)\right] + \left[\frac{\partial\hat{v}_z}{\partial t}(x,0) - i\omega\hat{v}_z(x,0)\right]\frac{d\rho_0}{dx}\right\}, \tag{7}$$

where the vorticity, $\Omega(x,t)\hat{\mathbf{y}} = \hat{\Omega}(x,t)e^{ikz}\hat{\mathbf{y}} = \nabla\times\mathbf{v}(\mathbf{x},t)$, is given by

$$\hat{\Omega}(\mathbf{x},t) = \frac{i}{k}\left(\frac{\partial^2\hat{v}_x}{\partial x^2} - k^2\hat{v}_x\right). \tag{8}$$

Consider equilibrium magnetic field and density profiles given by

$$B(x) = \begin{cases} B_1 & \text{if } x < x_0, \\ B_0 & \text{if } |x| \le x_0, \\ B_2 & \text{if } x > x_0, \end{cases} \quad \text{and} \quad \rho(x) = \begin{cases} \rho_1 & \text{if } x < x_0, \\ \rho_0 & \text{if } |x| \le x_0, \\ \rho_2 & \text{if } x > x_0, \end{cases} \tag{9}$$

where B_i and ρ_i are uniform for $i = 0, 1, 2$. This establishes the plasma as magnetic slab embedded in an asymmetric magnetic background.

In this equilibrium, transverse velocity perturbations are related to initial perturbations in the following way:

$$\frac{d^2\tilde{v}_x}{dx^2} - k^2\tilde{v}_x = \begin{cases} f(x)/\epsilon_1, & \text{if } x < x_0, \\ f(x)/\epsilon_0, & \text{if } |x| < x_0, \\ f(x)/\epsilon_2, & \text{if } x > x_0. \end{cases} \tag{10}$$

Sturm-Liouville Theory tells us that with the aid of the Green's function, $G(x; s)$, Equation (10) can be solved to give

$$\tilde{v}_x(x) = \begin{cases} A(\cosh kx + \sinh kx) - \dfrac{1}{\epsilon_1}\displaystyle\int_{-\infty}^{-x_0} G(x; s)f(x)ds, & \text{if } x < x_0, \\ B\cosh kx + C\sinh kx - \dfrac{1}{\epsilon_0}\displaystyle\int_{-x_0}^{x_0} G(x; s)f(x)ds, & \text{if } |x| < x_0, \\ D(\cosh kx - \sinh kx) - \dfrac{1}{\epsilon_2}\displaystyle\int_{x_0}^{\infty} G(x; s)f(x)ds, & \text{if } x > x_0, \end{cases} \tag{11}$$

where

$$G(x; s) = \frac{1}{2k}[e^{ks}e^{-kx}H(x-s) + e^{-ks}e^{kx}H(s-x)] \tag{12}$$

and H is the Heaviside step function. Ensuring continuity of both transverse velocity and total pressure across the boundaries at $x = \pm x_0$ gives us the following system of linear algebraic equations for the constants A, B, C, and D:

$$\begin{pmatrix} c_0 - s_0 & -c_0 & s_0 & 0 \\ 0 & c_0 & s_0 & s_0 - c_0 \\ \epsilon_1(c_0 - s_0) & \epsilon_0 s_0 & -\epsilon_0 c_0 & 0 \\ 0 & \epsilon_0 s_0 & \epsilon_0 c_0 & -\epsilon_2(s_0 - c_0) \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \frac{1}{2k} \begin{pmatrix} e^{kx_0}/\epsilon_1 & -e^{-kx_0}/\epsilon_0 & 0 & 0 \\ 0 & 0 & -e^{-kx_0}/\epsilon_0 & e^{kx_0}/\epsilon_2 \\ -e^{kx_0} & -e^{-kx_0}/\epsilon_0 & 0 & 0 \\ 0 & 0 & -e^{-kx_0}/\epsilon_0 & e^{kx_0}/\epsilon_2 \end{pmatrix} \begin{pmatrix} I_1 \\ I_0^- \\ I_0^+ \\ I_2 \end{pmatrix}, \tag{13}$$

where $c_0 = \cosh kx_0$, $s_0 = \sinh kx_0$, and the functionals I_1, I_0^-, I_0^+, and I_2 are given by

$$I_1 = \int_{-\infty}^{-x_0} e^{ks}f(s)ds, \quad I_0^- = \int_{-x_0}^{x_0} e^{-ks}f(s)ds, \quad I_0^+ = \int_{-x_0}^{x_0} e^{ks}f(s)ds, \quad I_2 = \int_{x_0}^{\infty} e^{-ks}f(s)ds. \tag{14}$$

Solving this system of equations gives

$$A = \frac{e^{2kx_0}}{2k\epsilon_1 D_R}\left\{\left[I_1 e^{4kx_0}(\epsilon_0 - \epsilon_1)(\epsilon_0 + \epsilon_2) - 4I_2 e^{2kx_0}\epsilon_0\epsilon_1 - I_1(\epsilon_0 + \epsilon_1)(\epsilon_0 - \epsilon_2)\right]\right. \tag{15}$$

$$\left. - 2\epsilon_1\left[I_0^- e^{2kx_0}(\epsilon_0 + \epsilon_2) + I_0^+(\epsilon_0 - \epsilon_2)\right]\right\}, \tag{16}$$

$$B = \frac{1}{2k\epsilon_0 D_R}\left\{\left[((I_0^+ - I_0^-)(\epsilon_2 - \epsilon_1)\epsilon_0 - (I_0^+ + I_0^-)(\epsilon_0^2 - \epsilon_1\epsilon_2))e^{2kx_0} - (I_0^+ + I_0^-)(\epsilon_0 - \epsilon_1)(\epsilon_0 - \epsilon_2)\right]\right. \tag{17}$$

$$\left. - 2\epsilon_0 e^{2kx_0}\left[((I_1 + I_2)\epsilon_0 + I_1\epsilon_2 + I_2\epsilon_1)e^{2kx_0} + (I_1 + I_2)\epsilon_0 - I_1\epsilon_2 - I_2\epsilon_1\right]\right\}, \tag{18}$$

$$C = \frac{1}{2k\epsilon_0 D_R}\left\{\left[((I_0^+ + I_0^-)(\epsilon_2 - \epsilon_1)\epsilon_0 - (I_0^+ - I_0^-)(\epsilon_0^2 - \epsilon_1\epsilon_2))e^{2kx_0} - (I_0^+ - I_0^-)(\epsilon_0 - \epsilon_1)(\epsilon_0 - \epsilon_2)\right]\right. \tag{19}$$

$$\left. - 2\epsilon_0 e^{2kx_0}\left[((I_2 - I_1)\epsilon_0 - I_1\epsilon_2 + I_2\epsilon_1)e^{2kx_0} + (I_1 - I_2)\epsilon_0 - I_1\epsilon_2 + I_2\epsilon_1\right]\right\}, \tag{20}$$

$$D = \frac{e^{2kx_0}}{2k\epsilon_2 D_R}\left\{\left[I_2 e^{4kx_0}(\epsilon_0 + \epsilon_1)(\epsilon_0 - \epsilon_2) - 4I_1 e^{2kx_0}\epsilon_0\epsilon_2 - I_2(\epsilon_0 - \epsilon_1)(\epsilon_0 + \epsilon_2)\right]\right. \tag{21}$$

$$\left. - 2\epsilon_2\left[(I_0^+ e^{2kx_0}(\epsilon_0 + \epsilon_1) + I_0^-(\epsilon_0 - \epsilon_1)\right]\right\}, \tag{22}$$

where

$$D_R = (\epsilon_0 + \epsilon_1)(\epsilon_0 + \epsilon_2)e^{4kx_0} - (\epsilon_0 - \epsilon_1)(\epsilon_0 - \epsilon_2). \tag{23}$$

3. INITIAL VALUE PROBLEM - COLD MAGNETIC SLAB

Employing the same procedure as Ruderman & Roberts (2006a) to reduce the linearised MHD equations governing cold (zero-beta) plasma leads us to the following equations for the perturbations in magnetic pressure, $P = B_0 b_z/\mu$, and transverse velocity, v_x:

$$\frac{\partial^2 P}{\partial t^2} = v_A^2 \nabla^2 P, \quad \frac{\partial^2 v_x}{\partial t^2} = v_A^2 \nabla^2 v_x. \tag{24}$$

Taking Fourier components in z and Laplace transform in t, such that $\hat{P}(x) = \int_0^\infty P(x,z,t)e^{i(\omega t - kz)}dt$, and introducing transverse wavenumber Λ, such that $\Lambda^2 = (v_A^2 k^2 - \omega^2)/v_A^2$, reduces these equations to

$$\frac{d^2\hat{P}}{dx^2} - \Lambda^2 \hat{P} = F_P, \quad \frac{d^2\hat{v}_x}{dx^2} - \Lambda^2 \hat{v}_x = F_v, \tag{25}$$

where $F_{P_0} = (i\omega P_0 - \dot{P}_0)/v_A^2$ and $F_v = (i\omega u_0 - \dot{u}_0)/v_A^2$, where we have defined the initial values by

$$P|_{t=0} = P_0, \quad \left.\frac{\partial P}{\partial t}\right|_{t=0} = \dot{P}_0, \quad v_x|_{t=0} = u_0, \quad \left.\frac{\partial v_x}{\partial t}\right|_{t=0} = \dot{u}_0. \tag{26}$$

3.1. *Solution within the slab*

Consider an asymmetric magnetic slab as defined in Section 2. For the solution inside the slab, $|x| < x_0$, $\hat{P}(x)$ satisfies

$$\frac{d^2\hat{P}}{dx^2} - \Lambda_0^2 \hat{P} = F_{P_0}, \tag{27}$$

under the boundary conditions $\hat{P}(x_0) = \hat{P}(-x_0) = \hat{A}$. To solve this we construct the Green's function, $G_0(x;s)$ that satisfies

$$\frac{d^2G_0}{dx^2} - \Lambda_0^2 G_0 = \delta(x-s), \quad G_0(-x_0;s) = G_0(x_0;s) = 0, \tag{28}$$

where δ denotes the Dirac Delta function. The general solution of this equation is

$$G_0(x;s) = c_1 \sinh(\Lambda_0(x - x_0)) + c_2 \sinh(\Lambda_0(x + x_0)), \tag{29}$$

where $c_1 = 0$ for $x < s$ and $c_2 = 0$ for $x > s$. Ensuring G_0 and $\partial G_0/\partial x$ have jumps of 0 and 1 at $x = s$, respectively, determines c_1 and c_2 so that $G_0(x;s)$ is

$$G_0(x;s) = \frac{1}{\Lambda_0 \sinh(2\Lambda_0 x_0)} \begin{cases} \sinh(\Lambda_0(s - x_0))\sinh(\Lambda_0(x + x_0)), & \text{if } -x_0 < x < s, \\ \sinh(\Lambda_0(x - x_0))\sinh(\Lambda_0(s + x_0)), & \text{if } s < x < x_0, \end{cases} \tag{30}$$

and the solution of system of Equations (28) is

$$\hat{P}(x) = \hat{A}\frac{\cosh \Lambda_0 x}{\cosh \Lambda_0 x_0} + \int_{-x_0}^{x_0} G_0(x;s) F_{P_0}(s) ds. \tag{31}$$

3.2. *Solution outside the slab*

For the solution outside and to the left of the slab, $x < -x_0$, $\hat{P}(x)$ satisfies

$$\frac{d^2\hat{P}}{dx^2} - \Lambda_1^2 \hat{P} = F_{P_1}, \tag{32}$$

and the boundary conditions $\hat{P}(-\infty) = 0$, $\hat{P}(-x_0) = \hat{A}$. By following a Green's function method, the solution of the aforementioned Sturm-Liouville system is

$$\hat{P}(x) = \hat{A}e^{\Lambda_1(x_0 + x)} + \int_{-\infty}^{-x_0} G_1(x;s) F_{P_1}(s) ds, \tag{33}$$

where the Green's function, G_1, is defined by

$$G_1(x;s) = \frac{1}{2\Lambda_1}\begin{cases} e^{\Lambda_1(x+s+2x_0)} - e^{\Lambda_1(x-s)}, & \text{if } -\infty < x < s, \\ e^{\Lambda_1(x+s+2x_0)} - e^{\Lambda_1(s-x)}, & \text{if } s < x < -x_0. \end{cases} \tag{34}$$

Similarly, for the solution outside and to the right of the slab, $x > x_0$, $\hat{P}(x)$ satisfies

$$\hat{P}(x) = \hat{A}e^{\Lambda_2(x_0-x)} + \int_{x_0}^{\infty} G_2(x;s)F_{P_2}(s)ds, \tag{35}$$

where the Green's function, G_2, is defined by

$$G_2(x;s) = \frac{1}{2\Lambda_2}\begin{cases} e^{-\Lambda_2(x+s-2x_0)} - e^{\Lambda_2(x-s)}, & \text{if } x_0 < x < s, \\ e^{-\Lambda_2(x+s+2x_0)} - e^{\Lambda_2(s-x)}, & \text{if } s < x < \infty. \end{cases} \tag{36}$$

3.3. *Matching solutions*

For physically relevant solutions, we require that the transverse displacement (equivalently, the perturbation in transverse velocity) and the total pressure (equivalently, the magnetic pressure, since we are assuming that the plasma is cold) be continuous across the interfaces at $x = \pm x_0$.

Continuity in magnetic pressure, P, is satisfied automatically by considering the solutions inside and outside the slab given by Equations (31), (33), and (35), respectively, and our definition of $\hat{A} = \hat{P}(x = \pm x_0)$.

Continuity in transverse velocity, v_x can be dealt with as follows. If we make the simplification to the prescribed initial conditions such at $u_0 = 0$ and $\dot{u}_0 + \frac{1}{\rho_0}\frac{dP_0}{dx} = 0$, so that the initial plasma is at rest and it's initial acceleration transverse to the slab is due to the initial pressure gradient, then this boundary condition is equivalent to

$$\left[\left[\frac{1}{\rho_0(\omega^2 - k^2 v_A^2)}\frac{\partial \hat{P}}{\partial x}\right]\right]_{x=\pm x_0} = 0. \tag{37}$$

Substituting the solutions given by Equations (31), (33), and (35) into this boundary condition at $x = -x_0$ gives

$$\hat{A}(\omega) = \frac{T_1(\omega)}{D_1(\omega)}, \tag{38}$$

where

$$T_1(\omega) = \frac{\Lambda_1^2}{\sin 2\Lambda_0 x_0}\int_{-x_0}^{x_0} \sin(\Lambda_0(s-x_0))F_{p_0}(s)ds - \Lambda_0^2\int_{-x_0}^{x_0} e^{\Lambda_1(s+x_0)}F_{p_1}(s)ds, \tag{39}$$

$$D_1(\omega) = i\Lambda_0^2\Lambda_1 - \Lambda_1^2\Lambda_0\tan(\Lambda_0 x_0). \tag{40}$$

Substitution into this boundary condition at $x = x_0$ gives

$$\hat{A}(\omega) = \frac{T_2(\omega)}{D_2(\omega)}, \tag{41}$$

where

$$T_2(\omega) = -\frac{\Lambda_2^2}{\sin 2\Lambda_0 x_0}\int_{-x_0}^{x_0} \sin(\Lambda_0(s+x_0))F_{p_0}(s)ds - \Lambda_0^2\int_{-x_0}^{x_0} e^{\Lambda_2(s-x_0)}F_{p_2}(s)ds, \tag{42}$$

$$D_2(\omega) = i\Lambda_0^2\Lambda_2 - \Lambda_2^2\Lambda_0\tan(\Lambda_0 x_0). \tag{43}$$

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Software: Mayavi

APPENDIX

A. APPENDIX INFORMATION

REFERENCES

Andries, J., & Goossens, M. 2007, Physics of Plasmas, 14, 052101

Cally, P. S. 2003, SoPh, 217, 95

—. 2006, SoPh, 233, 79

Ruderman, M. S., & Roberts, B. 2006a, SoPh, 237, 119

—. 2006b, Journal of Plasma Physics, 72, 285

Terradas, J., Andries, J., & Goossens, M. 2007, SoPh, 246, 231

Terradas, J., Oliver, R., & Ballester, J. L. 2006, ApJ, 642, 533