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EVOLUTION OF ASYMMETRIC SLAB MAGNETOHYDRODYNAMIC WAVES

Matthew Allcock[1] and Robert Erdélyi[1]

[1] Solar Physics and Space Plasma Research Centre, School of Mathematics and Statistics, University of Sheffield, Hicks Building, Hounsfield Road, Sheffield, S3 7RH, UK

ABSTRACT

Abstract (250 word limit for ApJ)

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Corresponding author: Robert Erdélyi
robertus@sheffield.ac.uk

1. INTRODUCTION

Numerical results: Terradas et al. (2006)

Physicality of the principle leaky kink mode: Cally (2003) solved the initial value problem of transverse waves in a cold magnetic flux tube. Ruderman & Roberts (2006a) repeated it showing PLK modes are on an unphysical banch of the complex plane (?). Commented on by Cally (2006) and in return by Ruderman & Roberts (2006b). Settled (?) by considering numerical solution by Terradas et al. (2007) and analytically by Andries & Goossens (2007). So PLK modes are not physical.

2. INITIAL VALUE PROBLEM - INCOMPRESSIBLE MAGNETIC SLAB

Consider an equilibrium plasma with magnetic field $B_0(x)\hat{\mathbf{z}}$, density $\rho_0(x)$, and pressure $p_0(x)$, without gravity. In the absence of structuring in the z-direction and considering perturbations in the (x, z)-plane only, we can take Fourier components for velocity and other parameters like $\mathbf{v}(\mathbf{x}, t) = \hat{\mathbf{v}}(x)e^{i(kz+ly-\omega t)}$. The velocity perturbation amplitude in the inhomogeneous direction of an ideal plasma is given by

$$\frac{d}{dx}\left(\frac{\epsilon(x)}{l^2 + m_0^2(x)}\frac{d\hat{v}_x}{dx}\right) - \epsilon(x)\hat{v}_x = 0, \tag{1}$$

where

$$\epsilon(x) = \rho_0(x)[k^2 v_A^2(x) - \omega^2], \quad m_0^2(x) = \frac{(k^2 c_0^2(x) - \omega^2)(k^2 v_A^2(x) - \omega^2)}{(c_0^2(x) + v_A^2(x))(k^2 c_T^2(x) - \omega^2)}, \tag{2}$$

and

$$c_0(x) = \sqrt{\frac{\gamma p_0(x)}{\rho_0(x)}}, \quad v_A(x) = \frac{B_0(x)}{\sqrt{\mu\rho_0(x)}}, \quad c_T(x) = \frac{c_0(x)v_A(x)}{\sqrt{c_0^2(x) + v_A^2(x)}} \tag{3}$$

are the sound, Alfvén, and tube speeds, respectively.

When the plasma is incompressible, so that $c_0 \to \infty$, we have $c_T^2 \to v_A^2$ and $m_0^2 \to k^2$. After restricting the analysis to propagation only parallel to the magnetic field ($l = 0$), Equation (1) reduces to

$$\frac{d}{dx}\left(\epsilon(x)\frac{d\hat{v}_x}{dx}\right) - k^2\epsilon(x)\hat{v}_x = 0, \tag{4}$$

which is commonly used for solving eigenmode problems in MHD wave physics. In the present work, the temporal evolution of linear asymmetric MHD waves is considered so we must dial back our assumptions about how the wave behaviour through time.

Above we used a Fourier decomposition in time, which is valid when the solutions are homogeneous in time, such as normal mode solutions (rewrite this). To investigate the temporal evolution of solutions, we take only Fourier components in the z-direction, that is $\mathbf{v}(\mathbf{x}, t) = \hat{\mathbf{v}}(x, t)e^{ikz}$, and we take the Laplace transform with respect to time, such that

$$\tilde{\mathbf{v}}(x) = \int_0^\infty \hat{\mathbf{v}}(x, t)e^{i\omega t}dt. \tag{5}$$

This gives us the initial-value form of Equation (4) to be

$$\frac{d}{dx}\left(\epsilon(x)\frac{d\tilde{v}_x}{dx}\right) - k^2\epsilon(x)\tilde{v}_x = f(x), \tag{6}$$

where

$$f(x) = ik\left\{\rho_0\left[\frac{\partial\hat{\Omega}}{\partial t}(x, 0) - i\omega\hat{\Omega}(x, 0)\right] - \left[\frac{\partial\hat{v}_z}{\partial t}(x, 0) - i\omega\hat{v}_z(x, 0)\right]\frac{d\rho_0}{dx}\right\}, \tag{7}$$

where the vorticity, $\Omega(x, t)\hat{\mathbf{y}} = \hat{\Omega}(x, t)e^{ikz}\hat{\mathbf{y}} = \nabla \times \mathbf{v}(\mathbf{x}, t)$, is given by

$$\hat{\Omega}(\mathbf{x}, t) = -\frac{i}{k}\left(\frac{\partial^2\hat{v}_x}{\partial x^2} - k^2\hat{v}_x\right). \tag{8}$$

(this differs from Rae & Roberts (1981) by a factor of -1 due to taking Fourier forms like e^{ikz} rather than e^{-ikz})

Consider equilibrium magnetic field and density profiles given by

$$B(x) = \begin{cases} B_1, & \text{if } x < -x_0, \\ B_0, & \text{if } |x| \leq x_0, \\ B_2, & \text{if } x > x_0, \end{cases} \quad \text{and} \quad \rho(x) = \begin{cases} \rho_1, & \text{if } x < -x_0, \\ \rho_0, & \text{if } |x| \leq x_0, \\ \rho_2, & \text{if } x > x_0, \end{cases} \tag{9}$$

where B_i and ρ_i are uniform for $i = 0, 1, 2$. This establishes the plasma as magnetic slab embedded in an asymmetric magnetic background.

In this equilibrium, transverse velocity perturbations are related to initial perturbations in the following way:

$$\frac{d^2\tilde{v}_x}{dx^2} - k^2\tilde{v}_x = \begin{cases} f(x)/\epsilon_1, & \text{if } x < -x_0, \\ f(x)/\epsilon_0, & \text{if } |x| < x_0, \\ f(x)/\epsilon_2, & \text{if } x > x_0. \end{cases} \tag{10}$$

2.1. *Attempt 1*

Sturm-Liouville Theory tells us that with the aid of a Green's function, $G(x; s)$, Equation (10) can be solved to give

$$\tilde{v}_x(x) = \begin{cases} A(\cosh kx + \sinh kx) - \frac{1}{\epsilon_1}\int_{-\infty}^{-x_0} G(x; s)f(s)ds, & \text{if } x < -x_0, \\ B\cosh kx + C\sinh kx - \frac{1}{\epsilon_0}\int_{-x_0}^{x_0} G(x; s)f(s)ds, & \text{if } |x| < x_0, \\ D(\cosh kx - \sinh kx) - \frac{1}{\epsilon_2}\int_{x_0}^{\infty} G(x; s)f(s)ds, & \text{if } x > x_0, \end{cases} \tag{11}$$

where

$$G(x; s) = \frac{1}{2k}[e^{ks}e^{-kx}H(x - s) + e^{-ks}e^{kx}H(s - x)] \tag{12}$$

and H is the Heaviside step function. Ensuring continuity of both transverse velocity and total pressure across the boundaries at $x = \pm x_0$ gives us the following system of linear algebraic equations for the constants A, B, C, and D:

$$\begin{pmatrix} c_0 - s_0 & -c_0 & s_0 & 0 \\ 0 & c_0 & s_0 & s_0 - c_0 \\ \epsilon_1(c_0 - s_0) & \epsilon_0 s_0 & -\epsilon_0 c_0 & 0 \\ 0 & \epsilon_0 s_0 & \epsilon_0 c_0 & -\epsilon_2(s_0 - c_0) \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \frac{1}{2k} \begin{pmatrix} e^{kx_0}/\epsilon_1 & -e^{-kx_0}/\epsilon_0 & 0 & 0 \\ 0 & 0 & e^{-kx_0}/\epsilon_0 & -e^{kx_0}/\epsilon_2 \\ -e^{kx_0} & -e^{-kx_0} & 0 & 0 \\ 0 & 0 & -e^{-kx_0} & -e^{kx_0} \end{pmatrix} \begin{pmatrix} I_1 \\ I_0^- \\ I_0^+ \\ I_2 \end{pmatrix}, \tag{13}$$

where $c_0 = \cosh kx_0$, $s_0 = \sinh kx_0$, and the functionals I_1, I_0^-, I_0^+, and I_2 are given by

$$I_1 = \int_{-\infty}^{-x_0} e^{ks}f(s)ds, \quad I_0^- = \int_{-x_0}^{x_0} e^{-ks}f(s)ds, \quad I_0^+ = \int_{-x_0}^{x_0} e^{ks}f(s)ds, \quad I_2 = \int_{x_0}^{\infty} e^{-ks}f(s)ds. \tag{14}$$

Solving this system of equations gives

$$A = \frac{e^{2kx_0}}{2k\epsilon_1 D_R}\left\{\left[I_1 e^{4kx_0}(\epsilon_0 - \epsilon_1)(\epsilon_0 + \epsilon_2) - 4I_2 e^{2kx_0}\epsilon_0\epsilon_1 - I_1(\epsilon_0 + \epsilon_1)(\epsilon_0 - \epsilon_2)\right]\right. \tag{15}$$

$$\left. - 2\epsilon_1\left[I_0^- e^{2kx_0}(\epsilon_0 + \epsilon_2) + I_0^+(\epsilon_0 - \epsilon_2)\right]\right\}, \tag{16}$$

$$B = \frac{1}{2k\epsilon_0 D_R}\left\{\left[((I_0^+ - I_0^-)(\epsilon_2 - \epsilon_1)\epsilon_0 - (I_0^+ + I_0^-)(\epsilon_0^2 - \epsilon_1\epsilon_2))e^{2kx_0} - (I_0^+ + I_0^-)(\epsilon_0 - \epsilon_1)(\epsilon_0 - \epsilon_2)\right]\right. \tag{17}$$

$$\left. - 2\epsilon_0 e^{2kx_0}\left[((I_1 + I_2)\epsilon_0 + I_1\epsilon_2 + I_2\epsilon_1)e^{2kx_0} + (I_1 + I_2)\epsilon_0 - I_1\epsilon_2 - I_2\epsilon_1\right]\right\}, \tag{18}$$

$$C = \frac{1}{2k\epsilon_0 D_R}\left\{\left[((I_0^+ + I_0^-)(\epsilon_2 - \epsilon_1)\epsilon_0 - (I_0^+ - I_0^-)(\epsilon_0^2 - \epsilon_1\epsilon_2))e^{2kx_0} - (I_0^+ - I_0^-)(\epsilon_0 - \epsilon_1)(\epsilon_0 - \epsilon_2)\right]\right. \tag{19}$$

$$\left. - 2\epsilon_0 e^{2kx_0}\left[((I_2 - I_1)\epsilon_0 - I_1\epsilon_2 + I_2\epsilon_1)e^{2kx_0} + (I_1 - I_2)\epsilon_0 - I_1\epsilon_2 + I_2\epsilon_1\right]\right\}, \tag{20}$$

$$D = \frac{e^{2kx_0}}{2k\epsilon_2 D_R}\left\{\left[I_2 e^{4kx_0}(\epsilon_0 + \epsilon_1)(\epsilon_0 - \epsilon_2) - 4I_1 e^{2kx_0}\epsilon_0\epsilon_2 - I_2(\epsilon_0 - \epsilon_1)(\epsilon_0 + \epsilon_2)\right]\right. \tag{21}$$

$$\left. - 2\epsilon_2\left[(I_0^+ e^{2kx_0}(\epsilon_0 + \epsilon_1) + I_0^-(\epsilon_0 - \epsilon_1)\right]\right\}, \tag{22}$$

where

$$D_R = (\epsilon_0 + \epsilon_1)(\epsilon_0 + \epsilon_2)e^{4kx_0} - (\epsilon_0 - \epsilon_1)(\epsilon_0 - \epsilon_2). \tag{23}$$

The solutions of equation $D_R = 0$ are precisely the solutions of the dispersion relation for an incompressible asymmetric magnetic slab. To confirm this, we follow notion from Zsámberger et al. (2018) (with an added superscript z) by observing that for an incompressible plasma, the sound speed, $c_s \to \infty$. Therefore,

$$m_j^2 = \frac{(k^2c_j^2 - \omega^2)(k^2v_{Aj}^2 - \omega^2)}{(c_j^2 + v_{Aj}^2)(k^2c_Tj^2 - \omega^2)} \to k^2, \tag{24}$$

$$\Lambda_j^z = \frac{i\rho_j}{\omega m_j}(k^2v_{Aj}^2 - \omega^2) \to \frac{i\rho_j}{\omega k}(k^2v_{Aj}^2 - \omega^2), \quad \text{for } j = 0, 1, 2. \tag{25}$$

Therefore, after multiplying Equation (23) by $\omega k/i(e^{4kx_0} + 1)$, we recover the dispersion relation in Zsámberger et al. (2018) for an incompressible plasma, namely

$$2(\Lambda_0^2 + \Lambda_1\Lambda_2) + \Lambda_0(\Lambda_1 + \Lambda_2)[\tanh kx_0 + \coth kx_0] = 0. \tag{26}$$

2.2. *Attempt 2*

The idea behind this second attempt is to choose the Green's function in such a way as to make the constant coefficients of the particular solution part are directly related to the boundary values of the transverse velocity.

We are trying to solve the equation

$$\frac{d^2\tilde{v}_x}{dx^2} - k^2\tilde{v}_x = \begin{cases} f(x)/\epsilon_1, & \text{if } x < -x_0, \\ f(x)/\epsilon_0, & \text{if } |x| < x_0, \\ f(x)/\epsilon_2, & \text{if } x > x_0, \end{cases} \tag{27}$$

under the boundary conditions $\tilde{v}_x(-x_0) = \tilde{A}_1$ and $\tilde{v}_x(x_0) = \tilde{A}_2$.

The Green's function, $G(x; s)$, corresponding to Equation (27) must satisfy

$$\frac{\partial^2 G}{\partial x^2} - k^2 G = \delta(x - s), \quad G(-x_0; s) = G(x_0; s) = 0, \tag{28}$$

where δ denotes the Dirac Delta function. It is instructive to piecewise define the Green's function as

$$G(x; s) = \begin{cases} G_1(x; s), & \text{if } x < -x_0, \\ G_0(x; s), & \text{if } |x| < x_0, \\ G_2(x; s), & \text{if } x_0 < x. \end{cases} \tag{29}$$

The general solution, for $|x| < x_0$, of the equation for G_0 is

$$G_0(x; s) = c_1 \sinh(k(x - x_0)) + c_2 \sinh(k(x + x_0)), \tag{30}$$

where $c_1 = 0$ for $x < s$ and $c_2 = 0$ for $x > s$. Ensuring G_0 and $\partial G_0/\partial x$ have jumps of 0 and 1 at $x = s$, respectively, determines c_1 and c_2, so that $G_0(x; s)$ is

$$G_0(x; s) = \frac{1}{k \sinh(2kx_0)} \begin{cases} \sinh(k(s - x_0)) \sinh(k(x + x_0)), & \text{if } -x_0 < x < s, \\ \sinh(k(x - x_0)) \sinh(k(s + x_0)), & \text{if } s < x < x_0. \end{cases} \tag{31}$$

Because the boundary conditions are inhomogeneous, we must add to the standard Green's function solution a term that is a solution to the homogeneous equation and the inhomogeneous boundary conditions. In this manner, we find that the solution within the slab is

$$\tilde{v}_x(x) = \frac{1}{\sinh 2kx_0}\left[\tilde{A}_1 \sinh(k(x_0 - x)) + \tilde{A}_2 \sinh(k(x_0 + x))\right] + \frac{1}{\epsilon_0}\int_{-x_0}^{x_0} G_0(x; s) f(s) ds. \tag{32}$$

Similarly, we find that the Green's function for the plasma outside the slab is

$$G_1(x;s) = \frac{1}{k}\begin{cases} e^{k(x+x_0)}\sinh(k(s+x_0)), & \text{if } x < s, \\ e^{k(s+x_0)}\sinh(k(x+x_0)), & \text{if } s < x < -x_0, \end{cases} \tag{33}$$

for $x < -x_0$, and

$$G_2(x;s) = -\frac{1}{k}\begin{cases} e^{-k(s-x_0)}\sinh(k(x-x_0)), & \text{if } x_0 < x < s, \\ e^{-k(x-x_0)}\sinh(k(s-x_0)), & \text{if } s < x, \end{cases} \tag{34}$$

for $x > x_0$. Therefore the solution is

$$\tilde{v}_x(x) = \tilde{A}_1 e^{k(x_0+x)} + \frac{1}{\epsilon_1}\int_{-\infty}^{-x_0} G_1(x;s)f(s)ds, \tag{35}$$

for $x < -x_0$, and

$$\tilde{v}_x(x) = \tilde{A}_2 e^{k(x_0-x)} + \frac{1}{\epsilon_2}\int_{x_0}^{\infty} G_2(x;s)f(s)ds, \tag{36}$$

for $x > x_0$.

2.2.1. *Matching solutions*

To establish physically relevant solutions, we require that the transverse velocity and the total pressure are continuous over each interface. Given the above solutions, the transverse velocity is automatically continuous over the boundaries. The perturbation in the total pressure, P_T, for a compressible plasma is given (for example, in Allcock & Erdélyi (2017)) by

$$\tilde{P}_T(x) = \frac{\Lambda}{m}\frac{d\tilde{v}_x}{dx}, \tag{37}$$

where

$$\Lambda = \frac{i\rho(\omega^2 - k^2 v_A^2)}{m\omega}, \quad m^2 = \frac{(k^2v_A^2 - \omega^2)(k^2c_0^2 - \omega^2)}{(c_0^2 + v_A^2)(k^2c_T^2 - \omega^2)}, \quad \text{and} \quad c_T^2 = \frac{c_0^2 v_A^2}{c_0^2 + v_A^2}. \tag{38}$$

When the plasma is incompressible, $m^2 \to k^2$, therefore continuity in total pressure is equivalent to continuity in $\epsilon(x)\tilde{v}_x'(x)$.

Applying this boundary condition gives

$$\tilde{A}_1(\omega) = \frac{T_1(\omega)}{kD(\omega)}, \quad \tilde{A}_2(\omega) = \frac{T_2(\omega)}{kD(\omega)}, \tag{39}$$

where

$$T_1(\omega) = (I_0^- - I_1)[\epsilon_0\cosh(2kx_0) + \epsilon_2\sinh(2kx_0)] - \epsilon_0(I_0^+ + I_2), \tag{40}$$

$$T_2(\omega) = \epsilon_0(I_0^- - I_1) - (I_0^+ + I_2)[\epsilon_0\cosh(2kx_0) + \epsilon_1\sinh(2kx_0)], \tag{41}$$

$$D(\omega) = \epsilon_0(\epsilon_1 + \epsilon_2)\cosh(2kx_0) + (\epsilon_0^2 + \epsilon_1\epsilon_2)\sinh(2kx_0), \tag{42}$$

and

$$I_0^{\pm} = \int_{-x_0}^{x_0}\frac{\sinh(k(s\pm x_0))}{\sinh(2kx_0)}f(s)ds, \quad I_1 = \int_{-\infty}^{-x_0} e^{k(s+x_0)}f(s)ds, \quad I_2 = \int_{x_0}^{\infty} e^{k(x_0-s)}f(s)ds. \tag{43}$$

2.2.2. *Asymptotic solution for large time*

To study the asymptotic behaviours of the velocity perturbation, we begin with the asymptotic behaviour of the boundary velocities, $A_1(t) = v_x(t, -x_0)$ and $A_2(t) = v_x(t, x_0)$. These variables can be determined, using the inverse Laplace transform (non-standard, discussed in Appendix A), to be

$$A_1(t) = \frac{1}{2\pi k}\lim_{L\to\infty}\int_{i\gamma-L}^{i\gamma+L}\frac{T_1(\omega)}{D(\omega)}e^{-i\omega t}d\omega, \quad A_2(t) = \frac{1}{2\pi k}\lim_{L\to\infty}\int_{i\gamma-L}^{i\gamma+L}\frac{T_2(\omega)}{D(\omega)}e^{-i\omega t}d\omega, \tag{44}$$

where γ is a real number such that all the singularities of the integrands are below the contour of integration. The integrals are evaluated along an infinite horizontal line in the upper half of the complex plane.

Since the problem of finding the solution is now reduced to solving a complex integral, it is dependent on the singularities (with respect to ω) of T_1, T_2, and D (so that we can construct the Bromwich contour such that it is contained to a single-valued branch) and the zeros of D (whose residues determine the value of the contour integral).

The functions $\epsilon_{0,1,2}$ are polynomial in ω, and are therefore entire. The integrals $I_{1,2}$ and $I_0^{\pm}$ are not functions of ω so also contribute no singularities in ω. Therefore T_1 and T_2 are entire functions.

The zeros of $D(\omega)$ are determined by firstly noting that $D = 0$ is the dispersion relation of the corresponding eigenvalue problem solved by Zsámberger et al. (2018). They show that the dispersion relation governing transverse wave propagation parallel to the magnetic field in an asymmetric slab of compressible plasma is given by

$$2(\Lambda_0^2 + \Lambda_1\Lambda_2) + \Lambda_0(\Lambda_1 + \Lambda_2)[\tanh(m_0x_0) + \coth(m_0x_0)] = 0, \tag{45}$$

where

$$\Lambda_j = -\frac{i\rho_j(k^2v_{Aj}^2 - \omega^2)}{\omega m_j}, \quad m_j^2 = \frac{(k^2c_j^2 - \omega^2)(k^2v_{Aj}^2 - \omega^2)}{(c_j^2 + v_{Aj}^2)(k^2c_{Tj}^2 - \omega^2)}, \quad \text{and} \quad c_{Tj}^2 = \frac{c_j^2v_{Aj}^2}{c_j^2 + v_{Aj}^2}, \tag{46}$$

for $j = 0, 1, 2$. When compressibility is neglected, such that the sound speeds, c_j, approach infinity, we have $c_{Tj}^2 \to v_{Aj}^2$, $m_j^2 \to k^2$, and therefore $\Lambda_j = -i\rho_j(k^2v_{Aj}^2 - \omega^2)/\omega k = -i\epsilon_j/\omega k$.Therefore, Equation (45) can be reduced to the dispersion relation for an incompressible magnetic slab, which is

$$2(\epsilon_0^2 + \epsilon_1\epsilon_2) + \epsilon_0(\epsilon_1 + \epsilon_2)[\tanh(m_0x_0) + \coth(m_0x_0)] = 0, \tag{47}$$

which can easily to shown to be equivalent to $D(\omega) = 0$, where $D(\omega)$ is given by Equation (42). It follows that the zeros of $D(\omega)$ are precisely the eigenvalues of the asymmetric incompressible magnetic slab.

The zeros of $D(\omega) = 0$ are found by writing this equation as

$$\epsilon_0(\epsilon_1 + \epsilon_2) + (\epsilon_0^2 + \epsilon_1\epsilon_2)\tanh(2kx_0) = 0. \tag{48}$$

Substituting expressions for $\epsilon(x)$ into this equation gives

$$\rho_0(k^2v_{A0}^2-\omega^2)[\rho_1(k^2v_{A1}^2-\omega^2)+\rho_2(k^2v_{A2}^2-\omega^2)]+[\rho_0^2(k^2v_{A0}^2-\omega^2)^2+\rho_1\rho_2(k^2v_{A1}^2-\omega^2)(k^2v_{A2}^2-\omega^2)]\tanh(2kx_0) = 0. \tag{49}$$

The above equation can be rewritten as a quadratic in $(\omega/k)^2$, namely

$$[(\rho_0^2 + \rho_1\rho_2)\tanh(2kx_0) + \rho_0(\rho_1 + \rho_2)]\left(\frac{\omega}{k}\right)^4 \tag{50}$$

$$-[(2\rho_0^2v_{A0}^2 + \rho_1\rho_2(v_{A1}^2 + v_{A2}^2))\tanh(2kx_0) + \rho_0v_{A0}^2(\rho_1 + \rho_2) + \rho_0(\rho_1v_{A1}^2 + \rho_2v_{A2}^2)]\left(\frac{\omega}{k}\right)^2 \tag{51}$$

$$+[(\rho_0^2v_{A0}^4 + \rho_1\rho_2v_{A1}^2v_{A2}^2)\tanh(2kx_0) + \rho_0(\rho_1v_{A1}^2 + \rho_2v_{A2}^2)] = 0, \tag{52}$$

which has solutions

$$\left(\frac{\omega_0}{k}\right)^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{53}$$

where

$$a = (\rho_0^2 + \rho_1\rho_2)\tanh(2kx_0) + \rho_0(\rho_1 + \rho_2), \tag{54}$$

$$b = (2\rho_0^2v_{A0}^2 + \rho_1\rho_2(v_{A1}^2 + v_{A2}^2))\tanh(2kx_0) + \rho_0v_{A0}^2(\rho_1 + \rho_2) + \rho_0(\rho_1v_{A1}^2 + \rho_2v_{A2}^2), \tag{55}$$

$$c = (\rho_0^2v_{A0}^4 + \rho_1\rho_2v_{A1}^2v_{A2}^2)\tanh(2kx_0) + \rho_0(\rho_1v_{A1}^2 + \rho_2v_{A2}). \tag{56}$$

Given that the function $T(\omega)$ is analytic, we can construct a closed Bromwich contour, $C = C_0 + C_1$, where C_0 is a straight line from $(-L, \gamma)$ to (L, γ), and C_1 connects $(-L, \gamma)$ and (L, γ) via a semi-circle to ensure that C encloses $\pm\omega_0$ (Figure 2.2.2 ¡ – WHY IS THIS HAPPENING?!).

First, we see that, as $|\omega| \to \infty$, the integrand in question behaves like $T_{1,2}(\omega) \to \mathcal{O}(|\omega|^{-1})$. Therefore,

$$\lim_{L\to\infty}\int_{C_1}\frac{T_{1,2}(\omega)}{D(\omega)}e^{-i\omega t}d\omega = 0. \tag{57}$$

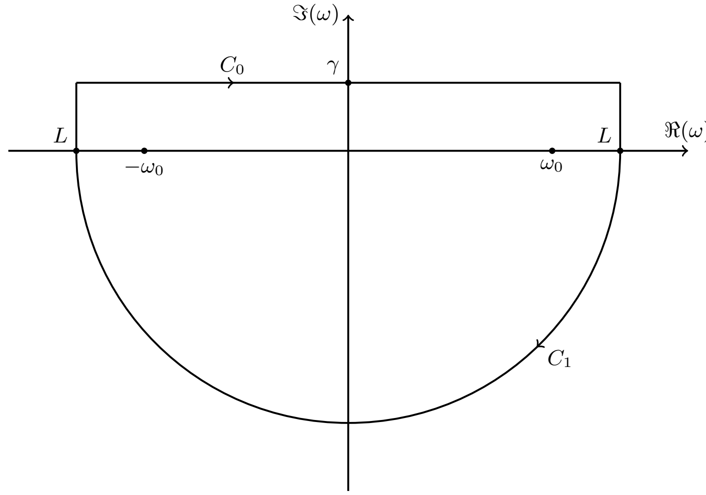

Figure 1. Bromwich contour for the complex integration of $\tilde{A}_{1,2}$.

Secondly, the contour C is integrated in the clockwise direction and therefore is equal to $-2\pi i$ multiplied by the sum of the residues of the integrand at $\omega = \pm\omega_0$. The residue at $\omega = \omega_0$ can be evaluated using L'Hopital's Rule, to within choice of initial condition, as

$$\mathrm{Res}\left[\frac{T_{1,2}(\omega)}{D(\omega)}e^{-i\omega t}, \omega = \omega_0\right] = \lim_{\omega\to\omega_0}\frac{(\omega-\omega_0)T_{1,2}(\omega)}{D(\omega)}e^{-i\omega t} \tag{58}$$

$$= \lim_{\omega\to\omega_0}\frac{1}{D'(\omega)}[T_{1,2}(\omega)e^{-i\omega t} + (\omega-\omega_0)T'_{1,2}(\omega)e^{-i\omega t} - it(\omega-\omega_0)T_{1,2}(\omega)e^{-i\omega t}] \tag{59}$$

$$= \chi_{1,2}e^{-i\omega_0 t}, \tag{60}$$

where $\chi_{1,2} = T_{1,2}(\omega_0)/D'(\omega_0)$. Similarly, the residue at $\omega = -\omega_0$ is

$$\mathrm{Res}\left[\frac{T_{1,2}(\omega)}{D(\omega)}e^{-i\omega t}, \omega = -\omega_0\right] = \lim_{\omega\to-\omega_0}\frac{(\omega+\omega_0)T_{1,2}(\omega)}{D(\omega)}e^{-i\omega t} \tag{61}$$

$$= -\chi_{1,2}e^{i\omega_0 t}. \tag{62}$$

where we have used the fact that D and $T_{1,2}$ are even functions and D' is an odd function of ω.

Putting all of the above results together, we find that the boundary velocities are

$$A_{1,2} = \frac{1}{2\pi k}\lim_{L\to\infty}\int_{C_0}\frac{T_{1,2}(\omega)}{D(\omega)}e^{-i\omega t}d\omega \tag{63}$$

$$= \frac{1}{2\pi k}\lim_{L\to\infty}\int_{C}\frac{T_{1,2}(\omega)}{D(\omega)}e^{-i\omega t}d\omega \tag{64}$$

$$= \frac{-i}{k}\sum\mathrm{Res}\left[\frac{T_{1,2}(\omega)}{D(\omega)}e^{-i\omega t}, \omega = \pm\omega_0\right] \tag{65}$$

$$= \frac{-i}{k}\chi_{1,2}(e^{-i\omega_0 t} - e^{i\omega_0 t}) \tag{66}$$

$$= -\frac{2}{k}\chi_{1,2}\sin(\omega_0 t). \tag{67}$$

3. INITIAL VALUE PROBLEM - ZERO-BETA MAGNETIC SLAB

Employing the same procedure as Ruderman & Roberts (2006a) to reduce the linearised MHD equations governing cold (zero-beta) plasma leads us to the following equations for the perturbations in magnetic pressure, $P = B_0 b_z/\mu$, and transverse velocity, v_x:

$$\frac{\partial^2 P}{\partial t^2} = v_A^2\nabla^2 P, \quad \frac{\partial^2 v_x}{\partial t^2} = v_A^2\nabla^2 v_x. \tag{68}$$

Taking Fourier components in z and Laplace transform (non-standard, see Appendix A) in t, such that $\hat{P}(x) = \int_0^\infty P(x,z,t)e^{i(\omega t - kz)}dt$, and introducing transverse wavenumber Λ, such that $\Lambda^2 = k^2 - \omega^2/v_A^2$, reduces these equations to

$$\frac{d^2\hat{P}}{dx^2} - \Lambda^2\hat{P} = F_P, \quad \frac{d^2\hat{v}_x}{dx^2} - \Lambda^2\hat{v}_x = F_v, \tag{69}$$

where $F_P(x) = (i\omega P_0 - \dot{P}_0)/v_A^2$ and $F_v(x) = (i\omega u_0 - \dot{u}_0)/v_A^2$, where we have defined the initial values by

$$P|_{t=0} = P_0, \quad \left.\frac{\partial P}{\partial t}\right|_{t=0} = \dot{P}_0, \quad v_x|_{t=0} = u_0, \quad \left.\frac{\partial v_x}{\partial t}\right|_{t=0} = \dot{u}_0. \tag{70}$$

3.1. *Solution in Laplace space*

3.1.1. *Solution within the slab*

Consider an asymmetric magnetic slab as defined in Section 2. For the solution inside the slab, $|x| < x_0$, $\hat{P}(x)$ satisfies

$$\frac{d^2\hat{P}}{dx^2} - \Lambda_0^2\hat{P} = F_P, \tag{71}$$

under the boundary conditions $\hat{P}(-x_0) = \hat{A}_1$ and $\hat{P}(-x_0) = \hat{A}_2$. To solve this we construct the Green's function, $G_0(x;s)$ that satisfies

$$\frac{d^2G_0}{dx^2} - \Lambda_0^2 G_0 = \delta(x-s), \quad G_0(-x_0;s) = G_0(x_0;s) = 0, \tag{72}$$

where δ denotes the Dirac Delta function. The general solution, for $|x| < x_0$, of this equation is

$$G_0(x;s) = c_1\sinh(\Lambda_0(x-x_0)) + c_2\sinh(\Lambda_0(x+x_0)), \tag{73}$$

where $c_1 = 0$ for $x < s$ and $c_2 = 0$ for $x > s$. Ensuring G_0 and $\partial G_0/\partial x$ have jumps of 0 and 1 at $x = s$, respectively, determines c_1 and c_2 so that $G_0(x;s)$ is

$$G_0(x;s) = \frac{1}{\Lambda_0\sinh(2\Lambda_0 x_0)}\begin{cases}\sinh(\Lambda_0(s-x_0))\sinh(\Lambda_0(x+x_0)), & \text{if } -x_0 < x < s,\\ \sinh(\Lambda_0(x-x_0))\sinh(\Lambda_0(s+x_0)), & \text{if } s < x < x_0.\end{cases} \tag{74}$$

Then the solution of Equation (71) is

$$\hat{P}(x) = \frac{1}{\sinh 2\Lambda_0 x_0}\left[\hat{A}_1\sinh(\Lambda_0(x_0-x)) + \hat{A}_2\sinh(\Lambda_0(x_0+x))\right] + \int_{-x_0}^{x_0} G_0(x;s)F_P(s)ds. \tag{75}$$

This is the sum of the Green's function term and a two terms that are independent solutions to the homogeneous version of Equation (71) that ensure that the inhomogeneous boundary conditions are satisfied.

3.1.2. *Solution outside the slab*

For the solution outside and to the left of the slab, $x < -x_0$, $\hat{P}(x)$ satisfies

$$\frac{d^2\hat{P}}{dx^2} - \Lambda_1^2\hat{P} = F_P, \tag{76}$$

and the boundary conditions $\hat{P}(-\infty) = 0$, $\hat{P}(-x_0) = \hat{A}_1$. By following a Green's function method, the solution of the aforementioned Sturm-Liouville system is

$$\hat{P}(x) = \hat{A}_1 e^{\Lambda_1(x_0+x)} + \int_{-\infty}^{-x_0} G_1(x;s)F_P(s)ds, \tag{77}$$

where the Green's function, G_1, is defined by

$$G_1(x;s) = \frac{1}{2\Lambda_1}\begin{cases}e^{\Lambda_1(x+s+2x_0)} - e^{\Lambda_1(x-s)}, & \text{if } -\infty < x < s,\\ e^{\Lambda_1(x+s+2x_0)} - e^{\Lambda_1(s-x)}, & \text{if } s < x < -x_0.\end{cases} \tag{78}$$

Similarly, for the solution outside and to the right of the slab, $x > x_0$, $\hat{P}(x)$ satisfies

$$\hat{P}(x) = \hat{A}_2 e^{\Lambda_2(x_0-x)} + \int_{x_0}^{\infty} G_2(x;s)F_P(s)ds, \tag{79}$$

where the Green's function, G_2, is defined by

$$G_2(x;s) = \frac{1}{2\Lambda_2}\begin{cases} e^{-\Lambda_2(x+s-2x_0)} - e^{\Lambda_2(x-s)}, & \text{if } x_0 < x < s,\\ e^{-\Lambda_2(x+s+2x_0)} - e^{\Lambda_2(s-x)}, & \text{if } s < x < \infty. \end{cases} \tag{80}$$

putting all of this together, the (Laplace transform of) the magnetic pressure is

$$\hat{P}(x) = \begin{cases} \hat{A}_1 e^{\Lambda_1(x_0+x)} + \int_{-\infty}^{-x_0} G_1(x;s)F_P(s)ds, & \text{if } 0 < x < -x_0,\\ \frac{1}{\sinh 2\Lambda_0 x_0}\left[\hat{A}_1\sinh(\Lambda_0(x_0-x)) + \hat{A}_2\sinh(\Lambda_0(x_0+x))\right] + \int_{-x_0}^{x_0} G_0(x;s)F_P(s)ds, & \text{if } -x_0 < x < x_0,\\ \hat{A}_2 e^{\Lambda_2(x_0-x)} + \int_{x_0}^{\infty} G_2(x;s)F_P(s)ds, & \text{if } x_0 < x < \infty, \end{cases} \tag{81}$$

where the Greens function is given by

$$G(x;s) = \begin{cases} G_1(x;s) = \\ \frac{1}{2\Lambda_1}[(e^{\Lambda_1(x+s+2x_0)} - e^{\Lambda_1(x-s)})H(x-s) + (e^{\Lambda_1(x+s+2x_0)} - e^{\Lambda_1(s-x)})H(s-x)], & \text{if } -\infty < x < -x_0,\\ G_0(x;s) = \\ \frac{1}{\Lambda_0\sinh(2\Lambda_0 x_0)}[\sinh(\Lambda_0(s-x_0))\sinh(\Lambda_0(x+x_0))H(x-s)+ \\ \sinh(\Lambda_0(x-x_0))\sinh(\Lambda_0(s+x_0))H(s-x)], & \text{if } -x_0 < x < x_0,\\ G_2(x;s) = \\ \frac{1}{2\Lambda_2}[(e^{-\Lambda_2(x+s-2x_0)} - e^{\Lambda_2(x-s)})H(x-s) + (e^{-\Lambda_2(x+s+2x_0)} - e^{\Lambda_2(s-x)})H(s-x)], & \text{if } x_0 < x < \infty. \end{cases} \tag{82}$$

3.1.3. *Matching solutions*

For physically relevant solutions, we require that the transverse displacement (equivalently, the perturbation in transverse velocity) and the total pressure (equivalently, the magnetic pressure, since we are assuming that the plasma is cold) be continuous across the interfaces at $x = \pm x_0$.

Continuity in magnetic pressure, P, is satisfied automatically by considering the solutions inside and outside the slab given by Equations (75), (77), and (79), respectively, and our definition of $\hat{A}_1 = \hat{P}(x = -x_0)$ and $\hat{A}_2 = \hat{P}(x = x_0)$.

Continuity in transverse velocity, v_x can be dealt with as follows. If we make the simplification to the prescribed initial conditions such at $u_0 = 0$ and $\dot{u}_0 + \frac{1}{\rho_0}\frac{dP_0}{dx} = 0$, so that the initial plasma is at rest and its initial acceleration transverse to the slab is due to the initial pressure gradient, then this boundary condition is equivalent to

$$\left[\left[\frac{1}{\rho_0(\omega^2 - k^2 v_A^2)}\frac{\partial\hat{P}}{\partial x}\right]\right]_{x=\pm x_0} = 0. \tag{83}$$

Substituting the solutions given by Equations (75), (77), and (79) into these boundary conditions gives

$$\hat{A}_1(\omega) = \frac{T_1(\omega)}{D(\omega)}, \quad \hat{A}_2(\omega) = \frac{T_2(\omega)}{D(\omega)}, \tag{84}$$

where

$$T_1(\omega) = (\Lambda_1^2 I_{s1} - \Lambda_0^2 I_{e1})[\Lambda_0\Lambda_2\sinh(2\Lambda_0 x_0) + \Lambda_2^2\cosh(2\Lambda_0 x_0)] - \Lambda_1^2(\Lambda_0^2 I_{e2} + \Lambda_2^2 I_{s2}), \tag{85}$$

$$T_2(\omega) = \Lambda_2^2(\Lambda_1^2 I_{s1} - \Lambda_0^2 I_{e1}) - (\Lambda_0^2 I_{e2} + \Lambda_2^2 I_{s2})[\Lambda_0\Lambda_1\sinh(2\Lambda_0 x_0) + \Lambda_1^2\cosh(2\Lambda_0 x_0)], \tag{86}$$

$$D(\omega) = \Lambda_0\Lambda_1\Lambda_2[\Lambda_0(\Lambda_1+\Lambda_2)\cosh(2\Lambda_0 x_0) + (\Lambda_0^2 + \Lambda_1\Lambda_2)\sinh(2\Lambda_0 x_0)], \tag{87}$$

and

$$I_{e1} = \int_{-\infty}^{-x_0} e^{\Lambda_1(s+x_0)}F_p(s)ds, \qquad I_{e2} = \int_{x_0}^{\infty} e^{\Lambda_2(x_0-s)}F_p(s)ds, \tag{88}$$

$$I_{s1} = \int_{-x_0}^{x_0}\frac{\sinh(\Lambda_0(s-x_0))}{\sinh(2\Lambda_0 x_0)}F_p(s)ds, \quad I_{s2} = \int_{-x_0}^{x_0}\frac{\sinh(\Lambda_0(s+x_0))}{\sinh(2\Lambda_0 x_0)}F_p(s)ds.$$

3.2. *Solution in time*

We aim to study the asymptotic behaviour of the magnetic pressure perturbation for various regimes in time.

3.2.1. *Asymptotic solution for large time*

To study the asymptotic behaviours of the magnetic pressure perturbation, we start with the asymptotic behaviours of $A_1(t) = P(t, -x_0)$ and $A_2(t) = P(t, x_0)$. These variables can be determined, using the inverse Laplace transform (non-standard, discussed in Appendix A), to be

$$A_1(t) = \frac{1}{2\pi} \lim_{L\to\infty} \int_{i\gamma-L}^{i\gamma+L} \frac{T_1(\omega)}{D(\omega)} e^{-i\omega t} d\omega, \quad A_2(t) = \frac{1}{2\pi} \lim_{L\to\infty} \int_{i\gamma-L}^{i\gamma+L} \frac{T_2(\omega)}{D(\omega)} e^{-i\omega t} d\omega, \tag{89}$$

where γ is a real number such that all the singularities of the integrands are below the contour of integration. The integrals are evaluated along an infinite horizontal line in the upper half of the complex plane.

Since the problem of finding the solution is now reduced to solving a complex integral, it is dependent on the singularities (with respect to ω) of T_1, T_2, and D (so that we can construct the Bromwich contour such that it is contained to a single-valued branch) and the zeros of D (whose residues determine the value of the contour integral).

To determine the singularities of these functions, we determine the singularities of the constituent functions, as follows.

- The functions $\Lambda^2_{0,1,2}$ are polynomial in ω, and are therefore entire.
- However, $\Lambda_{0,1,2}$ are involve radicals and have (algebraic) branch points of at $\omega = \pm k v_{A0}$, $\pm k v_{A1}$, and $\pm k v_{A2}$, respectively[1].
- The functions $\cosh(z)$ and $\sinh(z)$ are entire functions of z with only even and odd terms in their respective series expansions. Therefore $\cosh(z)$ and $z\sinh(z)$ are entire functions of z^2. Hence $\cosh(\Lambda_0 x_0)$ and $\Lambda_0 \sinh \Lambda_0 x_0$ are entire functions of ω.
- The integrands of $I_{s1,2}$ are integrated with respect to s. So, to determine the singularities of $I_{s1,2}$, we need consider only the singularities of the integrands. The initial condition, $F_P(s)$ is not a function of ω so does not influence the singularities. The function $f(z) = \sinh(az)/\sinh(bz)$, for constants a and $b \neq 0$ are entire functions of z, containing only even powers (once f has been trivially redefined as to remove the removable singularity at $z = 0$). Therefore, f is also an entire function of z^2. Hence, by letting $a = s \pm x_0$ and $b = 2x_0$, it follows that $\sinh(\Lambda_0(s - x_0))/\sinh(2\Lambda_0 x_0)$ is entire in ω.
- As above, to determine the singularities of $I_{e1,2}$, we need consider only the singularities of the integrands. The function $e^{a\sqrt{z}}$, for constant $a \neq 0$ has algebraic branch points at $z = 0$. Therefore, by setting $a = x_0 \pm s$, it follows that the functions $e^{\Lambda_{1,2}(x_0 \pm s)}$, and therefore $I_{e1,2}$, have algebraic branch points (of ramification index 2) at $\omega = \pm k v_{A1,2}$, respectively.

The above analysis determines that the singularities of each function $T_1(\omega)$, $T_2(\omega)$, and $D(\omega)$ are precisely the algebraic branch points at $\omega = \pm k v_{A1}$ and $\omega = \pm k v_{A2}$.

The zeros of $D(\omega)$ are determined by firstly noting that $D = 0$ is the dispersion relation of the corresponding eigenvalue problem solved by Zsámberger et al. (2018). To see this, observe that for a zero-beta plasma, the parameters $\Lambda^z_i = -i\rho_i\omega\sqrt{k^2 v^2_{Ai} - \omega^2}$, for $i = 0, 1, 2$, where we have added the superscript label z to differentiate it from Λ_i in the present text. Trivially,

$$\Lambda_i = \frac{i\mu\omega}{B^2_i} \Lambda^z_i. \tag{90}$$

Due to equilibrium total pressure balance (the total pressure in a cold plasma is merely the magnetic pressure), the strength of the magnetic field is uniform across the structure, *i.e.* $B^2_0 = B^2_1 = B^2_2$. Therefore, $D = 0$ is equivalent to

$$\Lambda^z_0(\Lambda^z_1 + \Lambda^z_2) + (\Lambda^{z2}_0 + \Lambda^z_1\Lambda^z_2)\tanh(2m_0x_0) = 0, \tag{91}$$

[1] More precisely, $\omega = \pm k v_{A0}$, $\pm k v_{A1}$, and $\pm k v_{A2}$ are the ramification points corresponding to the branch points $\Lambda_{0,1,2} = 0$, each with ramification index 2. However the language used in the main text is common shorthand that is considered synonymous, although less rigorous.

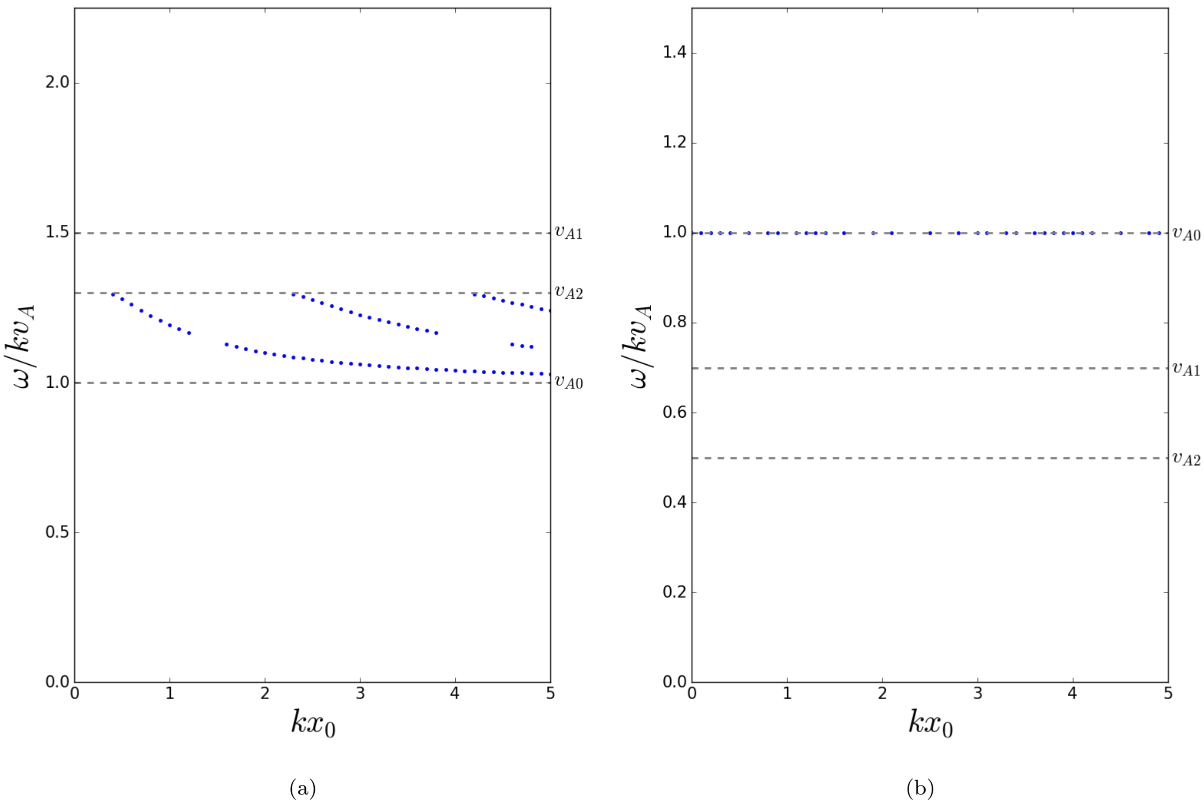

Figure 2. Solutions to the zero-beta dispersion relation in two different ordering of the characteristic speeds.

where $m_0^2 = k^2 - \omega^2/v_{A0}^2$, which is the dispersion relation of a zero-beta asymmetric magnetic slab (Zsámberger et al. 2018). It follows that the zeros of D are precisely the eigenvalues of the asymmetric magnetic slab.

Perhaps this marks the end of the road for the cold plasma IVP because there do not exist any surface modes of a cold magnetic slab, regardless of Alfvén speed ordering (Figure 2). Maybe there are surface leaky modes? Doesn't seem like it when you look at the imaginary part of the dispersion function. Back to incompressible??? It would be more tractable due to being analytically soluble and waves manifest only as surface modes, therefore allowing application of SMS techniques.

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Software: Mayavi

APPENDIX

A. NON-STANDARD LAPLACE TRANSFORM

Consider a function $f(t)$, whose standard Laplace transform, $F_1(\omega)$, and non-standard Laplace transform, $F_2(\omega)$, are

$$F_1(\omega) = \int_0^\infty f(t)e^{-\omega t}dt, \quad \text{and} \quad F_2(\omega) = \int_0^\infty f(t)e^{i\omega t}dt. \tag{A1}$$

Trivially, $F_1(-i\omega) = F_2(\omega)$. Using the standard inverse Laplace transform, and letting γ be real and greater than the real part of all the singularities of $F_1(\omega)$, the original function $f(t)$ can be written

$$f(t) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\gamma - iT}^{\gamma + iT} F_1(\omega) e^{\omega t} d\omega, \tag{A2}$$

$$= \frac{1}{2\pi i} \lim_{T\to\infty} \int_{i\gamma - T}^{i\gamma + T} F_1(-i\omega) e^{-i\omega t} (-i d\omega), \tag{A3}$$

$$= \frac{1}{2\pi} \lim_{T\to\infty} \int_{i\gamma - T}^{i\gamma + T} F_2(\omega) e^{-i\omega t} d\omega. \tag{A4}$$

Therefore, the inverse transform of the non-standard Laplace transform is

$$f(t) = \frac{1}{2\pi} \lim_{T\to\infty} \int_{i\gamma - T}^{i\gamma + T} F_2(\omega) e^{-i\omega t} d\omega. \tag{A5}$$

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