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EVOLUTION OF ASYMMETRIC SLAB MAGNETOHYDRODYNAMIC WAVES

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ABSTRACT

Abstract (250 word limit for ApJ)

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1. INITIAL VALUE PROBLEM - INCOMPRESSIBLE MAGNETIC SLAB

Consider an equilibrium plasma with magnetic field $B_0(x)\hat{\mathbf{z}}$, density $\rho_0(x)$, and pressure $p_0(x)$, without gravity. In the absence of structuring in the z-direction and considering perturbations in the (x, z)-plane only, we can take Fourier components for velocity and other parameters like $\mathbf{v}(\mathbf{x}, t) = \hat{\mathbf{v}}(x)e^{i(kz+ly-\omega t)}$. The velocity perturbation amplitude in the inhomogeneous direction of an ideal plasma is given by

$$\frac{d}{dx}\left(\frac{\epsilon(x)}{l^2 + m_0^2(x)}\frac{d\hat{v}_x}{dx}\right) - \epsilon(x)\hat{v}_x = 0, \tag{1}$$

where

$$\epsilon(x) = \rho_0(x)[k^2 v_A^2(x) - \omega^2], \quad m_0^2(x) = \frac{(k^2 c_0^2(x) - \omega^2)(k^2 v_A^2(x) - \omega^2)}{(c_0^2(x) + v_A^2(x))(k^2 c_T^2(x) - \omega^2)}, \tag{2}$$

and

$$c_0(x) = \sqrt{\frac{\gamma p_0(x)}{\rho_0(x)}}, \quad v_A(x) = \frac{B_0(x)}{\sqrt{\mu\rho_0(x)}}, \quad c_T(x) = \frac{c_0(x)v_A(x)}{\sqrt{c_0^2(x) + v_A^2(x)}} \tag{3}$$

are the sound, Alfvén, and tube speeds, respectively.

When the plasma is incompressible, so that $c_0 \to \infty$, we have $c_T^2 \to v_A^2$ and $m_0^2 \to k^2$. After restricting the analysis to propagation only parallel to the magnetic field ($l = 0$), Equation (1) reduces to

$$\frac{d}{dx}\left(\epsilon(x)\frac{d\hat{v}_x}{dx}\right) - k^2\epsilon(x)\hat{v}_x = 0, \tag{4}$$

which is commonly used for solving eigenmode problems in MHD wave physics. In the present work, the temporal evolution of linear asymmetric MHD waves is considered so we must dial back our assumptions about how the wave behaviour through time.

Above we used a Fourier decomposition in time, which is valid when the solutions are homogeneous in time, such as normal mode solutions (rewrite this). To investigate the temporal evolution of solutions, we take only Fourier components in the z-direction, that is $\mathbf{v}(\mathbf{x}, t) = \hat{\mathbf{v}}(x, t)e^{ikz}$, and we take the Laplace transform with respect to time, such that

$$\tilde{\mathbf{v}}(x) = \int_0^\infty \hat{\mathbf{v}}(x, t)e^{i\omega t}dt. \tag{5}$$

This gives us the initial-value form of Equation (4) to be

$$\frac{d}{dx}\left(\epsilon(x)\frac{d\tilde{v}_x}{dx}\right) - k^2\epsilon(x)\tilde{v}_x = f(x), \tag{6}$$

where

$$f(x) = ik\left\{\rho_0\left[\frac{\partial\hat{\Omega}}{\partial t}(x, 0) - i\omega\hat{\Omega}(x, 0)\right] - \left[\frac{\partial\hat{v}_z}{\partial t}(x, 0) - i\omega\hat{v}_z(x, 0)\right]\frac{d\rho_0}{dx}\right\}, \tag{7}$$

where the vorticity, $\Omega(x, t)\hat{\mathbf{y}} = \hat{\Omega}(x, t)e^{ikz}\hat{\mathbf{y}} = \nabla \times \mathbf{v}(\mathbf{x}, t)$, is given by

$$\hat{\Omega}(\mathbf{x}, t) = -\frac{i}{k}\left(\frac{\partial^2\hat{v}_x}{\partial x^2} - k^2\hat{v}_x\right). \tag{8}$$

(this differs from Rae & Roberts (1981) by a factor of -1 due to taking Fourier forms like e^{ikz} rather than e^{-ikz})

Consider equilibrium magnetic field and density profiles given by

$$B(x) = \begin{cases} B_1, & \text{if } x < -x_0, \\ B_0, & \text{if } |x| \leq x_0, \\ B_2, & \text{if } x > x_0, \end{cases} \quad \text{and} \quad \rho(x) = \begin{cases} \rho_1, & \text{if } x < -x_0, \\ \rho_0, & \text{if } |x| \leq x_0, \\ \rho_2, & \text{if } x > x_0, \end{cases} \tag{9}$$

where B_i and ρ_i are uniform for $i = 0, 1, 2$. This establishes the plasma as magnetic slab embedded in an asymmetric magnetic background.

In this equilibrium, transverse velocity perturbations are related to initial perturbations in the following way:

$$\frac{d^2\tilde{v}_x}{dx^2} - k^2\tilde{v}_x = \begin{cases} f(x)/\epsilon_1, & \text{if } x < -x_0, \\ f(x)/\epsilon_0, & \text{if } |x| < x_0, \\ f(x)/\epsilon_2, & \text{if } x > x_0. \end{cases} \tag{10}$$

1.1. *Attempt 1*

Sturm-Liouville Theory tells us that with the aid of a Green's function, $G(x;s)$, Equation (10) can be solved to give

$$\tilde{v}_x(x) = \begin{cases} \tilde{A}(\cosh kx + \sinh kx) - \frac{1}{\epsilon_1}\int_{-\infty}^{-x_0} G(x;s)f(s)ds, & \text{if } x < -x_0, \\ \tilde{B}\cosh kx + \tilde{C}\sinh kx - \frac{1}{\epsilon_0}\int_{-x_0}^{x_0} G(x;s)f(s)ds, & \text{if } |x| < x_0, \\ \tilde{D}(\cosh kx - \sinh kx) - \frac{1}{\epsilon_2}\int_{x_0}^{\infty} G(x;s)f(s)ds, & \text{if } x > x_0, \end{cases} \tag{11}$$

where

$$G(x;s) = \frac{1}{2k}[e^{ks}e^{-kx}H(x-s) + e^{-ks}e^{kx}H(s-x)] \tag{12}$$

and H is the Heaviside step function. Ensuring continuity of both transverse velocity and total pressure across the boundaries at $x = \pm x_0$ gives us the following system of linear algebraic equations for the constants A, B, C, and D:

$$\begin{pmatrix} c_0 - s_0 & -c_0 & s_0 & 0 \\ 0 & c_0 & s_0 & s_0 - c_0 \\ \epsilon_1(c_0 - s_0) & \epsilon_0 s_0 & -\epsilon_0 c_0 & 0 \\ 0 & \epsilon_0 s_0 & \epsilon_0 c_0 & -\epsilon_2(s_0 - c_0) \end{pmatrix} \begin{pmatrix} \tilde{A} \\ \tilde{B} \\ \tilde{C} \\ \tilde{D} \end{pmatrix} = \frac{1}{2k} \begin{pmatrix} e^{kx_0}/\epsilon_1 & -e^{-kx_0}/\epsilon_0 & 0 & 0 \\ 0 & 0 & e^{-kx_0}/\epsilon_0 & -e^{kx_0}/\epsilon_2 \\ -e^{kx_0} & -e^{-kx_0} & 0 & 0 \\ 0 & 0 & -e^{-kx_0} & -e^{kx_0} \end{pmatrix} \begin{pmatrix} I_1 \\ I_0^- \\ I_0^+ \\ I_2 \end{pmatrix}, \tag{13}$$

where $c_0 = \cosh kx_0$, $s_0 = \sinh kx_0$, and the functionals I_1, I_0^-, I_0^+, and I_2 are given by

$$I_1 = \int_{-\infty}^{-x_0} e^{ks}f(s)ds, \quad I_0^- = \int_{-x_0}^{x_0} e^{-ks}f(s)ds, \quad I_0^+ = \int_{-x_0}^{x_0} e^{ks}f(s)ds, \quad I_2 = \int_{x_0}^{\infty} e^{-ks}f(s)ds. \tag{14}$$

Solving this system of equations gives

$$\tilde{A} = \frac{e^{2kx_0}T_1}{2k\epsilon_1 D_R}, \quad \tilde{B} = \frac{e^{2kx_0}T_0^-}{2k\epsilon_0 D_R}, \quad \tilde{C} = \frac{e^{2kx_0}T_0^+}{2k\epsilon_0 D_R}, \quad \tilde{D} = \frac{e^{2kx_0}T_2}{2k\epsilon_2 D_R}, \tag{15}$$

where

$$T_1(\omega) = I_1[e^{4kx_0}(\epsilon_0 - \epsilon_1)(\epsilon_0 + \epsilon_2) - (\epsilon_0 + \epsilon_1)(\epsilon_0 - \epsilon_2)] - 4I_2 e^{2kx_0}\epsilon_0\epsilon_1 - 2\epsilon_1\left[I_0^- e^{2kx_0}(\epsilon_0 + \epsilon_2) + I_0^+(\epsilon_0 - \epsilon_2)\right], \tag{16}$$

$$\begin{aligned} T_0^-(\omega) = {} & [(I_0^+ - I_0^-)(\epsilon_2 - \epsilon_1)\epsilon_0 - (I_0^+ + I_0^-)(\epsilon_0^2 - \epsilon_1\epsilon_2)]e^{2kx_0} - (I_0^+ + I_0^-)(\epsilon_0 - \epsilon_1)(\epsilon_0 - \epsilon_2) \\ & - 2\epsilon_0 e^{2kx_0}\left[((I_1 + I_2)\epsilon_0 + I_1\epsilon_2 + I_2\epsilon_1)e^{2kx_0} + (I_1 + I_2)\epsilon_0 - I_1\epsilon_2 - I_2\epsilon_1\right], \end{aligned} \tag{17}$$

$$\begin{aligned} T_0^+(\omega) = {} & [(I_0^+ + I_0^-)(\epsilon_2 - \epsilon_1)\epsilon_0 - (I_0^+ - I_0^-)(\epsilon_0^2 - \epsilon_1\epsilon_2)]e^{2kx_0} - (I_0^+ - I_0^-)(\epsilon_0 - \epsilon_1)(\epsilon_0 - \epsilon_2) \\ & - 2\epsilon_0 e^{2kx_0}\left[((I_2 - I_1)\epsilon_0 - I_1\epsilon_2 + I_2\epsilon_1)e^{2kx_0} + (I_1 - I_2)\epsilon_0 - I_1\epsilon_2 + I_2\epsilon_1\right], \end{aligned} \tag{18}$$

$$T_2(\omega) = I_2[e^{4kx_0}(\epsilon_0 + \epsilon_1)(\epsilon_0 - \epsilon_2) - (\epsilon_0 - \epsilon_1)(\epsilon_0 + \epsilon_2)] - 4I_1 e^{2kx_0}\epsilon_0\epsilon_2 - 2\epsilon_2\left[I_0^+ e^{2kx_0}(\epsilon_0 + \epsilon_1) + I_0^-(\epsilon_0 - \epsilon_1)\right], \tag{19}$$

$$D_R(\omega) = (\epsilon_0 + \epsilon_1)(\epsilon_0 + \epsilon_2)e^{4kx_0} - (\epsilon_0 - \epsilon_1)(\epsilon_0 - \epsilon_2). \tag{20}$$

The solutions of equation $D_R(\omega) = 0$ are precisely the solutions of the dispersion relation for an incompressible asymmetric magnetic slab. To confirm this, we follow notion from Zsámberger et al. (2018) (with an added superscript z) by observing that for an incompressible plasma, the sound speed, $c_s \to \infty$. Therefore,

$$m_j^2 = \frac{(k^2c_j^2 - \omega^2)(k^2v_{Aj}^2 - \omega^2)}{(c_j^2 + v_{Aj}^2)(k^2c_Tj^2 - \omega^2)} \to k^2, \tag{21}$$

$$\Lambda_j^z = \frac{i\rho_j}{\omega m_j}(k^2 v_{Aj}^2 - \omega^2) \to \frac{i\rho_j}{\omega k}(k^2 v_{Aj}^2 - \omega^2), \quad \text{for } j = 0, 1, 2. \tag{22}$$

Therefore, after multiplying Equation (20) by $\omega k / i(e^{4kx_0} + 1)$, we recover the dispersion relation in Zsámberger et al. (2018) for an incompressible plasma, namely

$$2(\Lambda_0^2 + \Lambda_1\Lambda_2) + \Lambda_0(\Lambda_1 + \Lambda_2)[\tanh kx_0 + \coth kx_0] = 0. \tag{23}$$

The solution given by Equation (11), with constants (with respect to x) $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$, given by Equation (15) corroborates with those describing the initial value problem of surface waves on an interface between two incompressible plasmas. When the slab width, $2x_0$, approaches zero, it is expected that the solutions here approach those describing the interface. As $x_0 \to 0$, the dispersion function behaves like

$$D_R \to (\epsilon_0 + \epsilon_1)(\epsilon_0 + \epsilon_2) - (\epsilon_0 - \epsilon_1)(\epsilon_0 - \epsilon_2) = 2\epsilon_0(\epsilon_1 + \epsilon_2). \tag{24}$$

Therefore, in this limit, $\tilde{A}$ and $\tilde{D}$ behave like

$$\tilde{A} \to \frac{1}{4k\epsilon_0\epsilon_1(\epsilon_1 + \epsilon_2)}\{2I_1\epsilon_0(\epsilon_2 - \epsilon_1) - 4I_2\epsilon_0\epsilon_1\} = \frac{1}{k(\epsilon_1 + \epsilon_2)}\left\{-I_2 + \frac{1}{2\epsilon_1}(\epsilon_2 - \epsilon_1)I_1\right\}, \tag{25}$$

$$\tilde{D} \to \frac{1}{4k\epsilon_0\epsilon_2(\epsilon_1 + \epsilon_2)}\{2I_2\epsilon_0(\epsilon_1 - \epsilon_2) - 4I_1\epsilon_0\epsilon_2\} = \frac{1}{k(\epsilon_1 + \epsilon_2)}\left\{-I_1 + \frac{1}{2\epsilon_2}(\epsilon_1 - \epsilon_2)I_2\right\}, \tag{26}$$

$$\tag{27}$$

which are equal to (the corrected versions of) A_- and A_+, respectively, from Rae & Roberts (1981).

1.1.1. *Solution in time*

To recover the transverse velocity, $v_x(x,t)$, we employ the inverse Laplace transform, such that

$$v_x(x,t) = \frac{1}{2\pi}\lim_{L\to\infty}\int_{i\gamma - L}^{i\gamma + L} \tilde{v}_x(x,\omega)e^{-i\omega t}d\omega, \tag{28}$$

where γ is a real number such that all the singularities of the integrand is below the contour of integration. The integral is evaluated along an infinite horizontal line in the upper half of the complex plane. Since the problem is now reduced to solving a complex integral, it is dependent on the singularities (with respect to ω) of $\tilde{v}_x$, whose residues determine the value of the contour integral.

Focusing firstly on the region $x < -x_0$, the solution is

$$v_x = \frac{1}{2\pi}\lim_{L\to\infty}\int_{i\gamma - L}^{i\gamma + L}\left[\tilde{A}e^{k(x+x_0)} + \frac{1}{\epsilon_1}\int_{-\infty}^{-x_0} G(x;s)f(s)ds\right]e^{-i\omega t}d\omega, \tag{29}$$

$$= \frac{e^{k(x+x_0)}}{2\pi}\left(\lim_{L\to\infty}\int_{i\gamma - L}^{i\gamma + L}\tilde{A}e^{-i\omega t}ds\right) + \left(\int_{-\infty}^{-x_0} G(x;s)f(s)ds\right)\left(\lim_{L\to\infty}\int_{i\gamma - L}^{i\gamma + L}\frac{e^{-i\omega t}}{\epsilon_1}d\omega\right), \tag{30}$$

The first integral in the above solution is calculated as follows: The functions $\epsilon_{0,1,2}$ are polynomial in ω, and are therefore entire. The integrals $I_{1,2}$ and $I_0^{\pm}$ are not functions of ω so also contribute no singularities in ω. Therefore, $T_{1,2}$, and $T_0^{\pm}$ are entire functions.

The final integral in Equation (30) is calculated as follows:
the singularities of $\tilde{v}_x$ are precisely the singularities of A and $1/\epsilon_1$

Considering the functional form of $\tilde{v}_x(x,\omega)$, we see that, in each region, it is made up of a term involving

1.1.2. *Specific initial conditions*

The above approach using arbitrary initial conditions is forced to cease here due to mathematical intractability of the general inverse Laplace transform calculation (however, general asymptotic results are, just barely, tractable). Instead, we progress with specific initial conditions.

Choosing specific initial conditions requires a delicate balance between mathematical tractability and physical applicability.

REFERENCES

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