Initial value problem of a finite beta asymmetric magnetic slab

1 TO DO

- Calculate residues for fast body modes.
- Check that integral around semi-circle and vertical contours go to zero as semi-circle is extended to infinity.
- Calculate contributions of the contours around the branch points - corresponding to "improper modes". Including check that the integrals around the little semi-circles goes to zero.
- Check that there aren't any leaky modes on other Riemann sheets (I don't think there is because we have branch points from the square roots, not from logarithmic terms, so there are only two Riemann sheets).
- Try initial conditions.

2 Basic equations

From [1],

$$\frac{1}{\rho(\omega^2 - \omega_A^2)} \left(\frac{\partial^2}{\partial x^2} - m^2 \right) \hat{p}_T = f(\omega, x), \tag{1}$$

where

$$f(\omega,x) = \frac{\partial}{\partial x}\frac{\dot{\xi}_{x0} - i\omega\xi_{x0}}{\rho(\omega^2-\omega_A^2)} + ik_y\frac{\dot{\xi}_{y0} - i\omega\xi_{y0}}{\rho(\omega^2-\omega_A^2)} + ik_z\frac{c_0^2}{c_0^2+v_A^2}\frac{\dot{\xi}_{z0} - i\omega\xi_{z0}}{\rho(\omega^2-\omega_T^2)}, \tag{2}$$

$$m^2 = m(\omega)^2 = \frac{(\omega_1^2-\omega^2)(\omega_2^2-\omega^2)}{(c_0^2+v_A^2)(\omega_T^2-\omega^2)}, \tag{3}$$

$$\omega_1^2 = \frac{1}{2}(k_y^2+k_z^2)(c_0^2+v_A^2)\left(1-\sqrt{1-\frac{4\omega_T^2}{(k_y^2+k_z^2)(c_0^2+v_A^2)}}\right), \tag{4}$$

$$\omega_2^2 = \frac{1}{2}(k_y^2+k_z^2)(c_0^2+v_A^2)\left(1+\sqrt{1-\frac{4\omega_T^2}{(k_y^2+k_z^2)(c_0^2+v_A^2)}}\right). \tag{5}$$

Take $k_y \ll k_z$ and $\xi_{y0} = \dot{\xi}_{y0} = 0$, then

$$\left(\frac{\partial^2}{\partial x^2} - m^2\right)\hat{p}_T = f(\omega,x), \tag{6}$$

where

$$f(\omega,x) = \frac{\partial}{\partial x}(\dot{\xi}_{x0} - i\omega\xi_{x0}) + ik_z\frac{c_0^2}{c_0^2+v_A^2}\frac{(\omega^2-\omega_A^2)}{(\omega^2-\omega_T^2)}(\dot{\xi}_{z0} - i\omega\xi_{z0}), \tag{7}$$

$$m^2 = \frac{(\omega_0^2-\omega^2)(\omega_A^2-\omega^2)}{(c_0^2+v_A^2)(\omega_T^2-\omega^2)}, \tag{8}$$

where we have defined the initial values by

$$\xi_x(x,0) = \xi_{x0}, \quad \frac{\partial\xi_x}{\partial t}(x,0) = \dot{\xi}_{x0}, \quad \xi_z(x,0) = \xi_{z0}, \quad \frac{\partial\xi_z}{\partial t}(x,0) = \dot{\xi}_{z0}. \tag{9}$$

Considering a magnetic slab in a non-magnetic environment, we have

$$m_0^2 = -\frac{(\omega^2-\omega_0^2)(\omega^2-\omega_A^2)}{(c_0^2+v_A^2)(\omega^2-\omega_T^2)}, \quad m_{1,2}^2 = \frac{\omega_{1,2}^2-\omega^2}{c_{1,2}^2}, \tag{10}$$

$$f_{1,2}(\omega,x) = \frac{\partial}{\partial x}(\dot{\xi}_{x0} - i\omega\xi_{x0}) + ik(\dot{\xi}_{z0} - i\omega\xi_{z0}). \tag{11}$$

3 Solution in Laplace space

3.1 Solution within the slab

For the solution inside the slab, $|x| < x_0$, $\hat{p}_T(x)$ satisfies

$$\left(\frac{\partial^2}{\partial x^2} - m_0^2\right)\hat{p}_T = f_0(\omega,x), \tag{12}$$

under the boundary conditions $\hat{p}_T(-x_0) = \hat{A}_1$ and $\hat{p}_T(-x_0) = \hat{A}_2$. To solve this we construct the Green's function, $G_0(x;s)$ that satisfies

$$\frac{d^2G_0}{dx^2} - m_0^2 G_0 = \delta(x-s), \quad G_0(-x_0;s) = G_0(x_0;s) = 0, \tag{13}$$

where δ denotes the Dirac delta function. The general solution of this equation is

$$G_0(x;s) = c_1 \sinh(m_0(x-x_0)) + c_2 \sinh(m_0(x+x_0)), \tag{14}$$

where $c_1 = 0$ for $x < s$ and $c_2 = 0$ for $x > s$. Ensuring G_0 and $\partial G_0/\partial x$ have jumps of 0 and 1 at $x = s$, respectively, determines c_1 and c_2 so that $G_0(x;s)$ is

$$G_0(x;s) = \frac{-1}{m_0 \sinh(2m_0x_0)} \begin{cases} \sinh(m_0(x_0-s))\sinh(m_0(x_0+x)), & \text{if } -x_0 < x < s, \\ \sinh(m_0(x_0-x))\sinh(m_0(x_0+s)), & \text{if } s < x < x_0. \end{cases} \tag{15}$$

Then the solution of Equation (**??**) is

$$\hat{p}_T(x) = \frac{1}{m_0 \sinh 2m_0x_0}\left[\hat{A}_1 \sinh(m_0(x_0-x)) + \hat{A}_2 \sinh(m_0(x_0+x))\right] + \int_{-x_0}^{x_0} G_0(x;s) f_0(\omega,s) ds. \tag{16}$$

This is the sum of the Green's function term and a two terms that are independent solutions to the homogeneous version of Equation (**??**) that ensure that the inhomogeneous boundary conditions are satisfied.

3.2 Solution outside the slab

For the solution outside and to the left of the slab, $x < -x_0$, $\hat{p}_T(x)$ satisfies

$$\left(\frac{d^2}{dx^2} - m_1^2\right)\hat{p}_T = f_1(\omega,x), \tag{17}$$

and the boundary conditions $\hat{p}_T(-\infty) = 0$, $\hat{p}_T(-x_0) = \hat{A}_1$. By following a Green's function method, the solution of this Sturm-Liouville system is

$$\hat{p}_T(x) = \hat{A}_1 e^{m_1(x_0+x)} + \int_{-\infty}^{-x_0} G_1(x;s) f_1(\omega,s) ds, \tag{18}$$

where $m_1 > 0$ and the Green's function, G_1, is defined by

$$G_1(x;s) = \frac{1}{m_1}\begin{cases} e^{m_1(x_0+x)} \sinh(m_1(x_0+s)), & \text{if } x < s, \\ e^{m_1(x_0+s)} \sinh(m_1(x_0+x)), & \text{if } s < x < -x_0. \end{cases} \tag{19}$$

Similarly, for the solution outside and to the right of the slab, $x > x_0$, $\hat{p}_T(x)$ satisfies

$$\hat{p}_T(x) = \hat{A}_2 e^{m_2(x_0 - x)} + \int_{x_0}^{\infty} G_2(x; s) f_2(\omega, s) ds, \tag{20}$$

where $m_2 > 0$ and the Green's function, G_2, is defined by

$$G_2(x; s) = \frac{1}{m_2} \begin{cases} e^{m_2(x_0 - s)} \sinh(m_2(x_0 - x)), & \text{if } x_0 < x < s, \\ e^{m_2(x_0 - x)} \sinh(m_2(x_0 - s)), & \text{if } s < x. \end{cases} \tag{21}$$

Putting all of this together, the (Laplace transform of) total pressure is

$$\hat{p}_T(x) = \begin{cases} \hat{A}_1 e^{m_1(x_0 + x)} + \int_{-\infty}^{-x_0} G_1(x; s) f_1(\omega, s) ds, & \text{if } - \\ \frac{1}{\sinh 2m_0 x_0} \left[\hat{A}_1 \sinh(m_0(x_0 - x)) + \hat{A}_2 \sinh(m_0(x_0 + x)) \right] + \int_{-x_0}^{x_0} G_0(x; s) f_0(\omega, s) ds, & \text{if } - \\ \hat{A}_2 e^{m_2(x_0 - x)} + \int_{x_0}^{\infty} G_2(x; s) f_2(\omega, s) ds, & \text{if } x_0 \end{cases} \tag{22}$$

3.3 Matching solutions

For physically relevant solutions, we require that the transverse displacement (equivalently, the perturbation in transverse velocity) and the total pressure be continuous across the interfaces at $x = \pm x_0$.

Continuity in total pressure perturbation, $\hat{p}_T$, is satisfied automatically by considering the solutions inside and outside the slab given by Equations (16), (18), and (20), respectively, and our definition of $\hat{A}_1 = \hat{p}_T(-x_0)$ and $\hat{A}_2 = \hat{p}_T(x_0)$.

Continuity in transverse displacement, ξ_x, can be dealt with as follows. If we make the simplification to the prescribed initial conditions such that $\dot{\xi}_{x0} - i\omega \xi_{x0} = 0$, then this boundary condition is equivalent to

$$\left[\left[\frac{1}{\rho(\omega^2 - \omega_A^2)} \frac{\partial \hat{p}_T}{\partial x} \right]\right]_{x = \pm x_0} = 0. \tag{23}$$

(Equation (8) in [1]).

Substituting the solutions given by Equations (16), (18), and (20) into these boundary conditions gives

$$\hat{A}_1(\omega) = \frac{T_1(\omega)}{D(\omega)}, \quad \hat{A}_2(\omega) = \frac{T_2(\omega)}{D(\omega)}, \tag{24}$$

where

$$T_1(\omega) = T_1[f](\omega) = -(\Lambda_2 \cosh 2m_0x_0 + \Lambda_0 \sinh 2m_0x_0)(\Lambda_1 I_0^- + \Lambda_0 I_1) - \Lambda_1(\Lambda_0 I_2 + \Lambda_2 I_0^+), \tag{25}$$

$$T_2(\omega) = T_2[f](\omega) = -\Lambda_2(\Lambda_1 I_0^- + \Lambda_0 I_1) - (\Lambda_1 \cosh 2m_0x_0 + \Lambda_0 \sinh 2m_0x_0)(\Lambda_0 I_2 + \Lambda_2 I_0^+), \tag{26}$$

$$D(\omega) = \Lambda_0(\Lambda_1 + \Lambda_2)\cosh(2m_0x_0) + (\Lambda_0^2 + \Lambda_1\Lambda_2)\sinh(2m_0x_0), \tag{27}$$

where $\Lambda_j = \rho_j(\omega^2 - \omega_{Aj}^2)/m_j$, for $j = 0, 1, 2$, and

$$I_0^{\pm} = I_0^{\pm}[f] = \frac{1}{m_0}\int_{-x_0}^{x_0} \frac{\sinh(m_0(x_0 \pm s))}{\sinh(2m_0x_0)} f(\omega, s)ds, \tag{28}$$

$$I_1 = I_1[f] = \frac{1}{m_1}\int_{-\infty}^{-x_0} e^{m_1(s+x_0)} f(\omega, s)ds, \quad I_2 = I_2[f] = \frac{1}{m_2}\int_{x_0}^{\infty} e^{m_2(x_0-s)} f(\omega, s)ds, \tag{29}$$

where

$$f(\omega, x) = \begin{cases} f_1(\omega, x), & \text{if } x < -x_0, \\ f_0(\omega, x), & \text{if } -x_0 < x < x_0, \\ f_2(\omega, x), & \text{if } x_0 < x. \end{cases} \tag{30}$$

4 Solution in time

To recover the transverse velocity, $v_x(x,t)$, we employ the inverse Laplace transform (non-standard, discussed in Appendix A), such that

$$p_T(x,t) = \mathcal{L}^{-1}\{\hat{p}_T(x)\} = \frac{1}{2\pi}\lim_{L\to\infty}\int_{i\gamma-L}^{i\gamma+L} \hat{p}_T(x)e^{-i\omega t}d\omega, \tag{31}$$

where γ is a real number such that all the singularities of the integrand are below the contour of integration to ensure that all singularities contribute to the integral. The integral is evaluated along an infinite horizontal line in the upper half of the complex plane and is dependent on the singularities (with respect to ω) of $\hat{p}_T$, whose residues determine the value of the contour integral.

Focusing firstly on the region $x < -x_0$, the solution is

$$p_T(x,t) = \mathcal{L}^{-1}\left\{\hat{A}_1 e^{m_1(x+x_0)} + \int_{-\infty}^{-x_0} G_1(x;s) f_1(\omega,s)ds\right\}, \tag{32}$$

$$= \mathcal{L}^{-1}\left\{\hat{A}_1 e^{m_1(x+x_0)}\right\} + \mathcal{L}^{-1}\left\{\int_{-\infty}^{-x_0} G_1(x;s) f_1(\omega,s)ds\right\}. \tag{33}$$

4.1 Asymptotic solution for large time

To study the asymptotic behaviours of the total pressure perturbation, we start with the asymptotic behaviours of $A_1(t) = p_T(-x_0,t)$ and $A_2(t) = p_T(x_0,t)$. These variables can be determined, using the inverse Laplace transform (non-standard, discussed in Appendix A), to be

$$A_1(t) = \frac{1}{2\pi}\lim_{L\to\infty}\int_{i\gamma-L}^{i\gamma+L}\frac{T_1(\omega)}{D(\omega)}e^{-i\omega t}d\omega, \quad A_2(t) = \frac{1}{2\pi}\lim_{L\to\infty}\int_{i\gamma-L}^{i\gamma+L}\frac{T_2(\omega)}{D(\omega)}e^{-i\omega t}d\omega, \tag{34}$$

where γ is a real number such that all the singularities of the integrands are below the contour of integration. The integrals are evaluated along an infinite horizontal line in the upper half of the complex plane.

Since the problem of finding the solution is now reduced to solving a complex integral, it is dependent on the singularities (with respect to ω) of T_1, T_2, and D (so that we can construct a Bromwich contour such that it is confined to a single-valued branch) and the zeros of D (whose residues determine the value of the contour integral).

To determine the singularities of these functions, we determine the singularities of the constituent functions, as follows.

- The functions Λ_j^2 are rational functions of ω with simple poles at $\omega = \pm\omega_{0j}$, for $j = 0, 1, 2$.
- Λ_j, for $j = 0, 1, 2$, involve radicals and have (algebraic) branch points at $\omega = \pm\omega_{Aj}$, $\pm\omega_{0j}$, and $\pm\omega_{Tj}$, respectively.[1]
- The functions $\cosh(z)$ and $\sinh(z)$ are entire functions of z with only even and odd terms in their respective series expansions. Therefore,

[1]More precisely, $\omega = \pm\omega_{Aj}$, $\pm\omega_{0j}$, and $\pm\omega_{Tj}$ are the ramification points corresponding to the branch points $\Lambda_j(\omega)$, each with ramification index 2. However, the language used in the main text is common shorthand that is considered synonymous.

$\cosh(z)$ and $z\sinh(z)$ are entire functions of z^2. Hence, $\cosh(2m_0x_0)$ and $\Lambda_0 \sinh 2m_0x_0$ have only simple poles at $\omega = \omega_{T0}$.

- The integrands of $I_0^{\pm}$ are integrated with respect to s. Therefore, the singularities of $I_{1,2}$ are precisely the singularities of the integrands. The function $g(z) = \sinh(az)/\sinh(bz)$, for constants a and $b \neq 0$ are entire functions of z, containing only even powers (once g has been redefined as to remove the removable singularity at $z = 0$). Therefore, for another complex function h, the singularities of the composition $g \cdot h$ are precisely the singularities of the function $h(z^2)$. Hence, by letting $h(\omega) = m_0$, $a = s \pm x_0$, and $b = 2x_0$, it follows that $\sinh(m_0(s - x_0))/\sinh(2m_0x_0)$ has simple poles at $\omega = \pm\omega_{T0}$.

- To determine the singularities of $I_{1,2}$, we need consider the singularities of the integrands. The functions $e^{a\sqrt{z}}$ and $e^{\frac{a}{\sqrt{z}}}$, for constant $a \neq 0$ have branch points at $z = 0$ that are algebraic (of ramification index 2) and transcendental, respectively. Therefore, by setting $a = x_0 \pm s$, it follows that the functions $e^{m_j(x_0 \pm s)}$, and therefore I_j, have algebraic branch points at $\omega = \pm\omega_{Aj}$ and $\pm\omega_{0j}$ and transcendental branch points at $\pm\omega_{Tj}$.

By the algebra of branch points, the set of branch points of a sum of functions is the union of the branch points of the constituent functions. Therefore, the branch points of both T_1 and T_2 are $\omega = \pm\omega_{A0,1,2}$ (algebraic), $\pm\omega_{00,1,2}$ (algebraic), $\pm\omega_{T0}$ (algebraic), and $\pm\omega_{T1,2}$ (transcendental). The dispersion function D has branch points at $\omega = \pm\omega_{A0,1,2}$, $\pm\omega_{00,1,2}$, and $\pm\omega_{T0,1,2}$, all of which are algebraic.

TO DO: Redo the above to show that we have algebraic branch points at all of the $\omega_{A,0,T}$ corresponding to the Alfven, slow and fast continua, corresponding to leaky modes. So we have 18 branch points. How should be choose the branch cuts? We need to ensure we are confined to a single Riemann sheet i.e. single valued. See e.g. Fig 6 in Sedlacek 1970.

The above analysis determines that the singularities of each function $T_1(\omega)$, $T_2(\omega)$, and $D(\omega)$ are precisely the algebraic branch points at $\omega = \pm kv_{A1}$ and $\omega = \pm kv_{A2}$.

4.2 Asymptotic solution - non-magnetic external plasma

If the external plasma is non-magnetic, *i.e.* $v_{A1} = v_{A2} = 0$, then the integral has (algebraic) branch points at $\omega = \pm\omega_{A0}$, $\pm\omega_{00,1,2}$, and $\pm\omega_{T0}$, where, without loss of generality, we let $\omega_1 < \omega_2$. The contour, $C = C_1+C_2+C_3+C_4+C_5$ for the inverse Laplace transform can be modified as shown in Figure 1. To ensure that the contour remains on a single Riemann surface, it is modified around the branch cuts so as to encircle the poles. The closed contour C is a sum of the following sub-contours:

- C_0: the horizontal line with imaginary part γ.
- C_1: the horizontal line from $L + \delta i$ to $\omega_0 + \delta i$, round the semicircle of radius δ and back along the horizontal line from $\omega_0 - \delta i$ to $L - \delta i$.
- C_2: the vertical lines from $\pm L + \gamma i$ to $\pm L + \delta i$ and the arcs of the large semicircle centred at the origin with radius Δ.
- C_3: the vertical line up to, around, and back down from ω_T.
- C_4: the vertical line up to, around, and back down from $-\omega_T$.
- C_5: the horizontal line from $-L - \delta i$ to $-\omega_0 - \delta i$, round the semicircle of radius δ and back along the horizontal line from $-\omega_0 + \delta i$ to $-L + \delta i$.

We use the integral along C as $\delta \to 0$ and $L \to \infty$ to determine the inverse Laplace transform from Equation (34). In the limit of $L \to \infty$, the integral along C_0 is identical to the desired integral in Equation (34). The integrals along each of the sub-contours $C_1 - C_5$ are calculated in the following subsections.

4.2.1 Integral along C_1 and C_5

In the limit as $\delta \to 0$, the integral along the semicircular part of C_1 approaches 0 because the integrand is analytic in this limit and the length of the contour approaches zero.

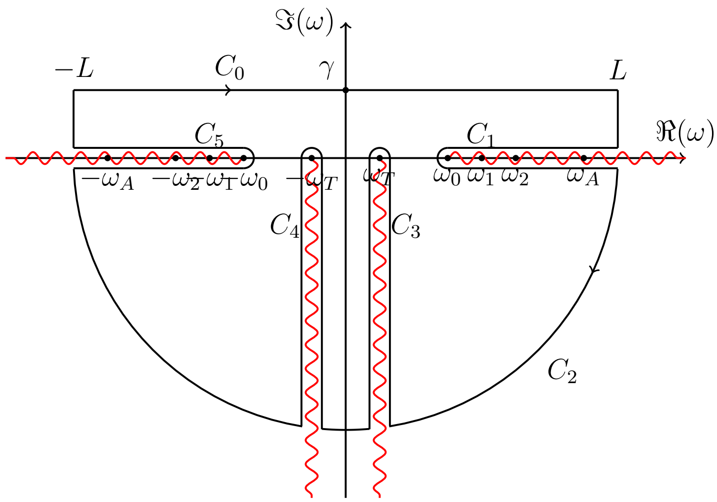

Figure 1: The Bromwich contour, $C = C_1+C_2+C_3+C_4+C_5$, for the complex integration of $\tilde{A}_{1,2}$ in the inverse Laplace transform in Equation (34). The radius of the large semicircle is L and the radius of the small semicircles around the points $\pm\omega_0$ and $\pm\omega_T$ is δ.

As $L \to \infty$ the integrals along C_1 become

$$I_{C_1} = \int_{C_1} \frac{T_{1,2}}{D} e^{-i\omega t} d\omega \tag{35}$$

$$= \int_{\infty}^{\omega_0} \frac{T_{1,2}^{+}}{D^{+}} e^{-i\omega t} d\omega + \int_{\omega_0}^{\infty} \frac{T_{1,2}^{-}}{D^{-}} e^{-i\omega t} d\omega \tag{36}$$

$$= \int_{\omega_0}^{\infty} \left(\frac{T_{1,2}^{-}}{D^{-}} - \frac{T_{1,2}^{+}}{D^{+}} \right) e^{-i\omega t} d\omega, \tag{37}$$

where superscripts $+$ and $-$ indicate the value of the function above and below the horizontal branch cut $[\omega_0, \infty)$, respectively.

For values of ω between the branch points at ω_0, ω_1, ω_2, and ω_A, the integrand is analytic (except at the endpoints), therefore we can use integration by parts in these sub-intervals. For example, the integral over the sub-interval (ω_0, ω_1) is

$$I_{(\omega_0,\omega_1)} = \int_{\omega_0}^{\omega_1} \left(\frac{T_{1,2}^-}{D^-} - \frac{T_{1,2}^+}{D^+} \right) e^{-i\omega t} d\omega \tag{38}$$

$$= \frac{i}{t} \left\{ \left[\left(\frac{T_{1,2}^-}{D^-} - \frac{T_{1,2}^+}{D^+} \right) e^{-i\omega t} \right]_{\omega_0}^{\omega_1} - \int_{\omega_0}^{\omega_1} \frac{d}{d\omega} \left(\frac{T_{1,2}^-}{D^-} - \frac{T_{1,2}^+}{D^+} \right) e^{-i\omega t} d\omega \right\}. \tag{39}$$

Given that $T_{1,2}^+(\omega_0)/D^\pm(\omega_0) = T_{1,2}^-(\omega_0)/D^\pm(\omega_0)$, the slowest decaying term is the first term on the right hand side evaluated at ω_1, which decays like $\mathcal{O}(t^{-1})$ as $t \to \infty$. Repeating this for the other three intervals over which the integrand is analytic, and using the fact that $T_{1,2}^\pm(\omega)/D^\pm(\omega) \to 0$ as $|\omega| \to \infty$, we find that the slowest decaying terms decay like $\mathcal{O}(t^{-1})I_{C_1} = \mathcal{O}(t^{-1})$ as $t \to \infty$. Therefore $I_{C_1} = \mathcal{O}(t^{-1})$ (and similarly $I_{C_5} = \mathcal{O}(t^{-1})$) as $t \to \infty$.

4.2.2 Integral along C_2

As $L \to \infty$, points that occupy the curve C_2 will behave like $|\omega| \to \infty$. When $|\omega| \to \infty$, $T_{1,2} = \mathcal{O}(|\omega|)$ and $D = \mathcal{O}(|\omega|^2)$, therefore the integrands behave like $T_{1,2}/D = \mathcal{O}(1/|\omega|)$. Therefore the integral around C_2 approaches 0 as $L \to \infty$.

4.2.3 Integral along C_3 and C_4

In the limit as $\delta \to 0$, the integral along the semicircular part of C_3 approaches 0 because the integrand is analytic in this limit and the length of the contour approaches zero. As $L \to \infty$ and $\delta \to 0$, the integrals along C_3 become

$$I_{C_3} = \int_{C_3} \frac{T_{1,2}}{D} e^{-i\omega t} d\omega \tag{40}$$

$$= \int_{\omega_T}^{\omega_T - \infty i} \left(\frac{T_{1,2}^-}{D^-} - \frac{T_{1,2}^+}{D^+} \right) e^{-i\omega t} d\omega, \tag{41}$$

where superscripts + and − indicate the value of the function to the right and left of the vertical branch cut $\omega_T + [0, \infty)i$, respectively. The integrand is analytic in the integration interval (except at the endpoints), so by integration by parts

$$I_{C_3} = \frac{i}{t}\left\{\left[\left(\frac{T_{1,2}^-}{D^-} - \frac{T_{1,2}^+}{D^+}\right)e^{-i\omega t}\right]_{\omega_T}^{\omega_T-\infty i} - \int_{\omega_T}^{\omega_T-\infty i}\frac{d}{d\omega}\left(\frac{T_{1,2}^-}{D^-} - \frac{T_{1,2}^+}{D^+}\right)e^{-i\omega t}d\omega\right\}. \tag{42}$$

the first term is zero because $T_{1,2}^{\pm}(\omega)/D^{\pm}(\omega) \to 0$ as $|\omega| \to \infty$ and $T_{1,2}^{+}(\omega_T)/D^{\pm}(\omega_T) = T_{1,2}^{-}(\omega_T)/D^{\pm}(\omega_T)$. Therefore, using integration by parts for a second time,

$$I_{C_3} = -\frac{i}{t}\int_{\omega_T}^{\omega_T-\infty i}\frac{d}{d\omega}\left(\frac{T_{1,2}^-}{D^-} - \frac{T_{1,2}^+}{D^+}\right)e^{-i\omega t}d\omega \tag{43}$$

$$= \frac{1}{t^2}\left\{\left[\frac{d}{d\omega}\left(\frac{T_{1,2}^-}{D^-} - \frac{T_{1,2}^+}{D^+}\right)e^{-i\omega t}\right]_{\omega_T}^{\omega_T-\infty i} - \int_{\omega_T}^{\omega_T-\infty i}\frac{d^2}{d\omega^2}\left(\frac{T_{1,2}^-}{D^-} - \frac{T_{1,2}^+}{D^+}\right)e^{-i\omega t}d\omega\right\} \tag{44}$$

$$= \mathcal{O}(t^{-2}) \text{ as } t \to \infty. \tag{45}$$

Similarly, $I_{C_5} = \mathcal{O}(t^{-2})$ as $t \to \infty$.

4.2.4 Poles (AKA eigenfrequencies)

Let $v_A > c_i$ for $i = 0, 1, 2$, then there can exist, in general:

1. 2 slow surface modes,
2. an infinite number of slow body modes,
3. 2 fast surface modes.

1 and 2 exist for all values of kx_0. 3 exist only for kx_0 values greater than a cut-off value. Analytical expressions for the behaviour of the eigenmodes do not exist in general. It is at this point that we are forced to make two simplifications to the model:

- The thin-slab approximation, so that $kx_0 \ll 1$
- The approximate symmetry simplification, so that $\rho_1 \approx \rho_2$.

The eigenmodes of an asymmetric magnetic slab under these simplifications are given by the following expressions:

- Slow quasi-sausage surface modes:

$$\omega^2 = k^2 c_{\mathrm{T}}^2 \left[1 - \frac{2(kx_0)(c_0^2 - c_{\mathrm{T}}^2)}{(c_0^2 + v_{\mathrm{A}}^2)\left(\frac{\rho_0}{\rho_1}\frac{(c_1^2 - c_{\mathrm{T}}^2)^{1/2}}{c_1} + \frac{\rho_0}{\rho_2}\frac{(c_2^2 - c_{\mathrm{T}}^2)^{1/2}}{c_2}\right)} \right], \tag{46}$$

  which is less than $k^2 c_{\mathrm{T}}^2$ and exists only when $c_1 > c_{\mathrm{T}}$ and $c_2 > c_{\mathrm{T}}$.

- If $c_1 = c_2 = c_{\mathrm{e}}$ (and therefore $\rho_1 = \rho_2 = \rho_{\mathrm{e}}$), then there exists a fast sausage surface mode:

$$\omega^2 = k^2 c_{\mathrm{e}}^2 \left(1 - \left[\frac{\rho_{\mathrm{e}}}{\rho_0} \frac{c_{\mathrm{e}}^2 (c_0^2 - c_{\mathrm{e}}^2)(kx_0)}{(c_0^2 + v_{\mathrm{A}}^2)(c_{\mathrm{T}}^2 - c_{\mathrm{e}}^2)} \right]^2 \right) \tag{47}$$

  in the thin slab limit.

- Slow quasi-kink surface mode:

$$\omega^2 = \frac{k^2 x_0 v_{\mathrm{A}}^2 \left(\frac{\rho_0}{\rho_1} m_1 + \frac{\rho_0}{\rho_2} m_2 \right)}{\left(\frac{\rho_0}{\rho_1} m_1 x_0 + \frac{\rho_0}{\rho_2} m_2 x_0 \right) + 2} \approx \frac{1}{2} k^2 v_{\mathrm{A}}^2 \left(\frac{\rho_0}{\rho_1} + \frac{\rho_0}{\rho_2} \right) kx_0. \tag{48}$$

- Quasi-sausage body solutions:

$$\omega^2 = k^2 c_{\mathrm{T}}^2 \left(1 + \frac{c_{\mathrm{T}}^4 (kx_0)^2}{c_0^2 v_{\mathrm{A}}^2 \pi^2 j^2} \right), \quad j = 1, 2, \ldots. \tag{49}$$

- Quasi-kink body solutions:

$$\omega^2 = k^2 c_{\mathrm{T}}^2 \left(1 + \frac{c_{\mathrm{T}}^4 (kx_0)^2}{c_0^2 v_{\mathrm{A}}^2 \pi^2 (j - \frac{1}{2})^2} \right), \quad j = 1, 2, \ldots. \tag{50}$$

Solution of the inverse Laplace transform is the sum of the residues at these points.

As shown in Appendix B, under the approximate symmetry simplification, the dispersion function, $D(\omega)$, becomes

$$D(\omega) = \frac{\sinh 2m_0 x_0}{4\Lambda_1 \Lambda_2} D_k(\omega) D_s(\omega), \tag{51}$$

where

$$D_k(\omega) = \Lambda_0(\Lambda_1 + \Lambda_2) + 2\Lambda_1\Lambda_2 \coth(m_0x_0), \tag{52}$$
$$D_s(\omega) = \Lambda_0(\Lambda_1 + \Lambda_2) + 2\Lambda_1\Lambda_2 \tanh(m_0x_0). \tag{53}$$

This can be further simplified for surface modes under the thin-slab approximation, to first order in kx_0, to

$$D_k(\omega) = \Lambda_0(\Lambda_1 + \Lambda_2) + 2\Lambda_1\Lambda_2\frac{1}{m_0x_0}, \tag{54}$$
$$D_s(\omega) = \Lambda_0(\Lambda_1 + \Lambda_2) + 2\Lambda_1\Lambda_2 m_0x_0. \tag{55}$$

4.2.5 Residues

In order to calculate the residue of each pole, we much determine their order. Firstly, for the slow kink surface mode, $\omega = \omega_{sks}$, it can be shown that $D_s(\omega_{sks}) \neq 0$, and after some algebra,

$$D_k(\omega) = \frac{\rho_1\rho_2\omega^2 R}{m_0m_1m_2x_0}(\omega^2 - \omega_{sks}^2), \tag{56}$$

where

$$R = 2 + \left(\frac{\rho_0}{\rho_1} + \frac{\rho_0}{\rho_2}\right) kx_0, \tag{57}$$

so each of $\omega = \pm\omega_{sks}$ are simple zeros of D_k and hence of D, therefore, they are simple poles of the integrand in question.

Therefore, the residue of the integrand at $\omega = \omega_{sks}$ is

$$\mathrm{Res}\left\{\frac{T_1}{D}e^{i\omega t}; \omega = \omega_{sks}\right\} = \lim_{\omega\to\omega_{sks}}\left\{(\omega - \omega_{sks})\frac{T_1(\omega)}{D(\omega)}e^{i\omega t}\right\} \tag{58}$$
$$= \frac{2m_0m_1m_2x_0\Lambda_1\Lambda_2}{\rho_1\rho_2\omega_{sks}^2Rm_0x_0}|_{\omega=\omega_{sks}}\frac{T_1(\omega_{sks})}{2\omega_{sks}D_s(\omega_{sks})}e^{i\omega_{sks}t} \tag{59}$$
$$= \frac{\omega_{sks}}{R}\frac{T_1(\omega_{sks})}{D_s(\omega_{sks})}e^{i\omega_{sks}t}. \tag{60}$$

Similarly, the residue at $\omega = -\omega_{sks}$ is

$$\mathrm{Res}\left\{\frac{T_1}{D}e^{i\omega t}; \omega = -\omega_{sks}\right\} = \frac{\omega_{sks}}{R}\frac{T_1(\omega_{sks})}{D_s(\omega_{sks})}e^{-i\omega_{sks}t} \tag{61}$$

since T_1 and D_s are even functions of ω.

Similarly, the residues for the slow sausage surface mode $\omega = \pm\omega_{sss}$ are

$$\text{Res}\left\{\frac{T_1}{D}e^{i\omega t}; \omega = \omega_{sss}\right\} = \lim_{\omega\to\omega_{sss}}\left\{(\omega-\omega_{sss})\frac{T_1(\omega)}{D(\omega)}e^{i\omega t}\right\} \tag{62}$$

$$= \frac{2m_0m_1m_2\Lambda_1\Lambda_2}{\rho_1\rho_2\omega_{sss}^2 m_0 x_0}|_{\omega=\omega_{sss}} \frac{(\omega_{sss}^2-\omega_T^2)}{(\omega_{sss}^2-\omega_A^2)k(\frac{\rho_0}{\rho_1}\sqrt{1-\frac{c_T^2}{c_1^2}}+\frac{\rho_0}{\rho_2}\sqrt{1-\frac{c_T^2}{c_2^2}})}\frac{T_1(\omega_{sss})}{2\omega_{sss}D_k(\omega_s} \tag{63}$$

$$= \frac{\omega_{sss}}{kx_0}\frac{(\omega_{sss}^2-\omega_T^2)}{(\omega_{sss}^2-\omega_A^2)(\frac{\rho_0}{\rho_1}\sqrt{1-\frac{c_T^2}{c_1^2}}+\frac{\rho_0}{\rho_2}\sqrt{1-\frac{c_T^2}{c_2^2}})}\frac{T_1(\omega_{sss})}{D_k(\omega_{sss})}e^{i\omega_{sss}t}. \tag{64}$$

Similarly, the residue at $\omega = -\omega_{sss}$ is

$$\text{Res}\left\{\frac{T_1}{D}e^{i\omega t}; \omega = -\omega_{sss}\right\} = \frac{\omega_{sss}}{kx_0}\frac{(\omega_{sss}^2-\omega_T^2)}{(\omega_{sss}^2-\omega_A^2)(\frac{\rho_0}{\rho_1}\sqrt{1-\frac{c_T^2}{c_1^2}}+\frac{\rho_0}{\rho_2}\sqrt{1-\frac{c_T^2}{c_2^2}})}\frac{T_1(\omega_{sss})}{D_k(\omega_{sss})}e^{-i\omega_{sss}t}. \tag{65}$$

Regarding, the fast kink body modes $\omega = \pm\omega_{fkbj}$, for $j = 1, 2, ...$, it can be shown that $D_s(\omega_{fkbn}) \neq 0$ and that by letting $\omega_{fkbj}^2 = \omega_T^2(1+\nu(kx_0)^2)$, we can show that

$$D_k(\omega) = A_0\cos\left(\frac{c_T}{\sqrt{\nu(c_0^2+v_A^2)}}\right) + \mathcal{O}(kx_0), \tag{66}$$

for some constant A_0, whose form is irrelevant at present. Zeros of $\cos(z)$ are of order 2, therefore zeros of $\cos(\sqrt{z})$ are simple. Therefore, letting $\nu = c_T^4/c_0^2v_A^2\pi^2(j-\frac{1}{2})^2$ for $j \in \mathcal{N}$, $\omega_{fkbj} = \pm\omega_T\sqrt{1+\nu(kx_0)^2}$ are simple zeros of D_k. Therefore, they are simple poles of $T_{1,2}/D$, so the residue at $\omega = \omega_{fkbj}$ is

$$\text{Res}\left\{\frac{T_1}{D}e^{i\omega t}; \omega = \omega_{fkbj}\right\} = \lim_{\omega\to\omega_{fkbj}}\left\{(\omega-\omega_{fkbj})\frac{T_1(\omega)}{D(\omega)}e^{i\omega t}\right\} \tag{67}$$

$$= \tag{68}$$

Similarly, the residue at $\omega = -\omega_{fkbj}$ is

$$\text{Res}\left\{\frac{T_1}{D}e^{i\omega t}; \omega = -\omega_{fkbj}\right\} = \tag{69}$$

The residues for the integrand of A_2 are found by replacing subscript 2 with 1.

A Non-standard Laplace transform

Consider a function $f(t)$, whose standard Laplace transform, $F_1(\omega)$, and non-standard Laplace transform, $F_2(\omega)$, are

$$F_1(\omega) = \int_0^\infty f(t) e^{-\omega t} dt, \quad \text{and} \quad F_2(\omega) = \int_0^\infty f(t) e^{i\omega t} dt. \tag{70}$$

Trivially, $F_1(-i\omega) = F_2(\omega)$. Using the standard inverse Laplace transform, and letting γ be real and greater than the real part of all the singularities of $F_1(\omega)$, the original function $f(t)$ can be written

$$f(t) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\gamma - iT}^{\gamma + iT} F_1(\omega) e^{\omega t} d\omega, \tag{71}$$

$$= \frac{1}{2\pi i} \lim_{T\to\infty} \int_{i\gamma - T}^{i\gamma + T} F_1(-i\omega) e^{-i\omega t} (-i d\omega), \tag{72}$$

$$= \frac{1}{2\pi} \lim_{T\to\infty} \int_{i\gamma - T}^{i\gamma + T} F_2(\omega) e^{-i\omega t} d\omega. \tag{73}$$

Therefore, the inverse transform of the non-standard Laplace transform is

$$f(t) = \frac{1}{2\pi} \lim_{T\to\infty} \int_{i\gamma - T}^{i\gamma + T} F_2(\omega) e^{-i\omega t} d\omega. \tag{74}$$

B Approximately symmetric slab dispersion relation

Consider the product of the following functions

$$D_k(\omega) = \Lambda_0(\Lambda_1 + \Lambda_2) + 2\Lambda_1\Lambda_2 \coth m_0 x_0, \tag{75}$$

$$D_s(\omega) = \Lambda_0(\Lambda_1 + \Lambda_2) + 2\Lambda_1\Lambda_2 \tanh m_0 x_0. \tag{76}$$

$$D_k(\omega) D_s(\omega) = \left[\Lambda_0(\Lambda_1 + \Lambda_2) + 2\Lambda_1\Lambda_2 \coth m_0 x_0\right] \left[\Lambda_0(\Lambda_1 + \Lambda_2) + 2\Lambda_1\Lambda_2 \tanh m_0 x_0\right] \tag{77}$$

$$= \Lambda_0^2(\Lambda_1 + \Lambda_2)^2 + 2\Lambda_0(\Lambda_1 + \Lambda_2)\Lambda_1\Lambda_2(\tanh m_0 x_0 + \coth m_0 x_0) + 4\Lambda_1^2\Lambda_2^2 \tag{78}$$

$$= \frac{4\Lambda_1\Lambda_2}{\sinh 2m_0 x_0} \left[\Lambda_0(\Lambda_1 + \Lambda_2)\cosh 2m_0 x_0 + (\Lambda_0^2 F + \Lambda_1\Lambda_2)\sinh 2m_0 x_0\right], \tag{79}$$

where

$$F = \frac{(\Lambda_1 + \Lambda_2)^2}{4\Lambda_1\Lambda_2}, \tag{80}$$

which, when the conditions on each side of the slab are approximately symmetric, *i.e.* $\Lambda_2 = \Lambda_1(1+\epsilon L)$, with $\epsilon \ll 1$ and $L \approx 1$, becomes

$$F = \frac{(2+\epsilon L)^2}{4(1+\epsilon L)} = 1 + \mathcal{O}(\epsilon^2). \tag{81}$$

Therefore,

$$D(\omega) = \Lambda_0(\Lambda_1 + \Lambda_2)\cosh(2m_0x_0) + (\Lambda_0^2 + \Lambda_1\Lambda_2)\sinh(2m_0x_0), \tag{82}$$

$$= \frac{\sinh 2m_0x_0}{4\Lambda_1\Lambda_2} D_k(\omega)D_s(\omega) + \mathcal{O}(\epsilon^2). \tag{83}$$

C Approximately symmetric slab parameter expansions

For an approximately symmetric slab, that is, when $\rho_2 = \rho_1(1+\epsilon)$ where $\epsilon \ll 1$, then the first order expansions in ϵ of other relevant parameters are given below.

- $c_2^2 = c_1^2\frac{\rho_1}{\rho_2} = c_1^2\left(\frac{1}{1+\epsilon}\right) \approx c_1^2(1-\epsilon)$.
- $m_2 = \sqrt{k^2 - \frac{\omega^2}{c_2^2}} = \sqrt{k^2 - \frac{\omega^2}{c_1^2}(1+\epsilon)} \approx m_1(1 - \frac{\omega^2}{2c_1^2m_1^2}\epsilon)$.
- $\Lambda_2 = \frac{\rho_2\omega^2}{m_2} = \frac{\rho_1(1+\epsilon)\omega^2}{m_1\left(1-\frac{1}{2m_1^2}\epsilon\right)} \approx \frac{\rho_1(1+\epsilon)\omega^2}{m_1}\left(1 + \frac{1}{2m_1^2}\epsilon\right) = \Lambda_1\left(1 + \epsilon\left[1 + \frac{\omega^2}{2c_1^2m_1^2}\right]\right)$.
- $I_2 = \int_{x_0}^{\infty} e^{m_2(x_0-s)}f(\omega,s)ds = \int_{-\infty}^{-x_0} e^{m_2(x_0+s)}f(\omega,s)ds$, if $f(\omega,-x) = f(\omega,x)\ \forall x \in (-\infty,-x_0)$. Therefore,

$$I_2 = \int_{-\infty}^{-x_0} e^{\epsilon m_1 M(x_0+s)}e^{m_1(x_0+s)}f(\omega,s)ds, \text{ where } M = -\frac{\omega^2}{2c_1^2m_1^2}, \tag{84}$$

$$= I_1 + \epsilon m_1 M \int_{-\infty}^{-x_0} (x_0+s)e^{m_1(x_0+s)}f(\omega,s)ds. \tag{85}$$

It doesn't seem that this simplifies into $I_1(1+J\epsilon)$ without making f less general.

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