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EVOLUTION OF ASYMMETRIC SLAB MAGNETOHYDRODYNAMIC WAVES: ZERO-BETA

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ABSTRACT

Abstract (250 word limit for ApJ)

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1. INITIAL VALUE PROBLEM - ZERO-BETA MAGNETIC SLAB

Employing the same procedure as Ruderman & Roberts (2006) to reduce the linearised MHD equations governing cold (zero-beta) plasma leads us to the following equations for the perturbations in magnetic pressure, $P = B_0 b_z/\mu$, and transverse velocity, v_x:

$$\frac{\partial^2 P}{\partial t^2} = v_A^2 \nabla^2 P, \quad \frac{\partial^2 v_x}{\partial t^2} = v_A^2 \nabla^2 v_x. \tag{1}$$

Taking Fourier components in z and Laplace transform (non-standard, see Appendix A) in t, such that $\hat{P}(x) = \int_0^\infty P(x,z,t)e^{i(\omega t - kz)}dt$, and introducing transverse wavenumber Λ, such that $\Lambda^2 = k^2 - \omega^2/v_A^2$, reduces these equations to

$$\frac{d^2\hat{P}}{dx^2} - \Lambda^2 \hat{P} = F_P, \quad \frac{d^2\hat{v}_x}{dx^2} - \Lambda^2 \hat{v}_x = F_v, \tag{2}$$

where $F_P(x) = (i\omega P_0 - \dot{P}_0)/v_A^2$ and $F_v(x) = (i\omega u_0 - \dot{u}_0)/v_A^2$, where we have defined the initial values by

$$P|_{t=0} = P_0, \quad \left.\frac{\partial P}{\partial t}\right|_{t=0} = \dot{P}_0, \quad v_x|_{t=0} = u_0, \quad \left.\frac{\partial v_x}{\partial t}\right|_{t=0} = \dot{u}_0. \tag{3}$$

1.1. *Solution in Laplace space*

1.1.1. *Solution within the slab*

Consider an asymmetric magnetic slab as defined in Section 2. For the solution inside the slab, $|x| < x_0$, $\hat{P}(x)$ satisfies

$$\frac{d^2\hat{P}}{dx^2} - \Lambda_0^2 \hat{P} = F_P, \tag{4}$$

under the boundary conditions $\hat{P}(-x_0) = \hat{A}_1$ and $\hat{P}(-x_0) = \hat{A}_2$. To solve this we construct the Green's function, $G_0(x;s)$ that satisfies

$$\frac{d^2 G_0}{dx^2} - \Lambda_0^2 G_0 = \delta(x-s), \quad G_0(-x_0;s) = G_0(x_0;s) = 0, \tag{5}$$

where δ denotes the Dirac Delta function. The general solution, for $|x| < x_0$, of this equation is

$$G_0(x;s) = c_1 \sinh(\Lambda_0(x - x_0)) + c_2 \sinh(\Lambda_0(x + x_0)), \tag{6}$$

where $c_1 = 0$ for $x < s$ and $c_2 = 0$ for $x > s$. Ensuring G_0 and $\partial G_0/\partial x$ have jumps of 0 and 1 at $x = s$, respectively, determines c_1 and c_2 so that $G_0(x;s)$ is

$$G_0(x;s) = \frac{1}{\Lambda_0 \sinh(2\Lambda_0 x_0)} \begin{cases} \sinh(\Lambda_0(s - x_0))\sinh(\Lambda_0(x + x_0)), & \text{if } -x_0 < x < s, \\ \sinh(\Lambda_0(x - x_0))\sinh(\Lambda_0(s + x_0)), & \text{if } s < x < x_0. \end{cases} \tag{7}$$

Then the solution of Equation (4) is

$$\hat{P}(x) = \frac{1}{\sinh 2\Lambda_0 x_0}\left[\hat{A}_1 \sinh(\Lambda_0(x_0 - x)) + \hat{A}_2 \sinh(\Lambda_0(x_0 + x))\right] + \int_{-x_0}^{x_0} G_0(x;s)F_P(s)ds. \tag{8}$$

This is the sum of the Green's function term and a two terms that are independent solutions to the homogeneous version of Equation (4) that ensure that the inhomogeneous boundary conditions are satisfied.

1.1.2. *Solution outside the slab*

For the solution outside and to the left of the slab, $x < -x_0$, $\hat{P}(x)$ satisfies

$$\frac{d^2\hat{P}}{dx^2} - \Lambda_1^2 \hat{P} = F_P, \tag{9}$$

and the boundary conditions $\hat{P}(-\infty) = 0$, $\hat{P}(-x_0) = \hat{A}_1$. By following a Green's function method, the solution of the aforementioned Sturm-Liouville system is

$$\hat{P}(x) = \hat{A}_1 e^{\Lambda_1(x_0 + x)} + \int_{-\infty}^{-x_0} G_1(x;s)F_P(s)ds, \tag{10}$$

where the Green's function, G_1, is defined by

$$G_1(x;s) = \frac{1}{2\Lambda_1} \begin{cases} e^{\Lambda_1(x+s+2x_0)} - e^{\Lambda_1(x-s)}, & \text{if } -\infty < x < s, \\ e^{\Lambda_1(x+s+2x_0)} - e^{\Lambda_1(s-x)}, & \text{if } s < x < -x_0. \end{cases} \tag{11}$$

Similarly, for the solution outside and to the right of the slab, $x > x_0$, $\hat{P}(x)$ satisfies

$$\hat{P}(x) = \hat{A}_2 e^{\Lambda_2(x_0-x)} + \int_{x_0}^{\infty} G_2(x;s) F_P(s) ds, \tag{12}$$

where the Green's function, G_2, is defined by

$$G_2(x;s) = \frac{1}{2\Lambda_2} \begin{cases} e^{-\Lambda_2(x+s-2x_0)} - e^{\Lambda_2(x-s)}, & \text{if } x_0 < x < s, \\ e^{-\Lambda_2(x+s+2x_0)} - e^{\Lambda_2(s-x)}, & \text{if } s < x < \infty. \end{cases} \tag{13}$$

putting all of this together, the (Laplace transform of) the magnetic pressure is

$$\hat{P}(x) = \begin{cases} \hat{A}_1 e^{\Lambda_1(x_0+x)} + \int_{-\infty}^{-x_0} G_1(x;s) F_P(s) ds, & \text{if } 0 < x < -x_0, \\ \frac{1}{\sinh 2\Lambda_0 x_0} \left[\hat{A}_1 \sinh(\Lambda_0(x_0-x)) + \hat{A}_2 \sinh(\Lambda_0(x_0+x)) \right] + \int_{-x_0}^{x_0} G_0(x;s) F_P(s) ds, & \text{if } -x_0 < x < x_0, \\ \hat{A}_2 e^{\Lambda_2(x_0-x)} + \int_{x_0}^{\infty} G_2(x;s) F_P(s) ds, & \text{if } x_0 < x < \infty, \end{cases} \tag{14}$$

where the Greens function is given by

$$G(x;s) = \begin{cases} G_1(x;s) = & \\ \frac{1}{2\Lambda_1}[(e^{\Lambda_1(x+s+2x_0)} - e^{\Lambda_1(x-s)})H(x-s) + (e^{\Lambda_1(x+s+2x_0)} - e^{\Lambda_1(s-x)})H(s-x)], & \text{if } -\infty < x < -x_0, \\ G_0(x;s) = & \\ \frac{1}{\Lambda_0 \sinh(2\Lambda_0 x_0)}[\sinh(\Lambda_0(s-x_0))\sinh(\Lambda_0(x+x_0))H(x-s) + & \\ \sinh(\Lambda_0(x-x_0))\sinh(\Lambda_0(s+x_0))H(s-x)], & \text{if } -x_0 < x < x_0, \\ G_2(x;s) = & \\ \frac{1}{2\Lambda_2}[(e^{-\Lambda_2(x+s-2x_0)} - e^{\Lambda_2(x-s)})H(x-s) + (e^{-\Lambda_2(x+s+2x_0)} - e^{\Lambda_2(s-x)})H(s-x)], & \text{if } x_0 < x < \infty. \end{cases} \tag{15}$$

1.1.3. *Matching solutions*

For physically relevant solutions, we require that the transverse displacement (equivalently, the perturbation in transverse velocity) and the total pressure (equivalently, the magnetic pressure, since we are assuming that the plasma is cold) be continuous across the interfaces at $x = \pm x_0$.

Continuity in magnetic pressure, P, is satisfied automatically by considering the solutions inside and outside the slab given by Equations (8), (10), and (12), respectively, and our definition of $\hat{A}_1 = \hat{P}(x = -x_0)$ and $\hat{A}_2 = \hat{P}(x = x_0)$.

Continuity in transverse velocity, v_x can be dealt with as follows. If we make the simplification to the prescribed initial conditions such at $u_0 = 0$ and $\dot{u}_0 + \frac{1}{\rho_0}\frac{dP_0}{dx} = 0$, so that the initial plasma is at rest and its initial acceleration transverse to the slab is due to the initial pressure gradient, then this boundary condition is equivalent to

$$\left[\left[\frac{1}{\rho_0(\omega^2 - k^2 v_A^2)} \frac{\partial \hat{P}}{\partial x} \right]\right]_{x=\pm x_0} = 0. \tag{16}$$

Substituting the solutions given by Equations (8), (10), and (12) into these boundary conditions gives

$$\hat{A}_1(\omega) = \frac{T_1(\omega)}{D(\omega)}, \quad \hat{A}_2(\omega) = \frac{T_2(\omega)}{D(\omega)}, \tag{17}$$

where

$$T_1(\omega) = (\Lambda_1^2 I_{s1} - \Lambda_0^2 I_{e1})[\Lambda_0\Lambda_2 \sinh(2\Lambda_0 x_0) + \Lambda_2^2 \cosh(2\Lambda_0 x_0)] - \Lambda_1^2(\Lambda_0^2 I_{e2} + \Lambda_2^2 I_{s2}), \tag{18}$$
$$T_2(\omega) = \Lambda_2^2(\Lambda_1^2 I_{s1} - \Lambda_0^2 I_{e1}) - (\Lambda_0^2 I_{e2} + \Lambda_2^2 I_{s2})[\Lambda_0\Lambda_1 \sinh(2\Lambda_0 x_0) + \Lambda_1^2 \cosh(2\Lambda_0 x_0)], \tag{19}$$
$$D(\omega) = \Lambda_0\Lambda_1\Lambda_2[\Lambda_0(\Lambda_1 + \Lambda_2)\cosh(2\Lambda_0 x_0) + (\Lambda_0^2 + \Lambda_1\Lambda_2)\sinh(2\Lambda_0 x_0)], \tag{20}$$

and

$$I_{e1} = \int_{-\infty}^{-x_0} e^{\Lambda_1(s+x_0)} F_p(s) ds, \qquad I_{e2} = \int_{x_0}^{\infty} e^{\Lambda_2(x_0-s)} F_p(s) ds, \tag{21}$$
$$I_{s1} = \int_{-x_0}^{x_0} \frac{\sinh(\Lambda_0(s-x_0))}{\sinh(2\Lambda_0 x_0)} F_p(s) ds, \quad I_{s2} = \int_{-x_0}^{x_0} \frac{\sinh(\Lambda_0(s+x_0))}{\sinh(2\Lambda_0 x_0)} F_p(s) ds.$$

1.2. *Solution in time*

We aim to study the asymptotic behaviour of the magnetic pressure perturbation for various regimes in time.

1.2.1. *Asymptotic solution for large time*

To study the asymptotic behaviours of the magnetic pressure perturbation, we start with the asymptotic behaviours of $A_1(t) = P(t, -x_0)$ and $A_2(t) = P(t, x_0)$. These variables can be determined, using the inverse Laplace transform (non-standard, discussed in Appendix A), to be

$$A_1(t) = \frac{1}{2\pi} \lim_{L\to\infty} \int_{i\gamma-L}^{i\gamma+L} \frac{T_1(\omega)}{D(\omega)} e^{-i\omega t} d\omega, \quad A_2(t) = \frac{1}{2\pi} \lim_{L\to\infty} \int_{i\gamma-L}^{i\gamma+L} \frac{T_2(\omega)}{D(\omega)} e^{-i\omega t} d\omega, \tag{22}$$

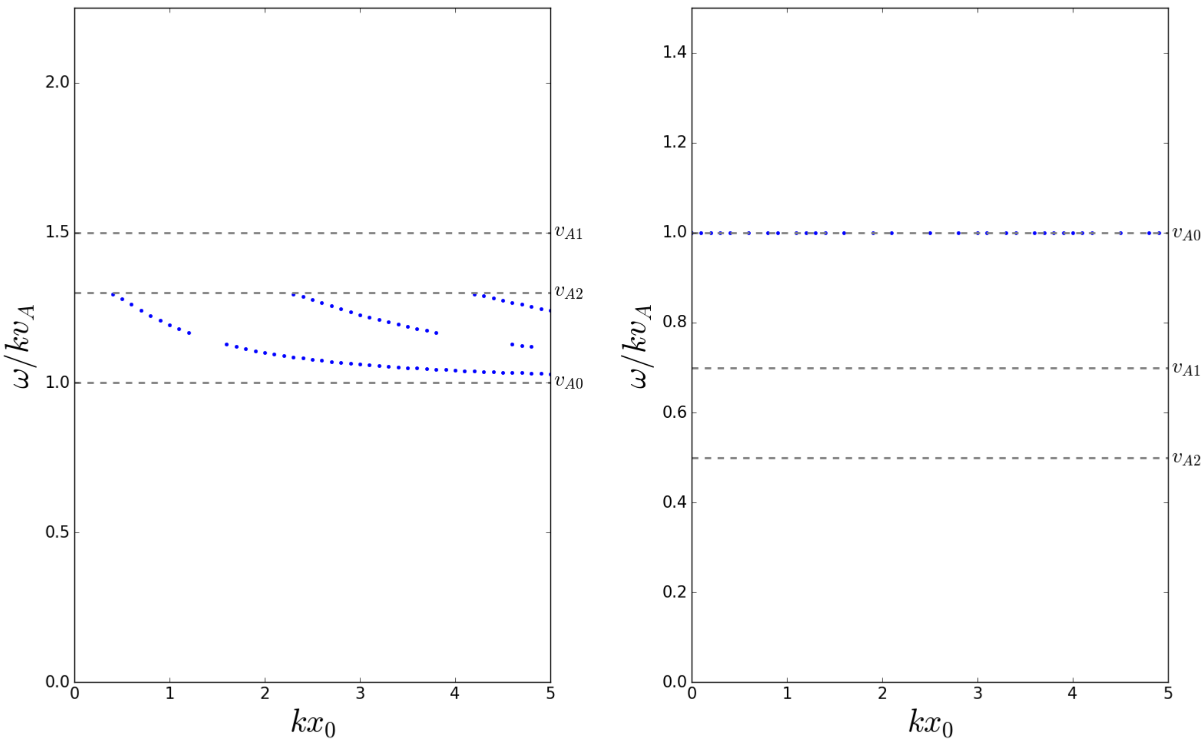
where γ is a real number such that all the singularities of the integrands are below the contour of integration. The integrals are evaluated along an infinite horizontal line in the upper half of the complex plane.

Since the problem of finding the solution is now reduced to solving a complex integral, it is dependent on the singularities (with respect to ω) of T_1, T_2, and D (so that we can construct the Bromwich contour such that it is contained to a single-valued branch) and the zeros of D (whose residues determine the value of the contour integral).

To determine the singularities of these functions, we determine the singularities of the constituent functions, as follows.

- The functions $\Lambda_{0,1,2}^2$ are polynomial in ω, and are therefore entire.
- However, $\Lambda_{0,1,2}$ are involve radicals and have (algebraic) branch points of at $\omega = \pm k v_{A0}$, $\pm k v_{A1}$, and $\pm k v_{A2}$, respectively[1].
- The functions $\cosh(z)$ and $\sinh(z)$ are entire functions of z with only even and odd terms in their respective series expansions. Therefore $\cosh(z)$ and $z\sinh(z)$ are entire functions of z^2. Hence $\cosh(\Lambda_0 x_0)$ and $\Lambda_0 \sinh \Lambda_0 x_0$ are entire functions of ω.
- The integrands of $I_{s1,2}$ are integrated with respect to s. So, to determine the singularities of $I_{s1,2}$, we need consider only the singularities of the integrands. The initial condition, $F_P(s)$ is not a function of ω so does not influence the singularities. The function $f(z) = \sinh(az)/\sinh(bz)$, for constants a and $b \neq 0$ are entire functions of z, containing only even powers (once f has been trivially redefined as to remove the removable singularity at $z = 0$). Therefore, f is also an entire function of z^2. Hence, by letting $a = s \pm x_0$ and $b = 2x_0$, it follows that $\sinh(\Lambda_0(s - x_0))/\sinh(2\Lambda_0 x_0)$ is entire in ω.
- As above, to determine the singularities of $I_{e1,2}$, we need consider only the singularities of the integrands. The function $e^{a\sqrt{z}}$, for constant $a \neq 0$ has algebraic branch points at $z = 0$. Therefore, by setting $a = x_0 \pm s$, it follows that the functions $e^{\Lambda_{1,2}(x_0 \pm s)}$, and therefore $I_{e1,2}$, have algebraic branch points (of ramification index 2) at $\omega = \pm k v_{A1,2}$, respectively.

[1] More precisely, $\omega = \pm k v_{A0}$, $\pm k v_{A1}$, and $\pm k v_{A2}$ are the ramification points corresponding to the branch points $\Lambda_{0,1,2} = 0$, each with ramification index 2. However the language used in the main text is common shorthand that is considered synonymous, although less rigorous.



(a) (b)

Figure 1. Solutions to the zero-beta dispersion relation in two different ordering of the characteristic speeds.

The above analysis determines that the singularities of each function $T_1(\omega)$, $T_2(\omega)$, and $D(\omega)$ are precisely the algebraic branch points at $\omega = \pm k v_{A1}$ and $\omega = \pm k v_{A2}$.

The zeros of $D(\omega)$ are determined by firstly noting that $D = 0$ is the dispersion relation of the corresponding eigenvalue problem solved by Zsámberger et al. (2018). To see this, observe that for a zero-beta plasma, the parameters $\Lambda_i^z = -i\rho_i\omega\sqrt{k^2 v_{Ai}^2 - \omega^2}$, for $i = 0, 1, 2$, where we have added the superscript label z to differentiate it from Λ_i in the present text. Trivially,

$$\Lambda_i = \frac{i\mu\omega}{B_i^2}\Lambda_i^z. \tag{23}$$

Due to equilibrium total pressure balance (the total pressure in a cold plasma is merely the magnetic pressure), the strength of the magnetic field is uniform across the structure, *i.e.* $B_0^2 = B_1^2 = B_2^2$. Therefore, $D = 0$ is equivalent to

$$\Lambda_0^z(\Lambda_1^z + \Lambda_2^z) + (\Lambda_0^{z2} + \Lambda_1^z\Lambda_2^z)\tanh(2m_0x_0) = 0, \tag{24}$$

where $m_0^2 = k^2 - \omega^2/v_{A0}^2$, which is the dispersion relation of a zero-beta asymmetric magnetic slab (Zsámberger et al. 2018). It follows that the zeros of D are precisely the eigenvalues of the asymmetric magnetic slab.

Perhaps this marks the end of the road for the cold plasma IVP because there do not exist any surface modes of a cold magnetic slab, regardless of Alfvén speed ordering (Figure 1). Maybe there are surface leaky modes? Doesn't seem like it when you look at the imaginary part of the dispersion function. Back to incompressible??? It would be more tractable due to being analytically soluble and waves manifest only as surface modes, therefore allowing application of SMS techniques.

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Software: Mayavi

APPENDIX

A. NON-STANDARD LAPLACE TRANSFORM

Consider a function $f(t)$, whose standard Laplace transform, $F_1(\omega)$, and non-standard Laplace transform, $F_2(\omega)$, are

$$F_1(\omega) = \int_0^\infty f(t)e^{-\omega t}dt, \quad \text{and} \quad F_2(\omega) = \int_0^\infty f(t)e^{i\omega t}dt. \tag{A1}$$

Trivially, $F_1(-i\omega) = F_2(\omega)$. Using the standard inverse Laplace transform, and letting γ be real and greater than the real part of all the singularities of $F_1(\omega)$, the original function $f(t)$ can be written

$$f(t) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT} F_1(\omega)e^{\omega t}d\omega, \tag{A2}$$

$$= \frac{1}{2\pi i}\lim_{T\to\infty}\int_{i\gamma-T}^{i\gamma+T} F_1(-i\omega)e^{-i\omega t}(-id\omega), \tag{A3}$$

$$= \frac{1}{2\pi}\lim_{T\to\infty}\int_{i\gamma-T}^{i\gamma+T} F_2(\omega)e^{-i\omega t}d\omega. \tag{A4}$$

Therefore, the inverse transform of the non-standard Laplace transform is

$$f(t) = \frac{1}{2\pi}\lim_{T\to\infty}\int_{i\gamma-T}^{i\gamma+T} F_2(\omega)e^{-i\omega t}d\omega. \tag{A5}$$

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