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Solar Magneto-Seismology with Asymmetric Slab Waveguides

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Abstract Abstract will go here.

Keywords: Some, Key, Words

1. Introduction

Include: Rayleigh-Ritz theorem about eigenfunctions being more affected by perturbation of parameters than eigenvalues.

2. Cross-slab amplitude ratio

The aim of this section is to derive an expression for the ratio of the oscillation on each interface of a magnetic slab in terms of the wave parameters and plasma parameters of the system.

In our previous work (Allcock and Erdélyi, 2017), it was shown that trapped MHD modes propagating along an asymmetric magnetic slab have velocity perturbation in the $\widehat{\mathbf{x}}$-direction given by $v_x(\mathbf{x},t) = \widehat{v}_x(x)e^{i(kz-\omega t)}$ where

$$\widehat{v}_x(x) = \begin{cases} A(\cosh m_1 x + \sinh m_1 x) & \text{if } x < -x_0, \\ B\cosh m_0 x + C \sinh m_0 x & \text{if } |x| \leq x_0, \\ D(\cosh m_2 x - \sinh m_2 x) & \text{if } x > x_0, \end{cases} \tag{2.1}$$

where

$$m_0^2 = \frac{(k^2 v_\mathrm{A}^2 - \omega^2)(k^2 c_0^2 - \omega^2)}{(c_0^2 + v_\mathrm{A}^2)(k^2 c_\mathrm{T}^2 - \omega^2)}, \qquad c_\mathrm{T}^2 = \frac{c_0^2 v_\mathrm{A}^2}{c_0^2 + v_\mathrm{A}^2}, \tag{2.2}$$

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$$m_j^2 = k^2 - \frac{\omega^2}{c_j^2}, \quad \text{for } j = 1, 2, \tag{2.3}$$

and A, B, C, and D are arbitrary constants (with respect to x). These constants can be determined, to within one degree of freedom, using the boundary conditions of continuity in total (kinetic plus magnetic) pressure and velocity component normal to the slab across the slab boundaries. Applying these four boundary conditions retrieves four linear homogeneous algebraic equations in the four unknowns, which can be row reduced to the following form:

$$\begin{pmatrix} c_1 - s_1 & -c_0 & s_0 & 0 \\ 0 & c_0 & s_0 & s_2 - c_2 \\ \Lambda_1(c_1 - s_1) & \Lambda_0 s_0 & -\Lambda_0 c_0 & 0 \\ 0 & \Lambda_0 s_0 & \Lambda_0 c_0 & -\Lambda_2(s_2 - c_2) \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{2.4}$$

where

$$\Lambda_0 = -\frac{i\rho_0(k^2 v_A^2 - \omega^2)}{m_0 \omega}, \quad \Lambda_1 = \frac{i\rho_1 \omega}{m_1}, \quad \text{and} \quad \Lambda_2 = \frac{i\rho_2 \omega}{m_2}, \tag{2.5}$$

and $c_i = \cosh m_i x_0$ and $s_i = \sinh m_i x_i$ for $i = 0, 1, 2$. To ensure the existence of non-trivial solutions of this equations, the determinant of the coefficient matrix must vanish. This gives us the dispersion relation

$$(\Lambda_0 c_0 + \Lambda_2 s_0)(\Lambda_0 s_0 + \Lambda_1 c_0) + (\Lambda_0 c_0 + \Lambda_1 s_0)(\Lambda_0 s_0 + \Lambda_2 c_0) = 0. \tag{2.6}$$

This allows one of the constants B or C to be arbitrary. These two types of solution correspond to the quasi-sausage and quasi-kink mode decoupling.

Firstly, for quasi-sausage modes, by letting C be arbitrary the other constants A, B, and D can be determined in terms of C as follows

$$A = \frac{1}{c_1 - s_1}(Bc_0 - Cs_0), \tag{2.7}$$

$$D = \frac{1}{c_2 - s_2}(Bc_0 + Cs_0), \tag{2.8}$$

where

$$B = \frac{\Lambda_0 c_0 + \Lambda_1 s_0}{\Lambda_0 s_0 + \Lambda_1 c_0} C = -\frac{\Lambda_0 c_0 + \Lambda_2 s_0}{\Lambda_0 s_0 + \Lambda_2 c_0} C. \tag{2.9}$$

The second formulation of B in equation (2.9) is found by substituting in the dispersion relation. A substitution of these values, using the first form of B, into the velocity solution, equation (2.1), evaluated at the slab boundaries ($x = \pm x_0$), yields

$$\begin{aligned}\widehat{v}_x(x_0) =& Bc_0 + Cs_0\\ =& \frac{2\Lambda_1 + \Lambda_0\left(\tau_0 + \frac{1}{\tau_0}\right)}{\Lambda_0 + \Lambda_1\frac{1}{\tau_0}}Cc_0,\end{aligned} \tag{2.10}$$

$$\begin{aligned}\widehat{v}_x(-x_0) =& Bc_0 - Cs_0\\ =& \frac{\Lambda_0}{\Lambda_0 + \Lambda_1\frac{1}{\tau_0}}C/s_0,\end{aligned} \tag{2.11}$$

where $\tau_0 = \tanh m_0 x_0$. Using the second form of B yields

$$\widehat{v}_x(x_0) = \frac{-\Lambda_0}{\Lambda_0 + \Lambda_2\frac{1}{\tau_0}}C/s_0, \tag{2.12}$$

$$\widehat{v}_x(-x_0) = \frac{-2\Lambda_2 - \Lambda_0\left(\tau_0 + \frac{1}{\tau_0}\right)}{\Lambda_0 + \Lambda_2\frac{1}{\tau_0}}Cc_0. \tag{2.13}$$

These forms are equivalent. The horizontal velocity perturbation amplitude, $\widehat{v}_x$ is, more specifically, a *signed* amplitude, where a positive (negative) value indicates perturbation in the positive (negative) $\widehat{\mathbf{x}}$-direction.

Secondly, for quasi-kink modes, by letting B be arbitrary the other constants A, C, and D can be determined in terms of B as follows

$$A = \frac{1}{c_1 - s_1}(Bc_0 - Cs_0), \tag{2.14}$$

$$D = \frac{1}{c_2 - s_2}(Bc_0 + Cs_0), \tag{2.15}$$

where

$$C = \frac{\Lambda_0 s_0 + \Lambda_1 c_0}{\Lambda_0 c_0 + \Lambda_1 s_0}B = -\frac{\Lambda_0 s_0 + \Lambda_2 c_0}{\Lambda_0 c_0 + \Lambda_2 s_0}B. \tag{2.16}$$

A substitution of these values, using the first form of B, into equation (2.1), evaluated at the slab boundaries ($x = \pm x_0$), yields

$$\widehat{v}_x(x_0) = \frac{2\Lambda_1 + \Lambda_0\left(\tau_0 + \frac{1}{\tau_0}\right)}{\Lambda_0 + \Lambda_1\tau_0}Bs_0, \tag{2.17}$$

$$\widehat{v}_x(-x_0) = \frac{\Lambda_0}{\Lambda_0 + \Lambda_1\tau_0}B/c_0. \tag{2.18}$$

Using the second form of C yields

$$\widehat{v}_x(x_0) = \frac{\Lambda_0}{\Lambda_0 + \Lambda_2\tau_0}B/c_0, \tag{2.19}$$

$$\widehat{v}_x(-x_0) = \frac{2\Lambda_2 + \Lambda_0\left(\tau_0 + \frac{1}{\tau_0}\right)}{\Lambda_0 + \Lambda_2\tau_0}Bs_0. \tag{2.20}$$

We now define the *cross-slab amplitude ratio (CSAR)*, $R_A := \frac{\widehat{v}_x(x_0)}{\widehat{v}_x(-x_0)}$ as the ratio of the amplitude of oscillation at the interface $x = x_0$ to that of the interface $x = -x_0$. Firstly, using equations (2.11) and (2.12), the CSAR for quasi-sausage modes is

$$\begin{aligned} R_A &= -\frac{\Lambda_0 + \Lambda_1 \frac{1}{\tau_0}}{\Lambda_0 + \Lambda_2 \frac{1}{\tau_0}} \\ &= -\frac{\rho_1 m_2}{\rho_2 m_1} \left[\frac{(k^2 v_A^2 - \omega^2) m_1 \frac{\rho_0}{\rho_1} - \omega^2 m_0 \frac{1}{\tanh m_0 x_0}}{(k^2 v_A^2 - \omega^2) m_2 \frac{\rho_0}{\rho_2} - \omega^2 m_0 \frac{1}{\tanh m_0 x_0}} \right]. \end{aligned} \tag{2.21}$$

Using equations (2.18) and (2.19), the corresponding expression for quasi-kink modes can be obtained, namely

$$\begin{aligned} R_A &= \frac{\Lambda_0 + \Lambda_1 \tau_0}{\Lambda_0 + \Lambda_2 \tau_0} \\ &= \frac{\rho_1 m_2}{\rho_2 m_1} \left[\frac{(k^2 v_A^2 - \omega^2) m_1 \frac{\rho_0}{\rho_1} - \omega^2 m_0 \tanh m_0 x_0}{(k^2 v_A^2 - \omega^2) m_2 \frac{\rho_0}{\rho_2} - \omega^2 m_0 \tanh m_0 x_0} \right]. \end{aligned} \tag{2.22}$$

As expected, equations (2.21) and (2.22) reduce to $R_A = -1$ and $R_A = 1$ for quasi-sausage and quasi-kink modes, respectively, when the slab is symmetric.

The cross-slab amplitude ratio has potential as a tool for solar magneto-seismology. The procedure goes as follows:

- Observe a surface mode in a slab-like structure,
- Determine whether the mode is sausage or kink,
- Directly measure the slab width, x_0, and the amplitude ratio, R_A, using plane-of-sky intensity (???) measurements,
- Estimate the wave period, $2\pi/\omega$, and wavelength, $2\pi/k$,
- Measure the density distribution, $\rho_{0,1,2}$, using (???), and use these to estimate the sound speeds, $c_{0,1,2}$.
- Solve (2.21) or (2.22) for the Alfvén speed, v_A, numerically or analytically under an additional approximation (Section **??**).

To solve equation (2.21) or (2.22) directly for v_A, a numerical procedure must be followed. The numerical procedure is simple and would involve using a standard root-finding method such as the secant method. To make analytical progress, an approximation must be made to simplify equation (2.21) or (2.22). The following subsections give the analytical solution for v_A of equations (2.21) and (2.22) under the thin slab, wide slab, incompressible plasma, and low-beta approximations.

2.1. Thin slab approximation

In the thin slab approximation, $kx_0 \ll 1$, it has been shown that $m_0 x_0 \ll 1$ for surface modes (Roberts, 1981b). Therefore, $\tanh m_0 x_0 \approx m_0 x_0$ and the amplitude ratio for a thin slab quasi-sausage surface mode reduces to

$$R_A = -\frac{\rho_1 m_2}{\rho_2 m_1}\left[\frac{(k^2v_A^2-\omega^2)m_1x_0\frac{\rho_0}{\rho_1}-\omega^2}{(k^2v_A^2-\omega^2)m_2x_0\frac{\rho_0}{\rho_2}-\omega^2}\right], \tag{2.23}$$

which has analytical solutions

$$v_A^2 = \frac{\omega^2}{k^2}\left[1+\frac{1}{x_0}\left(\frac{R_A\frac{\rho_2}{\rho_0 m_2}+\frac{\rho_1}{\rho_0 m_1}}{R_A+1}\right)\right]. \tag{2.24}$$

The amplitude ratio for a thin slab quasi-kink surface mode reduces to

$$R_A = \frac{\rho_1 m_2}{\rho_2 m_1}\left[\frac{(k^2v_A^2-\omega^2)m_1\frac{\rho_0}{\rho_1}-\omega^2m_0^2x_0}{(k^2v_A^2-\omega^2)m_2\frac{\rho_0}{\rho_2}-\omega^2m_0^2x_0}\right], \tag{2.25}$$

which has analytical solutions

$$v_A^2 = \frac{\omega^2}{k^2}\left[\frac{c_0^2}{c_0^2-\frac{\omega^2}{k^2}}+k^2x_0\left(\frac{R_A\frac{\rho_2}{\rho_0 m_2}-\frac{\rho_1}{\rho_0 m_1}}{R_A-1}\right)\right]. \tag{2.26}$$

In an asymmetric slab, there is only a slow quasi-kink surface mode because the fast version degenerates due to a cut-off by the external sound speeds splitting up (Allcock and Erdélyi, 2017). The slow quasi-kink surface mode has a phase speed that approaches zero in the thin slab limit. Therefore, to a good approximation, $\omega/k \ll c_0$, so that solution (2.26) simplifies to

$$v_A^2 = \frac{\omega^2}{k^2}\left[1+k^2x_0\left(\frac{R_A\frac{\rho_2}{\rho_0 m_2}-\frac{\rho_1}{\rho_0 m_1}}{R_A-1}\right)\right]. \tag{2.27}$$

2.2. Wide slab approximation

The wide slab approximation applies when the slab width is much larger than the wavelength, that is $kx_0 \gg 1$. To understand the properties of the eigenfunctions of the asymmetric slab system in the wide slab approximation, we must return to the dispersion relation, equation (2.6). For surface modes under than wide slab approximation, $m_0x_0 \gg 1$, and therefore $\sinh m_0x_0 \approx \cosh m_0x_0 \approx 1$ Roberts (1981b). Under this approximation, the dispersion relation, equation (2.6), becomes

$$(\Lambda_0+\Lambda_1)(\Lambda_0+\Lambda_2)=0, \tag{2.28}$$

which gives us two families of solutions, one satisfying $\Lambda_0+\Lambda_1=0$ and the other satisfying $\Lambda_0+\Lambda_2=0$. These are equivalent to

$$(k^2v_A^2-\omega^2)m_{1,2}\frac{\rho_0}{\rho_{1,2}}-\omega^2m_0=0, \tag{2.29}$$

respectively. This is the same as the dispersion relation governing surface waves along a single interface between a magnetised and a non-magnetised plasma

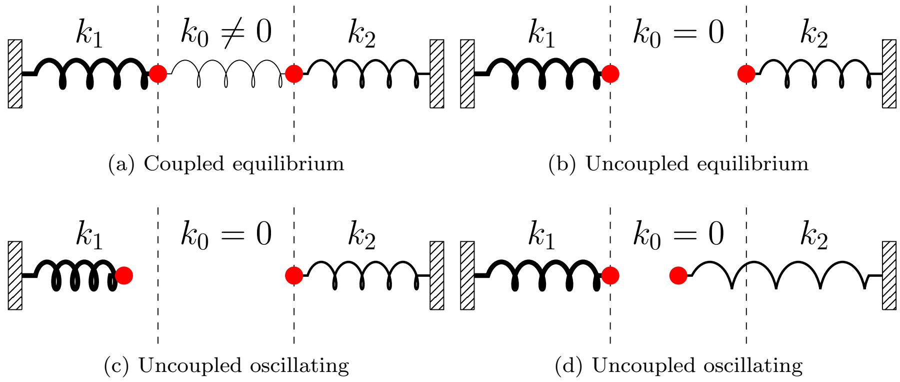

Figure 1. Mechanical example showing weak and zero coupling between the masses.

Roberts (1981a). Hence, the surface mode solutions of a wide asymmetric slab are just the surface modes that propagate along each interface independently. This makes intuitive sense because as the slab is widened the interfaces will have diminishing influence on each other, until each interface oscillates independently.

This has an analogy to the mechanical example introduced by Allcock and Erdélyi 2017. Consider two masses connected by a spring, with spring constant k_0, and each mass is also connected to a fixed wall on each side by springs with spring constants k_1 and k_2, respectively (Figure 1). When the middle spring has spring constant $k_0 \neq 0$, there are two modes, an in-phase mode (analogous to kink modes in a slab) and an in-antiphase mode (analogous to sausage modes in a slab) (Allcock and Erdélyi, 2017). When the two masses are decoupled by setting $k_0 = 0$ so that the middle spring is removed, each mass oscillated independently at the natural frequency of that side of the spring-mass system. This decoupling provides a good analogy to the wide slab limit for the magnetic slab. Each interface can oscillate at its own natural frequency, independent of the other interface. Given that we are considering magneto-acoustic waves, there are two restoring forces, the magnetic tension force and the pressure gradient force, which means that each independent interface has two natural frequencies (depending on the parameters of the system, there can be 0, 1, or 2 frequencies), corresponding to the fast and slow magneto-acoustic modes.

With this understanding of the modes in the wide slab limit, the amplitude ratio, R_A is either 0 or *undefined*, depending on which interface the wave is propagating.

2.3. Incompressible Approximation

If the plasma in each region is incompressible, the sound speeds become unbounded, so that $m_j \approx k$ for $j = 0, 1, 2$. Under this approximation, the CSAR for quasi-sausage modes (top) and quasi-kink modes (bottom) reduces to

$$R_A = \binom{-}{+} \frac{\rho_1}{\rho_2} \left[\frac{(k^2 v_\mathrm{A}^2 - \omega^2) k \frac{\rho_0}{\rho_1} - \omega^2 k \begin{pmatrix} \coth \\ \tanh \end{pmatrix} (k x_0)}{(k^2 v_\mathrm{A}^2 - \omega^2) k \frac{\rho_0}{\rho_2} - \omega^2 k \begin{pmatrix} \coth \\ \tanh \end{pmatrix} (k x_0)} \right]. \tag{2.30}$$

This equation has solutions for v_A given by

$$v_\mathrm{A}^2 = \frac{\omega^2}{k^2} \left[1 + \left(\frac{R_A \frac{\rho_2}{\rho_0} \binom{+}{-} \frac{\rho_1}{\rho_0}}{R_A \binom{+}{-} 1} \right) \begin{pmatrix} \coth \\ \tanh \end{pmatrix} (k x_0) \right] \tag{2.31}$$

2.4. Low-Beta Approximation

For a low-beta plasma, the Alfvén speed, v_A, dominates the internal sound speed, c_0, so that $m_0^2 \approx k^2 - \omega^2 / v_\mathrm{A}^2$. For waves with phase speed much less than the Alfvén speed, a further approximation of $m_0^2 \approx k^2$ can be made, which reduces the CSAR for quasi-sausage modes (top) and quasi-kink modes (bottom) to

$$R_A = \binom{-}{+} \frac{\rho_1 m_2}{\rho_2 m_1} \left[\frac{(k^2 v_\mathrm{A}^2 - \omega^2) m_1 \frac{\rho_0}{\rho_1} - \omega^2 k \begin{pmatrix} \coth \\ \tanh \end{pmatrix} (k x_0)}{(k^2 v_\mathrm{A}^2 - \omega^2) m_2 \frac{\rho_0}{\rho_2} - \omega^2 k \begin{pmatrix} \coth \\ \tanh \end{pmatrix} (k x_0)} \right]. \tag{2.32}$$

This equation can be solved for v_A to give

$$v_\mathrm{A}^2 = \frac{\omega^2}{k^2} \left[1 + k \left(\frac{\frac{\rho_1}{\rho_0 m_1} \binom{+}{-} R_A \frac{\rho_2}{\rho_0 m_2}}{1 \binom{+}{-} R_A} \right) \begin{pmatrix} \coth \\ \tanh \end{pmatrix} (k x_0) \right]. \tag{2.33}$$

2.5. Approximately Equal External Parameters

If the parameters of the non-magnetised external regions agree to first order approximation, the dispersion relation for magneto-acoustic waves along an asymmetric slab reduces to

$$\Lambda_0 (\Lambda_1 + \Lambda_2) + 2 \Lambda_1 \Lambda_2 \tau_0 = 0, \tag{2.34}$$

$$\Lambda_0 (\Lambda_1 + \Lambda_2) + 2 \Lambda_1 \Lambda_2 \frac{1}{\tau_0} = 0, \tag{2.35}$$

for quasi-sausage and quasi-kink modes, respectively. Manipulation of equation (2.34) yields

$$\Lambda_0 \Lambda_1 + \Lambda_1 \Lambda_2 \tau_0 = -(\Lambda_0 \Lambda_2 + \Lambda_1 \Lambda_2 \tau_0), \tag{2.36}$$

$$\implies \Lambda_1 (\Lambda_0 + \Lambda_2 \tau_0) = -\Lambda_2 (\Lambda_0 + \Lambda_1 \tau_0), \tag{2.37}$$

$$\implies \frac{\Lambda_0 + \Lambda_1 \tau_0}{\Lambda_0 + \Lambda_2 \tau_0} = -\frac{\Lambda_1}{\Lambda_2} = -\frac{\rho_1 m_2}{\rho_2 m_1}. \tag{2.38}$$

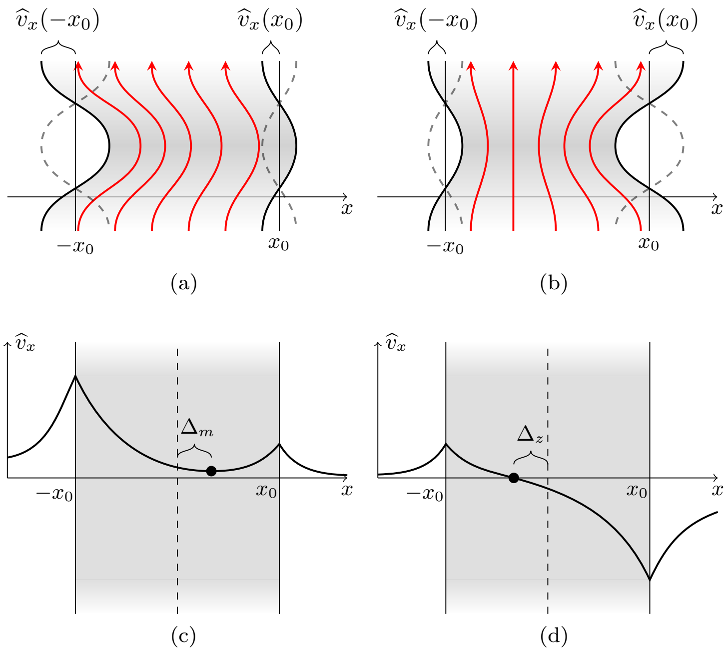

Figure 2. Shift of minimum perturbation.

Similar manipulation of equation (2.35) yields

$$\frac{\Lambda_0\tau_0+\Lambda_1}{\Lambda_0\tau_0+\Lambda_2}=-\frac{\Lambda_1}{\Lambda_2}=-\frac{\rho_1 m_2}{\rho_2 m_1}. \tag{2.39}$$

If the conditions on either side of the slab are equal to first order approximation, the cross-slab amplitude ratio reduces to

$$R_A=-\frac{\rho_1 m_2}{\rho_2 m_1} \tag{2.40}$$

for quasi-sausage modes, and

$$R_A=\frac{\rho_1 m_2}{\rho_2 m_1} \tag{2.41}$$

for quasi-kink modes.

3. Shift of minimum perturbation

An additional eigenfunction-based solar magneto-seismology technique uses the shift in position of minimum wave power from the centre of the slab towards a boundary due to the asymmetry in the external regions.

The position of minimum wave power for a symmetric sausage or kink mode is on the central line of the slab, at $x=0$. Define $\Delta_{\min}$ to be the displacement (from the central line) of the position of minimum wave power inside an asymmetric magnetic slab. For quasi-sausage modes, the value of $\Delta_{\min}$ will be the solution to $\widehat{v}_x(x) = 0$ under the constraint $|x| < x_0$, and for quasi-kink modes, the value of $\Delta_{\min}$ will be the solution to $\mathrm{d}\widehat{v}_x(x)/\mathrm{d}x = 0$ under the same constraint $|x| < x_0$. The constraint restrict the solutions to being within the slab.

Firstly, for quasi-sausage modes, using the solution for $\widehat{v}_x$ given by equation (2.1) and the expressions for the variables within given by equation (2.9), the shift of minimum perturbation can be calculated as follows:

$$\widehat{v}_x(x) = B \cosh m_0 x + C \sinh m_0 x = 0, \tag{3.1}$$

where B is given by equation (2.9) and C is arbitrary. This equation is solved to give

$$x = \frac{1}{m_0} \tanh^{-1} \left(-\frac{B}{C} \right). \tag{3.2}$$

therefore the shift of minimum perturbation is

$$\Delta_{min} = \frac{1}{m_0} \tanh^{-1} \left(-\frac{(k^2 {v_A}^2 - \omega^2) m_1 \frac{\rho_0}{\rho_1} - \omega^2 m_0 \tanh m_0 x_0}{(k^2 {v_A}^2 - \omega^2) m_1 \frac{\rho_0}{\rho_1} \tanh m_0 x_0 - \omega^2 m_0} \right). \tag{3.3}$$

Secondly, for quasi-kink modes, using equations (2.1) and (2.16) we calculate the shift of minimum perturbation to be

$$\Delta_{min} = \frac{1}{m_0} \coth^{-1} \left(-\frac{(k^2 {v_A}^2 - \omega^2) m_1 \frac{\rho_0}{\rho_1} - \omega^2 m_0 \tanh m_0 x_0}{(k^2 {v_A}^2 - \omega^2) m_1 \frac{\rho_0}{\rho_1} \tanh m_0 x_0 - \omega^2 m_0} \right). \tag{3.4}$$

It appears that expressions (3.3) and (3.4) for the shift in minimum perturbations depend on the parameters in the slab (subscript 0), the left external plasma (subscript 1), but not on the right external plasma (subscript 2). However, the dependence on the right external plasma is implicit in the determination of the eigenfrequency ω when solving the dispersion relation (Allcock and Erdélyi, 2017).

3.1. Thin slab approximation

3.2. Wide slab approximation

3.3. Incompressible approximation

3.4. Low-beta approximation

4. Discussion

Include: flow chart for solar magneto-seismology.

The concept of minimum perturbation shift is exclusive to surface modes. The eigenfunctions of surface modes in a magnetic slab depend much more on the plasma parameters, such as the density, than body modes (Allcock and Erdélyi, 2017). This makes intuitive sense given that the energy in a surface mode is localised to the boundaries of the slab whereas the energy in a body mode is largely isolated within the slab. There is a quantifiable shift in the spacial nodes and anti-nodes in body mode perturbations within a slab due to changing external plasma parameters, however it is so small that is would not be an effective observational tool. It is also much more complicated because it depends on the number of nodes that the particular body mode has, an observation that is yet to be made in solar structures in either slab or tube geometries.

References

Allcock, M., Erdélyi, R.: 2017, Magnetohydrodynamic Waves in an Asymmetric Magnetic Slab. *Solar Physics* **292**, 35. DOI. ADS.

Roberts, B.: 1981a, Wave propagation in a magnetically structured atmosphere I - Surface waves at a magnetic interface. *Solar Physics* **69**, 27.

Roberts, B.: 1981b, Wave Propagation in a Magnetically Structured Atmosphere II - Waves in a Magnetic Slab. *Solar Physics* **69**, 39.