Analysis of Surface and Body Waves along a Magnetic Slab in an Asymmetric Environment

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INTRODUCTION

The solar atmosphere has a complicated, inhomogeneous structure. Convection currents in the convective region, under the Sun's surface, drive waves in the outer layers (known as the photosphere and chromosphere) and atmosphere (known as the corona). These waves provide an indirect way to gain information about the structure of the Sun. If we can mathematically model a system which oscillates with similar properties to the waves on the Sun then we can deduce that this model may be an accurate description of the solar structure.

In a series of papers, Roberts (1981a,b) and Roberts and Edwin (1981) conducted a study into the waves which exist in the presence of various magnetic field and density stratifications in a symmetric plasma. My work involved extending Roberts' work to the asymmetric case.

MATHEMATICAL MODEL

My research looked at the properties of the waves which arise due to perturbations in the system illustrated below in figure 1. Roberts' model was identical but he retained symmetry by setting $\rho_1 = \rho_2$, $p_1 = p_2$ and $T_1 = T_2$.

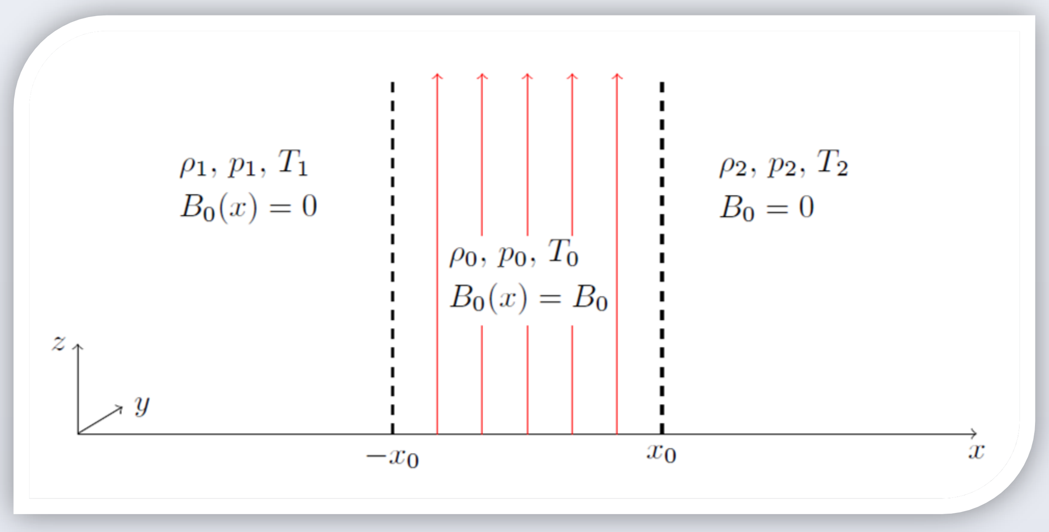

FIGURE 1: Equilibrium conditions of the system. The slab is the section between the dashed lines.

The system is an infinite 3-dimentional, compressible, inviscid plasma which is uniform in the y- and z-directions and stratified in the x-direction in such a way that for $x < -x_0$ and $x < -x_0$ there is no magnetic field, but on the 'slab', which is$|x| < x_0$, there is a constant magnetic field pointing in the positive z-direction with magnitude B_0. The variable parameters: density, pressure and temperature, are given as ρ, p, and T, and these are different for each subscript.

The equations which govern perturbations in the magnetic slab are the conservation of mass, the momentum equation, the induction equation and the energy equation:

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho_0 \mathbf{v}) = 0,$$
$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla(p + \frac{1}{\mu}\mathbf{B_0}.\mathbf{b}) + \frac{1}{\mu}(\mathbf{B_0}.\nabla)\mathbf{b} + \frac{1}{\mu}(\mathbf{b}.\nabla)\mathbf{B_0},$$
$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B_0}),$$
$$\frac{\partial p}{\partial t} + \mathbf{v}.\nabla p_0 = c_0^2 \left(\frac{\partial \rho}{\partial t} + \mathbf{v}.\nabla \rho_0 \right),$$

where μ is the magnetic permeability of the plasma, $c_0 = \sqrt{\gamma p_0 / \rho_0}$ is the sound speed inside the slab, and γ is the adiabatic index (see Priest (2014) for derivations and more information). From here we take ρ_0 and p_0 to be constants and seek waves which propagate in the z-direction. Then, following similar steps to those taken by Roberts (1981b), we can reduce these equations to a single differential equation which describes the way that $\hat{v}_x$, the amplitude of transverse oscillations in the x,z-plane, changes as you move in the x-direction:

$$\hat{v}_x'' - m_0^2 \hat{v}_x = 0$$

where

$$m_0^2 = \frac{(k^2 v_A^2 - \omega^2)(k^2 c_0^2 - \omega^2)}{(c_0^2 - v_A^2)(k^2 c_T^2 - \omega^2)}, \qquad c_T^2 = \frac{c_0^2 v_A^2}{c_0^2 + v_A^2}.$$

c_T is called the ***tube speed*** and m_0 is a conveniently defined parameter where k, and ω are the frequency and wavenumber of the waves, and $v_A = B_0 / \sqrt{\rho_0 \mu}$ is called the ***Alfven speed***.

By taking $B_0 = 0$ (and hence $v_A = 0$ and $c_T = 0$) in the differential equation governing the motion in the slab, we get the differential equations which govern the motion of the plasma outside the slab:

$$\hat{v}_x'' - m_{1,2}^2 \hat{v}_x = 0 \quad \text{where} \quad m_{1,2}^2 = k^2 - \frac{\omega^2}{c_{1,2}^2}$$

The differential equations combine to give us the dispersion relation (an equation relating the frequency, ω, to the other parameters of the system), which splits into 4 equations, 2 describing surface waves and 2 describing body waves. ***Body waves*** are oscillations which have several peaks within the slab as opposed to ***surface waves*** which only peak at the boundaries. The dispersion relations for surface waves and body waves respectively are:

$$(k^2 v_A^2 - \omega^2)\left(\frac{\rho_0}{\rho_1} m_1 + \frac{\rho_0}{\rho_2} m_2 \right) = 2\omega^2 m_0 \begin{pmatrix} \tanh \\ \coth \end{pmatrix} (m_0 x_0),$$
$$(k^2 v_A^2 - \omega^2)\left(\frac{\rho_0}{\rho_1} m_1 + \frac{\rho_0}{\rho_2} m_2 \right) = 2\omega^2 n_0 \begin{pmatrix} -\tan \\ \cot \end{pmatrix} (n_0 x_0),$$

where $n_0{}^2 = -m_0{}^2$ and tan, cot, $tanh$, and $coth$ are the tangent, cotangent, hyperbolic tangent and hyperbolic cotangent functions respectively. These 4 equations contain all the information about the properties of the waves in the system.

FINDINGS

The waves can be categorised by their phase speed, whether they are surface or body waves, and whether they are sausage or kink modes. Waves are called ***sausage modes*** when the oscillations along each side of the slab are in phase, and ***kink modes*** when they are in anti-phase. Waves are called ***fast*** if they have a phase speed greater than c_0 and ***slow*** if they have a phase speed less than c_0.

To obtain information from the dispersion relations, I simplified them by making several separate approximations about the waves. By making the thin and wide slab approximations ($kx_0 \ll 1$ and $kx_0 \gg 1$ respectively), I created figure 2, which illustrates the wave modes which are present when $v_A > c_0$ and $c_1, c_2 > c_T$. It is of interest to compare this with the corresponding plot in Roberts (1981b) where the notable difference is the existence of a fast surface wave which propagates at a speed greater than c_0 in Roberts(1981b). This mode degenerates in my model as c_1 and c_2 diverge.

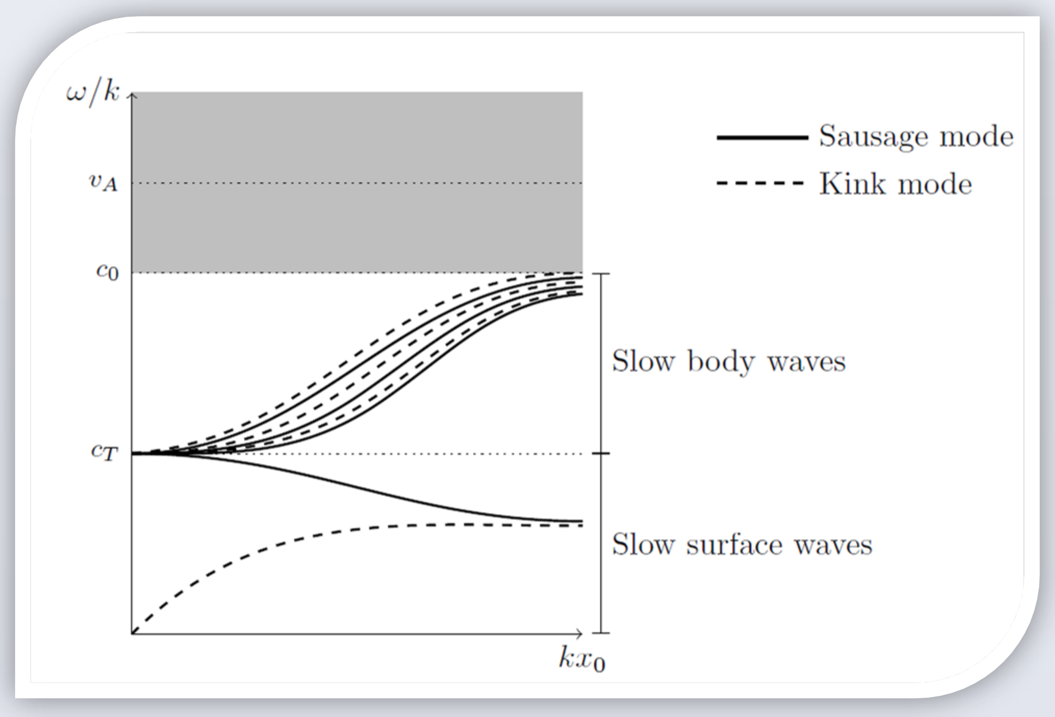

FIGURE 2: The grey area denotes the values of the phase speed, ω/k, for which modes do not occur. The area of slow body modes includes an infinite family of sausage and kink modes, only a few of which are shown here.

Another approximation I made was to assume the plasma was incompressible, i.e. it behaves more like a liquid than a gas. By manipulating the dispersion relations, it can be shown that in this case the slow body modes in figure 2 degenerate, so we are left with only slow surface waves.

Plasma beta, β, is a parameter which tells us which of magnetic or fluid pressure dominates the motion of the plasma. It is defined as $\beta = 2\mu p_0 / B_0$. In the limit as $\beta \to 0$, all the slow waves degenerate and we are left with an infinite family of fast body modes. This limit tells us what happens when the magnetic pressure dominates the fluid pressure in the system, which is similar to the conditions in the solar corona.

CONCLUSIONS AND FURTHER WORK

Breaking the symmetry in Roberts' research made the mathematics more difficult and meant that some of the techniques he used would not work in the analysis of my model, but this model can be studied without the need to introduce symmetry like Roberts did. The information I have gained about the waves in the asymmetric slab environment can be used in the future to see if waves observed on the Sun share the properties of the waves analysed in this research. If there are similarities, it would be an indication that the Solar environment is similar to the simplified model I have studied.

The obvious next step is to mirror Roberts' work by adding a magnetic field outside the slab environment so that the magnetic field on each side of the slab is different and different to the magnetic field inside. Another valuable addition would be to bound the slab so that rather than being infinite in two dimensions, it would be infinite in only one dimension. This would lead to standing waves being produced which may shine light on the difficult open problem of how the corona is heated.

REFERENCES

- Roberts, B., 1981, *Wave propagation in a magnetically structured atmosphere I: Surface Waves at a Magnetic interface.*
- Roberts, B., 1981, *Wave propagation in a magnetically structured atmosphere II: Waves in a Magnetic Slab.*
- Roberts, B. and Edwin, P. M., 1981, *Wave propagation in a magnetically structured atmosphere III: The Slab in a Magnetic Environment.*
- Roberts, B. and Edwin, P. M., 1983, *Wave propagation in a magnetic cylinder*.
- Priest, E., 2014, *Magnetohydrodynamics of the Sun*. Cambridge University Press, 560 p.