Magnetohydrodynamic Waves in Slab Waveguides

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What is Magnetohydrodynamics?

Magnetohydrodynamics (MHD)

The mathematical study of the motion of electrically conducting fluids.

On top of the familiar fluid phenomena:

- Compressibility and sound waves,
- Vortices,
- Boundary layers,

MHD introduces additional physical phenomena:

- MHD waves,
- Magnetic reconnection,
- Magnetic instabilities.

Magnetohydrodynamic equations

MHD equations:

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) &= 0, \\
\rho \frac{D\mathbf{v}}{Dt} &= -\nabla p - \frac{1}{\mu} \mathbf{B} \times (\nabla \times \mathbf{B}), \\
\frac{D}{Dt}\left(\frac{p}{\rho^\gamma}\right) &= 0, \\
\frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{v} \times \mathbf{B}),
\end{aligned}
$$

where $\mathbf{v}$ is the plasma velocity, $\mathbf{B}$ is the magnetic field strength, ρ is the density, p is the pressure, μ is the magnetic permeability, and γ is the adiabatic index.

Case I: Homogeneous plasma

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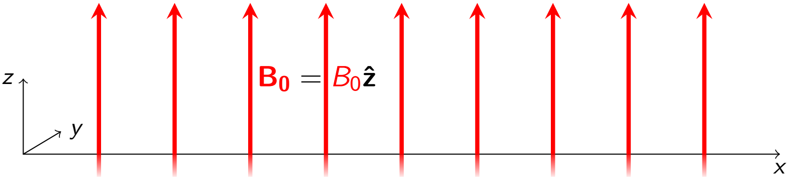
Equilibrium conditions



- Inviscid, compressible 3D plasma under a uniform magnetic field.
- Neglect gravity.

- Linearise the MHD equations about a stationary basic state,
- Seek wave-like solutions of the form $\rho(\mathbf{x}, t) = \hat{\rho} e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}$, where $\mathbf{k}$ is the wavenumber, $\omega = \omega(\mathbf{k})$ is the frequency.
- This reduces the MHD equations to a set of algebraic equations, which have non-trivial solutions when the dispersion relation is satisfied:

Dispersion relation

$$(k_z^2 {v_A}^2 - \omega^2)\left[k_x^2 k^2 c_0^2 {v_A}^2 - (k^2 c_0^2 - \omega^2)(k_z^2 {v_A}^2 - \omega^2)\right] = 0,$$

$c_0 = \sqrt{\frac{\gamma p_0}{\rho_0}}$ is the sound speed and $v_A = \frac{B_0}{\sqrt{\mu \rho_0}}$ is the Alfvén speed.

Case I: Homogeneous plasma

Solution

Dispersion relation

$$(k_z^2 v_A{}^2 - \omega^2)\left[k_x^2 k^2 c_0^2 v_A{}^2 - (k^2 c_0^2 - \omega^2)(k_z^2 v_A{}^2 - \omega^2)\right] = 0,$$

Solution 1:

- **Alfvén wave**: $\omega^2 = k_z^2 v_A{}^2$.
- Purely magnetic wave.
- Incompressible wave, i.e. no density perturbation.

Dispersion relation

$$(k_z^2 v_A{}^2 - \omega^2)\left[k_x^2 k^2 c_0^2 v_A{}^2 - (k^2 c_0^2 - \omega^2)(k_z^2 v_A{}^2 - \omega^2)\right] = 0,$$

Solution 2:

- **Fast** and **slow magnetoacoustic waves**:

$$k_x^2 k^2 c_0^2 v_A{}^2 - (k^2 c_0^2 - \omega^2)(k_z^2 v_A{}^2 - \omega^2) = 0$$

$$\Longrightarrow \omega^2 = \frac{k^2}{2}\left[(c_0^2 + v_A{}^2) \pm \sqrt{(c_0^2 + v_A{}^2)^2 - 4c_0^2 v_A{}^2 \left(\frac{k_z}{k}\right)^2}\right]$$

- Acoustic waves modified by the magnetic field.

Case II: Asymmetric magnetic slab

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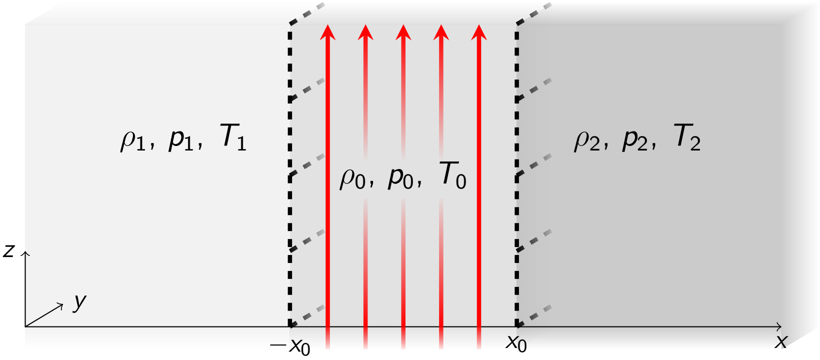
Equilibrium conditions



- Uniform vertical magnetic field in the slab.
- Non-magnetised plasma outside.
- **Different** density and pressure on each side.

Case II: Asymmetric magnetic slab

Governing equations

MHD equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,$$

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p - \frac{1}{\mu}\mathbf{B} \times (\nabla \times \mathbf{B}),$$

$$\frac{D}{Dt}\left(\frac{p}{\rho^{\gamma}}\right) = 0,$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}),$$

Linearise these equations about a stationary basic state, i.e. set

$$\mathbf{v} = \mathbf{v}', \quad \rho = \rho' + \rho_0, \quad p = p' + p_0, \quad \text{and} \quad \mathbf{B} = \mathbf{B}' + \mathbf{B_0},$$

where $\mathbf{v}' \ll 1$, $\rho' \ll \rho_0$, $p' \ll p_0$, and $\mathbf{B}' \ll \mathbf{B_0}$.

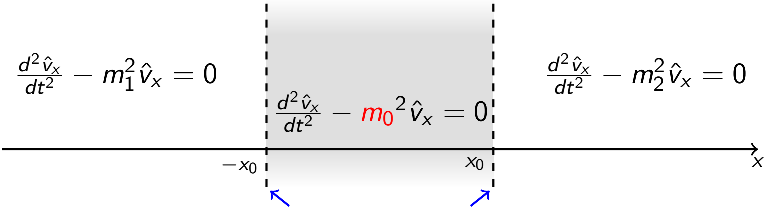
Case II: Asymmetric magnetic slab

Fourier decomposition

Combine these PDEs and look for *plane wave* solutions of the form:

$$v_x(\mathbf{x},t) = \hat{v}_x(x)e^{i(kz-\omega t)}, \quad v_y(\mathbf{x},t) = 0, \quad v_z(\mathbf{x},t) = \hat{v}_z(x)e^{i(kz-\omega t)},$$

to arrive at the following ODEs:



Boundary conditions:
pressure and **velocity** continuous across boundaries.

$${m_0}^2 = \frac{(k^2{v_A}^2-\omega^2)(k^2c_0^2-\omega^2)}{(c_0^2+{v_A}^2)(k^2{c_T}^2-\omega^2)}, \qquad {c_T}^2 = \frac{c_0^2{v_A}^2}{c_0^2+{v_A}^2}, \qquad v_A = \frac{B_0}{\sqrt{\mu\rho_0}},$$

$$m_{1,2}^2 = k^2 - \frac{\omega^2}{c_{1,2}^2}.$$

Using the conditions for the existence of non-trivial solutions gives us the dispersion relation:

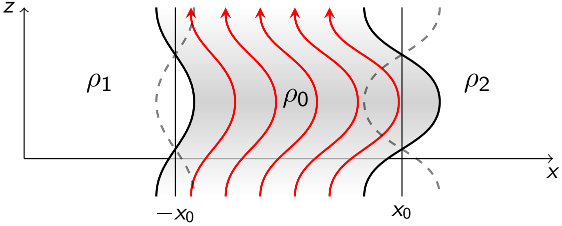
Dispersion relation

$$\frac{\omega^4 m_0{}^2}{k^2 v_A{}^2 - \omega^2} + \frac{\rho_0}{\rho_1} m_1 \frac{\rho_0}{\rho_2} m_2 (k^2 v_A{}^2 - \omega^2)$$
$$- \frac{1}{2} m_0 \omega^2 \left(\frac{\rho_0}{\rho_1} m_1 + \frac{\rho_0}{\rho_2} m_2 \right) (\tanh m_0 x_0 + \coth m_0 x_0) = 0,$$

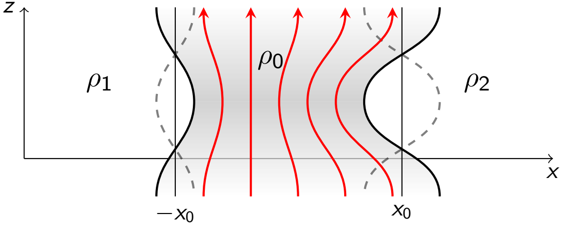
Case II: Asymmetric magnetic slab

Asymmetric sausage and kink modes

Asymmetric kink mode



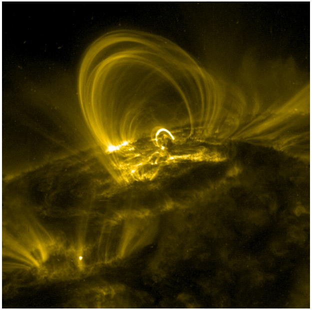
Asymmetric sausage mode



Solar magneto-seismology

Solar magneto-seismology

The method of using observations of waves propagating through the solar surface and atmosphere to deduce properties of the structure of the Sun.



Case II: Asymmetric magnetic slab

Cross-slab amplitude ratio

Evaluate the solution, $\hat{v}_x(x)$, at $x = \pm x_0$ to give the amplitude of transverse velocity oscillations at each boundary, which can be combined to yield the

Cross-slab amplitude ratio

$$R_v := \frac{\hat{v}_x(x_0)}{\hat{v}_x(-x_0)} = \frac{\rho_1 m_2}{\rho_2 m_1} \frac{(k^2 {v_A}^2 - \omega^2) m_1 \frac{\rho_0}{\rho_1} - \omega^2 m_0 \tanh(m_0 x_0)}{(k^2 {v_A}^2 - \omega^2) m_2 \frac{\rho_0}{\rho_2} - \omega^2 m_0 \tanh(m_0 x_0)}$$

This has a potential use as a diagnostic tool for *solar magneto-seismology*:

- Measure: R_v, ρ_0, ρ_1, ρ_2, ω, k, x_0, c_0.
- Solve to find: v_A which can give you B_0.

Summary

- Magnetic fields enrich the spectrum of waves present in an electrically conducting fluid, introducing the Alfvén wave, which can combine with sound waves to form magnetoacoustic waves.
- Structuring gives rise to eigenmodes, whose slab boundary oscillations can be in phase (kink mode) or in anti-phase (sausage mode).
- Asymmetry in the external plasma can potentially be used as a diagnostic tool to calculate difficult-to-measure plasma parameters.

NASA

What is next?

This project has involved...

- **MHD**: MHD equations, MHD waves, perturbation and asymptotic methods.
- **Solar physics**: Coronal wave observations, coronal heating, solar magneto-seismology.
- **Python programming**: for root finding and plotting in 2D and 3D.

Where to go from here...

- Generalise the asymmetric slab further: add **shear flow**, **external magnetic field**, **gravity**.
- Apply cross-slab amplitude ratio method to **solar observations**,
- Submit paper to peer-reviewed Solar Physics journal,
- I will be giving a talk at National Astronomy Meeting 2016,
- And continue with this work next year for my PhD.

Thank you