MHD Waves in Asymmetric Slab Waveguides

Matthew Allcock and Robert Erdélyi

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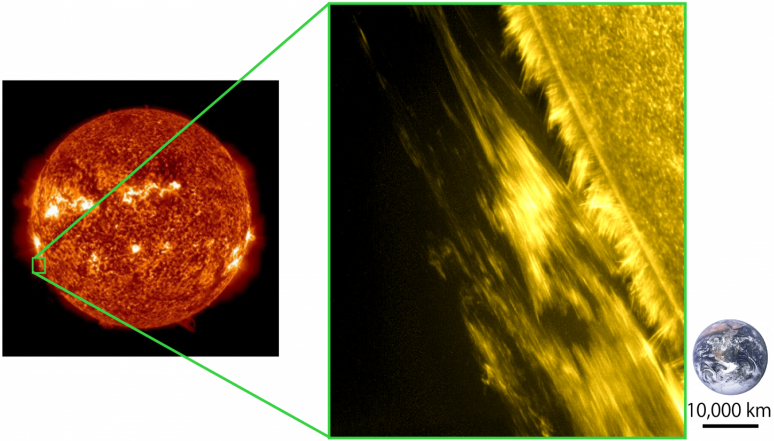
The University Of Sheffield.

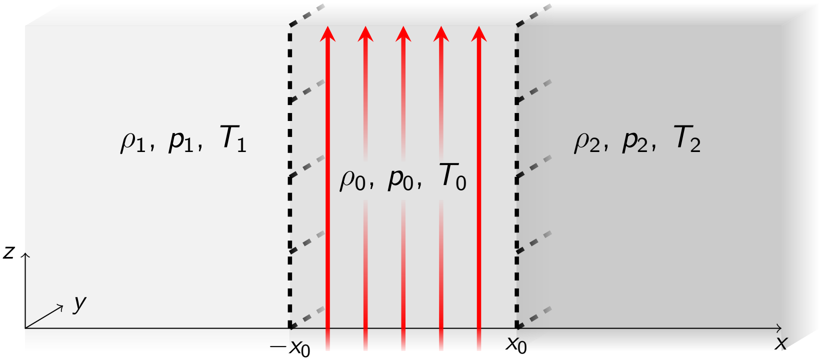
Research Centre

Motivation



- **Solar magneto-seismology**,
- **Numerical simulations**,
- **Coronal heating**.

Equilibrium conditions



- Uniform vertical magnetic field in the slab.
- Non-magnetised plasma outside.
- **Different** density and pressure on each side.

Governing equations

Ideal MHD equations:

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) &= 0, \\
\rho \frac{D\mathbf{v}}{Dt} &= -\nabla p - \frac{1}{\mu} \mathbf{B} \times (\nabla \times \mathbf{B}), \\
\frac{D}{Dt}\left(\frac{p}{\rho^{\gamma}}\right) &= 0, \\
\frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{v} \times \mathbf{B}),
\end{aligned}
$$

$\mathbf{v}$ = plasma velocity, $\mathbf{B}$ = magnetic field strength, ρ = density, p = pressure, μ = magnetic permeability, γ = adiabatic index.

Fourier decomposition

Look for *plane wave* solutions of the form:

$$v_x(\mathbf{x}, t) = \hat{v}_x(x)e^{i(kz-\omega t)}, \quad v_y(\mathbf{x}, t) = 0, \quad v_z(\mathbf{x}, t) = \hat{v}_z(x)e^{i(kz-\omega t)},$$

to arrive at the following ODEs:

$$\frac{d^2\hat{v}_x}{dx^2} - {m_1}^2\hat{v}_x = 0 \qquad \frac{d^2\hat{v}_x}{dx^2} - {m_0}^2\hat{v}_x = 0 \qquad \frac{d^2\hat{v}_x}{dx^2} - {m_2}^2\hat{v}_x = 0$$

$-x_0$ x_0 x

Boundary conditions:
pressure and **velocity** continuous across boundaries.

$${m_0}^2 = \frac{(k^2{v_A}^2 - \omega^2)(k^2c_0^2 - \omega^2)}{(c_0^2 + {v_A}^2)(k^2{c_T}^2 - \omega^2)}, \qquad m_{1,2}^2 = k^2 - \frac{\omega^2}{c_{1,2}^2},$$

$${c_T}^2 = \frac{c_0^2{v_A}^2}{c_0^2 + {v_A}^2}, \qquad v_A = \frac{B_0}{\sqrt{\mu\rho_0}},$$

Dispersion relation

The conditions for the existence of non-trivial solutions yield the dispersion relation (DR):

Dispersion relation

$$\frac{\omega^4 {m_0}^2}{k^2 {v_A}^2 - \omega^2} + \frac{\rho_0}{\rho_1} m_1 \frac{\rho_0}{\rho_2} m_2 (k^2 {v_A}^2 - \omega^2)$$
$$- \frac{1}{2} m_0 \omega^2 \left(\frac{\rho_0}{\rho_1} m_1 + \frac{\rho_0}{\rho_2} m_2 \right) (\tanh m_0 x_0 + \coth m_0 x_0) = 0,$$

Approximate dispersion relation

When each side of the slab have approximately equal parameters, the asymmetric slab DR decouples to:

Approximate dispersion relation

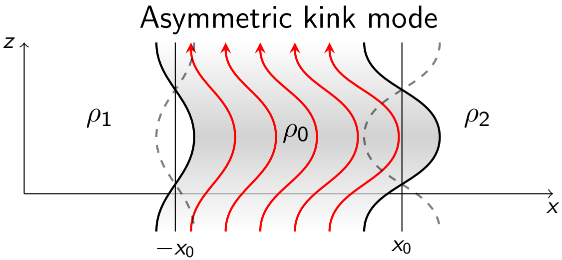
$$(k^2 v_A{}^2 - \omega^2)\left(\frac{\rho_0}{\rho_1}m_1 + \frac{\rho_0}{\rho_2}m_2\right) = 2\omega^2 m_0 \begin{pmatrix}\tanh\\ \coth\end{pmatrix}(m_0 x_0).$$

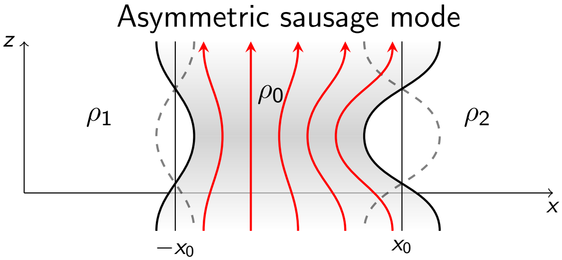
Compare with the symmetric slab DR, *Roberts 1981*:

Symmetric slab dispersion relation

$$(k^2 v_A{}^2 - \omega^2)\frac{\rho_0}{\rho_e}m_e = \omega^2 m_0 \begin{pmatrix}\tanh\\ \coth\end{pmatrix}(m_0 x_0).$$

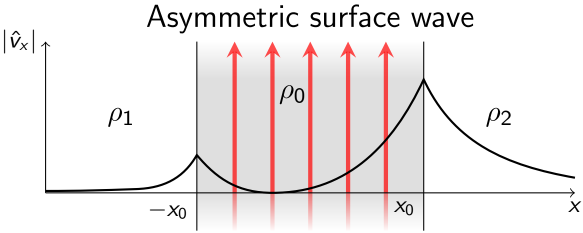
Asymmetric sausage and kink modes

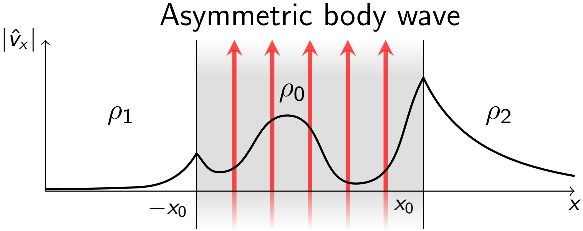




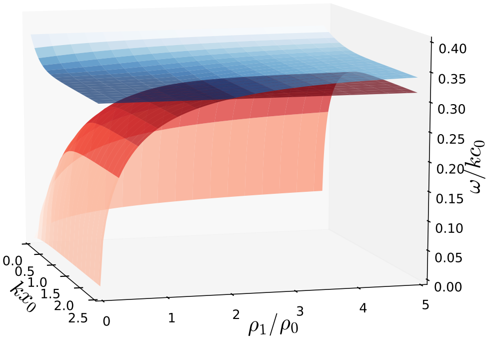
Asymmetric surface and body modes

- When $m_0^2 > 0$, the solution is **exponential**: *surface wave*.
- When $m_0^2 < 0$, the solution is **oscillatory**: *body wave*.



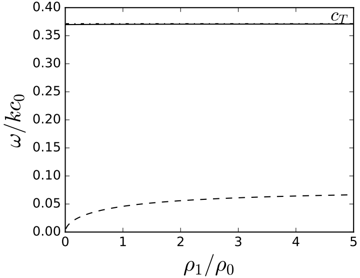


Density variation

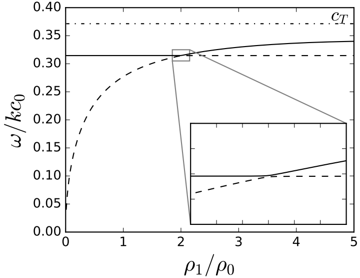


Slow surface modes of the **asymmetric** slab varying slab width (kx_0) and density ratio (ρ_1/ρ_0).

Density variation



(a) Small kx_0

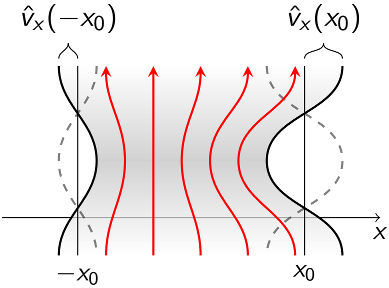


(b) Large kx_0

Avoided crossing

Can energy be transferred between sausage and kink surface modes of an asymmetric slab?

Cross-slab amplitude ratio



Cross-slab amplitude ratio

$$R := \frac{\hat{v}_x(x_0)}{\hat{v}_x(-x_0)} = \frac{\rho_1 m_2}{\rho_2 m_1} \frac{(k^2 {v_A}^2 - \omega^2) m_1 \frac{\rho_0}{\rho_1} - \omega^2 m_0 \tanh(m_0 x_0)}{(k^2 {v_A}^2 - \omega^2) m_2 \frac{\rho_0}{\rho_2} - \omega^2 m_0 \tanh(m_0 x_0)}$$

Solar magneto-seismology:

- Measure: R, ρ_i, c_i, ω, k, x_0.
- Solve to find: v_A and hence B_0.

Summary

- **Slab structuring** - kink and sausage modes.
- **Asymmetric** slab DR - does not decouple into sausage and kink eigenmodes.
- **Asymmetric** external plasma - difference in amplitude on each boundary.
- **Density ratio variation** - mode conversion between surface modes.
- **Cross-slab amplitude ratio** - potential use as a diagnostic tool to determine the magnetic field strength.

Future work

- Generalise the asymmetric slab further:
  - **Shear flow**
  - **External magnetic field**
  - **Gravity**
- Apply the cross-slab amplitude ratio method to **solar observations**.