CHAPTER 1

Introduction

1.1 The Sun

Its significance, structure, and magnetic phenomena.

From my confirmation report: *The Sun is hot. So hot, in fact, that many of the constituent particles cannot form neutral atoms, but exist as a soup of negatively charged electrons and positively charged nuclei. Matter in this state is known as plasma. Because many of the charges are dissociated in a plasma, it can conduct electricity and therefore induce a magnetic field. This magnetic field can interact with the fluid to produce a nonlinear coupling between the magnetic field and the plasma motion. This coupling is described by magnetohydrodynamic (MHD) theory.*

The atmosphere of the Sun is multi-layered. Starting in the lowest layer, known as the photosphere, and moving up through the chromosphere and transition region we reach the upper atmosphere, known as the corona. The solar atmosphere, particularly the corona, is dominated by a complex and dynamic magnetic field that makes it highly inhomogeneous. This gives rise to many high-energy events such as jets, eruptions, and flares. These dynamic solar events, amongst other things, drive MHD 1waves in the solar atmosphere which can be guided by the inhomogeneous magnetic field. MHD waves are similar to waves in terrestrial fluids, such as sound waves in air, but as well as a pressure gradient restoring force, MHD waves owe their existence to a restoring force due to perturbations of the magnetic field. MHD waves whose restoring force is a combination of the pressure gradient and this magnetic force, but not other forces such as gravity and Coriolis, are known as magneto-acoustic waves.

MHD waves in the Sun can be used to approximate plasma parameters, such as the magnetic field strength, that are difficult to measure using traditional methods (Nakariakov and Verwichte, 2005; De Moortel and Nakariakov, 2012). This is done by comparing observations of MHD waves in the solar atmosphere *to theoretical results from studying MHD wave propagation in waveguides that approximate those in the solar atmosphere, formed by a complex structuring of the magnetic field. This technique, known as solar magneto-seismology, has been an emerging field over the last few decades, and is on track to become a key tool for solar physics as the next generation of solar telescopes with greater spatial and temporal resolution are developed. The significance of making good approximations of plasma parameters in the solar atmosphere is so that we can use realistic parameters in numerical simulations and to gain an understanding of what conditions lead to instability and thus leading to solar flares and coronal mass ejections, which pose a significant threat to modern society on Earth (Cabinet Office, 2015).*

1.1.1 Solar plasma

Plasma - a quasineutral gas of charged particles exhibiting collective behaviour. $>$ 99 percent of baryonic matter in universe is in plasma state. Plasmas create magnetic fields and interact with magnetic fields.

1.2 Magnetohydrodynamics

1.2.1 The equations of ideal magnetohydrodynamics

To do: include charge neutrality? Single fluid? More discussion of amperes equation and the non-relativistic assumption?

To build up a mathematical description of the Sun's plasma dynamics, let's motivate some assumptions.

The Sun's plasma, just like all matter in the Universe, is made up of particles[1], but the phenomena such as MHD waves that we are concerned with in this Thesis operate on a *macroscopic* level. By this we mean on length-scales much larger than the *mean free path*[2]. This means that the *Knudsen number*, the dimensionless parameter defined by the ratio of the mean free path to a characteristic length scale, in the Sun is much less than unity. This motivates the **continuum assumption**, where the fluid is considered to "fill up" the space in which is is contained, so that small-scale inhomogeneities caused by particle dynamics are negligible. This gives us a coherent notion of fluid velocity, $\mathbf{v}(\mathbf{x}, t) = (v_1(\mathbf{x}, t), v_2(\mathbf{x}, t), v_3(\mathbf{x}, t))$, density, $\rho(\mathbf{x}, t)$, and pressure, $p(\mathbf{x}, t)$, as functions of continuous position, $\mathbf{x}$, and time, t.

[1] atoms or subatomic particles, depending on the temperature of its location in the Sun

[2] Approximately 1 cm - 1 km in the Sun. REFERENCE

The universe gifts us fundamental laws that are obeyed by all classical mechanics systems upon which we can build our framework. Firstly, the **conservation of mass** tells us that the change in density in a fixed volume is due only to mass entering or leaving the volume. The rate of change of density in a fixed volume V is

$$\frac{d}{dt}\iiint_V \rho \; d\mathbf{x} = \iiint_V \frac{\partial \rho}{\partial t} \; d\mathbf{x} \tag{1.1}$$

and the rate of mass flux into this volume, whose bounding surface we denote by S, which has infinitesimal surface normal component $d\mathbf{s}$, is

$$-\iint_S \rho\mathbf{v} \; d\mathbf{s} = \iiint_V -\nabla \cdot (\rho\mathbf{v}) \; d\mathbf{x}, \tag{1.2}$$

by use of the divergence theorem. Equations (1.1) and (1.2) must be equal for any volume V so the integrands must be equal, that is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{v}) = 0, \tag{1.3}$$

known as the **continuity equation**.

Secondly, the **conservation of momentum** tells us that the momentum in a volume V that moves with the fluid is only changed by forces exerted on the fluid. The rate of change of momentum in this volume is

$$\frac{d}{dt}\iiint_V \rho\mathbf{v} \; d\mathbf{x} = \iiint_V \rho\frac{D\mathbf{v}}{Dt} \; d\mathbf{x}, \tag{1.4}$$

where $D/Dt = \partial/\partial t + \mathbf{v}\cdot\nabla$ is derivative observed when moving with the fluid, known as the *material derivative*. The forces acting upon the fluid are either *surface forces* (such as pressure gradient force and viscosity) that act on an internal or external surface, or *body forces*, $\mathbf{b}$, (such as gravity and magnetic forces) that act on the whole volume. The surface forces form a stress tensor σ, so that the total force exerted on a volume of fluid is

$$\iint_S \sigma \cdot d\mathbf{s} + \iiint_V \mathbf{b} \; d\mathbf{x} = \iiint_V (\nabla \cdot \sigma + \mathbf{b}) \; d\mathbf{x}, \tag{1.5}$$

using the divergence theorem. Equations (1.4) and (1.5) must be equal for any volume V so the integrands must be equal, that is

$$\rho\frac{D\mathbf{v}}{Dt} = \nabla \cdot \sigma + \mathbf{b}. \tag{1.6}$$

Motivated by the large role they play in the dynamics of small to medium scale solar phenomena, in this Thesis we focus on the effects of magnetic forces and neglect commonly used other forces such as gravity and viscosity. Denoting the magnetic field and permeability by $\mathbf{B}$ and μ, respectively, the magnetic force felt by a (non-relativistic) fluid element is $(\nabla \times \mathbf{B}) \times \mathbf{B}/\mu$. By neglecting viscosity, we can write the stress tensor as $\sigma = -pI$, where I is the 3×3 identity matrix. This reduces Equation(1.6) to the **momentum equation**, namely

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \frac{1}{\mu}(\nabla \times \mathbf{B}) \times \mathbf{B}. \tag{1.7}$$

Finally, **conservation of entropy** occurs during processes that are *adiabatic* and *reversible*. The entropy per unit mass, s, for an *ideal fluid* is given by

$$s = C_v \ln\left(\frac{p}{\rho^\gamma}\right) + const, \tag{1.8}$$

where C_v and γ are the specific heat at constant volume and the adiabatic index, respectively. Entropy is conserved when moving with the fluid, which, using Equation (1.8), can be written in the form

$$\frac{D}{Dt}\left(\frac{p}{\rho^\gamma}\right) = 0, \tag{1.9}$$

which we call the **energy equation** because it can also be interpreted as the fundamental law of conservation of energy.

Equations (1.3), (1.9), and the three components of (1.7) are a system of five equations that relate eight unknowns (ρ, p, and three components of $\mathbf{v}$ and $\mathbf{B}$). Three additional equations are required to close the system. To establish these additional equations, we use **Ohm's Law**, which asserts that the current density, $\mathbf{j}$ is proportional to the total electric field when moving with the fluid,

$$\mathbf{j} = \frac{1}{\eta}(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{1.10}$$

where $\mathbf{E}$ is the electric field. In this Thesis, we are concerned with plasmas where resistive effects, including magnetic reconnection and diffusion, are unimportant. Therefore, we can neglect the left hand side of this equation to give

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0. \tag{1.11}$$

Faraday's law of electromagnetism relates the gradient of the electric field to the change in magnetic field:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{1.12}$$

Combining Equations (1.11) and (1.12) gives us the **induction equation** (for an ideal plasma)

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}). \tag{1.13}$$

Equations (1.3), (1.7), (1.9), and (1.13) constitute a complete set of equations that describe the evolution of an *ideal plasma* and are known as the **ideal MHD equations**.

In addition, **Gauss' Law**, which states that $\nabla \cdot \mathbf{B} = 0$, puts a constraint on the choice of initial magnetic field. Integrating Equation (1.13) shows us that initial satisfaction of Gauss' Law ensures its satisfaction for all later time.

1.2.2 Ideal magnetohydrodynamic behaviour

The ideal plasma assumption approximates the plasma to be perfectly conducting. Ideal plasmas behave in unique and surprisingly simple ways that will be discussed in this subsection. We will briefly discuss the decomposition of the Lorentz force into magnetic tension and pressure, the conservation of magnetic flux, and the conservation of magnetic field lines.

In this discussion it is helpful to define the notion of a *magnetic field line*. Magnetic field lines, or just "field lines", are lines parallel to the magnetic field, $\mathbf{B}$. The local strength of the magnetic field is proportional to the local field line density. Magnetic field lines are fictitious and are conceived of merely for ease of understanding and visualisation.

1.2.2.1 Magnetic tension and pressure

The Lorentz force in the momentum Equation (1.7) can be decomposed as

$$\frac{1}{\mu}(\nabla \times \mathbf{B}) \times \mathbf{B} = \frac{1}{\mu}(\mathbf{B} \cdot \nabla)\mathbf{B} - \nabla \left(\frac{\mathbf{B}^2}{2\mu} \right). \tag{1.14}$$

The first term on the right hand side is the *magnetic tension* force which acts normal to $\mathbf{B}$. It acts to "straighten out" magnetic field lines and it's strength is proportional ot the field line's curvature. The second term on the right hand side is the *magnetic pressure* force which acts along any negative gradient in magnetic field strength. It acts to "spread out" magnetic field lines.

1.2.2.2 Magnetic flux conservation

The magnetic flux through a surface S bounded by a simple closed curve C is

$$\Psi = \iint_S \mathbf{B} \cdot d\mathbf{s}. \tag{1.15}$$

The magnetic flux can change in two ways: when the magnetic field $\mathbf{B}$ changes with S held fixed, and the flux swept out by the C as it moved with the plasma.

Combining these, the rate of change of flux is

$$\frac{d\Psi}{dt} = \iint_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} + \oint_C \mathbf{B} \cdot \mathbf{v} \times d\mathbf{l}, \tag{1.16}$$

where $d\mathbf{l}$ is an element parallel to curve C. Using Stokes' Theorem on the second term on the right hand side, the above equation becomes

$$\frac{d\Psi}{dt} = \iint_S \left[\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B})\right] \cdot d\mathbf{s} = 0, \tag{1.17}$$

using Equation (1.13). Therefore, magnetic flux is conserved in an ideal plasma.

This result has the important corollary that the plasma and the magnetic field lines are "frozen in". That is, wherever the magnetic field moves, the plasma follows, and *vice versa*. In other words, plasma elements that initially occupy the same field line will always do so. This is known as **Alfvén's frozen flux theorem**.

1.3 Waves in the solar atmosphere

1.3.1 Magnetohydrodynamic waves in homogeneous plasma

Whilst the Sun's inhomogeneity is undeniable, it is instructive to first study the MHD waves that propagate in a homogeneous plasma. We start with the ideal MHD equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{1.18}$$

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \frac{1}{\mu}(\nabla \times \mathbf{B}) \times \mathbf{B}, \tag{1.19}$$

$$\frac{D}{Dt}\left(\frac{p}{\rho^\gamma}\right) = 0, \tag{1.20}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}). \tag{1.21}$$

The complexity of these equations is due to their non-linearity. Consider a stationary homogeneous plasma with equilibrium magnetic field given by $\mathbf{B_0} = (0, 0, B_0)$, without loss of generality. Each parameter can be written as a sum of its equilibrium quantity and a perturbation from that equilibrium, namely, $f = f_0 + f'$, where f is a placeholder for parameters $\rho, p, \mathbf{v}$, and $\mathbf{B}$. The equilibrium plasma is stationary and homogeneous, so $\mathbf{v}_0 = 0$, and each equilibrium parameter is uniform in space. By considering just small perturbations from equilibrium, *i.e.* $f' \ll f_0$ for each parameter, we can remove the non-linearity from the governing equations. Substituting this form of the parameters into the ideal MHD equations and neglecting terms of quadratic order in small perturbation parameters gives us the linearised ideal MHD equations

$$\frac{\partial \rho'}{\partial t} + \rho_0(\nabla \cdot \mathbf{v}') = 0, \tag{1.22}$$

$$\rho_0 \frac{\partial \mathbf{v}'}{\partial t} = -\nabla p' + \frac{1}{\mu}(\nabla \times \mathbf{B}') \times \mathbf{B}_0, \tag{1.23}$$

$$\frac{\partial p'}{\partial t} - c_0^2 \frac{\partial \rho'}{\partial t} = 0, \tag{1.24}$$

$$\frac{\partial \mathbf{B}'}{\partial t} = \nabla \times (\mathbf{v}' \times \mathbf{B}_0), \tag{1.25}$$

where $c_0 = \sqrt{\gamma p_0/\rho_0}$ is the *sound speed*. This system of equations can be combined into the generalised wave equation

$$\frac{\partial^2 \mathbf{v}}{\partial t^2} = c_0^2 \nabla(\nabla \cdot \mathbf{v}) + \frac{1}{\mu \rho_0}(\nabla \times (\nabla \times (\mathbf{v} \times \mathbf{B}_0))) \times \mathbf{B}_0, \tag{1.26}$$

where we have dropped the apostrophe on $\mathbf{v}'$ for brevity. The form of this equation motivates a search for solutions of the form

$$\mathbf{v}(\mathbf{x}, t) = \hat{\mathbf{v}}(\mathbf{x}) e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}, \tag{1.27}$$

corresponding to *plane-waves* with wavenumber vector $\mathbf{k}$, circular frequency ω, and amplitude $\hat{\mathbf{v}}(\mathbf{x})$. This reduces Equation (1.26) to an eigenvalue problem with eigenfrequency ω^2, namely

$$\omega^2 \hat{\mathbf{v}} = c_0^2 \mathbf{k}(\mathbf{k} \cdot \hat{\mathbf{v}}) + \frac{1}{\mu \rho_0}(\mathbf{k} \times (\mathbf{k} \times (\hat{\mathbf{v}} \times \mathbf{B}_0))) \times \mathbf{B}_0. \tag{1.28}$$

With the aim of first studying a limiting solution, the ratio of the first term to the second term on the right hand side fo the above equation is

$$\frac{|c_0^2 \mathbf{k}(\mathbf{k} \cdot \hat{\mathbf{v}})|}{|\frac{1}{\mu \rho_0}(\mathbf{k} \times (\mathbf{k} \times (\hat{\mathbf{v}} \times \mathbf{B}_0))) \times \mathbf{B}_0|} = \frac{c_0^2}{v_A^2}, \tag{1.29}$$

where $v_A = B_0/\sqrt{\mu \rho_0}$ is the *Alfvén speed.*

When the sound speed dominates the Alfvén speed[3], and assuming that $\mathbf{k} \cdot \hat{\mathbf{v}} \neq 0$ so that the fluid is compressible, taking the dot product of $\mathbf{k}$ and Equation (1.28) leads to $\omega = \pm k c_0$. These solutions correspond to forwards and backwards propagating *sound waves.* They are longitudinal waves that propagate isotropically in a homogeneous fluid.

[3]This is known as the *high beta limit.* Here, beta refers to the plasma beta parameter defined as the ratio of kinetic pressure to magnetic pressure and can be written as $\beta = \frac{2c_0^2}{\gamma v_A^2}$

When neither the sound speed or Alfvén speed dominates, we can write Equation (1.28) in component form as

$$\begin{pmatrix} \omega^2 - k_x^2 c_0^2 - (k_x^2 + k_z^2) v_A^2 & 0 & -k_x^2 k_z^2 c_0^2 \\ 0 & \omega^2 - k_z^2 v_A^2 & 0 \\ -k_x k_z c_0^2 & 0 & \omega^2 - k_z^2 c_0^2 \end{pmatrix} \begin{pmatrix} \hat{v}_x \\ \hat{v}_y \\ \hat{v}_z \end{pmatrix} = 0, \qquad (1.30)$$

where, without loss of generality, we have let $\mathbf{k} = (k_x, 0, k_z)$. For there to exist non-trivial solutions to this equation, the determinant of the matrix must vanish, that is

$$(\omega^2 - k_z^2 v_A^2) \left[\omega^4 - \omega^2 k^2 (c_0^2 + v_A^2) + k^2 k_z^2 c_0^2 v_A^2 \right] = 0, \qquad (1.31)$$

where we have defined $k^2 = k_x^2 + k_z^2$.

The first set of solutions to Equation (1.31) are $\omega = \pm k_z v_A$. These solutions correspond to forward and backwards propagating *Alfvén waves*. They are transverse oscillations of the magnetic field that propagate parallel to the magnetic field. They are described as purely magnetic waves because they are not associated with density perturbation.

The second set of solutions to Equation (1.31) are

1.3.1.1 Linearised ideal magnetohydrodynamic equations

1.3.2 Magnetohydrodynamic waves in inhomogeneous plasma

1.3.2.1 Tangential interface

1.3.2.2 Symmetric slab

1.3.2.3 Cylinder

1.4 Thesis outline

Bibliography