# COL351
# Assignment 1

Mallika Prabhakar, 2019CS50440
Sayam Sethi, 2019CS10399

August 2021

## Contents



## 1 Question 1

Let $G$ be an edge-weighted graph with $n$ vertices and $m$ edges satisfying the condition that all the edge weights in $G$ are distinct.

### 1.a Unique MST

Question 1.a

**Question.** *Prove that $G$ has a unique MST.*

*Proof.* We will prove this by induction on the size of $G$ using an idea similar to Kruskal's algorithm discussed in the class.
**Hypothesis:**

$$h(n) : \forall\ G = (V, E) :\ |V| = n \implies MST(G)\ is\ unique \tag{1}$$

**Base case:** $n = 1$ is true since there is no edge and $MST(G) = (V, \phi)$ is unique.

**Induction Step:** Assume $h(n-1)$ is true for $n \geq 2$, now for $h(n)$:
*(Note: This proof assumes each edge to be an unordered pair of vertices)*

Consider Kruskal's algorithm,

**Algorithm 1** Recursive MST Routine – Kruskal's algorithm

$e_0 \leftarrow (x, y)$ be edge with least weight
$H \leftarrow G$
remove $x, y$ from $H$ and add new vertex $z$
**for all** $v$ such that $v$ is neighbour of $x$ or $y$ **do**
    add $(v, z)$ to $H$
    $wt(v, z) \leftarrow \min(wt(v, x), wt(v, y))$
    **if** $wt(v, x) < wt(v, y)$ **then**
        $map(v, z) \leftarrow (v, x)$
    **else**
        $map(v, z) \leftarrow (v, y)$
    **end if**
**end for**
$T_H \leftarrow MST(H)$
$T_G \leftarrow (V, \{e_0\})$
**for all** $e \in T_H$ **do**
    **if** $e$ is not incident on $z$ **then**
        add $e$ to $T_G$
    **else**
        add $map(e)$ to $T_G$
    **end if**
**end for**

In the above algorithm, it is clear that $H$ has $n-1$ vertices. Thus, by our assumption, $h(n-1)$ is true and hence $T_H$ is unique. Also, we know that $T_G$ is a valid MST, from the correctness of Kruskal's algorithm. Now, assume by contradiction that $T_G$ is not unique. Then there exists an MST, say $T' \neq T_G$.

**Claim 1.1.** *$e_0$ cannot be in $T'$*

This is because, if $e_0$ were in $T'$, then $T \setminus \{e_0\} \neq T' \setminus \{e_0\}$ and thus, there would be two different MSTs for $H$ which would be a contradiction to our assumption. Thus, $e_0 \notin T'$.
Consider the path from $x$ to $y$ in $T'$. Since $e_0 = (x, y)$ is not present in $T'$, there exists a different path, say $P = (f_1, f_2 \cdots, f_k)$ where $f_i \in E(T'), 1 \leq i \leq k$. We know that $wt(f_i) > wt(e_0), 1 \leq i \leq k$.
Swap any of the $f_i$ with $e_0$ and let the subgraph formed be $T''$, i.e., $T'' = T' \setminus \{f_i\} \cup \{e_0\}$. We know $T''$ is a spanning tree of $G$ since $V(T'') = V(G)$ and there are no cycles formed on performing the swap operation (this can be proven using contradiction as discussed

in the lecture).
Now, consider the weight of $T''$:

$$\begin{aligned} & wt(T'') = wt(T') - wt(f_i) + wt(e_0) \\ \implies & wt(T'') < wt(T') \end{aligned} \tag{2}$$

We have shown that the total weight of $T''$ is lesser than the weight of $T'$. However, this is a contradiction to the fact that $T'$ is the MST of $G$. Thus our assumption that $T_G$ is not the unique MST of $G$ was wrong.Therefore, $h(n)$ is true.

This completes the induction and the proof that *if all edges in a graph are distinct, then its MST is unique.* □

### 1.b one point two

## 2 Question 2

### 2.a two point one

### 2.b two point two

## 3 Question 3

### 3.a three point one

### 3.b three point one