

Question 1a): By definition of the Poisson Distribution:

$$\mathbb{P}(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

Hence:

$$\mathbb{E}[X] = \sum_{k=0}^{\infty} k \mathbb{P}(X = k) = \sum_{k=1}^{\infty} k e^{-\lambda} \frac{\lambda^k}{k!}$$

Recognising that $\frac{k}{k!} = \frac{1}{(k-1)!}$:

$$\mathbb{E}[X] = e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!}$$

Separating a factor of k and reindexing such that $j = k - 1$:

$$\mathbb{E}[X] = e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} = e^{-\lambda} \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = e^{-\lambda} \lambda \sum_{j=0}^{\infty} \frac{\lambda^j}{j!}$$

By the definition of $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$:

$$\mathbb{E}[X] = e^{-\lambda} \lambda \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} = e^{-\lambda} \lambda e^{\lambda} = \lambda$$

Similarly, for $\mathbb{E}[X(X-1)]$:

$$\mathbb{E}[X(X-1)] = \sum_{k=2}^{\infty} k(k-1) \mathbb{P}(X = k) = \sum_{k=2}^{\infty} k(k-1) e^{-\lambda} \frac{\lambda^k}{k!}$$

Simplifying $\frac{k(k-1)}{k!} = \frac{1}{(k-2)!}$:

$$\mathbb{E}[X(X-1)] = e^{-\lambda} \sum_{k=2}^{\infty} \frac{\lambda^k}{(k-2)!}$$

Letting $j = k - 2$:

$$\mathbb{E}[X(X-1)] = e^{-\lambda} \sum_{k=2}^{\infty} \frac{\lambda^{j+2}}{j!} = e^{-\lambda} \lambda^2 \sum_{k=0}^{\infty} \frac{\lambda^j}{j!} = e^{-\lambda} \lambda^2 e^{\lambda} = \lambda^2$$

For $\mathbb{E}[X(X-1)(X-2)]$:

$$\mathbb{E}[X(X-1)(X-2)] = \sum_{k=3}^{\infty} k(k-1)(k-2) \mathbb{P}(X = k) = \sum_{k=3}^{\infty} k(k-1)(k-2) e^{-\lambda} \frac{\lambda^k}{k!}$$

Simplifying $\frac{k(k-1)(k-2)}{k!} = \frac{1}{(k-3)!}$:

$$\mathbb{E}[X(X-1)(X-2)] = e^{-\lambda} \sum_{k=3}^{\infty} \frac{\lambda^k}{(k-3)!}$$

Letting $j = k - 3$:

$$\mathbb{E}[X(X-1)(X-2)] = e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^{j+3}}{j!} = e^{-\lambda} \lambda^3 \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} = e^{-\lambda} \lambda^3 e^{\lambda} = \lambda^3$$

Therefore $\mathbb{E}[X] = \lambda$, $\mathbb{E}[X(X-1)] = \lambda^2$ **and** $\mathbb{E}[X(X-1)(X-2)] = \lambda^3$ **as required.**

Question 1b): To calculate the probability that X takes even integers, we can define an indicator function of the form:

$$I_{\text{even}} = \mathbf{1}_{\{X \text{ even}\}} = \begin{cases} 1, & X \text{ even} \\ 0, & X \text{ odd} \end{cases}$$

Then $\mathbb{P}(X \text{ even}) = \mathbb{E}[I_{\text{even}}]$ and we can relate the indicator to $(-1)^X$. We can observe that:

$$(-1)^X = \begin{cases} 1, & X \text{ even} \\ 0, & X \text{ odd} \end{cases} \Rightarrow (-1)^X = 2I_{\text{even}} - 1$$

Taking the expectation:

$$\mathbb{E}[(-1)^X] = 2\mathbb{E}[I_{\text{even}}] - 1 = 2\mathbb{P}(X_{\text{even}}) - 1.$$

Hence:

$$\mathbb{P}(X_{\text{even}}) = \frac{1 + \mathbb{E}[(-1)^X]}{2}$$

Taking $\mathbb{E}[(-1)^X]$ from the Definition of the Poisson distribution:

$$\mathbb{E}[(-1)^X] = \sum_{k=0}^{\infty} (-1)^k \mathbb{P}(X = k) = \sum_{k=0}^{\infty} (-1)^k e^{-\lambda} \frac{\lambda^k}{k!}$$

Factoring out $e^{-\lambda}$ and recognising that $\sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} = e^{-\lambda}$:

$$\mathbb{E}[(-1)^X] = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} = e^{-\lambda} e^{-\lambda} = e^{-2\lambda}$$

Therefore, when substituting back into the probability:

$$\begin{aligned} \mathbb{P}(X_{\text{even}}) &= \frac{1 + \mathbb{E}[(-1)^X]}{2} = \frac{1 + e^{-2\lambda}}{2} \\ \mathbb{P}(X_{\text{even}}) &= \frac{1}{2}(1 + e^{-2\lambda}) \end{aligned}$$

As required.

Question 2a): For X and XY to be independent, we must show by the definition of independence that:

$$\mathbb{P}(X = x, XY = z) = \mathbb{P}(X = x) \times \mathbb{P}(XY = z)$$

The joint event can be rewritten as follows:

$$\{X = x, XY = z\} = \{X = x, Y = xz\}$$

Because on the event $\{X = x\}$, the relation $XY = z$ is equivalent to $Y = xz$. By the definition of independence of random variables that each take ± 1 with probability $\frac{1}{2}$:

$$\mathbb{P}(X = x, Y = xz) = \mathbb{P}(X = x) \times \mathbb{P}(Y = xz) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

Therefore:

$$\mathbb{P}(X = x, Y = xz) = \frac{1}{4}$$

To compute $\mathbb{P}(XY = z)$, note that X and Y are independent and each combination of (X, Y) is equally likely:

X	Y	XY	Probability
1	1	1	$\frac{1}{4}$
1	-1	-1	$\frac{1}{4}$
-1	1	-1	$\frac{1}{4}$
-1	-1	1	$\frac{1}{4}$

From the table:

$$\begin{aligned}\mathbb{P}(XY = 1) &= \frac{1}{2} \\ \mathbb{P}(XY = -1) &= \frac{1}{2}\end{aligned}$$

Hence:

$$\mathbb{P}(X = x) \times \mathbb{P}(XY = z) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} = \mathbb{P}(X = x, XY = z)$$

And as this holds for all $x, z \in \{\pm 1\}$, **X and XY are independent. As required.**

Question 2b): For X, XY and Y to be mutually independent:

$$\mathbb{P}(X = a, Y = b, XY = c) = \mathbb{P}(X = a) \times \mathbb{P}(Y = b) \times \mathbb{P}(XY = c)$$

For all $a, b, c \in \{\pm 1\}$. Using the counterexample $a = b = c = 1$:

$$\mathbb{P}(X = 1, Y = 1, XY = 1) = \mathbb{P}(X = 1, Y = 1) = \frac{1}{4}$$

Because $(X, Y) = (1, 1)$ occurs with probability $\frac{1}{4}$ and then $XY = 1$ automatically. But the product of the marginals is:

$$\mathbb{P}(X = 1) \times \mathbb{P}(Y = 1) \times \mathbb{P}(XY = 1) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

Since $\frac{1}{4} \neq \frac{1}{8}$, the factorisation fails and thus **X, XY and Y are not independent. As required.**

Question 3a): We must show that if X is independent of itself, then its distribution only takes 1 value $\mathbb{P}(X = c) = 1$ for some constant c . As such, let $F(t) = \mathbb{P}(X \leq t)$ be the distribution of X . If X is independent of itself, then for every real t :

$$\mathbb{P}(X \leq t, X \leq t) = \mathbb{P}(X \leq t) \times \mathbb{P}(X \leq t)$$

Therefore, it can be seen that $F(t) = F(t)^2$. This only holds for either 0 or 1. Since $F(t)$ is a non-decreasing function that goes from $0 \rightarrow 1$, there must be a point c , where it transitions. As such:

$$\mathbb{P}(X \leq t) = \begin{cases} 0, & t < c, \\ 1, & t \geq c \end{cases}$$

From this we can get that $\mathbb{P}(X = c) = 1$, and thus **X takes c with probability 1. As required.**

Question 3b): To prove that X and Y are independent if $f, g: \mathbb{R} \rightarrow \mathbb{R}$ are Borel, and $f(X), g(Y)$ are independent, it is sufficient to show that for all Borel sets $A, B \subset \mathbb{R}$:

$$\mathbb{P}(f(X) \in A, g(Y) \in B) = \mathbb{P}(f(X) \in A) \times \mathbb{P}(g(Y) \in B).$$

Because f, g are Borel, their preimages $f^{-1}(A)$ and $g^{-1}(B)$ are Borel subsets of \mathbb{R} , and thus:

$$\begin{aligned} \{f(X) \in A\} &= \{X \in f^{-1}(A)\} \in \sigma(X) \\ \{g(Y) \in B\} &= \{Y \in g^{-1}(B)\} \in \sigma(Y) \end{aligned}$$

Independence of X and Y means the σ -algebras $\sigma(X)$ and $\sigma(Y)$ are independent. Hence:

$$\begin{aligned} \mathbb{P}(f(X) \in A, g(Y) \in B) &= \mathbb{P}(X \in f^{-1}(A), Y \in g^{-1}(B)) \\ &= \mathbb{P}(X \in f^{-1}(A)) \times \mathbb{P}(Y \in g^{-1}(B)) \\ &= \mathbb{P}(f(X) \in A) \times \mathbb{P}(g(Y) \in B) \end{aligned}$$

Since these holds, for all Borel sets **$A, B, f(X)$ and $g(Y)$ are independent. As required.**

Question 4a): To prove that $\text{Cov}(X_s, X_t) = \mathbb{E}[X_s X_t] = \mathbb{E} \int_0^{t \wedge s} f_r^2 dr$ for any $s, t \in [0, \infty)$, we take the stochastic process and reduce it with $s \leq t$ as the formula is symmetric in s, t :

$$X_t = \int_0^t f_r dW_r = \underbrace{\int_0^s f_r dW_r}_{X_s} + \underbrace{\int_s^t f_r dW_r}_{\text{future increment}}$$

Expanding $\mathbb{E}[X_s X_t]$ we get:

$$\mathbb{E}[X_s X_t] = \mathbb{E} \left[X_s \left(X_s + \int_s^t f_r dW_r \right) \right] = \mathbb{E}[X_s^2] + \mathbb{E} \left[X_s \int_s^t f_r dW_r \right]$$

As X_s is \mathcal{F}_s -measureable and only depends on the past, the 2nd term is:

$$\mathbb{E} \left[\int_s^t f_r dW_r \middle| \mathcal{F}_s \right] = 0$$

As the Brownian increment after s is independent of \mathcal{F}_s and the stochastic integral over $(s, t]$ has mean 0, then:

$$\mathbb{E} \left[X_s \int_s^t f_r dW_r \right] = \mathbb{E} \left[X_s \mathbb{E} \left[\int_s^t f_r dW_r \middle| \mathcal{F}_s \right] \right] = 0$$

And thus:

$$\mathbb{E}[X_s X_t] = \mathbb{E}[X_s^2]$$

By Itô isometry:

$$\mathbb{E}[X_s^2] = \mathbb{E} \left[\int_0^s f_r^2 dr \right]$$

Therefore:

$$\text{Cov}(X_s, X_t) = \mathbb{E}[X_s X_t] = \mathbb{E} \left[\int_0^s f_r^2 dr \right] = \mathbb{E} \left[\int_0^{s \wedge t} f_r^2 dr \right]$$

Question 4b): Given the Ornstein-Uhlenbeck process:

$$Y_t = e^{-\alpha t} \xi + \sigma \int_0^t e^{-\alpha(t-s)} dW_s$$

The mean of Y_t is the expectation:

$$\mathbb{E}[Y_t] = e^{-\alpha t} \mathbb{E}[\xi] + \sigma \mathbb{E} \left[\int_0^t e^{-\alpha(t-s)} dW_s \right] = e^{-\alpha t} \mu$$

Since an Itô integral has mean 0.

The central process is therefore:

$$\tilde{Y}_t = Y_t - \mathbb{E}[Y_t] = e^{-\alpha t} (\xi - \mu) + \sigma \int_0^t e^{-\alpha(t-s)} dW_s$$

We thus need to find $C(s, t) = \mathbb{E}[\tilde{Y}_s \tilde{Y}_t]$

Substituting \tilde{Y}_t and \tilde{Y}_s , into the covariance function $C(s, t) = \mathbb{E}[(Y_t - \mathbb{E}Y_t)(Y_s - \mathbb{E}Y_s)]$:

$$C(s, t) = \mathbb{E} \left[\left(e^{-\alpha t} (\xi - \mu) + \sigma \int_0^t e^{-\alpha(t-u)} dW_u \right) \left(e^{-\alpha s} (\xi - \mu) + \sigma \int_0^s e^{-\alpha(s-v)} dW_v \right) \right]$$

Where u and v are dummy variables. This can be expanded to:

$$\begin{aligned} C(s, t) = & e^{-\alpha(t+s)} \mathbb{E}[(\xi - \mu)^2] \\ & + \sigma e^{-\alpha t} \mathbb{E} \left[(\xi - \mu) \int_0^s e^{-\alpha(s-v)} dW_v \right] \\ & + \sigma e^{-\alpha s} \mathbb{E} \left[(\xi - \mu) \int_0^t e^{-\alpha(t-u)} dW_u \right] \\ & + \sigma^2 \mathbb{E} \left[\int_0^s e^{-\alpha(s-v)} dW_v \int_0^t e^{-\alpha(t-u)} dW_u \right] \end{aligned}$$

Using the given fact that ξ and $\int_0^t e^{\alpha r} dW_r$ are independent for all $t \geq 0$, this implies:

$$\mathbb{E} \left[(\xi - \mu) \int_0^t e^{-\alpha(t-u)} dW_u \right] = \mathbb{E}[\xi - \mu] \times \mathbb{E} \left[\int_0^t e^{-\alpha(t-u)} dW_u \right] = 0$$

Because $\mathbb{E}[\xi - \mu] = 0$ and the stochastic integral has mean 0. By symmetry:

$$\mathbb{E} \left[(\xi - \mu) \int_0^s e^{-\alpha(s-v)} dW_v \right] = \mathbb{E}[\xi - \mu] \times \mathbb{E} \left[\int_0^s e^{-\alpha(s-v)} dW_v \right] = 0$$

And as such both cross terms are eliminated and:

$$C(s, t) = e^{-\alpha(t+s)} \mathbb{E}[(\xi - \mu)^2] + \sigma^2 \mathbb{E} \left[\int_0^s e^{-\alpha(s-v)} dW_v \int_0^t e^{-\alpha(t-u)} dW_u \right]$$

Using the fact that $\mathbb{E}[(\xi - \mathbb{E}\xi)^2] = \mathbb{E}[(\xi - \mu)^2] = v^2$

$$C(s, t) = e^{-\alpha(t+s)} v^2 + \sigma^2 \mathbb{E} \left[\int_0^s e^{-\alpha(s-v)} dW_v \int_0^t e^{-\alpha(t-u)} dW_u \right]$$

Using Itô isometry for covariance, the expectation of the 2 Itô integrals for any deterministic functions $a(u), b(v)$ is:

$$\mathbb{E} \left[\int_0^t a(u) dW_u \int_0^s b(v) dW_v \right] = \int_0^{s \wedge t} a(r) b(r) dr$$

Therefore when $a(u) = e^{-\alpha(s-v)}$ and $b(v) = e^{-\alpha(t-u)}$ and assuming that $s \leq t$:

$$\begin{aligned} \mathbb{E} \left[\int_0^s e^{-\alpha(s-v)} dW_v \int_0^t e^{-\alpha(t-u)} dW_u \right] &= \int_0^s e^{-\alpha(s-u)} e^{-\alpha(t-u)} du \\ &= \int_0^s e^{-\alpha(t+s-2u)} du \\ &= e^{-\alpha(t+s)} \int_0^s e^{2\alpha u} du \\ &= e^{-\alpha(t+s)} \frac{e^{2\alpha s} - 1}{2\alpha} \end{aligned}$$

Therefore:

$$C(s, t) = v^2 e^{-\alpha(t+s)} + \frac{\sigma^2}{2\alpha} e^{-\alpha(t+s)} (e^{2\alpha s} - 1)$$

As such, the limit:

$$\begin{aligned} \lim_{s \rightarrow \infty} C(s, s+h) &= v^2 e^{-\alpha(s+h+s)} + \frac{\sigma^2}{2\alpha} e^{-\alpha(s+h+s)} (e^{2\alpha s} - 1) \\ &= v^2 e^{-\alpha(2s+h)} + \frac{\sigma^2}{2\alpha} (e^{-\alpha(2s+h)} e^{2\alpha s} - e^{-\alpha(2s+h)}) \\ &= v^2 e^{-\alpha(2s+h)} + \frac{\sigma^2}{2\alpha} (e^{-\alpha h} - e^{-\alpha(2s+h)}) \end{aligned}$$

Taking the limit, the behaviour as $s \rightarrow \infty$ is examined. The term $e^{-\alpha(2s+h)} \rightarrow 0$ because the exponential decays to 0, whereas the $e^{-\alpha h}$ term remains. Thus asymptotically:

$$\lim_{s \rightarrow \infty} C(s, s+h) = \frac{\sigma^2}{2\alpha} e^{-\alpha h}$$