Question 1a): By definition of the Poisson Distribution:

$$\mathbb{P}(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}$$

Hence:

$$\mathbb{E}[X] = \sum_{k=0}^{\infty} k\mathbb{P}(X = k) = \sum_{k=1}^{\infty} ke^{-\lambda}\frac{\lambda^k}{k!}$$

Recognising that $\frac{k}{k!} = \frac{1}{(k-1)!}$:

$$\mathbb{E}[X] = e^{-\lambda}\sum_{k=1}^{\infty}\frac{\lambda^k}{(k-1)!}$$

Separating a factor of k and reindexing such that $j = k - 1$:

$$\mathbb{E}[X] = e^{-\lambda}\sum_{k=1}^{\infty}\frac{\lambda^k}{(k-1)!} = e^{-\lambda}\lambda\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!} = e^{-\lambda}\lambda\sum_{j=0}^{\infty}\frac{\lambda^j}{j!}$$

By the definition of $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$:

$$\mathbb{E}[X] = e^{-\lambda}\lambda\sum_{j=0}^{\infty}\frac{\lambda^j}{j!} = e^{-\lambda}\lambda e^{\lambda} = \lambda$$

Similarly, for $\mathbb{E}[X(X-1)]$:

$$\mathbb{E}[X(X-1)] = \sum_{k=2}^{\infty} k(k-1)\mathbb{P}(X = k) = \sum_{k=2}^{\infty} k(k-1)e^{-\lambda}\frac{\lambda^k}{k!}$$

Simplifying $\frac{k(k-1)}{k!} = \frac{1}{(k-2)!}$:

$$\mathbb{E}[X(X-1)] = e^{-\lambda}\sum_{k=2}^{\infty}\frac{\lambda^k}{(k-2)!}$$

Letting $j = k - 2$:

$$\mathbb{E}[X(X-1)] = e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^{j+2}}{j!} = e^{-\lambda}\lambda^2\sum_{k=0}^{\infty}\frac{\lambda^j}{j!} = e^{-\lambda}\lambda^2 e^{\lambda} = \lambda^2$$

For $\mathbb{E}[X(X-1)(X-2)]$:

$$\mathbb{E}[X(X-1)(X-2)] = \sum_{k=3}^{\infty} k(k-1)(k-2)\mathbb{P}(X = k) = \sum_{k=3}^{\infty} k(k-1)(k-2)e^{-\lambda}\frac{\lambda^k}{k!}$$

Simplifying $\frac{k(k-1)(k-2)}{k!} = \frac{1}{(k-3)!}$:

$$\mathbb{E}[X(X-1)(X-2)] = e^{-\lambda}\sum_{k=3}^{\infty}\frac{\lambda^k}{(k-3)!}$$

Letting $j = k - 3$:

$$\mathbb{E}[X(X-1)(X-2)] = e^{-\lambda}\sum_{j=0}^{\infty}\frac{\lambda^{j+3}}{j!} = e^{-\lambda}\lambda^3\sum_{j=0}^{\infty}\frac{\lambda^j}{j!} = e^{-\lambda}\lambda^3 e^{\lambda} = \lambda^3$$

These are falling factorial moments of a Poisson Distribution where $\mathbb{E}[(X)_m] = \lambda^m$ and it is shown that $\mathbb{E}[\boldsymbol{X}] = \boldsymbol{\lambda}$, $\mathbb{E}[\boldsymbol{X}(\boldsymbol{X}-\mathbf{1})] = \boldsymbol{\lambda}^{\mathbf{2}}$ $\boldsymbol{and}$ $\mathbb{E}[\boldsymbol{X}(\boldsymbol{X}-\mathbf{1})(\boldsymbol{X}-\mathbf{2})] = \boldsymbol{\lambda}^{\mathbf{3}}$ **as required.**

Question 1b): To calculate the probability that X takes even integers, we can define an indicator function of the form:

$$I_{even} = \mathbf{1}_{\{X\ even\}} = \begin{cases} 1, & X\ even \\ 0, & X\ odd \end{cases}$$

Then $\mathbb{P}(X\ even) = \mathbb{E}[I_{even}]$ and we can relate the indicator to $(-1)^X$. We can observe that:

$$(-1)^X = \begin{cases} 1, & X\ even \\ -1, & X\ odd \end{cases} \quad \Rightarrow \quad (-1)^X = 2I_{even} - 1$$

Taking the expectation:

$$\mathbb{E}[(-1)^X] = 2\mathbb{E}[I_{even}] - 1 = 2\mathbb{P}(X_{even}) - 1.$$

Hence:

$$\mathbb{P}(X_{even}) = \frac{1 + \mathbb{E}[(-1)^X]}{2}$$

Taking $\mathbb{E}[(-1)^X]$ from the Definition of the Poisson distribution:

$$\mathbb{E}[(-1)^X] = \sum_{k=0}^{\infty}(-1)^k\,\mathbb{P}(X = k) = \sum_{k=0}^{\infty}(-1)^k e^{-\lambda}\frac{\lambda^k}{k!}$$

Factoring out $e^{-\lambda}$ and recognising that $\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!} = e^{-\lambda}$:

$$\mathbb{E}[(-1)^X] = e^{-\lambda}\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!} = e^{-\lambda}e^{-\lambda} = e^{-2\lambda}$$

Therefore, when substituting back into the probability:

$$\mathbb{P}(\boldsymbol{X}_{\boldsymbol{even}}) = \frac{\mathbf{1} + \mathbb{E}[(-\mathbf{1})^{\boldsymbol{X}}]}{\mathbf{2}} = \frac{\mathbf{1} + \boldsymbol{e}^{-\mathbf{2}\boldsymbol{\lambda}}}{\mathbf{2}}$$

$$\mathbb{P}(\boldsymbol{X}_{\boldsymbol{even}}) = \frac{\mathbf{1}}{\mathbf{2}}\left(\mathbf{1} + \boldsymbol{e}^{-\mathbf{2}\boldsymbol{\lambda}}\right)$$

As required.

Question 2a): For X and XY to be independent, we must show by the definition of independence that:

$$\mathbb{P}(X = x, XY = z) = \mathbb{P}(X = x) \times \mathbb{P}(XY = z)$$

The joint event can be rewritten as follows:

$$\{X = x, XY = z\} = \{X = x, Y = xz\}$$

Because on the event $\{X = x\}$, the relation $XY = z$ is equivalent to $Y = xz$. By the definition of independence of random variables that each take ± 1 with probability $\frac{1}{2}$:

$$\mathbb{P}(X = x, Y = xz) = \mathbb{P}(X = x) \times \mathbb{P}(Y = xz) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

Therefore:

$$\mathbb{P}(X = x, Y = xz) = \frac{1}{4}$$

To compute $\mathbb{P}(XY = z)$, note that $X$and $Y$are independent and each combination of $(X, Y)$is equally likely:

X	Y	XY	Probability
1	1	1	$\frac{1}{4}$
1	−1	-1	$\frac{1}{4}$
-1	1	-1	$\frac{1}{4}$
-1	−1	1	$\frac{1}{4}$

From the table:

$$\mathbb{P}(XY = 1) = \frac{1}{2}$$
$$\mathbb{P}(XY = -1) = \frac{1}{2}$$

Hence:

$$\mathbb{P}(X = x) \times \mathbb{P}(XY = z) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} = \mathbb{P}(X = x, XY = z)$$

More succinctly by the conditional expectation:

$$\mathbb{P}(XY = z \mid X = x) = \mathbb{P}(Y = xz) = \frac{1}{2} = \mathbb{P}(XY = z)$$

For all $x, z \in \{\pm 1\}$. As this holds for all $x, z \in \{\pm 1\}$, $\boldsymbol{X}$ **and** $\boldsymbol{XY}$ **are independent. As required.**

Question 2b): For X, XY and Y to be mutually independent:

$$\mathbb{P}(X = a, Y = b, XY = c) = \mathbb{P}(X = a) \times \mathbb{P}(Y = b) \times \mathbb{P}(XY = c)$$

For all $a, b, c \in \{\pm 1\}$. Using the counterexample "$\boldsymbol{a = b = c = 1}$":

$$\mathbb{P}(X = 1, Y = 1, XY = 1) = \mathbb{P}(X = 1, Y = 1) = \frac{1}{4}$$

Because $(X, Y) = (1,1)$ occurs with probability $\frac{1}{4}$ and then $XY = 1$ automatically. But the product of the marginals is:

$$\mathbb{P}(X = 1) \times \mathbb{P}(Y = 1) \times \mathbb{P}(XY = 1) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

Since $\frac{1}{4} \neq \frac{1}{8}$, the factorisation fails and thus **$\boldsymbol{X, XY}$ and $\boldsymbol{Y}$ are not independent. As required.**

Additionally, note that the 3 random variables satisfy the deterministic constraint:

$$(X)(XY)(Y) = X^2Y^2 = 1$$

This shows that the 3 random variables cannot be mutually independent, since three independent random variables would have $\mathbb{P}(X \cdot XY \cdot Y = 1) = \frac{1}{2} \neq 1$.

Question 3a): We must show that if X is independent of itself, then its distribution only takes 1 value $\mathbb{P}(X = c) = 1$ for some constant c. As such, let $F(t) = \mathbb{P}(X \leq t)$ be the distribution of X. If X is independent of itself, then for every real t:

$$\mathbb{P}(X \leq t, X \leq t) = \mathbb{P}(X \leq t) \times \mathbb{P}(X \leq t)$$

Therefore, it can be seen that $F(t) = F(t)^2$. This only holds for either 0 or 1. Since $F(t)$ is a non-decreasing function that goes from $0 \to 1$, there must be a point c, where it transitions. As such:

$$\mathbb{P}(X \leq t) = \begin{cases} 0, & t < c, \\ 1, & t \geq c \end{cases}$$

From this we can get that $\mathbb{P}(X = c) = 1$, and thus **$\boldsymbol{X}$ takes $\boldsymbol{c}$ with probability 1. As required.**

Question 3b): Assuming $X$and $Y$are independent, if $f, g: \mathbb{R} \to \mathbb{R}$ are Borel measurable, then to prove $f(X), g(Y)$ are independent, it is sufficient to show that for all Borel sets $A, B \subset \mathbb{R}$:

$$\mathbb{P}(f(X) \in A, g(Y) \in B) = \mathbb{P}(f(X) \in A) \times \mathbb{P}(g(Y) \in B).$$

Because f, g are Borel, their preimages $f^{-1}(A)$ and $g^{-1}(B)$ are Borel subsets of $\mathbb{R}$, and thus:

$$\{f(X) \in A\} = \{X \in f^{-1}(A)\} \in \sigma(X)$$
$$\{g(Y) \in B\} = \{Y \in g^{-1}(B)\} \in \sigma(Y)$$

Independence of X and Y means the σ-algebras $\sigma(X)$ and $\sigma(Y)$ are independent. Hence:

$$\begin{aligned}\mathbb{P}(f(X) \in A, g(Y) \in B) &= \mathbb{P}(X \in f^{-1}(A), Y \in g^{-1}(B)) \\ &= \mathbb{P}(X \in f^{-1}(A)) \times \mathbb{P}(Y \in g^{-1}(B)) \\ &= \mathbb{P}(f(X) \in A) \times \mathbb{P}(g(Y) \in B)\end{aligned}$$

Since these holds, for all Borel sets $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{f}(\boldsymbol{X})$ **and** $\boldsymbol{g}(\boldsymbol{Y})$ **are independent. As required.**

Question 4a): To prove that $Cov(X_s, X_t) = \mathbb{E}[X_s X_t] = \mathbb{E} \int_0^{t \wedge s} f_r^2 \, dr$ for any $s, t \in [0, \infty)$, we take the stochastic process and reduce it with $s \leq t$ as the formula is symmetric in s, t:

$$X_t = \int_0^t f_r \, dW_r = \underbrace{\int_0^s f_r dW_r}_{X_s} + \underbrace{\int_s^t f_r dW_r}_{future\ increment}$$

Expanding $\mathbb{E}[X_s X_t]$ we get:

$$\mathbb{E}[X_s X_t] = \mathbb{E}\left[X_s \left(X_s + \int_s^t f_r dW_r\right)\right] = \mathbb{E}[X_s^2] + \mathbb{E}\left[X_s \int_s^t f_r \, dW_r\right]$$

As X_s is $\mathcal{F}_s$- measureable and only depends on the past, the 2nd term is:

$$\mathbb{E}\left[\int_s^t f_r dW_r \middle| \mathcal{F}_s\right] = 0$$

As the Brownian increment after s in independent of $\mathcal{F}_s$ and the stochastic integral over $(s, t]$ has mean 0, then:

$$\mathbb{E}\left[X_s \int_s^t f_r \, dW_r\right] = \mathbb{E}\left[X_s \mathbb{E}\left[\int_s^t f_r dW_r \middle| \mathcal{F}_s\right]\right] = 0$$

And thus:

$$\mathbb{E}[X_s X_t] = \mathbb{E}[X_s^2]$$

By Itô isometry:

$$\mathbb{E}[X_s^2] = \mathbb{E}\left[\int_0^s f_r^2 \, dr\right]$$

Therefore:

$$Cov(X_s, X_t) = \mathbb{E}[X_s X_t] = \mathbb{E}\left[\int_0^s f_r^2 \, dr\right] = \mathbb{E}\left[\int_0^{s \wedge t} f_r^2 \, dr\right]$$

Question 4b): Given the Ornstein-Uhlenbeck process:

$$Y_t = e^{-\alpha t}\xi + \sigma \int_0^t e^{-\alpha(t-s)} dW_s$$

The mean of Y_t is the expectation:

$$\mathbb{E}[Y_t] = e^{-\alpha t}\mathbb{E}[\xi] + \sigma\mathbb{E}\left[\int_0^t e^{-\alpha(t-s)} dW_s\right] = e^{-\alpha t}\mu$$

Since an Itô integral has mean 0 and $\mathbb{E}[\xi] = \mu$.

The central process is therefore:

$$\widetilde{Y_t} = Y_t - \mathbb{E}[Y_t] = e^{-\alpha t}(\xi - \mu) + \sigma \int_0^t e^{-\alpha(t-s)} dW_s$$

We thus need to find $C(s,t) = \mathbb{E}\left[\widetilde{Y_s}\widetilde{Y_t}\right]$

Substituting $\widetilde{Y_t}$ and $\widetilde{Y_s}$, into the covariance function $C(s,t) = \mathbb{E}[(Y_t - \mathbb{E}Y_t)(Y_s - \mathbb{E}Y_s)]$:

$$C(s,t) = \mathbb{E}\left[\left(e^{-\alpha t}(\xi - \mu) + \sigma \int_0^t e^{-\alpha(t-u)} dW_u\right)\left(e^{-\alpha s}(\xi - \mu) + \sigma \int_0^s e^{-\alpha(s-v)} dW_v\right)\right]$$

Where u and v are dummy variables. This can be expanded to:

$$\begin{aligned} C(s,t) = {} & e^{-\alpha(t+s)}\mathbb{E}[(\xi - \mu)^2] \\ & +\sigma e^{-\alpha t}\mathbb{E}\left[(\xi - \mu)\int_0^s e^{-\alpha(s-v)}\, dW_v\right] \\ & +\sigma e^{-\alpha s}\mathbb{E}\left[(\xi - \mu)\int_0^t e^{-\alpha(t-u)}\, dW_u\right] \\ & +\sigma^2\mathbb{E}\left[\int_0^s e^{-\alpha(s-v)}\, dW_v \int_0^t e^{-\alpha(t-u)}\, dW_u\right] \end{aligned}$$

Using the given fact that ξ and $\int_0^t e^{\alpha r} dW_r$ are independent for all $t \geq 0$, this implies:

$$\mathbb{E}\left[(\xi - \mu)\int_0^t e^{-\alpha(t-u)}\, dW_u\right] = \mathbb{E}[\xi - \mu] \times \mathbb{E}\left[\int_0^t e^{-\alpha(t-u)}\, dW_u\right] = 0$$

Because $\mathbb{E}[\xi - \mu] = 0$ and the stochastic integral has mean 0. By symmetry:

$$\mathbb{E}\left[(\xi - \mu)\int_0^s e^{-\alpha(s-v)}\, dW_v\right] = \mathbb{E}[\xi - \mu] \times \mathbb{E}\left[\int_0^s e^{-\alpha(s-u)}\, dW_v\right] = 0$$

And as such both cross terms are eliminated and:

$$C(s,t) = e^{-\alpha(t+s)}\mathbb{E}[(\xi - \mu)^2] + \sigma^2\mathbb{E}\left[\int_0^s e^{-\alpha(s-v)}\, dW_v \int_0^t e^{-\alpha(t-u)}\, dW_u\right]$$

Using the fact that $\mathbb{E}[(\xi - \mathbb{E}\xi)^2] = \mathbb{E}[(\xi - \mu)^2] = v^2$

$$C(s,t) = e^{-\alpha(t+s)}v^2 + \sigma^2\mathbb{E}\left[\int_0^s e^{-\alpha(s-v)}\, dW_v \int_0^t e^{-\alpha(t-u)}\, dW_u\right]$$

Using Itô isometry for covariance, the expectation of the 2 Itô integrals for any deterministic functions $a(u), b(v)$ is:

$$\mathbb{E}\left[\int_0^t a(u)\,dW_u \int_0^s b(v)\,dW_v\right] = \int_0^{s\wedge t} a(r)b(r)dr$$

Therefore when $a(u) = e^{-\alpha(s-v)}$ and $b(v) = e^{-\alpha(t-u)}$ and assuming that $s \leq t$:

$$\begin{aligned}\mathbb{E}\left[\int_0^s e^{-\alpha(s-v)}\,dW_v \int_0^t e^{-\alpha(t-u)}\,dW_u\right] &= \int_0^s e^{-\alpha(s-u)}e^{-\alpha(t-u)}\,du \\ &= \int_0^s e^{-\alpha(t+s-2u)}\,du \\ &= e^{-\alpha(t+s)}\int_0^s e^{2u}\,du \\ &= e^{-\alpha(t+s)}\frac{e^{2\alpha s}-1}{2\alpha}\end{aligned}$$

Therefore:

$$\boldsymbol{C(s,t) = v^2 e^{-\alpha(t+s)} + \frac{\sigma^2}{2\alpha}e^{-\alpha(t+s)}\left(e^{2\alpha s}-1\right)}$$

As such, the limit:

$$\begin{aligned}\lim_{s\to\infty} C(s,s+h) &= v^2 e^{-\alpha(s+h+s)} + \frac{\sigma^2}{2\alpha}e^{-\alpha(s+h+s)}(e^{2\alpha s}-1) \\ &= v^2 e^{-\alpha(2s+h)} + \frac{\sigma^2}{2\alpha}\left(e^{-\alpha(2s+h)}e^{2\alpha s} - e^{-\alpha(2s+h)}\right) \\ &= v^2 e^{-\alpha(2s+h)} + \frac{\sigma^2}{2\alpha}\left(e^{-\alpha h} - e^{-\alpha(2s+h)}\right)\end{aligned}$$

Taking the limit, the behaviour as $s \to \infty$ is examined. The term $e^{-\alpha(2s+h)} \to 0$ because the exponential decays to 0, whereas the $e^{-\alpha h}$ term remains. Thus asymptotically the stationary covariance is:

$$\boldsymbol{\lim_{s\to\infty} C(s,s+h) = \frac{\sigma^2}{2\alpha}e^{-\alpha h}}$$