**Question 1a):** By definition of the Poisson Distribution:

Hence:

Recognising that :

Separating a factor of and reindexing such that :

By the definition of :

Similarly, for :

Simplifying :

Letting :

For :

Simplifying :

Letting :

These are falling factorial moments of a Poisson Distribution where and it is shown that  **as required.**

**Question 1b):** To calculate the probability that X takes even integers, we can define an indicator function of the form:

Then and we can relate the indicator to . We can observe that:

Taking the expectation:

Hence:

Taking from the Definition of the Poisson distribution:

Factoring out and recognising that :

Therefore, when substituting back into the probability:

**As required.**

**Question 2a):** For and to be independent, we must show by the definition of independence that:

The joint event can be rewritten as follows:

Because on the event , the relation is equivalent to . By the definition of independence of random variables that each take with probability :

Therefore:

To compute , note that and are independent and each combination of is equally likely:

|  |  |  |  |
| --- | --- | --- | --- |
|  | 𝑌 |  | Probability |
|  | 1 |  |  |
|  | −1 |  |  |
|  | 1 |  |  |
|  | −1 |  |  |

From the table:

Hence:

More succinctly by the conditional expectation:

For all . As this holds for all ,  **and are independent. As required.**

**Question 2b):** For and to be mutually independent:

For all . Using the counterexample “”:

Because occurs with probability and then automatically. But the product of the marginals is:

Since , thefactorisation fails and thus **and are not independent. As required.**

**Additionally,** note that the 3 random variables satisfy the deterministic constraint:

This shows that the 3 random variables cannot be mutually independent, since three independent random variables would have

**Question 3a):** We must show that if is independent of itself, then its distribution only takes 1 value for some constant . As such, let be the distribution of . If is independent of itself, then for every real :

Therefore, it can be seen that . This only holds for either or . Since is a non-decreasing function that goes from , there must be a point , where it transitions. As such:

From this we can get that , and thus  **takes with probability 1. As required.**

**Question 3b):** To prove that and are independent if are Borel, and are independent, it is sufficient to show that for all Borel sets :

Because are Borel, their preimages and are Borel subsets of and thus:

Independence of and means the -algebras and are independent. Hence:

Since these holds, for all Borel sets  **and are independent. As required.**

**Question 4a):** To prove that for any , we take the stochastic process and reduce it with as the formula is symmetric in

Expanding we get:

As is - measureable and only depends on the past, the 2nd term is:

As the Brownian increment after in independent of and the stochastic integral over has mean 0, then:

And thus:

By Itô isometry:

Therefore:

**Question 4b):** Given the Orstein-Uhlenbeck process:

The mean of is the expectation:

Since an Itô integral has mean .

The central process is therefore:

We thus need to find

Substituting and , into the covariance function :

Where and are dummy variables. This can be expanded to:

Using the given fact that and are independent for all , this implies:

Because and the stochastic integral has mean . By symmetry:

And as such both cross terms are eliminated and:

Using the fact that

Using Itô isometry for covariance, the expectation of the 2 Itô integrals for any deterministic functions is:

Therefore when and and assuming that :

Therefore:

As such, the limit:

Taking the limit, the behaviour as is examined. The term because the exponential decays to , whereas the term remains. Thus asymptotically: