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Introduction to Modern Geometry
Spring 2004, PSU

Lecture Notes 9

2.2 Definition of Gaussian Curvature

Let $M \subset \mathbf{R}^3$ be a regular embedded surface, as we defined in the previous lecture, and let $p \in M$. By the *tangent space* of M at p, denoted by T_pM, we mean the set of all vectors v in $\mathbf{R}^3$ such that for each vector v there exists a smooth curve $\gamma\colon (-\epsilon, \epsilon) \to M$ with $\gamma(0) = p$ and $\gamma'(0) = v$.

Exercise 1. Let $H \subset \mathbf{R}^3$ be a plane. Show that, for all $p \in H$, T_pH is the plane parallel to H which passes through the origin.

Exercise 2. Prove that, for all $p \in M$, T_pM is a 2-dimensional vector subspace of $\mathbf{R}^3$ (*Hint:* Let (U, X) be a proper regular patch centered at p, i.e., $X(0,0) = p$. Recall that $dX_{(0,0)}$ is a linear map and has rank 2. Thus it suffices to show that $T_pM = dX_{(0,0)}(\mathbf{R}^2)$).

Exercise 3. Prove that $D_1X(0,0)$ and $D_2X(0,0)$ form a basis for T_pM (*Hint:* Show that $D_1X(0,0) = dX_{(0,0)}(1,0)$ and $D_2X(0,0) = dX_{(0,0)}(0,1)$).

By a *local gauss map* of M centered at p we mean a pair (V, n) where V is an open neighborhood of p in M and $n\colon V \to \mathbf{S}^2$ is a continuous mapping such that $n(p)$ is orthogonal to T_pM for all $p \in M$. For a more explicit formulation, let (U, X) be a proper regular patch centered at p, and define $N\colon U \to \mathbf{S}^2$ by

$$N(u_1, u_2) := \frac{D_1X(u_1,u_2) \times D_2X(u_1,u_2)}{\|D_1X(u_1,u_2) \times D_2X(u_1,u_2)\|}.$$

Set $V := X(U)$, and recall that, since (U, X) is proper, V is open in M. Now define $n\colon V \to \mathbf{S}^2$ by

$$n(p) := N \circ X^{-1}(p).$$

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Exercise 4. Check that (V, n) is indeed a local gauss map.

Exercise 5. Show that $n : \mathbf{S}^2 \to \mathbf{S}^2$, given by $n(p) := p$ is a Gauss map (*Hint:* Define $f\colon \mathbf{R}^3 \to \mathbf{R}$ by $f(p) := \|p\|^2$ and compute its gradient. Note that $\mathbf{S}^2$ is a level set of f. Thus the gradient of f at p must be orthogonal to $\mathbf{S}^2$).

Let M_1 and M_2 be regular embedded surfaces in $\mathbf{R}^3$ and $f\colon M_1 \to M_2$ be a smooth map (recall that this means that f may be extended to a smooth map in an open neighborhood of M_1 in $\mathbf{R}^3$). Then for every $p \in M_1$, we define a mapping $df_p\colon T_pM_1 \to T_{f(p)}M_2$, known as the *differential* of M_1 at p as follows. Let $v \in T_pM_1$ and let $\gamma_v\colon (-\epsilon, \epsilon) \to M_1$ be a curve such that $\gamma(0) = p$ and $\gamma_v'(0) = v$. Then we set

$$df_p(v) := (f \circ \gamma_v)'(0).$$

Exercise 6. Prove that df_p is well defined (i.e. is independent of the smooth extension) and linear (*Hint:* Let $\tilde{f}$ be a smooth extension of f to an open neighborhood of M. Then $d\tilde{f}_p$ is well defined. Show that for all $v \in T_pM$, $df_p(v) = d\tilde{f}_p(v)$.

Let (V, n) be a local gauss map centered at $p \in M$. Then the *shape operator* of M at p with respect to n is defined as

$$S_p := -dn_p.$$

Note that the shape operator is determined up to two choices depending on the local gauss map, i.e., replacing n by $-n$ switches the sign of the shape operator.

Exercise 7. Show that S_p may be viewed as a linear operator on T_pM (*Hint:* By definition, S_p is a linear map from T_pM to $T_{n(p)}\mathbf{S}^2$. Thus it suffices to show that T_pM and $T_{n(p)}\mathbf{S}^2$ coincide).

Exercise 8. A subset V of M is said to be connected if any pairs of points p and q in V may be joined by a curve in V. Suppose that V is a connected open subset of M, and, furthermore, suppose that the shape operator vanishes throughout V, i.e., for every $p \in M$ and $v \in T_pM$, $S_p(v) = 0$. Show then that V must be flat, i.e., it is a part of a plane (*Hint:* It is enough to show that the gauss map is constant on V; or, equivalently, $n(p) = n(q)$ for all pairs of points p and q in V. Since V is connected, there exists a curve $\gamma\colon [0,1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$. Furthermore, since V is open, we may choose γ to be smooth as well. Define $f\colon [0,1] \to \mathbf{R}$ by $f(t) := n \circ \gamma(t)$, and differentiate. Then $f'(t) = dn_{\gamma(t)}(\gamma'(t)) = 0$. Justify the last step and conclude that $n(p) = n(q)$.

Exercise 9. Compute the shape operator of a sphere of radius r (*Hint:* Define $\pi\colon \mathbf{R}^3 - \{0\} \to \mathbf{S}^2$ by $\pi(x) := x/\|x\|$. Note that π is a smooth mapping and $\pi = n$ on $\mathbf{S}^2$. Thus, for any $v \in T_p\mathbf{S}^2$, $dn_p(v) = d\pi_p(v)$).

The *Gaussian curvature* of M at p is defined as the determinant of the shape operator:

$$K(p) := \det(S_p).$$

Exercise 10. Show that $K(p)$ does not depend on the choice of the local gauss map, i.e, replacing n by $-n$ does not effect the value of $K(p)$.

Exercise 11. Compute the curvature of a sphere of radius r (*Hint:* Use Exercise 9).

Next we derive an explicit formula for $K(p)$ in terms of local coordinates. Let (U, X) be a proper regular patch centered at p. For $1 \leqslant i, j \leqslant 2$, define the functions $g_{ij}\colon U \to \mathbf{R}$ by

$$g_{ij}(u_1, u_2) := \big\langle D_i X(u_1, u_2), D_j X(u_1, u_2)\big\rangle.$$

Note that $g_{12} = g_{21}$. Thus the above expression defines three functions. These are called the *coefficients of the first fundamental form* (a.k.a. *the metric tensor*) with respect to the given patch (U, X). In the classical notation, these functions are denoted by E, F, and G ($E := g_{11}$, $F := g_{12}$, and $G := g_{22}$). Next, define $l_{ij}\colon U \to \mathbf{R}$ by

$$l_{ij}(u_1, u_2) := \big\langle D_{ij} X(u_1, u_2), N(u_1, u_2)\big\rangle.$$

Thus l_{ij} is a measure of the second derivatives of X in a normal direction. l_{ij} are known as the *coefficients of the second fundamental form* of M with respect to the local patch (U, X) (the classical notation for these functions are $L := l_{11}$, $M := l_{12}$, and $N := l_{22}$). We claim that

$$K(p) = \frac{\det(l_{ij}(0,0))}{\det(g_{ij}(0,0))}.$$

To see the above, recall that $e_i(p) := D_i X(X^{-1}(p))$ form a basis for T_pM. Thus, since S_p is linear, $S_p(e_i) = \sum_{j=1}^2 S_{ij} e_j$. This yields that $\langle S_p(e_i), e_k \rangle = \sum_{j=1}^2 S_{ij} g_{jk}$. It can be shown that that

$$\langle S_p(e_i), e_k \rangle = l_{ik},$$

see the exercise below. Then we have $[l_{ij}] = [S_{ij}][g_{ij}]$, where the symbol $[\cdot]$ denotes the matrix with the given coefficients. Thus we can write $[S_{ij}] = [g_{ij}]^{-1}[l_{ij}]$ which yields the desired result.

Exercise 12. Show that $\langle S_p(e_i(p)), e_j(p) \rangle = l_{ij}(0,0)$ (*Hints:* First note that $\langle n(p), e_j(p) \rangle = 0$ for all $p \in V$. Let $\gamma \colon (-\epsilon, \epsilon) \to M$ be a curve with $\gamma(0) = p$ and $\gamma'(0) = e_i(p)$. Define $f \colon (-\epsilon, \epsilon) \to M$ by $f(t) := \langle n(\gamma(t)), e_j(\gamma(t)) \rangle$, and compute $f'(0)$.)

Exercise 13. Compute the Gaussian curvature of a surface of revolution, i.e., the surface covered by the patch

$$X(t, \theta) = \big(x(t) \cos \theta, x(t) \sin \theta, y(t)\big).$$

Next, letting

$$\big(x(t), y(t)\big) = (R + r \cos t, r \sin t),$$

i.e., a circle of radius r centered at $(R, 0)$, compute the curvature of a torus of revolution. Sketch the torus and indicate the regions where the curvature is postive, negative, or zero.

Exercise 14. Let (U, X) be a *Monge patch*, i.e,

$$X(u_1, u_2) := \big(u_1, u_2, f(u_1, u_2)\big),$$

centered at $p \in M$. Show that

$$K(p) := \frac{\det\big(\operatorname{Hess} f(0,0)\big)}{\big(1 + \|\operatorname{grad} f(0,0)\|^2\big)^2},$$

where $\operatorname{Hess} f := [D_{ij} f]$ is the Hessian matrix of f and $\operatorname{grad} f$ is its gradient.

Exercise 15. Compute the curvature of the graph of $z = ax^2 + by^2$, where a and b are constants. Note how the signs of a and b effect the curvature and shape of the surface. Also note the values of a and b for which the curvature is zero.