Math 4441 Sep 20, 2007[1]
Differential Geometry
Fall 2007, Georgia Tech

Lecture Notes 5

1.13 Osculating Circle and Radius of Curvature

Recall that in a previous section we defined the osculating circle of a planar curve $\alpha\colon I \to \mathbf{R}^2$ at a point a of nonvanishing curvature $t \in I$ as the circle with radius $r(t)$ and center

$$\alpha(t) + r(t)N(t)$$

where

$$r(t) := \frac{1}{\kappa(t)}$$

is called the *radius of curvature* of α. If we had a way to define the osculating circle independently of curvature, then we could define curvature simply as the reciprocal of the radius of the osculating circle, and thus obtain a more geometric definition for curvature.

Exercise 1. Let $r(s,t)$ be the radius of the circle which is tangent to α at $\alpha(t)$ and is also passing through $\alpha(s)$. Show that

$$\kappa(t) = \lim_{s\to t} \frac{1}{r(s,t)}.$$

To do the above exercise first recall that, as we showed in the previous lecture, curvature is invariant under rigid motions. Thus, after a rigid motion, we may assume that $\alpha(t) = (0,0)$ and $\alpha'(t)$ is parallel to the x-axis. Then, we may assume that $\alpha(t) = (t, f(t))$, for some function $f\colon \mathbf{R} \to \mathbf{R}$ with $f(0) = 0$ and $f'(0) = 0$. Further, recall that

$$\kappa(t) = \frac{|f''(t)|}{\big(1 + f'(t)^2\big)^{3/2}}.$$

[1]Last revised: September 9, 2025

Thus

$$\kappa(0) = |f''(0)|.$$

Next note that the center of the circle which is tangent to α at $(0,0)$ must lie on the y-axis at some point $(0,r)$, and for this circle to also pass through the point $(s, f(s))$ we must have:

$$r^2 = s^2 + (r - f(s))^2.$$

Solving the above equation for r and taking the limit as $s \to 0$, using L'Hopital's rule, we obtain

$$\lim_{s\to 0} \frac{2|f(s)|}{f^2(s) + s^2} = |f''(0)| = \kappa(0),$$

which is the desired result.

Note 2. The above limit can be used to define a notion of curvature for curves that are not twice differentiable. In this case, we may define the *upper curvature* and *lower curvature* respectively as the upper and lower limit of

$$\frac{2|f(s)|}{f^2(s) + s^2}.$$

as $s \to 0$. We may even distinguish between right handed and left handed upper or lower curvature, by taking the right handed or left handed limits respectively.

Exercise* 3. Let $\alpha\colon I \to \mathbf{R}^2$ be a planar curve and $t_0, t_1, t_2 \in I$ with $t_1 \leq t_0 \leq t_2$. Show that $\kappa(t_0)$ is the reciprocal of the limit of the radius of the circles which pass through $\alpha(t_0)$, $\alpha(t_1)$ and $\alpha(t_2)$ as $t_1, t_2 \to t_0$.

1.14 Kneser's Nesting Theorem

We say that the curvature of a curve is monotone if it is strictly increasing or decreasing. The following result shows that the osculating circles of a curve with monotone curvature are "nested", i.e., they lie inside each other:

Theorem 4 (Kneser's Nesting theorem)**.** *Let $\alpha\colon I \to \mathbf{R}^2$ be a C^4 curve with monotone nonvanishing curvature. Then the osculating circles of α are pairwise nested.*

To prove the above result we need the following Lemma. Note that if $\alpha\colon I \to \mathbf{R}^2$ is a curve with nonvanishing curvature, then the centers of the osculating circles of α for the curve

$$\beta(t) := \alpha(t) + r(t)N(t),$$

where $r(t) := 1/\kappa(t)$ is the radius of curvature of α. This curve β is known as the *evolute* of α.

Exercise 5. Show that if $\alpha\colon I \to \mathbf{R}^2$ is a C^4 curve with monotone nonvanishing curvature, then its evolute β is a regular curve which also has nonvanishing curvature. In particular β contains no line segments.

Now we are ready to prove the main result of this section:

Proof of Kneser's Theorem. We may suppose that $\|\alpha'\| = 1$, and the curvature κ is increasing. We need to show that for every t_0, $t_1 \in I$, with $t_0 < t_1$, the osculating circle at t_1 lies inside the osculating circle at t_0. To this end it suffices to show that

$$\|\beta(t_0) - \beta(t_1)\| + r(t_1) < r(t_0).$$

To see this, first note that, since β contains no line segments (see the previous exercise)

$$\|\beta(t_0) - \beta(t_1)\| < \int_{t_0}^{t_1} \|\beta'(t)\| dt.$$

Now a simple computation completes the proof:

$$\int_{t_0}^{t_1} \|\beta'(t)\| = \int_{t_0}^{t_1} |r'(t)|\, dt = \int_{t_0}^{t_1} -r'(t)\, dt \ = \ r(t_0) - r(t_1).$$

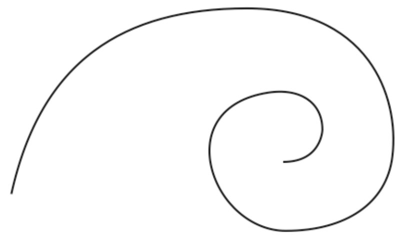
(Here $|r'| = -r'$, since κ is increasing by assumption and hence r is decreasing.) $\square$

Kneser's theorem has a number of interesting corollaries:

Exercise 6. Show that a curve with monotone curvature cannot have any self intersections.

Exercise 7. Show that a curve with monotone curvature cannot have any bitangent lines.

The last two exercises show that a curve with monotone curvature looks essentially as depicted in the following figure, i.e., it spirals around itself.



1.15 Total Curvature and Convexity

The *boundary* of $X \subset \mathbf{R}^n$ is defined as the intersection of the closure of X with the closure of its complement. A simple closed curve $\alpha\colon I \to \mathbf{R}^2$ is *convex* provided that its image lies on one side of every tangent line. A subset of $\mathbf{R}^n$ is convex if it contains the line segment joining each pair of its points. Clearly the intersection of convex sets is convex.

Exercise 8. Show that a simple closed planar curve $\alpha\colon I \to \mathbf{R}^2$ is convex only if it lies on the boundary of a convex set. (*Hint:* By definition, through each point p of Γ there passes a line ℓ_p with respect to which Γ lies on one side. Thus each ℓ_p defines a closed half plane H_p which contains Γ. Show that Γ lies on the boundary of the intersection of all these half planes).

The *total curvature* of a curve $\alpha\colon I \to \mathbf{R}^n$ is defined as

$$\int_I \kappa(t)dt,$$

where t is the arclength parameter.

Exercise 9. Show that the total curvature of any convex planar curve is 2π. (*Hint*: We only need to check that the exterior angles of polygonal approximations of a convex curve do not change sign. Recall that, as we showed in a previous section, the sum of these angles is the total signed curvature. So it follows that the signed curvature of any segment of α is either zero or has the same sign as any other segment. This in turn implies that the signed curvature of α does not change sign. So the total signed curvature of α is equal to its total curvature up to a sign. Since by definition the curve is simple, however, the total signed curvature is $\pm 2\pi$ by Hopf's theorem.)

Theorem 10. *For any closed planar curve $\alpha\colon I \to \mathbf{R}^2$,*

$$\int_I \kappa(t)dt \geq 2\pi,$$

with equality if and only if α is convex.

First we show that the total curvature of any curve is at least 2π. To this end recall that when t is the arclength parameter $\kappa(t) = \|T'(t)\|$. Thus the total curvature is simply the total length of the tantrix curve $T\colon I \to \mathbf{S}^2$. Since T is a closed curve, to show that its total length is bigger than 2π, it suffices to check that the image of T does not lie in any semicircle.

Exercise 11. Verify the last sentence.

To see that the image of T does not lie in any semicircle, let $u \in \mathbf{S}^1$ be a unit vector and note that

$$\int_a^b \langle T(t), u\rangle dt = \int_a^b \langle \alpha'(t), u\rangle dt = \langle \alpha(b) - \alpha(a), u\rangle = 0.$$

Since $T(t)$ is not constant (why?), the function $t \mapsto \langle T(t), u\rangle$ must change sign. So the image of T must lie on both sides of the line through the origin and orthogonal to u. Since u was chosen arbitrarily, it follows that the image of T does not lie in any semicircle, as desired.

Next we show that the total curvature is 2π if and only if α is convex. The "if" part has been established already in Exercise 9. To prove the "only if" part, suppose towards a contradiction that α is not convex. Then there exists a tangent line ℓ_0 of α, say at $\alpha(t_0)$, with respect to which the image of α lies on both sides. So α must have two more tangent lines parallel to ℓ_0.

Exercise 12. Verify the last sentence (*Hint*: Let u be a unit vector orthogonal to ℓ and note that the function $t \mapsto \langle \alpha(t) - \alpha(t_0), u\rangle$ must have a minimum and a maximum different from 0. Thus the derivative at these two points vanishes.)

Now that we have established that α has three distinct parallel lines, it follows that it must have at least two parallel tangents. This observation is worth recording:

Lemma 13. *If $\alpha\colon I \to \mathbf{R}^2$ is a closed curve which is* not *convex, then it has a pair of parallel tangent vectors which generate distinct parallel lines.*

Next note that

Exercise 14. If $\alpha \colon I \to \mathbf{R}^2$ is closed curve whose tantrix $T \colon I \to \mathbf{S}^1$ is not onto, then the total curvature is bigger than 2π. (*Hint*: This is immediate consequence of the fact that T is a closed curve and it does not lie in any semicircle)

So if T is not onto then we are done (recall that we are trying to show that if α is not convex, then its total curvature is bigger than 2π). We may assume, therefore, that T is onto. This together with the above lemma yields that the total curvature is bigger than 2π. To see this let t_1, $t_2 \in I$ be the two points such that $T(t_1)$ and $T(t_2)$ are parallel and the corresponding tangent lines are distinct. Then T restricted to $[t_1, t_2]$ is a closed nonconstant curve. So $T([t_1, t_2])$ either (i) covers some open segment of the circle twice or (ii) covers the entire circle. Since we have established that T is onto, the first possibility implies that the length of T is bigger than 2π and we are done. Furthermore, since T restricted to $I - (t_1, t_2)$ is not constant, the second possibility (ii) would again imply case (i). Hence we conclude that if α is not convex, then its total curvature is bigger than 2π, which completes the proof of Theorem 10.

Corollary 15. *Any simple closed curve* $\alpha \colon I \to \mathbf{R}^2$ *is convex if and only if its signed curvature does not change sign.*

Proof. Since α is simple, its total signed curvature is $\pm 2\pi$ by Hopf's theorem. After switching the orientation of α, if necessary, we may assume that the total signed curvature is 2π. Suppose, towards a contradiction, that the signed curvature does change sign. The integral of the signed curvature over the regions where it is positive must be bigger than 2π, which in turn implies that the total curvature is bigger than 2π, which contradicts the previous theorem. So if α is convex, then $\overline{\kappa}$ does not change sign.

Next suppose that $\overline{\kappa}$ does not change sign. Then the total signed curvature is equal to the total curvature (up to a sign), which, since the curve is simple, implies, via Hopf's theorem, that the total curvature is 2π. So by the previous theorem the curve is convex. $\square$