Math 4441 Aug 28, 2007[1]
Differential Geometry
Fall 2007, Georgia Tech

Lecture Notes 1

1 Curves

1.1 Definition and Examples

By an *interval* I we mean a connected subset of $\mathbf{R}$. We usually assume that $I = [a, b]$. A (parametrized) *curve* (in Euclidean space) is a continuous mapping $\alpha\colon I \to \mathbf{R}^n$. We also use the notation

$$I \ni t \stackrel{\alpha}{\longmapsto} \alpha(t) \in \mathbf{R}^n,$$

which emphasizes that α sends each element of the interval I to a certain point in $\mathbf{R}^n$. We say that α is (of the class of) C^k provided that it is k times continuously differentiable. Whenever we need to differentiate α we will assume that α is differentiable up to however many orders that are required.

Some standard examples of curves are a *line* which passes through a point $p \in \mathbf{R}^n$, is parallel to the vector $v \in \mathbf{R}^n$, and has constant speed $\|v\|$

$$[0, 2\pi] \ni t \stackrel{\alpha}{\longmapsto} p + tv \in \mathbf{R}^n;$$

a *circle* of radius $\mathbf{R}$ in the plane, which is oriented counterclockwise,

$$[0, 2\pi] \ni t \stackrel{\alpha}{\longmapsto} \big(r\cos(t), r\sin(t)\big) \in \mathbf{R}^2;$$

and the right handed *helix* (or corkscrew) given by

$$\mathbf{R} \ni t \stackrel{\alpha}{\longmapsto} \big(r\cos(t), r\sin(t), t\big) \in \mathbf{R}^3.$$

Other famous examples include the *figure-eight* curve

$$[0, 2\pi] \ni t \stackrel{\alpha}{\longmapsto} \big(\sin(t), \sin(2t)\big) \in \mathbf{R}^2,$$

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the *logarithmic spiral*

$$[0,\infty) \ni t \stackrel{\alpha}{\longmapsto} (e^{-t}\cos(t), e^{-t}\sin(t)) \in \mathbf{R}^2,$$

the *parabola*

$$\mathbf{R} \ni t \stackrel{\alpha}{\longmapsto} \big(t, t^2\big) \in \mathbf{R}^2,$$

and the *cubic curve*

$$\mathbf{R} \ni t \stackrel{\alpha}{\longmapsto} \big(t, t^2, t^3\big) \in \mathbf{R}^3.$$

Exercise 1. Sketch the cubic curve (*Hint:* First draw each of the projections into the xy, yz, and zx planes).

Exercise 2 (Cycloid). Find a formula for the curve which is traced by the motion of a fixed point on a circle of radius r rolling with constant speed on a line. This curve is known as the *cycloid*, which arises as the solution to the *brachistochrone* problem. (*Hint:* Add the formula for a circle to the formula for a line generated by the motion of the center of the wheel. You only need to make sure that the speed of the line correctly matches the speed of the circle).

Exercise 3. Let $\alpha\colon I \to \mathbf{R}^n$, and $\beta\colon J \to \mathbf{R}^n$ be a pair of differentiable *curves*. Show that

$$\big\langle \alpha(t), \beta(t)\big\rangle' = \big\langle \alpha'(t), \beta(t)\big\rangle + \big\langle \alpha(t), \beta'(t)\big\rangle$$

and

$$\|\alpha(t)\|' = \frac{\big\langle \alpha(t), \alpha'(t)\big\rangle}{\|\alpha(t)\|}.$$

(*Hint:* The first identity follows immediately from the definition of the inner-product, together with the ordinary product rule for derivatives. The second identity follows from the first once we recall that $\|\cdot\| := \langle\cdot,\cdot\rangle^{1/2}$).

Exercise 4. Show that if α has unit speed, i.e., $\|\alpha'(t)\| = 1$, then its velocity and acceleration are orthogonal, i.e., $\langle \alpha'(t), \alpha''(t)\rangle = 0$.

Exercise 5. Show that if the position vector and velocity of a planar curve $\alpha\colon I \to \mathbf{R}^2$ are always perpendicular, i.e., $\langle \alpha(t), \alpha'(t)\rangle = 0$, for all $t \in I$, then $\alpha(I)$ lies on a circle centered at the origin of $\mathbf{R}^2$.

Exercise 6. Use the fundamental theorem of Calculus for real valued functions to show:

$$\alpha(b) - \alpha(a) = \int_a^b \alpha'(t)\, dt.$$

Exercise 7. Prove that

$$\|\alpha(b) - \alpha(a)\| \leq \int_a^b \|\alpha'(t)\|\, dt.$$

(*Hint:* Use the fundamental theorem of calculus and the Cauchy-Schwarz inequality to show that for any unit vector $u \in \mathbf{R}^n$,

$$\big\langle \alpha(b) - \alpha(a), u \big\rangle = \int_a^b \langle \alpha'(t), u \rangle dt \leq \int_a^b \|\alpha'(t)\| dt.$$

Then set $u := (\alpha(b) - \alpha(a))/\|\alpha(b) - \alpha(a)\|$.

The previous exercise immediately yields the following theorem. Here 'sup' denotes supremum or the least upper bound.

Theorem 8 (Mean Value Theorem for curves). *If $\alpha\colon I \to \mathbf{R}^n$ is a C^1 curve, then for every t, $s \in I$,*

$$\|\alpha(t) - \alpha(s)\| \leq \sup_{[t,s]} \|\alpha'\| |t - s|.$$

1.2 Reparametrization

We say that $\beta\colon J \to \mathbf{R}^n$ is a *reparametrization* of $\alpha\colon I \to \mathbf{R}^n$ provided that there exists a smooth bijection $\theta\colon I \to J$ such that $\alpha(t) = \beta\big(\theta(t)\big)$. In other words, the following diagram commutes:

$$\begin{tikzcd} & \mathbf{R}^n & \\ I \arrow[ur, "\alpha"] \arrow[rr, "\theta"'] & & J \arrow[ul, "\beta"'] \end{tikzcd}$$

For instance $\beta(t) = (\cot(2t), \sin(2t))$, $0 \leq t \leq \pi$, is a reparametrization of $\alpha(t) = (\sin(t), \cos(t))$, $0 \leq t \leq 2\pi$, with $\theta\colon [0, 2\pi] \to [0, \pi]$ given by $\theta(t) = t/2$.

The *geometric quantities* associated to a curve do not change under reparametrization. These include length and curvature as we define below.

1.3 Length

By a *partition* P of an interval $[a,b]$ we mean a collection of points $\{t_0,\dots,t_n\}$ of $[a,b]$ such that

$$a = t_0 < t_1 < \cdots < t_n = b.$$

The *approximation of the length of* α *with respect to* P is defined as

$$\operatorname{length}[\alpha, P] := \sum_{i=1}^{n} \|\alpha(t_i) - \alpha(t_{i-1})\|,$$

and if Partition$[a,b]$ denotes the set of all partitions of $[a,b]$, then the *length* of α is given by

$$\operatorname{length}[\alpha] := \sup\big\{\ \operatorname{length}[\alpha, P] \mid P \in \operatorname{Partition}[a,b]\ \big\}.$$

Exercise 9. Show that the shortest curve between any pairs of points in $\mathbf{R}^n$ is the straight line segment joining them. (*Hint:* Use the triangle inequality).

We say that a curve is *rectifiable* if it has finite length.

Exercise* 10 (Nonrectifiable curves). Show that there exists a curve $\alpha\colon [0,1] \to \mathbf{R}^2$ which is not rectifiable (*Hint:* One such curve, known as the *Koch curve* (Figure 1), may be obtained as the limit of a sequence of curves $\alpha_i\colon [0,1] \to \mathbf{R}$ defined as follows. Let α_0 trace the line segment $[0,1]$. Consider an equilateral triangle of sides $1/3$ whose base rests on the middle third of $[0,1]$. Deleting this middle third from the interval and the triangle yields the curve traced by α_1.

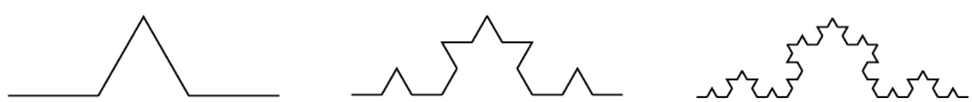

Figure 1:

Repeating this procedure on each of the 4 subsegments of α_1 yields α_2. Similarly α_{i+1} is obtained from α_i. You need to show that α_i converge to a (continuous) curve, which may be done using the Arzela-Ascoli theorem. It is easy to see that this limit has infinite length, because the length of α_i is $(4/3)^i$. Another example of a nonrectifiable curve $\alpha\colon [0,1] \to \mathbf{R}^2$ is given by $\alpha(t) := (t, t\sin(\pi/t))$, when $t \neq 0$, and $\alpha(t) := (0,0)$ otherwise. The difficulty here is to show that the length is infinite.)

If a curve is C^1, then its length may be computed as the following theorem shows. Note also that the following theorem shows that a C^1 curve over a compact domain is rectifiable. First we need the following fact. A function $f\colon X \to Y$, where X and Y are metric spaces, is called *uniformly continuous* provided that for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $d_X(u,v) < \delta$ then $d_Y(f(u), f(v)) < \epsilon$ for all $u, v \in X$.

Lemma 11 (**Heine-Cantor Theorem**). *If $f\colon X \to Y$ is continuous and X is compact then f is uniformly continuous.*

Proof. Fix $\epsilon > 0$. For each $x \in X$, continuity of f at x gives a number $\delta_x > 0$ such that if $d_X(y,x) < \delta_x$ then $d_Y\big(f(y), f(x)\big) < \epsilon/2$. Let $B_X(x,r)$ denote the open ball of radius r centered at the point x of X. Then $B_X(x, \delta_x/2)$ cover X. By compactness, there exist finitely many points $x_1, \dots, x_m \in X$ such that $B_X(x_i, \delta_{x_i}/2)$ cover X. Set $\delta = \min_{1\le i\le m} \delta_{x_i}/2$. Now let $u, v \in X$ with $d_X(u,v) < \delta$. Choose i such that $u \in B_X(x_i, \delta_{x_i}/2)$. Then

$$d_X(v, x_i) \le d_X(v,u) + d_X(u, x_i) < \delta + \delta_{x_i}/2 \le \delta_{x_i}.$$

Thus both u and v lie in $B_X(x_i, \delta_{x_i})$. Therefore,

$$d_Y\big(f(u), f(v)\big) \le d_Y\big(f(u), f(x_i)\big) + d_Y\big(f(x_i), f(v)\big) < \epsilon/2 + \epsilon/2 = \epsilon.$$

□

Theorem 12 (**Length of C^1 curves**). *If $\alpha\colon I \to \mathbf{R}^n$ is a C^1 curve, then*

$$\text{length}[\alpha] = \int_I \|\alpha'(t)\|\, dt. \tag{1}$$

Proof. [2] It suffices to show that (i) length$[\alpha, P]$ is not greater than the above integral, for any $P \in$ Partition$[a,b]$, and (ii) there exists a sequence P_N of partitions such that $\lim_{N\to\infty}$ length$[\alpha, P_N]$ is equal to the integral. The first part follows quickly from Exercise 7. To prove the second part, let P_N be a partition given by $t_i := a + i(b-a)/N$. Recall that, by the definition of integral, for any $\epsilon > 0$, we may choose N large enough so that

$$\left| \int_I \|\alpha'(t)\|\, dt - \sum_{i=1}^{N} \|\alpha'(t_i)\| \frac{b-a}{N} \right| \le \frac{\epsilon}{2}. \tag{2}$$

[2] I thank Patrick Massot for pointing out an error in an earlier draft of this proof

Next note that by the triangle inequality, and since $(b-a)/N = t_{i+1} - t_i$

$$\left|\text{length}[\alpha, P_N] - \sum_{i=1}^{N} \|\alpha'(t_i)\| \frac{b-a}{N}\right| = \left|\sum_{i=1}^{N} \left(\|\alpha(t_{i+1}) - \alpha(t_i)\| - \|\alpha'(t_i)\| \frac{b-a}{N} \right)\right|$$
$$\leq \frac{b-a}{N} \sum_{i=1}^{N} \left| \frac{\|\alpha(t_{i+1}) - \alpha(t_i)\|}{t_{i+1} - t_i} - \|\alpha'(t_i)\| \right|.$$

But, choosing N sufficiently large, we may assume that

$$\left| \frac{\|\alpha(t_{i+1}) - \alpha(t_i)\|}{t_{i+1} - t_i} - \|\alpha'(t_i)\| \right| \leq \frac{\epsilon}{2(b-a)},$$

since $\|\alpha(t_{i+1}) - \alpha(t_i)\|/(t_{i+1} - t_i) \to \|\alpha'(t_i)\|$ as $t_{i+1} \to t_i$. More precisely, the last inequality holds because since α is C^1, the function $F\colon [a,b] \times [a,b] \to \mathbf{R}$ given by $F(t,s) := \|\alpha(t) - \alpha(s)\|/(t-s)$ for $t \neq s$, and $F(t,t) := \|\alpha'(t)\|$ is continuous, and therefore is uniformly continuous by Lemma 11 since $[a,b] \times [a,b]$ is compact. So we conclude that

$$\left|\text{length}[\alpha, P_N] - \sum_{i=1}^{N} \|\alpha'(t_i)\| \frac{b-a}{N}\right| \leq \frac{\epsilon}{2}. \tag{3}$$

Finally (2) and (3) yield that,

$$\left| \int_I \|\alpha'(t)\|\, dt - \text{length}[\alpha, P_N] \right| \leq \epsilon$$

which completes the proof. □

Exercise 13. Compute the length of a circle of radius r, and the length of one cycle of the cycloid curve discussed in Exercise 2.

Exercise 14 (Invariance of length under reparametrization). Show that if $\beta\colon J \to \mathbf{R}^n$ is a reparametrization of a C^1 curve $\alpha\colon I \to \mathbf{R}^n$, then $\text{length}[\beta] = \text{length}[\alpha]$, i.e., length is invariant under reparametrization (*Hint*: you only need to recall the chain rule together with the integration by substitution.)

Let $L := \text{length}[\alpha]$. The *arclength* function of α is a mapping $s\colon [a,b] \to [0, L]$ given by

$$s(t) := \int_a^t \|\alpha'(u)\|\, du.$$

Thus $s(t)$ is the length of the subsegment of α which stretches from the initial time a to time t.

Exercise 15 (Regular curves). Show that if α is a *regular* curve, i.e., $\|\alpha'(t)\| \neq 0$ for all $t \in I$, then $s(t)$ is an invertible function, i.e., it is one-to-one (*Hint*: compute $s'(t)$).

Exercise 16 (Reparametrization by arclength). Show that every regular curve $\alpha\colon [a,b] \to \mathbf{R}^n$, may be reparametrized by arclength (*Hint:* Define $\beta\colon [0,L] \to \mathbf{R}^n$ by $\beta(t) := \alpha(s^{-1}(t))$, and use the chain rule to show that $\|\beta'\| = 1$; you also need to recall that since $f(f^{-1}(t)) = t$, then, again by the chain rule, we have $(f^{-1})'(t) = 1/f'(f^{-1}(t))$ for any smooth function f with nonvanishing derivative.)

1.4 Cauchy-Crofton formula and curves of constant width

Let $\alpha\colon I \to \mathbf{R}^2$ be a curve and $u(\theta) := (\cos(\theta), \sin(\theta))$ be a unit vector. The projection of α into the line passing through the origin and parallel to u is given by

$$\alpha_u(t) := \langle \alpha(t), u \rangle u.$$

For any function $f\colon \mathbf{S}^1 \to \mathbf{R}$ we define

$$\operatorname*{ave}_{\mathbf{S}^1}(f) := \frac{1}{2\pi} \int_0^{2\pi} f(u(\theta))\, d\theta.$$

Exercise 17 (Cauchy-Crofton formula). Show that if $\alpha\colon I \to \mathbf{R}^2$ is a rectifiable curve, then

$$\operatorname*{ave}_{\mathbf{S}^1}\big(\operatorname{length}[\alpha_u]\big) = \frac{2}{\pi} \operatorname{length}[\alpha].$$

(*Hint*: First prove this fact for the case when α traces a line segment. Then a limiting argument settles the general case, once you recall the definition of length. Also give a more direct proof in case α is C^1 which ensures that the length formula (1) holds.

As an application of the above formula we may obtain a sharp inequality involving the width of closed curves. The *width* of a set $X \subset \mathbf{R}^2$ is the distance between the closest pairs of parallel lines which contain X in between them. For instance the width of a circle of radius r is $2r$. A curve $\alpha\colon [a,b] \to \mathbf{R}^2$ is *closed* provided that $\alpha(a) = \alpha(b)$. We should also mention that α is a C^k closed curve provided that the (one-sided) derivatives of α match up at a and b.

Exercise 18 (**Width and length**)**.** Show that if $\alpha\colon I \to \mathbf{R}^2$ is a closed curve with width w and length L, then

$$w \leq \frac{L}{\pi}. \tag{4}$$

Note that the above inequality is sharp, since for circles $w = L/\pi$. Are there other curves satisfying this property? The answer may surprise you. For any unit vector $u \in \mathbf{S}^1$, the *width* of a set $X \subset \mathbf{R}^2$ in the direction u, w_u, is defined as the distance between the closest pairs of lines which contain X in between them. We say that a closed curve in the plane has *constant width* provided that w_u is constant for all directions $u \in \mathbf{S}^1$. For instance circles have constant width, but there are other examples:

Exercise 19 (**Reuleaux triangle**)**.** Consider three disks of radius r whose centers are on an equilateral triangle of sides r, see Figure 2. Show that

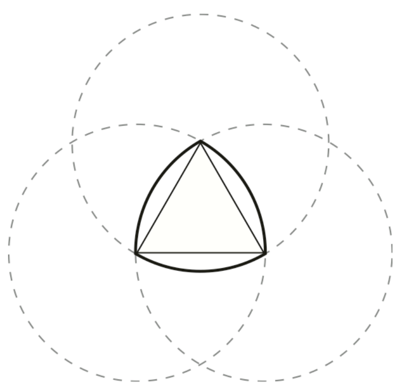

Figure 2:

the curve which bounds the intersection of these disks has constant width. Also show that similar constructions for any regular polygon yield curves of constant width.

Note that the Reuleaux triangle is not a C^1 regular curve for it has sharp corners. To obtain a C^1 example of a curve of constant width, we may take a curve which is a constant distance away from the Reuleaux triangle. Further, a C^∞ example may be constructed by taking an *evolute* of a *deltoid*, see [1, p. 174]. It can be shown that of all curves of constant width w, Reuleaux triangle has the least area. This is known as the Blaschke-Lebesque theorem. A more recent proof of this result has been obtained by Evans Harrell [2].

Exercise 20. Show that if equality holds in (4) then α has constant width.

The converse of the fact stated in the last exercise holds as well, that is for all curves of constant with $L = \pi w$. In other words, all curves with the same width have the same length, which is known as Barbier's theorem.

References

[1] A. Gray, E. Abbena, and S. Salamon, *Modern differential geometry of curves and surfaces with Mathematica®*, Third, Studies in Advanced Mathematics, Chapman & Hall/CRC, Boca Raton, FL, 2006. MR2253203 ↑9

[2] E. M. Harrell II, *A direct proof of a theorem of Blaschke and Lebesgue*, J. Geom. Anal. **12** (2002), no. 1, 81–88. MR1881292 (2002k:52009) ↑9