Math 4441 Sep 14, 2007[1]
Differential Geometry
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Lecture Notes 6

1.15 The four-vertex theorem for convex curves

A *vertex* of a planar curve $\alpha\colon I \to \mathbf{R}^2$ is a point where the signed curvature of α has a local max or min.

Exercise 1. Show that an ellipse has exactly 4 vertices, unless it is a circle.

Recall that a planar curve is *convex* if through each point in the image of it there passes a line with respect to which the curve lies on one side. The main aim of this section is to show that:

Theorem 2. *Any convex C^3 planar curve has (at least) four vertices.*

In fact any simple closed curve has 4 vertices, and it is not necessary to assume that κ is C^1, but the proof is harder. On the other hand if the curve is not simple, then the 4 vertex property may no longer be true:

Exercise 3. Sketch the limacon $\alpha\colon [0, 2\pi] \to \mathbf{R}^2$ given by

$$\alpha(t) := (2\cos t + 1)(\cos t, \sin t)$$

and show that it has only two vertices. (*Hint*: It looks like a loop with a smaller loop inside)

The proof of the above theorem is by contradiction. Suppose that α has fewer than 4 vertices, then it must have exactly 2.

Exercise 4. Verify the last sentence.

Suppose that these two vertices occur at t_0 and t_1. Then $\kappa'(t)$ will have one sign on (t_1, t_2) and the opposite sign on $I - [t_1, t_2]$. Let ℓ be the line passing through $\alpha(t_1)$ and $\alpha(t_2)$. Then, since α is convex, α restricted to (t_1, t_2) lies entirely in one of the closed half planes determined by ℓ and α restricted to $I - [t_1, t_2]$ lies in the other closed half plane.

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Exercise 5. Verify the last sentence, i.e., show that if $\alpha\colon I \to \mathbf{R}^2$ is a simple closed convex planar curve, and ℓ is any line in the plane which intersects $\alpha(I)$, then either ℓ intersects α in exactly two points, or $\alpha(I)$ lies on one side ℓ.(*Hint*: Show that if α intersects ℓ at 3 points, then it lies on one side of ℓ.)

Let p be a point of ℓ and v be a vector orthogonal to ℓ, then $f\colon I \to \mathbf{R}$, given by $f(t) := \langle \alpha(t) - p, v \rangle$ has one sign on (t_1, t_2) and has the opposite sign on $I - [t_1, t_2]$. Consequently, $\kappa'(t) f(t)$ is always nonnegative. So

$$0 < \int_I \kappa'(t) \langle \alpha(t) - p, v \rangle dt.$$

On the other hand

$$\begin{aligned} \int_I \kappa'(t) \langle \alpha(t) - p, v \rangle dt &= \kappa(t) \langle \alpha(t) - p, v \rangle |_a^b - \int_I \kappa(t) \langle T(t), v \rangle dt \\ &= 0 - \int_I \langle -N'(t), v \rangle dt \\ &= \langle N(t), v \rangle |_a^b \\ &= 0. \end{aligned}$$

So we have a contradiction, as desired.

Exercise 6. Justify each of the lines in the above computation.

1.16 Schur's Arm Lemma

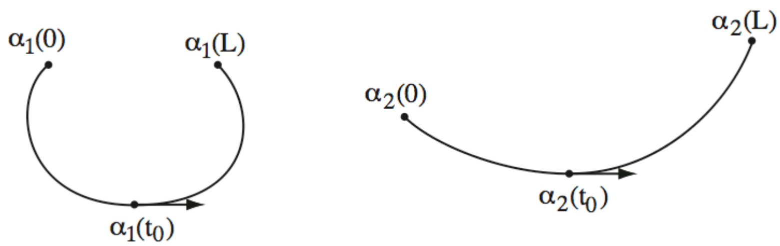
The following result describes how the distance between the end points of a planar curve is affected by its curvature:

Theorem 7 (Schur's Arm Lemma)**.** *Let α_1, $\alpha_2\colon [0, L] \to \mathbf{R}^3$ be unit speed C^1 curves such that the union of each α_i with the line segment from $\alpha_i(0)$ to $\alpha_i(L)$ is a convex curve. Suppose that for almost all $t \in [0, L]$, $\kappa_i(t)$ is well defined, e.g., α_i is piecewise C^2, and*

$$\kappa_1(t) \geq \kappa_2(t)$$

for almost all $t \in [0, L]$. Then

$$\operatorname{dist}\Big(\alpha_1(0), \alpha_1(L)\Big) \leq \operatorname{dist}\Big(\alpha_2(0), \alpha_2(L)\Big).$$



Proof. After a rigid motion we may assume that the segment $\alpha_1(0)\alpha_1(L)$ is parallel to the x-axis and α_1' is rotating counterclockwise, see the picture below. Then there exists $t_0 \in [0, L]$ such that $\alpha_1'(t_0)$ is horizontal. After a rigid motion, we may assume that $\alpha_2'(t_0)$ is horizontal as well. Now let θ_i be the angle that α_i' makes with the positive direction of the x-axis measured counterclockwise. Then $\theta_i \in [-\pi, \pi]$ (for θ_1 this follows from convexity of α_1, and for θ_2, this follows from the assumption that $\kappa_2 \leq \kappa_1$). Further note that

$$|\theta_i(t)| = |\theta_i(t) - \theta_i(t_0)| = \left| \int_{t_0}^{t} \theta_i'(s)\, ds \right| = \left| \int_{t_0}^{t} \kappa_i(s)\, ds \right|.$$

Thus $|\theta_1(t)| \geq |\theta_2(t)|$, and, since $|\theta_i(t)| \in [0, \pi]$, it follows that

$$\cos|\theta_1(t)| \leq \cos|\theta_2(t)|.$$

Finally note that, if we set $e_1 := (1, 0)$, then

$$\begin{aligned}
\|\alpha_1(L) - \alpha_1(0)\| &= \big\langle \alpha_1(L) - \alpha_1(0), e_1 \big\rangle \\
&= \int_0^L \langle \alpha_1'(t), e_1 \rangle \, dt \\
&= \int_0^L \cos|\theta_1(t)| \, dt \\
&\leq \int_0^L \cos|\theta_2(t)| \, dt \\
&= \int_0^L \langle \alpha_2'(t), e_1 \rangle \, dt \\
&= \big\langle \alpha_2(L) - \alpha_2(0), e_1 \big\rangle \leq \|\alpha_2(L) - \alpha_2(0)\|.
\end{aligned}$$

□

Exercise 8. Is Schur's arm lemma true for nonconvex arcs?

Exercise 9. Prove the four-vertex theorem for convex curves using the Schur's arm lemma.

Exercise* 10. Prove the polygonal version of the Schur's arm lemma: Suppose that we have a pair of polygonal arcs P_1 and P_2 in the plane, each of which is convex (i.e., when we connect the end points of each arc, then we obtain a closed convex curve). Further suppose that these curves have the same number of segments and the corresponding segments (if we order them consecutively) have the same length. Now show that if the exterior angles in P_1 are smaller than the corresponding angles in P_2, then the distance between the end points of P_1 is larger than the distance between the end points of P_2.

1.17 The four-vertex theorem for general curves

In this section we generalize the four-vertex theorem which was proved earlier. First we need the following result. An *inflection point* is a point where the signed curvature changes sign.

Lemma 11. *Let $\alpha\colon [a,b] \to \mathbf{R}^2$ be a simple C^2 curve. Suppose that $\alpha(a)$ and $\alpha(b)$ both lie on a line ℓ with respect to which the image of α lies on one side. Further suppose that $\alpha'(a)$ and $\alpha'(b)$ are parallel. Then either the image of α is a line segment, or else α has at least two inflection points.*

A *support line* of a set $A \subset \mathbf{R}^2$ is a line with respect to which A lies on one side and intersects A at some point.

Exercise* 12. Prove the above lemma.

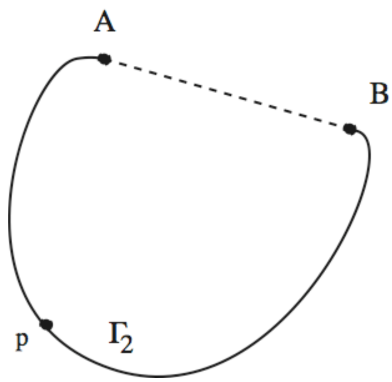
Lemma 13. *Let Γ be a simple closed C^2 curve in the plane. Suppose that every support line of Γ intersects Γ in a single point. Then Γ is convex.*

Exercise* 14. Prove the above lemma.

Theorem 15. *Every simple closed C^2 planar curve has four vertices.*

Proof. We may suppose that the signed curvature of our curve Γ changes sign at most twice, because there has to be a vertex between every pair of inflection points. Since Γ is not convex, there exists by the above lemma a support line ℓ which is tangent to Γ at two distinct points say A and B. Since Γ is simple there must be a portion of Γ, say Γ_1 bounded by A and

B so that the unit tangent vectors of Γ_1 at A and B, with respect to some parametrization, are parallel. Then by the above lemma Γ_1 must contain both inflection points of Γ. Consequently, the complement of Γ_1 say Γ_2 has no inflection points and it follows that the union of Γ_2 with the line segment AB is a closed convex curve, see the picture below.



It is enough to show that the interior of Γ_2 contains at least two vertices, because Γ_1 already contains at least one vertex (since a vertex must be between every pair of inflection points), and the total number of vertices must be even.

First we show that the interior of Γ_2 must contain at least one vertex. Suppose not. Then the curvature of Γ_2 is monotone, so its minimum must be either at A or B. Suppose that the minimum of curvature is at A. Let A' be the point in Γ_2 so that A and A' divide $\Gamma_2 \cup AB$ into portions of same length. In one of these portions the curvature is less than the other, which contradicts the Arm lemma proved earlier.

So there exists a point p in the interior of Γ_2 which is a vertex. Suppose that Γ has only two vertices. Then it follows that p must be a maximum point of curvature of Γ_2. Suppose that the length of the arc pA in Γ_2 is not bigger than the length of pB. Let $x(t)$ be a parametrization of pA from p to A, and let $y(t)$ be a point of pB, if one exists, so that the curvature at $y(t)$ is equal to the curvature at $x(t)$. Suppose there exists a time t_1 so that $x(t_1)$ and $y(t_1)$ divide the length of $\Gamma_2 \cup AB$ in half. Then one of the portions determined by $x(t_1)$ and $y(t_1)$ will have smaller curvature than other at every point, which contradicts the Arm Lemma.

So we may suppose that t_1 does not exist. This implies that there exists a point A' in the segment pB of Γ_2 such that the curvature at A' is equal to the curvature at A and the length of AA' in Γ_2 is less than half the length of $\Gamma_2 \cup AB$. In this case, let A'' be the point in the segment $A'B$ of Γ_2 so that A and A'' divide the length of $\Gamma_2 \cup AB$ in half. Then one of the portions determined by A and A'' will have everywhere bigger curvature than the other, which is again a contradiction. $\square$

1.18 Area of planar regions and the Isoperimetric inequality

The area of a rectangle is defined as the product of lengths of two of its adjacent sides. Let $X \subset \mathbf{R}^2$ be any set, R be a collection of rectangles which cover X, and $Area(X, R)$ be the sum of the areas of all rectangles in R. Then area of X is defined as the infimum of $Area(X, R)$ where R ranges over all different ways to cover X by rectangles. In particular it follows that, if $X \subset Y$, then $Area(X) \leq Area(Y)$, and if $X = X_1 \cup X_2$, then $Area(X) = Area(X_1) + Area(X_2)$. These in turn quickly yield the areas of triangles and polygons.

Exercise 16 (Invariance under isometry and the Special linear group)**.** Show that area is invariant under rigid motions of $\mathbf{R}^2$, and that dilation by a factor of r, i.e., multiplying each point $\mathbf{R}^2$ by r, changes the area by a factor of r^2. More generally show that any linear transformation $A\colon \mathbf{R}^2 \to \mathbf{R}^2$ changes the area by a factor of $\det(A)$.

Exercise 17 (Archimedes)**.** Compute the area of a circle (*Hint:* For any n compute the area of regular n-gons which are inscribed in the circle, and take the limit. Each of these areas is the sum of n isosceles triangles with an angle $2\pi/n$, and adjacent sides of length equal to the radius of the circle. This gives a lower bound for the area. An upper bound may also be obtained by taking the limit of regular polygons which circumscribe the circle.)

Recalling the definition of Riemann sums, one quickly observes that

$$Area(X) = \int\int_X 1\, dxdy.$$

We say that a subset X of $\mathbf{R}^n$ is *connected* provided that the only subsets of X which are both open and closed in X are the X and the empty set. Every subset of X which is connected and is different from X and the empty set is called a *component* of X.

Let $\alpha\colon I \to \mathbf{R}^2$ be a simple closed planar curve. By the Jordan curve theorem (which we will not prove here), $\mathbf{R}^2 - \alpha(I)$ consists of exactly two connected components, and the boundary of each component is $\alpha(I)$. Further, one of these components, which we call the *interior* of α, is contained in some large sphere, while the other is unbounded. By area of α we mean the area of its interior.

Theorem 18. *For any simple closed planar curve $\alpha\colon I \to \mathbf{R}^2$,*

$$\mathit{Area}[\alpha] \leq \frac{\mathit{Length}[\alpha]^2}{4\pi}.$$

Equality holds only when α is a circle.

Our proof of the above theorem hinges on the following subtle fact whose proof we leave out

Lemma 19. *Of all simple closed curves of fixed length L, there exists at least one with the biggest area. Further, every such curve is C^1.*

Exercise* 20. Show that the area maximizer (for a fixed length) must be convex. (*Hint*: It is enough to show that if the maximizer, say α, is not convex, then there exist a line ℓ with respect to which $\alpha(I)$ lies on one side, and intersects $\alpha(I)$ at two points p and q but not in the intervening open segment of ℓ determined by p and q. Then reflecting one of the segments of $\alpha(I)$, determined by p and q, through ℓ increases area while leaving the length unchanged.)

We say that α is symmetric with respect to a line ℓ provided that the image of α is invariant under reflection with respect to ℓ.

Exercise 21. Show that a C^1 convex planar curve $\alpha\colon I \to \mathbf{R}^2$ is a circle, if and only if for every unit vector $u \in \mathbf{S}^1$ there exists a line perpendicular to u with respect to which α is symmetric (*Hint:* Suppose that α has a line of symmetry in every direction. First show that each line of symmetry is unique in the corresponding direction. After a translation we may assume that α is symmetric with respect to both the x-axis and the y-axis. Show that this yields that α is symmetric with respect to the origin, i.e. rotation by 180°. From this and the uniqueness of the lines of symmetry conclude that every line of symmetry passes through the origin. Finally show that each line of symmetry must meet the curve orthogonally at the intersection points. This shows that $\langle \alpha(t), \alpha'(t) \rangle = 0$, which in turn yields that $\|\alpha(t)\| = \mathit{const}$.)

Now we are ready to prove the isoperimetric inequality. The proof we give here is based on Steiner's symmetrization technique.

Let $\alpha\colon I \to \mathbf{R}^2$ be an area maximizer. By Exercise 20 we may assume that α is convex. We claim that α must have a line of symmetry in every direction, which would show, by Exercise 21, that α is a circle, and hence would complete the proof.

Suppose, towards a contradiction, that there exists a direction $u \in \mathbf{S}^1$ such that α has no line of symmetry in that direction. After a rigid motion, we may assume that $u = (0, 1)$.

Let $[a, b]$ be the projection of $\alpha(I)$ to the x-axis. Then, since α is convex, every vertical line which passes through an interior point of (a, b) intersects $\alpha(I)$ at precisely two points. Let $f(x)$ be the y-coordinate of the higher point, and $g(x)$ be the y-coordinate of the other points. Then

$$Area[\alpha] = \int_a^b f(x) - g(x)\, dx.$$

Further note that if α is C^1 then f and g are C^1 as well, thus

$$Length[\alpha] = f(a)-g(a)+\int_a^b \sqrt{1+f'(x)^2}\, dx+\int_a^b \sqrt{1+g'(x)^2}\, dx+f(b)-g(b).$$

Now we are going to define a new curve $\overline{\alpha}$ which is composed of the graph of the function $\overline{f}\colon [a, b] \to \mathbf{R}$ given by

$$\overline{f}(x) := \frac{f(x) - g(x)}{2},$$

on top, the graph of $-\overline{f}$ in the bottom, and vertical segments, which may consist only of a single point, on right and left (We may think of this curve as the boundary of the region which is obtained when we move the segments with end points at $f(x)$ and $g(x)$ parallel to themselves until their centers lie on the x-axis). One immediately checks that

$$Area[\overline{\alpha}] = Area[\alpha].$$

Further, note that since by assumption α is not symmetric with respect to the x-axis, $\overline{f}$ is strictly positive on (a, b). This may be used to show that

$$Length[\overline{\alpha}] < Length[\alpha].$$

Exercise 22. Verify the last inequality above (*Hint*: It is enough to check that $\int_a^b \sqrt{1+\overline{f}'(x)^2}dx$ is strictly smaller than either of the integrals in the above formula for the length of α).