Math 497C
Curves and Surfaces
Fall 2004, PSU

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Lecture Notes 14

2.11 The Induced Lie Bracket on Surfaces; The Self-Adjointness of the Shape Operator Revisited

If V, W are tangent vectorfields on M, then we define

$$[V,W]_M := \nabla_V W - \nabla_W V,$$

which is again a tangent vector field on M. Note that since, as we had verified in an earlier exercise, S is self-adjoint, the Gauss's formula yields that

$$\begin{aligned}
[V,W] &= \overline{\nabla}_V W - \overline{\nabla}_W V \\
&= \nabla_W V - \nabla_V W + \Big(\big\langle V, S(W)\big\rangle - \big\langle W, S(V)\big\rangle\Big)n \\
&= [V,W]_M.
\end{aligned}$$

In particular if V and W are tangent vectorfields on M, then $[V,W]$ is also a tangent vectorfield.

Let us also recall here, for the sake of completeness, the proof of the self-adjointness of S. To this end it suffices to show that if E_i, $i = 1, 2$, is a basis for T_pM, then $\langle E_i, S_p(E_j)\rangle = \langle S_p(E_i), E_j\rangle$. In particular we may let $E_i = X_i(0,0)$, where $X\colon U \to M$ is a regular patch of M centered at p. Now note that

$$\big\langle X_i, S_p(X_j)\big\rangle = -\big\langle X_i, dn_p(X_j)\big\rangle = -\big\langle X_i, (n\circ X)_j\big\rangle = \big\langle X_{ij}, (n\circ X)\big\rangle.$$

Since the right hand side of the above expression is symmetric with respect to i and j, the right hand side must be symmetric as well, which completes the proof that S is self-adjoint.

Note that while the above proof is short and elegant one might object to it on the ground that it uses local coordinates. On the other hand, if we can

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give an independent proof that $[V,W]_M = [V,W]$, then we would have an alternative proof that S is self-adjoint. To this end note that

$$[V,W]^\top = \left(\overline{\nabla}_V W\right)^\top - \left(\overline{\nabla}_W V\right)^\top = \nabla_V W - \nabla_W V = [V,W]_M.$$

Thus to prove that $[V,W]_M = [V,W]$ it is enough to show that $[V,W]^\top = [V,W]$, i.e., $[V,W]$ is tangent to M. To see this note that if $f\colon M \to \mathbf{R}$ is any function, and $\overline{f}\colon U \to \mathbf{R}$ denoted an extension of f to an open neighborhood of M, then

$$[V,W]\overline{f} = [V,W]^\top \overline{f} + [V,W]^\perp \overline{f} = [V,W]^\top f + [V,W]^\perp \overline{f}.$$

So if we can show that the left hand side of the above expression depends only on f (not $\overline{f}$), then it would follow that the right hand side must also be independent of $\overline{f}$, which can happen only if $[V,W]^\perp$ vanishes. Hence it remains to show that $[V,W]\overline{f} = [V,W]f$. To see this recall that by a previous exercise

$$[V,W]\overline{f} = V(W\overline{f}) - W(V\overline{f}).$$

But since V and W are tangent to M, $V\overline{f} = Vf$ and $W\overline{f} = Wf$. Thus the right hand side of the above equality depends only on f, which completes the proof.

Exercise 1. Verifythe next to last statement.

2.12 The Riemann Curvature Tensor of Surfaces; The Gauss and Codazzi Mainardi Equations, and Theorema Egregium Revisited

If V, W, Z are tangent vectorfields on M, then

$$R(V,W)Z := \nabla_V \nabla_W Z - \nabla_W \nabla_V Z - \nabla_{[V,W]} Z$$

gives a tangent vectorfield on M. Note that this operation is well defined, because, as we verified in the previous section, $[V,W]$ is tangent to M. If Y is another tangent vectorfield on M, then we may also define an associated scalar quantity by

$$R(V,W,Z,Y) := \left\langle R(V,W)Z, Y \right\rangle,$$

which is the *Riemann curvature tensor* of M, and, as we show below, coincides with the quantity of the same name which we had defined earlier in terms of local coordinates. To this end first recall that

$$\overline{R}(V,W)Z := \overline{\nabla}_V\overline{\nabla}_W Z - \overline{\nabla}_W\overline{\nabla}_V Z - \overline{\nabla}_{[V,W]}Z = 0$$

as we had shown in an earlier exercise. Next note that, by Gauss's formula,

$$\begin{aligned}
\overline{\nabla}_V\overline{\nabla}_W Z &= \overline{\nabla}_V\big(\nabla_W Z + \langle S(W), Z\rangle n\big)\\
&= \overline{\nabla}_V\big(\nabla_W Z\big) + \overline{\nabla}_V\big(\langle S(W), Z\rangle n\big)\\
&= \nabla_V\nabla_W Z + \langle S(V), \nabla_W Z\rangle n + V\langle S(W), Z\rangle n + \langle S(W), Z\rangle \nabla_V n.
\end{aligned}$$

Also recall that, since $\langle n, n\rangle = 1$,

$$\nabla_V n := (\overline{\nabla}_V n)^\top = \overline{\nabla}_V n = dn(V) = S(V).$$

Thus

$$\begin{aligned}
\overline{\nabla}_V\overline{\nabla}_W Z = \nabla_V\nabla_W Z &+ \langle S(W), Z\rangle S(V)\\
&+ \Big(\langle S(V), \nabla_W Z\rangle + \langle \nabla_V S(W), Z\rangle + \langle S(W), \nabla_V Z\rangle\Big)n.
\end{aligned}$$

Similarly,

$$\begin{aligned}
-\overline{\nabla}_W\overline{\nabla}_V Z = -\nabla_W\nabla_V Z &- \langle S(V), Z\rangle S(W)\\
&- \Big(\langle S(W), \nabla_V Z\rangle + \langle \nabla_W S(V), Z\rangle + \langle S(V), \nabla_W Z\rangle\Big)n.
\end{aligned}$$

Also note that

$$-\overline{\nabla}_{[V,W]}Z = -\nabla_{[V,W]}Z - \langle S([V,W]), Z\rangle n.$$

Adding the last three equations yields

$$\begin{aligned}
\overline{R}(V,W)Z = R(V,W)Z &+ \langle S(W), Z\rangle S(V) - \langle S(V), Z\rangle S(W)\\
&+ \Big(\langle \nabla_V S(W), Z\rangle - \langle \nabla_W S(V), Z\rangle - \langle S([V,W]), Z\rangle\Big)n.
\end{aligned}$$

Since the left hand side of the above equation is zero, each of the tangential and normal components of the right hand side must vanish as well. These respectively yield:

$$R(V,W)Z = \langle S(W), Z\rangle S(V) - \langle S(V), Z\rangle S(W)$$

and

$$\nabla_V S(W) - \nabla_W S(V) = S([V, W]),$$

which are the Gauss and Codazzi-Mainardi equations respectively. In particular, in local coordinates they take on the forms which we had derived earlier.

Exercise 2. Verify the last sentence above.

Finally note that by Gauss's equation

$$\langle R(V, W)W, V\rangle = \langle S(V), V\rangle\langle S(W), W\rangle - \langle S(W), V\rangle\langle S(V), W\rangle$$

In particular, if V and W are orthonormal, then

$$\langle R(V, W)W, V\rangle = \det(S) = K.$$

Thus we obtain yet another proof of the Theorema Egregium, which, in this latest reincarnation, does not use local coordinates.

Exercise 3. Show that if V and W are general vectorfields (not necessarily orthonormal), then

$$K = \frac{R(V, W, W, V)}{\|V \times W\|^2}$$