SBA Center

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4th Grade

Operations Research

Lecture 2 after Mid - Term

- **Markov Chains**
- **Game Theory**

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Markov Chains

Markov Analysis: is used to analyze the current state and movement (makes transitions) to predict the future occurrences by the use of presently known probabilities.

الهدف من (Markov Chain) ان انا اتنبأ او اتوقع حجم ال (Market Share) او ال (Market Size) و هل هيزيد ولا هيقل.

و ده بيتم عن طريق تحليل كل منتج و تحليل نسبة (Probability) تحول او انتقال المستهلكين من منتج الى منتج اخر.

و خلي بالك علشان اتنبأ بحاجة في المستقبل بستخدم المعلومات المعايا في الحاضر.

<u>In Early 1960s. This Procedure has been used in:</u>

- Marketing Strategies.
- Maintenance Decisions.
- Stock Market.

Basic Assumption: the future depends only on the present state and not the past.

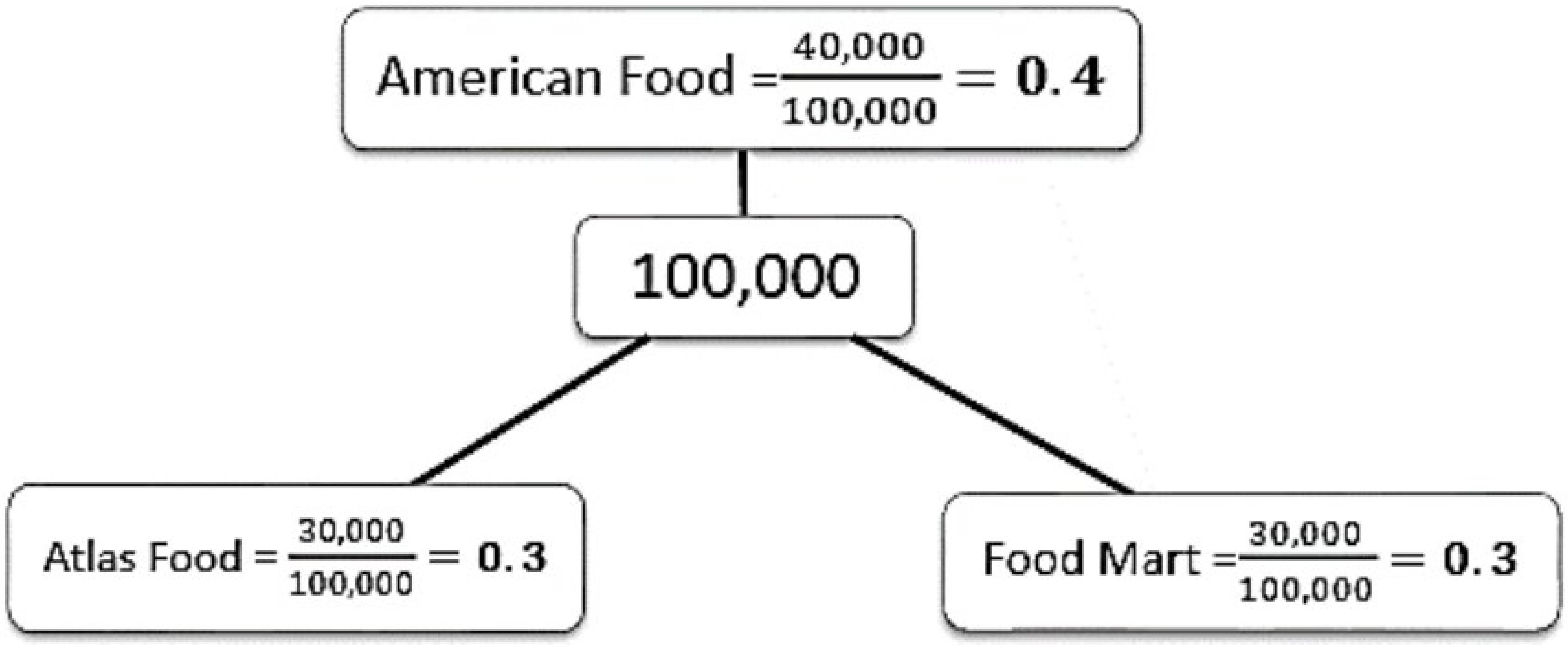
Example 1:

A Total of 100,000 people shop at the three groceries during a given month, **forty thousand** may be shopping at American Food Store, **Thirty Thousand** may be shopping at food mart and **thirty thousand** may be shopping at Atlas Foods. Compute the Stores **Market Share changes** in the **next month** by using the following matrix.

$$\begin{bmatrix} 0.8 & 0.1 & 0.2 \\ 0.1 & 0.7 & 0.2 \\ 0.1 & 0.2 & 0.6 \end{bmatrix}$$

Solution

Market Shares:



These probabilities can be placed in the following vector of state probabilities

$$\pi(0) = (0.4, 0.3, 0.3)$$

(Probability of current state or present) (State Matrix or vector)

Convert to transition matrix:

$$\begin{bmatrix} 0.8 & 0.1 & 0.2 \\ 0.1 & 0.7 & 0.2 \\ 0.1 & 0.2 & 0.6 \end{bmatrix} \longrightarrow \begin{bmatrix} \mathbf{0.8} & \mathbf{0.1} & \mathbf{0.1} \\ \mathbf{0.1} & \mathbf{0.7} & \mathbf{0.2} \\ \mathbf{0.2} & \mathbf{0.2} & \mathbf{0.6} \end{bmatrix}$$

احنا جبنا ال (transition matrix) عن طريق تحويل كل صف الى عمود او العكس

For knowing the new market share:

Multiplying a "**state matrix or state vector**" by the "**transition matrix**" gives the probability of each outcome **<u>(New market share)</u>** ⟶ **π (1)**

$\pi(1) = \pi(0) \times P$

$$[0.4 \quad 0.3 \quad 0.3] \times \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}$$

$$(1 \times 3) \qquad\qquad (3 \times 3)$$

صف واحد و ثلاث عواميد

لازم يكونوا =

ثلاث صفوف و ثلاث عواميد

Solution will be (1 × 3) matrix

=

[(0.4)(0.8) + (0.3)(0.1) + (0.3)(0.2) , (0.4)(0.1) + (0.3)(0.7) + (0.3)(0.2) , (0.4)(0.1) + (0.3)(0.2) + (0.3)(0.6)]

= (0.41, 0.31, 0.28) (Probability of new market share)

New Market Shares:

American Food	**= 100,000 x 0.41 = 41,000**
Food Mart	**= 100,000 x 0.31 = 31,000**
Atlas Foods	**= 100,000 x 0.28 = 28,000**
	100,000

As shown:

Market Shares Increased for American Food & Food Mart & Decreased for Atlas.

Example 2:

The Company Produces Three Products A, B & C, the Total Sales of them were 8,000 Units this year, 3,200, 2,800 & 2,000 Units Respectively. The Company decided to use Markov Chains to predict their Market Share. Use the following Matrix to Predict the Market Share Next Year in Units if:

a) **Market Size will not be changed.**

b) **Market Size will decrease by 30%.**

$$\begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.7 & 0.4 & 0.1 \\ 0.1 & 0.3 & 0.4 \end{bmatrix}$$

Solution

Market Shares:

$A = \frac{3,200}{8,000} = \mathbf{0.4}$ $\qquad B = \frac{2,800}{8,000} = \mathbf{0.35}$ $\qquad C = \frac{2,000}{8,000} = \mathbf{0.25}$

Convert to Transition Matrix:

$$\begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.7 & 0.4 & 0.1 \\ 0.1 & 0.3 & 0.4 \end{bmatrix} \longrightarrow \begin{bmatrix} 0.2 & 0.7 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.5 & 0.1 & 0.4 \end{bmatrix}$$

Probability of New Market Shares:

$$[0.4 \quad 0.35 \quad 0.25] \times \begin{bmatrix} 0.2 & 0.7 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.5 & 0.1 & 0.4 \end{bmatrix}$$

$$= \big[(0.4)(0.2) + (0.35)(0.3) + (0.25)(0.5), \quad (0.4)(0.7) + (0.35)(0.4) + (0.25)(0.1),$$

$$(0.4)(0.1) + (0.35)(0.3) + (0.25)(0.4)\big] = [0.31 \quad 0.445 \quad 0.245]$$

New Market Shares:

a) Market Size will not be changed	b) Market Size will Decreases by 30%
A = 8,000 x 0.31 = 2,480 B = 8,000 x 0.445 = 3,560 A = 8,000 x 0.245 = 1,960 8,000	New Market Size = 8,000 (100% - 30%) = 8,000 × (70%) = 5,600 A = 5,600 x 0.31 = 1,736 B = 5,600 x 0.445 = 2,492 A = 5,600 x 0.245 = 1,372 5,600

Example 3:

The Company Produces Three Products A, B & C. The Total Sales of them were 3,800 Units this Year, 950, 1330 & 1520 units respectively. The Company decided to use Markov Chains to predict their Market Share. Use the following Matrix to Predict the Market Share Next Year in Units if Market Size will **increase by 45%.**

$$\begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.4 & 0.5 & 0.2 \\ 0.4 & 0.2 & 0.3 \end{bmatrix}$$

Solution

Market Shares:

$A = \frac{950}{3{,}800} = 0.25$ $B = \frac{1{,}330}{3{,}800} = 0.35$ $C = \frac{1{,}520}{3{,}800} = 0.4$

Convert to Transition Matrix:

$$\begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.4 & 0.5 & 0.2 \\ 0.4 & 0.2 & 0.3 \end{bmatrix} \longrightarrow \begin{bmatrix} 0.2 & 0.7 & 0.4 \\ 0.3 & 0.5 & 0.2 \\ 0.5 & 0.2 & 0.3 \end{bmatrix}$$

Probability of new Market Shares:

$$[0.25 \quad 0.35 \quad 0.4] \times \begin{bmatrix} 0.2 & 0.4 & 0.4 \\ 0.3 & 0.5 & 0.2 \\ 0.5 & 0.2 & 0.3 \end{bmatrix}$$

$$= [(0.25)(0.2) + (0.35)(0.3) + (0.4)(0.5), \quad (0.25)(0.4) + (0.35)(0.5) + (0.4)(0.2),$$

$$(0.25)(0.4) + (0.35)(0.2) + (0.4)(0.3)] = [0.355 \quad 0.355 \quad 0.29]$$

New Market Shares: $\longrightarrow$ if Market Size will **increase by 45%.**

So New Market Size = 3,800 (100% + 45%) = 3,800 (145%) = 5,510

A = 5,510 x 0.355 = 1,956.05 ≈ 1,956

B = 5,510 x 0.355 = 1,956.05 ≈ 1,956

A = 5,510 x 0.29 = 1,597.9 ≈ 1,598

5,510

Games Theory

Why we are describing the Business world by a game?

Any Game consists of
A set of Players. اللي هما في عالم البيزنس بيمثلوا الشركة بتاعتي و المنافسين اللي معايا في السوق
A set of strategies for each Player. كل لاعب او كل شركة اكيد بتضع دراسة جدوى و خطط سنوية علشان يمشوا عليها و بنسميها ال plan او ال Strategy
A situation where the participants, Payoffs depend on not only their decisions, but also competitors' decisions, this is called strategic inter action. و اكيد كل استراتيجية بتوصلني الى نتيجة (Payoffs) (مكسب او عائد) و التي نطلق عليها التفاعل الاستراتيجي بين جميع الخطط

In Game Theory our target is to Find the *Game Value* for each player and it may be as follow
Equilibrium or Saddle point (the win – win situation for both players)
Or there is no Equilibrium (so we will use the mixed strategies) (depends on probabilities)

Example 1:

Find the best strategies for player P_1 and P_2; use the following **payoff matrix** to determine if the game has a saddle point (equilibrium)? In addition, **what is the value of the game?**

PLAYER 2

PLAYER 1

$$\begin{array}{c|cc} & Y_1 & Y_2 \\ \hline X1 & 3 & 2 \\ X2 & 4 & 8 \end{array}$$

قبل ما نحل عايزك تفهم المنظر اللي فوق ده معناه ايه!

اولا : انا عندي Player 1 بيستخدم (two strategies) اللي هما X1 و X2

ثانيا : كذلك انا عندي Player 2 بيستخدم (two strategies) اللي هما Y1 و Y2

ثالثا : كل الارقام اللي موجودة داخل المصفوفة بتمثل الربح بتاع PLAYER 1 ← يعني لو PLAYER 1 استخدم الاستراتيجية X1 ,,,, ثم قام ال PLAYER 2 بالرد عليه بالاستراتيجية Y1,,,, اذا (PLAYER 1) هيكسب 3 و (PLAYER 2) هيخسر 3 و هكذا...

الهدف دلوقتي انا عايز اشوف هل في توازن او تساوي او Equilibrium للعائد بتاع اللاعبين ولا لأ... و لو في هتكون قيمته بكام.

يلا نشوف هنحلها ازاي؟

Solution

		Player 2		Row Minimum
		Y_1	Y_2	
Player 1	X_1	3	2	**2**
	X_2	4	8	**4**
	Column Maximum	**4**	**8**	

Max – Min strategy = 4

Min – Max strategy = 4

The game has saddle point or equilibrium point, the game value is 4.

اولا : هدخل على كل صف و احدد اصغر رقم موجود في كل صف (يعني الصف الاول عنده 3 و 2) يبقى ال Row Minimum= 2

وكذلك مع الصف التاني ال row minimum = 4 ,,,,,,, (دلوقتي معايا 2 و 4) بعدين هحدد ال Max of Minimum اللي هو = 4

ثانيا : هدخل على كل عمود و احدد اكبر رقم موجود في كل عمود (يعني العمود الاول عنده 3 و 4) يبقى ال column Max= 4

وكذلك مع العمود التاني ال column Max = 8 ,,,,,,, (دلوقتي معايا 4 و 8) بعدين هحدد ال Min of Maximum اللي هو = 4

Since the MAX MIN Strategy = the MIN MAX Strategy = 4

Then the game is said to be equilibrium

Example 2:

PLAYER 2

$$\text{PLAYER 1} \quad \begin{array}{c c} & \begin{array}{cc} Y_1 & Y_2 \end{array} \\ \begin{array}{c} X1 \\ X2 \end{array} & \begin{bmatrix} -3 & -1 \\ 6 & 2 \end{bmatrix} \end{array}$$

Solution

		Player 2		Row Minimum
		Y_1	Y_2	
Player 1	X_1	– 3	– 1	**– 3**
	X_2	6	2	**2**
	Column Maximum	**6**	**2**	

Max – Min strategy = 2

Min – Max strategy = 2

The game has saddle point or equilibrium point, the game value is 2.

Example 3:

$$
\begin{array}{ccc}
 & & P_2 \\
 & & \begin{array}{cc} Y_1 & Y_2 \end{array} \\
P_1 & \begin{array}{c} X1 \\ X2 \end{array} & \begin{bmatrix} 18 & 25 \\ 30 & 22 \end{bmatrix}
\end{array}
$$

Solution

		P_2		Row Minimum
		Y_1	Y_2	
P_1	X_1	18	25	**18**
	X_2	30	22	**22**
	Column Maximum	**30**	**25**	

Max – Min strategy = 22

Min – Max strategy = 25

There is no Saddle point or Equilibrium point so we will use the mixed strategy and we have two Methods for it.

1st Method

	Y_1	Y_2	Difference	Replacement	%
X_1	18	25	25 – 18 = 7	8	$\frac{8}{15}$ = 0.53 x 100 = 53%
X_2	30	22	30 – 22 8	7	$\frac{7}{15}$ = 0.47 x 100 = 47%
Difference	30 – 18 = 12	25 – 22 = 3	(D) Total = 15		
Replacement	3	12			
%	$\frac{3}{15}$ = 0.2 x 100 = 20%	$\frac{12}{15}$ = 0.8 x 100 = 80%			

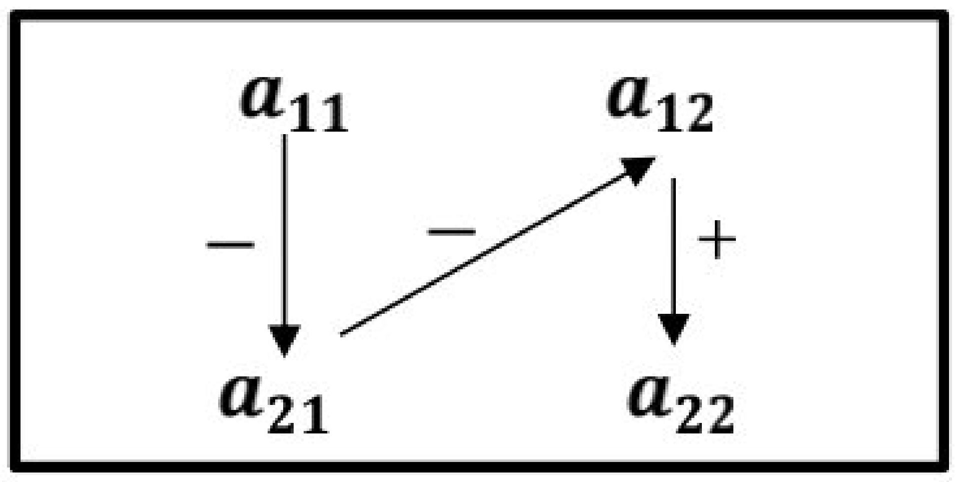
Strategies	Payoff	Probability of the mixed strategies	The value of the mixed strategies (value of game)
$X_1 Y_1$	18	0.53 x 0.2 = 0.106	18 x 0.106 = 1.908
$X_1 Y_2$	25	0.53 x 0.8 = 0.424	25 x 0.424 = 10.6
$X_2 Y_1$	30	0.47 x 0.2 = 0.094	30 x 0.094 = 2.82
$X_2 Y_2$	22	0.47 x 0.8 = 0.376	22 x 0.376 = 8.272
		لازم $\sum = 1$	$\sum = 23.6$ The value of the game = 23.6

2nd Method

$$\begin{array}{ccc} & & P_2 \\ & & \begin{array}{cc} Y_1 & Y_2 \end{array} \\ P_1 & \begin{array}{c} X1 \\ X2 \end{array} & \begin{bmatrix} 18 & 25 \\ 30 & 22 \end{bmatrix} \end{array} \longrightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ Payoffs Matrix}$$

1st) Devisor (d) $= a_{11} - a_{21} - a_{12} + a_{22}$

d = 18 – 30 – 25 + 22 = -15



2nd) Value of the game (g) $= \frac{(a_{11}\ x\ a_{22})-(a_{12}\ x\ a_{21})}{d}$

$= \frac{(18\ x\ 22)-(25\ x\ 30)}{-15} = 23.6$

3rd Optimum Strategies: نسبة نجاح كل استراتيجية

Row Player [X1 X2]	Column Player [Y1 Y2]
X1 = Play row 1 $= \frac{(a_{22} - a_{21})}{d}$ $= \frac{(22-30)}{-15} = 0.53\ x\ 100 = 53\%$	**Y1 = Play column 1** $= \frac{(a_{22} - a_{12})}{d}$ $= \frac{(22-25)}{-15} = 0.2\ x\ 100 = 20\%$
X2 = Play row 2 $= 1 - X1$ $= 1 - 0.53 = 0.47\ x\ 100 = 47\%$	**Y2 = Play column 2** $= 1 - Y1$ $= 1 - 0.2 = 0.8\ x\ 100 = 80\%$

Example 4:

$$\begin{array}{cc} & P_2 \\ P_1 & \begin{bmatrix} 29 & 4 \\ 15 & 25 \end{bmatrix} \end{array}$$

Solution

		P 2		Row Minimum
		Y_1	Y_2	
P 1	X_1	29	4	**4**
	X_2	15	25	**15**
	Column Maximum	**29**	**25**	

Max – Min strategy = 15

Min – Max strategy = 25

There is no Saddle point or Equilibrium point so we will use the mixed strategy

	Y_1	Y_2	Difference	Replacement	%
X_1	29	4	25	10	$\frac{10}{35}$ = 0.286 x 100 = 28.6%
X_2	15	25	10	25	$\frac{25}{35}$ = 0.714 x 100 = 71.4%
Difference	14	21	(D) Total = 35		
Replacement	21	14			
%	$\frac{21}{35}$ = 0.6 x 100 = 60%	$\frac{14}{35}$ = 0.4 x 100 = 40%			

Strategies	Payoff	Probability of the mixed strategies	The value of the mixed strategies (value of game)
$X_1 Y_1$	29	0.286 x 0.6 = 0.17	29 x 0.17 = 4.93
$X_1 Y_2$	4	0.286 x 0.4 = 0.11	4 x 0.11 = 0.44
$X_2 Y_1$	15	0.714 x 0.6 = 0.43	15 x 0.43 = 6.45
$X_2 Y_2$	25	0.714 x 0.4 = 0.29	25 x 0.29 = 7.25
		$\sum = 1$ لازم	$\sum = 19.07$ The value of the game = 19.07

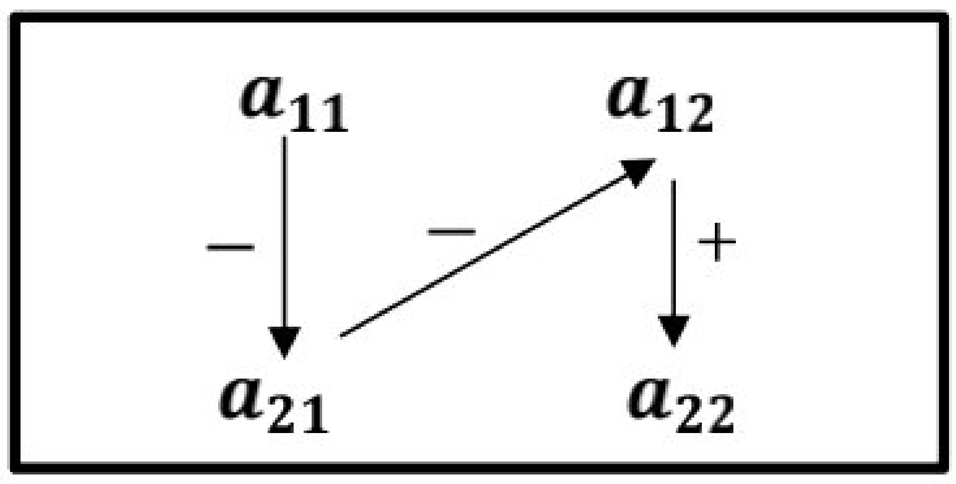
2nd Method

$$P_1 \quad \begin{array}{c c} & \begin{array}{cc} Y_1 & Y_2 \end{array} \\ \begin{array}{c} X1 \\ X2 \end{array} & \begin{bmatrix} 29 & 4 \\ 15 & 25 \end{bmatrix} \end{array} \longrightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

(column player P_2) Payoffs Matrix

1st) Devisor (d) $= a_{11} - a_{21} - a_{12} + a_{22}$

$d = 29 - 15 - 4 + 25 = 35$



2nd) Value of the game (g) $= \dfrac{(a_{11}\ x\ a_{22}) - (a_{12}\ x\ a_{21})}{d}$

$= \dfrac{(29\ x\ 25) - (4\ x\ 15)}{35} = 19$

3rd Optimum Strategies: نسبة نجاح كل استراتيجية

Row Player [X1 X2]	Column Player [Y1 Y2]
X1 = Play row 1 $= \dfrac{(a_{22} - a_{21})}{d}$ $= \dfrac{(25 - 15)}{35} = 0.2857$	**Y1 = Play column 1** $= \dfrac{(a_{22} - a_{12})}{d}$ $= \dfrac{(25 - 4)}{35} = 0.6$
X2 = Play row 2 $= 1 - X1$ $= 1 - 0.2857 = 0.714$	**Y2 = Play column 2** $= 1 - Y1$ $= 1 - 0.6 = 0.4$

Example 5:

Find the best strategies for player P_1 and P_2; use the following **payoff matrix** to determine if the game has a saddle point (equilibrium)? In addition, **what is the value of the game?**

$$\begin{array}{c c} & \begin{array}{cccc} Y_1 & Y_2 & Y_3 & Y_4 \end{array} \\ \begin{array}{c} X1 \\ X2 \\ X3 \\ X4 \end{array} & \begin{bmatrix} 10 & 0 & 24 & -14 \\ 12 & 5 & 6 & 18 \\ -4 & -3 & 16 & -2 \\ 12 & -7 & 6 & 5 \end{bmatrix} \end{array}$$

Solution

		P 2				Row Minimum
		Y_1	Y_2	Y_3	Y_4	
P 1	X_1	10	0	24	-14	-14
	X_2	12	5	6	18	5
	X_3	-4	-3	16	-2	-4
	X_4	12	-7	6	5	-7
	Column Maximum	12	5	24	18	

Max – Min strategy = 5

Min – Max strategy = 5

The game has saddle point or equilibrium point, the game value is 5.