CLL110 Transport Phenomena
Department of Chemical Engineering IIT Delhi
Semester 1 2024-25, D slot

Exercise

Reading assignment: Appendix A, Transport Phenomena, BSL

Tutorial 1

why do we have to use diff indices and when same indices

Prove

1. $\underline{u} \cdot (\underline{v} \times \underline{w}) = \underline{v} \cdot (\underline{w} \times \underline{u})$
2. $(\underline{u} \times \underline{v}) \cdot (\underline{w} \times \underline{z}) = (\underline{u} \cdot \underline{w})(\underline{v} \cdot \underline{z}) - (\underline{u} \cdot \underline{z})(\underline{v} \cdot \underline{w})$
3. $\nabla^2 \underline{v} = \underline{\nabla}(\underline{\nabla} \cdot \underline{v}) - \underline{\nabla} \times (\underline{\nabla} \times \underline{v})$ → Doubt?
4. $(\underline{\nabla} \cdot s\underline{v}) = \underline{\nabla} s \cdot \underline{v} + s\underline{\nabla} \cdot \underline{v}$
5. $\underline{\nabla} \cdot (\underline{v} \times \underline{w}) = \underline{w} \cdot (\underline{\nabla} \times \underline{v}) - \underline{v} \cdot (\underline{\nabla} \times \underline{w})$
6. $(\underline{u} \times \underline{v}) \times (\underline{w} \times \underline{z}) = \big((\underline{u} \times \underline{v}) \cdot \underline{z}\big)\underline{w} - \big((\underline{u} \times \underline{v}) \cdot w\big)\underline{z}$ → Good ques

4) $\left(\frac{\partial}{\partial x_i} s \,\underset{\sim}{\delta_i}\right) \cdot \left(v_j \underset{\sim}{\delta_j}\right) + s \frac{\partial}{\partial x_i} \underset{\sim}{\delta_i} \cdot (v_j \delta_j)$

$\Rightarrow \left(\frac{\partial}{\partial x_i} s\, v_j\right)\left(\delta_{ij}\right) + s \frac{\partial}{\partial x_i} v_j \; \delta_{ij}$

$\Rightarrow (i = j)$

$\left(\frac{\partial}{\partial x_i} s v_i\right) + s \frac{\partial}{\partial x_i} v_i = \frac{\partial}{\partial x_i} \cdot (s \cdot v_i) = \underset{\sim}{\nabla} \cdot (s \cdot \underset{\sim}{v})$

③ $\underset{\sim}{\nabla} = \underset{\sim}{\delta_i} \frac{\partial}{\partial x_i}$ $\quad \cdot \underset{\sim}{v} = \underset{\sim}{\delta_j} v_j$

$$\nabla^2 v = \underset{\sim}{\nabla}(\underset{\sim}{\nabla} \cdot \underset{\sim}{v}) - \underset{\sim}{\nabla} \times (\underset{\sim}{\nabla} \times \underset{\sim}{v})$$

$$= \underset{\sim}{\delta_i} \frac{\partial}{\partial x_i} \left(\underset{\sim}{\delta_i} \frac{\partial}{\partial x_i} \cdot \underset{\sim}{\delta_j} v_j \right) - \underset{\sim}{\delta_i} \frac{\partial}{\partial x_i} \times \left(\underset{\sim}{\delta_i} \frac{\partial}{\partial x_i} \times \underset{\sim}{\delta_j} v_j \right)$$

$$= \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i} v_j \underset{\sim}{\delta_i} (\delta_i \cdot \delta_j) - \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i} v_j \delta_i \times (\delta_i \times \delta_j)$$

$$\Rightarrow \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i} v_j \delta_i \delta_{ij} \quad - \quad \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i} v_j \underset{\sim}{\delta_i} \times \varepsilon_{ijk} \underset{\sim}{\delta_k}$$

$$\Rightarrow \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i} v_j \underset{\sim}{\delta_i} - \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i} v_j \varepsilon_{ijk} (\delta_i \times \delta_k)$$

$$- \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i} v_j \varepsilon_{ijk} \delta_{ikl} \underset{\sim}{\delta_l}$$

$l \leftrightarrow i$

$$- \frac{\partial}{\partial x_i} \times \frac{\partial}{\partial x_i} v_j \varepsilon_{ijk} \cancel{\varepsilon_{jki}}^{\,0} \underset{\sim}{\delta_i}$$

$$\Rightarrow \boxed{\nabla^2 v_i \delta_i = \nabla^2 \underset{\sim}{v}}$$

- $(\underset{\sim}{\nabla} \cdot s \underset{\sim}{v}) = \underset{\sim}{\nabla} s \cdot \underset{\sim}{v} + s \underset{\sim}{\nabla} \cdot \underset{\sim}{v}$

RHS $\Rightarrow$ $\delta_i \frac{\partial}{\partial x_i} \cdot s v_j \underset{\sim}{\delta_j} + s \delta_i \frac{\partial}{\partial x_i} \cdot v_j \delta_j$

Ans ①: $\underset{\sim}{u} \cdot (\underset{\sim}{v} \times \underset{\sim}{w}) = \underset{\sim}{v} \cdot (\underset{\sim}{w} \times \underset{\sim}{u})$

$$u_i \underset{\sim}{\delta_i} \cdot (v_j \underset{\sim}{\delta_j} \times w_k \underset{\sim}{\delta_k}) = v_j \underset{\sim}{\delta_j} \cdot (w_k \underset{\sim}{\delta_k} \times u_i \underset{\sim}{\delta_i})$$

$$u_i \delta_i \cdot (v_j w_k \varepsilon_{jki} \delta_i) = v_j \delta_j \cdot (w_k u_i \varepsilon_{kij} \delta_j)$$

$$u_i v_j w_k \varepsilon_{jki} \cancel{\delta_{ii}}^{\,1} = v_i v_j w_k \varepsilon_{jki} \cancel{\delta_{jj}}^{\,1}$$

H.P

Ans ②: $(\underset{\sim}{u} \times \underset{\sim}{v}) \cdot (\underset{\sim}{w} \times \underset{\sim}{z}) = (\underset{\sim}{u} \cdot \underset{\sim}{w})(\underset{\sim}{v} \cdot \underset{\sim}{z}) - (\underset{\sim}{u} \cdot \underset{\sim}{z})(\underset{\sim}{v} \cdot \underset{\sim}{w})$

LHS

$$(u_i \underset{\sim}{\delta_i} \times v_j \underset{\sim}{\delta_j}) \cdot (w_k \underset{\sim}{\delta_k} \times z_l \underset{\sim}{\delta_l}) \Rightarrow (u_i v_j \varepsilon_{ijm} \underset{\sim}{\delta_m}) \cdot (w_k z_l \varepsilon_{kln} \underset{\sim}{\delta_n})$$

$$\Rightarrow u_i v_j w_k z_l \underbrace{\varepsilon_{ijm} \varepsilon_{kl\cancel{n}}^{\,m}}\, \cancel{\underset{\sim}{\delta_{mn}}}^{\,1} \quad (m = n)$$

$$u_i v_j w_k z_l (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk})$$

$$\Rightarrow u_i v_j w_k z_l \underbrace{\delta_{ik}}_{i=k} \underbrace{\delta_{jl}}_{j=l} - u_i v_j w_k z_l \underbrace{\delta_{il}}_{i=l} \underbrace{\delta_{jk}}_{j=k}$$

$$\Rightarrow$$

$$\underbrace{u_i v_j w_i z_j} - u_i v_j w_j z_i$$

$$\underbrace{(\underset{\sim}{u} \cdot \underset{\sim}{w})(\underset{\sim}{v} \cdot \underset{\sim}{z}) - (\underset{\sim}{u} \cdot \underset{\sim}{z})(\underset{\sim}{v} \cdot \underset{\sim}{w})} = RHS$$

H.P

(2) $$\nabla^2 v = \underset{\sim}{\nabla}\left(\underset{\sim}{\nabla}\cdot\underset{\sim}{v}\right) - \underset{\sim}{\nabla}\times\left(\underset{\sim}{\nabla}\times\underset{\sim}{v}\right)$$

$$= \frac{\partial}{\partial x_i}\underset{\sim}{\delta}_i\left(\frac{\partial}{\partial x_j}\underset{\sim}{\delta}_j\cdot v_k\underset{\sim}{\delta}_k\right) - \frac{\partial}{\partial x_l}\underset{\sim}{\delta}_l\times\left(\frac{\partial}{\partial x_m}\underset{\sim}{\delta}_m\times v_n\underset{\sim}{\delta}_n\right)$$

$$= \frac{\partial}{\partial x_i}\underset{\sim}{\delta}_i\left(\frac{\partial v_j}{\partial x_j}\cancel{\delta_{jk}}\right)^{j=k} - \frac{\partial}{\partial x_l}\underset{\sim}{\delta}_l\times\left(\frac{\partial}{\partial x_m}v_n\,\varepsilon_{mno}\,\underset{\sim}{\delta}_o\right)$$

$$= \left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}v_j\right)\underset{\sim}{\delta}_i \qquad - \frac{\partial}{\partial x_l}\frac{\partial}{\partial x_m}v_n\,\varepsilon_{mno}\,\overset{\varepsilon_{plo}}{\widetilde{\varepsilon_{lop}}}\,\underset{\sim}{\delta}_p$$

$$- \frac{\partial}{\partial x_l}\frac{\partial}{\partial x_m}v_n\left(\delta_{mp}\delta_{nl} - \delta_{ml}\delta_{np}\right)\underset{\sim}{\delta}_p$$

$$- \frac{\partial}{\partial x_l}\frac{\partial}{\partial x_m}v_n\,\underset{m=p}{\delta_{mp}}\,\underset{n=l}{\delta_{nl}}\,\underset{\sim}{\delta}_p + \frac{\partial}{\partial x_l}\frac{\partial}{\partial x_m}v_n\,\underset{m=l}{\delta_{ml}}\,\underset{n=p}{\delta_{np}}\,\underset{\sim}{\delta}_p$$

$$\left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}v_j\right)\underset{\sim}{\delta}_i \qquad \underbrace{-\frac{\partial}{\partial x_l}\frac{\partial}{\partial x_p}v_l\,\underset{\sim}{\delta}_p + \frac{\partial}{\partial x_m}\frac{\partial}{\partial x_m}v_p\,\underset{\sim}{\delta}_p}_{p\to i}$$

$$\cancel{\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}v_j\,\delta_i} \qquad \cancel{-\frac{\partial}{\partial x_l}\frac{\partial}{\partial x_i}v_i\,\underset{\sim}{\delta}_i} + \frac{\partial}{\partial x_m}\frac{\partial}{\partial x_m}v_i\,\underset{\sim}{\delta}_i$$

$$\Rightarrow \frac{\partial}{\partial x_m}\frac{\partial}{\partial x_m}v_i\,\underset{\sim}{\delta}_i \Rightarrow \underline{\nabla^2\cdot v} \quad \text{LHS}$$

H.P

$$: (\underset{\sim}{u}\times\underset{\sim}{v})\times(\underset{\sim}{w}\times\underset{\sim}{z}) = ((\underset{\sim}{u}\times\underset{\sim}{v})\cdot\underset{\sim}{z})\underset{\sim}{w} - ((\underset{\sim}{u}\times\underset{\sim}{v})\cdot\underset{\sim}{w})\underset{\sim}{z}$$

$$(u_i \underset{\sim}{\delta}_i \times v_j \underset{\sim}{\delta}_j)\times(w_k\underset{\sim}{\delta}_k \times z_l \underset{\sim}{\delta}_l)$$

$$\Rightarrow (u_i v_j \varepsilon_{ijm}\underset{\sim}{\delta}_m)\times(w_k z_l \varepsilon_{kln}\underset{\sim}{\delta}_n)$$

$$u_iv_jw_kz_l\varepsilon_{ijm}\varepsilon_{kln}\varepsilon_{mno}\underset{\sim}{\delta}_o \Rightarrow u_iv_jw_kz_l\varepsilon_{ijm}\varepsilon_{kln}\varepsilon_{omn}\underset{\sim}{\delta}_o$$

$$u_iv_jw_kz_l\varepsilon_{ijm}(\delta_{ko}\delta_{lm}-\delta_{km}\delta_{lo})\underset{\sim}{\delta}_o$$

Compaction Rule

$l \leftrightarrow m$ $k \leftrightarrow m$

$$u_iv_jw_kz_l\varepsilon_{ijm}\underbrace{\delta_{ko}\cancel{\delta_{lm}}}_{\text{Replace } k\leftrightarrow o}\underset{\sim}{\delta}_o - u_iv_jw_kz_l\varepsilon_{ijm}\underbrace{\cancel{\delta_{km}}\delta_{lo}\underset{\sim}{\delta}_o}_{\text{Replace } o\leftrightarrow l}$$

$$u_iv_jw_oz_l\varepsilon_{ijl}\underset{\sim}{\delta}_o - u_iv_jw_mz_o\varepsilon_{ijm}\underset{\sim}{\delta}_o$$

$$((u\times v)\cdot z)w_o - ((u\times v)\cdot w)z \Longrightarrow \underline{\underline{\text{H.P}}}$$

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Tutorial 2

Prove

1. If $\varepsilon_{ijk}\sigma_{ij} = 0$, then $\underline{\sigma}$ is a symmetric tensor

   If $\underline{\tau}$ is a second order symmetric tensor (in following)

2. $\underline{\nabla} \cdot (\underline{\tau} \cdot \underline{v}) = \underline{v} \cdot (\underline{\nabla} \cdot \underline{\tau}) + \underline{\tau} : \underline{\nabla}\, \underline{v}$ $\quad \underline{\tau}$ is a second order symmetric tensor
3. $\underline{\tau} : \underline{u}\, \underline{v} = (\underline{\tau} \cdot \underline{u}) \cdot \underline{v}$
4. $\underline{\nabla} \cdot (\underline{v}\, \underline{w}) = \underline{v} \cdot \underline{\nabla}\, \underline{w} + \underline{w}\, (\underline{\nabla} \cdot \underline{v})$
5. Discuss the following
   (a) No slip boundary conditions
   (b) Continuum hypothesis for fluid
   (c) Lagrangian nature of a fluid
   (d) Substantial derivative
   (e) Laminar and turbulent flow
   (f) Boundary layer and potential flow
   (g) Momentum flux and shear forces

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Tut-2

1) $\varepsilon_{ijk}\sigma_{ij} = 0$

(k is a free index, hence k=1,2,3)

i,j can be 2,3 or 3,2

$k=1$ $\quad \varepsilon_{ij1}\sigma_{ij} = 0 \Rightarrow \varepsilon_{231}\sigma_{23} + \varepsilon_{321}\sigma_{32} = 0$

$\underbrace{-\varepsilon_{123}\sigma_{23} + \varepsilon_{321}\sigma_{32} = 0}$

$\boxed{\sigma_{23} = \sigma_{32}}$

$k=2$ $\quad \varepsilon_{ij2}\sigma_{ij} = 0$

$ij \Rightarrow 1,3$ or $3,1$

$k=3$ $\quad \varepsilon_{ij3}\sigma_{ij} = 0$

$ij \Rightarrow 1,2,$ or $2,1$

Similarly, $\sigma_{13} = \sigma_{31}$

$\sigma_{12} = \sigma_{21}$

Hence σ is symmetric.

2) $\underset{\sim}{\nabla}\cdot(\underset{\sim}{\tau}\cdot\underset{\sim}{v}) = \underset{\sim}{v}\cdot(\underset{\sim}{\nabla}\cdot\underset{\sim}{\tau}) + \underset{\sim}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{v}$

LHS. $\dfrac{\partial}{\partial x_k}\delta_k\cdot\left(\tau_{ij}\delta_i\delta_j\cdot v_l\delta_l\right)$

$\dfrac{\partial(\tau_{ij}v_l)}{\partial x_k}\left(\delta_k\cdot(\delta_i\delta_j\cdot\delta_l)\right) \Rightarrow \dfrac{\partial}{\partial x_k}(\tau_{ij}v_l)(\delta_k\cdot\delta_i)(\cancel{\delta_{jl}})$ $\quad j \Rightarrow l$

$= \dfrac{\partial}{\partial x_k}(\tau_{ij}v_j)(\cancel{\delta_k\cdot\delta_i})$ $\quad k=i$

$= \dfrac{\partial}{\partial x_i}(\tau_{ij}v_j) = v_j\dfrac{\partial\tau_{ij}}{\partial x_i} + \tau_{ij}\dfrac{\partial v_j}{\partial x_i}$

RHS: $\underset{\sim}{v}\cdot(\underset{\sim}{\nabla}\cdot\underset{\sim}{\tau}) + \underset{\sim}{\tau} : \nabla\underset{\sim}{v}$

$= v_l\delta_l\cdot\left(\dfrac{\partial}{\partial x_k}\delta_k\cdot\tau_{ij}\delta_i\delta_j\right) + \tau_{ij}\underset{\sim}{\delta_i}\underset{\sim}{\delta_j} : \delta_k\dfrac{\partial}{\partial x_k}v_l\delta_l$

$= v_l\delta_l\cdot\dfrac{\partial(\tau_{ij})}{\partial x_k}\left(\underbrace{\delta_k\cdot\delta_i}_{\cancel{\delta_{ki}}\ k=i}\delta_j\right) + \tau_{ij}\,\delta_{il}\,\underset{j=k}{\cancel{\delta_{jk}}}\dfrac{\partial v_l}{\partial x_k}$

$= v_l\dfrac{\partial\tau_{ij}}{\partial x_i}\underset{l=j}{\cancel{\delta_{lj}}} + \tau_{ij}\underset{i=l}{\cancel{\delta_{il}}}\dfrac{\partial v_l}{\partial x_j} \Rightarrow v_j\dfrac{\partial\tau_{ij}}{\partial x_i} + \tau_{ij}\dfrac{\partial v_i}{\partial x_j} \Rightarrow v_j\dfrac{\partial(\tau_{ij})}{\partial x_i} + \tau_{ij}\left(\dfrac{\partial v_j}{\partial x_i}\right)$

$\Rightarrow$ LHS

③ $\underline{T} : \underset{\sim}{u}\underset{\sim}{v} = (\underset{\sim}{T}\cdot\underset{\sim}{u})\cdot\underset{\sim}{v}$

$$T_{ij}\,\delta_i\,\delta_j : u_k\underset{\sim}{\delta}_k\,v_l\underset{\sim}{\delta}_l = (T_{ij}\,\delta_i\delta_j\cdot v_k\underset{\sim}{\delta}_k)\cdot(v_l\underset{\sim}{\delta}_l)$$

$$T_{ij}\,u_k\,v_l\,\delta_{jk}\,\delta_{il} = (T_{ij}\,u_k\,\delta_i\,\underset{j=k}{\cancel{\delta_{jk}}})\cdot(\underset{\sim}{v}_l\,\underset{\sim}{\delta}_l)$$

$$T_{ij}\,u_k v_l\,\underset{j=k}{\cancel{\delta_{jk}}}\,\underset{i=l}{\cancel{\delta_{il}}} = (T_{ij}\,u_j\,v_l\,\underset{i=l}{\cancel{\delta_{il}}})$$

$$\boxed{T_{ij}v_j v_i = T_{ij}\,u_j\,v_i} \longrightarrow \text{Prove}$$

④ $\underset{\sim}{\nabla}\cdot(\underset{\sim}{v}\underset{\sim}{w}) = \underset{\sim}{v}\cdot\underset{\sim}{\nabla}\underset{\sim}{w} + \underset{\sim}{w}(\underset{\sim}{\nabla}\cdot\underset{\sim}{v})$

$$\frac{\partial}{\partial x_i}\underset{\sim}{\delta}_i\cdot(v_j\underset{\sim}{\delta}_j\,w_k\underset{\sim}{\delta}_k) = v_j\underset{\sim}{\delta}_j\cdot\left(\frac{\partial}{\partial x_i}\underset{\sim}{\delta}_i\,w_k\underset{\sim}{\delta}_k\right) + w_k\underset{\sim}{\delta}_k\left(\frac{\partial}{\partial x_i}\underset{\sim}{\delta}_i\cdot v_j\delta_j\right)$$

$$\frac{\partial}{\partial x_i}v_j w_k\,(\underset{\sim}{\delta}_i\cdot\underset{\sim}{\delta}_j)\,\underset{\sim}{\delta}_k = v_j\frac{\partial}{\partial x_i}w_k\;\underset{\sim}{\delta}_k\;\underset{j=i}{\cancel{\delta_{ji}}}$$

$$= \frac{\partial}{\partial x_i}v_j\,\omega_k\,(\underset{i=j}{\cancel{\delta_{ij}}})\,\underset{\sim}{\delta}_k = v_i\frac{\partial}{\partial x_i}\,\omega_k\underset{\sim}{\delta}_k + w_k\,\underset{\sim}{\delta}_k\left(\frac{\partial}{\partial x_i}v_j\;\underset{i=j}{\cancel{\delta_{ij}}}\right)$$

$$= \frac{\partial}{\partial x_i}(v_i\;\omega_k)\,\underset{\sim}{\delta}_k \qquad = v_i\frac{\partial}{\partial x_i}\omega_k\,\underset{\sim}{\delta}_k + \omega_k\,\underset{\sim}{\delta}_k\,\frac{\partial}{\partial x_i}v_i$$

$$= \left(v_i\frac{\partial w_k}{\partial x_i} + \omega_k\frac{\partial v_i}{\partial x_i}\right)\underset{\sim}{\delta}_k = \frac{\partial}{\partial x_i}(v_i\;\omega_k\,\delta_k)$$

Same

$\underline{\underline{\text{LHS} = \text{RHS}}} = v_i\frac{\partial}{\partial x_i}\omega_k +$

Ans ①.

a) whenever fluid layer is in contact with solid boundary, their is no slipping b/w them i.e relative velocity is zero

Speed of fluid layer at boundary = speed of solid

b) Continnum hypothesis is the eulerian understanding of fluid transport. We assume the fluid to be a continous entity and assume uniformity and continuity between diff. particles

c) Langragian nature refers to when we assume the fluid to be discrete and observe a particular fluid particle and mole calculations and observations on it.

d) Substantial derivative, takes into account the diff. in time as well as space. Here we assume the observer is moving with the fluid's velocity

e) Laminar flow refers to a uniform fluid flow where there is no disturbance and velocity in z-direction depends only on radial distance.

Turbulent flow has disturbance and chaos and there are non-uniform velocities in all directions

f) Close to the boundary layer, velocity gradient can be observed and viscous forces are present while beyond it lies the potential flow region where there is no velocity gradient or viscous force.

g) Shear forces arise due to the velocity gradient and is an external explanation for viscosity while momentum flux refers to the flow from high velocity to low velocity just like heat flow from high temp to low temp.