SOLUTIONS TO THE PROBLEMS

in

# TRANSPORT PHEOMENA
# Second Edition
# (2002)

by

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**1A.1 Estimation of dense-gas viscosity.**

*a.* Table E.1 gives $T_c = 126.2$ K, $p_c = 33.5$ atm, and $\mu_c = 180 \times 10^{-6}$ g/cm·s for $N_2$. The reduced conditions for the viscosity estimation are then:

$$p_r = p/p_c = (1000 + 14.7)/33.5 \times 14.7 = 2.06$$

$$T_r = T/T_c = (273.15 + (68 - 32)/1.8)/126.2 = 2.32$$

At this reduced state, Fig. 1.3-1 gives $\mu_r = 1.15$. Hence, the predicted viscosity is $\mu = \mu_r/\mu_c = 1.15 \times 180 \times 10^{-6} = 2.07 \times 10^{-4}$ g/cm·s. This result is then converted into the requested units by use of Table F.3-4:

$$\mu = 2.07 \times 10^{-4} \times 6.7197 \times 10^{-2} = 1.4 \times 10^{-4} \text{ lb}_\text{m}/\text{ft·s}$$

**1A.2 Estimation of the viscosity of methyl fluoride.**

*a.* $CH_3F$ has $M = 16.04 - 1.008 + 19.00 = 34.03$ g/g-mole, $T_c = 4.55 + 273.15 = 277.70$ K, $p_c = 58.0$ atm, and $\widetilde{V}_c = 34.03/0.300 = 113.4$ cm$^3$/g-mole. The critical viscosity is then estimated as

$$\mu_c = 61.6(34.03 \times 277.70)^{1/2}(113.4)^{-2/3} = 255.6 \text{ micropoise}$$

from Eq. 1.3-1a, and

$$\mu_c = 7.70(34.03)^{1/2}(58.0)^{2/3}(277.7)^{-1/6} = 263.5 \text{ micropoise}$$

from Eq. 1.3-1b.

The reduced conditions for the viscosity estimate are $T_r = (370 + 273.15)/277.70 = 2.32$, $p_r = 120/58.0 = 2.07$, and the predicted $\mu_r$ from Fig. 1.3-1 is 1.1. The resulting predicted viscosity is

$$\mu = \mu_r \mu_c = 1.1 \times 255.6 \times 10^{-6} = 2.8 \times 10^{-4} \text{ g/cm·s via Eq.1.3-1a, or}$$
$$1.1 \times 263.5 \times 10^{-6} = 2.9 \times 10^{-4} \text{g/cm·s via Eq.1.3-1b.}$$

## 1A.3 Computation of the viscosities of gases at low density.

Equation 1.4-14, with molecular parameters from Table E.1 and collision integrals from Table E.2, gives the following results:

For $O_2$: $M = 32.00$, $\sigma = 3.433\,\text{Å}$, $\varepsilon/\kappa = 113$ K. Then at 20°C, $\kappa T/\varepsilon = 293.15/113 = 2.594$ and $\Omega_\mu = 1.086$. Equation 1.4-14 then gives

$$\begin{aligned}\mu &= 2.6693 \times 10^{-5} \frac{\sqrt{32.00 \times 293.15}}{(3.433)^2 \times 1.086} \\ &= 2.02 \times 10^{-4}\ \text{g/cm·s} \\ &= 2.02 \times 10^{-5}\ \text{Pa·s} \\ &= 2.02 \times 10^{-2}\ \text{mPa·s.}\end{aligned}$$

The reported value in Table 1.1-3 is $2.04 \times 10^{-2}$ mPa·s.

For $N_2$: $M = 28.01$, $\sigma = 3.667\,\text{Å}$, $\varepsilon/\kappa = 99.8$ K. Then at 20°C, $\kappa T/\varepsilon = 293.15/99.8 = 2.937$ and $\Omega_\mu = 1.0447$. Equation 1.4-14 then gives

$$\begin{aligned}\mu &= 2.6693 \times 10^{-5} \frac{\sqrt{28.01 \times 293.15}}{(3.667^2 \times 1.0447} \\ &= 1.72 \times 10^{-4}\ \text{g/cm·s} \\ &= 1.72 \times 10^{-5}\ \text{Pa·s} \\ &= 1.72 \times 10^{-2}\ \text{mPa·s.}\end{aligned}$$

The reported value in Table 1.1-3 is $1.75 \times 10^{-2}$ mPa·s.

For $CH_4$, $M = 16.04$, $\sigma = 3.780\,\text{Å}$, $\varepsilon/\kappa = 154$ K. Then at 20°C, $\kappa T/\varepsilon = 293.15/154 = 1.904$ and $\Omega_\mu = 1.197$. Equation 1.4-14 then gives

$$\begin{aligned}\mu &= 2.6693 \times 10^{-5} \frac{\sqrt{16.04 \times 293.15}}{(3.780)^2 \times 1.197} \\ &= 1.07 \times 10^{-4}\ \text{g/cm·s} \\ &= 1.07 \times 10^{-5}\ \text{Pa·s} \\ &= 1.07 \times 10^{-2}\ \text{mPa·s.}\end{aligned}$$

The reported value in Table 1.1-3 is $1.09 \times 10^{-2}$ mPa·s.

**1A.4 Gas-mixture viscosities at low density.**

The data for this problem are as follows:

| Component | M | $\mu$, poise $\times 10^6$ |
|---|---|---|
| $1(H_2)$ | 2.016 | 88.4 |
| $2(CCl_2F_2)$ | 120.92 | 124.0 |

Insertion of these data into Eq. 1.4-16 gives the foloowing coefficients for mixtures of $H_2$ and Freon-12 at this temperature:

$$\Phi_{11} = \Phi_{22} = 1.0$$

$$\Phi_{12} = \frac{1}{\sqrt{8}}\left(1 + \frac{2.016}{120.92}\right)^{-1/2}\left[1 + \left(\frac{88.4}{124.0}\right)^{1/2}\left(\frac{120.92}{2.016}\right)^{1/4}\right]^2$$

$$= 3.934$$

$$\Phi_{21} = \frac{1}{\sqrt{8}}\left(1 + \frac{120.92}{2.016}\right)^{-1/2}\left[1 + \left(\frac{124.0}{88.4}\right)^{1/2}\left(\frac{2.016}{120.92}\right)^{1/4}\right]^2$$

$$= 0.0920$$

Equation 1.4-15 then gives the predicted mixture viscosities:

| $x_1 =$ $1 - x_2$ | $\sum_1 =$ $\sum x_\beta \Phi_{1\beta}$ | $\sum_2 =$ $\sum x_\beta \Phi_{2\beta}$ | $A :=$ $x_1\mu_1/\sum_1$ | $B :=$ $x_2\mu_2/\sum_2$ | $A + B =$ $\mu_{\text{mix}} \times 10^6$ | $\mu_{\text{obs,poise}} \times 10^6$ |
|---|---|---|---|---|---|---|
| 0.00 | 3.934 | 1.000 | 0.0 | 124.0 | (124.0) | 124.0 |
| 0.25 | 3.200 | 0.773 | 6.9 | 120.3 | 127.2 | 128.1 |
| 0.50 | 2.467 | 0.546 | 18.1 | 113.6 | 131.7 | 131.9 |
| 0.75 | 1.734 | 0.319 | 38.2 | 97.2 | 135.4 | 135.1 |
| 1.00 | 1.000 | 0.092 | 88.4 | 0.0 | (88.4) | 88.4 |

**1A.5 Viscosities of chlorine-air mixtures at low density.**

Equation 1.4-14 and Tables E.1, E.2 give the following viscosities at 75°F(= 273.15 + (75 − 32)/1.8 = 297.03 K) and 1 atm:

For component 1, ($Cl_2$), $M_1 = 70.91$, $\sigma_1 = 4.115\,\text{Å}$, $\varepsilon_1/\kappa = 357\text{K}$; hence, $\kappa T/\varepsilon_1 = 297.03/357 = 0.832$ and $\Omega_{\mu,1} = 1.754$, and

$$\mu_1 = 2.6693 \times 10^{-5} \frac{\sqrt{70.91 \times 297.03}}{(4.115)^2 \times 1.754} = 1.304 \times 10^{-4}\ \text{g/cm·s} = 0.01304\ \text{cp.}$$

For component 2, (air), $M_2 = 28.97$, $\sigma = 3.617\,\text{Å}$, $\varepsilon_1/\kappa = 97.0\text{K}$; hence, $\kappa T/\varepsilon_1 = 297.03/97.0 = 3.062$ and $\Omega_{\mu,1} = 1.033$, and

$$\mu_2 = 2.6693 \times 10^{-5} \frac{\sqrt{28.97 \times 297.03}}{(3.617)^2 \times 1.033} = 1.832 \times 10^{-4}\ \text{g/cm·s} = 0.01832\ \text{cp.}$$

Eq. 1.4-16 then gives the following coefficients for Eq. 1.4-15 at this temperature:

$$\Phi_{11} = \Phi_{22} = 1.0$$

$$\Phi_{12} = \frac{1}{\sqrt{8}}\left(1 + \frac{70.91}{28.97}\right)^{-1/2}\left[1 + \left(\frac{0.01304}{0.01832}\right)^{1/2}\left(\frac{28.97}{70.91}\right)^{1/4}\right]^2$$
$$= 0.5339$$

$$\Phi_{21} = \frac{1}{\sqrt{8}}\left(1 + \frac{28.97}{70.91}\right)^{-1/2}\left[1 + \left(\frac{0.01832}{0.01304}\right)^{1/2}\left(\frac{70.91}{28.97}\right)^{1/4}\right]^2$$
$$= 1.8360$$

Equation 1.4-15 then gives the predicted mixture viscosities:

| $x_1 =$ $1 - x_2$ | $\sum_1 =$ $\sum x_\beta \Phi_{1\beta}$ | $\sum_2 =$ $\sum x_\beta \Phi_{2\beta}$ | $A :=$ $x_1\mu_1/\sum_1$ | $B :=$ $x_2\mu_2/\sum_2$ | $A + B =$ $\mu_{\text{mix,cp.}} \times 10^6$ |
|---|---|---|---|---|---|
| 0.00 | 0.5339 | 1.000 | 0.0 | 0.01832 | 0.0183 |
| 0.25 | 0.6504 | 1.2090 | 0.005012 | 0.011365 | 0.0164 |
| 0.50 | 0.7670 | 1.4180 | 0.008501 | 0.006460 | 0.0150 |
| 0.75 | 0.8835 | 1.6270 | 0.011070 | 0.002815 | 0.0139 |
| 1.00 | 1.000 | 1.8360 | 0.01304 | 0.0 | 0.0130 |

**1A.6 Estimation of liquid viscosity.**

*a.* The calculated values for Eq. 1.5-9 at 0°C and 100°C are as follows:

| | | |
|---|---|---|
| $T$,K | 273.15 | 373.15 |
| $\rho$, g/cm$^3$ | 0.9998 | 0.9584 |
| $\widetilde{V} = M/\rho$, cm$^3$/g-mole | 18.01 | 18.80 |
| $\Delta\widetilde{U}_{\text{vap},T_b}$, cal/g-mole$= 897.5 \times 18.016 \times 252.16/453.59$ | 8989. | 8989. |
| $\Delta\widetilde{U}_{\text{vap},T_b}/RT = 8989/1.98721/T$ | 16.560 | 12.120 |
| $\exp 0.408\Delta\widetilde{U}_{\text{vap},T_b}/RT$ | 859.6 | 140.5 |
| $\widetilde{N}h/\widetilde{V}$, g/cm·s | $2.22 \times 10^{-4}$ | $2.12 \times 10^{-4}$ |
| Predicted liquid viscosity, g/cm·s | 0.19 | 0.0298 |

*b.* The predicted values for Eq. 1.5-11 at 0°C and 100°C are:

| | | |
|---|---|---|
| $T$,K | 273.15 | 373.15 |
| $\widetilde{N}h/\widetilde{V}$, g/cm·s | $2.22 \times 10^{-4}$ | $2.12 \times 10^{-4}$ |
| $\exp(3.8T_b/T)$ | 179.7 | 44.70 |
| Predicted liquid viscosity, g/cm·s | 0.0398 | 0.0095 |

Summary of results:

| | | |
|---|---|---|
| Temperature, °C | 0 | 100 |
| Observed viscosity, centipoise[=]g/cm·s×100 | 1.787 | 0.2821 |
| Prediction of Eq. 1.5-9 | 19. | 2.98 |
| Prediction of Eq. 1.5-11 | 3.98 | 0.95 |

Both equations give poor predictions. This is not surprising, since the empirical formulas in Eqs. 1.5-8 *et seq.* are inaccurate for water and for other associated liquids.

**1A.7 Molecular velocity and mean free path.**

From eq. 1.4-1, the mean molecular velocity in $O_2$ at 273.2 K is

$$\overline{u} = \sqrt{\frac{8RT}{\pi M}} = \sqrt{\frac{8 \times 8.31451 \times 10^7 \times 273.2}{\pi \times 32.00}} = 4.25 \times 10^4 \text{ cm/s}$$

From eq. 1.4-3, the mean free path in $O_2$ at 1 atm and 273.2 K is

$$\lambda = \frac{RT}{\sqrt{2}\pi d^2 p \widetilde{N}} = \frac{82.0578 \times 273.2}{\sqrt{2}\pi(3 \times 10^{-8})^2 \times 1 \times 6.02214 \times 10^{23}} = 9.3 \times 10^{-6} \text{ cm}$$

Hence, the ratio of the mean free path to the molecular diameter is $(9.3 \times 10^{-4}/3 \times 10^{-8}) = 3.1 \times 10^4$ under these conditions. At liquid states, on the other hand, the corresponding ratio would be on the order of unity or even less.

**1B.1 Velocity profiles and stress components**

*a.* $\tau_{xy} = \tau_{yx} = -\mu b$, and all other $\tau_{ij}$ are zero.

$\rho v_x v_x = \rho b^2 y^2$, and all other $\rho v_i v_j$ are zero.

*b.* $\tau_{xy} = \tau_{yx} = -2\mu b$, and all other $\tau_{ij}$ are zero.

$\rho v_x v_x = \rho b^2 y^2$, $\rho v_x v_y = \rho v_y v_x = \rho b^2 xy$, $\rho v_y v_y = \rho b^2 x^2$, and all other $\rho v_i v_j$ are zero.

*c.* All $\tau_{ij}$ are zero

$\rho v_x v_x = \rho b^2 y^2$, $\rho v_x v_y = \rho v_y v_x = -\rho b^2 xy$, $\rho v_y v_y = \rho b^2 x^2$ and all other $\rho v_i v_j$ are zero.

*d.* $\tau_{xx} = \tau_{yy} = \mu b$, $\tau_{zz} = -2\mu b$, and all others are zero. the components of $\rho\mathbf{vv}$ may be given in the matrix:

$$\rho\mathbf{vv} = \begin{pmatrix} \rho v_x v_x = \frac{1}{4}\rho b^2 x^2 & \rho v_x v_y = \frac{1}{4}\rho b^2 xy & \rho v_x v_z = -\frac{1}{2}\rho b^2 xz \\ \rho v_y v_x = \frac{1}{4}\rho b^2 xy & \rho v_y v_y = \frac{1}{4}\rho b^2 y^2 & \rho v_y v_z = -\frac{1}{2}\rho b^2 yz \\ \rho v_z v_x = -\frac{1}{2}\rho b^2 xz & \rho v_z v_y = -\frac{1}{2}\rho b^2 yz & \rho v_z v_z = \rho b^2 z^2 \end{pmatrix}$$

**1B.2 A fluid in a state of rigid rotation**

*a*. A particle within a rigid body rotating with an angular velocity vector **w** has a velocity given by $\mathbf{v} = [\mathbf{w} \times \mathbf{r}]$. If the angular velocity vector is in the +z-direction, then there are two nonzero velocity components given by $v_x = -w_z y$ and $v_y = +w_z x$. Hence the magnitude of the angular velocity vector is $b$ in Problem 1B.1(*c*).

*b*. For the velocity components of Problem 1B.1(*c*),

$$\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} = 0 \qquad \text{and} \qquad \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = 2b$$

*c*. In Eq. 1.2-4, we selected only the linear *symmetric* combinations of derivatives of the velocity, so that in pure rotation there would be no viscous forces present. In (*b*) we see that the antisymmetric combination is nonzero in a purely rotational motion.

**1B.3 Viscosity of suspensions**

Expanding the Mooney expression, we get (with $\varepsilon = \phi/\phi_0$)

$$\frac{\mu_{\text{eff}}}{\mu_0} = 1 + \left(\frac{\frac{5}{2}\phi}{1-\varepsilon}\right) + \frac{1}{2!}\left(\frac{\frac{5}{2}\phi}{1-\varepsilon}\right)^2 + \frac{1}{3!}\left(\frac{\frac{5}{2}\phi}{1-\varepsilon}\right)^3 + \cdots$$

$$= 1 + \tfrac{5}{2}\phi\left(1+\varepsilon+\varepsilon^2+\cdots\right) + \tfrac{25}{8}\phi^2(1+2\varepsilon+\cdots) + \tfrac{125}{48}\phi^3(1+\cdots)+\cdots$$

$$= 1 + \frac{5}{2}\phi + \phi^2\left(\frac{25}{8} + \frac{5}{2}\frac{1}{\phi_0}\right) + \phi^3\left(\frac{125}{48} + \frac{25}{4}\frac{1}{\phi_0} + \frac{5}{2}\frac{1}{\phi_0^2}\right) + \cdots$$

The first two terms match exactly with the first two terms in Eq. 1B.3-1. We can make the third term match exactly, by setting

$$\frac{25}{8} + \frac{5}{2}\frac{1}{\phi_0} = 7.17 \quad \text{whence } \phi_0 = 0.618$$

and the coefficient of $\phi^3$ becomes

$$\frac{125}{48} + \frac{25}{4}\frac{1}{0.618} + \frac{5}{2}\frac{1}{0.382} = 20.26$$

If we try $\phi_0 = 0.70$, the coefficients of $\phi^2$ and $\phi^3$ become 6.70 and 17.6 respectively. This gives a somewhat better find of Vand's data.

**1C.1 Some consequences of the Maxwell-Boltzmann equation**

*a*. The mean speed is

$$\bar{u} = \frac{\int_0^\infty u^3 e^{-mu^2/2\kappa T} du}{\int_0^\infty u^2 e^{-mu^2/2\kappa T} du} = \sqrt{\frac{2\kappa T}{m}} \frac{\int_0^\infty \xi^3 e^{-\xi^2} d\xi}{\int_0^\infty \xi^2 e^{-\xi^2} d\xi} = \sqrt{\frac{2\kappa T}{m}} \frac{\frac{1}{2}}{\frac{1}{4}\sqrt{\pi}} = \sqrt{\frac{8\kappa T}{\pi m}}$$

*b*. First rewrite Eq. 1C.1-4 as

$$\bar{u}_x = \frac{\int_{-\infty}^\infty u_x e^{-mu_x^2/2\kappa T} du_x}{\int_{-\infty}^\infty e^{-mu_x^2/2\kappa T} du_x} \cdot \frac{\int_{-\infty}^\infty e^{-mu_y^2/2\kappa T} du_y}{\int_{-\infty}^\infty e^{-mu_y^2/2\kappa T} du_y} \cdot \frac{\int_{-\infty}^\infty e^{-mu_z^2/2\kappa T} du_z}{\int_{-\infty}^\infty e^{-mu_z^2/2\kappa T} du_z}$$

The integral over $u_x$ in the numerator of the first factor is zero because the integrand is the product of a factor "$u_x$" (an odd function of the integration variable about $u_x = 0$) and an exponential function (an even function), and the range of integration extends equally far in the positive and negative directions.

*c*. The mean kinetic energy per molecule is

$$\tfrac{1}{2} m\overline{u^2} = \tfrac{1}{2} m \frac{\int_0^\infty u^4 e^{-mu^2/2\kappa T} du}{\int_0^\infty u^2 e^{-mu^2/2\kappa T} du} = \tfrac{1}{2} m \frac{2\kappa T}{m} \frac{\int_0^\infty \xi^4 e^{-\xi^2} d\xi}{\int_0^\infty \xi^2 e^{-\xi^2} d\xi} = \kappa T \frac{\frac{3}{8}\sqrt{\pi}}{\frac{1}{4}\sqrt{\pi}} = \tfrac{3}{2}\kappa T$$

and is thus $\frac{1}{2}\kappa T$ for each degree of freedom.

**1C.2 The wall collision frequency**

When we change to dimensionless variables in the second line of Eq. 1C.2-1, we get

$$Z = n\left(\frac{m}{2\pi\kappa T}\right)^{3/2}\frac{2\kappa T}{m}\left(\int_0^\infty \xi e^{-\xi^2}d\xi\right)\sqrt{\frac{2\kappa T}{m}}\left(\int_0^\infty e^{-\xi^2}d\xi\right)\sqrt{\frac{2\kappa T}{m}}\left(\int_0^\infty e^{-\xi^2}d\xi\right)$$

$$= n\left(\frac{m}{2\pi\kappa T}\right)^{3/2}\left(\frac{2\kappa T}{m}\cdot\frac{1}{2}\right)\left(\sqrt{\frac{2\kappa T}{m}}\cdot\sqrt{\pi}\right)\left(\sqrt{\frac{2\kappa T}{m}}\cdot\sqrt{\pi}\right) = n\sqrt{\frac{\kappa T}{2\pi m}}$$

**1C.3 The pressure in an ideal gas**

*a*. The dimensions of the quantities in Eq. 1C.3-1 are

| | | |
|---|---|---|
| $S$ | $[=]$ | $L^2$ |
| $u_x$ | $[=]$ | $L/t$ |
| $\Delta t$ | $[=]$ | $t$ |
| $m$ | $[=]$ | $M$ |
| $f$ | $[=]$ | $(1/L^3)(L/t)^{-3}$ |
| $du_x du_y du_z$ | $[=]$ | $(L/t)^3$ |

Using these units, one finds that the expression on the right of Eq. 1C.3-1 has units of $M/Lt^2$ (which are the same as the units of force per area).

*b*. Combining Eqs. 1C.1-1 and 1C.3-1 we get

$$p = 2nm\left(\frac{m}{2\pi\kappa T}\right)^{3/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} u_x^2 e^{-m\left(u_x^2+u_x^2u_x^2\right)/2\kappa T}\,du_x du_y du_z$$

$$= 2nm\left(\frac{m}{2\pi\kappa T}\right)^{3/2}\int_{-\infty}^{\infty} u_x^2 e^{-mu_x^2/2\kappa T}du_x \cdot \int_{-\infty}^{\infty} e^{-mu_y^2/2\kappa T}du_y \cdot \int_{0}^{\infty} e^{-mu_z^2/2\kappa T}du_z$$

$$= 2nm\left(\frac{2\kappa T}{m}\right)\left(\frac{1}{\pi}\right)^{3/2}\int_0^{\infty}\xi^2 e^{-\xi^2}d\xi \cdot \int_{-\infty}^{\infty}\eta e^{-\eta^2}d\eta \cdot \int_{-\infty}^{\infty}\zeta^2 e^{-\zeta^2}d\zeta$$

$$= 2nm\left(\frac{2\kappa T}{m}\right)\left(\frac{1}{\pi}\right)^{3/2}\left(\tfrac{\sqrt{\pi}}{4}\right)\left(\sqrt{\pi}\right)\left(\sqrt{\pi}\right) = n\kappa T$$

**1D.1 Uniform rotation of a fluid**

*a*. For the special case that $\mathbf{w} = \boldsymbol{\delta}_z w$, we get

$$\mathbf{v} = [\mathbf{w} \times \mathbf{r}] = \Sigma_i \Sigma_j \Sigma_k \varepsilon_{ijk} \boldsymbol{\delta}_i w_j x_k = w(\boldsymbol{\delta}_1 \varepsilon_{132} y + \boldsymbol{\delta}_2 \varepsilon_{231} x) = w(-\boldsymbol{\delta}_1 y + \boldsymbol{\delta}_2 x)$$

Then using Eqs. A.6-1, 2, 13 and 14, we can get the velocity components in cylindrical coordinates

$$\begin{aligned} v_r &= (\mathbf{v} \cdot \boldsymbol{\delta}_r) = w\left(\left(-\boldsymbol{\delta}_x y + \boldsymbol{\delta}_y x\right) \cdot \boldsymbol{\delta}_r\right) = w(-y\cos\theta + x\sin\theta) \\ &= w(-r\sin\theta\cos\theta + x\cos\theta\sin\theta) = 0 \end{aligned}$$

$$\begin{aligned} v_\theta &= (\mathbf{v} \cdot \boldsymbol{\delta}_\theta) = w\left(\left(-\boldsymbol{\delta}_x y + \boldsymbol{\delta}_y x\right) \cdot \boldsymbol{\delta}_\theta\right) = w\left((-y)(-\sin\theta) + x\cos\theta\right) \\ &= w(r\sin\theta\sin\theta + r\cos\theta\cos\theta) = wr \end{aligned}$$

Therefore, the angular velocity of every point in the fluid is $v_\theta / r = w$, which is a constant, and there is no radial velocity. This is the way a rigid body rotates at constant angular velocity.

*b*. The vector operations are (using the abbreviated notation of §A.9 and the Einstein summation convention)

$$(\nabla \cdot \mathbf{v}) = \partial_i v_i = \partial_i \varepsilon_{imn} w_m x_n = \varepsilon_{imn} w_m \delta_{in} = \varepsilon_{nmn} w_m = 0$$

$$\begin{aligned} \{\nabla \mathbf{v}\}_{ij} &= \{\nabla[\mathbf{w} \times \mathbf{r}]\}_{ij} = \partial_i \varepsilon_{jmn} w_m x_n = \varepsilon_{jmn} w_m \delta_{in} = \varepsilon_{jmi} w_m \\ &= -\{\nabla \mathbf{v}\}_{ji} = -\{\nabla \mathbf{v}\}_{ij}^\dagger \end{aligned}$$

and from this last result we see that $\nabla \mathbf{v} + (\nabla \mathbf{v})^\dagger = 0$.

*c*. The results above indicate that for a fluid is a state of pure rotation, the tensor $\boldsymbol{\tau}$ is identically zero. That is, there are no viscous stresses present in the fluid. This was the assertion made just before Eq. 1.2-4.

**1D.2 Force on a surface of arbitrary orientation.**

*a*. We can specify the surface area *and* the orientation of the surface of $\Delta OBC$ as $\mathbf{n}dS$. To project this surface onto the $yz$-plane, we take the dot product with $\boldsymbol{\delta}_x$, so that the area of $\Delta OBC$ is $(\mathbf{n}\cdot\boldsymbol{\delta}_x)dS$.

*b*. The force per unit area on three triangles perpendicular to the three coordinate axes are

Force on $\Delta OBC = \boldsymbol{\delta}_x\pi_{xx} + \boldsymbol{\delta}_y\pi_{xy} + \boldsymbol{\delta}_z\pi_{xz}$

Force on $\Delta OCA = \boldsymbol{\delta}_x\pi_{yx} + \boldsymbol{\delta}_y\pi_{yy} + \boldsymbol{\delta}_z\pi_{yz}$

Force on $\Delta OAB = \boldsymbol{\delta}_x\pi_{zx} + \boldsymbol{\delta}_y\pi_{zy} + \boldsymbol{\delta}_z\pi_{zz}$

*c*. Force balance on the volume OABC is then

$$\begin{aligned}\boldsymbol{\pi}_n dS = &\left(\boldsymbol{\delta}_x\pi_{xx} + \boldsymbol{\delta}_y\pi_{xy} + \boldsymbol{\delta}_z\pi_{xz}\right)(\mathbf{n}\cdot\boldsymbol{\delta}_x)dS \\ &+\left(\boldsymbol{\delta}_x\pi_{yx} + \boldsymbol{\delta}_y\pi_{yy} + \boldsymbol{\delta}_z\pi_{yz}\right)(\mathbf{n}\cdot\boldsymbol{\delta}_y)dS \\ &+\left(\boldsymbol{\delta}_x\pi_{zx} + \boldsymbol{\delta}_y\pi_{zy} + \boldsymbol{\delta}_z\pi_{zz}\right)(\mathbf{n}\cdot\boldsymbol{\delta}_z)dS\end{aligned}$$

or

$$\begin{aligned}\boldsymbol{\pi}_n = &\left[\mathbf{n}\cdot\boldsymbol{\delta}_x\boldsymbol{\delta}_x\pi_{xx}\right] + \left[\mathbf{n}\cdot\boldsymbol{\delta}_x\boldsymbol{\delta}_y\pi_{xy}\right] + \left[\mathbf{n}\cdot\boldsymbol{\delta}_x\boldsymbol{\delta}_z\pi_{xz}\right] \\ &+\left[\mathbf{n}\cdot\boldsymbol{\delta}_y\boldsymbol{\delta}_x\pi_{yx}\right] + \left[\mathbf{n}\cdot\boldsymbol{\delta}_y\boldsymbol{\delta}_y\pi_{yy}\right] + \left[\mathbf{n}\cdot\boldsymbol{\delta}_y\boldsymbol{\delta}_z\pi_{yz}\right] \\ &+\left[\mathbf{n}\cdot\boldsymbol{\delta}_z\boldsymbol{\delta}_x\pi_{zx}\right] + \left[\mathbf{n}\cdot\boldsymbol{\delta}_z\boldsymbol{\delta}_y\pi_{zy}\right] + \left[\mathbf{n}\cdot\boldsymbol{\delta}_z\boldsymbol{\delta}_z\pi_{zz}\right] \\ &= \sum_i\sum_j\left[\mathbf{n}\cdot\boldsymbol{\delta}_i\boldsymbol{\delta}_j\pi_{ij}\right] = [\mathbf{n}\cdot\boldsymbol{\pi}]\end{aligned}$$

**2A.1 Thickness of a falling film.**

*a.* The volume flow rate $w/\rho$ per unit wall width $W$ is obtained from Eq. 2.2-25:

$$\frac{w}{\rho W} = \frac{\nu \mathrm{Re}}{4} = \frac{(1.0037 \times 10^{-2})(10)}{4} = 2.509 \times 10^{-2}\ \mathrm{cm^2/s}$$

Here the kinematic viscosity $\nu$ for liquid water at 20°C was obtained from Table 1.1-2. Since 1 ft=12×2.54 cm, 1 hr =3600 s, and 1 gal=231.00 in$^3$×(2.54cm/in)$^3$=3785.4 cm$^3$ (see Appendix F), the result in the requested units is

$$\begin{aligned}\frac{w}{\rho W} &= 0.02509\ \mathrm{cm^2/s} \times \frac{1}{3785.4}\ \mathrm{gal/cm^2} \times 30.48\ \mathrm{cm/ft} \times 3600\ \mathrm{s/hr} \\ &= 0.727\ \text{U.S. gal/hr·ft}\end{aligned}$$

*b.* The film thickness is calculated from Eqs. 2.2-25 and 2.2-22 as

$$\begin{aligned}\delta &= \left(\frac{3\nu}{g\cos\beta}\frac{w}{\rho W}\right)^{1/3} \\ &= \left(\frac{3\nu}{g\cos\beta}\frac{\nu \mathrm{Re}}{4}\right)^{1/3} \\ &= \left(\frac{3 \times 1.0037 \times 10^{-2}}{(980.665)(1.0)}(2.509 \times 10^{-2})\right)^{1/3} = 0.009167\ \mathrm{cm} \\ &= 0.00361\ \mathrm{in.}\end{aligned}$$

## 2A.2 Determination of capillary radius by flow measurement.

Assuming the flow to be laminar, we solve Eq. 2.3-21 for the capillary radius:

$$R = \sqrt[4]{\frac{8\mu L w}{\pi \rho \Delta \mathcal{P}}} = \sqrt[4]{\frac{8\nu L w}{\pi \Delta \mathcal{P}}}$$

Insertion of the data in mks units gives

$$\begin{aligned} R &= \sqrt[4]{\frac{8(4.03 \times 10^{-5})(0.5002)(2.997 \times 10^{-3})}{(3.1416)(4.829 \times 10^{5})}} \\ &= \sqrt[4]{3.186 \times 10^{-13}} \\ &= 7.51 \times 10^{-4} \text{ m} = 7.51 \times 10^{-2} \text{ cm} \end{aligned}$$

As a check on this result, we calculate the corresponding Reynolds number:

$$\begin{aligned} \mathrm{Re} &= \frac{D\langle v_z \rangle \rho}{\mu} = \frac{4w}{\pi D \mu} = \frac{2w}{\pi R \nu \rho} \\ &= \frac{2}{\pi} \frac{(2.997 \times 10^{-3})}{(7.51 \times 10^{-4})(4.03 \times 10^{-5})(0.9552 \times 10^{3})} = 66.0 \end{aligned}$$

This value supports our assumption of laminar flow. Since the entrance length, $L_e = 0.035 D\mathrm{Re} = 0.35$ cm is less than $L$, the entrance-effect correction to $R$ is at most of the order of $|[1-(L_e/L)]^{1/4} - 1|$, or 0.2 percent of $R$ in the present example.

Difficulties with this method include: (1) Inability to account for departures from a straight, circular cylindrical wall geometry. (2) Inability to account for inadvertent spatial and temporal variations of temperature, hence of the fluid density and viscosity.

A simpler method is to measure the length $L$ and mass $m$ of a small slug of liquid mercury (or another liquid of known density) injected into the tube, and calculate the mean radius $R$ of the slug as $(m/[\rho \pi L])^{1/2}$, on the assumption that the slug is a right circular cylinder. This method allows comparisons of mean $R$ values for various intervals of the tube length.

**2A.3 Volume rate of flow thrugh an annulus.**

Assuming the flow to be laminar, we use Eq. 2.4-17 to calculate the volume flow rate $w/\rho$, with the specifications

$$\begin{aligned}
\kappa &= 0.495/1.1 = 0.45 \\
\mu &= 136.8\ (\mathrm{lb}_m/\mathrm{ft{\cdot}hr})(1\ \mathrm{hr}/3600\mathrm{s}) = 3.80 \times 10^{-2}\ \mathrm{lb}_m/\mathrm{ft{\cdot}s} \\
(\mathcal{P}_0 - \mathcal{P}_L) &= (5.39\ \mathrm{psi})(4.6330 \times 10^3\ \mathrm{poundals/ft^2/psi}) = 2.497 \times 10^4\ \mathrm{lb}_m/\mathrm{ft}\cdots\mathrm{s}^2 \\
R &= 1.1\ \mathrm{in.} = 1.1/12\ \mathrm{ft}
\end{aligned}$$

Here Appendix F has been used for the conversions of units. With these specifications, Eq. 2.4-17 gives

$$\begin{aligned}
\frac{w}{\rho} &= \frac{(\pi)(2.497 \times 10^4)(1.1/12)^4}{(8)(3.80 \times 10^{-2})(27)} \left[(1 - (0.45)^4) - \frac{(1 - (0.45)^2)^2}{\ln(1/0.45)}\right] \\
&= (0.49242)\left[(1 - 0.04101) - \frac{(1 - 0.2025)^2}{\ln(1/0.495)}\right] \\
&= (0.6748)[0.1625] = 0.110\ \mathrm{ft}^3/\mathrm{s}
\end{aligned}$$

As a check on our assumption of laminar flow, we calculate the Reynolds number:

$$\begin{aligned}
\mathrm{Re} &= \frac{2R(1-\kappa)\langle v_z\rangle\rho}{\mu} = \frac{2w}{\pi R\mu(1+\kappa)} \\
&= \frac{2(0.110)(80.3)}{(3.1416)(1.1/12)(3.80 \times 10^{-2})(1.45)} = 1110
\end{aligned}$$

This value is well within the laminar range, so our assumption of laminar flow is confirmed.

**2A.4 Loss of catalyst particles in stack gas.**

*a.* Rearrangement of Eq. 2.6-17 gives the terminal velocity

$$v_t = D^4(\rho_s - \rho)g/18\mu$$

in which $D$ is the sphere diameter. Particles settling at $v_t$ greater than the centerline gas velocity will not go up the stack. Hence, the value of $D$ that corresponds to $v_t = 1.0$ ft/s will be the maximum diameter of particles that can be lost in the stack gas of the present system.

Conversions of data to cgs units give

$$\begin{aligned} v_t &= (1 \text{ ft/s})(12 \times 2.54 \text{ cm/ft}) = 30.48 \text{ cm/s} \\ \rho &= (0.045 \text{ lb}_m/\text{ft}^3)(453.59 \text{ g/lb}_m)((12 \times 2.54)^{-3} \text{ ft}^3/\text{cm}^3) \\ &= 7.2 \times 10^{-4} \text{ g/cm}^3 \end{aligned}$$

Hence,

$$\begin{aligned} D_{\max} &= \sqrt{\frac{18\mu v_t}{(\rho_s - \rho)g}} = \sqrt{\frac{(18)(0.00026)(30.48)}{(1.2 - 7.2 \times 10^{-4})(980.7)}} \\ &= \sqrt{1.21 \times 10^{-4}} = 1.1 \times 10^{-2} \text{ cm} = 110 \text{ microns} \end{aligned}$$

*b.* Equation 2.6-17 was derived for Re<< 1, but holds approximately up to Re=1. For the system at hand,

$$\text{Re} = \frac{Dv_t\rho}{\mu} = \frac{(1.1 \times 10^{-2})(30.5)(7.2 \times 10^{-4})}{(0.00026)} = 0.93$$

Hence, the result in *a.* is approximately correct. Methods are given in Chapter 6 for solving problems of this type without the creeping-flow assumption.

**2B.1 Different choice of coordinates for the falling film problem**

Set up a momentum balance as before, and obtain the differential equation

$$\frac{d\tau_{\bar{x}z}}{d\bar{x}} = \rho g \cos\beta$$

Since no momentum is transferred at $\bar{x} = \delta$, then at that plane $\tau_{\bar{x}z} = 0$. This boundary condition enables us to find that $C_1 = -\rho g \delta \cos\beta$, and the momentum flux distribution is

$$\tau_{\bar{x}z} = -\rho g \delta \cos\beta \left(1 - \frac{\bar{x}}{\delta}\right)$$

Note that the momentum flux is in the negative $\bar{x}$-direction.

Insertion of Newton's law of viscosity $\tau_{\bar{x}z} = -\mu(dv_z/d\bar{x})$ into the foregoing equation gives the differential equation for the velocity distribution:

$$\frac{dv_z}{d\bar{x}} = \left(\frac{\rho g \delta \cos\beta}{\mu}\right)\left(1 - \frac{\bar{x}}{\delta}\right)$$

This first-order differential equation can be integrated to give

$$v_z = \left(\frac{\rho g \delta^2 \cos\beta}{\mu}\right)\left(\frac{\bar{x}}{\delta} - \frac{1}{2}\left(\frac{\bar{x}}{\delta}\right)^2\right)$$

The constant $C_2$ is zero, because $v_z = 0$ at $\bar{x} = 0$.

We note that $\bar{x}$ and $x$ are related by $\bar{x}/\delta = 1 - (x/\delta)$. When this is substituted into the velocity distribution above, we get

$$v_z = \left(\frac{\rho g \delta^2 \cos\beta}{\mu}\right)\left(\left(1 - \frac{x}{\delta}\right) - \frac{1}{2}\left[1 - 2\frac{x}{\delta} + \left(\frac{\bar{x}}{\delta}\right)^2\right]\right)$$

which can be rearranged to give Eq. 2.2-18.

**2B.2 Alternate procedure for solving flow problems**

Substituting Eq. 2.2-14 into Eq. 2.2-10 gives

$$\frac{d}{dx}\left(-\mu\frac{dv_z}{dx}\right)=\rho g\cos\beta \qquad \text{or} \qquad \frac{d^2v_z}{dx^2}=-\frac{\rho g\cos\beta}{\mu}$$

Integrate twice with respect to $x$ (see Eq. C.1-10) and get

$$v_z=-\frac{\rho g\cos\beta}{2\mu}x^2+C_1x+C_2$$

Then use the no-slip boundary condition that $v_z=0$ at $x=\delta$, and the zero momentum flux boundary condition that $dv_z/dx=0$ at $x=0$. The second gives $C_1=0$, and the first gives $C_2=(\rho g\cos\beta/2\mu)\delta^2$. Substitution of these constants into the general solution and rearranging then gives Eq. 2.2-18.

**2B.3 Laminar flow in a narrow slit**

*a*. The momentum balance leads to

$$\frac{d}{dx}\tau_{xz} = \frac{(\mathcal{P}_0 - \mathcal{P}_L)}{L} \qquad \text{and} \qquad \tau_{xz} = \frac{(\mathcal{P}_0 - \mathcal{P}_L)}{L}x + C_1$$

Substitution of Newton's law $\tau_{xz} = -\mu\frac{dv_z}{dx}$ into the above gives

$$\frac{dv_z}{dx} = -\frac{(\mathcal{P}_0 - \mathcal{P}_L)}{\mu L}x + \frac{C_1}{\mu} \qquad \text{or} \qquad v_z = -\frac{(\mathcal{P}_0 - \mathcal{P}_L)x^2}{2\mu L} + \frac{C_1}{\mu}x + C_2.$$

Use of the no-slip boundary conditions at $x = \pm B$ gives the expressions in Eq. 2B.3-1 and 2. One can also see that $C_1 = 0$ directly, since we know that the velocity distribution must be symmetrical about the plane $x = 0$.

*b*. The maximum velocity is at the middle of the slit and is

$$v_{z,\max} = \frac{(\mathcal{P}_0 - \mathcal{P}_L)B^2}{2\mu L} \qquad \text{and hence} \qquad \frac{v_z}{v_{z,\max}} = 1 - \left(\frac{x}{B}\right)^2$$

The ratio of the average to the maximum velocity is then

$$\frac{\langle v_z \rangle}{v_{z,\max}} = \frac{\int_0^W \int_{-B}^{B}\left[1-(x/B)^2\right]dxdy}{\int_0^W \int_{-B}^{B} dxdy} = \frac{\int_0^1 (1-\xi^2)d\xi}{\int_0^1 d\xi} = \left(1 - \tfrac{1}{3}\right) = \tfrac{2}{3}$$

*c*. The mass rate of flow is

$$w = \rho(2BW)\langle v_z \rangle = \rho(2BW)\left(\tfrac{2}{3}\right)\frac{(\mathcal{P}_0 - \mathcal{P}_L)B^2}{2\mu L} = \frac{2}{3}\frac{(\mathcal{P}_0 - \mathcal{P}_L)\rho B^3 W}{\mu L}$$

*d*. In Eq. 2.5-22, set both viscosities equal to $\mu$. set $b$ equal to $B$, and multiply by $BW\rho$.

**2B.4 Laminar slit flow with a moving wall ("plane Couette flow")**

Start with the velocity distribution from part (*a*) of Problem 2B.3 (in terms of the integration constants). Determine $C_1$ and $C_2$ from the boundary conditions that $v_z = 0$ at $x = -B$, and $v_z = V$ at $x = B$. This leads to

$$v_z = -\frac{(\mathcal{P}_0 - \mathcal{P}_L)B^2}{2\mu L}\left[1 - \left(\frac{x}{B}\right)^2\right] + \frac{V}{2}\left(1 + \frac{x}{B}\right)$$

This expression can be differentiated with respect to $x$ and then Newton's law of viscosity $\tau_{xz} = -\mu(dv_z/dx)$ can be used to get the expression for the stress tensor. Notice that the velocity distribution is no longer symmetric about the midplane, so that $C_1 \neq 0$.

**2B.5 Interrelation of slit and annulus formulas**

From Eq. 2.4-17 we get

$$w\left(\frac{8\mu L}{\pi(\mathcal{P}_0-\mathcal{P}_L)R^4\rho}\right)=\left(1-(1-\varepsilon)^4\right)+\frac{\left(1-(1-\varepsilon)^2\right)^2}{\ln(1-\varepsilon)}$$

$$=\left(1-1+4\varepsilon-6\varepsilon^2+4\varepsilon^3-\varepsilon^4\right)+\frac{\left(1-1+2\varepsilon-\varepsilon^2\right)^2}{-\varepsilon-\frac{1}{2}\varepsilon^2-\frac{1}{3}\varepsilon^3-\frac{1}{4}\varepsilon^4-\cdots}$$

$$=\left(4\varepsilon-6\varepsilon^2+4\varepsilon^3-\varepsilon^4\right)-\frac{\left(4\varepsilon^2-4\varepsilon^3+\varepsilon^4\right)}{\varepsilon+\frac{1}{2}\varepsilon^2+\frac{1}{3}\varepsilon^3+\frac{1}{4}\varepsilon^4+\cdots}$$

$$=\left(4\varepsilon-6\varepsilon^2+4\varepsilon^3-\varepsilon^4\right)-\left(4\varepsilon-6\varepsilon^2+\tfrac{8}{3}\varepsilon^3-\tfrac{1}{3}\varepsilon^4+\cdots\right)$$

This gives, finally, a result in agreement with Eq. 2B.5-1

$$w=\frac{\pi(\mathcal{P}_0-\mathcal{P}_L)R^4\rho}{8\mu L}\left(\tfrac{4}{3}\varepsilon^3-\tfrac{2}{3}\varepsilon^4+\cdots\right)$$

**2B.6 Flow of a film on the outside of a circular tube**

*a*. A momentum balance on the film gives

$$-\frac{d(r\tau_{rz})}{dr}+\rho g r=0 \qquad \text{or} \qquad \mu\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)+\rho g r=0$$

The latter may be integrated to give

$$v_z=-\frac{\rho g r^2}{4\mu}+C_1\ln r+C_2$$

Next use the boundary conditions that at $r=R$, $v_z=0$ (no slip) and that at $r=aR$, $dv_z/dr=0$. When the integration constants have been found, we get for the velocity distribution

$$v_z=\frac{\rho g R^2}{4\mu}\left[1-\left(\frac{r}{R}\right)^2+2a^2\ln\frac{r}{R}\right]$$

*b*. The mass rate of flow in the film is then

$$w=\int_0^{2\pi}\int_R^{aR}\rho v_z r dr d\theta=2\pi R^2\rho\int_1^a v_z\xi d\xi$$

in which a dimensionless radial coordinate $\xi=r/R$ has been introduced. Then

$$\begin{aligned} w&=\frac{\pi\rho^2 g R^4}{2\mu}\int_1^a\left(1-\xi^2+2a^2\ln\xi\right)\xi d\xi \\ &=\frac{\pi\rho^2 g R^4}{2\mu}\left(\tfrac{1}{2}\xi^2-\tfrac{1}{4}\xi^4+2a^2\left[-\tfrac{1}{4}\xi^2+\tfrac{1}{2}\xi^2\ln\xi\right]\right)\Big|_1^a \\ &=\frac{\pi\rho^2 g R^4}{8\mu}\left(-1+4a^2-3a^4+4a^4\ln a\right)\end{aligned}$$

*c*. If we set $a=1+\varepsilon$ (where $\varepsilon$ is small) and expand in powers of $\varepsilon$ using §C.2, we get

$$w = \frac{\pi\rho^2 g R^4}{8\mu}\left(\tfrac{16}{3}\varepsilon^3 + O\left(\varepsilon^4\right)\right) \approx \frac{2\pi\rho^2 g R^4 \varepsilon^3}{3\mu}$$

This is in agreement with Eq. 2.2-21 if we make the identifications $W = 2\pi R$ and $\delta = \varepsilon R$ (and furthermore consider only the case that $\cos\beta = 1$.

**2B.7 Annular flow with inner cylinder moving axially**

*a*. The momentum balance is the same as that in Eq. 2.3-11 or Eq. 2.4-2, but with the pressure-difference term omitted. We can substitute Newton's law of viscosity into this equation to get

$$-\mu\frac{dv_z}{dr}=\frac{C_1}{r}, \text{ whence } \quad v_z=-\frac{C_1}{\mu}\ln r+C_2 \quad \text{or} \quad \frac{v_z}{v_0}=-D_1\ln\frac{r}{R}+D_2$$

That is, we select new integration constants, so that they are dimensionless. These integration constants are determined from the no-slip conditions at the cylindrical surfaces: $v_z(\kappa R)=v_0$ and $v_z(R)=0$. The constants of integration are $D_2=0$ and $D_1=-1/\ln\kappa$. This leads then directly to the result given in the book.

*b*. The mass rate of flow is

$$w=\int_0^{2\pi}\int_{\kappa R}^{R}\rho v_z r dr d\theta=2\pi\rho\frac{v_0R^2}{\ln\kappa}\int_\kappa^1(\ln\xi)\xi d\xi$$
$$=2\pi\rho\frac{v_0R^2}{\ln\kappa}\left(\tfrac{1}{2}\xi^2\ln\xi-\tfrac{1}{4}\xi^2\right)\Big|_\kappa^1=2\pi\rho\frac{v_0R^2}{\ln\kappa}\left(-\tfrac{1}{2}\kappa^2\ln\kappa-\tfrac{1}{4}\left(1-\kappa^2\right)\right)$$

which is equivalent to the answer in the text.

*c*. The force on a length $L$ of the rod

$$F=\int_0^L\int_0^{2\pi}\left(+\mu\frac{dv_z}{dr}\right)\Bigg|_{r=\kappa R}\kappa R d\theta dz=2\pi\kappa RL\mu v_0\frac{(1/\kappa R)}{\ln\kappa}$$

which gives the expression in the book.

*d*. When we replace $\kappa$ by $1-\varepsilon$ and expand in a Taylor series, we get

$$F=2\pi L(-\mu)v_0\frac{1}{-\left(\varepsilon+\frac{1}{2}\varepsilon^2+\frac{1}{3}\varepsilon^3+\frac{1}{4}\varepsilon^4\cdots\right)}=\frac{2\pi L\mu)v_0}{\varepsilon}\left(1-\tfrac{1}{2}\varepsilon-\tfrac{1}{12}\varepsilon^2\cdots\right)$$

To get this last result one has to do a long division involving the polynomial in the next-to-last step.

**2B.8 Analysis of a capillary flowmeter**

Designate the water by fluid "I" and the carbon tetrachloride by "II". Label the distance from $B$ to $C$ as "J". The mass rate of flow in the tube section "$AB$" is given by

$$w = \frac{\pi(\mathcal{P}_A - \mathcal{P}_B)R^4\rho_{\mathrm{I}}}{8\mu L} = \frac{\pi\left[(p_A - p_B) + \rho_{\mathrm{I}}gh\right]R^4\rho_{\mathrm{I}}}{8\mu L}$$

Since the fluid in the manometer is not moving, the pressures at $D$ and $E$ must be equal; hence

$$p_A + \rho_{\mathrm{I}}gh + \rho_{\mathrm{I}}gJ + \rho_{\mathrm{I}}gH = p_B + \rho_{\mathrm{I}}gJ + \rho_{\mathrm{II}}gH$$

from which we get

$$p_A - p_B + \rho_{\mathrm{I}}gh = (\rho_{\mathrm{II}} - \rho_{\mathrm{I}})gH$$

Insertion of this into the first equation above gives the expression for the mass rate of flow in terms of the difference in the densities of the two fluids, the acceleration of gravity, and the height $H$.

**2B.9 Low-density phenomena in compressible tube flow**

When we replace no-slip boundary condition of Eq. 2.3-17 by Eq. 2B.9-1, we get

$$C_2 = \frac{(p_0 - p_L)R^2}{4\mu L} + \frac{(p_0 - p_L)R\zeta}{2\mu L}$$

so that the velocity distribution in the tube is

$$v_z = \frac{(p_0 - p_L)R^2}{4\mu L}\left[1 - \left(\frac{r}{R}\right)^2\right] + \frac{(p_0 - p_L)R\zeta}{2\mu L}$$

Next we write the expression for $w$, but consider only the flow through a length $dz$ of the tube:

$$w = \int_0^{2\pi}\int_0^R \rho(z)v_z(r,z)rdrd\theta = 2\pi R^2\left(\frac{pM}{R_gT}\right)\int_0^1 v_z\xi d\xi$$

where we have introduced the ideal gas law, with $R_g$ being the gas constant (we use a subscript $g$ here to distinguish the gas constant from the tube radius). We have also introduced a dimensionless radial coordinate. When we introduce the velocity distribution above, we get

$$\begin{aligned} w &= \frac{2\pi R^4}{4\mu}\left(\frac{M}{R_gT}\right)\int_0^1\left(-p\frac{dp}{dz}\right)\left[(1-\xi^2) + \frac{2\zeta_0}{Rp}\right]\xi d\xi \\ &= \frac{\pi R^4}{8\mu}\left(\frac{M}{R_gT}\right)\left(-p\frac{dp}{dz}\right)\left(1 + \frac{4\zeta_0}{Rp}\right) \end{aligned}$$

This is now integrated over the length of the tube, keeping mind that the mass flow rate $w$ is constant over the entire length

$$\int_0^L w dz = -\frac{\pi R^4}{8\mu}\left(\frac{M}{R_gT}\right)\int_{p_0}^{p_L}\left(p + \frac{4\zeta_0}{R}\right)dp$$

$$= \frac{\pi R^4}{8\mu}\left(\frac{M}{R_g T}\right)\left(\frac{p_0^2 - p_L^2}{2} + \frac{4\zeta_0}{R}(p_0 - p_L)\right)$$

This gives

$$w = \frac{\pi(p_0 - p_L)R^4}{8\mu L}\left(\frac{M}{R_g T}\right)\left(\frac{p_0 + p_L}{2} + \frac{4\zeta_0}{R}\right)$$

$$= \frac{\pi(p_0 - p_L)R^4}{8\mu L}\left(\frac{p_{\text{avg}} M}{R_g T}\right)\left(1 + \frac{4\zeta_0}{R p_{\text{avg}}}\right)$$

which leads then to Eq. 2B.9-2.

**2B.10 Incompressible flow in a slightly tapered tube**

*a*. The radius at any downstream distance is

$$R(z) = R_0 + (R_L - R_0)(z/L)$$

*b*. Changing the independent variable proceeds as follows:

$$w = \frac{\pi R^4 \rho}{8\mu}\left(-\frac{d\mathcal{P}}{dR}\right)\left(\frac{dR}{dz}\right) = \frac{\pi R^4 \rho}{8\mu}\left(-\frac{d\mathcal{P}}{dR}\right)\left(\frac{R_L - R_0}{L}\right)$$

*c*. First we rearrange the equation in (*b*) to get

$$-\frac{d\mathcal{P}}{dR} = \left(\frac{8\mu w}{\pi\rho}\right)\left(\frac{L}{R_L - R_0}\right)\frac{1}{R^4}$$

Then we integrate this equation to get

$$-\int_{\mathcal{P}_0}^{\mathcal{P}_L} d\mathcal{P} = \left(\frac{8\mu w}{\pi\rho}\right)\left(\frac{L}{R_L - R_0}\right)\int_{R_0}^{R_L}\frac{1}{R^4}dR$$

whence we can get the pressure difference in terms of the mass rate of flow

$$\mathcal{P}_0 - \mathcal{P}_L = \left(\frac{8\mu w L}{3\pi\rho}\right)\left(\frac{R_L^{-3} - R_0^{-3}}{R_0 - R_L}\right)$$

Next we solve to get the mass flow rate

$$w = \left(\frac{3\pi(\mathcal{P}_0 - \mathcal{P}_L)\rho}{8\mu L}\right)\left(\frac{R_0 - R_L}{R_L^{-3} - R_0^{-3}}\right) = \left(\frac{\pi(\mathcal{P}_0 - \mathcal{P}_L)R_0^4\rho}{8\mu L}\right)\left(\frac{3}{R_0^4}\cdot\frac{R_0 - R_L}{R_L^{-3} - R_0^{-3}}\right)$$

This is the result, with the first factor being the solution for a straight tube, the second factor being a correction factor. It would be better to write the correction factor as "$1 - X$ ", so that the quantity $X$ gives the deviation from straight-tube behavior. The quantity $X$ is then

$$X = 1 - \frac{3}{R_0^4} \cdot \frac{R_0 - R_L}{R_L^{-3} - R_0^{-3}} = 1 - \frac{3\left[1 - \left(R_L/R_0\right)\right]}{\left(R_0/R_L\right)^3 - 1} = 1 - \frac{3\left[1 - \left(R_L/R_0\right)\right]\left(R_L/R_0\right)^3}{1 - \left(R_L/R_0\right)^3}$$

$$= 1 - \frac{3\left(R_L/R_0\right)^3}{1 + \left(R_L/R_0\right) + \left(R_L/R_0\right)^2} == \frac{1 + \left(R_L/R_0\right) + \left(R_L/R_0\right)^2 - 3\left(R_L/R_0\right)^3}{1 + \left(R_L/R_0\right) + \left(R_L/R_0\right)^2}$$

which then leads to the desired result in Eq. 2B.10-3.

**2B.11 The cone-and-plate viscometer**

*a*. In a parallel-plate system with rectilinear flow, the velocity distribution is just $v_x/v_0 = y/b$, where $b$ is the plate spacing and $v_0$ is the velocity of the upper plate. We now make the following correspondences between the parallel-plate system and the cone-plate system (using $\theta$ as the usual variable in spherical coordinates measured downward from the $z$-axis, and $\psi$ as the variable measured upward from the plate surface):

$$v_x \leftrightarrow v_\phi; \quad v_0 \leftrightarrow \Omega r; \quad b \leftrightarrow r\sin\psi_0; \quad y \leftrightarrow r\sin\psi \approx r\psi = r\left(\tfrac{1}{2}\pi - \theta\right)$$

When this correspondence is made, Eq. 2B.11-1 results.

*b*. From Eq. B.1-19, we get for the force per unit area in the $\phi$-direction on a face perpendicular to the $\theta$-direction

$$\tau_{\theta\phi} = -\mu\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\phi}{\sin\theta}\right) \approx -\mu\frac{1}{r}\frac{\partial v_\phi}{\partial\theta} = -\mu\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\Omega r\left(\frac{1}{2}\pi - \theta\right)}{\psi_0}\right)$$

Here we have used the fact that the angle between the cone and plate is so tiny that $\theta$ is very nearly $\frac{1}{2}\pi$ so that sin $\theta$ is very close to unity.

*c*. The torque is obtained by integrating the force times lever arm over the entire plate area:

$$T_z = \int_0^{2\pi}\int_0^R \left(\tau_{\theta\phi}\cdot r\right)\Big|_{\theta=\pi/2} r dr d\phi = = 2\pi\left(\frac{\mu\Omega}{\psi_0}\right)\int_0^R r^2 dr = 2\pi\left(\frac{\mu\Omega}{\psi_0}\right)\left(\frac{R^3}{3}\right)$$

which leads to Eq. 2B.11-3.

**2B.12 Flow of a fluid in a network of tubes**

At $A$ the pipe splits into three pipes, and at the next set of junctions the fluid flows equally in six pipes, and then at the next set of junctions the fluid flows back into three pipes, and finally at $B$ the fluid is all returned to a single pipe. Call the modified pressure at the junctions where three pipes split into six pipes $\mathcal{P}_{3\to 6}$, and that where six pipes join to form three pipes $\mathcal{P}_{6\to 3}$.

Then in each of the first set of three pipes

$$\frac{w}{3} = \frac{\pi(\mathcal{P}_A - \mathcal{P}_{3\to 6})R^4\rho}{8\mu L} \qquad \text{or} \qquad \mathcal{P}_A - \mathcal{P}_{3\to 6} = \frac{8\mu L w}{3\pi R^4\rho}$$

In each of the batch of six pipes

$$\frac{w}{6} = \frac{\pi(\mathcal{P}_{3\to 6} - \mathcal{P}_{6\to 3})R^4\rho}{8\mu L} \qquad \text{or} \qquad \mathcal{P}_{3\to 6} - \mathcal{P}_{6\to 3} = \frac{8\mu L w}{6\pi R^4\rho}$$

and in each of the final batch of 3 pipes

$$\frac{w}{3} = \frac{\pi(\mathcal{P}_{6\to 3} - \mathcal{P}_B)R^4\rho}{8\mu L} \qquad \text{or} \qquad \mathcal{P}_{6\to 3} - \mathcal{P}_B = \frac{8\mu L w}{3\pi R^4\rho}$$

When all the pressure differences are added together, the unknown quantities $\mathcal{P}_{3\to 6}$ and $\mathcal{P}_{6\to 3}$ cancel out, and we get

$$\mathcal{P}_A - \mathcal{P}_B = \frac{8\mu L w}{\pi R^4\rho}\left(\frac{5}{6}\right) \qquad \text{or} \qquad w = \frac{3\pi(\mathcal{P}_A - \mathcal{P}_B)R^4\rho}{20\mu L}$$

**2C.1 Performance of an electric dust collector**

*a*. First we solve the problem of the vertical motion of the particle as it falls under the action of the electromagnetic field. The equation of motion for the particle (without gravitational acceleration or Stokes drag) is

$$e\mathcal{E} = m\frac{d^2x}{dt^2}$$

This equation may be integrated with the initial conditions that $x = x_0$ and $dx/dt = 0$ at $t = 0$, to give

$$x = x_0 - \frac{e\mathcal{E}t^2}{2m}$$

From this we can get the time $t_f$ required for the particle to fall to $x = -B$:

$$t_f = \sqrt{\frac{2m(B + x_0)}{e\mathcal{E}}}$$

Next we look at the horizontal motion. From Eq. 2B.3-2 and the expression for $x(t)$, we find that (with $v_m = (p_0 - p_L)B^2/2\mu L$

$$\frac{dz}{dt} = v_m\left[1 - \left(\frac{x}{B}\right)^2\right] = \frac{dz}{dt} = v_m\left[1 - \left(\frac{x_0 - (e\mathcal{E}t^2/2m)}{B}\right)^2\right]$$

This may be integrated to give

$$L = \int_0^L dz = v_m\int_0^{t_f}\left[1 - \frac{1}{B^2}\left(x_0^2 - \frac{x_0 e\mathcal{E}t^2}{m} + \frac{e^2\mathcal{E}^2t^4}{4m^2}\right)dt\right]$$
$$= \frac{(p_0 - p_L)}{2\mu L}\left[\left(B^2 - x_0^2\right)t_f + \frac{x_0 e\mathcal{E}t_f^3}{3m} - \frac{e^2\mathcal{E}^2t_f^5}{20m^2}\right]$$

Next, square this expression and then insert the expression for $t_f$ above to get

$$L^2 = \frac{2(p_0 - p_L)}{15\mu}\sqrt{\frac{2m(B+x_0)}{e\mathcal{E}}}(3B-2x_0)(B+x_0)$$

Then, in order to remove the radical, we square this, thereby getting

$$L^4 = \frac{8(p_0 - p_L)^2 m}{225\mu^2 e\mathcal{E}}(3B-2x_0)^2(B+x_0)^3$$

Next, we set $dL^4/dx_0$ equal to zero, and this yields 4 values for $x_0$: $\frac{1}{2}B$, $\frac{3}{2}B$, $B$, and $B$. It is only the first of these that is physically acceptable. When that value is put back into the expression for $L$, we get finally

$$L_{\min} = \left[\frac{12}{35}\frac{(p_0 - p_L)^2 mB^5}{\mu^2 e\mathcal{E}}\right]^{1/4}$$

**2C.2 Residence-time distribution in tube flow**

*a*. A fluid element at a radius $r$ within the tube will require a time $t$ to reach the tube outlet

$$t(r) = \frac{L}{v_z(r)} = \frac{L}{v_{\max}\left[1-(r/R)^2\right]}$$

All the fluid with a radius less than $r$ will have left the tube at this time. Hence the fraction of the flow that will have left the tube is

$$F(t) = \frac{\int_0^{2\pi}\int_0^r v_z r dr d\theta}{\int_0^{2\pi}\int_0^R v_z r dr d\theta} = 2\left(\frac{r}{R}\right)^2 - \left(\frac{r}{R}\right)^4$$

When the first equation is solved for $(r/R)^2$ and substituted into the expression for $F(t)$, we get

$$F(t) = 2\left(1-\frac{L}{v_{\max}t}\right) - \left(1-\frac{L}{v_{\max}t}\right)^2 = 1-\left(\frac{L}{v_{\max}t}\right)^2 = 1-\left(\frac{L}{2\langle v_z\rangle t}\right)^2$$

*b*, The mean residence time is then obtained by solving the last equation in (*a*) for $t$. and substituting into the Eq. 2C.2-1:

$$t_m = \int_0^1 t dF = \frac{L}{v_{\max}}\int_0^1 \frac{dF}{\sqrt{1-F}} = \frac{2L}{v_{\max}} = \frac{L}{\langle v_z\rangle}$$

**2C.3 Velocity distribution in a tube**

The derivation in §2.3 is valid up through Eq. 2.3-15. If the viscosity is dependent on the radial coordinate, however, Eq. 2.3-16 is inappropriate. Instead we get

$$v_z = -\frac{(\mathcal{P}_0 - \mathcal{P}_L)}{2L}\int_0^r \frac{\bar{r}}{\mu(\bar{r})}d\bar{r} + C_2$$

Application of the no-slip boundary condition at the tube wall gives

$$0 = -\frac{(\mathcal{P}_0 - \mathcal{P}_L)}{2L}\int_0^R \frac{\bar{r}}{\mu(\bar{r})}d\bar{r} + C_2$$

This may be solved for the integration constant, and the velocity distribution is then

$$v_z = \frac{(\mathcal{P}_0 - \mathcal{P}_L)}{2L}\int_r^R \frac{\bar{r}}{\mu(\bar{r})}d\bar{r} \qquad \text{or} \qquad v_z = \frac{(\mathcal{P}_0 - \mathcal{P}_L)R^2}{2L}\int_y^1 \frac{\bar{y}}{\mu(\bar{y})}d\bar{y}$$

This is the same as Eq. 2.3-18 if the viscosity is a constant.

Next we get an expression for the average velocity

$$\langle v_z \rangle = \frac{\int_0^{2\pi}\int_0^R v_z r dr d\theta}{\int_0^{2\pi}\int_0^R r dr d\theta} = \frac{2}{R^2}\int_0^R v_z r dr = 2\int_0^1 v_z y dy$$

$$= \frac{(\mathcal{P}_0 - \mathcal{P}_L)R^2}{L}\int_0^1\int_y^1 \frac{\bar{y}}{\mu(\bar{y})}d\bar{y} y dy = \frac{(\mathcal{P}_0 - \mathcal{P}_L)R^2}{L}\int_0^1 \frac{\bar{y}}{\mu(\bar{y})}\int_0^{\bar{y}} y dy d\bar{y}$$

$$= \frac{(\mathcal{P}_0 - \mathcal{P}_L)R^2}{L}\int_0^1 \frac{\bar{y}}{\mu(\bar{y})}\frac{\bar{y}^2}{2}d\bar{y}$$

Then we find the dimensionless ratio $\dfrac{v_z}{\langle v_z \rangle} = \dfrac{\int_y^1 (\bar{y}/\mu)d\bar{y}}{\int_0^1 (\bar{y}^3/\mu)d\bar{y}}$.

**2C.4 Falling-cylinder viscometer**

*a.* Equation 2.4-2 is valid for this problem, but the pressure difference is not known. When Newton's law of viscosity is substituted into Eq. 2.4-2 we get

$$v_z = -\left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{4\mu L}\right) r^2 - C_1 \ln r + C_2$$

The two constants of integration and the (unknown) pressure difference can be obtained from two boundary conditions and a mass-conservation condition: At $r = \kappa R$, $v_z = -v_0$; at $r = R$, $v_z = 0$; and $\int_0^{2\pi}\int_{\kappa R}^{R} v_z r dr d\theta = \pi(\kappa R)^2 v_0$. This states that the fluid displaced by the falling cylinder must be compensated for by a net motion upward through the annular slit. These three unknown constants may be obtained from these conditions (lengthy!) and the result is

$$\frac{v_z}{v_0} = -\frac{\left(1-\xi^2\right) - \left(1+\kappa^2\right)\ln\left(1/\xi\right)}{\left(1-\kappa^2\right) - \left(1+\kappa^2\right)\ln\left(1/\kappa\right)}$$

*b.* The force acting downward on the cylindrical slug of height $H$ is $(\rho_0 - \rho)g \cdot \pi(\kappa R)^2 H$. The difference in the pressures acting on the top and bottom of the slug is an upward force

$$\begin{aligned}\left(\mathcal{P}_0 - \mathcal{P}_H\right)\cdot \pi(\kappa R)^2 &= -\int_0^H \left(d\mathcal{P}/dz\right)dz \cdot \pi(\kappa R)^2 \\ &= -\frac{4 v_0 H \cdot \pi(\kappa R)^2}{R^2\left[\left(1-\kappa^2\right) - \left(1+\kappa^2\right)\ln\left(1/\kappa\right)\right]}\end{aligned}$$

In addition, there is an upward force associated with the frictional drag by the fluid

$$\begin{aligned}2\pi(\kappa R)H\left(-\tau_{rz}\right)\Big|_{r=\kappa R} &= 2\pi(\kappa R)H\mu\left(\frac{dv_z}{dr}\right)\Bigg|_{r=\kappa R} \\ &= 2\pi\kappa v_0 H\mu\left(\frac{2\kappa - \left(1+\kappa^2\right)\left(1/\kappa\right)}{\left(1-\kappa^2\right) - \left(1+\kappa^2\right)\ln\left(1/\kappa\right)}\right)\end{aligned}$$

When these are equated and the result solved for the viscosity, the expression in Eq. 22C.4-2 is obtained.

*c*. Next put $\kappa = 1 - \varepsilon$, and expand in powers of $\varepsilon$, keeping terms up to $\varepsilon^5$. Use §C.2, and obtain Eq. 2C.4-3.

**2C.5 Falling film on a conical surface**

*a*. A mass balance on a ring of liquid contained between $s$ and $s+\Delta s$ gives

$$\left(2\pi s(\sin\beta)\delta(s)\langle v\rangle\right)\Big|_s - \left(2\pi s(\sin\beta)\delta(s)\langle v\rangle\right)\Big|_{s+\Delta s} = 0$$

Letting $\Delta s \to 0$ then gives

$$\frac{d}{ds}\left(s\delta\langle v\rangle\right) = 0 \qquad \text{whence, from Eq. 2.2-20} \qquad \frac{d}{ds}\left(s\delta^3\right) = 0$$

Equation 2.2-20 is valid strictly for a flat plate with constant film thickness, but we apply it here *approximately* to a different geometry.

*b*. When the equation in (*a*) is integrated, we get $s\delta^3 = C$, in which $C$ is an integration constant. This constant is determined by requiring that the mass flow down the conical surface be the same as that flowing up the central tube (i.e., $w$). We hence write (width of film) x (thickness of film) x (mass flow rate), and then use Eq. 2.2-20:

$$w = (2\pi s\sin\beta)\cdot\delta\cdot\rho\langle v\rangle = (2\pi s\sin\beta)\cdot\left(\frac{C}{s}\right)^{1/3}\cdot\rho\left(\frac{\rho g(C/s)^{2/3}\cos\beta}{3\mu}\right)$$

From this we get $C$;

$$C = \frac{3\mu w}{(2\pi\sin\beta)\cdot(\rho^2 g\cos\beta)} = \frac{3\mu w}{\pi\rho^2 g\sin 2\beta}$$

The film thickness as a function of the distance down the cone from the apex is thus

$$\delta = \sqrt[3]{\frac{3\mu w}{\pi\rho^2 g s\sin 2\beta}}$$

**2C.6 Rotating cone pump**

*a. Inner cone not rotating.*

For sufficiently small values of $B$, the flow will resemble very much that for a thin slit (see Problem 2B.3), for which the mass rate of flow is given as the answer to part (b). This formula may be adapted to the flow in the annular space of height $dz$. as follows, if the inner cone is not rotating and if the gravitational force is not included:

$$w \approx \frac{2}{3}\left(-\frac{dp}{dz}\right)\frac{B^3(2\pi z\sin\theta)\rho}{L}$$

where we have made the identification $(p_0 - p_L)/L \to -dp/dz$ and $W \to 2\pi r = 2\pi z\sin\theta$. Across any plane $z$ = constant, the mass flow rate will be a constant. Hence the above equation can be integrated to give

$$-\int_{p_1}^{p_2} dp = \frac{3}{4\pi}\frac{\mu w}{B^3\rho\sin\theta}\int_{L_1}^{L_2}\frac{dz}{z} \quad \text{or} \quad p_1 - p_2 = \frac{3}{4\pi}\frac{\mu w}{B^3\rho\sin\theta}\ln\frac{L_2}{L_1}$$

*b. Effect of the gravitational force and the centrifugal force*

The result in (*a*) may be modified to include the effect of gravitational acceleration $g$ and the angular velocity $\Omega$ of the inner cone.

The gravity force in the $z$-direction (per unit volume) is given by $F_{\text{grav},z} = -\rho g\cos\beta$. The centrifugal force (per unit volume) acting in the middle of the slit will be, approximately, $F_{\text{cent}} = \rho(\frac{1}{2}\Omega r)^2/r = \frac{1}{4}\rho\Omega^2 z\sin\beta$, where $r$ is the distance from the centerline of the cones to the middle of the slit. The component of this force in the $z$-direction is then $F_{\text{cent},z} = \frac{1}{4}\rho\Omega^2 z\sin^2\beta$. Then the first equation in (*a*) can be modified to give

$$w \approx \frac{2}{3}\left(-\frac{dp}{dz} + \tfrac{1}{4}\rho\Omega^2 z\sin^2\beta - \rho g\cos\beta\right)\frac{B^3(2\pi z\sin\beta)\rho}{L}$$

This equation can be integrated to give

$$w = \frac{4\pi B^3 \rho \sin\beta}{3\mu \ln(L_2/L_1)}\left[(p_1 - p_2) + \left(\tfrac{1}{8}\rho\Omega^2 \sin^2\beta\right)\left(L_2^2 - L_1^2\right) - (\rho g \cos\beta)(L_2 - L_1)\right]$$

Many assumptions have been used to get this solution: (1) laminar flow (turbulent flow analog is not difficult to work out); (2) curvature effects have been neglected (correction for this is easy to do); (3) entrance effects have been ignored (this can probably be handled approximately by introducing an "equivalent length"); (4) instantaneous accommodation of velocity profiles to the changing cross-section (it would be difficult to correct for this in a simple way).

**2C.7 A simple rate-of-climb indicator**

*a*. Consider two planes of area $S$ parallel to the earth's surface at heights $z$ and $z+\Delta z$. The pressure force in the $z$-direction acting on the plane at height $z$ and that acting at the plane at height $z+\Delta z$ will be just the mass of air in the layer of height $\Delta z$:

$$Sp\big|_z - Sp\big|_{z+\Delta z} = \rho g S \Delta z$$

Division by $S\Delta z$ and then letting $\Delta z \to 0$ gives the differential equation

$$\frac{dp}{dz} = -\rho g \quad \text{or} \quad \frac{dp}{dz} = -\left(\frac{pM}{R_g T}\right) g$$

which describes the decrease in the atmospheric pressure with increased elevation; here $R_g$ is the gas constant.

*b*. Let $p_i$ be the pressure inside the Bourdon element and $p_o$ be the pressure outside (i.e., the ambient atmospheric pressure). We now write an equation of conservation of mass for the entire instrument:

$$\frac{d}{dt} m_{\text{tot}} = -w_o; \quad V\frac{d}{dt}\rho_i = \frac{\pi(p_i - p_o)R^4}{8\mu L}\rho_{\text{avg}}; \quad V\frac{d}{dt}p_i = \frac{\pi(p_i - p_o)R^4}{8\mu L}p_{\text{avg}}$$

Here $m_{\text{tot}}$ is the total mass of air within the system (Bourdon element plus capillary tube), $w_o$ is the mass rate of flow of the air exiting to the outside, $\rho_i$ is the density of the air inside the Bourdon elements, and $\rho_{\text{avg}}$ is the arithmetic average of the inlet and outlet densities within the capillary tube (see Eq. 2.3-29). The third form of the mass balance written above has made use of the ideal gas law, $p = \rho R_g T/M$.

If we neglect changes in the arithmetic average pressure $p_{\text{avg}}$ and use the abbreviation $B = \pi R^4 p_{\text{avg}}/8\mu L V$, we can integrate the mass balance above and get

$$p_i = e^{-Bt}\left(\int B p_o e^{Bt} dt + C\right)$$

To get $p_o(t)$ we make use of the fact that there is a constant upward velocity, so that

$$\frac{dp_o}{dt} = \frac{dp_o}{dz}\frac{dz}{dt} = -\left(\frac{p_o M}{R_g T}\right) g \cdot v_z \equiv -Ap_o \qquad \text{whence} \qquad p_o = p_o^0 e^{-At}$$

Then the mass-balance equation becomes

$$p_i = e^{-Bt}\left(\int Bp_o^0 e^{-At} e^{Bt} dt + C\right) = \frac{Bp_o^0}{B-A} e^{-At} + Ce^{-Bt}$$

Determine the constant of integration, C, from the initial condition that $p_i^0 = p_o^0$ at $t = 0$. Then

$$p_i = p_o^0 \frac{Be^{-At} - Ae^{-Bt}}{B-A} \qquad \text{and} \qquad P = \frac{p_i - p_o}{p_o} = \frac{A}{B-A}\left(1 - e^{-(B-A)t}\right)$$

In the limit that $t \to \infty$, we get fir $B>>A$

$$P \to \frac{A}{B} = \frac{v_z Mg}{R_g T} \cdot \frac{8V\mu L}{\pi R^4 p_{\text{avg}}}$$

Hence for $p_o \approx p_{\text{avg}}$, the gauge pressure is

$$p_i - p_o = v_z\left(\frac{8\mu L}{\pi R^4}\right)\left(\frac{MgV}{R_g T}\right)$$

Hence the pressure difference approaches an asymptotic value that varies only slightly with altitude.

*c.* To get the relaxation time, note that

$$P = \frac{A}{B-A}\left(1 - e^{-(B-A)t}\right) \xrightarrow{B>>A} P_\infty\left(1 - e^{-Bt}\right)$$

whence

$$\frac{P_\infty - P}{P_\infty} = e^{-Bt} \equiv e^{-t/t_{\text{rel}}} \quad \text{so that} \quad t_{\text{rel}} = \frac{1}{B} = \frac{8\mu LV}{\pi R^4 p_{\text{avg}}}$$

It is necessary to have $t_{\text{rel}} << 100$ to insure the plausibility of the assumption that $B>>A$.

## 2D.1 Rolling-ball viscometer

The rolling-ball viscometer consists of an inclined tube containing a sphere whose diameter is but slightly smaller than the internal diameter of the tube. The fluid viscosity is determined by observing the speed with which the ball rolls down the tube, when the latter is filled with liquid. We want to interrelate the viscosity and the terminal velocity of the rolling ball.

The flow between the sphere and the cylinder can be treated locally as slit flow (see Problem 2B.3) and hence the only hydrodynamic result we need is

$$\frac{dp}{dz} = \frac{12\mu\langle v_z\rangle}{\sigma^2} \qquad (*)$$

But we must allow the slit width $\sigma$ to vary with $\theta$ and $z$. From the figure we see that

$$R^2 = (R-r)^2 + (r'+\sigma)^2 - 2(R-r)(r'+\sigma)\cos\theta$$

where $r' = \sqrt{r^2 - z^2}$. Solving for $\sigma$ we get

$$\sigma = -r' + (R-r)\cos\theta + R\sqrt{1 + \left[(R-r)/R\right]^2\left(-\sin^2\theta\right)}$$

The second term under the square-root sign will be very small for the tightly fitting sphere-cylinder system and will hence be neglected. Furthermore we replace $\sqrt{r^2 - z^2}$ by $\sqrt{R^2 - z^2}$ and add compensating terms

$$\begin{aligned}
\sigma &= (R-r)\left[\cos\theta + \frac{R - \sqrt{R^2 - z^2} + \left(\sqrt{R^2 - z^2} - \sqrt{r^2 - z^2}\right)}{R-r}\right] \\
&= (R-r)\left[\cos\theta + \frac{R - \sqrt{R^2 - z^2} + (R-r) - \frac{1}{2}(R-r)(z/R)^2\cdots}{R-r}\right] \\
&\approx (R-r)\left[(\cos\theta + 1) + \frac{R - \sqrt{R^2 - z^2}}{R-r}\right]
\end{aligned}$$

$$= 2(R-r)\left(\cos^2\frac{\theta}{2} + \frac{R-\sqrt{R^2-z^2}}{2(R-r)}\right)$$

The omission of the term containing $(z/R)^2$ and the higher-order terms is possible, since the greatest contribution to the viscous drag occurs at the plane $z = 0$, and hence less accuracy is required for regions of larger $z$. Note that the above result gives correctly $\sigma = 0$ at $z = 0$, $\theta = \pi$, and $\sigma = 2(R-r)$ at $z = 0$, $\theta = 0$.

Next we assert that $dp/dz$ will be independent of $\theta$, which is probably a good approximation. Then according to (*) $\langle v_z \rangle$ must have the form

$$\langle v_z \rangle = B(z)\sigma^2 \qquad (**)$$

Next, the volume rate of flow across any plane $z$ will be

$$Q = \int_{-\pi}^{+\pi} \langle v_z \rangle \sigma(\theta, z) R d\theta = RB(z)\int_{-\pi}^{+\pi} [\sigma(\theta, z)]^3 d\theta$$
$$= 8RB(z)(R-r)^3 \int_{-\pi}^{+\pi} \left[\cos^2 \tfrac{1}{2}\theta + \alpha\right]^3 d\theta = 8RB(z)(R-r)^3 I(\alpha)$$

in which $\alpha = (R - \sqrt{R^2 - z^2}) / 2(R-r)$.

The volume rate of flow $Q$ at all cross-sections will be the same, and its value will be, to a very good approximation $Q = \pi R^2 v_0$, where $v_0$ is the translational speed of the rolling ball. Equating the two expressions for $Q$ gives

$$B(z) = \frac{\pi R v_0}{4z(R-r)^3 I(\alpha)} \qquad (***)$$

Combining (*), (**), and (***) we get

$$\frac{dp}{dz} = \frac{3\pi\mu R v_0}{2(R-r)^3 I(\alpha)}$$

The total pressure drop across the slit is then

$$\Delta p = \int_{-R}^{+R} \frac{dp}{dz} dz \approx 2\int_0^\infty \frac{dp}{dz}\frac{dz}{d\alpha} d\alpha$$

into which we have to insert $dz/d\alpha$. Virtually no error is introduced by making the upper limit infinite. From the definition of $\alpha$

$$z^2 = -4(R-r)^2\alpha^2 + 4R(R-r)\alpha$$

The first term on the right is smaller than the second, at least for small $z$. Then $dz \approx \sqrt{R(R-r)}d\alpha / \sqrt{\alpha}$, and the pressure drop expression becomes (with $\xi^2 = \alpha$)

$$\Delta p = 2\sqrt{R(R-r)}\int_0^\infty \frac{dp}{dz}\frac{1}{\sqrt{\alpha}} d\alpha = 4\sqrt{R(R-r)}\int_0^\infty \frac{dp}{dz} d\xi$$
$$= 4\sqrt{R(R-r)}\frac{3\pi\mu v_0 R}{2(R-r)^3}\int_0^\infty \frac{1}{I(\xi^2)} d\xi = \frac{3\pi\mu v_0 R^{3/2}}{2(R-r)^{5/2}} \cdot J$$

where

$$J = 2\int_0^\infty \frac{1}{I(\xi^2)} d\xi = \frac{4}{3}\left[\sqrt{2} - \frac{1}{\sqrt{5}}\left(\sqrt{10}+2\right)^{1/2}\right] = 0.531$$

The pressure drop multiplied by the tube cross-section must, according to an overall force balance, be equal to the net force acting on the sphere by gravity and buoyancy

$$\tfrac{4}{3}\pi R^3(\rho_s - \rho)g\sin\beta = \pi R^2 \Delta p$$

where $\rho_s$ and $\rho$ are the densities of the sphere and fluid respectively. Combining the last three results gives the equation for the viscosity

$$\mu = \frac{4}{9\pi J}\frac{R^2(\rho_s - \rho)g\sin\beta}{v_0}\left(\frac{R-r}{R}\right)^{5/2}$$

**2D.2 Drainage of liquids**

*a*. The unsteady mass balance is

$$\frac{\partial}{\partial t}(\rho\delta W) = \left(\rho\langle v_z\rangle W\delta\right)\Big|_z - \left(\rho\langle v_z\rangle W\delta\right)\Big|_{z+\Delta z}$$

Divide by $\rho W\Delta z$ and take the limit as $\Delta z \to 0$, to get Eq. 2D.2-1.

*b*. Then use Eq. 2.2-22 to get Eq. 2D.2-1:

$$\frac{\partial\delta}{\partial t} = -\frac{\rho g}{3\mu}\frac{\partial\delta^3}{\partial z} = -\frac{\rho g\delta^2}{\mu}\frac{\partial\delta}{\partial z}$$

which is a first-order partial differential equation.

*c*. First let $\Delta = \sqrt{\rho g/\mu}$, so that the equation in (*b*) becomes:

$$\frac{\partial\Delta}{\partial t} = \Delta^2\frac{\partial\Delta}{\partial z}$$

Inspection of the equation suggests that $\Delta = \sqrt{z/t}$, which can be seen to satisfy the differential equation exactly. Therefore Eq. 2D.2-3 follows at once. This equation has a reasonable form, since for long times the boundary layer is thin, whereas for short times the boundary layer is thick.

### 3A.1 Torque required to turn a friction bearing.

Equation 3.6-31 describes the torque required to turn an *outer* cylinder at an angular velocity $\Omega_o$. The corresponding expression for the torque required to turn an *inner* rotating cylinder at an angular velocity $\Omega_i$ is given by a formally similar expression,

$$T_z = 4\pi\mu\Omega_i R^2 L \left(\frac{\kappa^2}{1-\kappa^2}\right)$$

derivable in like manner from the corresponding velocity profile in Eq. 3.6-32.

The specifications for this problem (converted into SI units via Appendix **F**) are:

$$\kappa = \frac{1.000}{1.002} = 0.998004; \qquad \kappa^2 = 0.996012$$

$$\left(\frac{\kappa^2}{1-\kappa^2}\right) = \frac{0.996012}{0.003988} = 249.8$$

$$\mu = (200 \text{ cp})(10^{-3} \text{ kg/m·s/cp} = 0.200 \text{ kg/m·s}$$

$$\Omega_i = (200 \text{ rpm})(1 \text{ min}/60 \text{ s})(2\pi \text{ radians/revolution}) = 20\pi/3 \text{ radians/s}$$

$$R^2 = (1 \text{ in}^2)(1 \text{ m}/39.37 \text{ in})^2 = 0.000645 \text{ m}^2$$

$$L = 2 \text{ in} = 2/39.37 \text{ m} = 0.0508 \text{ m}$$

$$\rho = (50 \text{ lb}_m/\text{ft}^3)(0.45359 \text{ kg/lb}_m(39.370/12 \text{ ft/m})^3 = 800.9 \text{ kg/m}^3$$

Hence, the required torque is

$$\begin{aligned} T_z &= (4\pi)(0.200 \text{ kg/m·s})(20\pi/3\text{radians/s})(0.000645\text{m}^2)(0.0508 \text{ m})(249.8) \\ &= 0.431 \text{ kg·m}^2/\text{s}^2 = 0.32 \text{ ft·lb}_f \end{aligned}$$

and the power required is

$$\begin{aligned} P = T_z\Omega_i &= (0.32 \text{ lb}_f\text{·ft})(20\pi/3 \text{ s}^{-1})(3600 \text{ s/hr})(5.0505 \times 10^{-7} \text{ hp·hr/lb}_f\text{·ft}) \\ &= 0.012 \text{ hp} \end{aligned}$$

In these calculations we have tacitly assumed the flow to be stable and laminar. To test this assumption, we formulate a transition criterion based on the critical angular velocity expression given under Fig. 3.6-2:

$$\text{Re} := \frac{\Omega_i \rho R^2 (1-\kappa)^{3/2}}{\mu} < \text{ about 41.3 for } \kappa \approx 1.$$

Insertion of numerical values for the present system gives

$$\text{Re} = \frac{(20\pi/3 \text{ radians/s})(800.9 \text{ kg/m}^3)(0.000645 \text{ m}^2)(1-0.998004)^{3/2}}{(0.200 \text{ kg/m·s}} = 0.0048$$

This Re value is well below the transition value of 41.3 for this geometry; therefore, the foregoing predictions of $T_z$ and $P$ are realistic.

**3A.2 Friction loss in bearings.**

The power expended to overcome the bearing friction is

$$P = T_z \Omega_i = 4\pi\mu\Omega_i^2 R^2 L \left(\frac{\kappa^2}{1-\kappa^2}\right)$$

in which $L$ is the total bearing length of $2 \times 20 \times 1 = 40$ ft for the two shafts. The specifications for this problem (converted to SI units via Appendix F) are:

$$\kappa = \frac{16}{16 + 2 \times 0.005} = 0.999375; \qquad \kappa^2 = 0.998751$$

$$\left(\frac{\kappa^2}{1-\kappa^2}\right) = \frac{0.998751}{0.001249} = 799.6$$

$$\mu = (5000 \text{ cp})(10^{-3} \text{ kg/m·s/cp}) = 5 \text{ kg/m·s}$$

$$\Omega_i = (50/60 \text{ rev/s})(2\pi \text{ radians/rev}) = 5\pi/3 \text{ radians/s}$$

$$R^2 = (8/39.37 \text{ m})^2 = 0.04129 \text{ m}^2$$

$$L = 40 \text{ ft} = (40 \times 12/39.37 \text{ m}) = 12.2 \text{ m}$$

With these values, the calculated power requirement is

$$\begin{aligned} P &= (4\pi)(5 \text{ kg/m·s})(5\pi/3 \text{ rad/s})^2(0.04129 \times 12.2 \text{ m}^3)(799.6) \\ &= 6.938 \times 10^5 \text{ kg·m}^2/\text{s}^3 \end{aligned}$$

This result is then expressed in horsepower by use of Table F.3-3:

$$\begin{aligned} P &= (6.938 \times 10^5 \text{ kg·m}^2/\text{s}^3)(3.7251 \times 10^{-7} \text{ hp·hr·[kg·m}^2/s^2]^{-1})(3600 \text{ s/hr}) \\ &= 930 \text{ hp} \end{aligned}$$

Thus, the fraction of the available power that is lost in bearing friction is 930/(4000+4000) = 0.116.

### 3A.3 Effect of altitude on air pressure.

For a stationary atmosphere (i.e., no wind currents), the vertical component of the equation of motion gives

$$\frac{dp}{dz} = -\rho g$$

The air is treated as an ideal gas,

$$\rho = \frac{pM}{RT}$$

with $M \approx 29$, and with temperature in °R given by

$$T(z) = 530 - 0.003z$$

at elevation $z$ ft above Lake Superior. The pressure $p_2$ at $z_2 = 2023 - 602 = 1421$ ft above lake level is to be calculated, given that $p_1 = 750$ mm Hg at $z = 0$.

The foregoing equations give

$$\frac{d\ln p}{dz} = -\frac{Mg}{R(530 - 0.003z)}$$

Integration gives

$$\begin{aligned}
\ln(p_2/p_1) &= \frac{Mg}{R}\int_0^{1421} \frac{dz}{(530 - 0.003z} \\
&= \frac{Mg}{R}\frac{1}{0.003}\ln[530 - 0.003z]\Big|_{z=0}^{z=1421} \\
&= \frac{Mg}{0.003R}\ln\left[\frac{525.737}{530}\right]
\end{aligned}$$

Insertion of numerical values in $\text{lb}_m$-ft-s units gives

$$\begin{aligned}
\ln(p_2/p_1) &= \frac{(29\ \text{lb}_m/\text{lb-mol})(32.17\ \text{ft/s}^2)}{(0.003\ \text{R/ft})(4.9686 \times 10^4\ \text{lb}_m\cdot\text{ft}^2/\text{s}^2\cdot\text{lb-mol}\cdot\text{R})}\ln\left[525.737/over530\right] \\
&= -0.0505
\end{aligned}$$

Hence,

$$p_2 = p_1 \exp(-0.0505) = 750 \times 0.9507 = 713 \text{ mm Hg}$$

Since the fractional change in $P$ is small, one gets a good approximation (and a quicker solution) by neglecting it. That method gives $p_2 = 712$ mm Hg.

**3A.4 Viscosity determnation with a rotating-cylinder viscometer.**

Here it is desirable to use a sufficiently high torque that the precision of viscosity determinations is limited mainly by that of the measurement of angular velocity. A torque of $10^4$ dyn·cm, corresponding to a torque uncertainty of 1%, appears reasonable if the resulting Reynolds number is in the stable laminar range.

The geometric specifications of the viscometer are:

$$R = 2.25 \text{ cm}; \qquad \kappa R = 2.00 \text{ cm}$$
$$\kappa = 2.00/2.25 = 0.888889; \qquad \kappa^2 = 0.790123$$
$$1 - \kappa^2 = 0.209877; \qquad (\kappa R)^2 = 4.00 \text{ cm}^2$$
$$L = 4 \text{ cm}; \qquad R^2 = 5.0625 \text{ cm}^2$$

The angular velocity corresponding to this torque value is:

$$\Omega_o = \frac{T_z(1-\kappa^2)}{4\pi\mu(\kappa R)^2 L} = \frac{(10^4 \text{ g·cm}^2/\text{s}^2)(0.209877)}{4\pi(0.57 \text{ g/cm·s})(4 \text{ cm}^2)(4 \text{ cm})} = 18.3 \text{ radians/s}$$

The Reynolds number at this condition is:

$$\text{Re} = \frac{\Omega_o R^2 \rho}{\mu} = \frac{(18.3)(5.0625)(1.29)}{0.57} = 210$$

Accordng to Fig. 3.6-2, this Re value is well within the stable laminar range; therefore, a torque of $10^4$ dyn·cm is acceptable.

### 3A.5 Fabrication of a parabolic mirror.

Equation 3.6-44 gives the shape of the free surface as

$$z - z_0 = \left(\frac{\Omega^2}{2g}\right) r^2$$

The required derivatives of this function at the axis of rotation are

$$\frac{dz}{dr} = 0 \quad \text{and} \quad \frac{d^2 z}{dr^2} = \frac{\Omega^2}{g}$$

Setting the desired focal length equal to half the radius of curvature of the mirror surface at $r = 0$, and using Eq. 3A.5-1, we obtain

$$f = \frac{1}{2} g/\Omega^2$$

Thus, the required angular velocity to produce a mirror with focal length $f = 100$ cm at standard terrestrial gravity is

$$\begin{aligned} \Omega &= \sqrt{\frac{g}{2f}} \\ &= \sqrt{\frac{980.665 \text{ cm/s}^2}{(2)(100 \text{ cm})}} \\ &= 2.214 \text{ radians/s} \end{aligned}$$

which corresponds to $60\Omega/2\pi = 21.1$ revolutions per minute.

### 3A.6 Scale-up of an agitated tank.

The specifications for the operation in the large tank (Tank I) are

$$N_{\mathrm{I}} = 120 \text{ rpm}; \qquad \mu_{\mathrm{I}} = 13.5 \text{ cp}; \qquad \rho_{\mathrm{I}} = 0.9 \text{ g/cm}^3$$

and the tank is to be operated with an uncovered liquid surface.

To allow direct prediction of the operation of Tank I from experiments in the smaller system (Tank II), the systems must be geometrically similar and must run at the same values of Re and Fr. To meet the latter requirement, Eqs. 3.7-40,41 must be satisfied.

Equation 3.7-41 requires

$$D_{\mathrm{II}} N_{\mathrm{II}}^2 = D_{\mathrm{I}} N_{\mathrm{I}}^2$$

when, as usual, the gravitational fields for the two systems are equal. Then the model must operate at

$$N_{\mathrm{II}} = N_{\mathrm{I}} \sqrt{D_{\mathrm{I}}/D_{\mathrm{II}}} = 120\sqrt{10} = 380 \text{ rpm}$$

and Eq. 3.7-40 requires

$$\begin{aligned} \nu_{\mathrm{II}} &= \nu_{\mathrm{I}} \left(\frac{D_{\mathrm{II}}}{D_{\mathrm{I}}}\right)^2 \left(\frac{N_{\mathrm{II}}}{N_{\mathrm{I}}}\right)^2 \\ &= (13.5/0.9)(0.1)^2(\sqrt{10}) = 0.474 \text{ cp} \end{aligned}$$

From Table 1.1-1, we see that this value of $\nu_{\mathrm{II}}$ corresponds closely to the value for liquid water at 60°C. Thus, the model should opeerate at 380 rpm, with liquid water at very nearly 60°C.

**3A.7 Air entrainment in a draining tank.**

As this system is too complex for analytic treatment, we use dimensional analysis. We must establish operating conditions such that both systems satisfy the same dimensionless differential equations and boundary conditions. This means that the large and small systems must be geometrically similar, and that the Froude and Reynolds numbers must be respectively the same for each.

Choose $D$ (tank diameter) as characteristic length, and $(4Q/\pi D^2)$ as characteristic velocity, where $Q$ is the volumetric draw-off rate. Then

$$\mathrm{Re} = \frac{4Q\rho}{\pi D\mu} \qquad \text{and} \qquad \mathrm{Fr} = \frac{16Q^2}{\pi^2 D^5 g}$$

Subscripts L and S will be used to identify quantities associated with the large and small tanks, respectively. We take the gravitational field $g$ to be the same for both. Then the requirement of equal Reynolds numbers gives

$$\left(\frac{Q_S}{Q_L}\right)\left(\frac{D_L}{D_S}\right) = \left(\frac{\rho_L}{\mu_L}\right)\left(\frac{\mu_S}{\rho_S}\right) = \left(\frac{1.286}{56.7}\right)(1.0037) = 0.02277,$$

and the requirement of equal Froude numbers gives

$$\left(\frac{Q_S}{Q_L}\right) = \left(\frac{D_S}{D_L}\right)^{5/2}$$

Combining these requirements, we obtain

$$\left(\frac{D_S}{D_L}\right) = (0.02277)^{2/3} = 0.080$$

Hence,

$$D_S = (0.080)(60\text{ ft}) = 4.8\text{ ft}$$
$$Q_S = (0.080)^{5/2}(800\text{ gal/min}) = 1.46\text{ gal/min}$$

Therefore:

*a.* The model tank should be 4.8 ft in diameter.

*b.* Its draw-off tube should be 0.080 ft in diameter and 0.080 ft high.

*c.* Its draw-off tube should have its axis 0.32 ft from the wall of the tank.

Furthermore, if water at 68°F (20°C) is withdrawn from the model tank at 1.46 gal/min, air entrainment will begin when the liquid level is (4.8/60) of the level at which entrainment would begin in the large tank at its withdrawal rate of 800 gal/min.

**3B.1 Flow between concentric cylinders and spheres**

*a*. The derivation proceeds as in Example 3.6-3 up to Eq. 3.6-26, which we choose to rewrite as

$$\frac{v_\theta}{r} = D_1 + D_2\left(\frac{R}{r}\right)^2$$

The boundary conditions are that $v_\theta(\kappa R) = \Omega_i \kappa R$ and $v_\theta(R) = \Omega_o R$. Putting these boundary conditions into the above equation for the angular velocity gives

$$\Omega_i = D_1 + D_2\frac{1}{\kappa^2} \qquad \text{and} \qquad \Omega_o = D_1 + D_2$$

These equations can be solved for the integration constants

$$D_1 = \frac{\begin{vmatrix} \Omega_i & 1/\kappa^2 \\ \Omega_o & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1/\kappa^2 \\ 1 & 1 \end{vmatrix}} = \frac{\begin{vmatrix} \Omega_i & 1 \\ \Omega_o & \kappa^2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & \kappa^2 \end{vmatrix}} = \frac{\Omega_i\kappa^2 - \Omega_o}{\kappa^2 - 1}$$

$$D_2 = \frac{\begin{vmatrix} 1 & \Omega_i \\ 1 & \Omega_o \end{vmatrix}}{\begin{vmatrix} 1 & 1/\kappa^2 \\ 1 & 1 \end{vmatrix}} = \frac{\begin{vmatrix} \kappa^2 & \Omega_i \\ \kappa^2 & \Omega_o \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & \kappa^2 \end{vmatrix}} = \frac{\kappa^2(\Omega_o - \Omega_i)}{\kappa^2 - 1}$$

Hence the solution to the differential equation is

$$\frac{v_\theta}{r} = \frac{(\Omega_o - \Omega_i\kappa^2)}{1-\kappa^2} + \frac{(\Omega_i - \Omega_o)}{1-\kappa^2}\left(\frac{\kappa R}{r}\right)^2$$

The z-components of the torques on the outer and inner cylinders are

$$T_z = \int_0^L\int_0^{2\pi}(-\tau_{r\theta}R)_{r=R}\,R d\theta dz = 2\pi L R^2\left[+\mu r\frac{d}{dr}\left(\frac{v_\theta}{r}\right)\right]_{r=R}$$

$$= 2\pi L R^2 \mu R \frac{(\Omega_i - \Omega_o)}{1-\kappa^2}(\kappa R)^2\left(-2\frac{1}{R^3}\right) = -4\pi\mu L(\Omega_i - \Omega_o)\frac{(\kappa R)^2}{1-\kappa^2}$$

$$T_z = \int_0^L\int_0^{2\pi}\left(+\tau_{r\theta}\kappa R\right)_{r=\kappa R}\kappa R d\theta dz = 2\pi L(\kappa R)^2\left[-\mu r\frac{d}{dr}\left(\frac{v_\theta}{r}\right)\right]_{r=\kappa R}$$

$$= +4\pi\mu L(\Omega_i - \Omega_o)\frac{(\kappa R)^2}{1-\kappa^2}$$

*b*. In Example 3.6-5 it is shown how to get Eq. 3.6-53 for the velocity distribution. The boundary conditions are : $v_\phi(\kappa R) = \kappa R\Omega_i \sin\theta$ and $v_\phi(R) = R\Omega_i \sin\theta$. Equation 3.6-53 can be written in the form

$$\frac{v_\phi}{r\sin\theta} = D_1 + D_2\left(\frac{R}{r}\right)^3$$

The constants can be obtained according to the method of (*a*) and the final expression is

$$\frac{v_\phi}{r\sin\theta} = \frac{\left(\Omega_o - \Omega_i\kappa^3\right)}{1-\kappa^3} + \frac{(\Omega_i - \Omega_o)}{1-\kappa^3}\left(\frac{\kappa R}{r}\right)^3$$

The torques at the outer and inner cylinders are then

$$T_z = \int_0^{2\pi}\int_0^{\pi}\left(-\tau_{r\phi}\right)_{r=R}(R\sin\theta)R^2\sin\theta d\theta d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi}\left[+\mu r\frac{d}{dr}\left(\frac{v_\phi}{r}\right)\right]_{r=R}(R\sin\theta)R^2\sin\theta d\theta d\phi$$

$$= -8\pi\mu(\Omega_i - \Omega_o)\frac{(\kappa R)^3}{1-\kappa^3}$$

$$T_z = \int_0^{2\pi}\int_0^{\pi}\left(+\tau_{r\phi}\right)_{r=\kappa R}(\kappa R\sin\theta)(\kappa R)^2\sin\theta d\theta d\phi = +8\pi\mu(\Omega_i - \Omega_o)\frac{(\kappa R)^3}{1-\kappa^3}$$

**3B.2 Laminar flow in a triangular duct**

*a*. It is clear that the boundary conditions that $v_z = 0$ at $y = H$ and at $y = \pm\sqrt{3}x$. Therefore the no-slip boundary conditions are satisfied. Next it has to be shown that the equation of motion

$$0 = \frac{\mathcal{P}_0 - \mathcal{P}_L}{L} + \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2}\right)$$

is satisfied. Substituting the solution into the second-derivative terms, we get

$$\mu\left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{4\mu LH}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(3x^2 y - y^3 - 3Hx^2 + Hy^2\right)$$
$$= \left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{4LH}\right)\left(6y - 6H - 6y + 2H\right)$$

and this just exactly cancels the pressure-difference term.

*b*. To get the mass rate of flow we integrate over half the cross-section and multiply by 2:

$$w = 2\rho\left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{4\mu LH}\right)\int_0^H \int_0^{y/\sqrt{3}} (y - H)\left(3x^2 - y^2\right) dx dy$$
$$= 2\rho\left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{4\mu LH}\right)\int_0^H (y - H)\left(x^3 - y^2 x\right)\Big|_0^{y/\sqrt{3}} dy$$
$$= 2\rho\left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{4\mu LH}\right)\int_0^H (y - H)\left(\frac{-2y^3}{3^{3/2}}\right) dy$$
$$= -\rho\left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{3^{3/2}\mu LH}\right)\left(-\frac{H^5}{20}\right) = \rho\left(\frac{\sqrt{3}(\mathcal{P}_0 - \mathcal{P}_L)H^4}{180\mu L}\right)$$

The average velocity is then the volume rate of flow $(w/\rho)$, divided by the cross-sectional area $H^2/\sqrt{3}$ so that

$$\langle v_z \rangle = \frac{(\mathcal{P}_0 - \mathcal{P}_L)H^2}{60\mu L}$$

The maximum velocity will be at the tube center, or at $x = 0$ and $y =$ $y = 2H/3$, so that

$$v_{z,\max} = \frac{(\mathcal{P}_0 - \mathcal{P}_L)H^2}{27\mu L} = \tfrac{20}{9}\langle v_z \rangle$$

**3B.3 Laminar flow in a square duct**

*a.* The boundary conditions at $x = \pm B$ and $y = \pm B$ are seen to be satisfied by direct substitution into Eq. 3B.3-1. Next we have to see whether the differential equation

$$0 = \frac{(\mathcal{P}_0 - \mathcal{P}_L)}{L} + \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2}\right)$$

is satisfied. Substituting the derivatives from Eq. 3B.3-1 into this differential equation gives

$$0 = \frac{(\mathcal{P}_0 - \mathcal{P}_L)}{L} - 2\mu\left(\frac{(\mathcal{P}_0 - \mathcal{P}_L)}{4\mu L}\left(2 - \left(\frac{x}{B}\right)^2 - \left(\frac{y}{B}\right)^2\right)\right)$$

Hence the differential equation is not satisfied.

*b.* The expression for the mass flow rate from Eq. 3B.3-1 is given by 4 times the flow rate for one quadrant:

$$\begin{aligned} w &= 4 \cdot \frac{(\mathcal{P}_0 - \mathcal{P}_L)B^2\rho}{4\mu L}\int_0^B\int_0^B\left[1 - \left(\frac{x}{B}\right)^2\right]\left[1 - \left(\frac{y}{B}\right)^2\right]dxdy \\ &= \frac{(\mathcal{P}_0 - \mathcal{P}_L)B^4\rho}{\mu L}\int_0^1\int_0^1(1-\xi^2)(1-\eta^2)d\xi d\eta \\ &= \frac{(\mathcal{P}_0 - \mathcal{P}_L)B^4\rho}{\mu L}\left[\int_0^1(1-\xi^2)d\xi\right]^2 \\ &= \frac{(\mathcal{P}_0 - \mathcal{P}_L)B^4\rho}{\mu L}\left(\tfrac{2}{3}\right)^2 = \frac{0.444(\mathcal{P}_0 - \mathcal{P}_L)B^4\rho}{\mu L} \end{aligned}$$

**3B.4 Creeping flow between two concentric spheres**

*a*. From Eq. B.4-3, there is only surviving term on the left side

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(rv_\theta\sin\theta\right)=0 \qquad \text{whence} \qquad v_\theta\sin\theta=u(r)$$

*b* . From Eq. B.6-8 (omitting the left side for *creeping flow*) the only surviving terms are

$$0=-\frac{1}{r}\frac{d\mathcal{P}}{d\theta}+\mu\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dv_\theta}{dr}\right)\right] \quad \text{or} \quad 0=-\frac{1}{r}\frac{d\mathcal{P}}{d\theta}+\mu\left[\frac{1}{r^2\sin\theta}\frac{d}{dr}\left(r^2\frac{du}{dr}\right)\right]$$

*c*. When the equation in (*b*) is multiplied by $r\sin\theta$, the left side is a function of $\theta$ alone, and the right side contains only $r$. This means that both sides must be equal to some constant, which we call $B$. This gives Eqs. 3B.4-2 and 3.

Integration of the pressure equation proceeds as follows:

$$\int_{\mathcal{P}_1}^{\mathcal{P}_2} d\mathcal{P}=\int_\varepsilon^{\pi-\varepsilon}\frac{d\theta}{\sin\theta} \qquad \text{or} \qquad \mathcal{P}_2-\mathcal{P}_1=B\ln\frac{\tan\frac{1}{2}(\pi-\varepsilon)}{\tan\frac{1}{2}\varepsilon}=B\ln\frac{\cot\frac{1}{2}\varepsilon}{\tan\frac{1}{2}\varepsilon}$$

From this we get the constant $B$

$$B=\frac{\mathcal{P}_2-\mathcal{P}_1}{\ln\cot^2\frac{1}{2}\varepsilon}=\frac{\mathcal{P}_2-\mathcal{P}_1}{2\ln\cot\frac{1}{2}\varepsilon}$$

Next we integrate the velocity equation

$$\frac{d}{dr}\left(r^2\frac{du}{dr}\right)=\frac{Br}{\mu} \quad \text{or} \quad u=\frac{BR}{2\mu}\left(\frac{r}{R}-\frac{C_1R}{r}+C_2\right)$$

where we have selected the constants of integration in such as way that they will be dimensionless: $C_1=-\kappa$ and $C_2=-\kappa-1$ (from the no-slip condition at the walls). When this solution is combined with the expression for $B$ we get:

$$u=\frac{BR}{2\mu}\left[-\left(1-\frac{r}{R}\right)-\kappa\left(1-\frac{R}{r}\right)\right]=\frac{\mathcal{P}_1-\mathcal{P}_2}{2\ln\cot\frac{1}{2}\varepsilon}\cdot\frac{R}{2\mu}\left[\left(1-\frac{r}{R}\right)+\kappa\left(1-\frac{R}{r}\right)\right]$$

which leads to Eq. 3B.4-5.

*d*. The mass rate of flow must be the same through any cross-section. It is easiest to get $w$ at $\theta = \frac{1}{2}\pi$ where $v_\theta = u(r)/\sin\theta = u(r)/\sin\frac{1}{2}\pi = u(r)$:

$$
\begin{aligned}
w &= \int_0^{2\pi}\int_{\kappa R}^{R} \rho v_\theta\big|_{\theta=\frac{1}{2}\pi} r dr d\phi = 2\pi R^2 \rho \int_\kappa^1 u\xi d\xi \\
&= 2\pi R^2 \rho \cdot \frac{(\mathcal{P}_1 - \mathcal{P}_2)R}{4\mu \ln\cot\frac{1}{2}\varepsilon} \int_\kappa^1 \left[(1-\xi) + \kappa\left(1 - \frac{1}{\xi}\right)\right]\xi d\xi \\
&= 2\pi R^2 \rho \cdot \frac{(\mathcal{P}_1 - \mathcal{P}_2)R}{4\mu \ln\cot\frac{1}{2}\varepsilon} \cdot \frac{1}{6}(1-\kappa)^3 \qquad \text{(see Eq. 3B.4-6)}
\end{aligned}
$$

**3B.5 Parallel-disk viscometer**

*a*. The equation of continuity (Eq. B.6-5) just gives 0 = 0. Equation B.6-5 gives for the $\theta$-component of the equation of motion

$$0 = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \frac{\partial^2 v_\theta}{\partial z^2}$$

*b*. When the postulated form for the tangential component of the velocity is inserted into this, we get simply $d^2f/dz^2 = 0$, which has to be solved with the boundary conditions that $f(0) = 0$ and $f(B) = \Omega$, which are just statements of the no-slip conditions on the wetted surfaces of the disks. It is easily shown that $f = \Omega z/B$. and therefore that $v_\theta = \Omega rz/B$.

*c*. The $z$-component of the torque exerted on the fluid by the upper rotating disk is

$$T_z = \int_0^{2\pi}\int_0^R (-\tau_{z\theta} r)\Big|_{z=B} r dr d\theta = 2\pi\int_0^R \left[+\mu\left(\frac{\partial v_\theta}{\partial z}\right)r\right]\Bigg|_{z=B} r dr = 2\pi\mu\cdot\frac{\Omega}{B}\int_0^R r^3 dr$$

We can now do the $r$-integration and solve for the viscosity to get the desired formula, $\mu = 2BT_z/\pi\Omega R^4$.

**3B.6 Circulating flow in an annulus**

*a*. Neglect of curvature gives for the *z*-component of the equation of motion

$$0 = -\frac{d\mathcal{P}}{dz} + \mu \frac{d^2 v_z}{dr^2} \qquad \text{or} \qquad \mu \frac{d^2 v_z}{dr^2} = \frac{d\mathcal{P}}{dz}$$

The left side is a function of $r$ alone and the right side is a function of $z$ alone, so that both sides must equal a constant. Therefore

$$\frac{d^2 v_z}{dr^2} = C' \quad \text{with boundary conditions } v_z(\kappa R) = v_0 \text{ and } v_z(R) = 0$$

Use the dimensionless variables $\phi = v_z/v_0$ and $\xi = r/R$; instead of the latter, it is more convenient to use $\zeta = (\xi - \kappa)/(1-\kappa)$ in this part of the problem. Then

$$\frac{d^2\phi}{d\zeta^2} = 2C_1 \quad \text{with boundary conditions } \phi(0) = 1 \text{ and } \phi(1) = 0$$

Integration of this equation gives

$$\phi = C_1\zeta^2 + C_2\zeta + C_3$$

The boundary conditions give $C_3 = 1$ and $C_2 - (1 + C_1)$ so that

$$\phi = C_1\zeta^2 - (1 + C_1)\zeta + 1$$

The mass-balance condition $\int_0^1 \phi d\zeta = 0$ gives $C_1 = 3$, and the velocity distribution is

$$\phi = 3\zeta^2 - 4\zeta + 1$$

*b*. We use exactly the same procedure when the thin-slit approximation is not made, but the algebraic manipulations are messier. The equation to be solved is

$$\mu \frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = \frac{d\mathcal{P}}{dz}$$

with the same no-slip boundary conditions as before. Use the same dimensionless variables as in (*a*) and find that the solution has the form

$$\phi = C_1\xi^2 + C_2 \ln \xi + C_3$$

The three constants of integration are determined from the no-slip boundary conditions $\phi(\kappa) = 1$ and $\phi(1) = 0$, along with the mass-conservation condition $\int_0^{2\pi}\int_\kappa^1 \phi\xi d\xi d\theta = 0$. The results are

$$C_1 = -C_3 = -\frac{1-\left[2\kappa^2/\left(1-\kappa^2\right)\right]\ln\left(1/\kappa\right)}{\left(1-\kappa^2\right)-\left(1+\kappa^2\right)\ln\left(1/\kappa\right)};\ C_2 = \frac{1-\kappa^2}{\left(1-\kappa^2\right)-\left(1+\kappa^2\right)\ln\left(1/\kappa\right)}$$

With these expressions for the integration constants, Eq. 3B.6-2 follows.

**3B.7 Momentum fluxes for creeping flow into a slot**

*a*. Inside the slot, the nonzero component of $\rho\mathbf{vv}$ is

$$\rho v_x v_x = \rho\left(\frac{3w}{4BW\rho}\right)^2\left[1-\left(\frac{y}{B}\right)^2\right]^2$$

Outside the slot, the nonzero component of $\rho\mathbf{vv}$ are

$$\rho v_x v_x = \rho\left(\frac{2w}{\pi W\rho}\right)^2 \frac{x^6}{(x^2+y^2)^4}; \qquad \rho v_y v_y = \rho\left(\frac{2w}{\pi W\rho}\right)^2 \frac{x^4 y^2}{(x^2+y^2)^4}$$

$$\rho v_x v_y = \rho v_y v_x = \rho\left(\frac{2w}{\pi W\rho}\right)^2 \frac{x^5 y}{(x^2+y^2)^4}$$

*b*. At $x = -a$, $y = 0$

$$\rho v_x v_x = \rho\left(\frac{2w}{\pi W\rho}\right)^2 \frac{1}{a^2}$$

This quantity is positive as we would expect, since positive $x$-momentum is being transported in the $x$-direction.

*c*. At $x = -a$, $y - +a$.

$$\rho v_x v_y = -\rho\left(\frac{2w}{\pi W\rho}\right)^2 \frac{1}{2^4 a^2}$$

This is negative, since at the point in question the $y$-momentum is negative and being transported in the $x$-direction.

*d*. The total flow of kinetic energy in the slot is (if we use $\eta = y/B$):

$$\int_0^W \int_{-B}^{+B} (\tfrac{1}{2}\rho v^2) v_x dy dz = 2W\int_0^B (\tfrac{1}{2}\rho v_x^3) dy = \rho BW\left(\frac{3w}{4BW\rho}\right)\int_0^1 (1-\eta^3) d\eta$$

$$= \frac{16}{35}\rho BW\left(\frac{3w}{4BW\rho}\right)^3$$

The total flow of kinetic energy across the plane at $x = -a$ is

$$\int_0^W \int_{-\infty}^{+\infty} (\tfrac{1}{2}\rho v^2) v_x dy dz = \tfrac{1}{2}\rho W \int_{-\infty}^{+\infty} (v_x^2 + v_y^2) v_x dy$$

$$= \tfrac{1}{2}\rho W \left( \frac{2w}{\pi W \rho} \right)^3 \int_{-\infty}^{+\infty} \left[ \frac{a^9}{(a^2+y^2)^6} + \frac{a^7 y^2}{(a^2+y^2)^6} \right] dy$$

$$= \frac{105}{768} \rho W \left( \frac{2w}{\pi W \rho} \right)^3 \frac{\pi}{a^2}$$

The integrals appearing here can be found in integral tables.

We conclude that the total flow of kinetic energy across the plane at $x = -a$ is not the same as that in the slit. As $a \to \infty$, the flow of kinetic energy tends toward zero, since the fluid velocity tends to zero as $x \to -\infty$. This emphasizes that kinetic energy is not conserved.

*e*. Eq. 3B.7-1 clearly satisfies the equation of continuity, since for incompressible flow $(\partial v_x/\partial x) + (\partial v_y/\partial y) + (\partial v_z/\partial z) = 0$. When the derivatives are calculated from Eqs. 3B.7-2, 3, and 4, it is found that these expressions also satisfy the incompressible equation of continuity as well.

*f*. From Eqs. 3B.7-2 and B.1-1 we get

$$\tau_{xx} = -2\mu \left. \frac{\partial v_x}{\partial x} \right|_{y=0} = 2\mu \left( \frac{2w}{\pi W \rho} \right) \left[ \frac{3x^2}{(x^2+y^2)^2} - \frac{4x^4}{(x^2+y^2)^3} \right]_{y=0} = -\frac{4w\mu}{\pi W \rho x^2}$$

$$\tau_{xx} = -2\mu \left. \frac{\partial v_x}{\partial x} \right|_{x=0} = 2\mu \left( \frac{2w}{\pi W \rho} \right) \left[ \frac{3x^2}{(x^2+y^2)^2} - \frac{4x^4}{(x^2+y^2)^3} \right]_{x=0} = 0$$

The second of these is an illustration of Example 3.1-1.

*g*. From Eqs. 3B.7-2, 3B.7-3, and B.1-4, we get, after evaluating the derivatives

$$\tau_{yx} = -\mu \left. \left( \frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right) \right|_{x=0} = 0$$

**3B.8 Velocity distribution for creeping flow toward a slot**

*a*. For the given postulates, the equation of continuity gives

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r)=0 \qquad \text{from which it follows that } v_r=\frac{1}{r}f(\theta)$$

Since the flow is symmetric about $\theta=0$, $df/d\theta=0$ at $\theta=0$; and since the fluid velocity is zero at $\theta=\pm\frac{1}{2}\pi$, it follows that $f=0$ at $\theta=\pm\frac{1}{2}\pi$.

*b*. The components of the equation of motion given in Eqs. B.6-4 and 5, appropriately simplified are

$$0=-\frac{\partial\mathfrak{P}}{\partial r}+\frac{\mu}{r^3}\frac{d^2f}{d\theta^2} \qquad \text{and} \qquad 0=-\frac{\partial\mathfrak{P}}{\partial\theta}+\frac{2\mu}{r^2}\frac{df}{d\theta}$$

*c*. When the first equation is differentiated with respect to $\theta$ and the second with respect to $r$ and the two results subtracted we get Eq. 3B.8-1.

*d*. The equation in Eq. 3B.8-1 can be integrated once to get

$$\frac{d^2f}{d\theta^2}+4f=C_1$$

A particular integral is $f_{\mathrm{PI}}=\frac{1}{4}C_1$, and the complementary function is (according to Eq. C.1-3) $f_{\mathrm{CF}}=C_2\cos 2\theta+C_3\sin 2\theta$ . The complete solution is then the sum of these two functions.

*e*. The integration constants are determined from the boundary conditions. It is found that $C_3=0$, and that $\frac{1}{4}C_1=C_2$. Then from

$$w=-W\rho\int_{-\pi/2}^{+\pi/2}v_r r\,d\theta=-W\rho\int_{-\pi/2}^{+\pi/2}f\,d\theta=-2W\rho C_2\int_{-\pi/2}^{+\pi/2}\cos^2\theta\,d\theta=-W\rho C_2\pi$$

we get $C_2=-w/W\rho\pi$ and the velocity distribution is given by Eq. 3B.8-2.

*f*. From the velocity distribution and the equations obtained in (*b*) we can get

$$\frac{\partial \mathfrak{P}}{\partial r} = \frac{\mu}{r^3}\left(\frac{4w}{\pi W \rho}\right)\left(\cos^2\theta - \sin^2\theta\right)$$

and hence

$$\mathfrak{P} = -\frac{\mu}{r^2}\left(\frac{2w}{\pi W \rho}\right)\left(\cos^2\theta - \sin^2\theta\right) + F(\theta) \quad (*)$$

Furthermore

$$\frac{\partial \mathfrak{P}}{\partial \theta} = \frac{2\mu}{r^2}\frac{df}{d\theta} \qquad \text{or} \qquad \mathfrak{P} = -\left(\frac{2\mu}{r^2}\right)\left(\frac{2w}{\pi W \rho}\right)\cos^2\theta + G(r)$$

Here $F$ and $G$ are arbitrary functions of their arguments. The second expression for the modified pressure can be rewritten as

$$\begin{aligned}\mathfrak{P} &= -\frac{\mu}{r^2}\left(\frac{2w}{\pi W \rho}\right)\left(\cos^2\theta + \left(1 - \sin^2\theta\right)\right) + G(r) \\ &= -\frac{\mu}{r^2}\left(\frac{2w}{\pi W \rho}\right)\left(\cos^2\theta - \sin^2\theta\right) + H(r) \quad (*)\end{aligned}$$

By comparing the two (*) expressions for the modified pressure, we see that they are the same except for the functions $F$ and $H$. Since the first is a function of $\theta$ alone and the second a function of $r$ alone, they must both be equal to a constant, which we call $\mathfrak{P}_\infty$. This is the value of the modified pressure at $r = \infty$.

*g*. The total normal stress exerted on the wall at $\theta = \pi/2$ is (when one uses the result of Example 3.1-1)

$$\begin{aligned}\pi_{\theta\theta}\big|_{\theta=\pi/2} &= \left(p + \tau_{\theta\theta}\right)\big|_{\theta=\pi/2} = \left(\mathfrak{P} - \rho g h\right)\big|_{\theta=\pi/2} \\ &= \mathfrak{P}_\infty + \frac{2\mu w}{\pi W \rho r^2} - \rho g h = p_\infty + \frac{2\mu w}{\pi W \rho r^2}\end{aligned}$$

*h*. From Eq. B.1-11

$$\tau_{\theta r}\Big|_{\theta=\pi/2} = -\mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)+\frac{1}{r}\frac{\partial v_r}{\partial\theta}\right]\Bigg|_{\theta=\pi/2}$$

$$= -\mu\left[0-\left(\frac{2w}{\pi W\rho r^2}\right)(-2\cos\theta\sin\theta)\right]\Bigg|_{\theta=\pi/2} = 0$$

The first term is zero since $v_\theta/r = f$ since $v_\theta = 0$ was one of the postulates. The second term is zero, as can be seen by using Eq. 3B.8-2, and the fact that $\cos\frac{\pi}{2} = 0$. This is agreement with the result in Problem 3B.7(*g*).

*i.* Since the *z*-component of the velocity is zero, we can expand the velocity vector in either the cylindrical coordinate system or the Cartesian system thus

$$\mathbf{v} = \boldsymbol{\delta}_r v_r + \boldsymbol{\delta}_\theta v_\theta = \boldsymbol{\delta}_x v_x + \boldsymbol{\delta}_y v_y$$

Since $v_\theta = 0$ was one of the postulates, when we take the dot product of this equation with $\boldsymbol{\delta}_x$, we get the *x*-component of the velocity

$$v_x = (\boldsymbol{\delta}_x\cdot\boldsymbol{\delta}_r) = v_r\cos\theta = -\frac{2w}{\pi W\rho r}\cos^3\theta = -\frac{2wx^3}{\pi W\rho r^4} = -\frac{2wx^3}{\pi W\rho\left(x^2+y^2\right)^2}$$

Similarly for the *y*-component of the velocity

$$v_y = (\boldsymbol{\delta}_y\cdot\boldsymbol{\delta}_r) = v_r\sin\theta = -\frac{2w}{\pi W\rho r}\cos^2\theta\sin\theta = -\frac{2wx^2y}{\pi W\rho r^4}$$

$$= -\frac{2wx^2y}{\pi W\rho\left(x^2+y^2\right)^2}$$

These results are in agreement with Eqs. 3B.7-2 and 3.

**3B.9 Slow transverse flow around a cylinder**

*a*. At the cylinder surface we get by using Eq. B.1-11

$$p\big|_{r=R} = p_\infty - \frac{C\mu v_\infty \cos\theta}{R} - \rho g R \sin\theta$$

$$\tau_{r\theta}\big|_{r=R} = -\mu\left( r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)\Bigg|_{r=\theta} = \frac{C\mu v_\infty \sin\theta}{R}$$

Also from Example 3.1-1 we know that $\tau_{rr}\big|_{r=R} = 0$.

*b*. If $\mathbf{n}$ is the outwardly directed unit normal vector for the cylinder, then the force per unit area acting on the surface is $-[\mathbf{n}\cdot\boldsymbol{\pi}]$ evaluated at the surface. But $\mathbf{n} = \boldsymbol{\delta}_r$, so that the $x$-component of the force per unit area at every point is

$$-\left(\boldsymbol{\delta}_x \cdot [\boldsymbol{\delta}_r \cdot \boldsymbol{\pi}]\right)\Big|_{r=R} = -\left(\boldsymbol{\delta}_x \cdot [\boldsymbol{\delta}_r \cdot (p\boldsymbol{\delta} + \boldsymbol{\tau})]\right)\Big|_{r=R}$$

The pressure and stress terms are evaluated thus (using Eq. A.6-13):

$$-\left(\boldsymbol{\delta}_x \cdot [\boldsymbol{\delta}_r \cdot p\boldsymbol{\delta}]\right)\Big|_{r=R} = -\left(\boldsymbol{\delta}_x \cdot \boldsymbol{\delta}_r p\right)\Big|_{r=R} = -p\big|_{r=R}\cos\theta$$

$$\begin{aligned}-\left(\boldsymbol{\delta}_x \cdot [\boldsymbol{\delta}_r \cdot \boldsymbol{\tau}]\right)\Big|_{r=R} &= -\left(\boldsymbol{\delta}_x \cdot [\boldsymbol{\delta}_r \cdot (\cdots + \boldsymbol{\delta}_r\boldsymbol{\delta}_\theta \tau_{r\theta} + \cdots)]\right)\Big|_{r=R} \\ &= -\left(\boldsymbol{\delta}_x \cdot \boldsymbol{\delta}_\theta\right)\tau_{r\theta}\big|_{r=R} = \tau_{r\theta}\big|_{r=R}\sin\theta\end{aligned}$$

These expressions lead to Eq. 3B.9-5.

*c*. The total force on a length $L$ of the cylinder is then

$$\begin{aligned}F_x &= \int_0^L\int_0^{2\pi}\left(-p\big|_{r=R}\cos\theta + \tau_{r\theta}\big|_{r=R}\sin\theta\right)R\,d\theta\,dz \\ &= RL\int_0^{2\pi}\left(-p_\infty\cos\theta + \frac{C\mu v_\infty\cos^2\theta}{R} + \rho g R\sin\theta\cos\theta + \frac{C\mu v_\infty \sin^2\theta}{R}\right)d\theta \\ &= RL\left(\frac{C\mu v_\infty}{R}\right)\int_0^{2\pi} d\theta = 2\pi C\mu v_\infty L\end{aligned}$$

The first and third integrals in the next-to-last line are zero since the integrands are odd.

**3B.10 Radial flow between parallel disks**

*a*. The continuity equation is, for $v_r = v_r(r,z)$ , from Eq. B.4-2

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) = 0 \qquad \text{from which} \qquad rv_r = \phi(z)$$

The equation of motion is obtained from Eq. B.6-4

$$\rho v_r \frac{\partial v_r}{\partial r} = -\frac{d\mathcal{P}}{dr} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right) + \frac{\partial^2 v_r}{\partial z^2}\right]$$

*b*. When the results for the equation of continuity in (*a*) are used in the equation of motion, we get Eq. 3B.10-1.

*c*. With the creeping flow assumption, Eq. 3B.10-1 gives

$$r\frac{d\mathcal{P}}{dr} = \mu\frac{d^2\phi}{dz^2} \qquad \text{from which} \qquad r\frac{d\mathcal{P}}{dr} = B \quad \text{and} \quad \mu\frac{d^2\phi}{dz^2} = B$$

since the left side is a function of $r$ alone and the right side a function of $z$ alone, and therefore both sides must be equal to a constant, $B$. When the pressure equation is integrated from $r_1$ to $r_2$, we get

$$\int_{\mathcal{P}_1}^{\mathcal{P}_2} d\mathcal{P} = B\int_{r_1}^{r_2}\frac{dr}{r} \quad \text{whence} \quad B = \frac{\mathcal{P}_2 - \mathcal{P}_1}{\ln(r_2/r_1)}$$

*d*. When this result is substituted into the $\phi$ equation we get

$$\frac{d^2\phi}{dz^2} = -\frac{\mathcal{P}_1 - \mathcal{P}_2}{\mu\ln(r_2/r_1)} \qquad \text{from which} \qquad \phi = -\frac{\mathcal{P}_1 - \mathcal{P}_2}{\mu\ln(r_2/r_1)}\frac{z^2}{2} + C_1 z + C_2$$

The integration constants are obtained from applying two boundary conditions. We could require that $\phi = 0$ at $z = \pm b$, and thereby determine the integration constants. Another method is to recognize that the flow is symmetric about $z = 0$, and use as one of the boundary conditions $d\phi/dz = 0$ at $z = 0$. Either method will give $C_1 = 0$, and then $C_2$ is easily obtained. The final result is

$$\phi = \frac{(\mathcal{P}_1 - \mathcal{P}_2)b^2}{2\mu \ln(r_2/r_1)}\left[1 - \left(\frac{z}{b}\right)^2\right]$$

Division by $r$ then gives Eq. 3B.10-3.

*e*. The mass rate of flow at any cylindrical surface in the system must be the same. Select the surface at $r = r_2$ and obtain

$$w = \int_0^{2\pi}\int_{-b}^{+b} \rho v_r\big|_{r=r_2}\, dz r_2 d\theta = 2\pi\rho b \cdot 2 \cdot \frac{(\mathcal{P}_1 - \mathcal{P}_2)b^2}{2\mu \ln(r_2/r_1)}\int_0^1 (1-\xi^2)d\xi$$

The integral gives 2/3, so that Eq. 3B.10-4 is obtained.

**3B.11 Radial flow between two coaxial cylinders**

*a*. From Eq. B.4-2 we get for this flow, with $v_r(r)$,

$$\frac{1}{r}\frac{d}{dr}(rv_r)=0 \qquad \text{whence} \qquad v_r=\frac{C}{r} \qquad \text{where C is a constant}$$

At $r = R$, $v_r(R) = C/R$ so that $C = Rv_r(R)$.

*b*. The relations in Eq. 3B.11-1 follow immediately from Eqs. B.6-4, 5, and 6 for the velocity profile $v_r(r)$ in (*a*).

*c*. Integration from $r$ to $R$ *gives*

$$\mathcal{P}(R)-\mathcal{P}(r)=\rho C^2\left(-\frac{1}{2r^2}\right)\Bigg|_r^R=-\tfrac{1}{2}\rho C^2\left(\frac{1}{R^2}-\frac{1}{r^2}\right)$$

This gives, making use of the meaning of $C$ obtained in (*a*)

$$\mathcal{P}(r)-\mathcal{P}(R)=\tfrac{1}{2}\rho\left[Rv_r(R)\right]^2\left(\frac{1}{R^2}-\frac{1}{r^2}\right)=\tfrac{1}{2}\rho\left[v_r(R)\right]^2\left[1-\left(\frac{R}{r}\right)^2\right]$$

*d*. The only nonzero components are (from Eqs. B.1-8 to 13)

$$\tau_{rr}=-2\mu\frac{dv_r}{dr}=+2\mu C\frac{1}{r^2}; \qquad \tau_{\theta\theta}=-2\mu\frac{v_r}{r}=-2\mu C\frac{1}{r^2}$$

**3B.12 Pressure distribution in incompressible fluids**

The equation of motion method to get the pressure distribution is correct. On the other hand, the second method gives nonsense, as one can see from Fig. 3.5-1. For an incompressible fluid (the vertical straight line), specifying the density does not give any information about the pressure.

**3B.13 Flow of a fluid through a sudden contraction**

*a*. For an incompressible fluid, Eq. 3.5-12 becomes

$$\tfrac{1}{2}\rho\left(v_2^2 - v_1^2\right) + \left(p_2 - p_1\right) + \rho g\left(h_2 - h_1\right) = 0 \quad \text{or} \quad \tfrac{1}{2}\rho\left(v_2^2 - v_1^2\right) + \left(\mathcal{P}_2 - \mathcal{P}_1\right) = 0$$

If "1" is the large tube and "2" the small one, then the fluid velocity in "2" must be greater than in "1." Then the modified pressure in "2" must be less than that in "1." Thus the modified pressure decreases as the fluid moves from the large cross-section region to the small cross-section region, in agreement with experimental observations.

*b*. For an ideal gas, Eq. 3.5-12 becomes

$$\tfrac{1}{2}\rho\left(v_2^2 - v_1^2\right) + \frac{RT}{M}\ln\frac{p_2}{p_1} + \rho g\left(h_2 - h_1\right) = 0$$

The pressure and elevation terms may no longer be combined. If the elevation does not change, the pressure decreases as the fluid moves into the contracted part of the tube.

**3B.14 Torricelli's equation for efflux from a tank**

From Eq. 3.5-12 we get

$$\tfrac{1}{2}\left(v_{\text{efflux}}^2 - 0\right) + \frac{1}{\rho}\left(p_{\text{atm}} - p_{\text{atm}}\right) + g(0 - h) = 0$$

Here it has been assumed that the velocity at the surface is virtually zero, that the pressure is atmospheric at both "1" and "2", and that the datum plane for the height is at the exit tube. When the above equation is solved for the efflux velocity we get Torricelli's equation.

**3B.15 Shape of free surface in tangential annular flow**

*a*. The velocity distribution is given by Eq. 3.6-32, and the equations corresponding to Eqs. 3.6-38 and 39 are:

$$\frac{\partial p}{\partial r} = \rho\left(\frac{\Omega_i \kappa^2 R}{1-\kappa^2}\right)^2 \frac{1}{r}\left[\left(\frac{R}{r}\right)^2 - 2 + \left(\frac{r}{R}\right)^2\right] \quad \text{and} \quad \frac{\partial p}{\partial z} = -\rho g$$

Integration of these equations gives (see Eqs. 3.6-40, 41, and 42)

$$p = \tfrac{1}{2}\rho\left(\frac{\Omega_i \kappa^2 R}{1-\kappa^2}\right)^2\left[-\left(\frac{R}{r}\right)^2 - 4\ln r + \left(\frac{r}{R}\right)^2\right] - \rho g z + C$$

Now let $p = p_{\text{atm}}$ at $r = R$ and $z = z_R$, where $z_R$ is the height of the liquid at the outer-cylinder surface. Then we can write at $r = R$ and $z = z_R$

$$p_{\text{atm}} = \tfrac{1}{2}\rho\left(\frac{\Omega_i \kappa^2 R}{1-\kappa^2}\right)^2 [-1 - 4\ln R + 1] - \rho g z_R + C$$

which is the equation that determines C. When we subtract the last equation from the equation for $p$, we get

$$p - p_{\text{atm}} = \tfrac{1}{2}\rho\left(\frac{\Omega_i \kappa^2 R}{1-\kappa^2}\right)^2\left(-\frac{1}{\xi^2} - 4\ln\xi + \xi^2\right) - \rho g(z - z_R)$$

The equation of the liquid surface is then the locus of all points for which $p = p_{\text{atm}}$, or

$$z_R - z = \frac{1}{2g}\left(\frac{\Omega_i \kappa^2 R}{1-\kappa^2}\right)^2\left(\frac{1}{\xi^2} + 4\ln\xi - \xi^2\right)$$

*b*. When the outer cylinder is rotating, we can use Eq. 3.6-29 for the velocity distribution to get

$$p = \rho\left(\frac{\Omega_o \kappa R}{1-\kappa^2}\right)^2\left[\frac{1}{2}\left(\frac{r}{\kappa R}\right)^2 - 2\ln r - \frac{1}{2}\left(\frac{\kappa R}{r}\right)^2\right] - \rho g z + C$$

Then, we select $r = R$ and $z = z_R$ as the point to determine $C$.

$$p_{\text{atm}} = \rho\left(\frac{\Omega_o \kappa R}{1-\kappa^2}\right)^2\left[\frac{1}{2}\left(\frac{1}{\kappa}\right)^2 - 2\ln R - \frac{1}{2}(\kappa)^2\right] - \rho g z_R + C$$

Subtracting, we get

$$p - p_{\text{atm}} = \frac{\rho}{2}\left(\frac{\Omega_o \kappa R}{1-\kappa^2}\right)^2\left[\left(\frac{\xi}{\kappa}\right)^2 - \left(\frac{1}{\kappa}\right)^2 - 4\ln\xi - \left(\frac{\kappa}{\xi}\right)^2 + (\kappa)^2\right] + \rho g(z_R - z)$$

From this equation we can get Eq. 3B.15-2 by setting the left side of the equation equal to zero and solving for $z_R - z$.

**3B.16 Flow in a slit with uniform cross flow**

From Eq. B.6-1, for this problem we have

$$\rho v_0 \frac{dv_x}{dy} = \frac{(\mathcal{P}_0 - \mathcal{P}_L)}{L} + \mu \frac{d^2 v_x}{dy^2} \qquad \text{or} \qquad \frac{d^2\phi}{d\eta^2} - A\frac{d\phi}{d\eta} + 1 = 0$$

in which $\eta = y/B$, $\phi = v_x \big/ \left[(\mathcal{P}_0 - \mathcal{P}_L)B^2/\mu L\right]$, and $A = Bv_0\rho/\mu$. This equation has to be solved with the no-slip condition at $\eta = 0,1$. We write the solution as the sum of a complementary function and a particular integral. The equation for the complementary function is

$$\frac{d^2\phi_{\text{CF}}}{d\eta^2} - A\frac{d\phi_{\text{CF}}}{d\eta} = 0 \qquad \text{with solution} \qquad \phi_{\text{CF}} = \frac{C_1}{A}e^{A\eta} + C_2$$

By inspection, the particular integral is $\phi_{\text{PI}} = \eta/A$. Application of the boundary conditions then gives the constants of integration. The final solution is then (with $A = Bv_0\rho/\mu$)

$$\phi = \frac{1}{A}\left(\eta - \frac{e^{A\eta} - 1}{e^A - 1}\right) \qquad \text{or} \qquad v_x = \frac{(\mathcal{P}_0 - \mathcal{P}_L)B^2}{\mu L}\frac{1}{A}\left(\frac{y}{B} - \frac{e^{Ay/B} - 1}{e^A - 1}\right)$$

*b*. The mass rate of flow in the $x$-direction is then

$$w = \int_0^W \int_0^B \rho v_x dy dz = \frac{(\mathcal{P}_0 - \mathcal{P}_L)B^3 W\rho}{\mu L}\int_0^1 \phi d\eta$$
$$= \frac{(\mathcal{P}_0 - \mathcal{P}_L)B^3 W\rho}{\mu L}\frac{1}{A}\left(\frac{1}{2} - \frac{1}{A} + \frac{1}{e^A - 1}\right)$$

*c* By making a Taylor-series expansion about $A = 0$, from (*a*) we get $\phi = \frac{1}{2}(\eta - \eta^2) + O(A)$. When $A \to 0$, this result can be shown to be equivalent to Eq. 2B.3-2. Similarly, A Taylor-series expansion about $A = 0$ yields from the result in (*b*)

$$\frac{w}{\left[(\mathcal{P}_0 - \mathcal{P}_L)WB^3\rho/\mu L\right]} = \frac{1}{A}\left(\frac{1}{2} - \frac{1}{A} + \frac{1}{e^A - 1}\right)$$

$$
\begin{aligned}
&= \frac{1}{A}\left(\frac{1}{2} - \frac{1}{A} + \frac{1}{\left(1 + A + \frac{1}{2}A^2 + \frac{1}{3}A^3 + \cdots\right) - 1}\right) \\
&= \frac{1}{A}\left(\frac{1}{2} - \frac{1}{A} + \frac{1}{A\left(1 + \frac{1}{2}A + \frac{1}{3}A^2 + \cdots\right)}\right) \\
&= \frac{1}{A}\left(\frac{1}{2} - \frac{1}{A} + \frac{1 - \frac{1}{2}A + \frac{1}{12}A^2 + \cdots}{A}\right) \\
&= \tfrac{1}{12} + O(A)
\end{aligned}
$$

But the "$B$" in this problem is twice the "$B$" of problem 2B.3. If we switch to the "$B$" of Problem 2B.3, it is found that the answers agree exactly.

*d*. For the coordinate system here, we select as the dimensionless quantities

$$
Y = \frac{y}{b}; \quad V = \frac{v_x}{(\mathcal{P}_0 - \mathcal{P}_L)b^2/\mu L}; \; a = \frac{bv_0\rho}{\mu}
$$

Then the differential equation and boundary conditions are

$$
a\frac{dV}{dY} = 1 + \frac{d^2V}{dY^2} \quad \text{with} \qquad V(\pm 1) = 0
$$

The solution is then the sum of a complementary function and a particular integral (as before in (*c*))

$$
V = \frac{C_1}{a}e^{aY} + C_2 + \frac{Y}{a}
$$

Application of the boundary conditions then leads to

$$
V = \frac{-e^{aY} + \cosh a + Y \sinh a}{a \sinh a}
$$

Then the average value of this over the cross-section of flow is

$$\langle V\rangle = \frac{\int_{-1}^{+1} V dY}{\int_{-1}^{+1} dY} = \frac{1}{2}\frac{\int_{-1}^{+1}\left(-e^{aY} + \cosh a + Y\sinh a\right)dY}{a\sinh a} = \frac{-(1/a)\sinh a + \cosh a}{a\sinh a}$$

Then we can form the ratio given in Eq. 3B.16-3:

$$\frac{V}{\langle V\rangle} = \frac{e^{aY} - Y\sinh a - \cosh a}{(1/a)\sinh a - \cosh a}$$

As a check on this we can go directly from Eqs. 3B.16-1 and 2 to Eq. 3B.16-3. From the first two equations we get

$$\frac{v_x}{\langle v_x\rangle} = \frac{v_x}{w/WB} = \frac{\dfrac{y}{B} - \dfrac{e^{Ay/B} - 1}{e^A - 1}}{\dfrac{1}{2} - \dfrac{1}{A} + \dfrac{1}{e^A - 1}} = \frac{2A\left[\left(e^A - 1\right)y - B\left(e^{Ay/B} - 1\right)\right]}{B\left[(A-2)\left(e^A - 1\right) + 2A\right]}$$

Next, we make the connections between the notations in the two different approaches:

$$y = z + b\ ; \qquad B = 2b; \qquad A = 2\alpha$$

(the "$y$" of part (c) is called "z" here, and $\zeta = z/b$). Then

$$\frac{v_x}{\langle v_x\rangle} = \frac{\alpha\left[\left(e^{2\alpha} - 1\right)(z+b) - 2b\left(e^{(\alpha/b)(z+b)} - 1\right)\right]}{b\left[(\alpha - 1)\left(e^{2\alpha} - 1\right) + 2\alpha\right]}$$

$$= \frac{\left(e^{2\alpha} - 1\right)(\zeta + 1) - 2\left(e^{\alpha\zeta}e^{\alpha} - 1\right)}{(1/\alpha)\left[(\alpha - 1)\left(e^{2\alpha} - 1\right) + 2\alpha\right]} = \frac{\left(e^{\alpha} - e^{-\alpha}\right)(\zeta + 1) - 2\left(e^{\alpha\zeta} - e^{-\alpha}\right)}{(1/\alpha)\left[(\alpha - 1)\left(e^{\alpha} - e^{-\alpha}\right) + 2\alpha e^{-\alpha}\right]}$$

$$= \frac{\zeta\left(e^{\alpha} - e^{-\alpha}\right) + \left(e^{\alpha} + e^{-\alpha}\right) - 2e^{\alpha\zeta}}{(1/\alpha)\left[\alpha\left(e^{\alpha} + e^{-\alpha}\right) - \left(e^{\alpha} - e^{-\alpha}\right)\right]} = \frac{\zeta\sinh\alpha + \cosh\alpha - e^{\alpha\zeta}}{\cosh\alpha - (1/\alpha)\sinh\alpha}$$

**3C.1 Parallel-disk compression viscometer**

*a*. The equation of continuity of Eq. B.4-2 for incompressible fluids, taking into account the symmetry about the *z*-axis is just Eq. 3C.1-6. The equation of motion in Eq. 3C.1-7 comes from Eq. B.6-4 ignoring the hydrostatic pressure, the inertial force terms, and omitting the terms that are small.

*b*. Equation 3C.1-7 can be integrated with respect to $z$ to give

$$v_r = \frac{1}{2\mu}\frac{dp}{dr}z^2 + C_1 z + C_2$$

The constant $C_2$ is found to be zero from the boundary condition in Eq. 3C.1-8, and $C_1$ is found from Eq. 3C.1-9.

*c*. Integrating of Eq. 3C.1-6 with respect to $z$ from 0 to $H$

$$\int_0^H \frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\frac{1}{2\mu}\frac{dp}{dr}z(z-H)\right)dz = -\int_0^{-v_0} dv_z$$

Performing the integrations then gives

$$-\frac{H^3}{12\mu}\frac{1}{r}\frac{d}{dr}\left(r\frac{dp}{dr}\right) = v_0$$

*d*. Integration of the equation in (*c*) then yields

$$p = -\frac{12\mu v_0}{H^3}\frac{r^2}{4} + C_1 \ln r + C_2$$

The integration constant $C_1$ must be zero, since the pressure is finite at the center of the disks, and $C_2$ is determined from Eq. 3C.1-10. Equation 3C.1-13 is thus obtained.

*e*. The force on the upper plate is then

$$F(t) = \int_0^{2\pi}\int_0^R \frac{3\mu v_0 R^2}{[H(t)]^3}\left[1-\left(\frac{r}{R}\right)^2\right] r\,dr\,d\theta = 2\pi R^2 \frac{3\mu v_0 R^2}{[H(t)]^3}\int_0^1 (1-\xi^2)\xi\,d\xi$$

The integral is 1/4, and this leads to the result in Eq. 3C.1-14.

*f*. In this situation, the radius of the glob of liquid $R(t)$ and the instantaneous disk separation $H(t)$ are related to the sample volume $V$ by $V = \pi[R(t)]^2 H(t)$. Then the force acting on the upper disk is

$$F(t) = \int_0^{2\pi}\int_0^{R(t)} \frac{3\mu v_0 [R(t)]^2}{[H(t)]^3}\left[1-\left(\frac{r}{R(t)}\right)^2\right] r dr d\theta$$
$$= \frac{3\pi\mu v_0 [R(t)]^4}{2[H(t)]^3} = \frac{3\mu v_0 V^2}{2\pi[H(t)]^5}$$

*g*. If, in Eq. 3C.1-14 we replace $v_0$ by $-dH/dt$, we then have an ordinary, separable differential equation for $H(t)$. Integration gives

$$\frac{2F_0}{3\pi\mu R^4}\int_0^t dt = -\int_{H_0}^{H} \frac{dH}{H^3}$$

whence

$$\frac{1}{[H(t)]^2} = \frac{1}{H_0^2} + \frac{4F_0 t}{3\pi\mu R^4}$$

**3C.2 Normal stresses at solid surfaces for compressible fluids**

First write the equation of continuity for a compressible fluid as

$$\frac{\partial}{\partial t}\ln\rho = -(\nabla\cdot\mathbf{v}) - \frac{1}{\rho}(\mathbf{v}\cdot\nabla\rho)$$

The normal stress on a surface perpendicular to the $z$-axis is

$$\begin{aligned}
\tau_{zz}\big|_{z=0} &= \left(-2\mu\frac{\partial v_z}{\partial z} + \left(\tfrac{2}{3}\mu - \kappa\right)(\nabla\cdot\mathbf{v})\right)\Bigg|_{z=0} \\
&= \left(2\mu\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial}{\partial t}\ln\rho + \frac{1}{\rho}(\mathbf{v}\cdot\nabla\rho)\right) + \left(\tfrac{2}{3}\mu - \kappa\right)\left(-\frac{\partial}{\partial t}\ln\rho + \frac{1}{\rho}(\mathbf{v}\cdot\nabla\rho)\right)\right)\Bigg|_{z=0} \\
&= \left(\tfrac{4}{3}\mu + \kappa\right)\left(\frac{\partial}{\partial t}\ln\rho\right)\Bigg|_{z=0}
\end{aligned}$$

The terms containing $\mathbf{v}$ drop out by the no-slip condition at the surface, and their derivatives with respect to $x$ and $y$ drop out on the surface as explained in Example 3.1-1. This result shows that the normal stresses at surfaces are zero for compressible flow if the flow is at steady state.

### 3C.3 Deformation of a fluid line

The curve at any time $t$ is $\theta(r,t) = (v_\theta/r)t$, which in tangential annular flow is (from Eq. 3.6-32)

$$\theta(r,t) = \left(\frac{(R/r)^2 - 1}{(1/\kappa)^2 - 1}\right)\Omega_i t \qquad \text{and} \quad d\theta = -\frac{2(R^2/r^3)\Omega_i t}{(1/\kappa)^2 - 1}dr$$

The differential element of length along the curve is given by

$$(dl)^2 = (dr)^2 + (rd\theta)^2 = (dr)^2\left[1 + \frac{4(R/r)^4(\Omega_i t)^2}{\left((1/\kappa)^2 - 1\right)^2}\right]$$

The total length of the curve is then

$$l = \int_0^l d\bar{l} = \int_{\kappa R}^{R}\sqrt{1 + \frac{4(R/r)^4(2\pi N)^2}{\left((1/\kappa)^2 - 1\right)^2}}dr = R\int_\kappa^1\sqrt{1 + \frac{16\pi^2 N^2}{\left((1/\kappa)^2 - 1\right)^2\xi^4}}d\xi$$

To get a rough, order-of-magnitude estimate assume that $N$ is large and then the "1" can be neglected and the integral performed analytically

$$\frac{l}{R} \approx \frac{4\pi N\kappa}{1+\kappa} \qquad \text{(limit of large } N\text{)}$$

## 3C.4 Alternative methods of solving the Couette viscometer problem by use of angular momentum concepts

*a*. By making an angular momentum balance (actually the *z*-component of the angular momentum balance) over an annular region of thickness $\Delta r$ and height $L$ we obtain

$$(2\pi rL)\cdot(r\tau_{r\theta})\big|_r - (2\pi(r+\Delta r)L)\cdot(r\tau_{r\theta})\big|_{r+\Delta r} = 0$$

Dividing through by $2\pi L\Delta r$ and rearranging we get

$$\frac{(r^2\tau_{r\theta})\big|_{r+\Delta r} - (r^2\tau_{r\theta})\big|_r}{\Delta r} = 0$$

whence

$$\frac{d}{dr}(r^2\tau_{r\theta}) = 0$$

the second form resulting from taking the limit as $\Delta r \to 0$. Then using Eq. B.1-11 for the stress-tensor component, we have

$$\frac{d}{dr}\left(r^3\frac{d}{dr}\left(\frac{v_\theta}{r}\right)\right) = 0$$

whence

$$v_\theta = -\frac{C_1}{2r} + C_2 r$$

From this Eq 3.6-20 follows.

*b*. Here we start with Eq. 3.4-1, which simplifies to the following for the symmetric stress tensor

$$\left[\nabla\cdot\{\mathbf{r}\times\boldsymbol{\tau}\}^{\dagger}\right] = 0$$

The *z*-component of this equation is

$$\frac{1}{r}\frac{d}{dr}\left(r\{\mathbf{r}\times\boldsymbol{\tau}\}^{\dagger}_{rz}\right)=0 \qquad \text{or} \qquad \frac{1}{r}\frac{d}{dr}\left(r\{\mathbf{r}\times\boldsymbol{\tau}\}_{zr}\right)=0$$

where, in cylindrical coordinates, $\mathbf{r}=\boldsymbol{\delta}_r r+\boldsymbol{\delta}_z z$. We now work out the cross product, which is

$$\{\mathbf{r}\times\boldsymbol{\tau}\}_{zr}=\sum_i\sum_j \varepsilon_{zij}\left(\boldsymbol{\delta}_r r+\boldsymbol{\delta}_z z\right)_i \tau_{jr}=\varepsilon_{zr\theta} r\tau_{\theta r}+\varepsilon_{z\theta r}(0)\tau_{rr}$$

Hence the equation of change for angular momentum simplifies to

$$\frac{1}{r}\frac{d}{dr}\left(r^2\tau_{r\theta}\right)=0$$

and the development proceeds further as in (*a*).

**3C.5 Two-phase interfacial boundary conditions**

*a*. This result follows at once from Eq. 3C.5-1, when the viscous-stress-tensor terms are omitted.

*b*. To get the right side of Eq. 3C.5-3, it is evident that Eq. 3C.5-1 had to be multiplied by $1/\rho^{\mathrm{I}} v_0^2$. The interfacial-tension term in Eq. 3C.5-3 is then

$$\mathbf{n}^{\mathrm{I}}\left(\frac{l_0}{R_1}+\frac{l_0}{R_2}\right)\left[\left[\frac{\sigma}{l_0 v_0^2 \rho^{\mathrm{I}}}\right]\right]=\mathbf{n}^{\mathrm{I}}\left(\frac{1}{\breve{R}_1}+\frac{1}{\breve{R}_2}\right)\left[\left[\frac{\sigma}{l_0 v_0^2 \rho^{\mathrm{I}}}\right]\right]$$

The terms involving the viscous stress tensor are

$$\frac{\left[\mathbf{n}^{\mathrm{I}}\cdot\boldsymbol{\tau}^{\mathrm{I}}\right]}{\rho^{\mathrm{I}} v_0^2}=-\frac{\mu^{\mathrm{I}}\left[\mathbf{n}^{\mathrm{I}}\cdot\boldsymbol{\gamma}^{\mathrm{I}}\right]}{\rho^{\mathrm{I}} v_0^2}=-\frac{\mu^{\mathrm{I}}\left[\mathbf{n}^{\mathrm{I}}\cdot\breve{\boldsymbol{\gamma}}^{\mathrm{I}}\right]}{\rho^{\mathrm{I}} v_0^2}\frac{v_0}{l_0}=-\left[\mathbf{n}^{\mathrm{I}}\cdot\breve{\boldsymbol{\gamma}}^{\mathrm{I}}\right]\left[\left[\frac{\mu^{\mathrm{I}}}{l_0 v_0 \rho^{\mathrm{I}}}\right]\right]$$

$$\frac{\left[\mathbf{n}^{\mathrm{I}}\cdot\boldsymbol{\tau}^{\mathrm{II}}\right]}{\rho^{\mathrm{I}} v_0^2}=-\frac{\mu^{\mathrm{II}}\left[\mathbf{n}^{\mathrm{I}}\cdot\boldsymbol{\gamma}^{\mathrm{II}}\right]}{\rho^{\mathrm{I}} v_0^2}=-\frac{\mu^{\mathrm{II}}\left[\mathbf{n}^{\mathrm{I}}\cdot\breve{\boldsymbol{\gamma}}^{\mathrm{II}}\right]}{\rho^{\mathrm{I}} v_0^2}\frac{v_0}{l_0}=-\left[\mathbf{n}^{\mathrm{I}}\cdot\breve{\boldsymbol{\gamma}}^{\mathrm{II}}\right]\left[\left[\frac{\mu^{\mathrm{II}}}{l_0 v_0 \rho^{\mathrm{II}}}\right]\right]\left[\left[\frac{\rho^{\mathrm{II}}}{\rho^{\mathrm{I}}}\right]\right]$$

And finally, the pressure terms are converted to modified pressure terms plus terms involving the gravitational acceleration

$$\mathbf{n}^{\mathrm{I}}\left(\frac{p^{\mathrm{I}}-p_0+\rho^{\mathrm{I}} g(h-h_0)}{\rho^{\mathrm{I}} v_0^2}-\frac{p^{\mathrm{II}}-p_0+\rho^{\mathrm{II}} g(h-h_0)}{\rho^{\mathrm{I}} v_0^2}\right)-\mathbf{n}^{\mathrm{I}}\frac{g(h-h_0)\left(\rho^{\mathrm{I}}-\rho^{\mathrm{II}}\right)}{\rho^{\mathrm{I}} v_0^2}$$

$$\mathbf{n}^{\mathrm{I}}\left(\breve{\mathcal{P}}^{\mathrm{I}}-\breve{\mathcal{P}}^{\mathrm{II}}\left[\left[\frac{\rho^{\mathrm{II}}}{\rho^{\mathrm{I}}}\right]\right]\right)-\mathbf{n}^{\mathrm{I}}\left[\left[\frac{g l_0}{v_0^2}\right]\right]\left[\left[\frac{\rho^{\mathrm{I}}-\rho^{\mathrm{II}}}{\rho^{\mathrm{I}}}\right]\right]\breve{h}$$

We see that the Reynolds numbers for the two phases, the Weber number (Eq. 3.7-12), and the Froude number (Eq. 3.7-11) appear as well as the density ratios for the two phases.

## 3D.1 Derivation of the equation of motion from Newton's second law of motion

*a*. Equation 3D.1-1 is the statement that the time rate-of-change of momentum is equal to the sum of the surface forces and the gravity forces acting on a small blob of fluid.

When the Leibniz formula (Eq. A.5-5) is applied to the left side of Eq. 3D.1-1, we get

$$\frac{d}{dt}\int_{V(t)}\rho\mathbf{v}dV = \int_{V(t)}\frac{\partial}{\partial t}\rho\mathbf{v}dV + \int_{S(t)}(\rho\mathbf{v})(\mathbf{v}\cdot\mathbf{n})dS = \int_{V(t)}\frac{\partial}{\partial t}\rho\mathbf{v}dV + \int_{S(t)}[\mathbf{n}\cdot\rho\mathbf{v}\mathbf{v}]dS$$

$$= \int_{V(t)}\frac{\partial}{\partial t}\rho\mathbf{v}dV + \int_{V(t)}[\nabla\cdot\rho\mathbf{v}\mathbf{v}]dV \qquad \text{(using Eq. A.5-3)}$$

The term containing the stress tensor in Eq. 3D.1-1 can also be rewritten as a volume integral using Eq. A.5-3 to give

$$\int_{V(t)}\frac{\partial}{\partial t}\rho\mathbf{v}dV = -\int_{V(t)}[\nabla\cdot\rho\mathbf{v}\mathbf{v}]dV - \int_{V(t)}[\nabla\cdot\boldsymbol{\pi}]dV + \int_{V(t)}\rho\mathbf{g}dV$$

Since the choice of the blob volume was arbitrary, all the volume integral operations may be removed, and we obtain the equation of motion of Eq. 3.2-9.

*b*. If the blob is fixed, then we can write a momentum balance over the blob as follows:

$$\frac{d}{dt}\int_{V}\rho\mathbf{v}dV = -\int_{S}[\mathbf{n}\cdot\rho\mathbf{v}\mathbf{v}]dS - \int_{S}[\mathbf{n}\cdot\boldsymbol{\pi}]dS + \int_{V}\rho\mathbf{g}dV$$

This states that the rate of increase of momentum within the fixed volume equals the rate of increase because of convective transport, the rate of increase because of molecular momentum transport, and the force acting on the system by gravity. The time derivative can be taken inside, since the volume is fixed, and the surface integrals can be converted to volume integrals. The result is an equation containing only volume integrals over the fixed volume:

$$\int_{V}\frac{\partial}{\partial t}\rho\mathbf{v}dV = -\int_{V}[\nabla\cdot\rho\mathbf{v}\mathbf{v}]dV - \int_{V}[\nabla\cdot\boldsymbol{\pi}]dV + \int_{V}\rho\mathbf{g}dV$$

Since the volume was chosen arbitrarily, the volume integrals can be removed, and, once again, the result in Eq. 3.2-9 is obtained.

**3D.2 The equation of change for vorticity**

Method I:

Start with the Navier-Stokes equation in the $D/Dt$-form, but rearranged thus:

$$\frac{\partial \mathbf{v}}{\partial t} = -[\mathbf{v}\cdot\nabla\mathbf{v}] - \frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v} + \mathbf{g}$$

$$= -\nabla\tfrac{1}{2}v^2 + [\mathbf{v}\times[\nabla\times\mathbf{v}]] - \frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v} + \mathbf{g}$$

Next we take the curl and introduce the vorticity $\mathbf{w} = [\nabla\times\mathbf{v}]$

$$\frac{\partial \mathbf{w}}{\partial t} = [\nabla\times[\mathbf{v}\times\mathbf{w}]] + \nu\nabla^2\mathbf{w}$$

or

$$\frac{\partial \mathbf{w}}{\partial t} = [\nabla\cdot\mathbf{wv}] - [\nabla\cdot\mathbf{vw}] + \nu\nabla^2\mathbf{w}$$

Then using Eq. A.4-24 and the fact that $(\nabla\cdot\mathbf{v}) = 0$ for incompressible fluids and $(\nabla\cdot\mathbf{w}) = 0$ always (since the divergence of a curl is always zero, we get Eq. 3D.2-1.

Method II:

Start now with the Navier-Stokes equation in $\partial/\partial t$-form

$$\frac{\partial \mathbf{v}}{\partial t} = -[\nabla\cdot\mathbf{vv}] - \frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v} + \mathbf{g}$$

Take the curl of this equation and introduce the vorticity to get

$$\frac{\partial \mathbf{w}}{\partial t} = -[\nabla\times[\nabla\cdot\mathbf{vv}]] + \nu\nabla^2\mathbf{w}$$

or

$$\frac{\partial \mathbf{w}}{\partial t} = [\boldsymbol{\varepsilon}:\{\nabla\mathbf{v}\cdot\nabla\mathbf{v}\}] - [\mathbf{v}\cdot\nabla\mathbf{w}] + \nu\nabla^2\mathbf{w}$$

Details of the manipulations involved in this last step are given here using the abbreviated notation of §A.9 with the Einstein summation convention:

$$
\begin{aligned}
-\left[\nabla\times\left[\nabla\cdot\mathbf{vv}\right]\right]_i &= -\varepsilon_{ijk}\partial_j\left(\partial_l v_l v_k\right) = -\varepsilon_{ijk}\partial_j\left(v_l\partial_l v_k - v_k\partial_l v_l\right) \text{ but } \partial_l v_l = 0 \\
&= -\varepsilon_{ijk}\left[\left(\partial_j v_l\right)\left(\partial_l v_k\right) + \left(v_l\partial_j\partial_l v_k\right)\right] \\
&= \varepsilon_{ikj}\left(\partial_j v_l\right)\left(\partial_l v_k\right) - \left(v_l\partial_l\left(\varepsilon_{ijk}\partial_j v_k\right)\right) \\
&= \left[\boldsymbol{\varepsilon}:\left\{\nabla\mathbf{v}\cdot\nabla\mathbf{v}\right\}\right]_i - \left[\mathbf{v}\cdot\nabla\left[\nabla\times\mathbf{v}\right]\right]_i
\end{aligned}
$$

in which $\boldsymbol{\varepsilon} = \Sigma\Sigma\Sigma\varepsilon_{mnr}\boldsymbol{\delta}_m\boldsymbol{\delta}_n\boldsymbol{\delta}_r$ is a third-order tensor.

### 3D.3 Alternate form of the equation of motion

Take the divergence of the equation of motion for an incompressible fluid in the form of Eq. 3.2-9, but with the stress-tensor term written in terms of the viscosity and the Laplacian of the velocity. This gives

$$0 = -\left(\nabla \cdot [\nabla \cdot \mathbf{vv}]\right) - \frac{1}{\rho}\nabla^2 p$$

or

$$0 = -\left(\nabla \mathbf{v} : (\nabla \mathbf{v})^{\dagger}\right) - \frac{1}{\rho}\nabla^2 p$$

Then use the definitions in Eq. 3D.3-2 to get Eq. 3D.3-1.

**4A.1 Time for attainment of steady state in tube flow.**

*a.* In Figure 4D.2, the centerline velocity comes within 10% of its final value when $\nu t/R^2 = 0.45$, giving

$$t = 0.45R^2/\nu = (0.45)(0.49 \times 10^{-4}\ \mathrm{m}^2)/(3.45 \times 10^{-4}\ \mathrm{m}^2/\mathrm{s}) = 0.064\ \mathrm{s}$$

*b.* If water at (68°F=20°C) is used, with $\nu = 1.0037 \times 10^{-6}\ \mathrm{m}^2/\mathrm{s}$ from Tables 1.1-1 and F.3-6, the time required is

$$t = (0.45)(0.49 \times 10^{-4}\ \mathrm{m}^2)/(1.0037 \times 10^{-6}\ \mathrm{m}^2/\mathrm{s}) = 22\ \mathrm{s}$$

**4A.2 Velocity near a moving sphere.**

From Eq. 4.2-14 at $\theta = \pi/2$, the fluid velocity relative to the approach velocity falls to 1% of $v_\infty$ at $v_\theta = -0.99 v_\infty$ relative to the sphere, giving

$$-0.99 = -\left[1 - \frac{3}{4}\left(\frac{R}{r}\right) - \frac{1}{4}\left(\frac{R}{r}\right)^3\right]$$

with $R/r < 1$. If $R/r << 1$, the cubic term will be unimportant, giving

$$0.01 = \frac{3}{4}\left(\frac{R}{r}\right)$$

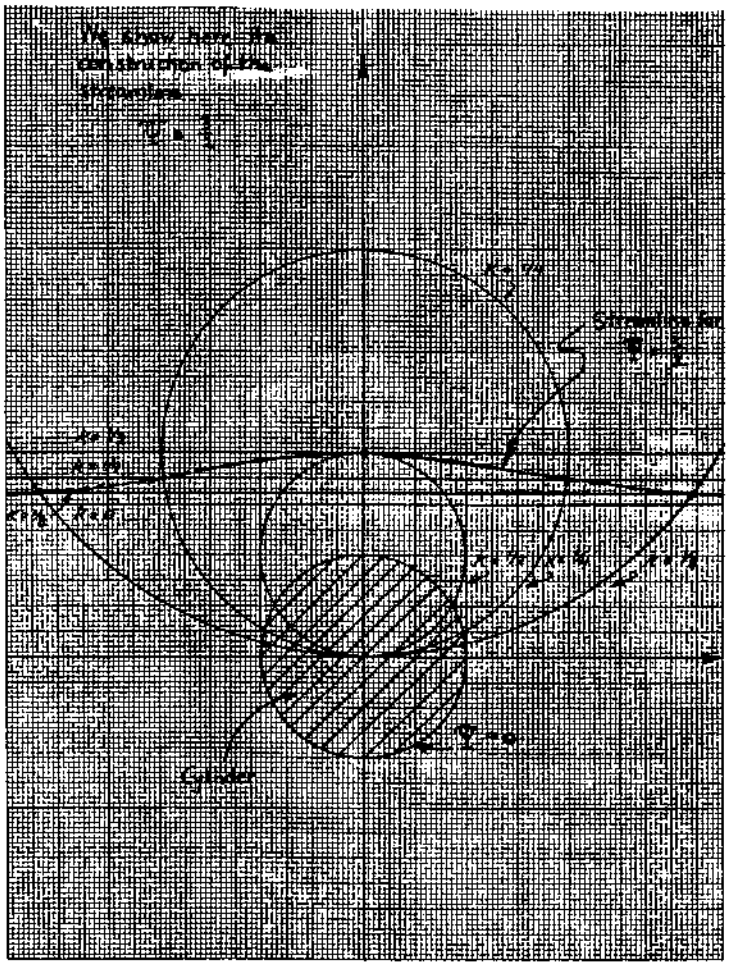
or

$$\frac{r}{R} = \frac{300}{4} = 75$$

Clearly, the neglect of the cubic term at this distance is justifiable.

## 4A.3 Construction of streamlines for the potential flow around a cylinder.

In the following drawing we show the construction of the streamline $\Psi = \frac{3}{2}$ by the method described in the problem statement.



**4A.4 Comparison of exact and aproximate profiles for flow along a flat plate.**

Let $\Pi_v = v_x/v_\infty$ and $Y = y\sqrt{v_\infty/\nu x}$; then Eq. 4.4-18 gives the approximation

$$\begin{aligned}\Pi_v &= \frac{3}{2}\sqrt{\frac{13}{280}}Y - \frac{1}{2}\left(\frac{13}{280}\right)^{3/2} Y^3 \\ &= 0.323209Y - 0.005002Y^3\end{aligned}$$

and Fig. 4.4-3 shows Blasius' "exact" $\Pi_v$ vs. $Y$. These two velocity representations will now be compared.

| Location, $Y$ | Approx. $\Pi_v$, Eq. 4.4-18 | "Exact" $\Pi_v$, Fig. 4.4-3 | Approx. $\Pi_v$/Exact $\Pi_v$ |
|---|---|---|---|
| *a.* 1.5 | 0.468 | 0.49 | 0.96 |
| *b.* 3.0 | 0.835 | 0.84 | 0.99 |
| *c.* 4.0 | 0.973 | 0.96 | 1.01 |

**4A.5 Numerical demonstration of the von Karman momentum balance.**

*a.* The integrals in Eq. 4.4-13 are

$$I_1 = \int_0^\infty \rho v_x(v_e - v_x)dy \qquad \text{and} \qquad I_2 = \int_0^\infty \rho(v_e - v_x)dy$$

Figure 4.4-3 gives $f' = v_x/v_\infty$ as a function of a dimensionless coordinate,

$$Y = y\sqrt{v_\infty/(\nu x)}$$

and $v_e = v_\infty$ in this geometry. Thus, $v_x = v_\infty f'$ and $\rho dy = \sqrt{\rho\mu x/v_\infty}dY$, so that these integrals take the forms

$$I_1 = \sqrt{\rho\mu v_\infty x}\int_0^\infty f'(1-f')dY \text{ and } I_2 = \sqrt{\rho\mu v_\infty x}\int_0^\infty (1-f')dY$$

Numerical evaluation of the integrals over $Y$ then gives

$$\int_0^\infty f'(1-f')dY = 0.664 \text{ and } \int_0^\infty (1-f')dY = 1.73$$

if an accurate table of the solution is used. The following calculation was made from Fig. 4.4-3:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $Y$ | 0 | 1.0 | 2.0 | 3.0 | 4.0 | 5.0 | 6.0 |
| $f'$ | 0 | 0.34 | 0.63 | 0.844 | 0.955 | 0.983 | 1.00 |
| $(1-f')$ | 1. | 0.66 | 0.37 | 0.156 | 0.045 | 0.017 | 0.000 |
| $f'(1-f')$ | 0 | 0.2244 | 0.2331 | 0.1309 | 0.0430 | 0.0167 | 0.000 |

Application of the trapezoidal rule gives the values

$$I_1 = [1/2 + 0.66 + 0.37 + 0.156 + 0.045 + 0.017 + 0/2] \times 1.0 = 1.74$$

$$I_2 = [0/2 + 0.2244 + 0.2331 + 0.1309 + 0.043 + 0.0167 + 0/2] \times 1.0 = 0.65$$

which agree, within their uncertainty, with the accepted values 1.73 and 0.664.

*b.* Use of Eq. 1.1-2 and the results of *a.* in Eq. 4.4-18 gives

$$\begin{aligned}\tau_{yx}\big|_{y=0} &= \frac{dI_1}{dx} + (0)I_2 \\ &= 0.65\sqrt{\rho\mu v_\infty^3}\frac{x^{-1/2}}{2} \\ &= 0.325\sqrt{\rho\mu v_\infty^3/x}\end{aligned}$$

*c.* The force in the x-direction on a plate of width W and length L, wetted on both sides, according to the result in *b*, is

$$\begin{aligned}F_x &= 2W\int_0^L \tau_{yx}\big|_{y=0}dx \\ &= 2W(0.325\sqrt{\rho\mu v_\infty^3})\int_0^L x^{-1/2}dx \\ &= 1.30\sqrt{\rho\mu v_\infty^3 LW^2}; \qquad \text{The recommended coefficient is 1.328.}\end{aligned}$$

**4A.6 Use of boundary-layer formulas.**

The data for this problem are:

$$W = 10 \text{ ft}$$
$$L = 3 \text{ ft}$$
$$v_\infty = 20 \text{ ft/s}$$

From Table 1.1-2 and Appendix F:

$$\nu = (0.1505 \text{ cm}^2/\text{s})/(12 \times 2.54 \text{ cm/ft})^2$$
$$= 1.62 \times 10^{-4} \text{ ft}^2/\text{s}$$
$$\mu = (0.01813 \text{ mPa·s})(10^{-3} \text{ Pa/mPa})(6.7197 \times 10^{-1} \text{ lb}_m/\text{ft·s/Pa})$$
$$= 1.218 \times 10^{-5} \text{ lb}_m/\text{ft·s}$$
$$\rho = \mu/\nu = 1.218 \times 10^{-5}/1.62 \times 10^{-4} = 7.5 \times 10^{-2} \text{ lb}_m/\text{ft}^3$$

*a.* The local Reynolds number at the trailing edge ($x = L = 20$ ft) is:

$$\text{Re} = Lv_\infty/\nu$$
$$= (3\text{ft})(20 \text{ ft/s}/(1.62 \times 10^{-4} \text{ ft}^2/\text{s})$$
$$= 3.7 \times 10^5$$

*b.* According to Eq. 4.4-17, the boundary layer thickness at the trailing edge is

$$\delta(L) = 4.64\sqrt{\frac{\nu L}{v_\infty}}$$
$$= 4.64\sqrt{\frac{(1.62 \times 10^{-4} \text{ ft}^2/\text{s})(3 \text{ ft})}{20 \text{ ft/s}}}$$
$$= 2.4 \times 10^{-5} \text{ ft}$$

*c.* According to Eq. 4.4-30, the total drag force of the fluid on both sides of the plate is

$$F_x = 1.328\sqrt{\rho\mu L W^2 v_\infty^3}$$
$$= 1.328\sqrt{(0.075 \text{ lb}_m/\text{ft}^3)(1.22 \times 10^{-5} \text{ lb}_m/\text{ft·s})(3 \text{ ft})(10 \text{ ft})^2(20 \text{ ft/s})^3}$$
$$= 0.62 \text{ lb}_m\text{·ft/s}^2$$
$$= 0.019 \text{ lb}_f$$

**4A.7 Entrance flow in conduits.**

*a.* With the indicated substitutions, Eq. 4.4-17 gives

$$\frac{1}{2}D = 4.64\sqrt{\frac{\nu L_e}{v_{\max}}}$$

Setting $v_{\max} = 2\langle v\rangle$ at the end of the entrance region, we obtain the following estimate of $L_e$:

$$\frac{\nu L_e}{2\langle v\rangle} = \left(\frac{D}{9.28}\right)^2$$

or

$$\begin{aligned} L_e &= \frac{2}{(9.28)^2}\frac{D^2\langle v\rangle}{\nu} \\ &= 0.023D\mathrm{Re} \end{aligned}$$

which is similar to the expression given in §2.3, except that the coefficient is about 2/3 as large.

*b.* At the typical transition locus $xv_\infty/\nu \approx 3\times 10^5$ for flow along a flat plate, Eq. 4.4-17 gives

$$\begin{aligned} \delta/x &= 4.64\sqrt{\frac{\nu}{v_\infty x}} \\ &= 4.64(3.5\times 10^5)^{-1/2} = 0.00847 \end{aligned}$$

and the transition Reynolds number based on the characteristic length $\delta$ is

$$\begin{aligned} \frac{v_\infty\delta}{\nu} &= \frac{v_\infty x}{\nu}\frac{\delta}{x} \\ &= (3\times 10^5)(0.00847) = 2542 \end{aligned}$$

For flow in tubes, with transition occurring when $\delta = D/2$ and with $v_\infty = 2\langle v\rangle$, the latter result gives $D\langle v\rangle/\nu$=2542 as the minimum transition Reynolds number, in fair agreement with the reported value of 2100.

*c.* For laminar flow between parallel planes, the method in Problem 4.C gives $\delta = B$ and $v_{\max} = (3/2)\langle v_x\rangle$ at the end of the entrance region. Insertion of these results into Eq. 4.4-17 gives

$$B = 4.64\sqrt{\frac{\nu L_e}{1.5\langle v_x\rangle}}$$

whence

$$\begin{aligned} L_e &= \frac{1.5}{(4.64)^2}\frac{B^2\langle v_x\rangle}{\nu} \\ &= 0.070B^2\langle v_x\rangle/\nu \\ &= 0.070B\mathrm{Re} \text{ with } \mathrm{Re}=B\langle v_x\rangle/\nu \end{aligned}$$

**4B.1 Flow of a fluid with a suddenly applied constant wall stress**

*a*. Differentiation of Eq. 4.1-1 with respect to $y$ gives

$$\frac{\partial^2 v_x}{\partial y \partial t} = \nu \frac{\partial}{\partial y}\frac{\partial^2 v_x}{\partial y^2} \qquad \text{or} \qquad \frac{\partial}{\partial t}\left(-\mu \frac{\partial v_x}{\partial y}\right) = \nu \frac{\partial^2}{\partial y^2}\left(-\mu \frac{\partial v_x}{\partial y}\right)$$

Then using Newton's law of viscosity, we get

$$\frac{\partial \tau_{yx}}{\partial t} = \nu \frac{\partial^2 \tau_{yx}}{\partial y^2}$$

*b*. This equation is to be solved with the initial condition that $\tau_{yx} = 0$ for $t \le 0$, and the boundary conditions that $\tau_{yx} = \tau_0$ at $y = 0$, and that $\tau_{yx} = 0$ at $t = \infty$.

*c*. The solution is exactly as in Example 4.1-1 with appropriate changes of notation, and the solution is given in Eq. 4B.1-1.

*d*. To get the velocity profile, we integrate Newton's law of viscosity:

$$\int_{v_x}^{0} d\bar{v}_x = -\frac{1}{\mu}\int_y^\infty \tau_{yx} dy \qquad \text{or} \qquad v_x = \frac{\tau_0}{\mu}\int_y^\infty \left(1 - \text{erf}\frac{\bar{y}}{\sqrt{4\nu t}}\right) d\bar{y}$$

Changing variables we get

$$v_x = \frac{\tau_0}{\mu}\sqrt{4\nu t}\int_{y/\sqrt{4\nu t}}^\infty (1 - \text{erf}u) du$$

$$= \frac{\tau_0}{\mu}\sqrt{4\nu t}\left[\frac{1}{\sqrt{\pi}}e^{-y^2/4\nu t} - \frac{\bar{y}}{\sqrt{4\nu t}}\left(1 - \text{erf}\frac{y}{\sqrt{4\nu t}}\right)\right]$$

The velocity at $y = 0$ is then

$$v_x(0,t) = \frac{\tau_0}{\mu}\sqrt{\frac{4\nu t}{\pi}} = 2\tau_0\sqrt{\frac{t}{\pi\mu\rho}}$$

**4B.2 Flow near a wall suddenly set in motion (approximate solution)**

*a*. Integration of Eq. 4.1-1 over $y$ gives

$$\rho\int_0^\infty \frac{\partial v_x}{\partial t}dz = \mu\int_0^\infty \frac{\partial^2 v_x}{\partial y^2}dz \qquad \text{whence} \qquad \frac{d}{dt}\int_0^\infty \rho v_x dz = \mu\frac{\partial v_x}{\partial y}\bigg|_{y=0}^{y=\infty}$$

Since the velocity gradient is zero at infinite distance from the plate, we end up with Eq. 4B.2-2.

*b*. We introduce the variable $\eta = y/\delta(t)$. Then when Eqs. 4B.2-3 and 4 are substituted in Eq. 4B.2-2, we find

$$\frac{d}{dt}\delta(t)\int_0^1 \rho v_\infty\left(1-\tfrac{3}{2}\eta+\tfrac{1}{2}\eta^3\right)d\eta = -\mu v_\infty\left(-\tfrac{3}{2}+\tfrac{3}{2}\eta^2\right)\Big|_{\eta=0}\frac{1}{\delta(t)}$$

Then after dividing by $\rho v_\infty$ and evaluating of the integral, we get

$$\frac{3}{8}\frac{d}{dt}\delta(t) = \frac{3}{2}\nu\frac{1}{\delta(t)} \qquad \text{(see Eq. 4B.2-5)}$$

*c*. Eq. 4B.2-5 when integrated gives

$$\int_0^\delta \delta d\delta = 4\nu\int_0^t dt \quad \text{or} \quad \delta(t) = \sqrt{8\nu t}$$

Then the velocity profile is given by

$$\frac{v_x}{v_\infty} = 1-\frac{3}{2}\left(\frac{y}{\sqrt{8\nu t}}\right)+\frac{1}{2}\left(\frac{y}{\sqrt{8\nu t}}\right)^3 \qquad \text{for} \quad 0 \le y \le \sqrt{8\nu t}$$

$$\frac{v_x}{v_\infty} = 1 \qquad \text{for} \quad y \ge \sqrt{8\nu t}$$

**4B.3 Creeping flow around a spherical bubble**

*a*. According to Eq. B.1-18, the vanishing of the shear stress is

$$r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)+\frac{1}{r}\frac{\partial v_r}{\partial\theta}=0 \text{ or } r\frac{\partial}{\partial r}\left(\frac{1}{r}\left\{\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}\right\}\right)+\frac{1}{r}\frac{\partial}{\partial\theta}\left\{-\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\right\}=0$$

In the second form, we have inserted the expressions for the velocity components in terms of the stream function from the last line in Table 4.2-1. Next we insert $\psi = f(r)\sin^2\theta$ and obtain Eq. 4B.3-1.

*b*. Equations 4.2-7 through 10 are still valid for this problem, as well as the values of $C_3 = -\frac{1}{2}v_\infty$ and $C_4 = 0$. Hence we have to determine the remaining constants in $f(r) = C_1 r^{-1} + C_2 r$ by requiring that Eq. 4B.3-1 be valid at $r = R$, as well as $f = 0$ at $r = R$ (Eq. 4.2-3). These boundary conditions lead to $C_1 = 0$ and $C_2 = \frac{1}{2}v_\infty R$. Then Eqs. 4B.3-2 and 3 follow directly.

*c*. When the velocity distributions in Eq. 4B.3-2 and 3 are substituted into the equations of motion (Eqs. B.6-7 and 8), we get

$$\frac{\partial\mathcal{P}}{\partial r}=2\left(\frac{\mu v_\infty}{R^2}\right)\left(\frac{R}{r}\right)^3\cos\theta \qquad \text{and} \qquad \frac{\partial\mathcal{P}}{\partial\theta}=\left(\frac{\mu v_\infty}{R}\right)\left(\frac{R}{r}\right)^2\sin\theta$$

Integration of these two equations gives

$$\mathcal{P}=-\left(\frac{\mu v_\infty}{R}\right)\left(\frac{R}{r}\right)^2\cos\theta+F(\theta) \text{ and } \mathcal{P}=-\left(\frac{\mu v_\infty}{R}\right)\left(\frac{R}{r}\right)^2\cos\theta+G(r)$$

In order for the solution to be unique, $F(\theta)$ and $G(r)$ must be equal to a constant. If we require that the modified pressure be equal to $p_0$ at $z = 0$ infinitely far from the sphere, we then get

$$p=p_0-\rho gz-\left(\frac{\mu v_\infty}{R}\right)\left(\frac{R}{r}\right)^2\cos\theta$$

*d*. The $z$-component of the force acting on the sphere is

$$F_z=-\int_0^{2\pi}\int_0^{\pi}\left(\boldsymbol{\delta}_z\cdot\left[\boldsymbol{\delta}_r\cdot\{p\boldsymbol{\delta}+\boldsymbol{\tau}\}\right]\right)\Big|_{r=R}R^2\sin\theta d\theta d\phi$$

$$= -2\pi R^2 \int_0^\pi \left(p\cos\theta + \tau_{rr}\cos\theta - \tau_{r\theta}\sin\theta\right)\Big|_{r=R} \sin\theta d\theta$$

where Eqs. A.6-28 and 29 have been used for getting the dot products of the unit vectors. Then we use the result of (*c*) and Eqs. B.1-15 and 18 for the components of the stress tensor (along with Eqs. 4B.3-2 and 3) to get the three contributions to the force:

$$F_{z,p} = -2\pi R^2 \int_0^\pi \left( p_0 - \rho g z - \left(\frac{\mu v_\infty}{R}\right)\left(\frac{R}{r}\right)^2 \cos\theta \right)\Bigg|_{r=R} \cos\theta \sin\theta d\theta$$
$$= \tfrac{4}{3}\pi R^3 \rho g + \tfrac{4}{3}\pi \mu R v_\infty$$

$$F_{z,\tau_{rr}} = -2\pi R^2 \int_0^\pi \left( -2\mu \frac{\partial v_r}{\partial r} \cdot \cos\theta \right)\Bigg|_{r=R} \sin\theta d\theta$$
$$= +4\pi\mu R^2 v_\infty \int_0^\pi \left( \frac{R}{r^2} \cdot \cos^2\theta \right)\Bigg|_{r=R} \sin\theta d\theta = \tfrac{8}{3}\pi\mu R v_\infty$$

$$F_{z,\tau_{r\theta}} = -2\pi R^2 \int_0^\pi (-\mu)\left( r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} \right)\Bigg|_{r=R} (-\sin\theta)\sin\theta d\theta = 0$$

When these are added together, we get Eq. 4B.3-5.

**4B.4 Use of the vorticity equation**

*a*. For the postulate $v_z = v_z(x)$ the vorticity $\mathbf{w} = [\nabla \times \mathbf{v}]$ has but one component $w_y = -dv_z/dx$. Then at steady state, the $y$-component of Eq. 3D.2-1 is

$$(\mathbf{v}\cdot\nabla)\left(\frac{dv_z}{dx}\right) = \nu\nabla^2\left(\frac{dv_z}{dx}\right) + (\mathbf{w}\cdot\nabla)v_y$$

But $v_y$ is postulated to be zero so that the last term drops out. Also the first term drops out, because $v_x$ and $v_y$ are zero, and $v_z$ is postulated to have no $z$-dependence. Consequently the equation simplifies to $d^3v_z/dx^3 = 0$. Integration gives

$$v_z = \tfrac{1}{2}C_1x^2 + C_2x + C_3 \quad \text{or} \quad v_z/v_{z,\max} = D_1\xi^2 + D_2\xi + D_3$$

where we have redefined the integration constants. From the three boundary conditions it is found that $D_1 = -1$, $D_2 = 0$, and $D_3 = 1$, Eq. 4B.4-1 results.

Then the $z$-component of the equation of motion becomes at steady state

$$\rho(\mathbf{v}\cdot\nabla)v_z = -\frac{d\mathcal{P}}{dz} + \mu\nabla^2 v_z \quad \text{or} \quad 0 = -\frac{d\mathcal{P}}{dz} + \mu\frac{d^2v_z}{dx^2}$$

Knowing the velocity distribution we can evaluate the second derivative of the velocity and get the pressure distribution thus

$$0 = -\frac{d\mathcal{P}}{dz} + \mu v_{z,\max}\left(-\frac{2}{B^2}\right) \quad \text{and} \quad \mathcal{P}_0 - \mathcal{P}(z) = \frac{2\mu v_{z,\max}}{B^2}z$$

For $z = L$, this gives $v_{z,\max} = (\mathcal{P}_0 - \mathcal{P}_L)B^2/2\mu L$ in agreement with Eq. 2B.3-2.

*b*. Since it is postulated that $v_z = v_z(r)$, the only nonzero component of of the vorticity vector is $w_\theta = -dv_z/dr$. Then the $\theta$-component of the steady-state vorticity equation is:

$$\left(v_r\frac{\partial}{\partial r}+\frac{v_\theta}{r}\frac{\partial}{\partial\theta}+v_z\frac{\partial}{\partial z}\right)w_\theta =$$
$$\nu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rw_\theta)\right)+\frac{1}{r^2}\frac{\partial^2 w_\theta}{\partial\theta^2}+\frac{\partial^2 w_\theta}{\partial z^2}+\frac{2}{r^2}\frac{\partial w_r}{\partial\theta}\right)$$
$$+\left(w_r\frac{\partial v_\theta}{\partial r}+w_\theta\left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r}{r}\right)+w_z\frac{\partial v_\theta}{\partial z}\right)$$

This simplifies to

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)\right)=0 \quad \text{or} \quad v_z=\tfrac{1}{4}C_1r^2+C_2\ln r+C_3$$

This is the same solution found in the soution to Problem 3B.6. When the two boundary conditions and the conservation-of-mass condition are used, we finally get the solution in Eq. 3B.6-2.

To get the pressure distribution we use the $z$-component of the equation of motion, which is

$$0=-\frac{d\mathcal{P}}{dr}+\mu\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)$$

When the velocity distribution of Eq. 3B.6-2 is inserted, we get

$$0=-\frac{d\mathcal{P}}{dr}-\frac{4\mu v_0}{R^2(1-\kappa^2)}\frac{(1-\kappa^2)+2\kappa^2\ln(1/\kappa)}{(1-\kappa^2)-2(1+\kappa^2)\ln(1/\kappa)}$$

This may then be integrated to get the pressure distribution.

**4B.5 Steady potential flow around a stationary sphere**

*a*. The boundary conditions are:

(i) as $r \to \infty$, $\mathbf{v} \to v_\infty \boldsymbol{\delta}_z$, or by using Eqs. A.6-28 and 29

as $r \to \infty$, $v_r = v_\infty \cos\theta$ and $v_\theta = -v_\infty \sin\theta$

(ii) at $r = R$, $v_r = 0$

(iii) at $z = 0$ as $r \to \infty$, $\mathfrak{P} \to \mathfrak{P}_0$

*b*. Since $\mathbf{v} = -\nabla\phi$, we have as $r \to \infty$

$$-\partial\phi/\partial r = v_\infty \cos\theta \quad \text{and} \quad -(1/r)\partial\phi/\partial\theta = -v_\infty \sin\theta$$

When these equations are integrated we find that $\phi = -v_\infty r\cos\theta$, Thus we feel that $\phi = f(r)\cos\theta$ may be an appropriate trial function.

*c*. We next write the 3-dimensional Laplace equation in spherical coordinates (for a system with symmetry about the *z*-axis)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) = 0$$

Into this equation we substitute the trial velocity potential and get

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df}{dr}\right) - 2\frac{f}{r^2} = 0 \quad \text{which as the solution} \quad f = C_1 r + C_2 r^{-2}$$

since this is an "equidimensional" equation of the form of Eq. C.1-14.

*d*. Application of the boundary conditions then gives

$$f = -v_\infty R\left[\frac{r}{R} + \frac{1}{2}\left(\frac{R}{r}\right)^2\right] \text{ and } \quad \phi = -v_\infty R\left[\frac{r}{R} + \frac{1}{2}\left(\frac{R}{r}\right)^2\right]\cos\theta$$

*e*. Then from the components of $\mathbf{v} = -\nabla\phi$. we get Eqs. 4B.5-2 and 3 by differentiation.

*f*. Then from the equation of motion for steady potential flow

$$\rho\left(\nabla\tfrac{1}{2}v^2\right) = -\nabla\mathfrak{P} \qquad \text{(see Eq. 4.3-2)}$$

By integrating the components of this equation we get

$$-\mathcal{P} = \rho v_\infty^2 \left[ \left[ 1 - \left( \frac{R}{r} \right)^3 \right] \cos^2\theta + \left[ 1 + \frac{1}{2} \left( \frac{R}{r} \right)^3 \right] \sin^2\theta \right] + C_3$$

The integration constant is then obtained from the boundary condition given in (*a*): $C_3 = -\mathcal{P}_0 - \rho v_\infty^2$. Then when the modified pressure is evaluated at the surface of the sphere, Eq. 4B.5-4 is obtained directly.

**4B.6 Potential flow near a stagnation point**

*a*. At the origin of coordinates ($z = 0$) the complex velocity $dw/dz = -2v_0 z$ is zero, which is a stagnation point.

*b*. By taking the real and imaginary parts of the complex velocity, we get from Eq. 4.3-12: $v_x = 2v_0 x$ and $v_y = -2v_0 y$.

*c*. When $v_0$ is positive, the fluid is flowing toward the surface $y = 0$ in the upper half plane. The magnitude of $v_0$ specifies the speed with which the fluid is flowing: $v = \sqrt{v_x^2 + v_y^2} = 2v_0 r$.

**4B.7 Vortex flow**

*a*. By using Eq. 4.3-12, we find that

$$\frac{dw}{dz} = \frac{i\Gamma}{2\pi}\left(\frac{1}{z}\right) = \frac{i\Gamma}{2\pi}\left(\frac{\bar{z}}{z\bar{z}}\right) = \frac{i\Gamma}{2\pi}\left(\frac{x - iy}{x^2 + y^2}\right) = -v_x + iv_y$$

in which the overbar indicates the complex conjugate. Equating the real and imaginary parts gives

$$v_x = -\frac{\Gamma}{2\pi}\left(\frac{y}{x^2 + y^2}\right) = -\frac{\Gamma}{2\pi}\left(\frac{\sin\theta}{r}\right); \; v_y = \frac{\Gamma}{2\pi}\left(\frac{x}{x^2 + y^2}\right) = \frac{\Gamma}{2\pi}\left(\frac{\cos\theta}{r}\right)$$

The components in cylindrical coordinates are

$$v_r = v_x \cos\theta + v_y \sin\theta = 0; \; v_\theta = -v_x \sin\theta + v_y \cos\theta = \frac{\Gamma}{2\pi}\left(\frac{1}{r}\right)$$

*b*. The forced vortex is given by $v_\theta = \Omega r$ in Eq. 3.6-37.

**4B.8 The flow field around a line source**

*a*. For this purely radial flow $v_r = v_r(r)$ and the other two components are zero. Then $\nabla^2\phi = 0$ simplifies at once to Eq. 4B.8-1.

*b*. Integration of Eq. 4B.8-1 gives $\phi = C_1 \ln r + C_2$. Then, since $v_r = -d\phi/dr$, we find $v_r = -C_1/r$. Next we calculate the volumetric flow rate per unit length thus

$$\Gamma = \int_0^{2\pi} v_r r d\theta = -2\pi C_1 \quad \text{whence} \quad v_r = \frac{\Gamma}{2\pi r}$$

The pressure distribution is then obtained from the radial component of the equation of motion

$$\rho v_r \frac{dv_r}{dr} = -\frac{d\mathfrak{P}}{dr} \quad \text{or} \quad \rho\left(\frac{\Gamma}{2\pi r}\right)\left(-\frac{\Gamma}{2\pi r^2}\right) = -\frac{d\mathfrak{P}}{dr}$$

Integration then gives

$$\int_{\mathfrak{P}}^{\mathfrak{P}_\infty} d\mathfrak{P} = \rho\left(\frac{\Gamma}{2\pi}\right)^2 \int_r^\infty \left(-\frac{1}{r^3}\right) dr \qquad \text{or} \qquad \mathfrak{P}_\infty - \mathfrak{P} = \frac{\rho}{2}\left(\frac{\Gamma}{2\pi r}\right)^2$$

**4B.9 Checking solutions to unsteady flow problems**

*a*. Substituting the solution of Eq. 4.1-14 (or 15) into Eq. 4.1-5, we have to verify that

$$\frac{\partial}{\partial t}\left(1-\frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-\bar{\eta}^2}\,d\bar{\eta}\right)=\nu\frac{\partial^2}{\partial y^2}\left(1-\frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-\bar{\eta}^2}\,d\bar{\eta}\right)$$

We have to use the chain rule of partial differentiation along with the Leibniz formula

$$\frac{d}{d\eta}\left(\int_0^{\eta} e^{-\bar{\eta}^2}\,d\bar{\eta}\right)\cdot\frac{\partial\eta}{\partial t}=\nu\frac{d^2}{d\eta^2}\left(\int_0^{\eta} e^{-\bar{\eta}^2}\,d\bar{\eta}\right)\cdot\left(\frac{\partial\eta}{\partial y}\right)^2$$

or

$$\left(e^{-\eta^2}\right)\left(-\frac{1}{2t}\frac{y}{\sqrt{4\nu t}}\right)=\nu\left(e^{-\eta^2}\cdot(-2\eta)\right)\left(\frac{1}{\sqrt{4\nu t}}\right)^2$$

When the definition of $\eta$ is used, the above is found to be an identity.

**4C.1 Laminar entrance flow in a duct.**

*a.* Calculation of the mass flow rate according to Eqs. 4C.1-1 and 4C.1-2 gives

$$\begin{aligned} w &= \rho\langle v_x\rangle WB \\ &= \rho v_e W \int_0^\delta \left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right] dy + \rho v_e W \int_\delta^B [1]dy \\ &= \rho v_e W \left[\delta - \frac{\delta}{3}\right] + \rho v_e W[B-\delta] = \rho v_e W \left[-\frac{\delta}{3} + B\right] \end{aligned}$$

Equating the first and last expressions for $w$, we get (with $\delta < B$ and $\Delta = \delta/B$),

$$v_e(x) = \langle v_x\rangle \frac{B}{B - \delta/3} = \langle v_x\rangle \frac{3}{3-\Delta}$$

The following related equations will prove useful in part *b:*

$$\frac{dv_e}{dx} = \langle v_x\rangle \frac{3}{(3-\Delta)^2}\frac{d\Delta}{dx}$$

$$v_e\delta\frac{dv_e}{dx} = \langle v_x\rangle^2 \frac{9B\Delta}{(3-\Delta)^3}\frac{d\Delta}{dx}$$

$$\begin{aligned} \frac{d}{dx}(v_e^2\delta) &= \frac{d}{dx}\left[\left(\langle v_x\rangle\frac{3}{3-\Delta}\right)^2 B\Delta\right] \\ &= B\langle v_x\rangle^2\left[\frac{27+9\Delta}{(3-\Delta)^3}\right]\frac{d\Delta}{dx} \end{aligned}$$

*b.* The boundary layer in this system lies between $y = 0$ and $y = \delta$, so those limits suffice for the integrals in Eq. 4.4-13. Evaluation of the terms in Eq. 4.4-13 (divided by $\rho$) according to the results in *a* then gives:

$$\nu\frac{\partial v_x}{\partial y}\Big|_{y=0} = \frac{2\nu v_e}{\delta} = \frac{6\nu\langle v_x\rangle}{B\Delta(3-\Delta)}$$

$$\begin{aligned} \frac{d}{dx}\int_0^\delta v_x(v_e - v_x)dy &= \frac{d}{dx}(v_e^2\delta)\int_0^1 [(2u-u^2)(1-2u+u^2)]du \text{ with } u = y/\delta \\ &= \frac{d}{dx}(v_e^2\delta)\int_0^1 [2u - 4u^2 + 2u^3 - u^2 + 2u^3 - u^4]du \\ &= \frac{d}{dx}(v_e^2\delta)[1 - (4/3) + (2/4) - (1/3) + (2/4) - (1/5)] \\ &= B\langle v_x\rangle^2\left[\frac{27+9\Delta}{(3-\Delta)^3}\right]\frac{d\Delta}{dx}[2/15] \end{aligned}$$

$$\frac{dv_e}{dx}\int_0^\delta (v_e - v_x)dy = v_e\delta\frac{dv_e}{dx}\int_0^1 [1-2u+u^2]du$$
$$= v_e\delta\frac{dv_e}{dx}[1-1+(1/3)]du$$
$$= 9\langle v_x\rangle^2 B\frac{\Delta}{(3-\Delta)^3}\frac{d\Delta}{dx}[1/3]$$

With these substitutions, Eq. 4.4-13 gives

$$\frac{6\nu x}{\langle v_x\rangle B^2} = B\langle v_x\rangle^2\left[\frac{27+9\Delta}{(3-\Delta)^3}\right]\frac{d\Delta}{dx}[2/15] + 9\langle v_x\rangle^2 B\frac{\Delta}{(3-\Delta)^3}\frac{d\Delta}{dx}[1/3]$$
$$= \frac{B\langle v_x\rangle^2}{(3-\Delta)^3}\frac{d\Delta}{dx}\left[\frac{54+18\Delta+45\Delta}{15}\right]$$
$$= \frac{B\langle v_x\rangle^2}{(3-\Delta)^3}\frac{d\Delta}{dx}\left[\frac{54+63\Delta}{15}\right]$$

Multiplication of both members by $(5/3)\Delta(3-\Delta)/(B\langle v_x\rangle^2$ gives

$$10\left(\frac{\nu}{\langle v_x\rangle B^2}\right) = \frac{6\Delta+7\Delta^2}{(3-\Delta)^2}\frac{d\Delta}{dx}$$

in agreement with Eq. 4C.1-4.

*c.* Integration of the last equation with the initial condition that $\Delta = 0$ at $x = 0$ gives

$$\frac{\nu x}{\langle v_x\rangle B^2} = \frac{1}{10}\int_0^\Delta \frac{6s+7s^2}{(3-s)^2}ds$$

From integral tables we get the formulas

$$\int \frac{s}{(a+bs)^2}ds = \frac{1}{b^2}\left[\ln(a+bs) + \frac{a}{a+bs}\right] + C$$

$$\int \frac{s^2}{(a+bs)^2}ds = \frac{1}{b^3}\left[a+bx-2a\ln(a+bx) - \frac{a^2}{a+bx}\right] + C$$

which yield the definite integrals

$$\int_0^\Delta \frac{s}{(3-s)^2}ds = \left[\ln\left(\frac{3-\Delta}{3}\right) + \frac{\Delta}{3-\Delta}\right]$$

$$\int_0^\Delta \frac{s^2}{(3-s)^2}ds = \left[(3-\Delta) - 6\ln\left(\frac{3-\Delta}{3}\right) - \frac{9}{3-\Delta}\right]$$

and the solution

$$
\begin{aligned}
10\frac{\nu x}{\langle v_x\rangle B^2} &= 6\left[\ln\left(\frac{3-\Delta}{3}\right)+\frac{\Delta}{3-\Delta}\right]+7\left[(3-\Delta)-6\ln\left(\frac{3-\Delta}{3}\right)-\frac{9}{3-\Delta}\right] \\
&= 7\Delta+48\ln\left(\frac{3-\Delta}{3}\right)+\frac{6\Delta}{3-\Delta}+21-\frac{63}{3-\Delta} \\
&= 7\Delta+48\ln\left(\frac{3-\Delta}{3}\right)+\frac{6\Delta-21(3-\Delta)-63}{3-\Delta} \\
&= 7\Delta+48\ln\left(\frac{3-\Delta}{3}\right)+\frac{27\Delta}{3-\Delta}
\end{aligned}
$$

in agreement with Eq. 4C.1-5.

*d.* Setting $\Delta = 1$ and $x = L_e$ in the last equation gives

$$
\begin{aligned}
L_e &= \frac{1}{10}\left[7+48\ln(2/3)+\frac{27}{2}\right] \\
&= (0.1)\left[7-19.462325+13.5\right] = 0.104
\end{aligned}
$$

in agreement with answer (d).

*e.* Application of Eq. 4.3-5 to the region $\delta(x) < y < B$, with $v_y$ neglected so that $v = v_e$ there, gives

$$\frac{1}{2}\rho v_e^2 + \mathcal{P} = \text{ constant}$$

Insertion of the result of part *a* gives

$$\frac{1}{2}\rho\langle v_x\rangle^2\left(\frac{3}{3-\Delta}\right)^2 + \mathcal{P} = \text{ constant}$$

Evaluating the constant at $x = 0$, where $\Delta = 0$ and $\mathcal{P} = \mathcal{P}_t$, we get

$$\frac{1}{2}\rho\langle v_x\rangle^2\left(\frac{3}{3-\Delta}\right)^2 + \mathcal{P} = \frac{1}{2}\rho\langle v_x\rangle^2 + \mathcal{P}_0$$

or

$$\mathcal{P} - \mathcal{P}_0 = \frac{1}{2}\rho\langle v_x\rangle^2\left[1-\left(\frac{3}{3-\Delta}\right)^2\right]$$

**4C.2 Torsional oscillatory viscometer**

*a*. The equation for the rotating bob in a vacuum is just "moment-of-inertia times angular acceleration equals the the sum of the torques." In mathematical terms this is

$$I\frac{d^2\theta_R}{dt^2} = -k\theta_R$$

in which $-k\theta_R$ is the "restoring torque." The solution to this equation is

$$\theta_R = C_1 \cos\sqrt{\frac{k}{I}}t + C_2 \sin\sqrt{\frac{k}{I}}t = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t$$

This states that the bob is oscillating with a frequency $\omega_0 = \sqrt{k/I}$, which is called the natural frequency.

*b*. If there is a fluid in the gap, the equation of motion of the bob must contain both the "restoring torque" and the torque exerted by the fluid in the $\theta$ direction on the solid surface that is perpendicular to the *r* direction:

$$I\frac{d^2\theta_R}{dt^2} = -k\theta_R - (2\pi RL)(R)(-\tau_{r\theta}|_{r=R})$$

Here $(2\pi RL)(-\tau_{r\theta}|_{r=R})$ is the force, and $R$ is the lever arm. Next one uses Eq. B.1-11 to replace the shear stress by the appropriate product of viscosity and velocity gradient. This gives Eq. 4C.2-1.

*c*. Equation 4C.2-3 is obtained from simplifying Eq. B.6-5.

*d*. The choice of the variable $x$ is convenient, since $x = 0$ at the surface of the bob, and $x = 1$ is the inner surface of the cup. There are many ways to select the other dimensionless variables, but the choices we have made allow the viscosity to appear only in the parameter called $M$. Equations 4C.2-11 and 12 follow immediately.

*e*. From Eq. 4C.2-14 we get for the bob angular displacement

$$\begin{aligned}\theta_R &= \Re\left\{\theta_R^{\circ}\exp(i\overline{\omega}\tau)\right\} = \left(\theta_{Rr}^{\circ} + i\theta_{Ri}^{\circ}\right)(\cos\overline{\omega}\tau + i\sin\overline{\omega}\tau)\\ &= \theta_{Rr}^{\circ}\cos\overline{\omega}\tau - \theta_{Ri}^{\circ}\sin\overline{\omega}\tau\end{aligned}$$

Here $\theta_{Rr}^{o}$ and $\theta_{Ri}^{o}$ are the real and imaginary parts of the complex amplitude $\theta_R^o$. The angular displacement of the bob can also be written in terms of an amplitude and a phase shift:

$$\theta_R = A\cos(\overline{\omega}\tau - \alpha) == A\cos\overline{\omega}\tau\cos\alpha - A\sin\overline{\omega}\tau\sin\alpha$$

When the two expressions for the bob displacement are equated, we find that $A\cos\overline{\omega}\tau = \theta_{Rr}^{o}$ and $A\sin\overline{\omega}\tau = \theta_{Ri}^{o}$ from which we get

$$A = \sqrt{\theta_{Rr}^{o2} + \theta_{Ri}^{o2}} = \left|\theta_R^o\right| \text{ and } \tan\alpha = \frac{\theta_{Ri}^{o}}{\theta_{Rr}^{o}} = \frac{\Im\left\{\theta_R^o\right\}}{\Re\left\{\theta_R^o\right\}}$$

for the amplitude and the phase angle respectively. The ratio of the amplitudes of the cup and bob is then $\left|\theta_R^o\right| / \theta_{aR}^{o}$.

*f.* $i\overline{\omega}\phi^o = M\dfrac{d^2\phi^o}{dx^2}$ and $(i\overline{\omega})^2\theta_R^o = -\theta_R^o + M\left.\dfrac{d\phi^o}{dx}\right|_{x=0}$

*g.* The differential equation for $\phi^o$ is then

$$\frac{d^2\phi^o}{dx^2} - \sqrt{\frac{i\overline{\omega}}{M}}\phi^o = 0 \text{ with } \phi^o(0) = A\theta_R^o i\overline{\omega} \text{ and } \phi^o(1) = A\theta_{aR}^o i\overline{\omega}$$

The solution to this equation with the boundary conditions is then

$$\phi^o = Ai\overline{\omega}\theta_R^o\cosh\sqrt{\frac{i\overline{\omega}}{M}}x + Ai\overline{\omega}\left(\theta_{aR}^o - \theta_R^o\cosh\sqrt{\frac{i\overline{\omega}}{M}}\right)\frac{\sinh\sqrt{i\overline{\omega}/M}x}{\sinh\sqrt{i\overline{\omega}/M}}$$

Differentiation of this with respect to $x$ gives Eq. 4C.2-16.

*h.* From Eqs. 4C.2-12, 14, and 16 it follows that

$$(i\overline{\omega})^2\theta_R^o = -\theta_R^o + \frac{MAi\overline{\omega}}{\sinh\sqrt{i\overline{\omega}/M}}\sqrt{\frac{i\overline{\omega}}{M}}\left(\theta_{aR}^o - \theta_R^o\cosh\sqrt{\frac{i\overline{\omega}}{M}}\right)$$

This equation may be solved for $\theta_R^o$, and Eq. 4C.2-17 results.

*i.* When the hyperbolic functions are expanded in Taylor series and the terms arranged in powers of $1/M$, we get

$$\frac{\theta_{aR}^{\circ}}{\theta_R^{\circ}} = \cosh\sqrt{\frac{i\overline{\omega}}{M}} + \frac{\left(1-\overline{\omega}^2\right)}{MAi\overline{\omega}}\frac{\sinh\sqrt{i\overline{\omega}/M}}{\sqrt{i\overline{\omega}/M}}$$

$$= \left[1 + \frac{1}{2!}\left(\frac{i\overline{\omega}}{M}\right) + \frac{1}{4!}\left(\frac{i\overline{\omega}}{M}\right)^2 + \cdots\right] - i\frac{\left(1-\overline{\omega}^2\right)}{MA\overline{\omega}}\left[1 + \frac{1}{3!}\left(\frac{i\overline{\omega}}{M}\right)\cdots\right]$$

$$= 1 - \frac{i}{M}\left[\frac{\left(1-\overline{\omega}^2\right)}{A\overline{\omega}} - \frac{\overline{\omega}}{2!}\right] + \frac{1}{M^2}\left[\frac{\left(1-\overline{\omega}^2\right)\overline{\omega}}{3!A\overline{\omega}} - \frac{\overline{\omega}^2}{4!}\right] + \cdots$$

From this, Eq. 4C.2-18 follows. The results of part (*e*) can then be used to get the amplitude and phase angle.

**4C.3 Darcy's equation for flow through porous media.**

**(a)** In Case 1, the equation of state is $\rho = \rho_0$. Hence, Eq. (4C.3-1) gives

$$(\nabla \bullet \boldsymbol{v}_0) = 0$$

Insertion of Eq. (4C.3-2) for $\boldsymbol{v}_0$, with $\rho = \rho_0$ and $\boldsymbol{g} = -\nabla\hat{\Phi}$, gives

$$\left(\nabla \bullet [\nabla p + \rho_0 \nabla \hat{\Phi}]\right) = 0$$

or

$$\nabla^2 \mathcal{P} = 0$$

in which $\mathcal{P} = p + \rho_0\hat{\Phi}$ is the "modified pressure" defined in Chapters 2 and 3 for systems with constant $\rho$.

**(b)** In Case 2, the equation of state is $\rho = \rho_0 e^{\beta p}$. Hence,

$$\nabla\rho = \rho_0 \beta e^{\beta p} \nabla p = \rho\beta\nabla p, \text{ so that } \rho\nabla p = \frac{1}{\beta}\nabla\rho$$

Expressing the divergence term of Eq. (4C.3-1) via Darcy's equation, we get

$$\begin{aligned}
-(\nabla \bullet \rho \boldsymbol{v}_0) &= +\frac{\kappa}{\mu}(\nabla \bullet \rho[\nabla p - \rho \boldsymbol{g}]) \\
&= \frac{\kappa}{\mu}(\nabla \bullet [\rho\nabla p - \rho^2 \boldsymbol{g}]) \\
&= \frac{\kappa}{\mu}[\frac{1}{\beta}\nabla^2\rho - (\nabla \bullet \rho^2 \boldsymbol{g})]
\end{aligned}$$

Inserting this result, multiplied by $\mu\beta/\kappa$, into the smoothed continuity equation, we get

$$\frac{\varepsilon\mu\beta}{\kappa}\frac{\partial\rho}{\partial t} = \nabla^2\rho - \beta(\nabla \bullet \rho^2 \boldsymbol{g})$$

**(c)** In Case 3, the equation of state is $\rho = \rho_0 p$. Hence, $\nabla\rho = \rho_0\nabla p$ and $\nabla p = (1/\rho_0)\nabla\rho$. Then Eqs. (4C.3-1,2), with the $\rho\boldsymbol{g}$ term neglected, give

$$\begin{aligned}
\varepsilon\frac{\partial\rho}{\partial t} &= -(\nabla \bullet \rho[-\frac{\kappa}{\mu}\nabla p]) \\
&= +(\nabla \bullet \rho\frac{\kappa}{\mu\rho_0}\nabla\rho) \\
&= \frac{\kappa}{\mu\rho_0}(\nabla \bullet \rho\nabla\rho) \\
&= \frac{\kappa}{\mu\rho_0}(\nabla \bullet \nabla(\frac{1}{2}\rho^2) \\
&= \frac{\kappa}{2\mu\rho_0}\nabla^2\rho^2
\end{aligned}$$

Hence,

$$\frac{2\varepsilon\mu\rho_0}{\kappa}\frac{\partial\rho}{\partial t} = \nabla^2\rho^2$$

**(d)** In Case 4, the equation of state is $\rho = \rho_0 p^m$, giving $p = (\rho/\rho_0)^{1/m}$. Thus,

$$\nabla p = \frac{1}{m}\rho_0^{-1/m}\rho^{(1/m)-1}\nabla\rho$$

whence

$$\rho\boldsymbol{v}_0 = -\frac{\kappa}{\mu m}\rho_0^{-1/m}\rho^{1/m}\nabla\rho$$

so that

$$\begin{aligned}-(\nabla\bullet\rho\boldsymbol{v}_0) &= +\frac{\kappa}{\mu m}\rho_0^{-1/m}(\nabla\bullet\rho^{1/m}\nabla\rho)\\ &= \frac{\kappa}{\mu m}\rho_0^{-1/m}(\nabla\bullet\left[\frac{m}{m+1}\nabla\rho^{(m+1)/m}\right]\end{aligned}$$

With this result, Eq. 4C-1 gives

$$\left(\frac{(m+1)\varepsilon\mu\rho_0^{1/m}}{\kappa}\right)\frac{\partial\rho}{\partial t} = \nabla^2\rho^{(m+1)/m}$$

**4C.4 Radial flow through a porous medium**

*a.* For the radial flow (in cylindrical coordinates) of an incompressible fluid Eqs. 4C.3-1 and 2 become

$$0 = -\frac{1}{r}\frac{d}{dr}(rv_{0r}) \qquad \text{and} \qquad v_{0r} = -\frac{\kappa}{\mu}\frac{d\mathcal{P}}{dr}$$

Integrating the first of these equations gives $v_{0r} = C_1/r$. Substituting this into the second equation and integrating gives

$$C_1 \ln r + C_2 = -\frac{\kappa}{\mu}\mathcal{P} \qquad \text{or} \qquad D_1 \ln r + D_2 = -\mathcal{P}$$

where new constants of integration have been introduced. These constants are determined from the boundary conditions:

B. C. 1: $D_1 \ln R_1 + D_2 = -\mathcal{P}_1$
B. C. 2: $D_1 \ln R_2 + D_2 = -\mathcal{P}_2$

When the integration constants have been determined, the pressure distribution is found to be

$$\frac{\mathcal{P} - \mathcal{P}_1}{\mathcal{P}_2 - \mathcal{P}_1} = \frac{\ln(r/R_1)}{\ln(R_2/R_1)}$$

*b.* The velocity distribution is then given by using Darcy's law

$$v_{0r} = -\frac{\kappa}{\mu}\frac{d\mathcal{P}}{dr} = -\frac{\kappa}{\mu}(\mathcal{P}_2 - \mathcal{P}_1)\frac{1}{\ln(R_2/R_1)}\frac{1}{r} = -\frac{\kappa}{\mu r}\frac{(\mathcal{P}_2 - \mathcal{P}_1)}{\ln(R_2/R_1)}$$

*c.* The mass rate of flow through the system is

$$w = \rho|v_{0r}(R_1)|2\pi R_1 h = \frac{2\pi\kappa h(\mathcal{P}_2 - \mathcal{P}_1)\rho}{\mu \ln(R_2/R_1)}$$

**4D.1. Flow near an oscillating wall**

The problem is to solve Eq. 4.1-1, with the initial condition that $v_x = 0$ at $t = 0$, and the boundary conditions that $v_x = 0$ at $y = \infty$ and $v_x = v_0 \cos \omega t$ at $y = 0$. When we take the Laplace transform of Eq. 4.1-1 and the boundary and initial conditions, we get

$$p\bar{v}_x = \nu \frac{d^2 \bar{v}_x}{dy^2} \qquad \text{with } \bar{v}_x = v_0 \frac{p}{p^2 + \omega^2} \text{ at } y = 0 \text{ and } \bar{v}_x = 0 \text{ at } y = \infty$$

This ordinary differential equation is easily solved with the boundary conditions to give

$$\bar{v}_x = v_0 \frac{p}{p^2 + \omega^2} \exp\left(\sqrt{p/\nu}\, y\right)$$

This may be inverted by using the convolution theorem, or else by consulting a table of transforms (see, for example, Formula #11, on p. 246 of Vol. 1 of A. Erdélyi, et al., *Table of Integral Transforms*, McGraw-Hill, New York (1954)). The use of the table leads directly to Eq. 4D.1-1.

**4D.2 Start-up of laminar flow in a circular tube**

*a*. The partial differential equation, initial condition, and boundary conditions are

$$\rho\frac{\partial v_z}{\partial t}=\frac{\mathcal{P}_0-\mathcal{P}_L}{L}+\mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)$$

with $v_z(r,0)=0$, $v_z(0,t)=$ finite, and $v_z(R,t)=0$. We now introduce the following dimensionless variables

$$\phi=\frac{v_z}{(\mathcal{P}_0-\mathcal{P}_L)R^2/4\mu L};\qquad \xi=\frac{r}{R}\qquad \tau=\frac{\mu t}{\rho R^2}$$

Then Eq. 4D.2-1 is obtained, along with the initial and boundary conditions: $\phi(\xi,0)=0$, $\phi(0,\tau)=$ finite, and $\phi(R,\tau)=0$.

*b*. The asymptotic solution is obtained by setting the time derivative equal to zero and solving the ordinary differential equation with the boundary conditions. Then the partial differential equation for $\phi_t(\xi,\tau)$ is

$$\frac{\partial\phi_t}{\partial\tau}=\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\phi_t}{\partial\xi}\right)$$

with ancillary conditions: $\phi_t(\xi,0)=\phi_\infty(\xi)$, $\phi_t(0,\tau)=$ finite, and $\phi_t(1,\tau)=0$. We now try a solution of the form $\phi_t(\xi,\tau)=\Xi(\xi)T(\tau)$. This leads to two ordinary differential equations

$$\frac{dT}{d\tau}=-\alpha^2T\qquad\text{and}\qquad\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{d\Xi}{d\xi}\right)+\alpha^2\Xi=0$$

which have as their solutions

$$T=C_0\exp\left(-\alpha^2\tau\right)\qquad\text{and}\qquad\Xi=C_1J_0(\alpha\xi)+C_2Y_0(\alpha\xi)$$

in which the $C$s are constants, and $J_0$ and $Y_0$ are zero-order Bessel functions. Since $Y_0$ is not finite at the tube axis, we must set $C_2$ equal to zero. Since $\Xi$ must be zero at the tube wall, this will occur only if

$J_0(\alpha)=0$.This will happen only at $\alpha_1, \alpha_2, \alpha_3, \cdots$, that is, at the zeroes of $J_0$. Hence there are many solutions to the $\Xi$-equation that will satisfy the boundary conditions: $\Xi_n = C_{1n} J_0(\alpha_n \xi)$. Therefore, the general solution to the partial differential equation must be

$$\phi_t = \sum_{n=1}^{\infty} B_n \exp\left(-\alpha_n^2 \tau\right) J_0(\alpha_n \xi)$$

The constants $B_n$ are to be determined from the initial condition,

$$\left(1-\xi^2\right) = \sum_{n=1}^{\infty} B_n J_0(\alpha_n \xi)$$

This is done by multiplying both sides of the last equation by $J_0(\alpha_m \xi)\xi$ and integrating from 0 to 1:

$$\int_0^1 J_0(\alpha_m \xi)\xi\left(1-\xi^2\right)d\xi = \sum_{n=1}^{\infty} B_n \int_0^1 J_0(\alpha_n \xi) J_0(\alpha_m \xi)\xi d\xi$$

Because of the orthogonality properties of the Bessel functions, the only term on the right side that contributes is the term for $m = n$. The integrals may then be evaluated using some standard relations for Bessel functions. This gives

$$\frac{4J_1(\alpha_m)}{\alpha_m^3} = B_m \cdot \tfrac{1}{2}\left[J_1(\alpha_m)\right]^2 \qquad \text{from which} \qquad B_m = \frac{8}{\alpha_m^3 J_1(\alpha_m)}$$

and this leads directly to Eq. 4D.2-2.

**4D.3 Flows in the disk-and-tube system**

*a*. If the tangential component of the velocity depends only on $r$ and $z$, then the equation of motion simplifies to

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right)+\frac{\partial^2 v_\theta}{\partial z^2}=0$$

We try a solution of the product form: $v_\theta(r,\theta)=f(r)g(z)$. When this is substituted into the partial differential equation above, and the resulting equation divided by $f(r)g(z)$ we get

$$\frac{1}{f}\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rf)\right)=-\frac{1}{g}\frac{d^2g}{dz^2}$$

The left side is a function of $r$ only and the right side a function of $z$, only and therefore both sides must equal a constant; we call the constant $-c^2$. Then we have two ordinary differential equations to solve:

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rf)\right)+c^2f=0 \quad \text{and} \quad \frac{d^2g}{dz^2}-c^2g=0$$

The second equation has the solution $g=A\cosh cz+B\sinh cz$, where $A$ and $B$ are constants. The requirement that the velocity be zero at $z = L$, gives

$$g=-B\frac{\sinh[cL(1-[z/L])]}{\cosh cL}$$

where $B$ is a constant. The first equation can also be written as

$$\frac{d^2f}{dr^2}+\frac{1}{r}\frac{df}{dr}+\left(c^2-\frac{1}{r^2}\right)f=0$$

which is now recognized as a Bessel equation of first order, with the solution $f=MJ_1(cr)+NY_1(cr)$, where $M$ and $N$ are constants. But since $Y_1$ becomes infinite at $r=0$, only the $J_1$ term is needed, Since $f =$

0 at at $r = R$, we must have $J_1(c_n) = 0$. There are infinitely many $c_n$ that satisfy this equation. Hence the solution is of the form

$$v_\theta(\xi,\zeta) = \sum_{n=1}^{\infty} B_n J_1(c_n\xi)\frac{\sinh c_n L(1-\zeta)}{\cosh c_n L}$$

The coefficients $B_n$ are determined from the initial condition

$$R\Omega\xi = \sum_{n=1}^{\infty} B_n J_1(c_n\xi)$$

This is done by multiplying the equation by $J_1(c_m\xi)\xi$ and integrating over $\xi$ from 0 to 1 and making use of the orthogonality relation. This gives

$$R\Omega\int_0^1 J_1(c_m\xi)\xi^2 d\xi = \sum_{n=1}^{\infty} B_n \int_0^1 J_1(c_n\xi)J_1(c_m\xi)\xi d\xi \quad \text{or}$$

$$R\Omega\frac{J_2(c_m)}{c_m} = B_m \cdot \tfrac{1}{2}\left[J_2(c_m)\right]^2 \quad \text{or} \quad B_m = \frac{2R\Omega}{c_m J_2(c_m)}$$

Thus the steady-state velocity distribution is

$$v_\theta(\xi,\zeta) = 2R\Omega\sum_{n=1}^{\infty}\frac{J_1(c_n\xi)}{c_n J_2(c_n)}\frac{\sinh c_n L(1-\zeta)}{\cosh c_n L}$$

**4D.4 Unsteady annular flows**

For the tangential annular flow (part (*b*)), the equation of motion for $v_\theta(r,t)$ is

$$\rho\frac{\partial v_\theta}{\partial t}=\mu\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right]$$

with $v_\theta(r,0)=0$, $v_\theta(\kappa R,t)=\kappa R\Omega_i$, $v_\theta(R,t)=R\Omega_o$. We introduce these dimensionless variables:

$$\xi=\frac{r}{R},\qquad \tau=\frac{\mu t}{\rho R^2};\quad \phi=\frac{v_\theta}{R(\Omega_o-\Omega_i)}\quad -\alpha=\frac{\Omega_i}{\Omega_o-\Omega_i}$$

Then the partial differential equation for $\phi(\xi,\tau)$ becomes

$$\frac{\partial\phi}{\partial\tau}=\frac{\partial}{\partial\xi}\left[\frac{1}{\xi}\frac{\partial}{\partial\xi}(\xi\phi)\right]$$

with $\phi(\xi,0)=0$, $\phi(\kappa,\tau)=-\kappa\alpha$, $\phi(1,\tau)=1-\alpha$.

For the steady state (i.e., at infinite time), we have the solution

$$\phi_\infty=\left(\frac{1-\alpha(1-\kappa^2)}{1-\kappa^2}\right)\xi-\left(\frac{\kappa^2}{1-\kappa^2}\right)\frac{1}{\xi}\equiv A\xi-B\frac{1}{\xi}$$

Hence the time-dependent solution is

$$\phi(\xi,\tau)=\phi_\infty(\xi)-\phi_t(\xi,\tau)\text{-}$$

Here $\phi_t(\xi,\tau)$ is the transient contribution, which satisfies the partial differential equation for $\phi(\xi,\tau)$ , but with boundary and initial conditions: $\phi_t(\kappa,\tau)=\phi_t(1,\tau)=0$ and $\phi_t(\xi,0)=\phi_\infty(\xi)$

Application of the method of separation of variables with $\phi_t(\xi,\tau)=f(\xi)g(\tau)$ gives

$$\frac{1}{g}\frac{dg}{d\tau} = -b^2 = \frac{1}{f}\frac{d}{d\xi}\left(\frac{1}{\xi}\frac{d}{d\xi}(\xi f)\right)$$

where $-b^2$ is the separation constant. Thus we arrive at two ordinary differential equations which can be solved (the second-order equation is a Bessel function ). Since this is a Sturm-Liouville equation, the complete solution is a sum of eigenfunctions multiplied by the appropriate exponential function:

$$\phi_t(\xi,\tau) = \sum_{n=1}^{\infty} C_n Z_1(b_n\xi)e^{-b_n^2\tau}$$

in which $Z_1(b_n\xi)$ is the superposition of the two solutions to the Bessel equation:

$$Z_1(b_n\xi) = J_1(b_n\xi)Y_1(b_n\kappa) - J_1(b_n\kappa)Y_1(b_n\xi)$$

These functions satisfy the boundary conditions in that $Z_1(b_n\xi)=0$ at $\xi = \kappa$. The conditions that $Z_1(b_n\xi)=0$ at $\xi=1$ determine the eigenvalues $b_n$. The $C_n$ can then be determined from the initial condition, which is

$$\phi_\infty(\xi) = \sum_{n=1}^{\infty} C_n Z_1(b_n\xi)$$

When this equation is multiplied by $Z_1(b_m\xi)\xi d\xi$ and then integrated over the domain of interest, we get

$$C_n = \frac{\int_\kappa^1 \left(A\xi - B\xi^{-1}\right)Z_1(b_n\xi)\xi d\xi}{\int_\kappa^1 \left[Z_1(b_n\xi)\right]^2 \xi d\xi}$$
$$= \frac{-Ab_n^{-1}\left[Z_0(b_n) - \kappa^2 Z_0(b_n\kappa)\right] + Bb_n^{-1}\left[Z_0(b_n) - Z_0(b_n\kappa)\right]}{\frac{1}{2}\left[Z_0^2(b_n) - \kappa^2 Z_0^2(b_n\kappa)\right]}$$

The integrals are performed by making use of a mathematical handbook. The expression above may be simplified by using

i. The expressions for $A$ and $B$
ii. The defining equation for the $b_n$
iii. The relation $J_0(x)Y_1(x) - J_1(x)Y_0(x) = -2/\pi x$

Then it may be shown that

$$Z_0(b_n) = -\frac{2}{\pi b_n}\frac{J_1(b_n\kappa)}{J_1(b_n)} \qquad \text{and} \qquad Z_0(b_n\kappa) = -\frac{2}{\pi b_n}\frac{1}{\kappa}$$

and also

$$C_n(\kappa,\alpha) = \frac{\pi J_1(b_n)\left[(1-\alpha)J_1(b_n\kappa) + \alpha\kappa J_1(b_n)\right]}{\left[J_1^2(b_n\kappa) - J_1^2(b_n)\right]}$$

in which $\alpha$ is the dimensionless angular velocity. Then the complete expression for the transient behavior in tangential annular flow of a Newtonian fluid is

$$\phi(\xi,\tau) = \left(\frac{1-\alpha(1-\kappa^2)}{1-\kappa^2}\right)\xi - \left(\frac{\kappa^2}{1-\kappa^2}\right)\frac{1}{\xi}$$
$$-\pi\sum_{n=1}^{\infty}\frac{J_1(b_n)\left[(1-\alpha)J_1(b_n\kappa) + \alpha\kappa J_1(b_n)\right]Z_1(b_n\xi)}{\left[J_1^2(b_n\kappa) - J_1^2(b_n)\right]}e^{-b_n^2\tau}$$

Complete tables are given in the original reference. as well as some typical velocity profiles. Also, a Laplace-transform solution is given for small times, for which the expression above converges too slowly to be of much value.

**4D.5 Stream functions for three-dimensional flow.**

**(a)** The divergence of the curl of $\boldsymbol{A}$ is $\sum_i(\partial/\partial x_i)[\nabla\times\boldsymbol{A}]_i$ which gives, according to Eq. A.4-10,

$$\sum_i(\partial/\partial x_i)\sum_j\sum_k\varepsilon_{ijk}(\partial/\partial x_j)A_k = \sum_i\sum_j\sum_k\varepsilon_{ijk}\frac{\partial^2 A_k}{\partial x_i\partial x_j}$$
$$= \sum_k\sum_i\sum_{j\neq i}\varepsilon_{ijk}\left[\frac{\partial^2 A_k}{\partial x_i\partial x_j}-\frac{\partial^2 A_k}{\partial x_j\partial x_i}\right]=0$$

Thus, the mass flux function $\rho\boldsymbol{v} = [\nabla\times\boldsymbol{A}]$ satisfies the continuity equation for steady flow (or for unsteady flow with constant density). Here use has been made of the relations $\varepsilon_{ijk} = -\varepsilon_{jik}$ and $\varepsilon_{iik} = \varepsilon_{ikk} = 0$ contained in Eq. A.2-15.

The divergence of the product expression for $\rho\boldsymbol{v}$ is

$$(\nabla\bullet[(\nabla\psi_1)\times(\nabla\psi_2)]) = \sum_i\left(\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\bullet\left[\sum_j\boldsymbol{\delta}_j\frac{\partial\psi_1}{\partial x_j}\times\sum_k\boldsymbol{\delta}_k\frac{\partial\psi_2}{\partial x_k}\right]\right)$$
$$= \sum_i\sum_j\sum_k(\boldsymbol{\delta}_i\bullet[\boldsymbol{\delta}_j\times\boldsymbol{\delta}_k])\frac{\partial}{\partial x_i}\left(\frac{\partial\psi_1}{\partial x_j}\frac{\partial\psi_2}{\partial x_k}\right)$$
$$= \sum_i\sum_j\sum_k(\boldsymbol{\delta}_i\bullet\varepsilon_{jki}\boldsymbol{\delta}_i)\frac{\partial}{\partial x_i}\left(\frac{\partial\psi_1}{\partial x_j}\frac{\partial\psi_2}{\partial x_k}\right)$$
$$= \sum_i\sum_j\sum_k\varepsilon_{ijk}\partial_i(\partial_j\psi_1\partial_k\psi_2)$$
$$= \sum_i\sum_j\sum_k\varepsilon_{ijk}(\partial_{ij}\psi_1\partial_k\psi_2+\partial_j\psi_1\partial_{ik}\psi_2)$$
$$= \varepsilon_{123}(\partial_{12}\psi_1\partial_3\psi_2+\partial_2\psi_1\partial_{13}\psi_2)$$
$$+\varepsilon_{231}(\partial_{23}\psi_1\partial_1\psi_2+\partial_3\psi_1\partial_{21}\psi_2)$$
$$+\varepsilon_{312}(\partial_{31}\psi_1\partial_2\psi_2+\partial_1\psi_1\partial_{32}\psi_2)$$
$$+\varepsilon_{132}(\partial_{13}\psi_1\partial_2\psi_2+\partial_3\psi_1\partial_{12}\psi_2)$$
$$+\varepsilon_{321}(\partial_{32}\psi_1\partial_1\psi_2+\partial_2\psi_1\partial_{31}\psi_2)$$
$$+\varepsilon_{213}(\partial_{21}\psi_1\partial_3\psi_2+\partial_1\psi_1\partial_{23}\psi_2)$$
$$= \partial_{12}\psi_1\partial_3\psi_2(\varepsilon_{123}+\varepsilon_{213})+\partial_2\psi_1\partial_{13}\psi_2(\varepsilon_{123}+\varepsilon_{321})$$
$$+\partial_{23}\psi_1\partial_1\psi_2(\varepsilon_{231}+\varepsilon_{321})+\partial_3\psi_1\partial_{21}\psi_2(\varepsilon_{231}+\varepsilon_{132})$$
$$+\partial_{31}\psi_1\partial_2\psi_2(\varepsilon_{312}+\varepsilon_{132})+\partial_1\psi_1\partial_{32}\psi_2(\varepsilon_{312}+\varepsilon_{213})$$
$$=0$$

Thus, the product expression for $\rho\boldsymbol{v}$ is divergenceless. Here Eq. A.2-15 and the symmetry relations $\partial_{ij} = \partial_{ji}$ have been used. Stream functions of this form have been used by several authors; see footnote 1 of §4.2.

**(b)** In each of the coordinate systems shown in Table 4.2-1, the two nontrivial velocity components for incompressible flow are the corresponding components of $\text{curl}(-\boldsymbol{\delta}_3 A_3)/\rho = \text{curl}(-\boldsymbol{\delta}_3 \psi/h_3)$, as may be verified by use of Eqs. G, H and I in Tables A.7-1,2,3. Thus, for Cartesian coordinates with $v_z = 0$ and no $z$-dependence, $h_3 = h_z = 1$, and Eq. A.7-18 (with $\boldsymbol{v}$ read as $\boldsymbol{\delta}_3\psi/h_3$) gives the velocity components

$$v_x = \text{curl}_x(-\boldsymbol{A})/\rho = -\frac{\partial\psi}{\partial y} \quad \text{and} \quad v_y = \text{curl}_y(-\boldsymbol{A})/\rho = +\frac{\partial\psi}{\partial x}$$

For cylindrical coordinates with $v_z = 0$ and no $z$-dependence, $h_3 = h_z = 1$ and Eq. A.7-18 (with $\boldsymbol{v}$ read as $\boldsymbol{\delta}_3\psi/h_3$) gives the velocity components

$$v_r = \text{curl}_r(-\boldsymbol{A})/\rho = -\frac{1}{r}\frac{\partial\psi}{\partial\theta} \qquad \text{and} \qquad v_\theta = \text{curl}_\theta(-\boldsymbol{A})/\rho = +\frac{\partial\psi}{\partial r}$$

For cylindrical coordinates with $v_\theta = 0$ and no $\theta$-dependence, $h_3 = h_\theta = r$ and Eq. A.7-18 (with $\boldsymbol{v}$ read as $\boldsymbol{\delta}_3\psi/h_3$) gives the velocity components

$$v_z = \text{curl}_z(-\boldsymbol{A})/\rho = -\frac{1}{r}\frac{\partial\psi}{\partial r} \qquad \text{and} \qquad v_r = \text{curl}_r(-\boldsymbol{A})/\rho = +\frac{1}{r}\frac{\partial\psi}{\partial z}$$

For spherical coordinates with $v_\phi = 0$ and no $\phi$-dependence, $h_3 = h_\phi = r\sin\theta$ and Eq. A.7-18 (with $\boldsymbol{v}$ read as $\boldsymbol{\delta}_3\psi/h_3$) gives the velocity components

$$v_r = \text{curl}_r(-\boldsymbol{A})/\rho = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi} \qquad \text{and} \qquad v_\theta = \text{curl}_\theta(\boldsymbol{A})/\rho = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

**(c)** Consider two surfaces, $\psi_1(x_1, x_2, x_3) = C_1$ and $\psi_2(x_1, x_2, x_3) = C_2$, which intersect along a line $\mathcal{L}$. At each point on $\mathcal{L}$, the vectors $\nabla\psi_1$ and $\nabla\psi_2$ are normal to both surfaces, and the velocity vector $\boldsymbol{v} = [(\nabla\psi_1)\times(\nabla\psi_2)]$ is consequently tangent to $\mathcal{L}$. Thus, the intersection of any such pair of surfaces is a streamline. In Fig. 4.3-1, we may choose $\psi_1 = \Psi(r,\theta)$ and $\psi_2 = z$; the resulting streamlines in a plane of constant $\psi_2$ are shown in the figure.

**(d)** Read $\boldsymbol{v}$ in Eq. A.5-4 as the vector $\boldsymbol{A}$ whose curl is the local mass flux $\rho\boldsymbol{v}$. Then the net mass flow through $S$ is

$$\int_S (\boldsymbol{n} \bullet [\nabla\times\boldsymbol{A}])dS = \oint_C (\boldsymbol{t} \bullet \rho\boldsymbol{v})dC$$

for steady flow, or for unsteady incompressible flow. A no-slip condition $\boldsymbol{v} = 0$ on $C$ requires $[\nabla\times\boldsymbol{A}] = 0$ there, but this derivative condition does not require the vanishing of $\boldsymbol{A}$ nor of the net mass flow.

**5A.1 Pressure drop needed for laminar-turbulent transition.**

The minimum value of Re= $4w/(\pi D\mu)$ needed to produce turbulent flow in a long, smooth tube is about 2100. Poiseuille's law, Eq. 2.3-21, holds until this critical Re value, giving

$$w = \frac{\pi(\mathcal{P}_0 - \mathcal{P}_L)R^4\rho}{8\mu L} \text{ for Re<Re}_{\text{crit}}$$

Hence, the pressure gradient needed to initiate the laminar-turbulent transition is

$$\begin{aligned} |dp/dz| &= \frac{8\mu w}{\pi R^4 \rho} \\ &= \left(\frac{8\mu}{\pi R^4}\right)\left(\frac{\pi D\mu \text{Re}_{\text{crit}}}{4}\right) \\ &\approx \frac{4\mu^2}{\rho R^3}(2.1 \times 10^3) \end{aligned}$$

The other specifications for this problem are:

$$\mu = 18.3 \text{ cp} = 0.183 \text{ g/cm·s}; \qquad \rho = 1.32 \text{ g/cm}^2; \qquad R = 2.67 \text{ cm}$$

The pressure gradient required to initiate turbulence at the given conditions is then:

$$\begin{aligned} \left|\frac{dp}{dz}\right| &\approx \frac{(4)(0.183 \text{ g/cm·s})^2}{(1.32 \text{ g/cm}^3)(2.67 \text{ cm})^3}(2.1 \times 10^3) \\ &\approx (11.2 \text{ dyne/cm}^3)(0.1 \text{ Pa/(dyne/cm}^2))(10^5 \text{ cm/km}) \\ &\approx 1.1 \times 10^5 \text{ Pa/km} \\ &\approx (1.1 \times 10^5 \text{ Pa/km})(1.45 \times 10^{-4} \text{ psi/Pa})(1.602 \text{ km/mile}) = 26 \text{ psi/mile} \end{aligned}$$

**5A.2 Velocity distribution in turbulent pipe flow.**

*a.* Application of Eq. 5.4-4 gives

$$\tau_0 = \frac{(p_0 - p_L)R}{2L} = \left(\frac{(1.0 \text{ psi})(0.25 \text{ ft})}{(2)(5280 \text{ ft})}\right) = 2.367 \times 10^{-5} \text{ psi}$$
$$= (2.367 \times 10^{-5} \text{ psi})(6.8947 \times 10^3 \text{ Pa/psi}) = 0.1633 \text{ Pa}$$

*b.* For use of Fig. 5.5-3 we need the following additional values:

$$\mu = 1.0019 \times 10^{-3} \text{ Pa·s}$$
$$\nu = 1.0037 \times 10^{-}2 \text{ cm}^2/\text{s} = 1.0037 \times 10^{-6} \text{ m}^2/\text{s}$$
$$\rho = 0.9992 \text{ g/cm}^3 = 0.9992 \times 10^3 \text{ kg/m}^3$$
$$v_* = \sqrt{\tau_0/\rho} = \sqrt{(0.1633 \text{ kg/m·s}^2)/(0.9992 \times 10^3 \text{ kg/m}^3)} = 0.01278 \text{ m/s}$$
$$v_*/\nu = (0.01278 \text{ m/s})/(1.0037 \times 10^{-6} \text{ m}^2/\text{s} = 1.273 \times 10^4 \text{ m}^{-1})$$
$$R = (3 \text{ in}/(39.37 \text{ in/m}) = 0.0762 \text{ m}$$

At the tube center, $y^+ = Rv_*/\nu = (0.0762)(1.27 \times 10^4) = 970$ and Fig. 5.5-3 gives $v_*\big|_{y=R} = 22.7$. Consequently,

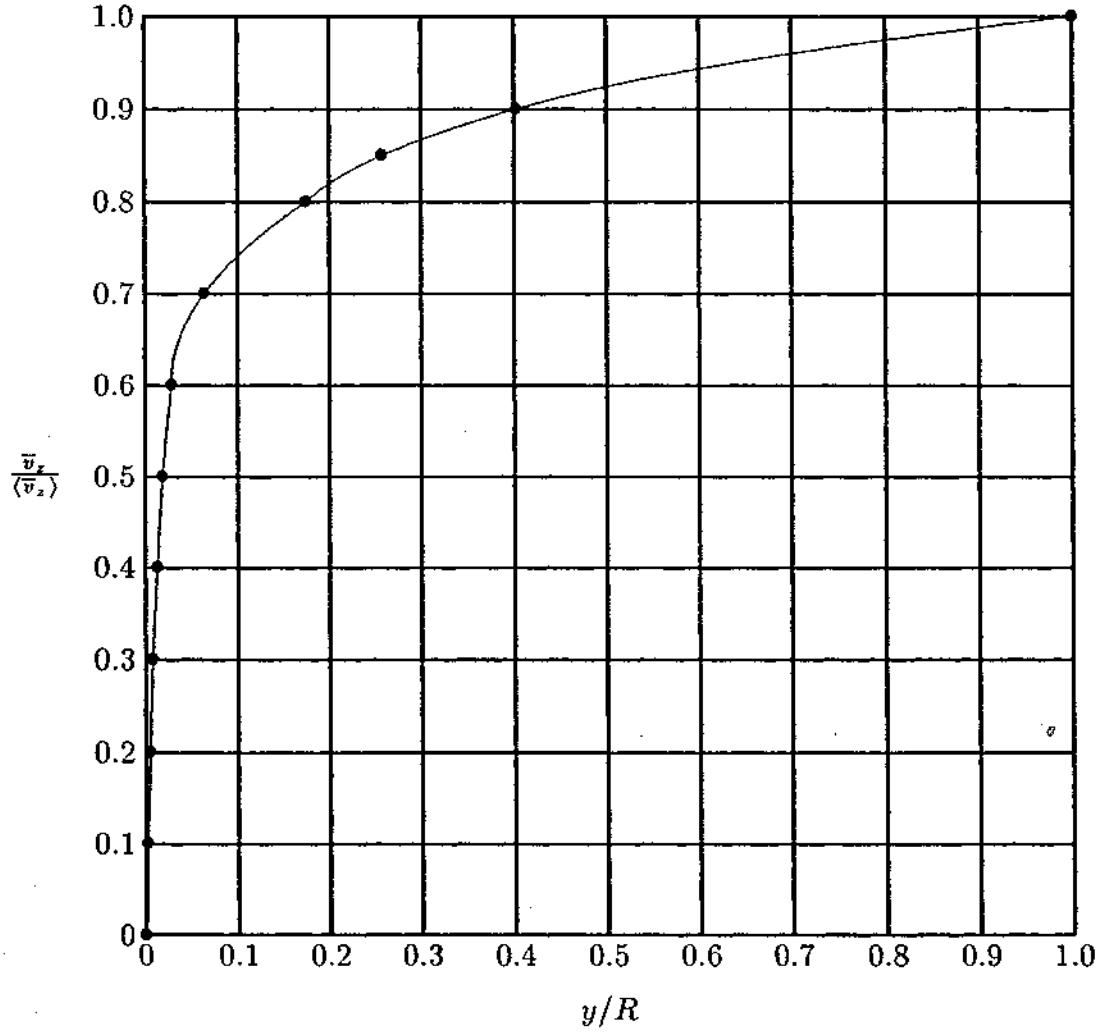
$$\overline{v}_{z\max} = 22.7 v_* = 0.290 \text{ m/s}, \qquad v^+ = 22.7\overline{v}_z/\overline{v}_{z\max}, \qquad \text{and} \quad y/R = s^+/970$$

We can now tabulate the time-smoothed velocity profile. Asterisked values of $\overline{v}_z/\overline{v}_{z\max}$ are added here to give a better calculation of the mass flow rate in part (d).

| $\overline{v}_z/\overline{v}_{z\max}$ | $v^+$ | $y^+$ (from Fig. 5.5-3) | $y/R$ | $r/R$ |
|---|---|---|---|---|
| 0.0 | 0.0 | 0.0 | 0.0 | 1.0 |
| 0.1 | 2.27 | 2.27 | 0.0023 | 0.9977 |
| 0.2 | 4.54 | 4.55 | 0.0047 | 0.9953 |
| 0.3* | 6.81 | 7.3 | 0.0075 | 0.9925 |
| 0.4 | 9.08 | 12.0 | 0.0124 | 0.9876 |
| 0.5* | 11.35 | 18.0 | 0.0186 | 0.9814 |
| 0.6* | 13.62 | 27.5 | 0.0284 | 0.9716 |
| 0.7 | 15.89 | 62.0 | 0.064 | 0.936 |
| 0.8* | 18.16 | 170 | 0.175 | 0.825 |
| 0.85 | 19.30 | 250 | 0.258 | 0.742 |
| 0.9* | 20.43 | 392 | 0.404 | 0.596 |
| 1.00 | 22.7 | 970 | 1.00 | 0.000 |

*c.* See the graph on the following page.

Velocity profile in Problem 5A.2



*d.* The estimate $\langle \overline{v}_z \rangle \approx 0.85\overline{v}_{z\max}$ gives $\langle \overline{v}_z \rangle \approx (0.85)(0.290\ \mathrm{m/s}$. Hence,

$$\mathrm{Re} = \frac{D\langle \overline{v}_z \rangle}{\nu} = \frac{(2 \times 0.0762\ \mathrm{m})(0.85)(0.290\ \mathrm{m/s})}{(1.0037 \times 10^{-6}\ \mathrm{m^2/s})} = 3.7 \times 10^6$$

so the flow is certainly turbulent.

*e.* The mean value of $v^+$ over the flow cross-section is

$$\langle v^+ \rangle = \int_0^1 v^+ d(r/R)^2$$

Applying trapezoidal quadrature to the values calculated in (b), we get

$$\begin{aligned}
\int_0^1 v^+ d(r/R)^2 = &\left(\frac{0+2.27}{2}\right)[1.0000^2 - 0.9977^2] \\
&+ \left(\frac{2.27+4.54}{2}\right)[0.9977^2 - 0.9953^2] \\
&+ \left(\frac{4.54+6.81}{2}\right)[0.9953^2 - 0.9925^2] \\
&+ \left(\frac{6.81+9.08}{2}\right)[0.9925^2 - 0.9876^2] \\
&+ \left(\frac{9.08+11.35}{2}\right)[0.9876^2 - 0.9814^2] \\
&+ \left(\frac{11.35+13.62}{2}\right)[0.9814^2 - 0.9716^2] \\
&+ \left(\frac{13.62+15.89}{2}\right)[0.9716^2 - 0.936^2] \\
&+ \left(\frac{15.89+18.16}{2}\right)[0.936^2 - 0.825^2] \\
&+ \left(\frac{18.16+19.30}{2}\right)[0.825^2 - 0.742^2] \\
&+ \left(\frac{19.30+20.43}{2}\right)[0.742^2 - 0.596^2] \\
&+ \left(\frac{20.43+22.7}{2}\right)[0.596^2 - 0.000^2] \\
&= 18.8\ \mathrm{m/s}
\end{aligned}$$

Hence,

$$\langle \overline{v}_z \rangle = \langle v^+ \rangle v_* = (18.8)(0.01278\ \mathrm{m/s}) = 0.241\ \mathrm{m/s}$$

and

$$\begin{aligned}
w &= \pi R^2 \rho \langle \overline{v}_z \rangle \\
&= \pi(0.0762\ \mathrm{m})^2(999.2\ \mathrm{kg/m^3})(0.241\ \mathrm{m/s}) \\
&= 4.4\ \mathrm{kg/s} = 9.7\ \mathrm{lb}_m/\mathrm{s}
\end{aligned}$$

**5B.1 Average flow velocity in turbulent tube flow**

*a*. The ratio of average to maximum velocity is obtained as follows for the power-law expression:

$$\frac{\langle \bar{v}_z \rangle}{\bar{v}_{z,\max}} = \frac{\int_0^{2\pi}\int_0^R \left(\bar{v}_z/\bar{v}_{z,\max}\right) r dr d\theta}{\int_0^{2\pi}\int_0^R r dr d\theta} = \frac{2}{R^2}\int_0^R \left(1-\frac{r}{R}\right)^{1/n} r dr$$

$$= 2\int_0^1 (1-\xi)^{1/n}\,\xi d\xi = 2\int_0^1 \zeta^{1/n}(1-\zeta)d\zeta$$

$$= 2\left(\frac{\zeta^{(1/n)+1}}{(1/n)+1} - \frac{\zeta^{(1/n)+2}}{(1/n)+2}\right)\Bigg|_0^1$$

$$= 2\frac{n\left[(2n+1)-(n+1)\right]}{(n+1)(2n+1)} = \frac{2n^2}{(n+1)(2n+1)}$$

*b*. For the logarithmic profile we have

$$\frac{\langle \bar{v}_z \rangle}{\bar{v}_{z,\max}} = \frac{v_*}{\bar{v}_{z,\max}} \frac{\int_0^{2\pi}\int_0^R \left(\bar{v}_z/v_*\right) r dr d\theta}{\int_0^{2\pi}\int_0^R r dr d\theta} == \frac{v_*}{\bar{v}_{z,\max}} \frac{2}{R^2}\int_0^R \left(\frac{1}{\kappa}\ln\frac{yv_*}{\nu} + \lambda\right) r dr$$

$$= \frac{v_*}{\bar{v}_{z,\max}} \frac{2}{R^2}\int_0^R \left(\frac{1}{\kappa}\ln\frac{yv_*}{\nu} + \lambda\right)(R-y)dy \quad \text{where} \quad y = R-r$$

$$= \frac{2v_*}{\bar{v}_{z,\max}}\left(\frac{\nu}{Rv_*}\right)^2 \int_0^{Rv_*/\nu}\left(\frac{1}{\kappa}\ln y^+ + \lambda\right)\left(\frac{Rv_*}{\nu} - y^+\right)dy^+ \qquad \left(y^+ = \frac{yv_*}{\nu}\right)$$

$$= \frac{2v_*}{\bar{v}_{z,\max}}\left(\frac{\nu}{Rv_*}\right)^2 \left[\frac{1}{\kappa}\frac{Rv_*}{\nu}\left(y^+\ln y^+ - y^+\right) - \frac{1}{\kappa}\left(\frac{1}{2}y^{+2}\ln y^+ - \frac{1}{4}y^{+2}\right)\right.$$

$$\left. + \lambda\frac{Rv_*}{\nu}y^+ - \lambda\frac{1}{2}y^{+2}\right]_0^{Rv_*/\nu}$$

$$= \frac{2v_*}{\bar{v}_{z,\max}}\left[\frac{1}{\kappa}\left(\ln\frac{Rv_*}{\nu} - 1\right) - \frac{1}{\kappa}\left(\frac{1}{2}\ln\frac{Rv_*}{\nu} - \frac{1}{4}\right) + \frac{1}{2}\lambda\right]$$

$$= \frac{v_*}{\bar{v}_{z,\max}}\left[\frac{1}{\kappa}\ln\frac{Rv_*}{\nu} - \frac{3}{2}\frac{1}{\kappa} + \lambda\right]$$

In the above we have used the fact that in the limit as $x$ goes to zero, $x\ln x$ vanishes.

**5B.2 Mass flow rate in a turbulent circular jet**

*a*. Immediately starting to differentiate the velocity components with respect to the position coordinates is *not* the way to solve this problem. It is much easier to solve if one introduces some abbreviations in order to minimize writing:

$$Z = \frac{\bar{v}_z}{\nu^{(t)}} \qquad R = \frac{\bar{v}_r}{\nu^{(t)}} \qquad C_3 = C \qquad u = \frac{Cr}{z} \qquad [\,] = \left[1 + \tfrac{1}{4}u^2\right]$$

In terms of these quantities, the velocity components are

$$Z = \frac{2C^2}{z[\,]^2} \quad \text{and} \quad R = \frac{C\left(u - \frac{1}{3}u^3\right)}{z[\,]^2}$$

Then we can calculate the derivatives appearing in the equation of continuity:

$$\frac{\partial Z}{\partial z} = -\frac{2C^2}{z^2[\,]^2} - 2\frac{2C^2}{z[\,]^3}\left(\tfrac{1}{2}u\right)\left(-\frac{Cr}{z^2}\right) = -\frac{2C^2}{z^2[\,]^2} + \frac{2C^2u^2}{z^2[\,]^3}$$
$$= -\frac{2C^2}{z^2[\,]^3}\left(\left(1 + \tfrac{1}{4}u^2\right) - u^2\right) = -\frac{2C^2}{z^2[\,]^3}\left(1 - \tfrac{3}{4}u^2\right)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rR) = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{u^2 - \frac{1}{4}u^4}{[\,]^2}\right) = \frac{1}{r}\left(\frac{\left(2u - u^3\right)}{[\,]^2}\left(\frac{C}{z}\right) - 2\left(u^2 - \tfrac{1}{4}u^4\right)\frac{\frac{1}{2}u}{[\,]^3}\left(\frac{C}{z}\right)\right)$$
$$= \frac{1}{r[\,]^3}\left(\frac{C}{z}\right)\left(\left(1 + \tfrac{1}{4}u^4\right)\left(2u - u^3\right) - u\left(u^2 - \tfrac{1}{4}u^4\right)\right) = \frac{2C^2}{z^2[\,]^3}\left(1 - \tfrac{3}{4}u^2\right)$$

Thus the equation of continuity is satisfied. Next we get the expressions for the terms on the left side of the equation of motion:

$$R\frac{\partial Z}{\partial r} = \left(\frac{C\left(u - \frac{1}{4}u^3\right)}{z[\,]^2}\right)\left(-\frac{2C^3u}{z^2[\,]^3}\right) = -\frac{2C^4u^2\left(1 - \frac{1}{4}u^2\right)}{z^3[\,]^5}$$

$$Z\frac{\partial Z}{\partial z}=\left(\frac{2C^2}{z[\,]^2}\right)\left(-\frac{2C^2\left(1-\frac{3}{4}u^2\right)}{z^2[\,]^3}\right)=-\frac{4C^4\left(1-\frac{3}{4}u^2\right)}{z^3[\,]^5}$$

$$R\frac{\partial Z}{\partial r}+Z\frac{\partial Z}{\partial z}=-\frac{4C^4\left(1-\frac{1}{4}u^2-\frac{1}{8}u^4\right)}{z^3[\,]^5}=-\frac{4C^4\left(1+\frac{1}{4}u^2\right)\left(1-\frac{1}{2}u^2\right)}{z^3[\,]^5}$$

$$=-\frac{4C^3u\left(1-\frac{1}{2}u^2\right)}{rz^2[\,]^4}$$

The term on the right side is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial Z}{\partial r}\right)=\frac{1}{r}\frac{\partial}{\partial r}\left(r\left(-\frac{2C^3u}{z^2[\,]^3}\right)\right)=-\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{2C^2u^2}{z[\,]^3}\right)$$

$$=-\frac{1}{r}\left(\frac{4C^2u}{z[\,]^3}\cdot\frac{C}{z}-\frac{3C^2u^3}{z[\,]^4}\cdot\frac{C}{z}\right)=-\frac{1}{r}\left(\frac{4C^3u\left(1-\frac{1}{2}u^2\right)}{z^2[\,]^4}\right)$$

Therefore the equation of motion is satisfied.

*b*. The mass flow rate is

$$w(z)=\int_0^{2\pi}\int_0^{\infty}\rho\bar{v}_z(r,z)rdrd\theta=2\pi\rho\cdot\frac{2C_3^2v^{(t)}}{z}\int_0^{\infty}\frac{rdr}{\left[1+\frac{1}{4}\left(C_3r/z\right)^2\right]^2}$$

$$=2\pi\rho\cdot\frac{2C_3^2v^{(t)}}{z}\cdot\frac{1}{2}\int_0^{\infty}\frac{ds}{\left[1+\frac{1}{4}\left(C_3/z\right)^2s\right]^2}$$

$$=2\pi\rho\cdot\frac{2C_3^2v^{(t)}}{z}\cdot\frac{1}{2}\left(-\frac{1}{\frac{1}{4}\left(C_3/z\right)^2\cdot\left[1+\frac{1}{4}\left(C_3/z\right)^2s\right]}\right)\Bigg|_0^{\infty}$$

$$=2\pi\rho\cdot\frac{2C_3^2v^{(t)}}{z}\cdot\frac{1}{2}\cdot\frac{4z^2}{C_3^2}=8\pi\rho v^{(t)}z$$

**5B.3 The eddy viscosity expression in the viscous sublayer**

We start with Eq. 5.3-13 is dimensionless form

$$v^+ = y^+\left[1-\frac{1}{2}\left(\frac{\nu}{v_*B}\right)y^+ - \frac{1}{4}\left(\frac{y^+}{14.5}\right)^3 + \cdots\right]$$

from which

$$\frac{dv^+}{dy^+} = 1-\left(\frac{\nu}{v_*B}\right)y^+ - \left(\frac{y^+}{14.5}\right)^3 + \cdots$$

For slit flow with thickness $2B$, we have

$$\bar{\tau}_{yx} = \bar{\tau}_{yx}^{(v)} + \bar{\tau}_{yx}^{(t)} = -\tau_0\left(1-\frac{y}{B}\right) = -\left(\mu+\mu^{(t)}\right)\frac{d\bar{v}_x}{dy}$$

where the definition of turbulent viscosity in Eq. 5.4-1 has been used. If now we write this last equation in dimensionless form, we have

$$\frac{\bar{\tau}_{yx}}{\rho v_*^2} = -1+\left(\frac{\nu}{v_*B}\right)y^+ = -\left(1+\frac{\mu^{(t)}}{\mu}\right)\frac{dv^+}{dy^+}$$

Substituting the dimensionless velocity gradient obtained from Eq. 5.3-13 into this expression gives

$$-1+\left(\frac{\nu}{v_*B}\right)y^+ = -1+\left(\frac{\nu}{v_*B}\right)y^+ + \left(\frac{y^+}{14.5}\right)^3 + \cdots$$
$$-\frac{\mu^{(t)}}{\mu} + \frac{\mu^{(t)}}{\mu}\left(\frac{\nu}{v_*B}\right)y^+ + \frac{\mu^{(t)}}{\mu}\left(\frac{y^+}{14.5}\right)^3 + \cdots$$

Terms 1 and 2 on the left exactly balance terms 1 and 2 on the right. Term 3 on the right will exactly cancel term 4 if the expression for the turbulent viscosity given in Eq. 5.4-2 is used. With the same substitution, terms 5 and 6 on the right side are of higher order and can be neglected in the vicinity of the wall.

**5C.1 Two-dimensional turbulent jet**

*a.* The total flow of $z$ momentum in the jet is given by

$$J = \int_0^W \int_{-\infty}^{+\infty} \rho \bar{v}_z^2 dx dy = W \rho \bar{v}_{z,\max}^2 z \int_{-\infty}^{+\infty} f^2 d\xi$$

Since $J$ is constant for all values of $z$, we conclude that $\bar{v}_{z,\max} \propto 1/\sqrt{z}$.

*b.* By integrating one of the stream-function equations we get

$$\psi \propto \int \bar{v}_z dx \propto \bar{v}_{z,\max} \int f dx \propto z^{-1/2} \int f d\xi \cdot z \propto z^{1/2}$$

The momentum flow $J$ [=] $Lm/t^2$ and $\sqrt{J/\rho}[=]L^2/t$. But $\psi[=]L^2/t$ also. This suggests that

$$\psi(x,z) = \sqrt{\frac{J}{\rho}} \sqrt{\frac{z}{W}} F(\xi)$$

is the only form that can be put together for the stream function.

*c.* According to Eq. 5.4-3, the kinematic viscosity is the product of $b$ and $\bar{v}_{z,\max}$, in which $b$ has dimensions of length and must be some function of $z$. This leads to the fact that the kinematic viscosity (which has dimensions of $L^2/t$) can be constructed thus:

$$\psi(x,z) \propto \sqrt{\frac{J}{\rho}} \sqrt{\frac{z}{W}} F(\xi) \nu^{(t)} \propto z \cdot \frac{1}{\sqrt{z}} \sqrt{\frac{J}{\rho}} \sqrt{\frac{1}{W}}$$

The third factor has been included in order to make the dimensions come out correctly. The dimensionless constant $\lambda$ is then included, but this quantity must be determined experimentally.

*d.* The $z$ component of the equation of motion is

$$\bar{v}_x \frac{\partial \bar{v}_z}{\partial x} + \bar{v}_z \frac{\partial \bar{v}_z}{\partial z} = \frac{\partial}{\partial x}\left( \nu^{(t)} \frac{\partial \bar{v}_z}{\partial x} \right)$$

or, in terms of the stream function,

$$-\frac{\partial\psi}{\partial z}\frac{\partial^2\psi}{\partial x^2}+\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x\partial z}=-\frac{\partial}{\partial x}\left(\nu^{(t)}\frac{\partial^2\psi}{\partial x^2}\right)$$

Inserting the expressions for dimensionless stream function from (*b*) and the kinematic viscosity from (*c*), we get

$$-\left(\frac{\partial}{\partial z}\sqrt{z}F\right)\left(\sqrt{z}\frac{\partial^2 F}{\partial x^2}\right)+\left(\sqrt{z}\frac{\partial F}{\partial x}\right)\left(\frac{\partial}{\partial x}\frac{\partial}{\partial z}\sqrt{z}F\right)=-\lambda\frac{\partial}{\partial x}\left(\sqrt{z}\frac{\partial^2\psi}{\partial x^2}\sqrt{z}F\right)$$

where $f$ is a function of $\xi$. We next convert the derivatives with respect to $x$ and $z$ to derivatives with respect to $\xi$ (indicated by primes), to get

$$-\left(\frac{F}{2\sqrt{z}}-\sqrt{z}F'\frac{\xi}{z}\right)\left(\frac{\sqrt{z}}{z^2}F''\right)$$
$$+\left(\frac{\sqrt{z}}{z}F'\right)\left(\frac{1}{2\sqrt{z}\,z}F'-\frac{\sqrt{z}}{z^2}\{\xi F''+F'\}\right)=-\lambda\frac{z}{z^3}F'''$$

Multiplying by $z^2$ and canceling two terms in the equation gives

$$\tfrac{1}{2}FF''+\tfrac{1}{2}F'^2=\lambda F'''$$

When we change to a new variable $\eta=\xi/4\lambda$, this equation becomes

$$F\frac{d^2F}{d\eta^2}+\left(\frac{dF}{d\eta}\right)^2=\frac{1}{2}\frac{d^3F}{d\eta^3}\qquad\text{or}\qquad\frac{d}{d\eta}\left(F\frac{dF}{d\eta}\right)=\frac{1}{2}\frac{d^3F}{d\eta^3}$$

*f*. The final result in (*d*) can be integrated at once to give

$$F\frac{dF}{d\eta}=\frac{1}{2}\frac{d^2F}{d\eta^2}+C'\qquad\text{or}\qquad\frac{d}{d\eta}F^2=\frac{d^2F}{d\eta^2}+2C'$$

But at $\eta=0$, both $F$ and and its second derivative are zero so that $C'=0$.

*g*. Integration of the result iin (*f*) gives

$$F^2 = \frac{dF}{d\eta} + C^2 \qquad \text{or} \qquad \frac{dF}{d\eta} = F^2 - C^2$$

*h*. Integrating this last result gives

$$\int_0^F \frac{dF}{F^2 - C^2} = \int_0^\eta d\eta \quad \text{or} \quad -\frac{1}{C}\operatorname{arctanh}\frac{F}{C} = \eta \quad \text{or} \quad F = -C\tanh C\eta$$

Then the time-smoothed axial velocity is

$$\begin{aligned}
\bar{v}_z(x,z) &= -\frac{\partial\psi}{\partial x} = -\frac{\partial}{\partial x}\left(\sqrt{\frac{Jz}{\rho W}}F(\xi)\right) \\
&= -\sqrt{\frac{Jz}{\rho W}}\frac{dF}{d\xi}\frac{1}{z} = -\sqrt{\frac{J}{\rho W}}\frac{dF}{d\eta}\frac{1}{4\lambda} \\
&= \sqrt{\frac{J}{\rho Wz}}\frac{C^2}{4\lambda}\operatorname{sech}^2 C\eta = \sqrt{\frac{J}{\rho Wz}}\frac{C^2}{4\lambda}\operatorname{sech}^2\frac{Cx}{4\lambda z}
\end{aligned}$$

*i*. The mass rate of flow at plane *z* is

$$\begin{aligned}
w &= \int_0^W \int_{-\infty}^{+\infty} \rho\bar{v}_z dx dy \\
&= W\rho\sqrt{\frac{J}{\rho Wz}}\frac{C^2}{4\lambda}\int_{-\infty}^{+\infty}\operatorname{sech}^2 C\eta d\eta \cdot 4\lambda z \\
&= W\rho z\sqrt{\frac{J}{\rho Wz}}C \cdot 2\tanh x\Big|_0^\infty = 2CW\rho z\sqrt{\frac{J}{\rho Wz}} \\
&= 2\sqrt[3]{3\lambda}\sqrt{J\rho Wz}
\end{aligned}$$

Thus we see that the mass rate of flow increases with the distance from the exit of the slit. This is a result of the fact that additional fluid is dragged along by the jet.

**5C.2 Axial turbulent flow in an annulus**

*a*. We divide the annular region into two parts, depending on whether or not $r$ is less than or greater than $bR$:

Region <: $aR < r < bR$

$$\text{stress at inner wall} = |\tau_{rz}| = \frac{(\mathcal{P}_0 - \mathcal{P}_L)R}{2L}\left(\frac{b^2 - a^2}{a}\right)$$

Region >: $bR < r < R$

$$\text{stress at outer wall} = |\tau_{rz}| = \frac{(\mathcal{P}_0 - \mathcal{P}_L)R}{2L}\left(1 - b^2\right)$$

To get the magnitudes of the wall shear stresses, we use the expression given in Eq. 2.4-4, which is valid for laminar and turbulent flow alike. Then, the friction velocity in the two regions is

Region <: $$v_*^< = \sqrt{\frac{\tau_0^<}{\rho}} = \sqrt{\frac{(\mathcal{P}_0 - \mathcal{P}_L)R}{2L\rho}\left(\frac{b^2 - a^2}{a}\right)} = v_{**}\sqrt{\frac{b^2 - a^2}{a}}$$

Region >: $$v_*^> = \sqrt{\frac{\tau_0^>}{\rho}} = \sqrt{\frac{(\mathcal{P}_0 - \mathcal{P}_L)R}{2L\rho}\left(1 - b^2\right)} = v_{**}\sqrt{1 - b^2}$$

where $v_{**} = \sqrt{(\mathcal{P}_0 - \mathcal{P}_L)R/2L\rho}$. Then the velocity distribution in the two regions may be obtained from Eq. 5.3-3:

Region <: $$\frac{\bar{v}_z^<}{v_*^<} = \frac{1}{\kappa^<}\ln\left[\frac{(r - aR)v_*^<}{\nu}\right] + \lambda^< = \frac{1}{\kappa^<}\ln\left(\frac{yv_*^<}{\nu}\right) + \lambda^<$$

Region >: $$\frac{\bar{v}_z^>}{v_*^>} = \frac{1}{\kappa^>}\ln\left[\frac{(R - r)v_*^>}{\nu}\right] + \lambda^> = \frac{1}{\kappa^>}\ln\left(\frac{yv_*^>}{\nu}\right) + \lambda^>$$

$y$ being defined as $y = r - aR$ in Region < and $y = R - r$ in Region >.

*b*. The continuity of the velocity profile at $r = bR$ gives a relation between $\lambda^<$ and $\lambda^>$:

$$\frac{1}{\kappa^<}\ln\left[\left(\frac{Rv_{**}}{\nu}\right)(b-a)\sqrt{\frac{b^2-a^2}{a}}\right]+\lambda^< = \frac{1}{\kappa^>}\ln\left[\left(\frac{Rv_{**}}{\nu}\right)(1-b)\sqrt{1-b^2}\right]+\lambda^>$$

*c*. The mass rate of flow through the annulus is

$$w = 2\pi\rho\int_{aR}^{bR}\overline{v}_z^< r dr + 2\pi\rho\int_{bR}^{R}\overline{v}_z^> r dr$$

The first integral is:

$$\begin{aligned}
&2\pi\rho v_*^<\int_{aR}^{bR}\left[\frac{1}{\kappa^<}\ln\left(\frac{yv_*^<}{\nu}\right)+\lambda^<\right]r dr \\
&\quad = 2\pi\rho v_*^<\int_0^{(b-a)R}\left[\frac{1}{\kappa^<}\ln\left(\frac{yv_*^<}{\nu}\right)+\lambda^<\right](y+aR)dy \\
&\quad = 2\pi\rho v_*^<\left(\frac{\nu}{v_*^<}\right)^2\int_0^{(b-a)(Rv_*^</\nu)}\left(\frac{1}{\kappa^<}\ln x+\lambda^<\right)\left(x+\frac{aRv_*^<}{\nu}\right)dx \\
&\quad = 2\pi\rho v_*^<\left(\frac{\nu}{v_*^<}\right)^2\left[\frac{1}{\kappa^<}\left(\frac{x^2}{2}\ln x-\frac{x^2}{4}\right)+\frac{1}{\kappa^<}\left(\frac{aRv_*^<}{\nu}\right)(x\ln x-x)\right. \\
&\qquad \left.+\frac{1}{2}\lambda^< x^2+\lambda^<\left(\frac{aRv_*^<}{\nu}\right)x\right]_0^{(b-a)(Rv_*^</\nu)} \\
&\quad = \pi R^2\rho v_*^<\left[\frac{1}{\kappa^<}\left[(b-a)^2+2a(b-a)\right]\ln(b-a)\left(\frac{Rv_*^<}{\nu}\right)\right. \\
&\qquad \left.+\lambda^<\left[(b-a)^2+2a(b-a)\right]-\frac{1}{\kappa^<}\left[\frac{1}{2}(b-a)^2+2a(b-a)\right]\right] \\
&\quad = \pi R^2\rho v_*^<\left(b^2-a^2\right)\left[\frac{1}{\kappa^<}\ln(b-a)\left(\frac{Rv_*^<}{\nu}\right)+\lambda^<-\frac{1}{2\kappa^<}-\frac{1}{\kappa^<}\frac{a}{b+a}\right]
\end{aligned}$$

The second integral in the expression for the mass flow rate is then:

$$2\pi\rho v_*^>\int_{bR}^{R}\left[\frac{1}{\kappa^>}\ln\left(\frac{yv_*^>}{\nu}\right)+\lambda^>\right]r dr$$

$$= 2\pi\rho v_*^> \int_0^{R(1-b)} \left[ \frac{1}{\kappa^>} \ln\left( \frac{y v_*^>}{\nu} \right) + \lambda^> \right] (R-y)\,dy$$

$$= 2\pi\rho v_*^> \left( \frac{\nu}{v_*^>} \right)^2 \int_0^{(1-b)(Rv_*^>/\nu)} \left( \frac{1}{\kappa^>} \ln x + \lambda^> \right) \left( \frac{R v_*^>}{\nu} - x \right) dx$$

$$= 2\pi\rho v_*^> \left( \frac{\nu}{v_*^>} \right)^2 \left[ \frac{1}{\kappa^>} \frac{R v_*^>}{\nu} (x \ln x - x) - \frac{1}{\kappa^>} \left( \frac{x^2}{2} \ln x - \frac{x^2}{4} \right) \right.$$

$$\left. + \lambda^> \left( \frac{R v_*^>}{\nu} \right) x - \lambda^> \frac{1}{2} x^2 \right]_0^{(1-b)(Rv_*^>/\nu)}$$

$$= \pi R^2 \rho v_*^> \left[ \frac{2}{\kappa^>} \left[ (1-b)\ln(1-b) \left( \frac{R v_*^>}{\nu} \right) - (1-b) \right] \right.$$

$$- \frac{1}{\kappa^>} \left[ (1-b)^2 \ln(1-b) \left( \frac{R v_*^>}{\nu} \right) - \frac{1}{2}(1-b) \right]$$

$$\left. + 2\lambda^> (1-b) - \lambda^> (1-b)^2 \right]$$

$$= \pi R^2 \rho v_*^> \left(1-b^2\right) \left[ \frac{1}{\kappa^>} \ln(1-b) \left( \frac{R v_*^>}{\nu} \right) + \lambda^> - \frac{1}{\kappa^>} \left( \frac{1}{2} + \frac{1}{1+b} \right) \right]$$

We now combine the two integrals and use the result in (*b*) to eliminate $\lambda^<$ in favor of $\lambda^>$; furthermore, we use the result in (*a*) to introduce $v_{**}$. This leads to:

$$w = \pi R^2 \rho v_{**} \left\{ A \left[ \frac{1}{\kappa^>} \ln\left( \frac{R v_{**}}{\nu} (1-b)\sqrt{1-b^2} \right) + \lambda^> \right] - B \right\}$$

in which

$$A = a^{-1/2} \left(b^2 - a^2\right)^{3/2} + \left(1-b^2\right)^{3/2}$$

$$B = a^{-1/2} \left(b^2 - a^2\right)^{3/2} \frac{1}{\kappa^<} \left( \frac{1}{2} + \frac{a}{b+a} \right) + \left(1-b^2\right)^{3/2} \frac{1}{\kappa^>} \left( \frac{1}{2} + \frac{1}{1+b} \right)$$

The expression for $A$ does not agree with that of Meter and Bird, and we conclude that their expression is incorrect.

**5C.3 Instability in a simple mechanical system**

*a*. The centrifugal force acting on the mass is $m\Omega^2 r = m\Omega^2 L\sin\theta$. The gravitational force acting downward is $mg$. These forces must have a resultant in the direction of the rod, and therefore

$$\tan\theta = \frac{m\Omega^2 L\sin\theta}{mg} \qquad \text{or} \qquad \cos\theta = \frac{g}{\Omega^2 L}$$

If the angular velocity goes to zero, it would appear that $\cos\theta$ would go to infinity!

*b*. However, this formula describes the relation among the various quantities when the system is rotating and $\theta \geq 0$. When as $\Omega$ decreases, the right side of the equation attains a value of 1 when $\Omega = \Omega_{\text{thr}}$, and then $\cos\theta = 1$ and $\theta$ is zero. For for value of $\Omega$ less than its "threshold value", the value of $\theta$ remains at 0.

When one starts up the system from rest, $\theta$ will always be zero. However, if $\Omega > \Omega_{\text{thr}}$ and there is any disturbance on the system, then the system will move up to the stable curve (given by the equation in part (*a*)). It has to be understood that the graph we have given is only for the steady state, and that to understand the system fully, it is necessary to examine the full unsteady-state equation.

*c*. According to p. 12 of L. D. Landau and E. M. Lifshitz, *Mechanics* (Pergamon, 1990) the Lagrangian for this problem is

$$L = \tfrac{1}{2} mL^2(\dot{\theta}^2 + \Omega^2 \sin^2\theta) + mgL\cos\theta$$

Then Lagrange's equation of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

gives for the system we are considering the following equation of motion

$$mL\frac{d^2\theta}{dt^2} = m\Omega^2 L\sin\theta\cos\theta - mg\sin\theta$$

*d*. Consider the lower (unstable) branch in the diagram. For very small perturbations ($\theta_1$) to the steady state ($\theta_0$), we have then $\sin\theta = \sin\theta_1 \approx \theta_1$ and $\cos\theta \approx 1$. Then the equation of motion becomes

$$\frac{d^2\theta_1}{dt^2} = \left(\Omega^2 - \frac{g}{L}\right)\theta_1$$

Now we try a small perturbation of the form $\theta_1 = A\Re\{e^{-i\omega t}\}$ When we substitute this function into the differential equation we get $\omega_\pm = \pm i\sqrt{\Omega^2 - (g/L)}$. If $\Omega^2 < g/L$, then both roots $\omega_\pm$ are real and $\theta_1$ oscillates. If $\Omega^2 > g/L$, $\omega_+$ is positive imaginary and $e^{-i\omega t}$ will increase indefinitely with time. Hence the branch $\theta = 0$ is unstable with respect to infinitestimal disturbances.

*e*. Next we consider the upper branch for which $\cos\theta_0 = g/\Omega^2 L$ and $\sin\theta_0 = \sqrt{1-(g/\Omega^2 L)^2}$. Then the equation of motion becomes

$$\begin{aligned} mL\frac{d^2\theta_1}{dt^2} &= m\Omega^2 L(\sin\theta_0 + \theta_1\cos\theta_0)(\cos\theta_0 - \theta_1\sin\theta_0) \\ &\qquad - mg(\sin\theta_0 + \theta_1\cos\theta_0) \\ &\approx -m\Omega^2 L\theta_1\sin^2\theta_0 \end{aligned}$$

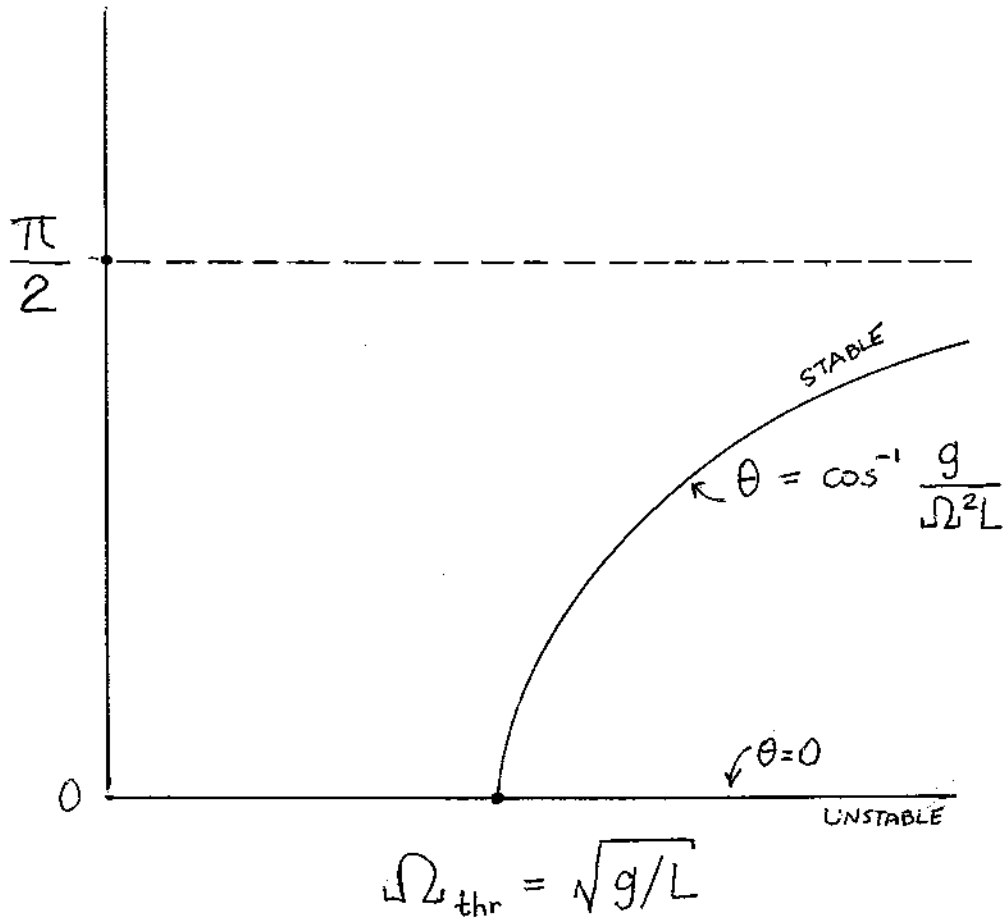
where we have neglected terms quadratic in $\theta_1$. Hence the equation of motion becomes

$$\frac{d^2\theta_1}{dt^2} + \frac{1}{\Omega^2}\left(\Omega^2 + (g/L)\right)\left(\Omega^2 - (g/L)\right)\theta_1$$

We now try a solution of the form $\theta_1 = A\Re\{e^{-i\omega t}\}$ and get

$$\omega_\pm = \pm\frac{1}{\Omega}\sqrt{\left(\Omega^2 + (g/L)\right)\left(\Omega^2 - (g/L)\right)}$$

For the upper branch, $\Omega^2 > (g/L)$, and hence both quantities $\omega_\pm$ are real. Hence the system is stable to small perturbations.


$\frac{\pi}{2}$
STABLE
$\theta = \cos^{-1} \frac{g}{\Omega^2 L}$
$\theta = 0$
0
UNSTABLE
$\Omega_{thr} = \sqrt{g/L}$

**5D.1 Derivation of the equations of change for the Reynolds stress**

Multiplication of the $i$th component of Eq. 5.25 by $v_j'$ and time-smoothing gives for constant $\rho$ gives (in the Cartesian tensor notation of §A.9, with the Einstein summation convention and with the shorthand notation $\partial_t \equiv \partial/\partial t$):

$$\underset{(1)}{\rho\overline{v_i'\partial_t v_j'}} = \underset{(2)}{-\overline{v_i'\partial_j p'}} \underset{(3)}{-\rho\left(\overline{v_i'\partial_k \overline{v}_k v_j'}\right.} \underset{(4)}{+\overline{v_i'\partial_k v_k' \overline{v}_j}} \underset{(5)}{\left.+\overline{v_i'\partial_k v_k' v_j'}\right)} \underset{(6)}{+\mu\overline{v_i'\partial_k\partial_k v_j'}}$$

Then we write the same equation with $i$ and $j$ interchanged. When the two equations are added we get, term by term:

(1) $$\rho\left(\overline{v_i'\partial_t v_j'} + \overline{v_j'\partial_t v_i'}\right) = \rho\partial_t\left(\overline{v_i'v_j'} + \overline{v_j'v_i'}\right) - \rho\left(\overline{v_j'\partial_t v_i'} + \overline{v_i'\partial_t v_j'}\right)$$

The second term on the right side is the negative of the left side. The first term on the right side can also be written as $2\rho\partial_t\overline{v_i'v_j'}$. Therefore the left side is just $\rho\partial_t\overline{v_i'v_j'}$.

(2) $$-\left(\overline{v_i'\partial_j p'} + \overline{v_j'\partial_i p'}\right)$$

(3) $$-\rho\left(\overline{v_i'\partial_k\overline{v}_k v_j'} + \overline{v_j'\partial_k\overline{v}_k v_i'}\right) = -\rho\left(\overline{v_i'\overline{v}_k\partial_k v_j'} + \overline{v_j'\overline{v}_k\partial_k v_i'}\right) \text{ (used 5.2-10)}$$
$$= -\rho\left(\overline{\overline{v}_k\partial_k v_j' v_i'}\right)$$

(4) $$-\rho\left(\overline{v_i'\partial_k v_k'\overline{v}_j} + \overline{v_j'\partial_k v_k'\overline{v}_i}\right) = -\rho\left(\overline{v_i'v_k'\partial_k\overline{v}_j} + \overline{v_j'v_k'\partial_k\overline{v}_i}\right) \text{ (used 5.2-11)}$$

(5) $$-\rho\left(\overline{v_i'\partial_k v_k' v_j'} + \overline{v_j'\partial_k v_k' v_i'}\right)$$
$$= -\rho\left(\overline{\partial_k v_k' v_j' v_i'} - \overline{v_j' v_k'\partial_k v_i'}\right) - \rho\left(\overline{v_i' v_j'\partial_k v_k'} + \overline{v_j' v_k'\partial_k v_i'}\right)$$

Note that the third term is zero by Eq. 5.2-11, and that the second and fourth terms cancel giving $-\rho\left(\overline{\partial_k v_k' v_j' v_i'}\right)$

(6) $$+\mu\left(\overline{v_i'\partial_k\partial_k v_j'} + \overline{v_j'\partial_k\partial_k v_i'}\right)$$

Combining the above gives:

$$\rho\partial_t \overline{v_i'v_j'} = -\left(\overline{v_i'\partial_j p' + v_j'\partial_i p'}\right) - \rho\left(\overline{\overline{v}_k\partial_k v_j'v_i'}\right) - \rho\left(\overline{v_i'v_k'\partial_k\overline{v}_j} + \overline{v_j'v_k'\partial_k\overline{v}_i}\right)$$
$$-\rho\left(\overline{\partial_k v_k'v_j'v_i'}\right) + \mu\left(\overline{v_i'\partial_k\partial_k v_j'} + \overline{v_j'\partial_k\partial_k v_i'}\right)$$

or

$$\rho\partial_t \overline{v_i'v_j'} + \rho\left(\overline{\overline{v}_k\partial_k v_j'v_i'}\right) = -\rho\left(\overline{v_i'v_k'\partial_k\overline{v}_j} + \overline{v_j'v_k'\partial_k\overline{v}_i}\right) - \rho\left(\overline{\partial_k v_k'v_j'v_i'}\right)$$
$$-\left(\overline{v_i'\partial_j p' + v_j'\partial_i p'}\right) + \mu\left(\overline{v_i'\partial_k\partial_k v_j'} + \overline{v_j'\partial_k\partial_k v_i'}\right)$$

The two terms on the left side are the substantial derivative term in Eq. 5D.1-1. The remainder of the terms are set out in the same order as in Eq. 5D.1-1.

## 5D.2 Kinetic energy of turbulence

Taking the trace of Eq. 5D.1-1, we get

$$\rho\frac{D}{Dt}\overline{(\mathbf{v}'\cdot\mathbf{v}')} = -2\rho\left(\overline{\mathbf{v}'\mathbf{v}'}:\nabla\overline{\mathbf{v}}\right) - \rho\left(\nabla\cdot\overline{(\mathbf{v}'\cdot\mathbf{v}')\mathbf{v}'}\right)$$
$$-2\overline{\left(\mathbf{v}'\cdot\nabla p'\right)} + 2\mu\overline{\left(\mathbf{v}'\cdot\nabla^2\mathbf{v}'\right)}$$

Modify the third term on the right as follows:

$$-2\overline{\left(\mathbf{v}'\cdot\nabla p'\right)} = -2\overline{\left(\nabla\cdot p'\mathbf{v}'\right)} + 2\overline{\left(p'(\nabla\cdot\mathbf{v}')\right)}$$

The last term on the right is zero according to Eq. 5.2-11. Then divide the first equation above by 2 to get

$$\frac{D}{Dt}\overline{\tfrac{1}{2}\rho v'^2} = -\rho\left(\overline{\mathbf{v}'\mathbf{v}'}:\nabla\overline{\mathbf{v}}\right) - \left(\nabla\cdot\overline{\left(\tfrac{1}{2}\rho v'^2\right)\mathbf{v}'}\right)$$
$$-\overline{\left(\nabla\cdot p'\mathbf{v}'\right)} + \mu\overline{\left(\mathbf{v}'\cdot\nabla^2\mathbf{v}'\right)}$$

which is Eq. 5D.2-1.

For an interpretation of this equation, see pp. 63 et seq. in the book by Tennekes and Lumley cited on p. 176.

**6A.1 Pressure drop required for a pipe with fittings.**

The average velocity at the given conditions is

$$\langle v \rangle = \frac{4(w/\rho)}{\pi D^2} = \frac{(4)(197\ \mathrm{m^3/s})}{(\pi)(0.25\ \mathrm{m})^2} = 40.1\ \mathrm{m/s}$$

and the Reynolds number is

$$\frac{D\langle v \rangle}{\nu} = (0.25\ \mathrm{m}(40.1\ \mathrm{m/s})/(1.0037 \times 10^{-6}\ \mathrm{m^2/s}) = 9.99 \times 10^6$$

Thus, the flow is turbulent, and Fig. 6.2-1 gives f=0.0020 for hydraulically smooth pipe. The total equivalent length of the pipe and fittings is

$$\begin{aligned} L_e &= 1234\ \mathrm{m\ of\ pipe} \\ &\quad + (4)(32)(0.25)\ \mathrm{m\ equivalent\ for\ 4\ 90^\circ\ elbows} \\ &\quad + (2)(15)(0,25)\ \mathrm{m\ equivalent\ for\ 2\ 90^\circ\ elbows} \\ &\quad = 1274.5\ \mathrm{m} \end{aligned}$$

The required pressure drop, according to Eq. 6.1-4 with $L$ replaced by $L_e$, is then

$$\begin{aligned} (p_0 - p_L) &= 2\frac{L_e}{D} \cdot \rho \langle v \rangle^2 f \\ &= 2\frac{1273.5\ \mathrm{m}}{0.25\ \mathrm{m}}(998\ \mathrm{kg/m^3})(40.1\ \mathrm{m/s})^2(0.0020) \\ &= 3.3 \times 10^7\ \mathrm{Pa} = 4.7 \times 10^3\ \mathrm{psi} \end{aligned}$$

**6A.2 Pressure difference required for flow in pipe with elevation change.**

In this problem the pipe diameter is (3.068 in.)/(39.37 in./m) = 0.07793 m, the mass flow rate is

$$w = (18/60 \text{ gal/s})(3.7853 \text{ lit/gal})(0.9982 \text{ kg/lit}) = 1.13 \text{ kg/s},$$

the average velocity is

$$\langle v \rangle = \frac{4w}{\pi D^2 \rho} = \frac{(4)(1.13 \text{ kg/s})}{\pi(0.07793 \text{ m})^2(998.2 \text{ kg/m}^3)} = 0.237 \text{ m/s},$$

and the Reynolds number is

$$\text{Re} = \frac{D\langle v \rangle \rho}{\mu} = \frac{(0.07793 \text{ m})(0.237 \text{ m/s})(998.2 \text{ kg/m}^3)}{(1.002 \times 10^{-3} \text{ kg/m·s})} = 1.84 \times 10^4$$

From Fig. 6.2-2 we find that for this Re value, $f = 0.0066$ for smooth tubes. Hence,

$$\begin{aligned} p_0 - p_L &= -\rho g(h_0 - h_L) + 2\frac{L_e}{D} \cdot \rho \langle v \rangle^2 f \\ &= -(998.2 \text{ kg/m}^3)(9.807 \text{ m/s}^2)((-50 \times 0.3048/\sqrt{2}) \text{ m}) \\ &\quad + 2[(50 \times 12/3.068) + (2 \times 15)] \cdot (998.2 \text{ kg/m}^3)(0.237 \text{ m/s})^2(0.0066) \\ &= 1.055 \times 10^5 \text{ Pa} + 167 \text{ Pa} \\ &= 1.055 \times 10^5 \text{ Pa} = 15.3 \text{ psi} \end{aligned}$$

The corresponding calculation in terms of $\text{lb}_m$, ft and s, neglecting the small friction term, is

$$\begin{aligned} p_0 - p_L &= (62.4 \times 0.9982 \text{ lb}_m/\text{ft}^3)(32.174 \text{ ft/s}^2)((50/\sqrt{2}) \text{ ft}) \\ &= 7.09 \times 10^4 \text{ lb}_m/\text{ft/s}^2 = 15.3 \text{ psia} \end{aligned}$$

**6A.3 Flow rate for a given pressure drop.**

The quantities needed for this calculation are as follows, in units of $\text{lb}_m$, ft, and s:

$$\begin{aligned} p_0 - p_L &= (0.25\ \text{lb}_f/\text{in}^2)(144\ \text{in}^2/\text{ft}^2)(32.174\ \text{lb}_m\text{ft/s}^2\cdot\text{lb}_f) \\ &= 1.16 \times 10^3\ \text{lb}_m/\text{ft}\cdot\text{s}^2; \\ D &= 0.5\ \text{ft}; \qquad \rho = 62.4\ \text{lb}_m/\text{ft}^2; \\ L &= 1320\ \text{ft}; \qquad \mu = 6.73 \times 10^{-4}\ \text{lb}_m/\text{ft}\cdot\text{s} \end{aligned}$$

Hence, the solution must lie on the locus

$$\begin{aligned} \text{Re}\sqrt{f} &= \frac{D\rho}{\mu}\sqrt{\frac{(p_0 - p_L)D}{2L\rho}} \\ &= \frac{(0.5)(62.4)}{6.73 \times 10^{-4}}\sqrt{\frac{(1.15 \times 10^{-3})(0.5)}{2(1320)(62.4)}} \\ &= (4.64 \times 10^4)\sqrt{3.52 \times 10^{-3}} = 2.74 \times 10^3 \end{aligned}$$

Method B: The last equation gives a straight line on the logarithmic plot of $f$ vs. Re, passing through $f = 1$ at Re$= 2.74 \times 10^3$ and through $f = 0.01$ at Re$= 2.74 \times 10^4$, and intersecting the $f$ curve for smooth tubes at Re$= 3.6 \times 10^4$. Hence, the average velocity is

$$\langle v \rangle = \frac{\text{Re}}{(D\rho/\mu)} = \frac{3.6 \times 10^4}{4.64 \times 10^4} = 0.78\ \text{ft/s}$$

and the volume rate of flow is

$$\begin{aligned} Q &= \frac{\pi D^2}{4}\langle v \rangle \\ &= \frac{(3.1416)(0.5^2)}{4}(0.78) = 0.152\ \text{ft}^3/\text{s} \\ &= 68\ \text{U.S. gal/hr} \end{aligned}$$

Method A gives the same result if the plot of $f$ vs. Re$\sqrt{f}$ is accurately drawn.

**6A.4 Motion of a sphere in a liquid.**

The force of gravity on the sphere is

$$F_{\text{grav}} = mg = (0.0500 \text{ g})(980.665 \text{ cm/s}^2) = 49.0_3 \text{ dynes}$$

The buoyant force of the fluid on the sphere is

$$\begin{aligned} F_{\text{buoy}} &= (4/3)\pi R^3 \rho g \\ &= (4/3)\pi(0.25 \text{ cm})^3(0.900 \text{ g/cm}^3)(980.665 \text{ cm/s}^2) = 57.7_7 \text{ dynes} \end{aligned}$$

*a.* The resultant upward force is

$$F_{\text{buoy}} - F_{\text{grav}} = 57.7_7 - 49.0_3 = 8.7_4 \text{ dynes}$$

and is balanced, at steady state, by an equal and opposite drag force $F_k = 8.7_4$ dynes.

*b.* The friction factor is defined by

$$F_k = (\pi R^2)(\tfrac{1}{2})\rho v_\infty^2)f$$

Thus, for this system,

$$\begin{aligned} f &= \frac{F_k}{\left(\frac{\pi D^2}{4}\right)\left(\frac{1}{2}\rho v_\infty^2\right)} = \frac{8F_k}{\pi D^2 \rho v_\infty^2} \\ &= \frac{8(8.7_4 \text{ dynes})}{\pi(0.500 \text{ cm})^2(0.900 \text{ g/cm}^3)(0.500 \text{ cm/s})^2} \\ &= 3.9_6 \times 10^2 \end{aligned}$$

*c.* From Fig. 6.3-1 we see that $f$ is very close to its creeping-flow asymptote, 24/Re. To the same approximation,

$$\frac{Dv_\infty \rho}{\mu} \approx 24/f = 0.061$$

Hence,

$$\begin{aligned} \mu &= \frac{Dv_\infty \rho f}{24} = \frac{(0.5 \text{ cm})(0.5 \text{ cm/s})(0.9\text{g/cm}^3)(3.9_6 \times 10^2)}{24} \\ &= 3.7 \text{ g/cm·s} = 3.7 \times 10^2 \text{ cp} \end{aligned}$$

**6A.5 Sphere diameter for a given terminal velocity.**

a. Method A: Replot the $f$-curve of Fig. 6.3-1 as $f/\mathrm{Re}$ (which does not contain $D$) vs. Re. Then from this curve we can find the value of Re for any calculated value of $f/\mathrm{Re}$, and determine $D$ as $\mathrm{Re}\mu/\rho v_\infty >$.

*a.* Method B: On the log-log plot of $f = f(\mathrm{Re})$, plot also the locus $f = (f/\mathrm{Re})\mathrm{Re}$, which will be a line of slope 1, and find the desired Re at the intersection of the two loci. This method avoids any need to prepare an auxiliary plot.

*b.* The data of Problem 2A.4 give

$$\begin{aligned}
v_\infty &= (1\ \mathrm{ft/s})(12 \times 2.54\ \mathrm{cm/ft}) = 30.48\ \mathrm{cm/s} \\
\rho &= 0.045\ \mathrm{lb}_m/\mathrm{ft}^3)(453.59\ \mathrm{g/lb}_m)(12 \times 2.54\ \mathrm{cm/ft})^{-3} \\
&= 7.2 \times 10^{-4}\ \mathrm{g/cm}^{-3} \\
\rho_s &= 1.2\ \mathrm{g/cm}^3 \\
\mu &= 2.6 \times 10^{-4}\ \mathrm{g/cm{\cdot}s} \\
g &= 980.7\ \mathrm{cm/s}^2
\end{aligned}$$

from which we calculate

$$\begin{aligned}
f/\ \mathrm{Re} &= \frac{4}{3}\frac{g\mu}{\rho v_\infty^2}\left(\frac{\rho_s - \rho}{\rho}\right) \\
&= \frac{4}{3}\frac{(980.7)(0.00026)}{1.2 v_\infty^3}\left(\frac{1.2 - 7.2 \times 10^{-4}}{7.2 \times 10^{-4}}\right) \\
&= 27.8
\end{aligned}$$

We therefore draw a line of slope 1 through $f = 27.8$, Re= 1 on Fig. 6.3-1. This line intersects the $f$ vs. Re curve at Re= 0.95. The particle diameter is then calculated as

$$D = \frac{\mathrm{Re}\mu}{v_\infty \rho} = \frac{(0.95)(2.6 \times 10^{-4})}{(30.48)(7.2 \times 10^{-4})} = (0.0112\ \mathrm{cm})(10^4\ \mathrm{microns/cm}) = 112\ \mathrm{microns}$$

*c.* Here $v_\infty$ is 10 times larger, giving $f/\mathrm{Re} = 27.8 \times 10^{-3}$. This locus intersects the $f(\mathrm{Re})$ curve at Re= 75. Hence, at this gas velocity the diameter of the largest particle that can be lost is

$$D = \frac{(75)(2.6 \times 10^{-4})}{(30.48)(7.2 \times 10^{-4}} = (0.89\ \mathrm{cm})(10^4\ \mathrm{microns/cm}) = 0.89 \times 10^4\ \mathrm{microns}$$

**6A.6 Estimation of void fraction of a packed column.**

The superficial velocity is

$$v_0 = \frac{(244 \text{ lb/min})(1 \text{ min}/60 \text{ s})(453.59 \text{ g/lb})}{(1.2865 \text{ g/cm}^3)(146 \text{ in}^2)(2.54 \text{ cm/in})^2} = 1.522 \text{ cm/s}$$

According to Eq. 6.4-9 (the Blake-Kozeny equation, developed for laminar flows),

$$\begin{aligned}\frac{\varepsilon^3}{(1-\varepsilon)^2} &= \frac{150\mu L v_0}{D_p^2 \Delta p} \\ &= \frac{150(0.565 \text{ g/cm·s})(73 \times 2.54 \text{ cm})(1.522 \text{ cm/s})}{(0.2 \text{ cm})^2(158 \text{ psi})(68947 \text{ dynes/cm}^2\text{·psi})} \\ &= 0.0549\end{aligned}$$

Solving this equation for $\varepsilon$, we find $\varepsilon = 0.30$; hence,

$$\frac{D_p v_0 \rho}{\mu} \cdot \frac{1}{1-\varepsilon} = \frac{(0.2 \text{ cm})(1.522 \text{ cm/s})(1.287 \text{ g/cm}^3)}{(0.565 \text{ g/cm·s})} \frac{1}{(1-0.30)} = 1.00$$

which indicates (see Fig. 6.4-2) that it was appropriate to use the Blake-Kozeny equation here.

**6A.7 Estimation of pressure drops in annular flow.**

*a.* The definition of $f$ in Eq. 6.1-1 gives

$$f = \frac{F_k}{2\pi RL(1+\kappa)\rho\langle\overline{v}_z\rangle^2/2} = \frac{\pi R^2(1-\kappa^2)(\mathcal{P}_0-\mathcal{P}_L)}{2\pi RL(1+\kappa)\rho\langle\overline{v}_z\rangle^2/2} = \frac{R(1-\kappa)(\mathcal{P}_0-\mathcal{P}_L)}{L\rho\langle\overline{v}_z\rangle^2}$$

Equations 6A.7-1 and 3 give

$$K = \frac{16}{f}\frac{\mu}{D(1-\kappa)\langle\overline{v}_z\rangle\rho}$$

Insertion of the previous result for $f$ gives

$$K = \frac{16L\rho\langle\overline{v}_z\rangle^2}{R(1-\kappa)(\mathcal{P}_0-\mathcal{P}_L)}\cdot\frac{\mu}{D(1-\kappa)\langle\overline{v}_z\rangle\rho} = \frac{8L\mu\langle\overline{v}_z\rangle}{R^2(1-\kappa)^2(\mathcal{P}_0-\mathcal{P}_L)}$$

Equation 2.4-16 gives

$$\frac{\langle\overline{v}_z\rangle}{(\mathcal{P}_0-\mathcal{P}_L)} = \frac{R^2}{8\mu L}\left[\frac{1-\kappa^4}{1-\kappa^2} - \frac{1-\kappa^2}{\ln(1/\kappa)}\right]$$

Combining the last two results, we get the relation

$$K = \frac{1}{1-\kappa^2}\left[\frac{1-\kappa^4}{1-\kappa^2} - \frac{1-\kappa^2}{\ln(1/\kappa)}\right]$$

needed to make Eqs. 6A.7-1 and 3 consistent with Eq. 2.4-16 for laminar flow. The $K$ values given by this formula are in excellent agreement with those tabulated on page 194 and recommended for turbulent flow as well.

*b.* The data for this operation are

$$\mu = 1.139 \text{ cp}(=\text{mPa·s}) = 7.66\times10^{-4} \text{ lb}_m/\text{ft·s}; \qquad \rho = 62.2 \text{ lb}_m/\text{ft}^3;$$
$$D = 15 \text{ in} = 1.25 \text{ ft}; \qquad \kappa = 6 \text{ in}/15 \text{ in} = 0.4; \qquad G = 3.801;$$
$$H = 0.131; \qquad K = 0.6759; \qquad w/\rho = 1500 \text{ ft}^3/\text{s};$$
$$\langle\overline{v}_z\rangle = \frac{4(w/\rho)}{\pi[D^2-(\kappa D)^2]} = \frac{4(1600 \text{ ft}^3/\text{s})}{\pi[1.25^2-0.5^2 \text{ ft}^2]} = 388 \text{ ft/s};$$
$$\text{Re}_\kappa = K\frac{D(1-\kappa)\langle\overline{v}_z\rangle\rho}{\mu}$$
$$= 0.6759\frac{(1.25 \text{ ft})(1-0.4)(388 \text{ ft/s})(62.3 \text{ lb}_m/\text{ft}^3)}{(7.66\times10^{-4})} = 1.6\times10^7$$

Equation 6A.7-2 then gives:

$$\frac{1}{\sqrt{f}} = 3.801\log_{10}(1.6\times10^7\sqrt{f}) - 0.131$$

Solving by iteration, we get $f = 0.00204$. The longitudinal pressure gradient in a horizontal flow is then

$$\begin{aligned}\frac{(p_0 - p_L)}{L} &= \frac{2f\rho\langle\overline{v}_z\rangle^2}{D(1-\kappa)} \\ &= \frac{2(0.00204)(62.3\ \mathrm{lb}_m/\mathrm{ft}^3)(388\ \mathrm{ft/s})^2}{(1.25\ \mathrm{ft})(1-0.4)} \\ &= 5.06 \times 10^4\ \mathrm{poundals/ft}^3 = 10.9\ \mathrm{psi/ft}\end{aligned}$$

*c.* The mean hydraulic radius for this system is

$$R_h = \frac{S}{Z} = \frac{\pi D^2(1-\kappa^2)/4}{\pi D(1-\kappa)} = \frac{D(1-\kappa))}{4}$$

and the correspnding Reynolds number is

$$\mathrm{Re}_h = \frac{4R_h\langle\overline{v}_z\rangle\rho}{\mu} = \frac{D(1-\kappa)\langle\overline{v}_z\rangle\rho}{\mu} = \mathrm{Re}/K = 2.37 \times 10^7$$

Equation 6.2-15 then gives $f = 0.00181$, which is 0.885 times the value found in part (b). The predicted pressure gradient is reduced correspondingly, giving

$$\frac{(p_0 - p_L)}{L} = 4.48\ \mathrm{poundals/ft}^3 = 9.7\ \mathrm{psi/ft}$$

**6A.8 Force on a water tower in a gale.**

The data for this problem are:

$D = 40$ ft for spherical tank; $\quad v_\infty = (100 \text{ mi/hr})(5280/3600) = 147$ ft/s;

$\rho = 0.08 \text{ lb}_m/\text{ft}^3$; $\quad \mu = 0.017 \text{ cp} = 1.14 \times 10^{-5} \text{ lb}_m/\text{ft·s}$

The Reynolds number for the tank is

$$\text{Re} = \frac{Dv_\infty \rho}{\mu} = \frac{(40 \text{ ft})(147 \text{ ft/s})(0.08 \text{ lb}_m/\text{ft}^3)}{(1.14 \times 10^{-5} \text{ lb}_m/\text{ft·s})} = 4.1 \times 10^7$$

The friction factor is approximated as 0.5 by extrapolation of Fig. 6.3-1.

The horizontal force of the wind on the tank is then

$$\begin{aligned} F_k &= (\frac{\pi D^2}{4})(\frac{1}{2}\rho v_\infty^2)f \\ &= \left(\frac{\pi(40 \text{ ft})^2}{4}\right)\left(\frac{1}{2}(0.08 \text{ lb}_m/\text{ft}^2)(147 \text{ ft/s})^2\right)(0.5) \\ &= 5.4 \times 10^5 \text{ poundals} = 1.7 \times 10^4 \text{ lb}_f = 7.5 \times 10^4 \text{ Newtons} \end{aligned}$$

**6A.9 Flow of gas through a packed column.**

For this compressible-flow problem, we write Eq. 6.4-12 in differential form with $v_0 = G_0/\rho$:

$$-\frac{dp}{dz} = 150\frac{\mu G_0}{\rho D_p^2}\frac{(1-\varepsilon)^2}{\varepsilon^3} + \frac{7}{4}\frac{G_0^2}{\rho D_p}\frac{(1-\varepsilon)}{\varepsilon^3}$$

Inserting the ideal gas formula

$$\rho = \frac{pM}{RT}$$

and integrating from $z = 0$ to $z = L$, we get the following implicit expression for the superficial mass flux $G_0$:

$$\left[150\frac{\mu G_0}{D_p^2}\frac{(1-\varepsilon)^2}{\varepsilon^3} + \frac{7}{4}\frac{G_0^2}{D_p}\frac{(1-\varepsilon)}{\varepsilon^3}\right]L = \frac{M}{2RT}(p_0^2 - p_L^2)$$

The terms are then calculated for this system in cgs units:

$$150\frac{\mu G_0}{D_p^2}\frac{(1-\varepsilon)^2}{\varepsilon^3}L$$

$$= 150\frac{(1.495\times10^{-4}\ \text{g/cm·s})(G_0\ \text{g/cm}^2\text{·s})}{(2.54/16\ \text{cm})^2}\frac{(1-0.41)^2}{(0.41)^3}(5.5\times30.48\ \text{cm})$$

$$= (753.4G_0)\ \text{g}^2/\text{cm}^4\text{·s}^2$$

$$\frac{7}{4}\frac{G_0^2}{D_p}\frac{(1-\varepsilon)}{\varepsilon^3}L$$

$$= \frac{7}{4}\frac{(G_0\ \text{g/cm}^2\text{·s})^2}{(2.54/16\ \text{cm}}\frac{(1-0.41)}{0.41^3}(5.5\times30.48\ \text{cm})$$

$$= (15820G_0^2)\ \text{g}^2/\text{cm}^4\text{·s}^2$$

$$\frac{M}{2RT}(p_0^2 - p_L^2) = \frac{44.01\ \text{g/g-mol}}{(8.31451\times10^7\times300\ \text{g·cm}^2/\text{s}^2\text{·g-mol})}$$

$$\times[(25\times1.0133\times10^6)^2 - (3\times1.0133\times10^6)^2\ \text{g}^2/\text{cm}^2\text{·s}^4]$$

$$= (1.116\times10^6)\ \text{g}^2/\text{cm}^4\text{·s}^4$$

Combining these results, we get the following quadratic equation for $G_0$,

$$15820G_0^2 + 753.4G_0 = 1.116\times10^6$$

which has the roots

$$G_0 = \frac{-753.46 \pm \sqrt{753.46^2 + 4(15820)(1.116\times10^6)}}{2\times15820}$$

$$= \frac{-753.46 \pm 265746.2}{2\times15820}$$

The positive root is 8.375 g/cm$^2$/s. the mass flow rate is then

$$w = (\pi/4)D^2G_0 = (0.7854)(4\times2.54\ \text{cm})^2(8.375\ \text{g/cm}^2/\text{s} = 679\ \text{g/s}$$

An identical result for $w$ is obtained if $\rho$ is assumed constant at the value $\overline{\rho} = (p_0 + p_L)M/RT$.

**6B.1 Effect of error in friction factor calculations**

The Blasius formula for the friction factor for turbulent flow in circular tubes is Eq. 6.2-12:

$$f = \frac{0.0791}{\mathrm{Re}^{1/4}}$$

We now form the differential of $f$, thus

$$df = -\frac{1}{4}\frac{0.0791}{\mathrm{Re}^{5/4}}d\mathrm{Re}$$

and regard the differentials as the errors in the relevant quantities. This result may be rearranged thus

$$df = -\frac{0.0791}{\mathrm{Re}^{1/4}}\frac{1}{4\mathrm{Re}}d\mathrm{Re} = -f\cdot\frac{1}{4\mathrm{Re}}d\mathrm{Re}$$

From this we find

$$\frac{df}{f} = -\frac{1}{4}\frac{d\mathrm{Re}}{\mathrm{Re}}$$

The quantity $df/f$ represents the fractional error in $f$, and $d\mathrm{Re}/\mathrm{Re}$ is the fractional error in Re.

Therefore, if the Reynolds number is too low by 4%, then the friction factor will be too high by 1%.

**6B.2 Friction factor for flow along a flat plate**

*a*. Equation 4.4-30 gives the kinetic force acting on both sides of the flat plate for *laminar flow*:

$$F_k = 1.328\sqrt{\rho\mu L W^2 v_\infty^3}$$

and an appropriate Reynolds number for a plate of length $L$ is

$$\mathrm{Re}_L = \frac{L v_\infty \rho}{\mu}$$

Then, using the definition of the friction factor in Eq. 6.1-1, we write

$$f = \frac{F_k}{AK} = \frac{1.328\sqrt{\rho\mu L W^2 v_\infty^3}}{2WL \cdot \frac{1}{2}\rho v_\infty^2} = 1.328\sqrt{\frac{\mu}{L v_\infty \rho}} = \frac{1.328}{\sqrt{\mathrm{Re}_L}}$$

*b*. For *turbulent flow*, the force acting on the plate is given by Eq. 6B.2-1 as

$$F_k = 0.074\rho v_\infty^2 W L \left(L v_\infty \rho/\mu\right)^{-1/5}$$

Then the friction factor is

$$f = \frac{F_k}{AK} = \frac{0.074\rho v_\infty^2 W L \left(L v_\infty \rho/\mu\right)^{-1/5}}{2WL \cdot \frac{1}{2}\rho v_\infty^2} = \frac{0.074}{\mathrm{Re}_L^{1/5}}$$

Here we have used the same definition of Reynolds number as in part (*a*).

**6B.3 Friction factor for laminar flow in a slit**

*a*. The mass flow rate through a slit of width $W$, thickness $2B$, and length $L$ is, for laminar flow

$$w = \frac{2}{3}\frac{(\mathcal{P}_0 - \mathcal{P}_L)B^3 W\rho}{\mu L} = \rho\langle v_z\rangle(2BW)$$

and the kinetic force acting on the walls by the fluid is

$$F_k = (\mathcal{P}_0 - \mathcal{P}_L)(2BW)$$

Then the friction factor is

$$f = \frac{F_k}{AK} = \frac{(\mathcal{P}_0 - \mathcal{P}_L)(2BW)}{(2WL)\left(\frac{1}{2}\rho\langle v_z\rangle^2\right)}$$

We now replace one of the $\langle v_z\rangle$ in the denominator of this expression by using the above expression for the mass flow rate. This gives

$$f = \frac{(\mathcal{P}_0 - \mathcal{P}_L)(2BW)}{(2WL)\left(\frac{1}{2}\rho\langle v_z\rangle\right)} \cdot \frac{3\mu L}{(\mathcal{P}_0 - \mathcal{P}_L)B^2} = \frac{12}{2B\langle v_z\rangle\rho/\mu} = \frac{12}{\mathrm{Re}}$$

*b* Next we try using the mean hydraulic radius empiricism suggested on p. 183. For the plane slit, $R_h$ is

$$R_h = \frac{S}{Z} = \frac{2BW}{2W + 4B} \approx B$$

In the last step we have made the assumption (consistent with the derivation of the velocity profile for the slit) that $B << W$. We now replace the $D$ in the tube flow friction factor by $4R_h$. Then

$$f = \frac{16}{D\langle v_z\rangle\rho/\mu} \rightarrow \frac{16}{4R_h\langle v_z\rangle\rho/\mu} = \frac{16}{4B\langle v_z\rangle\rho/\mu} = \frac{8}{2B\langle v_z\rangle\rho/\mu} = \frac{8}{\mathrm{Re}}$$

This poor result emphasizes the statement made just after Eq. 6.2-18.

**6B.4 Friction factor for a rotating disk**

*a*. Laminar flow

$$f = \frac{T_z}{AKR} = \frac{0.616\pi\rho R^4\sqrt{\mu\Omega^3/\rho}}{\left(2\pi R^2\right)\left(\frac{1}{2}\rho\Omega^2R^2\right)R} = 0.616\sqrt{\frac{\mu}{R^2\Omega\rho}} = \frac{0.616}{\mathrm{Re}^{1/2}}$$

*b*. Turbulent flow

$$f = \frac{T_z}{AKR} = \frac{0.073\rho\Omega^2R^5\left(\mu/R^2\Omega\rho\right)^{1/5}}{\left(2\pi R^2\right)\left(\frac{1}{2}\rho\Omega^2R^2\right)R}\,0.073\left(\frac{\mu}{R^2\Omega\rho}\right)^{1/5} = \frac{0.073}{\mathrm{Re}^{1/5}}$$

**6B.5 Turbulent flow in horizontal pipes.**

*a.* Equations 6.1-4 and 6.2-12 give, for horizontal flow in smooth pipe with fixed $w$, $\rho$, $\mu$, and $L$,

$$\begin{aligned}(p_0 - p_L) &= 2\frac{L}{D}\rho\langle v\rangle^2 \cdot f \\ &= 2\frac{L}{D}(\rho)(\frac{4w}{\rho\pi D^2})^2(0.0791)\left(\frac{4w}{\pi D\mu}\right)^{-1/4} \\ &\propto D^{-5+1/4}\end{aligned}$$

According to this formula, the replacement of the smooth pipe of diameter $D$ by a smooth one of diameter $D/2$, with all other right-hand variables unchanged, will alter $(p_0 - p_L)$ by a factor of $(1/2)^{-5+1/4} \approx 27$.

*b.* If the Blasius expression for $F$ is not appropriate (e.g., because the range of Re is higher or non-smooth pipes are used), then one must calculate Re$= 4w/\pi D\mu$ for the present operation, read $f_{\text{old}}$ from Fig. 6.2-2 at that Re with the applicable roughness measure $k/D$, and then read $f_{\text{new}}$ at the new values of $4w/\pi D\mu$ and $k/D$. Then the calculation in (a) needs to be modified only by using $f_{\text{new}}/f_{\text{old}}$ in place of $(D_{\text{new}}/D_{\text{old}})^{1/4}$. The inverse 5th-power of $D$ will still be dominant in the calculation of the change in $(p_0 - p_L)$.

**6B.6 Inadequacy of mean hydraulic radius for laminar flow**

*a*. The mean hydraulic radius for the annulus is (Eq. 6.2-16)

$$R_h = \frac{S}{Z} = \frac{\pi R^2 - \pi(\kappa R)^2}{2\pi R + 2\pi(\kappa R)} = \frac{R(1-\kappa^2)}{2(1+\kappa)} = \frac{R}{2}(1-\kappa)$$

Then, according to Eq. 6.2-18, the Reynolds number is

$$\mathrm{Re}_h = \frac{4R_h\langle v_z\rangle\rho}{\mu} = \frac{2R(1-\kappa)\langle v_z\rangle\rho}{\mu}$$

Then, according to Eq. 6.2-17 the friction factor is

$$f = \left(\frac{R_h}{L}\right)\left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{\frac{1}{2}\rho\langle v_z\rangle^2}\right) = \frac{16}{\mathrm{Re}_h} = \frac{4\mu}{R_h\langle v_z\rangle\rho}$$

or

$$\langle v_z\rangle = \frac{(\mathcal{P}_0 - \mathcal{P}_L)R_h^2}{2\mu L} = \frac{(\mathcal{P}_0 - \mathcal{P}_L)R^2(1-\kappa)^2}{8\mu L}$$

This is the mean-hydraulic-radius analog of Eq. 2.4-16.

*b*. When $\kappa = \frac{1}{2}$, the above result for the mean-hydraulic-radius empiricism gives:

$$\langle v_z\rangle = \frac{(\mathcal{P}_0 - \mathcal{P}_L)R^2}{8\mu L}(0.250)$$

whereas Eq. 2.4-16 (the exact result) gives

$$\langle v_z\rangle = \frac{(\mathcal{P}_0 - \mathcal{P}_L)R^2}{8\mu L}\left[\frac{1-\kappa^4}{1-\kappa^2} - \frac{1-\kappa^2}{\ln(1/\kappa)}\right]$$

$$= \frac{(\mathcal{P}_0 - \mathcal{P}_L)R^2}{8\mu L}\left[\frac{1-\left(\frac{1}{2}\right)^4}{1-\left(\frac{1}{2}\right)^2} - \frac{1-\left(\frac{1}{2}\right)^2}{\ln 2}\right]$$

$$= \frac{(\mathcal{P}_0 - \mathcal{P}_L)R^2}{8\mu L}\left[\frac{\left(\frac{15}{16}\right)}{\left(\frac{3}{4}\right)} - \frac{\left(\frac{3}{4}\right)}{0.6931}\right]$$

$$= \frac{(\mathcal{P}_0 - \mathcal{P}_L)R^2}{8\mu L}\left[1.25 - \frac{0.750}{0.6931}\right]$$

$$= \frac{(\mathcal{P}_0 - \mathcal{P}_L)R^2}{8\mu L}[1.25 - 1.0821]$$

$$= \frac{(\mathcal{P}_0 - \mathcal{P}_L)R^2}{8\mu L}(0.1679)$$

The percentage error in the mean-hydraulic-radius empiricism is then

$$\frac{0.2500 - 0.1679}{0.1679} \times 100 = \frac{0.0821}{0.1679} \times 100 = 49\%$$

This confirms the inadequacy of the mean-hydraulic-radius empiricism for laminar flow.

**6B.7 Falling sphere in Newton's drag-law region.**

Let $z$ be the (downward) displacement of the sphere from its initial position, and $v = v_z = dz/dt$ denote the instantaneous velocity of the sphere. Then the $z$-component of Newton's second law gives, for a falling sphere of mass $m$,

$$m\frac{dv}{dt} = F_z$$

in which

$$\begin{aligned} F_z &= mg_z - F_{kz} \\ &= mg - (\pi R^2)(\tfrac{1}{2}\rho v^2)(0.44) \\ &= mg(1 - c^2 v^2) \end{aligned}$$

with $c^2$ defined by the last equation. Hence,

$$\frac{dv}{dt} = g(1 - c^2 v^2)$$

and integration from the initial condition $v = 0$ at $t = 0$ gives

$$\int_0^v \frac{dv}{1 - c^2 v^2} = g\int_0^t dt$$

or

$$\frac{1}{c}\tanh^{-1} cv = gt$$

whence

$$\frac{dz}{dt} \equiv v = \frac{1}{c}\tanh cgt$$

Note that as $t \to \infty$, $\tanh cgt \to 1$. Hence, the "terminal velocity" is $1/c$. A second integration then gives (for $z = 0$ at $t = 0$):

$$\int_0^z dz = \frac{1}{c}\int_0^t \tanh cgt dt$$

or

$$\begin{aligned} z &= \frac{1}{c^2 g}\int_0^{cgt} \tanh \beta d\beta \\ &= \frac{1}{c^2 g}\ln\cosh cgt \end{aligned}$$

Now $mgc^2 = 0.22\pi R^2\rho$, so that

$$\begin{aligned} c &= \sqrt{0.22\pi R^2\rho/mg} \\ &= \sqrt{\frac{0.22\pi R^2\rho}{\frac{4}{3}\pi R^2\rho_{\text{sph}}g}} \\ &= \sqrt{\left(\tfrac{3}{4}\right)(0.22)\left(\frac{\rho}{\rho_{\text{sph}}}\right)\left(\frac{1}{Rg}\right)} \end{aligned}$$

Note that this solution assumes that the particle is in the Newton's drag law region during its entire trajectory. However, the initial condition that $v = 0$ at $t = 0$ is clearly outside the Newton's drag law region.

**6B.8 Design of an experiment to verify the $f$ vs Re chart for spheres**

*a*. We start with the friction-factor expression in Eq. 6.1-7. Since we need to have a $D$ expression containing the Reynolds number (but not the terminal velocity), we eliminate the velocity in favor of Re. This gives

$$f = \frac{4}{3}\frac{gD}{v_\infty^2}\left(\frac{\rho_{\text{sph}} - \rho}{\rho}\right) = \frac{4}{3}\frac{gD}{\text{Re}^2}\left(\frac{D\rho}{\mu}\right)^2\left(\frac{\rho_{\text{sph}} - \rho}{\rho}\right) = \frac{4}{3}\frac{gD^3}{\text{Re}^2\mu^2}\left(\rho_{\text{sph}} - \rho\right)\rho$$

We now solve this expression for $D$ to get

$$D = \sqrt[3]{\frac{3}{4}\frac{f\,\text{Re}^2\mu^2}{\left(\rho_{\text{sph}} - \rho\right)\rho g}}$$

*b*. Next, substituting numerical values into the formula, we get

$$D = \sqrt[3]{\frac{3}{4}\frac{(1)(100)^2\left(10^{-2}\right)^2}{(8-1)(1)(980.7)}} = 0.048 \text{ cm}$$

**6B.9 Friction factor for flow past an infinite cylinder**

We select the area $A$ of Eq. 6.1-1 to be the area "seen" by the approaching fluid, which is $DL$, and $K$ to be $\frac{1}{2}\rho v_\infty^2$. Then combining Eqs. 6.1-1 and 6B.9-1 gives

$$F_k = fAK = \frac{4\pi\mu v_\infty L}{\ln(7.4/\mathrm{Re})}$$

We now solve for $f$ and insert the expressions for $A$ and $K$ to get

$$f = \frac{4\pi\mu v_\infty L}{\ln(7.4/\mathrm{Re})}\frac{1}{(DL)\left(\frac{1}{2}\rho v_\infty^2\right)} = \frac{8\pi}{\ln(7.4/\mathrm{Re})}\frac{1}{(Dv_\infty\rho/\mu)}$$

Therefore

$$f = \frac{8\pi}{\mathrm{Re}\ln(7.4/\mathrm{Re})}$$

**6C.1 Two-dimensional particle directories**

*a.* We start with Newton's law of motion for a particle

$$\frac{d}{dt}m\mathbf{v} = \sum \mathbf{F}$$

which states that the rate of change of momentum of the particle is equal to the sum of the various forces acting on the particle. In this instance, the forces are those associated with friction and those related to gravity. Specifically, since $m$ is a constant, we may write

$$\left(\tfrac{4}{3}\pi R^3\rho_{\text{sph}}\right)\frac{d}{dt}\mathbf{v} = -\left(\pi R^2\right)\left(\tfrac{1}{2}\rho_{\text{air}}\left(v_x^2+v_y^2\right)f\right)\frac{\mathbf{v}}{|\mathbf{v}|} + \left(\tfrac{4}{3}\pi R^3\rho_{\text{sph}} - \tfrac{4}{3}\pi R^3\rho_{\text{air}}\right)g\boldsymbol{\delta}_y$$

Here the positive direction for the $y$ axis has been taken to be in the direction of gravity (i.e., "downward"). The frictional force has been taken from Eqs. 6.1-5 and 5a, and the gravitational force from Eq. 6.1-6. Dividing $\mathbf{v}$ by its absolute value $|\mathbf{v}| = \sqrt{v_x^2+v_y^2}$ gives the vector $\mathbf{n}$ of Eq. 6.1-5a. When the above equation is divided by the mass of the sphere, we get

$$\frac{d}{dt}\mathbf{v} = -\frac{3}{8R}\frac{\rho_{\text{air}}}{\rho_{\text{sph}}}\sqrt{v_x^2+v_y^2}f\mathbf{v} + \left(1-\frac{\rho_{\text{air}}}{\rho_{\text{sph}}}\right)g\boldsymbol{\delta}_y$$

Next we take the $x$ and $y$ components of this equation to get:

$$\frac{dv_x}{dt} = -\frac{3}{8R}\frac{\rho_{\text{air}}}{\rho_{\text{sph}}}v_x\sqrt{v_x^2+v_y^2}f$$

$$\frac{dv_y}{dt} = -\frac{3}{8R}\frac{\rho_{\text{air}}}{\rho_{\text{sph}}}v_y\sqrt{v_x^2+v_y^2}f + \left(1-\frac{\rho_{\text{air}}}{\rho_{\text{sph}}}\right)g$$

in which it is understood that $f$ is a function of the Reynolds number based on the instantaneous velocity: $\text{Re} = 2R|\mathbf{v}|\rho_{\text{air}}/\mu_{\text{air}}$.

*b*. For the sphere that is dropped vertically, the $y$ component of the equation of motion becomes

$$\frac{dv_y}{dt} = -\frac{3}{8R}\frac{\rho_{\text{air}}}{\rho_{\text{sph}}}v_y^2 f + \left(1 - \frac{\rho_{\text{air}}}{\rho_{\text{sph}}}\right)g$$

Thus the particle that is fired horizontally has a frictional force that is $\sqrt{v_x^2 + v_y^2}/v_y$ times greater than that of the particle that moves vertically. Therefore the horizontally fired particle will be slowed down in its descent and the vertically moving particle will land first.

*c*. For the particle moving in the Stokes' law region, the friction factor is given by Eq. 6.3-15:

$$f = \frac{24}{\text{Re}} = \frac{24}{2R\sqrt{v_x^2 + v_y^2}\rho_{\text{air}}/\mu_{\text{air}}}$$

Then, the $x$ and $y$ components of the equation of motion given in (*a*) become:

$$\frac{dv_x}{dt} = -\frac{9}{2R^2}\frac{\mu_{\text{air}}}{\rho_{\text{sph}}}v_x$$

$$\frac{dv_y}{dt} = -\frac{9}{2R^2}\frac{\mu_{\text{air}}}{\rho_{\text{sph}}}v_y + \left(1 - \frac{\rho_{\text{air}}}{\rho_{\text{sph}}}\right)g$$

Thus, the equations of motion are "uncoupled." Therefore the motion in the $y$ direction is completely independent of the motion in the $x$ direction, and both particles will arrive at their destination simultaneously.

**6D.1 Friction factor for a bubble in a clean liquid**

The rate of energy dissipation in any region is, for a Newtonian fluid with negligible dilatational viscosity

$$E_v = \iiint (-\boldsymbol{\tau}:\nabla\mathbf{v})dV = \mu\iiint\left(\nabla\mathbf{v}+(\nabla\mathbf{v})^\dagger\right):\nabla\mathbf{v}dV$$
$$= \tfrac{1}{2}\iiint\left(\nabla\mathbf{v}+(\nabla\mathbf{v})^\dagger\right):\left(\nabla\mathbf{v}+(\nabla\mathbf{v})^\dagger\right)dV$$

For potential flow, $[\nabla\times\mathbf{v}]=0$ or $\nabla\mathbf{v}=(\nabla\mathbf{v})^\dagger$, and hence

$$E_v = 2\mu\iiint\left(\nabla\mathbf{v}:(\nabla\mathbf{v})^\dagger\right)dV = 2\mu\iiint\left(\nabla\cdot\left[\mathbf{v}\cdot(\nabla\mathbf{v})^\dagger\right]\right)dV - 2\mu\iiint\left(\mathbf{v}\cdot\nabla^2\mathbf{v}\right)dV$$

But for irrotational flow of an incompressible fluid

$$\left(\nabla^2\mathbf{v}\right) = \nabla(\nabla\cdot\mathbf{v}) - [\nabla\times[\nabla\times\mathbf{v}]] = 0$$

Then using the Gauss divergence theorem on the remaining term

$$E_v = 2\mu\iint\left(\mathbf{n}\cdot\left[\mathbf{v}\cdot(\nabla\mathbf{v})^\dagger\right]\right)dS = \mu\iint\left(\mathbf{n}\cdot\nabla v^2\right)dS$$
$$= -\mu\iint_{\text{surface at } R}\frac{\partial v^2}{\partial r}R^2\sin\theta d\theta d\phi$$

The surface integral includes the spherical surface at $r=R$ at which $\mathbf{n}=-\boldsymbol{\delta}_r$, and the surface at $r=\infty$ at which $\mathbf{n}=\boldsymbol{\delta}_r$.

For the potential flow around the sphere, we get from Eqs. 4B.5-2 and 3

$$v^2 = v_\infty^2\left(\left[1-\left(\frac{R}{r}\right)^3\right]^2\cos^2\theta + \left[1-\frac{1}{2}\left(\frac{R}{r}\right)^3\right]^2\sin^2\theta\right)$$

From this we find

$$\left.\frac{\partial}{\partial r}v^2\right|_{r=R} = -\frac{9}{2}\frac{v_\infty^2}{R}\sin^2\theta$$

Then, when this is substituted into the expression for $E_v$, we get

$$E_v = -2\pi\mu \cdot \left( -\frac{9}{2}\frac{v_\infty^2}{R} \right) R^2 \int_0^\pi \sin^3\theta d\theta = 12\pi\mu R v_\infty^2 = F_k v_\infty$$

Hence, from $F_k = \left(\pi R^2\right)\left(\frac{1}{2}\rho v_\infty^2\right) f$ we conclude that

$$f = \frac{12\pi\mu R v_\infty}{\left(\pi R^2\right)\left(\frac{1}{2}\rho v_\infty^2\right)} = \frac{48}{2Rv_\infty\rho/\mu} = \frac{48}{\mathrm{Re}}$$

**7A.1 Pressure rise in a sudden enlargement.**

According to Eq. 7.6-4

$$p_2 - p_1 = \rho v_2^2 \left(\frac{1}{\beta} - 1\right) \text{ in which } \beta = \frac{S_1}{S_2}$$

We compute $\beta$ and $v_2$ thus:

$$\beta = \frac{S_1}{S_2} = \left(\frac{D_1}{D_2}\right)^2 = \left(\frac{5}{9}\right)^2 = 0.309$$

$$v_2 = \frac{Q}{S_2} = \frac{(450/60 \text{ gal/s})(0.13368 \text{ ft}^3/\text{gal})}{((\pi/4)(25/144) \text{ ft}^2)} = 7.35 \text{ ft/s}$$

Hence,

$$\begin{aligned} p_2 - p_1 &= \rho v_2^2 \left(\frac{1}{\beta} - 1\right) \\ &= (63 \text{ lb}_m/\text{ft}^2)(7.35 \text{ ft/s})^2 \left(\frac{1}{0.309} - 1\right) \\ &= 7610 \text{ poundals/ft}^2) = 1.64 \text{ psi} \end{aligned}$$

**7A.2 Pumping a hydrochloric acid solution.**

Equation 7.5-10 reduces to

$$\widehat{W}_m = \frac{p_2 - p_1}{\rho} + \frac{1}{2}v^2\frac{L}{R_h}f \text{ (pipe friction)} + \frac{1}{2}v_2^2 \text{ (exit loss)}$$

for the liquid between free surface "1" and free surface "2", when the liquid is regarded as incompressible and the negligibility of kinetic and potential energy differences between those locations is noted. The major right-hand term is

$$\frac{p_2 - p_1}{\rho} = \frac{(4 - 1 \text{ atm})(6.8087 \times 10^4 \text{ lb}_m/\text{ft}\cdot \text{ s2/atm})}{62.4 \text{ lb}_m/\text{ft}^3} = 3273 \text{ ft}^2/\text{s}^2,$$

the pipe friction term (with $f = 0.0048_4$ from Eq. 6.2-12) is

$$\frac{1}{2}v^2\frac{L}{D/4}f = \frac{1}{2}(2.30 \text{ ft/s})^2\frac{300 \text{ ft}}{0.5/12 \text{ ft})}(0.0048_4) = 92 \text{ ft}^2/\text{s}^2,$$

and the exit loss is

$$\frac{1}{2}(2.30 \text{ ft/s})^2 = 2.6 \text{ ft}^2/\text{s}^2.$$

Thus, the required work input per unit mass of fluid pumped is

$$\widehat{W}_m = 3273 + 92 + 2.6 = 3368 \text{ ft}^2/\text{s}^2$$

Multiplication by the mass flow rate of

$$w = (\pi R^2)\rho\langle v\rangle = (\pi)(\frac{2^2}{144} \text{ ft}^2)(62.4 \text{ lb}_m/\text{ft}^3)(2.30\text{ft/s}) = 12.5 \text{ lb}_m/\text{s}$$

gives the power required from the pump:

$$\begin{aligned} W = \widehat{W}w &= (3368 \text{ ft}^2/\text{s}^2)(12.5 \text{ lb}_m/\text{s}) \\ &= (4.22 \times 10^4 \text{ ft poundals/s}) \times (3600 \text{ s/hr})(1.5698 \times 10^{-8} \text{ hp-hr/ft poundal}) \\ &= 2.4 \text{ hp} = 1.8 \text{ kw} \end{aligned}$$

**7A.3 Compressible gas flow in a cylindrical pipe.**

For steady isothermal, horizontal pipe flow of an ideal gas at $T = 25°\text{C}=536.7$ R with flat velocity profiles and $\hat{W}_m = 0$, Eq. 7.4-7 gives

$$\begin{aligned}\hat{E}_v &= -\int_1^2 \frac{1}{\rho} dp - \left(\frac{v_2^2}{2} - \frac{v_1^2}{2}\right) \\ &= \frac{RT}{M} \ln \frac{p_1}{p_2} + \tfrac{1}{2}\left(\frac{w}{S}\right)^2 \left(\frac{1}{\rho_1^2} - \frac{1}{\rho_2^2}\right)\end{aligned}$$

Calculation of the terms in $\text{ft}^2/\text{s}^2$ gives

$$\frac{RT}{M} \ln \frac{p_1}{p_2} = \frac{(4.9686 \times 10^4 \text{ lb}_m\text{ft}^2/\text{s}^2\cdot\text{lb-mol}\cdot\text{R})(536.7 \text{ R})}{(28.01 \text{ lb}_m/\text{lb-mol})} \ln 2 = 6.60 \times 10^5 \text{ ft}^2/\text{s}^2$$

and with $\rho = pM/RT$ in corresponding units,

$$\begin{aligned}\rho_1 &= (2 \times 2116.2 \text{ lb}_f/\text{ft}^2)(28.01 \text{ lb}_m/\text{lb-mol})/(1544.3 \text{ ft}\cdot\text{lb}_f/\text{lb-mol}\cdot\text{R})(536.7 \text{ R}) \\ &= 0.1430 \text{ lb}_m/\text{ft}^3 = 2\rho_2\end{aligned}$$

Then

$$\begin{aligned}&\tfrac{1}{2}\left(\frac{w}{S}\right)^2 \left(\frac{1}{\rho_1^2} - \frac{1}{\rho_2^2}\right) \\ &\quad = \tfrac{1}{2}\left(\frac{(0.28 \text{ lb}_m/\text{s})}{(\pi/64 \text{ ft}^2)}\right)^2 \left(\frac{1}{(0.1430 \text{ lb}_m/\text{ft}^3)^2} - \frac{1}{(0.0715 \text{ lb}_m/\text{ft}^3)^2}\right) \\ &\quad = -2.39 \times 10^3 \text{ ft}^2/\text{s}^2\end{aligned}$$

Combining these results, we get

$$\hat{E}_v = 6.60 \times 10^5 - 2.39 \times 10^3 = 6.58 \times 10^5 \text{ ft}^2/\text{s}^2$$

With the aid of Table F.3-3 we get the corresponding values in other units:

$$\hat{E}_v = 26.3 \text{ Btu/lb}_m = 6.11 \times 10^4 \text{ J/kg}$$

**7A.4 Incompressible flow in an annulus.**

(i) The mean hydraulic radius is (see Eq. 6.2-16):

$$R_h = \frac{S}{Z} = \frac{\pi(R_o^2 - R_i^2)}{2\pi(R_o + R_i)} = \frac{1}{2}(R_o - R_i)$$

The mass flow rate is

$$w = (241/60 \text{ U.S. gal/s})(231/1728 \text{ ft}^3/\text{gal})(62.4 \text{ lb}_m/\text{ft}^3) = 33.5 \text{ lb}_m/\text{s}$$

and the average flow velocity is

$$\begin{aligned}\langle v \rangle &= \frac{w}{\rho S} = \frac{w/\rho}{\pi(R_o^2 - R_i^2)} \\ &= \frac{(33.5 \text{ lb}_m/\text{s})/(62.4 \text{ lb}_m/\text{ft}^3)}{\pi((7/12)^2 - (3/12)^2 \text{ ft}^2}) = 0.615 \text{ ft/s}\end{aligned}$$

(ii) The Reynolds number is (see Eq. 6.2-18):

$$\begin{aligned}\text{Re} &= \frac{4R_h\langle v \rangle \rho}{\mu} = \frac{2(R_o - R_i)w}{\pi(R_o^2 - R_i^2)\mu} = \frac{2w}{\pi(R_o + R_i)\mu} \\ &= \frac{2(33.5 \text{ lb}_m/\text{s})}{\pi((7+3)/12 \text{ ft})(1.14 \times 0.000672 \text{ lb}_m/\text{ft}\cdot\text{s})} \\ &= 3.34 \times 10^4\end{aligned}$$

Hence, the flow is turbulent and $f = 0.0059$ according to either Eq. 6.2-15 or Fig. 6.2-2.

(iii) From Eq. 7.5-10 we calculate the work required $W$ in ft-poundals per pound [=] $\text{ft}^2/\text{s}^2$:

$$\begin{aligned}\hat{W} &= g(h_2 - h_1) + \frac{1}{2}\langle v \rangle^2 \frac{L}{R_h} f \\ &= g(h_2 - h_1) + \langle v \rangle^2 \frac{L}{(R_o - R_i)} f \\ &= (32.2 \text{ ft/s}^2)(5 \text{ ft}) + (0.615 \text{ ft/s})^2 \left(\frac{20.3}{(7-3)/12}\right)(0.0059) \\ &= (161 + 0.175) = 161.1 \text{ ft}^2/\text{s}^2\end{aligned}$$

The power output required from the pump is then

$$\begin{aligned}w\hat{W} &= (33.5 \text{ lb}_m/\text{s})(161.1 \text{ ft}^2/\text{s}^2 = 5.4 \times 10^3 \text{ ft}\cdot\text{poundals/s} \\ &= 1.9 \times 10^7 \text{ ft}\cdot\text{poundals/hr} = 0.31 \text{ hp} = 0.23 \text{ kw}\end{aligned}$$

**7A.5 Force on a U-bend.**

According to Eq. 7.2-3, if the $x$-axis is taken in the direction of the downstream unit vector $\boldsymbol{u}_1$ at $S_1$, which is the negative of the downstream unit vector $\boldsymbol{u}_2$ at $S_2$ and perpendicular to the gravity vector $\boldsymbol{g}$, then the force exerted by the water on the U-bend has the $x$-component

$$(\boldsymbol{\delta}_x \bullet \boldsymbol{F}_{f\to s}) = (v_1 w + p_1 S_1)(\boldsymbol{\delta}_x \bullet \boldsymbol{u}_1) - (v_2 w + p_2 S_2)(\boldsymbol{\delta}_x \bullet \boldsymbol{u}_2) + m_{\text{tot}}(\boldsymbol{\delta}_x \bullet \boldsymbol{g})$$

when the velocity profiles at Planes 1 and 2 are approximated as flat. With $(\boldsymbol{\delta}_x \bullet \boldsymbol{u}_1) = 1$, $(\boldsymbol{\delta}_x \bullet \boldsymbol{u}_2) = -1$, $(\boldsymbol{\delta}_x \bullet \boldsymbol{g}) = 0$, $S_1 = S_2$, and $v_1 = v_2 = w/\rho = Q/S$, this gives

$$\begin{aligned} F_{x,f\to s} &= (v_1 + v_2)w + (p_1 + p_2)S \\ &= (2Q/S)(Q\rho) + (p_1 + p_2)S \\ &= 2Q^2\rho/S + (p_1 + p_2)S \end{aligned}$$

with $Q = 3\ \text{ft}^3/\text{s}$. Thus, the horizontal force of the water on the U-bend is

$$\begin{aligned} F_{x,f\to s} &= 2(3\ \text{ft}^3/\text{s})^2(62.4\ \text{lb}_m/\text{ft}^3)/(4\pi/144\ \text{ft}^2) \\ &\quad + (21 + 19\ \text{lb}_f/\text{in}^2)(4\pi\ \text{in}^2)(32.174\ \text{lb}_m\text{ft}/\text{s}^2\cdot\text{lb}_f) \\ &= (12871 + 16172\ \text{lb}_m\text{ft}/\text{s}^2) = 29043\ \text{lb}_m\text{ft}/\text{s}^2 = 903\ \text{lb}_f \end{aligned}$$

**7A.6 Flow-rate calculation.**

As our system for the mechanical energy balance, we consider the water between plane 1 (the upper liquid surface in the constant-head tank) and plane 2 (just inside the outlet of the exit piping). We neglect the velocity at plane 1, set $p = p_{\text{atm}}$ at planes 1 and 2, and treat the water as incompressible. Then Equation 7.5-10 takes the form

$$\frac{1}{2}v_2^2 + g(z_2 - z_1) = -\frac{1}{2}v^2\frac{L}{R_h}f - \frac{1}{2}v^2\sum_i e_{vi}$$

in which the kinetic energy and friction terms all are calculated with the same average velocity $v$.

Collecting the coefficients of $v^2$, we get

$$v^2\left[1 + \frac{L}{R_h}f + 0.45 + 0.4 + 0.4\right] = 2g(z_1 - z_2)$$

in which the coefficients 0.45, 0.4, and 0.4 are the $e_{vi}$ values for the sudden contraction at the pipe inlet and for two smooth 90° elbows. There is no enlargement loss in the system considered, since plane 2 is just inside the exit of the piping. Setting $R_h = D/4$ and solving for $v$, we get

$$v = \sqrt{\frac{2g(z_1 - z_2)}{2.25 + 4fL/D}}$$

This result is explicit except for the friction factor $f$, which depends on Re and thus on $v$.

Multiplying the last equation by $v/\nu$ and evaluating all the dimensional constants in cgs, we get the working formula

$$\begin{aligned}
\text{Re} = \frac{Dv}{\nu} &= \frac{D}{\nu}\sqrt{\frac{2g(z_1 - z_2)}{2.25 + 4L/Df}} \\
&= \frac{12.7\text{ cm}}{0.010037\text{ cm}^2/\text{s}}\sqrt{\frac{2(980.7\text{ cm/s}^2)(38 \times 30.48\text{ cm})}{2.25 + 355.2f}} \\
&= \frac{1.907 \times 10^6}{\sqrt{2.25 + 355.2f}} \qquad (A)
\end{aligned}$$

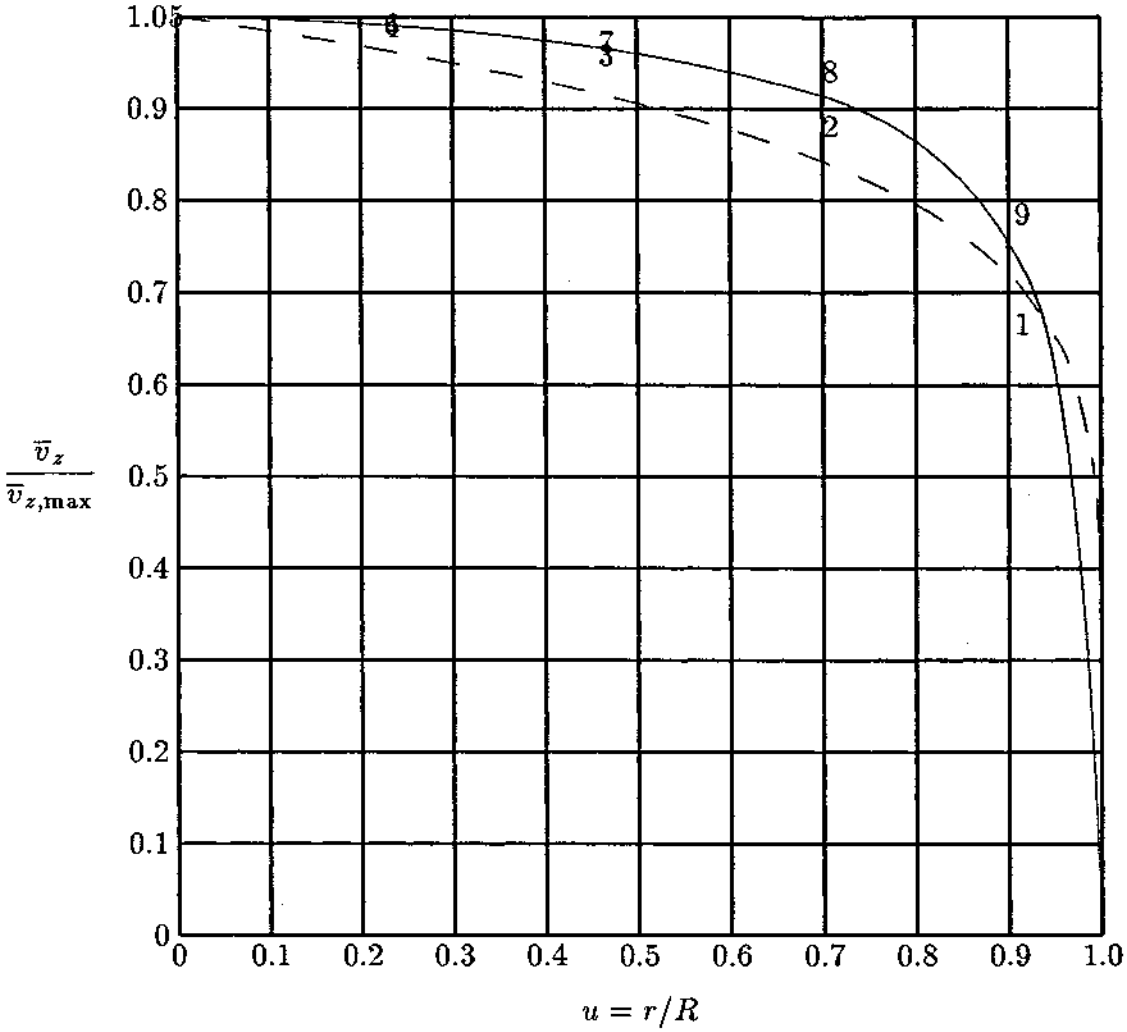
The last equation is solvable rapidly by iteration. A first trial value $1\times10^5$ for Re gives $f = 0.0045$ from Fig. 6.2.2, and a new value Re$= 9.72 \times 10^5$ from Eq. (A). At this new trial Re value, we get $F = 0.00285$ from Fig. 6.2-2, and Eq. (A) then gives Re$= 1.056 \times 10^6$. Returning to Fig. 6.2-2 with this new trial Re value, we get $f = 0.0028$, and Eq. (A) then gives the value Re$= 1.059 \times 10^6$, which we accept as final.

The mass flow rate can then be calculated as $w = \mathrm{Re}\pi D\mu/4$, or the volume flow rate as

$$
\begin{aligned}
Q &= \frac{w}{\rho} = \frac{\mathrm{Re}\pi D\mu}{4} \\
&= \frac{(1.056 \times 10^6)(\pi)(12.7\ \mathrm{cm})(0.010037\ \mathrm{cm}^2/\mathrm{s}}{4} = 1.1 \times 10^5\ \mathrm{cm}^3/\mathrm{s} = 0.11\ \mathrm{m}^3/\mathrm{s}
\end{aligned}
$$

Calibration is necessary to get accurate measurements of flow rates in such an apparatus, because of the uncertainties in the geometric details and thus in the energy loss factors (see Table 7.5-1).

**7A.7 Evaluation of various velocity averages from Pitot tube data.**




This figure shows velocity data from the C. E. thesis of B. Bird (University of Wisconsin, 1915). Each measurement is labelled with its position number from the table in Problem 7A.7. The flow is evidently turbulent, but not fully developed; if it were, the measured velocities at positions 1, 2, and 3 would show better agreement with those at positions 7, 8, and 9 for the same $r$ values. The solid curve in the figure is a curve-fit of the data, and the dashed curve represents the 1/7th-power model given in Eq. 5.1-4, which is based on extensive experiments in fully developed flow.

Application of Simpson's rule to 11 uniformly spaced points on the solid curve gives the following velocity averages. A finer grid would give more accurate approximations to the integrals.

$$
\begin{aligned}
\frac{\langle v \rangle}{v_{\max}} &= \int_0^1 \frac{v}{v_{\max}} 2u du \\
&= [(1 \times 1.0 \times 0) + (4 \times 0.995 \times 0.1) + (2 \times 0.986 \times 0.2) \\
&+ (4 \times 0.98 \times 0.3) + (2 \times 0.975 \times 0.4) + (4 \times 0.96 \times 0.5) \\
&+ (2 \times 0.94 \times 0.6) + (4 \times 0.92 \times 0.7) + (2 \times 0.87 \times 0.8) \\
&+ (4 \times 0.755 \times 0.9) + (1 \times 0 \times 1.0)] \times 2/30 \\
&= 0.832
\end{aligned}
$$

$$
\begin{aligned}
\frac{\langle v^2 \rangle}{v_{\max}^2} &= \int_0^1 \frac{v^2}{v_{\max}^2} 2u du \\
&= [(1 \times 1.0^2 \times 0) + (4 \times 0.995^2 \times 0.1) + (2 \times 0.986^2 \times 0.2) \\
&+ (4 \times 0.98^2 \times 0.3) + (2 \times 0.975^2 \times 0.4) + (4 \times 0.96^2 \times 0.5) \\
&+ (2 \times 0.94^2 \times 0.6) + (4 \times 0.92^2 \times 0.7) + (2 \times 0.87^2 \times 0.8) \\
&+ (4 \times 0.755^2 \times 0.9) + (1 \times 0^2 \times 1.0)] \times 2/30 \\
&= 0.749
\end{aligned}
$$

$$
\begin{aligned}
\frac{\langle v^3 \rangle}{v_{\max}^3} &= \int_0^1 \frac{v^3}{v_{\max}^3} 2u du \\
&= [(1 \times 1.0^3 \times 0) + (4 \times 0.995^3 \times 0.1) + (2 \times 0.986^3 \times 0.2) \\
&+ (4 \times 0.98^3 \times 0.3) + (2 \times 0.975^3 \times 0.4) + (4 \times 0.96^3 \times 0.5) \\
&+ (2 \times 0.94^3 \times 0.6) + (4 \times 0.92^3 \times 0.7) + (2 \times 0.87^3 \times 0.8) \\
&+ (4 \times 0.755^3 \times 0.9) + (1 \times 0^3 \times 1.0)] \times 2/30 \\
&= 0.680
\end{aligned}
$$

These integrals give the ratios $\langle v^2 \rangle / \langle v \rangle^2 = 1.08$ and $\langle v^3 \rangle / \langle v \rangle^3 = 1.18$. As one might expect from inspection of the plotted curves, these ratios differ significantly from the values 1.02 and 1.06 calculated for a 1/7th-power velocity profile.

**7B.1 Velocity averages from the $\frac{1}{7}$-power law**

*a.* Average of the velocity

$$\frac{\langle \bar{v}_z \rangle}{v_{\max}} = \frac{\int_0^{2\pi}\int_0^R \left[1-(r/R)\right]^{1/7} r dr d\theta}{\int_0^{2\pi}\int_0^R r dr d\theta} = 2\int_0^1 (1-\xi)^{1/7}\,\xi d\xi$$

$$= 2\int_0^1 x\left(1-x^7\right)\left(7x^6\right)dx = 14\left(\tfrac{1}{8}-\tfrac{1}{15}\right) = \tfrac{49}{60}$$

*b.* Average of the square of the velocity

$$\frac{\langle \bar{v}_z^2 \rangle}{v_{\max}^2} = \frac{\int_0^{2\pi}\int_0^R \left[1-(r/R)\right]^{2/7} r dr d\theta}{\int_0^{2\pi}\int_0^R r dr d\theta} = 2\int_0^1 (1-\xi)^{2/7}\,\xi d\xi$$

$$= 2\int_0^1 x\left(1-x^{7/2}\right)\left(\tfrac{7}{2}x^{5/2}\right)dx = 7\left(\tfrac{2}{9}-\tfrac{1}{8}\right) = \tfrac{49}{72}$$

so that $\langle \bar{v}_z^2 \rangle / \langle \bar{v}_z \rangle^2 = \left(\tfrac{49}{72}\right)\left(\tfrac{60}{49}\right)^2 = \tfrac{50}{49}$

*c.* Average of the cube of the velocity

$$\frac{\langle \bar{v}_z^3 \rangle}{v_{\max}^3} = \frac{\int_0^{2\pi}\int_0^R \left[1-(r/R)\right]^{3/7} r dr d\theta}{\int_0^{2\pi}\int_0^R r dr d\theta} = = 2\int_0^1 (1-\xi)^{3/7}\,\xi d\xi$$

$$= 2\int_0^1 x\left(1-x^{7/3}\right)\left(\tfrac{7}{3}x^{4/3}\right)dx = \tfrac{14}{3}\left(\tfrac{3}{10}-\tfrac{3}{17}\right) = \tfrac{49}{85}$$

so that $\langle \bar{v}_z^3 \rangle / \langle \bar{v}_z \rangle^3 = \left(\tfrac{49}{85}\right)\left(\tfrac{60}{49}\right)^3 = \tfrac{43200}{40817}$

**7B.2 Relation between force and viscous loss for flow in conduits of variable cross section**

For the variable cross-section, the macroscopic balances in Eqs. 7.5-5 and 6 have to be restated (when the gravitational forces can be neglected) thus:

(z-momentum) $$F_{f\to s} = (v_1 - v_2)w + (p_1S_1 - p_2S_2)$$

(mechanical energy) $$\hat{E}_v = \tfrac{1}{2}\left(v_1^2 - v_2^2\right) + \frac{1}{\rho}(p_1 - p_2)$$

To prove that Eq. 7B.2-1 is correct, we multiply the second equation above by $\rho S_m$ (defined in the problem statement) to get

$$\begin{aligned}\rho S_m \hat{E}_v &= \tfrac{1}{2}\rho S_m\left(v_1^2 - v_2^2\right) + S_m(p_1 - p_2)\\ &= \tfrac{1}{2}\rho S_m(v_1 - v_2)(v_1 + v_2) + S_m(p_1 - p_2)\end{aligned}$$

If Eq. 7B.2-1 is to be correct, then in the term containing the velocities, we must have $w = \tfrac{1}{2}\rho S_m(v_1 + v_2)$. That this is indeed true can be seen as follows:

$$\tfrac{1}{2}\rho S_m(v_1 + v_2) = \rho\left(\frac{w}{\rho S_1} + \frac{w}{\rho S_2}\right)\left(\frac{S_1S_2}{S_1 + S_2}\right) = \rho\left(\frac{w}{\rho}\right)\left(\frac{1}{S_1} + \frac{1}{S_2}\right)\left(\frac{S_1S_2}{S_1 + S_2}\right) = w$$

Next, we have to verify that $(p_1S_1 - p_2S_2) = S_m(p_1 - p_2) + p_m(S_1 - S_2)$. Substituting the definitions of $S_m$ and $p_m$ gives for the right side:

$$\begin{aligned}&\left(\frac{2S_1S_2}{S_1 + S_2}\right)(p_1 - p_2) + \left(\frac{p_1S_1 + p_2S_2}{S_1 + S_2}\right)(S_1 - S_2)\\ &\quad = \frac{2S_1S_2p_1 - 2S_1S_2p_2 + S_1^2p_1 - S_2^2p_2 - S_1S_2p_1 + S_1S_2p_2}{S_1 + S_2}\\ &\quad = \frac{p_1S_1(S_1 + S_2) - p_2S_2(S_1 + S_2)}{S_1 + S_2} = p_1S_1 - p_2S_2\end{aligned}$$

(Ref: R. B. Bird, *CEP Symposium Ser. #58*, Vol. 61 (1965), pp. 14-15.

**7B.3 Flow through a sudden enlargement**

The pressure rise is given by Eq. 7.6-5, and, in Eq. 7.6-6, in terms of the downstream velocity. We need a similar expression written in terms of the upstream velocity:

$$p_2 - p_1 = \rho(\beta v_1)[v_1 - (\beta v_1)] = \rho v_1^2 \beta(1-\beta)$$

Then, to get the extremum, we set the first derivative equal to zero:

$$\frac{d(p_2 - p_1)}{d\beta} = \rho v_1^2 (1 - 2\beta) = 0$$

Hence, we get

$$\beta = \frac{S_1}{S_2} = \frac{D_1^2}{D_2^2} = \frac{1}{2} \qquad \text{or} \qquad \frac{D_2}{D_1} = \sqrt{2}$$

We still have to verify that this is a maximum for the velocity. To do that, we need the second derivative:

$$\frac{d^2(p_2 - p_1)}{d\beta^2} = -2\rho v_1^2 < 0$$

Since this is negative, we have found the location of the maximum. The maximum pressure difference is then

$$p_2 - p_1 = \rho v_1^2 \beta(1-\beta) = \tfrac{1}{4}\rho v_1^2$$

**7B.4 Flow between two tanks**

The flow in a tube can be described by the steady-state macroscopic energy balance of Eq. 7.4-7, which for this problem simplifies to

$$\frac{p_B - p_A}{\rho} = -\hat{E}_v \qquad \text{or} \qquad p_A - p_B = \rho \hat{E}_v$$

Then using Eqs. 7.5-8, 6.2-16, and 6.2-12 we get

$$p_A - p_B = \tfrac{1}{2}\rho\langle v^2\rangle \frac{L}{R_h} f = \tfrac{1}{2}\rho\langle v^2\rangle \frac{4L}{D} f = \tfrac{1}{2}\rho\langle v^2\rangle \frac{4L}{D} \cdot \frac{0.0791}{\mathrm{Re}^{1/4}}$$

(Note that we could just as well have started with Eqs. 6.1-4 and 6.2-12.) The last equation can now be written for each of the two systems, and we omit the factors involving numerical constants, tube length, and the physical properties (since they are the same for the two systems):

$$(p_A - p_B)_{\mathrm{I}} \propto \left(\frac{w_{\mathrm{I}}}{S_{\mathrm{I}}}\right)^2 \frac{1}{D_{\mathrm{I}}}\left(\frac{D_{\mathrm{I}} w_{\mathrm{I}}}{S_{\mathrm{I}}}\right)^{-1/4} \propto \left(\frac{w_{\mathrm{I}}}{S_{\mathrm{I}}}\right)^2 \frac{1}{\sqrt{S_{\mathrm{I}}}}\left(\frac{w_{\mathrm{I}}}{\sqrt{S_{\mathrm{I}}}}\right)^{-1/4}$$

$$(p_A - p_B)_{\mathrm{II}} \propto \left(\frac{w_{\mathrm{II}}}{S_{\mathrm{II}}}\right)^2 \frac{1}{D_{\mathrm{II}}}\left(\frac{D_{\mathrm{II}} w_{\mathrm{II}}}{S_{\mathrm{II}}}\right)^{-1/4} \propto \left(\frac{w_{\mathrm{II}}}{S_{\mathrm{II}}}\right)^2 \frac{1}{\sqrt{S_{\mathrm{II}}}}\left(\frac{w_{\mathrm{II}}}{\sqrt{S_{\mathrm{II}}}}\right)^{-1/4}$$

$$\propto \left(\frac{w_{\mathrm{I}}}{S_{\mathrm{I}}}\right)^2 \frac{1}{\sqrt{\frac{1}{2}S_{\mathrm{I}}}}\left(\frac{\frac{1}{2}w_{\mathrm{I}}}{\sqrt{\frac{1}{2}S_{\mathrm{I}}}}\right)^{-1/4}$$

In the last step, we have used $S_{\mathrm{II}} = \frac{1}{2}S_{\mathrm{I}}$ and $w_{\mathrm{II}} = \frac{1}{2}w_{\mathrm{I}}$. Hence

$$\frac{(p_A - p_B)_{\mathrm{II}}}{(p_A - p_B)_{\mathrm{I}}} = \frac{1}{\sqrt{\frac{1}{2}}}\left(\frac{\frac{1}{2}}{\sqrt{\frac{1}{2}}}\right)^{-1/4} = \sqrt{2}\left(\sqrt{2}\right)^{1/4} = 2^{(1/2)+(1/8)} = 2^{5/8}$$

**7B.5 Revised design of an air duct**

*a*. The flow in a tube can be described by the steady-state macroscopic energy balance of Eq. 7.4-7, which for this problem simplifies to

$$\frac{p_2 - p_1}{\rho} = -\hat{E}_v \qquad \text{or} \qquad p_1 - p_2 = \rho \hat{E}_v$$

Then using Eqs. 7.5-8, 6.2-16, and 6.2-12 we get

$$p_1 - p_2 = \tfrac{1}{2}\rho\langle v^2\rangle \frac{L}{R_h} f = \tfrac{1}{2}\rho\langle v^2\rangle \frac{4L}{D} f = \tfrac{1}{2}\rho\langle v^2\rangle \frac{4L}{D} \cdot \frac{0.0791}{\mathrm{Re}^{1/4}}$$

(Note that we could also have started with Eqs. 6.1-4 and 6.2-12.)

*b*. Label the two systems--the original and the revised--I and II. Then the pressure drop in the two systems will be

$$(p_1 - p_2)_{\mathrm{I}} = \frac{0.0791}{\left(4 R_{h,\mathrm{I}} \langle v\rangle_{\mathrm{I}} \rho/\mu\right)^{1/4}} \frac{\tfrac{1}{2}\rho\langle v^2\rangle_{\mathrm{I}} L}{R_{h,\mathrm{I}}}$$

$$(p_1 - p_2)_{\mathrm{II}} = \frac{0.0791}{\left(4 R_{h,\mathrm{II}} \langle v\rangle_{\mathrm{II}} \rho/\mu\right)^{1/4}} \frac{\tfrac{1}{2}\rho\langle v^2\rangle_{\mathrm{II}} L}{R_{h,\mathrm{II}}}$$

Since the numerical constant, the length $L$, and the values of the physical constants are exactly the same, we now omit those quantities from consideration and concentrate on the quantities that are different. Then we have (with $W$ and $H$ being used for the width and height of the ducts):

$$(p_1 - p_2)_{\mathrm{I}} \propto \frac{\langle v\rangle_{\mathrm{I}}^{7/4}}{R_{h,\mathrm{I}}^{5/4}} \propto \frac{(w/S_{\mathrm{I}})^{7/4}}{(S_{\mathrm{I}}/Z_{\mathrm{I}})^{5/4}} \propto \frac{Z_{\mathrm{I}}^{5/4}}{S_{\mathrm{I}}^3} = \frac{[2(W_{\mathrm{I}} + H_{\mathrm{I}})]^{5/4}}{[W_{\mathrm{I}} H_{\mathrm{I}}]^3}$$

$$(p_1 - p_2)_{\mathrm{II}} \propto \frac{\langle v\rangle_{\mathrm{II}}^{7/4}}{R_{h,\mathrm{II}}^{5/4}} \propto \frac{(w/S_{\mathrm{II}})^{7/4}}{(S_{\mathrm{II}}/Z_{\mathrm{II}})^{5/4}} \propto \frac{Z_{\mathrm{II}}^{5/4}}{S_{\mathrm{II}}^3} = \frac{[2(W_{\mathrm{II}} + H_{\mathrm{II}})]^{5/4}}{[W_{\mathrm{II}} H_{\mathrm{II}}]^3}$$

We have used the fact that $w$ is the same in both systems. Next we make use of the the fact that the pressure drop is the same in both systems:

$$\frac{[2(W_{\mathrm{I}}+H_{\mathrm{I}})]^{5/4}}{[W_{\mathrm{I}}H_{\mathrm{I}}]^{3}}=\frac{[2(W_{\mathrm{II}}+H_{\mathrm{II}})]^{5/4}}{[W_{\mathrm{II}}H_{\mathrm{II}}]^{3}}$$

*c*. Next we put in numerical values

$$\frac{[2(4+4)]^{5/4}}{[4\cdot 4]^{3}}=\frac{[2(W_{\mathrm{II}}+2)]^{5/4}}{[2W_{\mathrm{II}}]^{3}}$$

Taking the fourth power of both sides gives

$$\frac{[16]^{5}}{[16]^{12}}=\frac{[2(W_{\mathrm{II}}+2)]^{5}}{[2W_{\mathrm{II}}]^{12}}$$

Thus the equation for $2W_{\mathrm{II}}=x$ becomes

$$\frac{1}{16^{7}}=\frac{(x+4)^{5}}{x^{12}}$$

This equation can be solved by trial and error, and the solution is $x$ = 18.4 ft or $W_{\mathrm{II}}$ = 9.2 ft.

**7B.6 Multiple discharge into a common conduit**

*a.* Application of the macroscopic mass balance gives:

$$w_1 = w_2 \qquad \text{or} \qquad \langle v_1 \rangle S_1 = \langle v_2 \rangle S_2 \qquad \text{(for incompressible flow)}$$

Therefore, if the ratio of cross-sectional areas $\beta = S_1/S_2$ is specified, the relation between the average velocities is

$$\frac{\langle v_2 \rangle}{\langle v_1 \rangle} = \beta$$

We now define, for later use, a set of quantities $K_i^{(j)}$ as

$$K_i^{(j)} = \frac{\langle v_i^j \rangle}{\langle v_i \rangle^j}$$

where the index $i$ indicates inlet (1) and (2), and $j$ is the power to which the velocities are raised. For $j$ = 2 and 3, these quantities are (for the 1/7th power law of turbulent flow) $\frac{50}{49}$ and $\frac{43200}{40817}$ respectively (see Problem 7B.1)..

*b.* Application of the macroscopic momentum balance in the direction of flow gives (with no external forces):

$$\rho \langle v_1^2 \rangle S_1 - \rho \langle v_2^2 \rangle S_2 + p_1 S_1 - p_2 S_2 - F_{f \to s} = 0$$

The force of the fluid on the solid will consist of the viscous force acting tangentially on the walls (which we neglect) and the normal force $p_1(S_2 - S_1)$ acting in the direction opposite to the direction of the flow. Thus, when the quantity $\beta = S_1/S_2$ is introduced, we have

$$\rho \langle v_1^2 \rangle \left( \beta - \frac{\langle v_2^2 \rangle}{\langle v_1^2 \rangle} \frac{\langle v_1 \rangle^2}{\langle v_2 \rangle^2} \beta^2 \right) + (p_1 - p_2) = 0$$

Hence the pressure rise is given by

$$p_2 - p_1 = \rho\langle v_1\rangle^2\left(\beta K_1^{(2)} - \beta^2 K_2^{(2)}\right)$$

*c*. Finally we apply the steady-state mechanical energy balance (Eq. 7.4-7) for incompressible fluids to get

$$\hat{E}_v = \frac{1}{2}\left(\frac{\langle v_1^3\rangle}{\langle v_1\rangle} - \frac{\langle v_2^3\rangle}{\langle v_2\rangle}\right) + \frac{p_1 - p_2}{\rho}$$

The pressure difference is evaluated from part (*b*), so that we get

$$\begin{aligned}
\hat{E}_v &= \frac{1}{2}\langle v_1\rangle^2\left(\frac{\langle v_1^3\rangle}{\langle v_1\rangle^3} - \frac{\langle v_2^3\rangle}{\langle v_2\rangle^3}\frac{\langle v_2\rangle^2}{\langle v_1\rangle^2}\right) - \langle v_1\rangle^2\left(\beta K_1^{(2)} - \beta^2 K_2^{(2)}\right) \\
&= \frac{1}{2}\langle v_1\rangle^2 K_1^{(3)}\left(1 - \beta^2\frac{K_2^{(3)}}{K_1^{(3)}}\right) - \langle v_1\rangle^2\left(\beta K_1^{(2)} - \beta^2 K_2^{(2)}\right) \\
&= \frac{1}{2}\langle v_1\rangle^2 K_1^{(3)}\left(1 - 2\beta\frac{K_1^{(2)}}{K_1^{(3)}} + \beta^2\left(2\frac{K_2^{(2)}}{K_1^{(3)}} - \frac{K_2^{(3)}}{K_1^{(3)}}\right)\right)
\end{aligned}$$

For *turbulent flow*, the ratios of the $K_i^{(j)}$ in these expressions are of the order of unity.

**7B.7 Inventory variations in a gas reservoir**

*a*. The maximum, minimum, and average values of $w_2$ are

$$(w_2)_{\max} = A + B \qquad\qquad (w_2)_{\min} = A - B$$

$$(w_2)_{\text{avg}} = \frac{\int_{\omega t=0}^{2\pi}(A + B\cos\omega t)d(\omega t)}{\int_{\omega t=0}^{2\pi}d(\omega t)} = A$$

*b*. A mass balance over a 24-hour period is obtained by integrating the unsteady mass balance over the 24-hour period:

$$\int_0^{24}\left(\frac{dm_{\text{tot}}}{dt}\right)dt = \int_0^{24} w_1 dt - \int_0^{24} w_2 dt$$

If the amount of gas in the reservoir is to have the same average value over a 24-hour period, then

$$0 = 24w_1 - 24A$$

from which $w_1 = A$.

*c*. The total mass of gas in the reservoir as a function of time can be found by integrating the unsteady macroscopic mass balance:

$$\int_0^t\left(\frac{dm_{\text{tot}}}{d\bar{t}}\right)d\bar{t} = \int_0^t w_1 d\bar{t} - \int_0^t w_2 d\bar{t}$$

to get

$$m_{\text{tot}}(t) = m_{\text{tot}}^0 + \int_0^t(A - A - B\cos\omega\bar{t})d\bar{t} = m_{\text{tot}}^0 - \frac{B}{\omega}\sin\omega t$$

*d*. In this problem we are assuming that the density of the fluid does not appreciably change with time, and so we assume that it is constant. However, the volume of the gas. $V(t)$, will be a function of time and $m_{\text{tot}}(t) = \rho V(t)$, so that

$$\rho V(t) = m_{\text{tot}}^0 - \frac{B}{\omega}\sin\omega t$$

The criterion for uninterrupted operation of the system is that $V(t)$ must never go below zero. In this limiting situation, the quantity $m_{\text{tot}}(t) = \rho V(t)$ must then oscillate between the minimum value, $m_{\text{tot}}^0 - (B/\omega)$ ,and the maximum value, $m_{\text{tot}}^0 + (B/\omega)$. Therefore, the minimum total mass in the system that can accommodate this kind of oscillation is that

$$\rho V_{\min} = [m_{\text{tot}}^0 + (B/\omega)] - [m_{\text{tot}}^0 - (B/\omega)]$$

or

$$V_{\min} = \frac{2B}{\rho\omega} = \frac{(2)(2000)}{(0.044)(2\pi/24)} = 3.48\times10^5 \text{ ft}^3$$

3. Add a three-day supply to the amount found above

$$V_{\min} = \frac{2B}{\rho\omega} + (3\times24A) = \left(3.48\times10^5\right) + \frac{(72)(5000)}{(0.044)}$$
$$= \left(3.48\times10^5\right) + \left(8.18\times10^6\right) = 8.53\times10^6 \text{ ft}^3$$

**7B.8 Change in liquid height with time**

*a.* We want to calculate the volume between the liquid level at $h$ and the part of the sphere below the liquid level. The sphere is visualized as being generated by a circle in the $xz$ plane, with its bottom at the origin and its center at $x = 0, z = R$. Such a circle has the equation

$$x^2 + (z - R)^2 = R^2$$

or

$$x^2 = 2Rz - z^2$$

Next we visualize the liquid volume as being made up of thin circular disks of thickness $dz$, each disk with a volume

$$dV = \pi x^2 dz = \pi\left(2Rz - z^2\right)dz$$

Then the total volume of the liquid is

$$\begin{aligned} V &= \pi\int_0^h \left(2Rz - z^2\right)dz \\ &= \pi\left(Rz^2 - \tfrac{1}{3}z^3\right)\Big|_0^h \\ &= \pi\left(Rh^2 - \tfrac{1}{3}h^3\right) \end{aligned}$$

or

$$V = \pi R h^2\left(1 - \frac{1}{3}\frac{h}{R}\right)$$

This may be checked by verifying that when the tank is full, $h = 2R$, and the liquid volume is $V = \frac{4}{3}\pi R^3$; that when the tank is half full, $h = R$, and the liquid volume is $V = \frac{2}{3}\pi R^3$; and that when the tank is empty, $h = 0$, the liquid volume is $V = 0$

(Note: This method of obtaining the liquid volume was suggested by Professsor L. E. Wedgewood, University of Illinois at Chicago)

*b*. To get the liquid height as a function of the time, we start with the differential equation in Eq. 7.1-7 which may be rearranged to give:

$$\left(H - 2(L+R) + \frac{L(2R+L)}{H}\right)\frac{dH}{dt} = A$$

This separable, first-order equation can be integrated to give

$$\tfrac{1}{2}H^2 - 2(L+R)H + L(2R+L)\ln H = At + C$$

At $t = 0$, $h = 2R$, and $H = 2R + L$. Therefore

$$\tfrac{1}{2}(2R+L)^2 - 2(L+R)(2R+L) + L(2R+L)\ln(2R+L) = C$$

Subtraction and elimination of the integration constant then gives

$$\tfrac{1}{2}\left[H^2 - (2R+L)^2\right] - 2(L+R)\left[H - (2R+L)\right] + L(2R+L)\ln\frac{H}{2R+L} = At$$

or

$$\tfrac{1}{2}\left[(h+L)^2 - (2R+L)^2\right] - 2(L+R)(h-2R) + L(2R+L)\ln\frac{h+L}{2R+L} = At$$

When $h = 2R$, this equation gives $t = 0$, and when $h = 0$, it gives $t = t_{\text{efflux}}$ exactly as in Eq. 7.1-8. We now introduce dimensionless variables: $\eta = h/2R$ (dimensionless liquid height) and $\lambda = L/2R$ (dimensionless tube length). Then the above equation becomes:

$$\tfrac{1}{2}\left[(\eta+\lambda)^2 - (1+\lambda)^2\right] - (2\lambda+1)(\eta-1) + \lambda(1+\lambda)\ln\frac{\eta+\lambda}{1+\lambda} = \frac{At}{4R^2}$$

*c*. The parameter $\lambda = L/2R$ is fixed by the geometry of the system. Choose values of $\eta = h/2R$ from 0 to 1 and calculate $t$. from the above equation. These may be plotted to give the curve of the dimensionless liquid height versus the dimensionless time.

**7B.9 Draining of a cylindrical tank with exit pipe**

*a*. First we write the unsteady-state mass balance for the tank. The mass of fluid in the tank at any time is $\pi R^2 h\rho$, and hence the mass balance is (since there is no inflow stream)

$$\frac{d}{dt}\pi R^2 h\rho = -w_2$$

The quantity $w_2$ is the mass rate of flow out of the tank, and this is equal to the mass rate of flow in the tube. The latter is given by the Hagen-Poiseuille formula:

$$w_2 = \frac{\pi(\mathcal{P}_0 - \mathcal{P}_L)D^4\rho}{128\mu L} = \frac{\pi(\rho gh + \rho gL)D^4\rho}{128\mu L}$$

Here $\rho gh$ is the pressure $p_0$ exerted by the fluid above the tube entrance, and $\rho gL$ is the "$\rho gh$-term" in the expression for $\mathcal{P}$ in fn. 1 on p. 50 and discussed after Eq. 3.5-7 on p. 84. (It is unfortunate that we are dealing with two $h$'s here: the $h$ in the expression for $\mathcal{P}$ and the $h$ which is the height of the fluid in the cylindrical tank; these must not be confused with each other.)

The mass balance is then

$$\pi R^2\rho\frac{dh}{dt} = -\frac{\pi\rho g(h+L)D^4\rho}{128\mu L} \quad \text{or} \quad \frac{dh}{dt} = -\frac{g(h+L)D^4\rho}{128\mu LR^2}$$

This first-order, separable equation can be integrated thus:

$$\int_H^0 \frac{dh}{h+L} = -\int_0^{t_{\text{efflux}}} \frac{\rho gD^4}{128\mu LR^2}dt \quad \text{or} \quad t_{\text{efflux}} = \frac{128\mu LR^2}{\rho gD^4}\ln\left(1+\frac{H}{L}\right)$$

*b*. One has many mass-flow-rate vs. pressure-difference relations to choose from in turbulent flow. For purposes of illustration we use Eq. 5.1-6, which can be shown to be identical to the Blasius formula given in Eq. 6.2-12. Thus we replace the second equation in part (*a*) by

$$w = \frac{\pi}{2}\left(\frac{\rho R^{19/4}}{0.198\mu^{1/4}L}\right)^{4/7}(\mathcal{P}_0 - \mathcal{P}_L)^{4/7} = \frac{\pi}{2}\left(\frac{\rho R^{19/4}}{0.198\mu^{1/4}L}\right)^{4/7}[\rho g(h+L)]^{4/7}$$

Then the mass balance becomes

$$\frac{dh}{dt} = -\frac{1}{2}\left(\frac{\rho^{1/4}R^{5/4}}{0.198\mu^{1/4}L}\right)^{4/7}(h+L)^{4/7} \equiv -B(h+L)^{4/7}$$

This equation can be integrated to give

$$\tfrac{7}{3}(h+L)^{3/7} = -Bt + C$$

Application of the initial condition gives

$$\tfrac{7}{3}(H+L)^{3/7} = C$$

Subtraction then gives the expression for the instantaneous liquid level

$$\tfrac{7}{3}\left[(H+L)^{3/7} - (h+L)^{3/7}\right] = Bt$$

The efflux time is then

$$\tfrac{7}{3}\left[(H+L)^{3/7} - L^{3/7}\right] = Bt_{\text{efflux}}$$

or

$$t_{\text{efflux}} = \frac{14}{3}\left(\frac{0.198\mu^{1/4}L}{\rho^{1/4}R^{5/4}}\right)^{4/7}\left[(H+L)^{3/7} - L^{3/7}\right]$$

**7B.10 Efflux time for draining a conical tank**

*a*. From the unsteady mass balance we get, for the truncated cone system with no input stream

$$\frac{d}{dt}m_{\text{tot}} = -w_2 \qquad \text{or} \qquad \frac{d}{dt}\left(\tfrac{1}{3}\pi r^2 z\rho - \tfrac{1}{3}\pi r_2^2 z_2\right) = -\rho v_2\left(\pi r_2^2\right)$$

From Fig. 7B.10 we see that $r/r_2 = z/z_2$ so that the mass balance may be rewritten as

$$\frac{d}{dt}\left(\tfrac{1}{3}\pi z^3\rho(r_2/z_2)^2\right) = -\rho v_2\left(\pi r_2^2\right) \qquad \text{or} \qquad \frac{d}{dt}\left(\tfrac{1}{3}z^3\right) = -v_2 z_2^2$$

Therefore we get

$$z^2\frac{dz}{dt} = -v_2 z_2^2 \qquad \text{or} \qquad v_2 = -\frac{z^2}{z_2^2}\frac{dz}{dt}$$

*b*. We simplify the unsteady mechanical energy balance by (i) omitting the kinetic energy contribution on the left side, because it was shown in Example 7.7-1 to be unimportant, (ii) neglecting the viscous dissipation term $E_v$ because it is believed to be small, (iii) assuming incompressibility (which causes the $E_c$ term to drop out, (iv) omitting the work term, since there are no moving parts, (v) omitting the pressure terms on the right side, since the pressure is atmospheric at both ends, and (vi) assuming no vortex motion of te fluid. All potential energies are with respect to $z = 0$ as the datum plane. Hence we get:

$$\frac{d}{dt}\Phi_{\text{tot}} = -\left(\tfrac{1}{2}v_2^2 + gz_2\right)w_2$$

First we get the total potential energy in the fluid. This is done by integrating $\rho gz$ over the volume (regarded as a truncated cone):

$$\Phi_{\text{tot}} = \int_{V(t)} \rho\hat{\Phi}dV = \int_{V(t)} \rho gzdV = \rho g\int_{z_2}^{z} \bar{z}\cdot\pi\bar{r}^2 d\bar{z}$$

$$= \rho g \int_{z_2}^{z} \bar{z} \cdot \pi \left( \frac{r_2}{z_2} \bar{z} \right)^2 d\bar{z} = \rho g \pi \left( \frac{r_2}{z_2} \right)^2 \int_{z_2}^{z} \bar{z}^3 d\bar{z} = \frac{\rho g \pi}{4} \left( \frac{r_2}{z_2} \right)^2 \left( z^4 - z_2^4 \right)$$

Then the unsteady mechanical energy balance becomes

$$\frac{\rho g \pi}{4} \left( \frac{r_2}{z_2} \right)^2 \frac{d}{dt} \left( z^4 - z_2^4 \right) = -\left( \tfrac{1}{2} v_2^2 + g z_2 \right) \left( \rho v_2 \pi r_2^2 \right) \quad \text{or}$$

$$\rho g \pi \left( \frac{r_2}{z_2} \right)^2 z^3 \frac{dz}{dt} = -\left( \tfrac{1}{2} v_2^2 + g z_2 \right) \left( \rho v_2 \pi r_2^2 \right)$$

Then, using the result of the mass balance to eliminate the time derivatve, we get

$$\rho g \pi \left( \frac{r_2}{z_2} \right)^2 z^3 \left( -\frac{z_2^2}{z^2} v_2 \right) = -\left( \tfrac{1}{2} v_2^2 + g z_2 \right) \left( \rho v_2 \pi r_2^2 \right)$$

After dividing through by $-\rho \pi r_2^2 v_2$ we get Torricelli's law:

$$g z = \tfrac{1}{2} v_2^2 + g z_2 \qquad \text{or} \qquad v_2 = \sqrt{2g(z - z_2)}$$

*c*. Using the results of (*a*) and (*b*) we get

$$-\frac{dz}{dt} = \left( \frac{z_2}{z} \right)^2 \sqrt{2g(z - z_2)} \approx \frac{\sqrt{2g} z_2^2}{z^{3/2}}$$

By making the indicated simplification, we can perform the integration, along with the initial condition that at $t = 0$, $z = z_0$, to get

$$z_0^{5/2} - z^{5/2} = \tfrac{5}{2} \sqrt{2g} z_2^2 t \qquad \text{and} \qquad t_{\text{efflux}} = \frac{1}{5} \left( \frac{z_0}{z_2} \right)^2 \sqrt{\frac{2z_0}{g}}$$

To get the efflux time, we have assumed that $z = z_2 \approx 0$ when the container is empty.

**7B.11 Disintegration of wood chips**

We start with Eq. 7.5-10, taking plane 1 at the top of the slurry dispersion and plane 2 right at the outlet to the digester; we also take plane 2 to be the datum plane for the calculation of the potential energy:

$$\tfrac{1}{2}\left(v_2^2 - 0\right) + \left(0 - gz_1\right) + \frac{1}{\rho}\left(p_2 - p_1\right) = 0$$

Therefore the exit velocity is

$$v_2 = \sqrt{2\left(\frac{1}{\rho}\left(p_1 - p_2\right) + gz_1\right)}$$

$$= \sqrt{2\left(\frac{\left(100\ \text{lb}_f/\text{in}^2\right)\left(144\ \text{in}^2/\text{ft}^2\right)\left(32.2\ \text{poundals}/\text{lb}_f\right)}{65\ \text{lb}_m/\text{ft}^3} + \left(32.2\ \text{ft}/\text{s}^2\right)\left(20\ \text{ft}\right)\right)}$$

$$= \sqrt{15550} = 124\ \text{ft/s}$$

Therefore the mass rate of flow at the exit is

$$w_2 = \rho v_2 S_2 = \left(65\ \text{lb}_m/\text{ft}^3\right)\left(124\ \text{ft/s}\right)\left(\tfrac{1}{9}\pi\,\text{ft}^2\right) = 2910\ \text{lb}_m/\text{s}$$

Next, we apply Eq. 7.2-3 (the momentum balance) to get the impact force (assuming that the pressure and external force terms are quite negligible):

$$F_{f\to s} = v_2 w_2 = \left(124\ \text{ft/s}\right)\left(2910\ \text{lb}_m/\text{s}\right)\left(1/32\ \text{lb}_f/\text{poundals}\right)$$
$$= 10{,}900\ \text{lb}_f$$

**7B.12 Criterion for Vapor-Free Flow in a Piping System.**

In the piping system of Fig. 7.5-1, the criterion $p > p_{\text{vap}}$ might be violated either (i) in the pump, or (ii) at a plane "A" just downstream of the final elbow. Lacking data on the NPSH (net positive suction head) requirements of the pump at the given operating condition, we can only test for condition (ii); further information on (i) is available in *Perry's Chemical Engineers' Handbook* and in unit operations texts.

Applying Eq. 7.5-10 from plane "A" to just above plane "2", we find (since $v_A = v_2$ and no fittings or enlargement loss occurs in this vertical section),

$$g(z_2 - z_A) + \frac{p_2 - p_A}{\rho} = -\frac{v_2^2}{2}\frac{L_{A\to 2}}{R_h}f$$

which indicates the minimum pressure to be

$$p_A = p_2 + \rho\left[g(z_2 - z_A) + \frac{v_2^2}{2}\frac{L_{A\to 2}}{R_h}f\right]$$

Using values from page 208 and setting $p_A = 1$ atm $= 6.8087 \times 10^4$ poundals/ft$^2$, we find the pressure at plane "A" to be

$$\begin{aligned} p_A &= 68087 + 62.4\left[32.2(-20) + \frac{20}{5+300+100+120}(85)\right] \text{poundals/ft}^2 \\ &= 68087 - 40186 + 3 = 27905 \text{poundals/ft}^2 \\ &= 0.41 \text{ atm} \end{aligned}$$

Since the minimum pressure is well above the vapor pressure of water at the system conditions, the pipe will run full if the pump does. For mixtures, one must use the bubble-point pressure rather than the vapor pressure.

**7C.1 End corrections in tube viscometers.**

We apply the steady-state mechanical energy balance, in the form of Eq. 7.4-7, with the assumptions of incompressible fluid and no mechanical work:

$$\frac{1}{2}\left(\frac{\langle v_1^3\rangle}{\langle v_1\rangle}-\frac{\langle v_2^3\rangle}{\langle v_2\rangle}\right)+g(h_1-h_2)+\frac{1}{\rho}(p_1-p_2)=\hat{E}_v$$

where "1" and "2" are general labels for the input and output streams. We label the plane at the bottom of the tanks as Plane 5, and designate by $p_{\text{atm}}$ the pressure at the outlet plane.

Then for Run A, we apply the balance to the region between Plane 5 and Plane 2:

$$\frac{1}{2}\left(\frac{\langle v_5^3\rangle}{\langle v_5\rangle}-\frac{\langle v_2^3\rangle}{\langle v_2\rangle}\right)+gL_A+\frac{1}{\rho}(p_A+\rho g l_A-p_{\text{atm}})=\hat{E}_v(5\rightarrow 2)$$

Note that the kinetic energy term is nonzero, because the velocity distribution at the inlet and outlet are not the same.

For Run B, we apply the balance to the region between Plane 5 and Plane 0 to get:

$$\frac{1}{2}\left(\frac{\langle v_5^3\rangle}{\langle v_5\rangle}-\frac{\langle v_0^3\rangle}{\langle v_0\rangle}\right)+gL_A+\frac{1}{\rho}(p_B+\rho g l_B-p_0)=\hat{E}_v(5\rightarrow 0)$$

We now subtract the second equation from the first, noting that the kinetic energy terms will exactly cancel, as will the viscous dissipation terms, since the flow rates (and hence the velocity profiles) in the two systems are equal. This gives

$$\frac{1}{\rho}\left[(p_A-p_B)+\rho g(l_A-l_B)+(p_0-p_{\text{atm}})\right]=0 \qquad (*)$$

Next we apply the mechanical energy balance to the region between Plane 0 and Plane 4 in Run B:

$$g(L_B - L_A) + \frac{1}{\rho}(p_0 - p_{\text{atm}}) = \hat{E}_v(0 \to 4)$$

Here the kinetic energy terms do cancel, because the flow is fully developed at the inlet and outlet planes. The dissipation term can now be calculated by using Eq. 7.5-7 together with Eq. 2.3-22:

$$\hat{E}_v(0 \to 4) = \frac{F_{f \to s}(0 \to 4)}{\rho(\pi R^2)} = \frac{1}{\rho}\left[(p_0 - p_{\text{atm}}) + g(L_B - L_A)\right]$$

But, according to Fn. 1 on p 50,

$$\mathcal{P}_0 - \mathcal{P}_4 = (p_0 - p_4) + \rho g(h_0 - h_4) = (p_0 - p_{\text{atm}}) + \rho g(L_B - L_A)$$

Therefore, by combining the last three equations we get

$$g(L_B - L_A) + \frac{1}{\rho}(p_0 - p_{\text{atm}}) = \frac{1}{\rho}(\mathcal{P}_0 - \mathcal{P}_4) \qquad (**)$$

Now Eq. (**) can be rearranged and then combined with Eq. (*) to give:

$$\begin{aligned}
\frac{\mathcal{P}_0 - \mathcal{P}_4}{L_B - L_A} &= \rho g + \frac{p_0 - p_{\text{atm}}}{L_B - L_A} \\
&= \rho g - \frac{1}{L_B - L_A}\left[(p_A - p_B) + \rho g(l_B - l_A)\right] \\
&= \frac{p_B - p_A}{L_B - L_A} + \rho g\left(1 - \frac{l_A - l_B}{L_B - L_A}\right) \\
&= \frac{p_B - p_A}{L_B - L_A} + \rho g\left(1 + \frac{l_B - l_A}{L_B - L_A}\right)
\end{aligned}$$

in agreement with Eq. 7C.1-1.

## 7D.1 Derivation of the macroscopic balances from the equations of change

*a. Derivation of the macroscopic mass balance*

We start by integrating the equation of continuity, Eq.3.1-4 over the macroscopic flow system of Fig. 7.0-1:

$$\int_{V(t)} \frac{\partial \rho}{\partial t} dV = -\int_{V(t)} (\nabla \cdot \rho \mathbf{v}) dV$$

We write $V(t)$ to remind ourselves that the volume may be changing with time because of the presence of moving parts within the system. We now apply the Leibniz formula (Eq. A.5-5) to the left side and the Gauss divergence theorem (Eq. A.5-2) to the right side to get

$$\frac{d}{dt}\int_{V(t)} \rho dV - \int_{S(t)} \rho(\mathbf{n}\cdot\mathbf{v}_S) dS = -\int_{S(t)} (\mathbf{n}\cdot\rho\mathbf{v}) dS$$

We now combine the two surface integrals thus:

$$\frac{d}{dt} m_{\text{tot}} = -\int_{S(t)} (\mathbf{n}\cdot\rho(\mathbf{v}-\mathbf{v}_S)) dS$$

Next we divide the surface up into four parts as indicated on p. 221, right after Eq. 7.8-2. We also introduce the assumptions (i) and (ii) listed in §§7.1 and 7.2. Then we divide the surface into four nonoverlapping parts, $S = S_1 + S_2 + S_f + S_m$, and write the right side of the above equation as the sum of four contributions:

$$\frac{d}{dt} m_{\text{tot}} = -\int_{S_1} (\mathbf{n}\cdot\rho(\mathbf{v}-\mathbf{v}_S)) dS - \int_{S_2} (\mathbf{n}\cdot\rho(\mathbf{v}-\mathbf{v}_S)) dS - \int_{S_f} (\mathbf{n}\cdot\rho(\mathbf{v}-\mathbf{v}_S)) dS - \int_{S_m} (\mathbf{n}\cdot\rho(\mathbf{v}-\mathbf{v}_S)) dS$$

We now evaluate seriatim the terms on the right side:

The surface $S_1$ is the inlet plane, which is not moving so that $\mathbf{v}_S = 0$, and the outwardly directed unit normal vector $\mathbf{n}$ is the negative of the vector $\mathbf{u}_1$, which indicates the direction of the flow. Since $\mathbf{v} = \mathbf{v}_1$ at this surface is assumed (assumption (i)) to be exactly in the

direction of flow, we may say that $\mathbf{v} = \mathbf{u}_1 v_1$. The evaluation of the integral proceeds as follows:

$$\int_{S_1} (\mathbf{n} \cdot \rho(\mathbf{v} - \mathbf{v}_S)) dS = +\int_{S_1} (\mathbf{u}_1 \cdot \mathbf{u}_1 v_1 \rho) dS = +\int_{S_1} \rho v_1 dS = \rho_1 \langle v_1 \rangle S_1 = w_1$$

where it has been assumed that the density is constant over the cross section (assumption (ii)). The integral over the exit plane is evaluated in the same way, the only difference being that $\mathbf{n}$ is the same as $\mathbf{u}_2$, so that

$$-\int_{S_2} (\mathbf{n} \cdot \rho(\mathbf{v} - \mathbf{v}_S)) dS = -\int_{S_2} (\mathbf{u}_2 \cdot \mathbf{u}_2 v_2 \rho) dS = +\int_{S_2} \rho v_2 dS = \rho_2 \langle v_2 \rangle S_2 = w_2$$

On the fixed surfaces both $\mathbf{v}$ and $\mathbf{v}_S$ are zero, so that $S_f = 0$. Also, on the moving surfaces, $\mathbf{v} = \mathbf{v}_S$, with the result that $S_m$. Therefore, Eq. 7.1-1 follows at once:

$$\frac{d}{dt} m_{\text{tot}} = \rho_1 \langle v_1 \rangle S_1 - \rho_2 \langle v_2 \rangle S_2$$

and the definitions of $w_1$ and $w_2$ lead to Eq. 7.1-2.

*b. Derivation of the macroscopic momentum balance*

First we integrate Eq. 3.2-9 over the volume of the system:

$$\int_{V(t)} \left( \frac{\partial}{\partial t} \rho \mathbf{v} \right) dV = -\int_{V(t)} (\nabla \cdot \rho \mathbf{v} \mathbf{v}) dV - \int_{V(t)} (\nabla p) dV - \int_{V(t)} [\nabla \cdot \boldsymbol{\tau}] dV + \int_{V(t)} \rho \mathbf{g} dV$$

We now manipulate seriatim the terms in this equation so that they can be interpreted. In so doing we make use of assumptions i-iv in §§7.1 and 2, and follow the procedures in (*a*) above. The first term is rearranged by using the Leibniz formula (as applied to a vector function), and then $\mathbf{P}_{\text{tot}}$ is introduced as the definition of the integral over the momentum per unit volume, and the second term is rearranged, thus:

$$\int_{V(t)} \left( \frac{\partial}{\partial t} \rho \mathbf{v} \right) dV = \frac{d}{dt} \int_{V(t)} \rho \mathbf{v} dV - \int_{S(t)} \rho \mathbf{v} (\mathbf{v}_S \cdot \mathbf{n}) dS = \frac{d}{dt} \mathbf{P}_{\text{tot}} - \int_{S(t)} [\mathbf{n} \cdot \rho \mathbf{v} \mathbf{v}_S] dS$$

The first term and third terms on the right of the momentum balance are treated by using Eq. A.5-3, and the pressure term is treated by using Eq. A.5-2:

$$- \int_{V(t)} (\nabla \cdot \rho \mathbf{v} \mathbf{v}) dV = - \int_{S(t)} [\mathbf{n} \cdot \rho \mathbf{v} \mathbf{v}] dS$$

$$- \int_{V(t)} [\nabla \cdot \boldsymbol{\tau}] dV = - \int_{S(t)} [\mathbf{n} \cdot \boldsymbol{\tau}] dS$$

$$- \int_{V(t)} (\nabla p) dV = - \int_{S(t)} \mathbf{n} p dS$$

Finally, the last term is integrated (with $\mathbf{g}$ constant) and the definition of the total mass within the system (given above):

$$+ \int_{V(t)} \rho \mathbf{g} dV = + m_{\text{tot}} \mathbf{g}$$

When the above contributions to the momentum balance are combined we get the following equation:

$$\frac{d}{dt} \mathbf{P}_{\text{tot}} = - \int_{S(t)} [\mathbf{n} \cdot \rho \mathbf{v} (\mathbf{v} - \mathbf{v}_S)] dS - \int_{S(t)} \mathbf{n} p dS - \int_{S(t)} [\mathbf{n} \cdot \boldsymbol{\tau}] dS + m_{\text{tot}} \mathbf{g}$$

The surface integrals are interpreted by partitioning the surface of the system as before: $S = S_1 + S_2 + S_f + S_m$. The first integral on the right is zero on the fixed and moving surfaces, and the integrations on the inlet and outlet surfaces give the following:

$$\begin{aligned} - \int_{S(t)} [\mathbf{n} \cdot \rho \mathbf{v} (\mathbf{v} - \mathbf{v}_S)] dS &= - \int_{S_1} [-\mathbf{u}_1 \cdot \rho (\mathbf{u}_1 v_1)(\mathbf{u}_1 v_1)] - \int_{S_2} [\mathbf{u}_2 \cdot \rho (\mathbf{u}_2 v_2)(\mathbf{u}_2 v_2)] \\ &= + \mathbf{u}_1 \rho_1 \int_{S_1} v_1^2 dS - \mathbf{u}_2 \rho_2 \int_{S_2} v_2^2 dS \\ &= + \mathbf{u}_1 \rho_1 \langle v_1^2 \rangle S_1 - \mathbf{u}_2 \rho_2 \langle v_2^2 \rangle S_2 \end{aligned}$$

where assumption (ii) has been used--that the density of the fluid is constant over the cross-section. Similarly the contributions of the pressure integral at the entry and exit planes are:

$$-\int_{S_1+S_2} \mathbf{n} p dS = +\mathbf{u}_1 p_1 S - \mathbf{u}_2 p_2 S_2$$

where, according to assumption (iv), the pressure is assumed constant over the cross-section. The integral of the pressure over the fixed and moving surfaces contributes to the force of the fluid on the solid surfaces, and the negative of that is the pressure force of the solid on the fluid: $\mathbf{F}^{(p)}_{s\to f}$. Similarly the integral over the $[\mathbf{n}\cdot\boldsymbol{\tau}]$ term over the fixed and moving surfaces gives a contribution $\mathbf{F}^{(\tau)}_{s\to f}$. Therefore

$$-\int_{S_1+S_2} \mathbf{n} p dS - \int_{S_1+S_2} [\mathbf{n}\cdot\boldsymbol{\tau}] dS = \mathbf{F}^{(p)}_{s\to f} + \mathbf{F}^{(\tau)}_{s\to f} = \mathbf{F}_{s\to f}$$

According to assumption (iii) we neglect the contribution of the $[\mathbf{n}\cdot\boldsymbol{\tau}]$ terms at the entry and exit planes. When all of the contributions above are collected, we get Eq. 7.2-1.

$$\frac{d}{dt}\mathbf{P}_{\text{tot}} = +\rho_1\langle v_1^2\rangle S_1\mathbf{u}_1 - \rho_2\langle v_2^2\rangle S_2\mathbf{u}_2 + p_1 S\mathbf{u}_1 - p_2 S_2\mathbf{u}_2 + \mathbf{F}_{s\to f} + m_{\text{tot}}\mathbf{g}$$

and the definitions of $w_1$ and $w_2$ then lead to Eq. 7.2-2.

**8A.1 Flow of a polyisoprene solution in a pipe**

We use Eq. 8.3-9, for the mass flow rate of a power-law fluid in a circular pipe:

$$w = \frac{\pi R^3 \rho}{(1/n)+3}\left(\frac{(\mathcal{P}_0 - \mathcal{P}_L)R}{2mL}\right)^{1/n}$$

Consider two systems, labeled I and II differing only in the radius $R$ and the length $L$. Then, for the same mass flow rate and the same pressure difference, and also the same values of the density and the power-law parameters, from the above equation we know that the dimensions of the two systems will be related by

$$\frac{R_{\mathrm{I}}^{3+(1/n)}}{L_{\mathrm{I}}^{1/n}} = \frac{R_{\mathrm{II}}^{3+(1/n)}}{L_{\mathrm{II}}^{1/n}}$$

Solving for the radius of the second system, we get

$$R_{\mathrm{II}} = R_{\mathrm{I}}\left(\frac{L_{\mathrm{II}}}{L_{\mathrm{I}}}\right)^{1/(3n+1)}$$

Inserting the numerical values for this problem gives

$$R_{\mathrm{II}} = \left(\frac{1.3\ \mathrm{cm}}{2}\right)\left(\frac{30.6\ \mathrm{m}}{10.2\ \mathrm{m}}\right)^{5/8} = 1.25\ \mathrm{cm}$$

**8A.2 Pumping of a polyethylene oxide solution**

The starting point is the same as for Problem 8A.1. The mass rate of flow must be the same for the two designs, so that

$$\frac{\pi R_1^3 \rho}{(1/n)+3}\left(\frac{(\mathcal{P}_0 - \mathcal{P}_L)R_1}{2mL}\right)^{1/n} = 2 \cdot \frac{\pi R_2^3 \rho}{(1/n)+3}\left(\frac{(\mathcal{P}_0 - \mathcal{P}_L)R_2}{2mL}\right)^{1/n}$$

Therefore

$$R_1^{3+(1/n)} = 2R_2^{3+(1/n)}$$

whence

$$\left(\frac{R_2}{R_1}\right)^{3+(1/n)} = \frac{1}{2}$$

Therefore

$$\frac{D_2}{D_1} = \frac{R_2}{R_1} = \left(\frac{1}{2}\right)^{n/(3n+1)}$$

and, for $n = 0.6$

$$D_2 = D_1\left(\frac{1}{2}\right)^{0.6/2.8} = (27\text{ cm})\left(\frac{1}{2}\right)^{0.2143} = 23.3\text{ cm}$$

**8B.1 Flow of a polymeric film**

*a*. The momentum flux distribution is the same as Eq. 2.2-13, inasmuch as this result depends only on momentum conservation, and not on the constitutive relation.

*b*. Next we write the power-law model as

$$\tau_{xz} = -\eta \frac{dv_z}{dx} = -m\dot{\gamma}^{n-1}\frac{dv_z}{dx} = -m\left(-\frac{dv_z}{dx}\right)^{n-1}\frac{dv_z}{dx} = m\left(-\frac{dv_z}{dx}\right)^{n}$$

which is the appropriate form, because $dv_z/dx$ is negative.

*c*. When the above expression for $\tau_{xz}$ is substituted into Eq. 2.2-13, we get a differential equation for the velocity

$$m\left(-\frac{dv_z}{dx}\right)^{n} = (\rho g\cos\beta)x \qquad \text{or} \qquad -\frac{dv_z}{dx} = \left(\frac{\rho g\cos\beta}{m}\right)^{1/n} x^{1/n}$$

This equation is integrated to give

$$-v_z = \left(\frac{\rho g\cos\beta}{m}\right)^{1/n}\frac{x^{(1/n)+1}}{(1/n)+1} + C_2$$

Application of the no-slip condition at the wall gives

$$0 = \left(\frac{\rho g\cos\beta}{m}\right)^{1/n}\frac{\delta^{(1/n)+1}}{(1/n)+1} + C_2$$

Subtraction of the last two equations gives

$$v_z = \left(\frac{\rho g\cos\beta}{m}\right)^{1/n}\frac{\delta^{(1/n)+1} - x^{(1/n)+1}}{(1/n)+1}$$

$$= \left(\frac{\rho g\delta\cos\beta}{m}\right)^{1/n}\frac{\delta}{(1/n)+1}\left[1-\left(\frac{x}{\delta}\right)^{(1/n)+1}\right]$$

*d*. Then the mass rate of flow in the film is

$$w = \int_0^W \int_0^\delta \rho v_z dx dy = \rho W \left( \frac{\rho g \delta \cos\beta}{m} \right)^{1/n} \frac{\delta^2}{(1/n)+1} \int_0^1 \left( 1 - \xi^{(1/n)+1} \right) d\xi$$

$$= \rho W \left( \frac{\rho g \delta \cos\beta}{m} \right)^{1/n} \frac{\delta^2}{(1/n)+2} \xrightarrow{n \to 1, m \to \mu} \frac{\rho^2 g W \delta^3 \cos\beta}{3\mu}$$

**8B.2 Power-law flow in a narrow slit**

Substituting Eq. 8.3-11 into Eq. 2B.3-1, we get for $-B \le x \le 0$

$$-m\left(\frac{dv_z}{dx}\right)^n = \frac{(\mathcal{P}_0 - \mathcal{P}_L)x}{2L} \qquad \text{or} \qquad \frac{dv_z}{d(-x)} = -\left(\frac{(\mathcal{P}_0 - \mathcal{P}_L)}{2mL}\right)^{1/n}(-x)^{1/n}$$

Integration then gives

$$v_z = \left(\frac{(\mathcal{P}_0 - \mathcal{P}_L)}{2mL}\right)^{1/n} \frac{B^{(1/n)+1} - (-x)^{(1/n)+1}}{(1/n)+1}$$

$$= \left(\frac{(\mathcal{P}_0 - \mathcal{P}_L)B}{2mL}\right)^{1/n} \frac{B}{(1/n)+1}\left[1 - \left(\frac{-x}{B}\right)^{(1/n)+1}\right] \qquad (-B \le x \le 0)$$

This may be combined with Eq. 8.3-13 to get a solution that is valid for $-B \le x \le B$

$$v_z = \left(\frac{(\mathcal{P}_0 - \mathcal{P}_L)B}{2mL}\right)^{1/n} \frac{B}{(1/n)+1}\left[1 - \left|\frac{x}{B}\right|^{(1/n)+1}\right]$$

**8B.3 Non-Newtonian flow in an annulus**

*a*. From the equation of motion, the momentum-flux distribution is

$$\frac{1}{r}\frac{d}{dr}\left(r\tau_{rz}\right)=0 \qquad \text{or} \qquad {}_{rz}=\frac{C_1}{r}$$

The appropriate expression for the momentum flux component is

$$\tau_{rz}=-m\dot{\gamma}^{n-1}\frac{dv_z}{dr}=-m\left(-\frac{dv_z}{dr}\right)^{n-1}\frac{dv_z}{dr}=+m\left(-\frac{dv_z}{dr}\right)^{n}$$

When the above two expressions are combined, one gets a differential equation for the velocity distribution, which can be integrated to give (after applying the boundary conditions)

$$\frac{v_z}{v_0}=\frac{\xi^{1-(1/n)}-1}{\kappa^{1-(1/n)}-1} \quad (\text{here } \xi=r/R)$$

*b*. In the limit that $n\to 1$ and $m\to\mu$, we have to use L'Hôpital's rule

$$\frac{v_z}{v_0}=\lim_{\varepsilon\to 0}\frac{\xi^{\varepsilon}-1}{\kappa^{\varepsilon}-1}=\lim_{\varepsilon\to 0}\frac{\xi^{\varepsilon}\ln\xi}{\kappa^{\varepsilon}\ln\kappa}=\frac{\ln\xi}{\ln\kappa} \qquad \text{(see Problem 2B.7)}$$

*c*. The mass flow rate is then

$$w=2\pi\int_{\kappa R}^{R}\rho v_z r dr=2\pi R^2\rho v_0\int_{\kappa}^{1}\left(\frac{\xi^{2-(1/n)}-\xi}{\kappa^{1-(1/n)}-1}\right)d\xi$$

$$=\frac{2\pi R^2\rho v_0}{\kappa^{1-(1/n)}-1}\left(\frac{1-\kappa^{3-(1/n)}}{3-(1/n)}-\frac{1-\kappa^2}{2}\right) \qquad (n\neq\tfrac{1}{3})$$

*d*. When $n=\frac{1}{3}$ use of L'Hôpital's rule gives

$$w=\frac{2\pi\kappa^2R^2\rho v_0}{1-\kappa^2}\left(\ln\frac{1}{\kappa}-\frac{1-\kappa^2}{2}\right)$$

*e*. When L'Hôpital's rule is used we get

$$w = \pi R^2 \rho v_0 \left( \frac{1-\kappa^2}{2\ln(1/\kappa)} - \kappa^2 \right)$$

**8B.4 Flow of a polymeric liquid in a tapered tube**

We consider a small region of the tapered tube to be a straight tube over a short distance *dz*; then we can write "locally"

$$w = \frac{\pi R^3 \rho}{(1/n)+3}\left[-\frac{d\mathcal{P}}{dz}\frac{R}{2m}\right]^{1/n}$$

Take the *n*th power of both sides to get

$$-\frac{d\mathcal{P}}{dz} = \frac{2m}{R}\left[\frac{w}{\pi R^3 \rho}\left(\frac{1}{n}+3\right)\right]^n$$

in which $R$ is a function of $z$:

$$R = R_0 + \left(\frac{R_L - R_0}{L}\right)z$$

It is easier to integrate the differential equation if we rewrite it as

$$-\frac{d\mathcal{P}}{dR}\frac{dR}{dz} = -\frac{d\mathcal{P}}{dR}\left(\frac{R_L - R_0}{L}\right) = \frac{2m}{R}\left[\frac{w}{\pi R^3 \rho}\left(\frac{1}{n}+3\right)\right]^n$$

Then when this equation is integrated with respect to $R$, we get

$$-\int_{\mathcal{P}_0}^{\mathcal{P}_L} d\mathcal{P} = \left(\frac{2mL}{R_L - R_0}\right)\left[\frac{w}{\pi\rho}\left(\frac{1}{n}+3\right)\right]^n \int_{R_0}^{R_L}\frac{1}{R^{3n+1}}dR$$

Therefore

$$\mathcal{P}_0 - \mathcal{P}_L = \left(\frac{2mL}{R_L - R_0}\right)\left[\frac{w}{\pi\rho}\left(\frac{1}{n}+3\right)\right]^n\left(\frac{R_L^{-3n} - R_0^{-3n}}{-3n}\right)$$

$$= \left(\frac{2mL}{3n}\right)\left[\frac{w}{\pi\rho}\left(\frac{1}{n}+3\right)\right]^n\left(\frac{R_L^{-3n} - R_0^{-3n}}{R_0 - R_L}\right)$$

This is the power-law analog of Eq. 2B.10-3.

**8B.5 Slit flow of a Bingham fluid**

*a*. For $|x| \le x_0$ (i.e., in the region where the yield stress is not exceeded), $\eta = \infty$ (according to the upper equation of Eq. 8B.5-1. But the expression for the shear stress is, from Eq. 8.3-2: $\tau_{xz} = -\eta dv_z/dx$. Since the shear stress is finite, the velocity gradient must be equal to zero. This is the plug-flow region.

For $|x| \ge x_0$ (i.e., in the region where the yield stress is exceeded), the lower equation of Eq. 8B.5-1 has to be used. This means that in the region where $x \ge x_0$, the velocity will be decreasing in the positive $x$ direction, so that $\gamma = -dv_z/dx$ is required so that $\gamma$ will be positive. Similarly, when $x \le -x_0$, the velocity will be increasing in the positive $x$ direction, so that $\gamma = +dv_z/dx$ is needed in order to guarantee that $\gamma$ be positive. Hence we have:

$$\tau_{xz} = -\mu_0 \frac{dv_z}{dx} - \tau_0 \qquad \text{for} \quad -B \le x \le -x_0$$

$$\tau_{xz} = -\mu_0 \frac{dv_z}{dx} + \tau_0 \qquad \text{for} \quad +x_0 \le x \le +B$$

Since the flow will be symmetric about the plane $x = 0$, we need solve for the velocity distribution only in one half of the slit. We choose to work with the region $0 \le x \le +B$.

*b*. Substitution of the upper relation of Eq. 8B.5-3 into Eq. 2B.3-1 gives

$$-\mu_0 \frac{dv_z}{dx} + \tau_0 = \frac{\mathcal{P}_0 - \mathcal{P}_L}{L} x \qquad \text{or} \qquad \frac{dv_z}{dx} = -\frac{\mathcal{P}_0 - \mathcal{P}_L}{\mu_0 L} x + \frac{\tau_0}{\mu_0}$$

Integration gives:

$$v_z = -\frac{\mathcal{P}_0 - \mathcal{P}_L}{2\mu_0 L} x^2 + \frac{\tau_0}{\mu_0} x + C$$

Applying the no-slip boundary condition at the slit wall ($z = B$), we get

$$0 = -\frac{\mathcal{P}_0 - \mathcal{P}_L}{2\mu_0 L} B^2 + \frac{\tau_0}{\mu_0} B + C$$

Subtracting these last two equations eliminates the integration constant and gives

$$v_z = \frac{(\mathcal{P}_0 - \mathcal{P}_L)B^2}{2\mu_0 L}\left[1 - \left(\frac{x}{B}\right)^2\right] - \frac{\tau_0 B}{\mu_0}\left(1 - \frac{x}{B}\right) \qquad (+x_0 \le x \le +B)$$

$$v_z = \frac{(\mathcal{P}_0 - \mathcal{P}_L)B^2}{2\mu_0 L}\left[1 - \left(\frac{x_0}{B}\right)^2\right] - \frac{\tau_0 B}{\mu_0}\left(1 - \frac{x_0}{B}\right) \qquad (0 \le x \le x_0)$$

*c*. The mass flow rate is then obtained from

$$w = W\rho\int_{-B}^{+B} v_z dx = 2 \cdot W\rho\int_0^{+B} v_z dx$$

One could split this integral up into two parts, one for the region $0 \le x \le x_0$ and the other for $+x_0 \le x \le +B$, and evaluate them using the velocity profiles worked out in (*b*). A better way to do the integral is to integrate by parts:

$$w = 2 \cdot W\rho\left[v_z x\Big|_0^B - \int_0^B x\left(\frac{dv_z}{dx}\right)dx\right]$$

Then the first term is zero at both limits, and the second term has no contribution from the portion from $x$ to $x_0$. Hence

$$\begin{aligned} w &= 2 \cdot W\rho\int_{x_0}^B x\left(-\frac{dv_z}{dx}\right)dx = 2 \cdot W\rho\int_{x_0}^B x\left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{\mu_0 L}x - \frac{\tau_0}{\mu_0}\right)dx \\ &= 2 \cdot W\rho\left[\frac{(\mathcal{P}_0 - \mathcal{P}_L)x^3}{3\mu_0 L} - \frac{\tau_0 x^2}{2\mu_0}\right]\Bigg|_{x_0}^B \\ &= 2 \cdot W\rho\left[\frac{(\mathcal{P}_0 - \mathcal{P}_L)B^3}{3\mu_0 L} - \frac{\tau_0 x^2}{2\mu_0} - \frac{(\mathcal{P}_0 - \mathcal{P}_L)x_0^3}{3\mu_0 L} + \frac{\tau_0 x_0^2}{2\mu_0}\right] \end{aligned}$$

$$= \frac{2}{3}\frac{(\mathcal{P}_0 - \mathcal{P}_L)B^3 W\rho}{\mu_0 L}\left[1 - \frac{3}{2}\left(\frac{\tau_0 L}{(\mathcal{P}_0 - \mathcal{P}_L)B}\right) + \frac{1}{2}\left(\frac{\tau_0 L}{(\mathcal{P}_0 - \mathcal{P}_L)B}\right)^3\right]$$

In going from the next-to-last step to the last step of this development, we have used the defining equation for $x_0$, given in Eq. 8B.5-2, to eliminate $x_0$ in favor of the yield stress.

When the yield stress goes to zero, The above result simplifies to the expression given in the answer to part (*c*) of Problem 2B.3 for the Newtonian fluid.

**8B.6 Derivation of the Buckingham-Reiner Equation**

The equation of motion for tube flow can be integrated to give $\tau_{rz} = \tau_R(r/R)$, where $\tau_R = (\mathcal{P}_0 - \mathcal{P}_L)R/2L$ (cf. Eq. 2.3-13). The outer boundary of the plug-flow region is $r_0 = (\tau_0/\tau_R)R$.

The Bingham fluid (see Problem 8B.5) gives for the shear stress component

$$\tau_{rz} = -\mu_0 \frac{dv_z}{dr} + \tau_0 \qquad (r \geq r_0) \qquad \text{and} \qquad \frac{dv_z}{dr} = 0 \qquad (r \leq r_0)$$

In the region $(r \geq r_0)$, we can combine the shear-stress expression with the Bingham formula to get

$$-\mu_0 \frac{dv_z}{dr} + \tau_0 = \tau_R \frac{r}{R}$$

Integration gives

$$v_z = -\frac{\tau_R}{2\mu_0 R} r^2 + \frac{\tau_0}{\mu_0} r + C$$

The integration constant is determined from the boundary condition that the velocity vanishes at the tube wall (the "no slip condition"). Then we get

$$v_z = \frac{\tau_R R}{2\mu_0}\left[1 - \left(\frac{r}{R}\right)^2\right] - \frac{\tau_0 R}{\mu_0}\left[1 - \left(\frac{r}{R}\right)\right] \qquad (r \geq r_0)$$

Since the velocity must be continuous at $r = r_0$, we can set $r = r_0$ in this last expression and get

$$v_z = \frac{\tau_R R}{2\mu_0}\left[1 - \left(\frac{r_0}{R}\right)^2\right] - \frac{(r_0 \tau_R/R)R}{\mu_0}\left[1 - \left(\frac{r_0}{R}\right)\right]$$
$$= \frac{\tau_R R}{2\mu_0}\left[1 - \left(\frac{r_0}{R}\right)^2 - 2\left(\frac{r_0}{R}\right) + 2\left(\frac{r_0}{R}\right)^2\right] = \frac{\tau_R R}{2\mu_0}\left[1 - \left(\frac{r_0}{R}\right)\right]^2 \quad (r \leq r_0)$$

The mass rate of flow is then

$$w = \int_0^{2\pi}\int_0^R \rho v_z r dr d\theta = 2\pi\rho\int_0^R v_z r dr$$

We then integrate by parts (cf. Problem 8B.5) to get

$$w = 2\pi\rho\left[\tfrac{1}{2}r^2 v_z\Big|_0^R - \tfrac{1}{2}\int_0^R r^2\frac{dv_z}{dr}dr\right]$$

The first term is zero at both limits, and the second contributes nothing in the range $(r \le r_0)$. Therefore we get

$$\begin{aligned} w &= \pi\rho\int_{r_0}^R r^2\left(\frac{\tau_R}{\mu_0}\frac{r}{R} - \frac{\tau_0}{\mu_0}\right)dr \\ &= \frac{\pi R^3\tau_R\rho}{4\mu_0}\left[1 - \frac{4}{3}\left(\frac{\tau_0}{\tau_R}\right) + \frac{1}{3}\left(\frac{\tau_0}{\tau_R}\right)^4\right] \end{aligned}$$

In getting the final expression, $r_0$ has been eliminated by using the relation $r_0 = (\tau_0/\tau_R)R$. When the yield stress is set equal to zero, this result simplifies to the Hagen-Poiseuille relation for Newtonian fluids.

**8B.7 The complex-viscosity components for the Jeffreys fluid**

*a*. The $yx$ component of Eq. 8.4-4 is

$$\tau_{yx} + \lambda_1 \frac{d\tau_{yx}}{dt} = -\eta_0 \left( \dot{\gamma}_{yx} + \lambda_2 \frac{d\dot{\gamma}_{yx}}{dt} \right)$$

Dividing by $\lambda_1$, rearranging, and introducing complex quantitites, we get

$$\frac{d\tau_{yx}}{dt} + \frac{1}{\lambda_1}\tau_{yx} = -\frac{\eta_0 \dot{\gamma}^0}{\lambda_1}\left( \Re\{e^{i\omega t}\} + \lambda_2 \frac{d}{dt}\Re\{e^{i\omega t}\} \right)$$
$$= -\frac{\eta_0 \dot{\gamma}^0}{\lambda_1}\Re\{e^{i\omega t}(1 + i\lambda_2\omega)\}$$

The only reason for using the complex representation of the trigonometric functions is that then we have to integrate only exponentials, and not products of exponentials and trigonometric functions.

Thus, we now have to integrate the above first-order differential equation to which the solution is

$$\tau_{yx} = -\frac{\eta_0 \dot{\gamma}^0}{\lambda_1} e^{-t/\lambda_1} \Re\left\{ \int_{-\infty}^{t} e^{+t'/\lambda_1} e^{i\omega t'} (1 + i\lambda_2\omega) dt' + C \right\}$$

If we assume that the fluid is in an unstressed state at $t = -\infty$, then the integration constant $C$ is zero. The integral can now be evaluated to give:

$$\tau_{yx} = -\frac{\eta_0 \dot{\gamma}^0}{\lambda_1} e^{-t/\lambda_1} \Re\left\{ \frac{(1 + i\lambda_2\omega)}{(1/\lambda_1) + i\omega} e^{+t/\lambda_1} e^{i\omega t} \right\} = -\eta_0 \dot{\gamma}^0 \Re\left\{ \frac{(1 + i\lambda_2\omega)}{(1 + i\lambda_1\omega)} e^{i\omega t} \right\}$$
$$= -\eta_0 \dot{\gamma}^0 \Re\left\{ \frac{(1 + i\lambda_2\omega)(1 - i\lambda_1\omega)}{1 + (\lambda_1\omega)^2} (\cos\omega t + i\sin\omega t) \right\}$$

Then we collect the real and imaginary parts within the braces, and discard the imaginary parts to get

$$\tau_{yx} = -\left[\eta_0 \frac{1+\lambda_1\lambda_2\omega^2}{1+(\lambda_1\omega)^2}\right]\dot{\gamma}^0 \cos\omega t - \left[\eta_0 \frac{(\lambda_1-\lambda_2)\omega}{1+(\lambda_1\omega)^2}\right]\dot{\gamma}^0 \sin\omega t$$

By comparing this result with the defining equation in Eq. 8.2-4, we see that the quantities in the brackets are $\eta'(\omega)$ and $\eta''(\omega)$ respectively. Equations (F) and (G) of Table 8.5-1 are obtained by setting $\lambda_2 = 0$ and $\lambda_1 = \lambda$. This shows that the Giesekus model is a nonlinear generalization of the linear Maxwell model. On the other hand, the Oldroyd model, as may be seen from Eq. 8.5-12 and 13, is a nonlinear generalization of the Maxwell model.

*b*. According to fn. 3 on p. 246, the Jeffreys model can be regarded as a superposition of the Newtonian model and the Maxwell model. For the Newtonian model. For the Newtonian fluid (here designated by *s*) we have

$$\eta'(\omega) = \eta_s \qquad \text{and} \qquad \eta''(\omega) = 0$$

That is, the Newtonian fluid shows no phase difference between the stress and the velocity gradient. For the Maxwell fluid (here designated by *p*) we have, by taking just one term in Eqs. 8.4-14 and 15

$$\eta'(\omega) = \frac{\eta_p}{1+(\lambda_1\omega)^2} \qquad \text{and} \qquad \eta''(\omega) = \frac{\eta_p\lambda_1\omega}{1+(\lambda_1\omega)^2}$$

In fn. 3, the connection between the constants of the Jeffreys model and the Newtonian and Maxwell models is given:

$$\eta_0 = \eta_s + \eta_p \qquad \text{and} \qquad \lambda_2 = \frac{\eta_s}{\eta_s + \eta_p}\lambda_1$$

These relations give $\eta_0$ and $\lambda_2$ in terms of $\eta_s$ and $\eta_p$. Presently we will need $\eta_s$ and $\eta_p$ in terms of $\eta_0$ and $\lambda_2$:

$$\eta_s = \frac{\eta_0\lambda_2}{\lambda_1} \qquad \text{and} \qquad \eta_p = \frac{\eta_0}{\lambda_1}(\lambda_1 - \lambda_2)$$

We now get the contributions to the complex viscosity for the Jeffreys fluid by summing the contributions of the Newtonian and Maxwell fluids:

$$\eta'(\omega) = \eta_s + \frac{\eta_p}{1+(\lambda_1\omega)^2} = \frac{\eta_s(1+\lambda_1\omega)^2 + \eta_p}{1+(\lambda_1\omega)^2}$$
$$= \frac{(\eta_s + \eta_p) + \eta_s(\lambda_1\omega)^2}{1+(\lambda_1\omega)^2} = \frac{\eta_0 + \eta_0\lambda_1\lambda_2\omega^2}{1+(\lambda_1\omega)^2}$$
$$= \eta_0 \frac{1+\lambda_1\lambda_2\omega^2}{1+(\lambda_1\omega)^2}$$

$$\eta''(\omega) = 0 + \frac{\eta_p\lambda_1\omega}{1+(\lambda_1\omega)^2} = \eta_0 \frac{(\lambda_1 - \lambda_2)\omega}{1+(\lambda_1\omega)^2}$$

This is in exact agreement with the results obtained above in (*a*).

**8B.8 Stress relaxation after cessation of shear flow**

*a*. If one writes out the 3-constant Oldroyd model for the special case of steady-state shearing flow, component by component, one gets (cf. Eqs. 8.5-5, 6, 7, and 8):

$$\tau_{yx} + \tfrac{1}{2}\lambda_1\dot{\gamma}\tau_{xx} - \tfrac{1}{2}\lambda_1\dot{\gamma}\tau_{yy} = -\eta_0\dot{\gamma}; \qquad \tau_{xx} - \lambda_1\dot{\gamma}\tau_{yx} = 0; \;\; \tau_{yy} + \lambda_1\dot{\gamma}\tau_{yx} = 0$$

and $\tau_{zz} = 0$. When these equations are solved simultaneously, we get

$$\tau_{yx} = -\eta_0\dot{\gamma}\frac{1}{1+(\lambda_1\dot{\gamma})^2} \quad \text{and} \quad \tau_{xx} = -\eta_0\lambda_1\dot{\gamma}^2\frac{1}{1+(\lambda_1\dot{\gamma})^2}$$

This expression agrees only somewhat with the data of Fig. 8.2-4. The equation shows the curve for viscosity starting out with the zero-shear-rate viscosity, and then decreasing--which is correct. However, the equation gives $\eta \propto \dot{\gamma}^{-2}$ for high shear rates, and this is nonsense. Then one would have $\tau_{yx} \propto \dot{\gamma}^{-1}$ at high shear rates; this would mean that as one applies more shear to the fluid, the velocity gradient decreases! Therefore the model has to be labeled as unusable at high shear rates. In other words, it can be used only to describe small deviations from Newtonian behavior.

*b*. For unsteady shear flow, Eq. 8.5-8 describes the shear stress. When the motion is stopped, $\dot{\gamma} = 0$ becomes zero, and the equation gives $\tau_{yx} + \lambda_1(d\tau/dt) = 0$, which can be solved to give $\tau_{yx} = Ce^{-t/\lambda_1}$. The constant is then determined by the requirement that the solution must match at $t = 0$ with the steady state expression in (*a*). This leads to Eq. 8B.8-2.

*c*. The normal stress in steady flow was given in (*a*). Arguments similar to those in (*b*) lead to

$$\tau_{xx} = -\eta_0\lambda_1\dot{\gamma}^2\frac{1}{1+(\lambda_1\dot{\gamma})^2}e^{-t/\lambda_1}$$

This predicts (correctly) that the normal stresses also relax after cessation of flow, but experimentally it is found that the normal stresses relax more slowly than the shear stresses.

**8B.9 Draining of a tank with an exit pipe**

We begin by writing the formula for the mass flow rate through the exit pipe (with radius $R_0$) for a power-law fluid thus

$$w = \frac{\pi R_0^3 \rho}{(1/n)+3}\left(-\frac{d\mathcal{P}}{dz}\frac{R_0}{2m}\right)^{1/n} = \frac{\pi R_0^3 \rho}{(1/n)+3}\left(\frac{\rho g(h+L)}{L}\frac{R_0}{2m}\right)^{1/n}$$

Note that this is a quasi-steady-state approximation. We now equate this to the rate of mass depletion in the tank

$$-\pi R^2 \rho \frac{dh}{dt} = \frac{\pi R_0^3 \rho}{(1/n)+3}\left(\frac{\rho g(h+L)}{L}\frac{R_0}{2m}\right)^{1/n}$$

We now divide through by $\pi R^2 \rho (h+L)^{1/n}$ and integrate (for $n \neq 1$)

$$-\int_H^0 \frac{dh}{(h+L)^{1/n}} = \frac{R_0^3}{[(1/n)+3]R^2}\left(\frac{\rho g R_0}{2mL}\right)^{1/n}\int_0^{t_{\text{efflux}}} dt$$

Performing the integration and solving for the efflux time we get

$$t_{\text{efflux}} = \left(\frac{2mL}{\rho g R_0}\right)^{1/n}\frac{[(1/n)+3]R^2}{R_0^3}\left[\frac{(H+L)^{1-(1/n)} - L^{1-(1/n)}}{1-(1/n)}\right]$$

To check this result we can let $n \to 1$ and $m \to \mu$; we make use of Lhopital's rule to get the result of Problem 7B.9(a):

$$t_{\text{efflux}} = \left(\frac{2\mu L}{\rho g R_0}\right)\frac{4R^2}{R_0^3}\lim_{n\to 1}\left[\frac{(d/dn)\left\{(H+L)^{1-(1/n)} - L^{1-(1/n)}\right\}}{(d/dn)\{1-(1/n)\}}\right]$$

$$= \left(\frac{2\mu L}{\rho g R_0}\right)\frac{4R^2}{R_0^3}\lim_{n\to 1}\left[\frac{(H+L)^{1-(1/n)}[\ln(H+L)](1/n^2) - L^{1-(1/n)}[\ln L](1/n^2)}{(1/n^2)}\right]$$

$$= \left(\frac{8\mu L R^2}{\rho g R_0^4}\right)\ln\left(1+\frac{H}{L}\right)$$

**8B.10 The Giesekus model**

*a*. We have to start by expanding the $\chi^2$ in Eq. (E) of Table 8.5-1 as a function of $(\lambda\dot{\gamma})^2$:

$$\chi^2 = \frac{1+\frac{1}{2}\left[16\alpha(1-\alpha)\right](\lambda\dot{\gamma})^2 - \frac{1}{8}\left[16\alpha(1-\alpha)\right]^2(\lambda\dot{\gamma})^4+\cdots-1}{8\alpha(1-\alpha)(\lambda\dot{\gamma})^2}$$
$$= 1-4\alpha(1-\alpha)(\lambda\dot{\gamma})^2+\cdots$$

and then, the series expansion of the square root of this will be

$$\chi = \sqrt{1-4\alpha(1-\alpha)(\lambda\dot{\gamma})^2+\cdots} = 1-2\alpha(1-\alpha)(\lambda\dot{\gamma})^2+\cdots$$

Next we get $f$ to the same order ($f$ is defined in Eq. (D) of Table 8.5-1)

$$f = \frac{1-\left[1-2\alpha(1-\alpha)(\lambda\dot{\gamma})^2+\cdots\right]}{1+(1-2\alpha)\left[1-2\alpha(1-\alpha)(\lambda\dot{\gamma})^2+\cdots\right]} = \alpha(\lambda\dot{\gamma})^2+\cdots$$

Then the viscosity expression (of Eq. (A) in Table 8.5-1) becomes

$$\frac{\eta}{\eta_0} = \frac{\left(1-\alpha(\lambda\dot{\gamma})^2+\cdots\right)^2}{1+(1-2\alpha)\left(\alpha(\lambda\dot{\gamma})^2+\cdots\right)} = 1-2\alpha(\lambda\dot{\gamma})^2+\cdots \underset{(\lambda\dot{\gamma})^2\to 0}{\to} 1$$

Similarly, the first normal stress coefficient (Eq. (B) of Table 8.5-1) becomes

$$\frac{\Psi_1}{2\eta_0\lambda} = \frac{\left(\alpha(\lambda\dot{\gamma})^2+\cdots\right)\left[1-\left(\alpha(\lambda\dot{\gamma})^2+\cdots\right)\right]}{\alpha\left[1-\left(\alpha(\lambda\dot{\gamma})^2+\cdots\right)\right]}\frac{1}{(\lambda\dot{\gamma})^2} \underset{(\lambda\dot{\gamma})^2\to 0}{\to} 1$$

and the second normal stress coefficient (Eq. (C) of Table 8.5-1) is $\Psi_2 = -\alpha\eta_0\lambda = -\frac{1}{2}\alpha\Psi_1$ which shows (correctly) that the second normal stress coefficient is smaller than the first normal stress coefficient and has the opposite sign.

*b*. We begin by dividing numerator and denominator of the expression for $\chi^2$ by $(\lambda\dot{\gamma})^2$ and then replacing $1/\lambda\dot{\gamma}$ by $\varepsilon$ (a small quantity):

$$\chi^2 = \frac{\varepsilon\sqrt{\varepsilon^2 + 16\alpha(1-\alpha)} - \varepsilon^2}{8\alpha(1-\alpha)} = \frac{\varepsilon\sqrt{16\alpha(1-\alpha)}\sqrt{1+[1/16\alpha(1-\alpha)]\varepsilon^2} - \varepsilon^2}{8\alpha(1-\alpha)}$$

We now expand this last expression in powers of $\varepsilon$:

$$\chi^2 = = \frac{\varepsilon\left(1+[1/32\alpha(1-\alpha)]\varepsilon^2+\cdots\right) - \varepsilon^2}{2\sqrt{\alpha(1-\alpha)}} = \frac{\varepsilon(1-\varepsilon+\cdots)}{2\sqrt{\alpha(1-\alpha)}}$$

Then we take the square root and get

$$\chi = \frac{\sqrt{\varepsilon}\left(1-\frac{1}{2}\varepsilon+\cdots\right)}{\sqrt{2}[\alpha(1-\alpha)]^{1/4}}$$

Then $f$ is given by

$$f = \frac{1-\chi}{1+(1-2\alpha)\chi} = (1-\chi)[1-(1-2\alpha)\chi+\cdots] = 1-2\chi(1-\alpha)+\cdots$$

Then the viscosity and normal-stress coefficients are

$$\frac{\eta}{\eta_0} = \frac{(1-f)^2}{1+(1-2\alpha)f} = \frac{(2\chi(1-\alpha)+\cdots)^2}{2(1-\alpha)+\cdots}$$
$$= \frac{(2(1-\alpha))^2\varepsilon}{[2(1-\alpha)]\cdot 2\sqrt{\alpha(1-\alpha)}} + \cdots = \sqrt{\frac{1-\alpha}{\alpha}}\frac{1}{\lambda\dot{\gamma}} + \cdots$$

$$\frac{\Psi_1}{2\eta_0\lambda} = \frac{f(1-\alpha f)}{\alpha(1-f)}\left(\frac{1}{\lambda\dot{\gamma}}\right)^2 = \frac{(1-\alpha)+\cdots}{\alpha[2\chi(1-\alpha)+\cdots]}\left(\frac{1}{\lambda\dot{\gamma}}\right)^2$$
$$= \frac{[\alpha(1-\alpha)]^{1/4}}{\sqrt{2}\alpha}\left(\frac{1}{\lambda\dot{\gamma}}\right)^{3/2} + \cdots$$

$$\frac{\Psi_2}{\eta_0\lambda} = -\left(\frac{1}{\lambda\dot{\gamma}}\right)^2 + \cdots$$

These expressions show (correctly) that the normal stress coefficient has a steeper slope at high shear rates than the viscosity. They also give a second normal stress coefficient that is smaller than the first normal stress coefficient, and that the two coefficients have opposite signs (this is in agreement with the experimental data for flexible polymers).

*c*. For elongational flow, we first get the limit as $\lambda\dot{\varepsilon} \to 0$ (please note that this $\dot{\varepsilon}$ is not related to the $\varepsilon$ in part (*b*)). In this limit we can expand the square root signs that appear in Eq. (H) of Table 8.5-1 thus:

$$\frac{\overline{\eta}}{3\eta_0} = \frac{1}{6\alpha}\left[3 + \frac{1}{\lambda\dot{\varepsilon}}\left(1 - 2(1-2\alpha)\lambda\dot{\varepsilon} + \cdots - 1 - (1-2\alpha)\lambda\dot{\varepsilon} - \cdots\right)\right]$$
$$= \frac{1}{6\alpha}\left[3 + \frac{1}{\lambda\dot{\varepsilon}}\left(-3(1-2\alpha)\lambda\dot{\varepsilon} + \cdots\right)\right] = 1 + \cdots$$

In the limit that $\lambda\dot{\varepsilon}$ becomes infinite, the quadratic terms under the square root signs dominate and we get

$$\frac{\overline{\eta}}{3\eta_0} = \frac{1}{6\alpha}(3 + 2 - 1) = \frac{2}{3\alpha}$$

Thus, the elongational viscosity remains finite, unlike the Oldroyd model for which the elongational viscosity becomes infinite.

**8C.1 The cone-and-plate viscometer**

*a*. According to Eq. 2B.11-1, the velocity distribution in the cone-and-plate system is given approximately by

$$\frac{v_\phi}{r} = \Omega\left(\frac{(\pi/2)-\theta}{\psi_0}\right)$$

The shear rate is given by the relation two lines above Eq. 8.3-2. In this problem the only nonzero components of the rate of strain tensor are $\dot{\gamma}_{\theta\phi}$ and $\dot{\gamma}_{\phi\theta}$. This can be seen in one of two ways: (a) Look at the right side of Eqs. B.1-15 to 21, but without the factor $-\mu$ and with the div **v** terms set equal to zero; these are the spherical components of the rate of strain tensor given in Eq. 8.3-1 or Eq. 8.4-1; or (b) add to the components of grad **v** in Eqs. (S) to (AA) in Table A.7-3 the corresponding transposes in order to construct the rate-of-strain tensor according to Eq. 8.3-1. Therefore, from Eq. B.1-19

$$\dot{\gamma}_{\theta\phi} = \dot{\gamma}_{\phi\theta} = \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\phi}{\sin\theta}\right) \approx \frac{1}{r}\frac{\partial v_\phi}{\partial\theta}$$

since the angle between the cone and plate is extremely small, which means that $\sin\theta \approx \sin\frac{1}{2}\pi = 1$. When the velocity distribution of Eq. 2B.11-1 is inserted in to the above approximate expression for the nonzero components of the rate-of-strain tensor we get

$$\dot{\gamma}_{\theta\phi} = \dot{\gamma}_{\phi\theta} = \frac{\partial}{\partial\theta}\left(\frac{v_\phi}{r}\right) = -\frac{\Omega}{\psi_0}$$

Hence the shear rate is

$$\dot{\gamma} = \sqrt{\tfrac{1}{2}\left(\dot{\gamma}_{\theta\phi}\dot{\gamma}_{\phi\theta} + \dot{\gamma}_{\phi\theta}\dot{\gamma}_{\theta\phi}\right)} = \pm\dot{\gamma}_{\theta\phi} = \mp\frac{\Omega}{\psi_0}$$

We now have to choose the proper sign. Both $\Omega$ and $\psi_0$ are positive quantities. Therefore the plus sign must be selected.

*b*. The non-Newtonian viscosity is obtained from the ratio $\eta = \left(\tau_{\theta\phi}/-\dot{\gamma}_{\theta\phi}\right)$. Hence we have to find a way to get the shear stress

from the measured quantities. As pointed out in (*a*), since the shear rate is constant throughout the gap, all stress components are also constant. This means that the torque can be calculated from the shear stress by integrating shear-stress times lever arm over the surface of the plate from zero to *R*:

$$T_z = \int_0^{2\pi}\int_0^R r \cdot \tau_{\theta\phi}\Big|_{\theta=\pi/2} r dr d\phi = \tfrac{2}{3}\pi R^3 \tau_{\theta\phi} \qquad \text{whence} \qquad \tau_{\theta\phi} = \frac{3T_z}{2\pi R^3}$$

Then the non-Newtonian viscosity is given by

$$\eta = \frac{\tau_{\theta\phi}}{-\dot{\gamma}_{\theta\phi}} = \frac{\left(3T_z/2\pi R^3\right)}{\left(\Omega/\psi_0\right)} = \frac{3T_z\psi_0}{2\pi R^3\Omega}$$

*c*. Equation 8C.1-2 follows directly from Eq. B.5-7, after dropping the terms on the left side and the $\tau_{\theta r}$ and $\tau_{\phi r}$ terms on the right side; the terms $\tau_{\theta\theta}$ and $\tau_{\phi\phi}$ must, however, be retained, since we know that normal stresses are nonzero in shear flow. Then Eq. 8C.1-3 follows from $\pi_{rr} = p + \tau_{rr}$ and some minor rearranging, and Eq. 8C.1-4 follows from the definitions of the normal stress coefficients and from the results of (*a*).

Next we integrate Eq. 8C.1-4 from the outer rim of the cone-plate system to some arbitrary position *r*:

$$\int_R^r d\pi_{\theta\theta} = -\left[\left(\Psi_1 + 2\Psi_2\right)\dot{\gamma}^2\right]\int_R^r d\ln r$$

The normal-stress coefficients are constants, because of the constancy of the shear rate over the gap. Therefore we get

$$\begin{aligned}\pi_{\theta\theta}(r) &= \pi_{\theta\theta}(R) - \left[\left(\Psi_1 + 2\Psi_2\right)\dot{\gamma}^2\right]\ln\frac{r}{R} \\ &= \pi_{rr}(R) + \left[\pi_{\theta\theta}(R) - \pi_{rr}(R)\right] - \left[\left(\Psi_1 + 2\Psi_2\right)\dot{\gamma}^2\right]\ln\frac{r}{R} \\ &= p_a - \Psi_2\dot{\gamma}^2 - \left[\left(\Psi_1 + 2\Psi_2\right)\dot{\gamma}^2\right]\ln\frac{r}{R}\end{aligned}$$

Here we have used the boundary condition that the normal stress at the rim is equal to the atmospheric pressure, and we have also used the definition of the second normal stress difference.

*d*. The force exerted by the fluid in the *z*-direction on the cone is then obtained from the result in (*c*) as follows:

$$
\begin{aligned}
F_z &= \int_0^{2\pi}\int_0^R \pi_{\theta\theta}(r) r dr d\phi - \pi R^2 p_a \\
&= 2\pi\int_0^R \left[ p_a - \Psi_2\dot{\gamma}^2 - \left[(\Psi_1 + 2\Psi_2)\dot{\gamma}^2\right]\ln\frac{r}{R}\right] r dr - \pi R^2 p_a \\
&= -\pi R^2\Psi_2\dot{\gamma}^2 - 2\pi R^2\left[(\Psi_1 + 2\Psi_2)\dot{\gamma}^2\right]\int_0^1 (\ln\xi)\xi d\xi \\
&= -\pi R^2\Psi_2\dot{\gamma}^2 - 2\pi R^2\left[(\Psi_1 + 2\Psi_2)\dot{\gamma}^2\right]\left(\tfrac{1}{2}\xi^2\ln\xi - \tfrac{1}{4}\xi^2\right)\Big|_0^1 \\
&= -\pi R^2\Psi_2\dot{\gamma}^2 + \tfrac{1}{2}\pi R^2\Psi_1\dot{\gamma}^2 + \pi R^2\Psi_2\dot{\gamma}^2 = \tfrac{1}{2}\pi R^2\Psi_1\dot{\gamma}^2
\end{aligned}
$$

Here we made use of the fact that $\lim_{\xi\to 0}\xi\ln\xi = 0$. Solving for the first normal stress difference, we get finally

$$\Psi_1(\dot{\gamma}) = \frac{2F_z}{\pi R^2\dot{\gamma}^2}$$

*e*. If one measures $\pi_{\theta\theta}(r) - p_a$ as a function of $r/R$ using flush-mounted pressure transducers, then, knowing $\Psi_1$ from (*d*), the second normal-stress coefficient can be calculated from Eq. 8C.1-5.

**8C.2 Squeezing flow between parallel disks**

This problem is solved by a quasi-steady-state method. Conservation of mass states that, for an incompressible fluid, the mass rate at which the fluid crosses the cylindrical surface at $r$ should equal the rate at which the mass between the plates within the cylindrical surface at $r$ decreases. The rate of mass displacement caused by the disk motion is:

$$w(r) = \pi r^2 \rho(-\dot{H}) \qquad \text{where} \qquad (\dot{H} = dh/dt)$$

To get the mass rate of flow between the two disks, we adapt the result in Eq. 8.3-14 by making the following correspondences locally: $W \to 2\pi r$, $2B \to H$, $(\mathcal{P}_0 - \mathcal{P}_L)/L \to -dp/dr$, and $w \to w(r)$. Then we get for the mass rate of flow emerging from between the disks:

$$w(r) = \frac{2(2\pi r)\left(\frac{1}{2}H\right)^2}{(1/n)+2}\left(-\frac{dp}{dr}\frac{(\frac{1}{2}H)}{m}\right)^{1/n}$$

Equating the two expressions for $w(r)$ we get a differential equation for for the pressure distribution $p(r)$ between the disks

$$-\frac{dp}{dr} = \frac{\left[r(-\dot{H})\right]^n\left[(1/n)+2\right]^n}{\left[H^2\right]^n}\left(\frac{2m}{H}\right)$$

This may be integrated to give

$$-\int_p^{p_{\text{atm}}} dp = \frac{2m(-\dot{H})^n\left[(1/n)+2\right]^n}{H^{2n+1}}\int_r^R \bar{r}^n d\bar{r}$$

or

$$p - p_{\text{atm}} = \frac{2m(-\dot{H})^n\left[(1/n)+2\right]^n}{H^{2n+1}}\frac{R^{n+1}}{n+1}\left[1-\left(\frac{r}{R}\right)^{n+1}\right]$$

which is the power-law equivalent of the Newtonian result in Eq. 3C.1-13.

When a constant force $F_0$ is applied to the upper disk, this force must be resisted by the pressure in the fluid at the upper disk, integrated over the disk (we include here also the normal stress $\tau_{zz}$, even though we know that it will not contribute for a generalized Newtonian fluid--the proof for this is similar to that given in Example 3.1-1 for Newtonian fluids):

$$
\begin{aligned}
F_0 &= \int_0^{2\pi}\int_0^R (p - p_0 + \tau_{zz}) r dr d\theta \\
&= 2\pi \cdot \frac{2m(-\dot{H})^n\left[(1/n)+2\right]^n}{H^{2n+1}} \frac{R^{n+3}}{n+1} \int_0^1 (1-\xi^2)\xi d\xi \\
&= \frac{2\pi m(-\dot{H})^n\left[(1/n)+2\right]^n}{H^{2n+1}} \frac{R^{n+3}}{n+3}
\end{aligned}
$$

This is now a differential equation for the motion of the upper plate as a function of time:

$$
-\frac{\dot{H}}{H^{(1/n)+2}} = \left(\frac{(n+3)}{2\pi m R^{n+3}}\right)^{1/n} \left(\frac{1}{(1/n)+2}\right) F_0^{1/n}
$$

This can be integrated to give:

$$
-\int_{H_0}^{H} \frac{dH}{H^{(1/n)+2}} = \left(\frac{(n+3)}{2\pi m R^{n+3}}\right)^{1/n} \left(\frac{1}{(1/n)+2}\right) F_0^{1/n} \int_0^t dt
$$

or

$$
\frac{1}{H^{(1/n)+1}} - \frac{1}{H_0^{(1/n)+1}} = \left(\frac{(n+3)}{2\pi m R^{n+3}}\right)^{1/n} \left(\frac{(1/n)+1}{(1/n)+2}\right) F_0^{1/n} t
$$

This simplifies properly to Eq. 3C.1-16 for the Newtonian fluid.

**8C.3 Verification of Giesekus viscosity function**

*a*. Eq. 8.5-4 is the same as Eq. 8.5-3 with $\lambda_1 = \mu_1 = \lambda$ and $\lambda_2 = \mu_2 = \mu_0 = 0$, but contains in addition a term $-(\lambda/\eta_0)\alpha\{\boldsymbol{\tau}\cdot\boldsymbol{\tau}\}$. Therefore, for steady shear flow, we may take over Eqs. 8.5-5 to 8 by making the appropriate changes in the constants, and by adding the extra term. To do the latter we need to calculate the components of the product $\{\boldsymbol{\tau}\cdot\boldsymbol{\tau}\}$; this is most easily done by matrix multiplication:

$$\begin{pmatrix} \tau_{xx} & \tau_{yx} & 0 \\ \tau_{yx} & \tau_{yy} & 0 \\ 0 & 0 & \tau_{zz} \end{pmatrix}\begin{pmatrix} \tau_{xx} & \tau_{yx} & 0 \\ \tau_{yx} & \tau_{yy} & 0 \\ 0 & 0 & \tau_{zz} \end{pmatrix} = \begin{pmatrix} \tau_{xx}^2 + \tau_{yx}^2 & \tau_{yx}\left(\tau_{xx} + \tau_{yy}\right) & 0 \\ \tau_{yx}\left(\tau_{xx} + \tau_{yy}\right) & \tau_{xx}^2 + \tau_{yx}^2 & 0 \\ 0 & 0 & \tau_{zz}^2 \end{pmatrix}$$

When we use this result, Eqs. 8.5-5 to 8 become modified as follows for the Giesekus model:

$$\begin{aligned}
&\tau_{xx} - 2\lambda\dot{\gamma}\tau_{yx} - (\lambda/\eta_0)\alpha\left(\tau_{xx}^2 + \tau_{yx}^2\right) = 0 \\
&\tau_{yy} - (\lambda/\eta_0)\alpha\tau_{xx}^2 + \tau_{yx}^2 = 0 \\
&\tau_{zz} - (\lambda/\eta_0)\alpha\tau_{zz}^2 = 0 \\
&\tau_{yx} - \lambda\dot{\gamma}\tau_{yy} - (\lambda/\eta_0)\alpha\tau_{yx}\left(\tau_{xx} + \tau_{yy}\right) = -\eta_0\dot{\gamma}
\end{aligned}$$

When these equations are multiplied by $\lambda/\eta_0$ and dimensionless variables are introduced, we obtain Eqs. 8C.3-1 plus an equation that gives $T_{zz} = 0$ (this equation gives another solutions which is physically unacceptable).

*b*. When these dimensionless equations are written in terms of the dimensionless normal stresses we get

$$\begin{aligned}
&N_1\left[1 - \alpha\left(N_1 + 2N_2\right)\right] = 2T_{yx}\dot{\Gamma} \\
&N_2 = \alpha\left(T_{yx}^2 + N_2^2\right) \\
&T_{yx}\left[1 - \alpha\left(N_1 + 2N_2\right)\right] = -\left(1 - N_2\right)\dot{\Gamma}
\end{aligned}$$

*c*. The second equation in (*b*) can be solved at once for the dimensionless shear stress:

$$T_{yx}^2 = \frac{N_2}{\alpha}(1-\alpha N_2) \qquad (*)$$

Division of the first of the equations in ($b$ ) by the third gives a relation for the dimensionless first normal stress difference in terms of the dimensionless second normal stress difference and the dimensionless shear stress; into that we can substitute the equation just obtained above and get

$$N_1 = -\frac{2N_2(1-\alpha N_2)}{\alpha(1-N_2)} \qquad (**)$$

Next, we square the third equation in ($b$), and then insert the expression for $T_{yx}^2$ from (*) and the expression for $N_1$ from (**), to get

$$\dot{\Gamma}^2 = \frac{N_2(1-\alpha N_2)[1+(1-2\alpha)N_2]^2}{\alpha(1-N_2)^4} \qquad (***)$$

*d*. To solve the final equation in ($c$) Giesekus (see p 87 of Ref. 5 on p. 262) suggested making the following change of variable:

$$N_2 = \frac{1-\chi}{1+(1-2\alpha)\chi}$$

Then the various factors in the final equation in ($c$) are:

$$1-N_2 = \frac{2\chi(1-\alpha)}{1+(1-2\alpha)\chi}$$

$$1-\alpha N_2 = \frac{(1+\chi)(1-\alpha)}{1+(1-2\alpha)\chi};$$

$$1+(1-2\alpha)N_2 = \frac{2(1-\alpha)}{1+(1-2\alpha)\chi}$$

When the last four equations are substituted into (***) we get

$$\dot{\Gamma}^2 = \frac{1-\chi^2}{4\alpha(1-\alpha)\chi^4}$$

This is a quadratic equation for $\chi^2$ which may be readily solved:

$$\chi^2 = \frac{\sqrt{1+16\alpha(1-\alpha)\dot{\Gamma}^2}-1}{8\alpha(1-\alpha)\dot{\Gamma}^2} = 1-4\alpha(1-\alpha)\dot{\Gamma}^2+\cdots$$

Then this result along with Eq. 8C.3-7 gives the second normal stress difference as a function of the shear rate. Having this, the shear stress and first normal stress difference can be obtained from Eqs. 8C.3-4 and 5.

As the shear rate goes to zero, one gets $\eta \to \eta_0$, $\Psi_1 \to 2\eta_0\lambda$, and $\Psi_2 \to -\alpha\eta_0\lambda$. The functions given in Table 8.5-1 are given in Giesekus's paper as well as in R. B. Bird, R. C. Armstrong, and O.Hassager, *Dynamics of Polymeric Liquids*, Vol. 1 (1987). pp. 361-368.

**8C.4 Tube flow for the Oldroyd 6-constant model**

The non-Newtonian viscosity for the Oldroyd model is given in Eq. 8.5-9. Therefore for tube flow

$$\tau_{rz} = -\eta\dot{\gamma}_{rz} = \eta_0\dot{\gamma}\left(\frac{1+\sigma_2\dot{\gamma}^2}{1+\sigma_1\dot{\gamma}^2}\right)$$

From Eq. 2.3-13, we have the expression valid for any kind of liquid

$$\tau_{rz} = \frac{(\mathcal{P}_0 - \mathcal{P}_L)r}{2L}$$

Combination of these two results gives

$$r = \frac{2\eta_0\dot{\gamma}L}{\mathcal{P}_0 - \mathcal{P}_L}\left(\frac{1+\sigma_2\dot{\gamma}^2}{1+\sigma_1\dot{\gamma}^2}\right)$$

The mass rate of flow through the tube is given by

$$w = \rho\int_0^{2\pi}\int_0^R v_z r dr d\theta = 2\pi\rho\int_0^R v_z r dr = 2\pi\rho\left[\tfrac{1}{2}v_z r^2\Big|_0^R - \int_0^R \tfrac{1}{2}r^2\frac{dv_z}{dr}dr\right]$$

The first term in the last result is zero at both limits. We next integrate by parts again

$$w = -\pi\rho\left[\tfrac{1}{3}r^3\frac{dv_z}{dr}\Big|_0^R - \int_0^R \tfrac{1}{3}r^3\frac{d}{dr}\left(\frac{dv_z}{dr}\right)dr\right] = \tfrac{1}{3}\pi\rho R^3\dot{\gamma}_R - \tfrac{1}{3}\pi\rho\int_0^{\dot{\gamma}_R}\left[r(\dot{\gamma})\right]^3 d\dot{\gamma}$$

in which $\dot{\gamma}_R$ is $-dv_z/dr$ evaluated at the tube wall. This result is good for any non-Newtonian fluid. We now specialize to the Oldroyd model:

$$w = \tfrac{1}{3}\pi\rho R^3\dot{\gamma}_R - \tfrac{1}{3}\pi\rho\left(\frac{2\eta_0 L}{\mathcal{P}_0 - \mathcal{P}_L}\right)^3\int_0^{\dot{\gamma}_R}\left[\left(\frac{1+\sigma_2\dot{\gamma}^2}{1+\sigma_1\dot{\gamma}^2}\right)\right]^3\dot{\gamma}^3 d\dot{\gamma}$$

$$= \tfrac{1}{3}\pi\rho R^3\dot{\gamma}_R - \tfrac{1}{3}\pi\rho\left(\frac{2\eta_0 L}{\mathcal{P}_0 - \mathcal{P}_L}\right)^3\int_0^{\sigma_1\dot{\gamma}_R^2}\left(\frac{1+nY}{1+Y}\right)^3 Y dY\frac{1}{2\sigma_1^2}$$

in which $Y = \sigma_1 \dot{\gamma}^2$ and $n = \sigma_2/\sigma_1$.

The expression for $r$ given earlier may be evaluated at the tube wall, and this gives

$$R = \frac{2\eta_0 \dot{\gamma}_R L}{\mathcal{P}_0 - \mathcal{P}_L}\left(\frac{1+\sigma_2\dot{\gamma}_R^2}{1+\sigma_1\dot{\gamma}_R^2}\right) = \left(\frac{2\eta_0 L}{\mathcal{P}_0 - \mathcal{P}_L}\right)\sqrt{X}\left(\frac{1+nX}{1+X}\right)\frac{1}{\sqrt{\sigma_1}} \qquad (*)$$

in which $X = \sigma_1 \dot{\gamma}_R^2$. Next, multiply the equation for $w$. by $3\sqrt{\sigma_1}/\pi\rho R^3$ and eliminate $\mathcal{P}_0 - \mathcal{P}_L$ by using the expression for $R$ just above to get

$$\Omega = \frac{3\sqrt{\sigma_1}w}{\pi\rho R^3} = \sqrt{X} - \frac{1}{2X^{3/2}}\left(\frac{1+X}{1+nX}\right)^3 f(X) \qquad (**)$$

in which

$$\begin{aligned} f(X) &= \int_0^X \left(\frac{1+nY}{1+Y}\right)^3 Y dY \\ &= \tfrac{1}{2}n^3X^2 - 3n^2(n-1)X + 3n(n-1)(2n-1)\ln(1+X) \\ &\quad -\tfrac{1}{2}X\left(\frac{n-1}{1+X}\right)^2\left[6n+(7n-1)X\right] \qquad (***) \end{aligned}$$

This gives, in dimensionless form, the mass rate of flow in terms of the wall shear rate. The latter may be eliminated in favor of the pressure drop by using the expression for $R$ above. That is, to get $\Omega$ in terms of the pressure drop, the asterisked equation have to be combined. The curves thus obtained may be found in the original publications (refs. 6 and 7).

**8C.5 Chain models with rigid connectors**

*a*. The paper by M. Gottlieb is the first example of a molecular dynamics calculation for polymer chain with rigid connectors. He followed the motions of a three-bead-two-rod model as it moves around in a liquid made up of 47 "solvent beads." The solution is presumed to be at macroscopic equilibrium. He showed that one does not get a Gaussian distribution of orientations for the polymer model, in agreement with the previously published theory of H. A. Kramers, *Physica*, **11**, 1-19 (1944). See also, R. B. Bird, C. F. Curtiss, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Liquids* (Vol. 2), Wiley, New York, 2nd edition (1987), pp. 40-41. This book will hereinafter be referred to as "DPL."

*b*. The publication by O. Hassager showed how to get the viscosity and normal stress coefficients for a three-bead-two-rod chain in a flow situation. In a subsequent publication [C. F. Curtiss and R. B. Bird, *J. Non-Newtonian Fluid Mechanics*, **2**, 392-396 (1977)] it was shown how to extend the calculations to models with 4 and 5 beads. Hassager showed that the viscosity and first normal stress coefficient for models with infinitely stiffened Fraenkel springs are different from those of the rigid rod models. This perplexing situation was what led Gottlieb to investigate the same problem by using molecular dynamics. See also DPL (§16.5).

*c*. The paper by X. J. Fan and T. W. Liu dealt with Kirkwood-Riseman chains of 3 to 8 beads. This model has fixed bond lengths and fixed bond angles, that is, two kinds of constraints. They were able to evaluate the equilibrium configurational distribution, and they found that it differs from the classical "Gaussian distribution." This calculation enabled the authors to relate rheological properties to "chain stiffness," by comparing their results with those for a Kramers freely jointed chain. See also DPL (§16.6).

*d*. The paper by T. W. Liu is a landmark contribution to the theory of Kramers chains. He showed how one can study the configurations and rheological properties of freely-jointed Kramers chains by the use of Brownian dynamics. By this technique he was able to generate movies showing the change of the configurations with respect to time. He also succeeded in taking appropriate averages over the configurations, in order to get the viscosity and first normal-stress coefficient as a function of shear rate, for chains with 3, 5, 10, and 20 beads. He also calculated the elongational viscosity in steady elongational flow and got information on how the

polymer molecules "unravel" under the influence of the elongational flow.

*e*. The paper by H. H. Saab, et al., is an extensive comparison between the Curtiss-Bird phase-space theory for polymer melts and the available experimental data on a host of rheological properties. An extensive comparison of the results with those of the theory of Doi and Edwards confirms that in almost every instance, the Curtiss-Bird theory is to be preferred over the Doi-Edwards theory. In the Curtiss-Bird theory, the polymer molecules are modeled as Kramers freely jointed bead-rod chains, whereas in the Doi-Edwards theory, mixed modeling is used. See also DPL Chapter 19.

*f*. The papers by J. D. Schieber show how the extension of the Curtiss-Bird theory to polydisperse polymer melts can be implemented. Both the log-normal (Wesslau) distribution of molecular weights and the Flory-Schultz distribution are used. The curves for the viscosity, first normal-stress coefficient, elongational stress growth viscosity, and steady-state elongational viscosity were obtained. These curves often differed appreciably from those for monodisperse samples. Comparisons with experimental data are given in the second of the two papers. See DPL, Example 19.6-1.

General Comments:

i. The rodlike connectors are generally much more difficult to deal with than the springlike connectors, because it becomes necessary to use nonorthogonal coordinates to describe the chain space. For Hookean springs, one can perform many of the kinetic theory derivations with relative ease. For non-Hookean springs, it is possible to make some assumptions that enable analytical results to be obtained. But even for rigid dumbbells and three-bead-two-rod models, analytical calculations become prohibitively time consuming.

ii. To overcome the problems associated with the use of rigid connectors in modeling, molecular dynamics and Brownian dynamics have proven to be useful. Also, Brownian dynamics can be useful when considering the flow in constrained channels, where the interaction with the containing walls have to be considered. Brownian dynamics proves to be particularly helpful in getting information about the actual motions of the polymer molecules during various types of flow.

**9A.1 Prediction of thermal conductivities of gases at low density.**

*a.* Since Argon is monatomic, we use Eq. 9.3-13 to predict its $k$ in the low-density gas region:

$$k = 1.9891 \times 10^{-4} \frac{\sqrt{T/M}}{\sigma^2 \Omega_\mu}$$

Here $T = 100 + 273.15 = 373.15$K, and Table E.1 gives $M = 39.948$, $\sigma = 3.432\,\text{Å}$, $\varepsilon/\kappa = 122.4$K for Argon. Then $\kappa T/\varepsilon = 373.15/122.4 = 3.049$, and interpolation in Table E.2 gives $\Omega_\mu = 1.0344$. Equation 9.3-13 then gives

$$k = 1.9891 \times 10^{-4} \frac{\sqrt{373.15/39.948}}{3.432^2 \times 1.0344} = 499 \times 10^{-7} \text{ cal/s·cm·K}$$

which is within 1.5 percent of the observed value.

*b.* Equation 9.3-15, Eucken's formula, gives

$$k = \left(\hat{C}_p + 1.25R/M\right)\mu = (\widetilde{C}_p + 1.25R)\mu/M$$

with $R = 1.987$ cal/g-mole·K. Insertion of the data for $\widetilde{C}_p$ and $\mu$, and for $M$ from Table E.1, gives

for NO, $$k = (7.15 + 1.25 * 1.987)\frac{1929 \times 10^{-7}}{30.01} = 620 \times 10^{-7} \text{ cal/s·cm·K}$$
$$= 0.02595 \text{ W/m·K}$$
vs. 0.02590 W/m·K from Table 9.1-2.

for $CH_4$, $$k = (8.55 + 1.25 \times 1.987)\frac{1116 \times 10^{-7}}{16.04} = 768 \times 10^{-7} \text{ cal/s·cm·K}$$
$$= 0.03212 \text{ W/m·K}$$
vs. 0.03427 W/m·K from Table 9.1-2.

**9A.2 Computation of the Prandtl number for gases at low density.**

Use of Eq. 9.3-16, with molar heat capacities $\widetilde{C}_p = M\hat{C}_p$ calculated from the given values $\hat{C}_p$ and molecular weights $M$ from Table E.2, along with $R = 8.31451 \times 10^3$ J/kg-mol·K from Appendix F, gives the predictions in column ($a$) of the following table. Computation of Pr from its definition, Eq. 9.1-9, and the tabulated $\hat{C}_p$, $\mu$, and $k$, gives the values in column ($b$). The predictions are closely confirmed for He and Ar, but are less successful for the polyatomic compounds.

| Gas | ($a$) Pr from Eq. 9.3-16 | ($b$) Pr from $\hat{C}_p$, $\mu$, and $k$ |
|---|---|---|
| He | 0.667 | 0.670 |
| Ar | 0.667 | 0.665 |
| $H_2$ | 0.735 | 0.714 |
| Air | 0.736 | 0.710 |
| $CO_2$ | 0.782 | 0.769 |
| $H_2O$ ($M = 18.016$) | 0.764 | 0.862 |

**9A.3 Estimation of the thermal conductivity of a dense gas.**

*a.* Table E.1 gives the following critical constants for methane ($CH_4$): $T_c = 191.1$ K, $p_c = 45.8$ atm, and $k_c = 158 \times 10^{-6}$ cal/cm·s·K. The reduced conditions for the prediction are then $T_r = (459.7 + 127)/(1.8 \times 191.1) = 1.71$ and $p_r = 110.4/45.8 = 2.41$. From Fig. 9.2-1 we find $k_r = 0.77$ at that state, giving

$$\begin{aligned} k = k_r k_c &= 0.77 \times 158 \times 10^{-6} \\ &= 1.22 \times 10^{-4} \text{ cal/cm·s·K} \\ &= 0.0294 \text{ Btu/hr·ft·F.} \end{aligned}$$

which is about 4% above the observed value.

*b.* For this calculation, we need to predict the viscosity of methane at 127 F (325.9 K) and low pressure from Eq. 1.4-18. We find $\sigma = 3.780$ and $\kappa T/\varepsilon = 325.9/154 = 2.116$ and $\Omega_\mu = 1.153$ by use of Tables E.1 and E.2. Hence,

$$\mu = 2.6693 \times 10^{-5} \frac{\sqrt{16.04 \times 325.9}}{(3.780)^2 \times 1.153} = 1171 \times 10^{-7} \text{ g/cm·s}$$

Next we use the Eucken formula, Eq. 9.3-15, to estimate the thermal conductivity $k^\circ$ at low pressure and 127 F, where $\widetilde{C}_p = 37.119$ J/g-mol·K according to the heat capacity polynomial given for methane in Reid, Prausnitz and Poling (1987):

$$\begin{aligned} k^\circ &= (37.119 + 1.25 \times 8.31451)\frac{1171 \times 10^{-7}}{16.04} = 0.000347 \text{ W/cm·K} \\ &= 0.0347 \text{ W/m·K} \\ &= 0.0200 \text{ Btu/hr·ft·F} \end{aligned}$$

Finally, we multiply $k^\circ$ by the ratio of $k_r$ at 110.4 atm to the asymptote $k_r = 0.52$ at $p_r = 0$ in Fig. 9.2-1. The resulting predicted $k$ at (110.4 atm, 127 F) is

$$k = 0.0200 \times 0.77/0.52 = 0.0297 \text{ Btu/hr·ft·F}$$

and is just 1% above the measured value. This is unusually good agreement.

**9A.4 Prediction of the thermal conductivity of a gas mixture.**

The data for this problem are as follows:

| Component | M | $\mu \times 10^5$, Pa·s | $k$, W/m·K | Mole fraction |
|---|---|---|---|---|
| 1($H_2$) | 2.016 | 0.8944 | 0.1789 | 0.80 |
| 2($CO_2$) | 44.01 | 1.506 | 0.01661 | 0.20 |

Insertion of these data into Eq. 1.4-16 gives the dimensionless coefficients

$$\Phi_{11} = \Phi_{22} = 1.0$$

$$\Phi_{12} = \frac{1}{\sqrt{8}}\left(1 + \frac{2.016}{44.01}\right)^{-1/2}\left[1 + \left(\frac{0.8944}{1.506}\right)^{1/2}\left(\frac{44.01}{2.016}\right)^{1/4}\right]^2$$
$$= 2.457$$

$$\Phi_{21} = \frac{1}{\sqrt{8}}\left(1 + \frac{44.01}{2.016}\right)^{-1/2}\left[1 + \left(\frac{1.506}{0.8944}\right)^{1/2}\left(\frac{2.016}{44.01}\right)^{1/4}\right]^2$$
$$= 0.1819$$

Substitution of these results into Eq. 9.3-17 gives

$$k_{\text{mix}} = \frac{0.8 \times 0.1789}{0.8 \times 1.0 + 0.2 \times 2.457} + \frac{0.2 \times 0.01661}{0.8 \times 0.1819 + 0.2 \times 1.0}$$
$$= 0.1204 \text{ W/m·K}$$

**9A.5 Estimation of the thermal conductivity of a pure liquid.**

We first calculate the derivative $(\partial p/\partial \rho)_T$ required for Eq. 9.3-4:

$$\begin{aligned}(\partial p/\partial \rho)_T &= \rho^{-1}/\left[\rho^{-1}(\partial \rho/\partial p)_T\right]^{-1} \\ &= (1/0.9938)[38\times 10^{-6}]^{-1} = 2.648\times 10^4 \text{ megabar cm}^3/\text{g} \\ &= 2.648\times 10^{10} \text{ cm}^2/\text{s}^2\end{aligned}$$

Inserting this result into Eq. 9.3-4 and setting $C_p \approx C_V$, we obtain

$$v_s = \sqrt{2.648\times 10^{10}} = 1.627\times 10^5 \text{ cm/s}$$

Equation 9.4-3 then gives the following estimate of the thermal conductivity:

$$\begin{aligned}k &= 2.80(\tilde{N}\rho/M)^{2/3}\mathrm{K}v_s \\ &= 2.80\left[\frac{6.02214\times 10^{23}\times 0.9938}{18.02}\right]^{2/3}\times 1.38066\times 10^{-16}\times 1.627\times 10^5 \\ &= 6.50\times 10^4 \text{ g cm/s}^3\cdot\text{K} \\ &= 0.650 \text{ W/m}\cdot\text{K} \\ &= 0.375 \text{ Btu/hr}\cdot\text{ft}\cdot\text{F}\end{aligned}$$

**9A.6 Calculation of the Lorenz number.**

*a.* When $\kappa$ and $e$ in Eq. 9A.6-1 are expressed in terms of the gas constant $R$ and Faraday constant $F$, the Lorenz number takes the form

$$L = \frac{\pi^2}{3}\left(\frac{R}{F}\right)^2$$

Insertion of numerical values for $R$ and $F$ from Appendix E gives

$$\begin{aligned} L &= \frac{\pi^2}{3}\left(\frac{8.31451}{96485.3}\right)^2 \\ &= 2.44 \times 10^{-8}\ \text{volt}^2/\text{K}^2 \end{aligned}$$

*b.* Insertion of the result just found, and the given $k_e$ and $T$, into Eq. 9.5-1 gives the thermal conductivity estimate

$$\begin{aligned} k &= Lk_eT \\ &= \frac{2.44 \times 10^{-8}\text{volt}^2/\text{K}^2 \times 293.15\text{K}}{1.72 \times 10^{-6}\text{ohm}\cdot\text{cm}} \\ &= 4.16\ \text{volt}^2/\text{K}\cdot\text{ohm}\cdot\text{cm} \\ &= 416\ \text{W/m}\cdot\text{K} \end{aligned}$$

for copper at 20°C.

**9A.7. Corroboration of the Wiedemann-Franz-Lorenz Law.**

Conversion of the tabulated data into SI units and insertion into Eq. 9.5-1 gives the following results at $T$=293.15K:

| Metal | $1/k_e$, ohm·m | $k$, W/m·K | Lorenz number, $L = k/k_eT$, volt$^2$/K$^2$ |
|---|---|---|---|
| Na | $4.6\times10^4$ | 133 | $2.1\times10^{-8}$ |
| Ni | $6.9\times10^4$ | 59 | $1.4\times10^{-8}$ |
| Cu | $1.69\times10^4$ | 385 | $2.2\times10^{-8}$ |
| Al | $2.62\times10^4$ | 209 | $1.9\times10^{-8}$ |

The approximate agreement of $L$ for these metals illustrates the Wiedemann-Franz-Lorenz law.

**9A.8 Thermal conductivity and Prandtl number of a polyatomic gas.**

*a.* To calculate $k$ for a polyatomic gas at moderate pressure we use Eq. 9.3-15,

$$k = (\hat{C}_p + 1.25R/M)\mu = (\widetilde{C}_p + 1.25R)\mu/M$$

along with the viscosity expression in Eq. 1.4-18:

$$\mu = 2.6693 \times 10^{-5} \frac{\sqrt{MT}}{\sigma^2 \Omega_\mu}$$

From Table E.1 we find, for $CH_4$, the values $M = 16.04$, $\sigma = 3.780$, $\varepsilon/\kappa = 154\text{K}$, and from Table E.2 at $\kappa T/\varepsilon = 1500/154 = 9.740$ we find $\Omega_\mu = 0.8280$. Equation 1.4-18 then gives the predicted viscosity

$$\begin{aligned}\mu &= 2.6693 \times 10^{-5} \frac{\sqrt{16.04 \times 1500}}{(3.780)^2 \times 0.8280} \\ &= 3500 \times 10^{-7} \text{ g/cm·s}\end{aligned}$$

and Eq. 9.315 gives the predicted $k$ value

$$\begin{aligned}k &= (20.71 + 1.25 \times 1.987) \times 3500 \times 10^{-7}/16.04 \\ &= 5.06 \times 10^{-4} \text{ cal/cm·s·K} \\ &= 2.12 \text{ W/m·K}\end{aligned}$$

*b.* The predicted Prandtl number according to Eq. 9.3-16 is

$$\text{Pr} = \frac{20.71}{20.71 + 1.25 \times 1.987} = 0.89, \text{ dimensionless}$$

**9A.9 Thermal conductivity of gaseous chlorine.**

For $Cl_2$, Table E.1 gives $M = 70.91$, $\sigma = 4.115\,\text{Å}$ and $\varepsilon/\kappa = 357\text{K}$. Equation 1.4-14 then gives

$$\mu = 2.6693 \times 10^{-5} \frac{\sqrt{MT}}{\sigma^2 \Omega_\mu} = 1.34274 \times 10^{-5} \sqrt{T}$$

with $\mu$[=]g/cm·s and $T$[=]K, and Eq. 9.3-15 gives $k = (\widetilde{C}_p + 1.25R)\mu/M$. The calculated results follow, with $\mu$, $\widetilde{C}_p$ and $k$ in the units of the problem statement:

| $T$, K | $T/357$ | $\Omega_\mu$ | $10^4\mu$ | $(\widetilde{C}_p + 1.25R)$ | $10^5 k_{\text{pred}}$ | $k_{\text{obs}}/k_{\text{pred}}$ |
|---|---|---|---|---|---|---|
| 198 | 0.5546 | 2.0915 | 0.8931 | 10.54 | 1.33 | 0.985 |
| 275 | 0.7703 | 1.8217 | 1.2044 | 10.59 | 1.80 | 1.056 |
| 276 | 0.7731 | 1.8239 | 1.2091 | 10.59 | 1.81 | 1.066 |
| 276 | 0.7731 | 1.8239 | 1.2091 | 10.59 | 1.81 | 1.061 |
| 363 | 1.017 | 1.5799 | 1.6008 | 10.81 | 2.44 | 1.074 |
| 363 | 1.017 | 1.5799 | 1.6008 | 10.81 | 2.44 | 1.070 |
| 395 | 1.106 | 1.5138 | 1.7427 | 10.91 | 2.68 | 1.13 |
| 453 | 1.269 | 1.4172 | 1.9935 | 11.02 | 3.10 | 1.14 |
| 453 | 1.269 | 1.4172 | 1.9935 | 11.02 | 3.10 | 1.10 |
| 495 | 1.3866 | 1.3609 | 2.1701 | 11.095 | 3.40 | 1.09 |
| 553 | 1.549 | 1.2974 | 2.406 | 11.17 | 3.79 | 1.09 |
| 583 | 1.633 | 1.2694 | 2.5249 | 11.2 | 3.99 | 1.11 |
| 583 | 1.633 | 1.2694 | 2.5249 | 11.2 | 3.99 | 1.12 |
| 676 | 1.884 | 1.1994 | 2.8775 | 11.32 | 4.59 | 1.10 |
| 676 | 1.884 | 1.1994 | 2.8775 | 11.32 | 4.59 | 1.07 |
| | | | | | Ave.= | 1.084 |

The predicted $k$ values exceed the observed values by an average of 8.4% in this temperature range.

**9A.10 Thermal conductivity of chlorine-air mixtures.**

Numbering chlorine and air as components 1 and 2, respectively, and inserting their given properties into Eq. 1.4-16, we obtain the following coefficients for Eq. 9.3-17:

$$\Phi_{11} = 1; \qquad \Phi_{22} = 1;$$

$$\Phi_{12} = \frac{1}{\sqrt{8}}\left(1 + \frac{70.91}{28.97}\right)^{-1/2}\left[1 + \sqrt{\frac{1.351}{1.854}}\left(\frac{28.97}{70.91}\right)^{1/4}\right]^2 = 0.53899;$$

$$\Phi_{21} = \frac{1}{\sqrt{8}}\left(1 + \frac{28.97}{70.91}\right)^{-1/2}\left[1 + \sqrt{\frac{1.854}{1.351}}\left(\frac{70.91}{28.97}\right)^{1/4}\right]^2 = 1.8105$$

Equation 9.3-17 gives, for binary mixtures:

$$k_{\text{mix}} = \frac{x_1 k_1}{x_1\Phi_{11} + x_2\Phi_{12}} + \frac{x_2 k_2}{x_2\Phi_{21} + x_2\Phi_{22}}$$

Insertion of the coefficients and compositions for this problem gives
At $x_1 = 0.25$,

$$k_{\text{mix}} = \frac{0.25 \times 0.0896}{0.25 + 0.75 \times 0.53899} + \frac{0.75 \times 0.02614}{0.25 \times 1.8105 + 0.75} = 0.0506 \text{ cal/cm·s·K}$$

At $x_1 = 0.5$,

$$k_{\text{mix}} = \frac{0.5 \times 0.0896}{0.5 + 0.5 \times 0.53899} + \frac{0.5 \times 0.02614}{0.5 \times 1.8105 + 0.5} = 0.0675 \text{ cal/cm·s·K}$$

At $x_1 = 0.75$,

$$k_{\text{mix}} = \frac{0.75 \times 0.0896}{0.75 + 0.25 \times 0.53899} + \frac{0.25 \times 0.02614}{0.75 \times 1.8105 + 0.25} = 0.0800 \text{ cal/cm·s·K}$$

**9A.11 Thermal conductivity of quartz sand.**

*a.* For spheres ($g_1 = g_2 = g_3 = 1/3$), Eq. 9A.11-2 reduces to

$$\alpha_i = 3/(2 + k_j/k_0)$$

The resulting $\alpha_i$ values from Eq. 9A.11-2 for the water-saturated sand are

$$\alpha_0 = \frac{3}{2 + (k_0/k_0)} = 1; \quad \alpha_1 = \frac{3}{2 + (k_1/k_0)} = 0.183; \quad \alpha_2 = \frac{3}{2 + (k_2/k_0)} = 0.433$$

and Eq. 9A.11-1 then gives

$$k_{\text{eff}} = \frac{(1)(0.427)(0.00142) + (0.183)(0.510)(0.0204) + (0.433)(0.063)(0.0070)}{(1)(0.427) + (0.183)(0.510) + (0.433)(0.063)}$$

which predicts $k_{\text{eff}} = 6.3 \times 10^{-3}$ cal/cm·s·K, vs. $6.2 \times 10^{-3}$ observed.

For the same sand when completely dry ($k_1/k_0 = 332$, $k_2/k_0 = 114$), Eq. 9A-11-2 for spheres gives

$$\alpha_0 = \frac{3}{2+1} = 1; \quad \alpha_1 = \frac{3}{2+332} = 0.00898; \quad \alpha_2 = \frac{3}{2+114} = 0.0.0259$$

and Eq. 9A.11-1 with de Vries' correction factor of 1.25 for dry sand gives

$$\frac{k_{\text{eff}}}{1.25} = \frac{(1)(0.427)(0.0000615) + 0.00898)(0.510)(0.0204) + (0.0259)(0.063)(0.0070)}{(1)(0.427) + (0.00898)(0.510) + (0.0259)(0.063)}$$

predicting $k_{\text{eff}} = 0.38 \times 10^{-3}$ cal/cm·s·K, vs. $0.58 \times 10^{-3}$ observed.

For the same sand when water-saturated at 20°C, de Vries' recommended $g_j$ values give

$$\alpha_0 = \frac{1}{3}\left[\frac{2}{1 + 0.125[(1.42/1.42) - 1]} + \frac{1}{1 + 0.750[(1.42/1.42) - 1]}\right] = 1.000$$

$$\alpha_1 = \frac{1}{3}\left[\frac{2}{1 + 0.125[(20.4/1.42) - 1]} + \frac{1}{1 + 0.750[(20.4/1.42) - 1]}\right] = 0.280$$

$$\alpha_2 = \frac{1}{3}\left[\frac{2}{1 + 0.125[(7.0/1.42) - 1]} + \frac{1}{1 + 0.750[(7.0/1.42) - 1]}\right] = 0.532$$

and Eq. 9A.11-1 gives

$$k_{\text{eff}} = \frac{(1)(0.427)(0.00142) + (0.280)(0.510)(0.0204) + (0.532)(0.063)(0.0070)}{(1)(0.427) + (0.280)(0.510) + (0.532)(0.063)}$$

predicting $k_{\text{eff}} = 6.2 \times 10^{-3}$ cal/cm·s·K, in still better agreement with the observed value $6.2 \times 10^{-3}$.

For the same sand when completely dry ($k_1/k_0 = 332$, $k_2/k_0 = 114$), Eq. 9A-11-2 with de Vries' recommended $g_j$ values gives

$$\alpha_0 = \frac{1}{3}\left[\frac{2}{1+0.125[1-1]} + \frac{1}{1+0.750[1-1]}\right] = 1.000$$
$$\alpha_1 = \frac{1}{3}\left[\frac{2}{1+0.125[332-1]} + \frac{1}{1+0.750[332-1]}\right] = 0.0171$$
$$\alpha_2 = \frac{1}{3}\left[\frac{2}{1+0.125[114-1]} + \frac{1}{1+0.750[114-1]}\right] = 0.0048$$

and Eq. 9A.11-1 with de Vries' correction factor for dry sand gives

$$\frac{k_{\text{eff}}}{1.25} = \frac{(1)(0.477)(0.00006) + (0.0171)(0.510)(0.0204) + (0.0048)(0.063)(0.0070)}{(1)(0.477) + (0.0171)(0.510) + (0.0048)(0.063)}$$

predicting $k_{\text{eff}} = 0.54 \times 10^{-3}$ cal/cm·s·K, vs. $0.58 \times 10^{-3}$ observed.

**(b)** Equation 9.6-1 gives

$$\frac{k_{\text{eff}}}{k_0} = 1 + \frac{3\phi}{\left(\dfrac{k_1 + 2k_0}{k_1 - k_0}\right) - \phi}$$

Insertion of $\phi = 0.573$ and $k_1 = 0.0189$ cal/cm·s·K for the solids gives, for the water-saturated sand,

$$\frac{k_{\text{eff}}}{k_0} = 1 + \frac{3(0.573)}{\left(\dfrac{0.0189 + (2)(0.00142)}{0.0189 - 0.00142}\right) - 0.573} = 3.563$$

predicting $k_{\text{eff}} = 5.1 \times 10^{-3}$ cal/cm·s·K, vs. $6.2 \times 10^{-3}$ observed. This is not as good as the prediction in **(a)** from Eq. 9A.6-11 with de Vries' $g_j$ values.

For the completely dry sand, insertion of the $k$ value for air as $k_0$ into Eq. 9.6-1 gives

$$\frac{k_{\text{eff}}}{k_0} = 1 + \frac{3(0.573)}{\left(\dfrac{0.0189 + 2(0.0000615)}{0.0189 - 0.0000615}\right) - 0.573} = 4.935$$

predicting $k_{\text{eff}} = 0.30 \times 10^{-3}$ cal/cm·s·K, vs. $0.58 \times 10^{-3}$ observed. The result in **(a)**, from Eq. 9A.6-11 with de Vries' $g_j$, is better.

Predictions of $k_{\text{eff}}$ are more difficult for dry sand than for water-saturated sand. An oblate-spheroidal model gives little advantage according to the present data.

**9A.12 Calculation of molecular diameters from transport properties.**

*a.* Equation 1.4-9 and the viscosity value from Problem 9A.2 yield the following molecular diameter calculation for Argon in cgs units:

$$
\begin{aligned}
d &= \sqrt{2/3\mu}\left(\frac{M\kappa T}{\widetilde{N}\pi^3}\right)^{1/4} \\
&= \sqrt{2/3/(2.278\times 10^{-4})}\left(\frac{39.948\times 1.38066\times 10^{-16}\times 300}{6.02214\times 10^{23}\times \pi^3}\right)^{1/4} \\
&= 2.95\times 10^{-8}\ \text{cm}
\end{aligned}
$$

*b.* Equation 9..3-12 and the $k$ value from Problem 9.A.2 give the following molecular diameter calculation for Argon in cgs units:

$$
\begin{aligned}
d &= \sqrt{\frac{1}{k}}\left(\frac{\kappa^3 T\widetilde{N}}{\pi^3 M}\right)^{1/4} \\
&= \sqrt{\frac{1}{0.01784\times 10^5}}\left((\frac{1.38066\times 10^{-16})^3\times 300\times 6.02214\times 10^{23}}{\pi^3\times 39.948}\right)^{1/4} \\
&= 1.86\times 10^{-8}\ \text{cm}
\end{aligned}
$$

*c.* Equation 1.4-14, Tables E.1 and E.2 and the viscosity from Problem 9A.2 give for Argon in cgs units,

$$
\begin{aligned}
\sigma &= \sqrt{\frac{5}{16\mu\Omega_\mu}}\left(\frac{M\kappa T}{\widetilde{N}\pi}\right)^{1/4} \\
&= \sqrt{\frac{5}{16\times 2.278\times 10^{-4}\times 1.1000}}\left(\frac{39.948\times 1.38066\times 10^{-16}\times 300}{6.02214\times 10^{23}\times \pi}\right)^{1/4} \\
&= 3.415\times 10^{-8}\ \text{cm}
\end{aligned}
$$

Equation 9.3-15, Tables E.1 and E.2 and the $k$ value from Problem 9A.2 give for Argon in cgs units,

$$
\begin{aligned}
\sigma &= \sqrt{\frac{75}{64k\Omega_k}}\left(\frac{\kappa^3 T\widetilde{N}}{\pi M}\right)^{1/4} \\
&= \sqrt{\frac{75}{64\times 1784\times 1.100}}\left(\frac{(1.38066\times 10^{-16})^3\times 300\times 6.02214\times 10^{23}}{\pi\times 39.948}\right)^{1/4} \\
&= 3.409\times 10^{-8}\ \text{cm}
\end{aligned}
$$

*d.* The excellent agreement between the results for $\sigma$, and the poor agreement for $d$, show that the data are represented much better by the Chapman-Enskog theory than by the simple hard-sphere kinetic theory.

**9C.1 Enskog theory for dense gases**

*a*. Equation 9C.1-4 can be written with pressure and temperature as the independent variables, thus:

$$y = \frac{\tilde{V}}{R}\left( -\frac{\left(\partial\tilde{V}/\partial T\right)_p}{\left(\partial\tilde{V}/\partial p\right)_T} \right) - 1$$

We may now use $\tilde{V} = ZRT/p$ and rewrite the derivatives appearing above as

$$\left(\frac{\partial\tilde{V}}{\partial T}\right)_p = \frac{R}{p}\left(\frac{\partial}{\partial T}ZT\right)_p = \frac{R}{p}\left(Z + T\left(\frac{\partial Z}{\partial T}\right)_p\right) = \frac{RZ}{p}\left(1 + \left(\frac{\partial \ln Z}{\partial \ln T}\right)_p\right)$$

$$\left(\frac{\partial\tilde{V}}{\partial p}\right)_T = RT\left(\frac{\partial}{\partial p}\frac{Z}{p}\right)_T = -\frac{RT}{p^2}\left(Z - p\left(\frac{\partial Z}{\partial p}\right)_T\right) = -\frac{RTZ}{p^2}\left(1 - \left(\frac{\partial \ln Z}{\partial \ln p}\right)_T\right)$$

When these expressions are substituted into the expression for $y$, we get

$$y = Z\left[\frac{1 + \left(\partial \ln Z/\partial \ln T\right)_p}{1 - \left(\partial \ln Z/\partial \ln p\right)_T}\right] - 1 \quad \text{or} \quad y = Z\left[\frac{1 + \left(\partial \ln Z/\partial \ln T_r\right)_{p_r}}{1 - \left(\partial \ln Z/\partial \ln p_r\right)_{T_r}}\right] - 1$$

*b*. First one would differentiate the Hougen-Watson Z chart to get the derivatives appearing in the result in (*a*) and hence $y(p_r, T_r)$. Then for a given reduced temperature and pressure, one would calculate the right sides of Eqs. 9C.1-1 and 2; call these quantities $f_\mu(p_r, T_r)$ and $f_k(p_r, T_r)$. Then

$$\frac{k}{k^\circ} = \frac{\mu}{\mu^\circ} \cdot \frac{f_k(p_r, T_r)}{f_\mu(p_r, T_r)}$$

One can then read off values of $\mu$ from the Uyehara-Watson chart and construct a chart for the thermal conductivity. This procedure is not recommended for polyatomic gases.

**10A.1 Heat loss from an insulated pipe.**

We use the notation of Fig. 10.6-2. When the temperatures at the inner and outer surfaces are known, Eq. 10.6-29 can be reduced to

$$\frac{Q_0}{L} = \frac{2\pi L(T_0 - T_3)}{\left[\frac{\ln(r_1/r_0)}{k_{01}} + \frac{\ln(r_2/r_1)}{k_{12}} + \frac{\ln(r_3/r_2)}{k_{23}}\right]}$$

The $r_i$ for this problem are:

$$\begin{aligned} r_0 &= 2.067/2 = 1.0335 \text{ in} \\ r_1 &= 1.0335 + 0.154 = 1.19 \text{ in} \\ r_2 &= 1.19 + 2 = 3.19 \\ r_3 &= 3.19 + 2 = 5.19 \end{aligned}$$

Insertion of numerical values into the above formula gives:

$$\begin{aligned} \frac{Q_0}{L} &= \frac{2\pi(250 - 90 \text{ F})}{\left[\frac{\ln(1.19/1.0335)}{26.1} + \frac{\ln(3.19/1.19)}{0.04} + \frac{\ln(5.19/3.19)}{0.03} \text{ F·hr·ft/Btu}\right]} \\ &= \frac{320\pi}{0.0054 + 24.7 + 16.2} = 24 \text{ Btu/hr per foot of pipe} \end{aligned}$$

**10A.2 Heat loss from a rectangular fin.**

From Eq. 10.7-14 we obtain the heat loss expression

$$Q = 2WLh(T_w - T_a) \cdot \eta$$

in which $\eta$ is given by Eq. 10.7-16:

$$\eta = \frac{\tanh N}{N} \qquad \text{with } N = \sqrt{\frac{hL^2}{kB}}$$

For the conditions of this problem,

$$N = \sqrt{\frac{hL^2}{kB}} = \sqrt{\frac{(120 \text{ Btu/hr·ft}^2\text{·F})(0.2 \text{ ft})^2}{(60 \text{ Btu/hr·ft·F})(0.08/12 \text{ ft})}} = \sqrt{12} = 3.4641$$

and

$$\eta = \tanh(3.4641)/3.4641 = 0.2881$$

The foregoing heat loss expression then gives:

$$\begin{aligned} Q &= 2WLh(T_0 - T_a) \cdot \eta \\ &= 2(1.0 \text{ ft})(0.2 \text{ ft})(120 \text{ Btu/hr·ft·F})(500 - 350 \text{ F})(0.2881) \\ &= 2074 \text{ Btu/hr} \end{aligned}$$

**10A.3 Maximum temperature in a lubricant.**

The parallel-plate approximation in §10.4 is used here to estimate the temperature rise; more accurate results will be presented in Chapter 11. Multiplication of Eq. 10.4-9 by $(T_b - T_0)$, and setting $T_b = T_0$, gives the temperature profile

$$T - T_0 = \frac{1}{2}\frac{\mu v_b^2}{k}(x/b)[1 - (x/b)]$$

The maximum temperature occurs at $x = b/2$; hence, with $v_b = \Omega R$,

$$\begin{aligned} T_{\max} - T_0 &= \frac{1}{8}\frac{\mu v_b^2}{k} = \frac{1}{8}\frac{\mu \Omega^2 R^2}{k} \\ &= \frac{1}{8}\frac{(0.923\ \text{g/cm·s})(7908 \times 2\pi/60\ \text{radians/s})^2(5.06\ \text{cm})^2}{(0.0055\ \text{cal/s·cm·C})(4.1840 \times 10^7\ \text{g·cm}^2/\text{s}^2/\text{cal})} \\ &= 8.8\ \text{C} = 16\ \text{F} \end{aligned}$$

Thus, the maximum temperature in the oil is $T_{\max} = 158 + 16 = 174°\text{F}$.

**10A.4 Current-carrying capacity of a wire.**

Consider a straight cylindrical wire of radius $r_2 = 0.040$ in, tightly covered with plastic insulation of outer radius $r_3 = 0.12/2 = 0.06$ in. Since the heat generated by electrical dissipation in the wire must flow radially outward, the maximum temperature in the plastic will occur at the wire-plastic interface $r = r_2$. The rate of heat loss from the wire reaches its largest permitted value when $T = 200°\text{F}$ at $r = r_2$ while the ambient temperature $T_4$ is 100°F. From Eq. 10.6-29 we calculate this rate as

$$\begin{aligned}
\frac{Q_{\max}}{L} &= 2\pi(T_{2,\max} - T_4)/\left[\frac{\ln(r_3/r_2)}{k_{23}} + \frac{1}{r_3 h_3}\right] \\
&= 2\pi(200 - 100\ \text{F})/\left[\frac{\ln(0.060/0.020)}{0.20} + \frac{1}{(0.060/12)(1.5)}\ \text{hr·ft·F/Btu}\right] \\
&= 200\pi/\,[5.5 + 133] = 4.54\ \text{Btu/hr·ft} \\
&= 0.0436\ \text{watt/cm}
\end{aligned}$$

Next, we equate this rate of heat loss to the electrical energy dissipation:

$$Q_{\max}/L = i_{\max}^2 R_e/L$$

in which

$$\begin{aligned}
R_e/L = (\pi r_2^2 k_e)^{-1} &= \text{ wire resistance } R_e \text{ per unit length} \\
&= \left(\pi(0.02 \times 2.54\ \text{cm})^2(5.1 \times 10^5\ \text{ohm}^{-1}\text{cm}^{-1})\right)^{-1} \\
&= 2.42 \times 10^{-4}\ \text{ohm/cm} \\
i_{\max} &= \text{ maximum permitted current}
\end{aligned}$$

The current-carrying capacity of the wire is then

$$\begin{aligned}
i_{\max} &= \sqrt{(Q_{\max}/L)/(R_e/L)} \\
&= \sqrt{(0.0435\ \text{watts/cm})/(2.42 \times 10^{-4}\ \text{ohm/cm})} \\
&= 13.4\ \text{amperes}
\end{aligned}$$

**10A.5 Free convection velocity.**

*a.* The solution for $v_z$ in Eq. 10.9-15 has the form

$$v_z = A\left[u^3 - u\right] \qquad \text{in which } A = \frac{(\overline{\rho} g \overline{\beta} \Delta T) B^2}{12\mu} \quad \text{and} \quad u = (y/B),$$

and its average over the upward-moving stream is

$$\begin{aligned}\langle v_z^{(\mathrm{up})} \rangle &= A \int_{u=-1}^{u=0} \left[u^3 - u\right] du = \frac{A}{4} \\ &= \frac{(\overline{\rho} g \overline{\beta} \Delta T) B^2}{48\mu}\end{aligned}$$

in agreement with Eq. 10.9-16.

*b.* For the conditions of this problem,

$$\begin{aligned}\overline{T} &= \frac{1}{2}(T_1 + T_2) = 60\ ^\circ\mathrm{C} = 333.2\ \mathrm{K}; \\ \overline{\beta} &= 1/\overline{T} = 3 \times 10^{-3}\ \mathrm{K}^{-1}.\end{aligned}$$

*c.* Insertion of known values into Eq. 10.9-16 (with $\mu/\overline{\rho}$ averaged as $\mu(\overline{T})/\rho(\overline{T})$ $= \nu(\overline{T}) = 0.1886\ \mathrm{cm}^2/\mathrm{s}$) then gives:

$$\begin{aligned}\langle v_z^{(\mathrm{up})} \rangle &= \frac{(980.7\ \mathrm{cm/s^2})((3 \times 10^{-3}\ \mathrm{K}^{-1})(80\ \mathrm{C})(0.3\ \mathrm{cm})^2}{(48)(0.1886\ \mathrm{cm^2/s})} \\ &= 2.3\ \mathrm{cm/s}\end{aligned}$$

**10A.6 Insulating power of a wall.**

*a.* Application of Eq. 10.6-9 to the data for the plastic panel gives

$$q_0 = \frac{k_{12}(T_1 - T_2)}{(x_2 - x_1)} = \frac{(0.075 \text{ Btu/hr·ft·F})(69 - 61 \text{ F})}{(0.502/12 \text{ ft})}$$
$$= 14.3 \text{ Btu/hr·ft}^2$$

*b.* The thermal resistance of the wall is therefore

$$R_{23} = \frac{(T_2 - T_3)}{q_0} = \frac{(61 - 0 \text{ F})}{14.3 \text{ Btu/hr·ft}^2}$$
$$= 4.2 \frac{\text{F}}{\text{Btu/hr·ft}^2}$$

**10A.7 Viscous heating in a ball-point pen.**

The paralel-plate approximation in §10.4 is used here to estimate the viscous heating of the fluid; more accurate results will be presented in Chapter 11. Multiplication of Eq. 10.4-9 by $(T_b - T_0)$, and setting $T_b = T_0$, gives the temperature profile in the ink,

$$T - T_0 = \frac{1}{2}\frac{\mu v_b^2}{k}(x/b)[1 - (x/b)]$$

valid when both adjoining surfaces are at temperature $T_0$.

At $x = b/2$ the temperature rise attains its maximum value

$$(T - T_0)_{\max} = \frac{1}{8}\frac{\mu v_b^2}{k}$$

Insertion of the data for this problem gives

$$\begin{aligned}(T - T_0)_{\max} &= \frac{1}{8}\frac{(10^4 \times 0.01\text{g/cm·s})(100 \times 2.54/60\ \text{cm/s})^2}{(5 \times 10^{-4} \times 4.1840 \times 10^7\ \text{g·cm/s}^3\text{·K})} \\ &= 0.011\ \text{K}\end{aligned}$$

as the maximum dissipative temperature rise in the ink. Thus, the warming of the ink by viscous dissipation will be negligible compared with the warming of the pen by contact with the hand of the user.

**10A.8 Temperature rise in an electrical wire.**

The maximum temperature in the wire occurs at the centerline. Equation 10.2-23 gives this value as

$$T_{\max} = T_{\text{air}} + \frac{S_e R^2}{4k} + \frac{S_e R}{2h}$$

in which

$$S_e = \frac{I^2}{k_e} = \left(\frac{k_e E}{L}\right)^2 \frac{1}{k_e} = \left(\frac{E^2 k_e}{L^2}\right)$$

The Wiedemann-Franz-Lorenz relation of Eq. 9.5-1, with the Lorenz constant for copper, gives

$$\frac{k}{k_e T} = 223 \times 10^{-8} \text{ volt}^2/\text{K}^2$$

and $k$ for copper ranges from 384.1 W/m·K at 25°C to 379.9 W/m·K at 100°C. Assuming the temperature rise to be small, we evaluate the conductivity values at 25°C=298.15K:

$$\begin{aligned} k &= 384.1 \text{ W/m·K} \\ k_e &= \frac{k}{(\text{Lorenz constant})(T)} \\ &= \frac{(384.1 \text{ W/m·K}}{(223 \times 10^{-8} \text{ volt}^2/\text{K})(298.15 \text{ K})} \\ &= 5.78 \times 10^5 \text{ ohm}^{-1}\text{m}^{-1}) \end{aligned}$$

We can then calculate $S_e$ as

$$\begin{aligned} S_e &= \left(\frac{E^2 k_e}{L^2}\right) \\ &= \frac{(0.6 \text{ volt})^2 (5.78 \times 10^5 \text{ ohm}^{-1}\text{m}^{-1})}{(15 \text{ ft} \times 0.3048 \text{ m/ft})^2} \\ &= 9.95 \times 10^3 \text{ W/m}^3 \end{aligned}$$

and the maximum temperature elevation in the wire as

$$\begin{aligned} T_{\max} - T_{\text{air}} &= \frac{(9.95 \times 10^3 \text{ W/m}^3)(0.005 \text{ m})^2}{4(384.1 \text{ W/m·K})} + \frac{(9.95 \times 10^3 \text{ W/m}^3)(0.005 \text{ m})}{2([5.7 \times 5.6782] \text{ W/m}^2\text{·K})} \\ &= 1.6 \times 10^{-4} + 0.77 \approx 0.77 \text{ K} \end{aligned}$$

*b.* The temperature difference across the wire is 4 orders of magnitude smaller than that in the surrounding air, as shown by the terms in the last line of the calculation.

**10B.1 Heat conduction from a sphere to a stagnant fluid**

*a*. The heat is being conducted in the $r$ direction only. Therefore, we select a shell of thickness $\Delta r$ over which we make the energy balance:

$$\Delta r q_r 4\pi r^2 q_r\Big|_r - 4\pi(r+\Delta r)^2 q_r\Big|_{r+\Delta r} = 0 \text{ or } 4\pi\left(r^2 q_r\right)\Big|_r - 4\pi\left(r^2 q_r\right)\Big|_{r+\Delta r} = 0$$

We now divide by $\Delta r$ and then take the limit as $\Delta r$ goes to zero

$$\lim_{\Delta r\to 0}\frac{\left(r^2 q_r\right)\Big|_{r+\Delta r} - \left(r^2 q_r\right)\Big|_r}{\Delta r} = 0$$

We then use the definition of the first derivative to get

$$\frac{d}{dr}\left(r^2 q_r\right) = 0 \qquad \text{and} \qquad \frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = 0$$

In the second equation we have inserted Fourier's law of heat conduction with constant thermal conductivity.

*b*. Integration of this equation twice with respect to $r$ gives

$$r^2\frac{dT}{dr} = C_1 \quad \text{and} \quad T = -\frac{C_1}{r} + C_2$$

The boundary conditions then gives $C_1 = -R(T_R - T_\infty)$, $C_2 = T_\infty$, and

$$\frac{T - T_\infty}{T_R - T_\infty} = \frac{R}{r}$$

*c*. The heat flux at the surface is

$$q_r\Big|_{r=R} = -k\frac{dT}{dr}\Big|_{r=R} = +kR(T_R - T_\infty)\frac{1}{r^2}\Big|_{r=R} = \frac{k(T_R - T_\infty)}{R} \equiv h(T_R - T_\infty)$$

so that $h = k/R = 2k/D$ and Nu = 2.

*d*. Bi contains $k$ of the solid; Nu contains $k$ of the fluid.

**10B.2 Viscous heating in slit flow**

Equation s 10.4-5 and 10.4-6 are still valid for this problem. Since at $x = b$, $q_x = -k(dT/dx) = 0$, we get from Eq. 10.4-5:

$$-\mu b\left(\frac{v_b}{b}\right)^2 = C_1 \quad \text{or} \quad C_1 = -\frac{\mu v_b^2}{b}$$

And from Eq. 10.4-6, we get $C_2 = T_0$.

Then substitution of these expressions for the integration constants into Eq. 10.4-6 yields

$$T = -\left(\frac{\mu}{k}\right)\left(\frac{v_b}{b}\right)^2 \frac{x^2}{2} + \frac{\mu v_b^2}{bk}x + T_0$$

When this is rearranged in dimensionless form we have

$$\frac{T - T_0}{\mu v_b^2/k} = \left(\frac{x}{b}\right) - \frac{1}{2}\left(\frac{x}{b}\right)^2$$

**10B.3 Heat conduction in a nuclear fuel rod assembly**

The differential equation may be set up following the procedure in §10.2, by replacing $S_e$ by $S_n$ in Eq. 10.2-6. Then, when Fourier's law with constant thermal conductivity is substituted into the thus modified Eq. 10.2-6, we get

$$-k_F \frac{d}{dr}\left( r \frac{dT_F}{dr} \right) = S_{n0} \left[ 1 + b \left( \frac{r}{R_F} \right)^2 \right] r$$

for the heat conduction equation in the fuel rod. In the cladding a similar equation, without the source term, is appropriate:

$$-k_C \frac{d}{dr}\left( r \frac{dT_C}{dr} \right) = 0$$

The boundary conditions in this problem are

| | | |
|---|---|---|
| B. C. 1: | At $r = 0$, | $T_F$ is finite |
| B. C. 2: | At $r = R_F$, | $T_F = T_C$ |
| B. C. 3: | At $r = R_F$, | $-k_F(dT_F/dr) = -k_C(dT_C/dr)$ |
| B. C. 4: | At $r = R_C$, | $-k_C(dT_C/dr) = h_L(T_C - T_L)$ |

Integrating the above differential equations twice gives

$$\frac{dT_F}{dr} = -\frac{S_{n0} r}{2k_F}\left( 1 + \frac{b}{R_F^2}\frac{r^2}{2} \right) + \frac{C_1}{r} \qquad \frac{dT_C}{dr} = \frac{C_3}{r}$$

$$T_F = -\frac{S_{n0} r^2}{4k_F}\left( 1 + \frac{b}{R_F^2}\frac{r^2}{4} \right) + C_1 \ln r + C_2; \qquad T_C = C_3 \ln r + C_4$$

The constant is zero by B. C. 1, since the temperature is not infinite at the axis of the fuel rod. From B. C. 3, we can find $C_3$:

$$C_3 = -\frac{S_{n0} R_F^2}{2k_C}\left( 1 + \frac{b}{2} \right)$$

From B. C. 4, we get $C_4$:

$$C_4 = T_L - \left( \frac{k_C}{R_C h_L} + \ln R_C \right) C_3 = T_L + \left( \frac{k_C}{R_C h_L} + \ln R_C \right) \frac{S_{n0} R_F^2}{2k_C} \left( 1 + \frac{b}{2} \right)$$

And finally $C_2$ can be obtained from B. C. 2:

$$C_2 = T_L + \frac{S_{n0} R_F^2}{4k_F} \left( 1 + \frac{b}{4} \right) + \frac{S_{n0} R_F^2}{2k_C} \left( 1 + \frac{b}{2} \right) \left( \ln \frac{R_C}{R_F} + \frac{k_C}{R_C h_L} \right)$$

Then we can get the maximum temperature at the axis of the fuel rod:

$$T_{F,\max} = T_L + \frac{S_{n0} R_F^2}{4k_F} \left( 1 + \frac{b}{4} \right) + \frac{S_{n0} R_F^2}{2k_C} \left( 1 + \frac{b}{2} \right) \left( \ln \frac{R_C}{R_F} + \frac{k_C}{R_C h_L} \right)$$

**10B.4 Heat conduction in an annulus**

*a*. The energy balance on a cylindrical shell of thickness $\Delta r$ and length $L$ is

$$2\pi r L q_r\big|_r - 2\pi(r+\Delta r)L q_r\big|_{r+\Delta r} = 0 \qquad \text{or} \qquad 2\pi L(rq_r)\big|_r - 2\pi L(rq_r)\big|_{r+\Delta r} = 0$$

When this equation is divided by $2\pi L$ and the limit is taken as $\Delta r$ goes to zero, we get

$$\frac{d}{dr}(rq_r) = 0$$

which may be integrated to give

$$rq_r = C_1 \qquad \text{or} \qquad -k\frac{dT}{dr} = \frac{C_1}{r}$$

The thermal conductivity varies linearly with temperature, so that

$$k = k_0 + (k_1 - k_0)\left(\frac{T - T_0}{T_1 - T_0}\right) \equiv k_0 + (k_1 - k_0)\Theta$$

Then

$$-\left[k_0 + (k_1 - k_0)\Theta\right]\frac{dT}{dr} = \frac{C_1}{r} \qquad \text{or} \qquad -(T_1 - T_0)\left[k_0 + (k_1 - k_0)\Theta\right]\frac{d\Theta}{dr} = \frac{C_1}{r}$$

This first-order, separable differential equation may be integrated:

$$-(T_1 - T_0)\left[k_0 + \tfrac{1}{2}(k_1 - k_0)\Theta\right]\Theta = C_1 \ln r + C_2$$

The constants of integration may be found from the boundary conditions: $\Theta(r_0) = 0$ and $\Theta(r_1) = 1$.

$$0 = C_1 \ln r_0 + C_2 \quad \text{and} \quad -(T_1 - T_0)\left[k_0 + \tfrac{1}{2}(k_1 - k_0)\right] = C_1 \ln r_1 + C_2$$

When these relations are subtracted, and equation for $C_1$ is obtained:

$$C_1 = -\frac{(T_1 - T_0)}{\ln(r_1/r_0)}\left[\tfrac{1}{2}(k_0 + k_1)\right]$$

and $C_2$ may also be obtained if desired.

The heat flow through the wall may then be obtained:

$$Q = 2\pi r_0 L q_r\big|_{r=r_0} = 2\pi L r_0\left(\frac{C_1}{r_0}\right) = 2\pi L\frac{(T_0 - T_1)}{\ln(r_1/r_0)}\left[\tfrac{1}{2}(k_0 + k_1)\right]$$

*b*. Let the ratio of the outer to the inner radius be written as $r_1/r_0 = 1 + \varepsilon$, where $\varepsilon$ is very small. Then use the Taylor series for the logarithm as given in Eq. C.2-3: $\ln(1+\varepsilon) = \varepsilon - \frac{1}{2}\varepsilon^2 + \frac{1}{3}\varepsilon^3 - \cdots$. If we keep just one term of the series, then this corresponds to

$$\varepsilon = (r_1/r_0) - 1 = (r_1 - r_0)/r_0$$

When this is substituted into the expression for $Q$ we get

$$Q = 2\pi L r_0\left[\tfrac{1}{2}(k_0 + k_1)\right]\frac{T_0 - T_1}{r_1 - r_0}$$

This is just: area times average thermal conductivity times a temperature gradient.

**10B.5 Viscous heat generation in a polymer melt**

We can start with a modification of Eq. 10.4-4:

$$-k\frac{dT}{dx} - v_z\left(m\left|\frac{dv_z}{dx}\right|^{n-1}\frac{dv_z}{dx}\right) = C_1$$

Since the velocity gradient is positive in this problem, the absolute value operation is not needed:

$$-k\frac{dT}{dx} - mv_z\left(\frac{dv_z}{dx}\right)^n = C_1$$

When the linear velocity profile is inserted, this becomes

$$-k\frac{dT}{dx} - mx\left(\frac{v_b}{b}\right)^{n+1} = C_1$$

Integration with respect to $x$ gives

$$-kT = C_1 x + \frac{1}{2}mx^2\left(\frac{v_b}{b}\right)^{n+1} + C_2$$

The constants are then determined by the boundary conditions given in Eqs. 104-7 and 8, and final result is

$$\frac{T-T_0}{T_b-T_0} = \frac{x}{b} + \frac{1}{2}\frac{mb^2}{k(T_b-T_0)}\left(\frac{v_b}{b}\right)^{n+1}\left[\left(\frac{x}{b}\right)-\left(\frac{x}{b}\right)^2\right]$$

or

$$\frac{T-T_0}{T_b-T_0} = \frac{x}{b} + \frac{1}{2}\mathrm{Br}_n\frac{x}{b}\left(1-\frac{x}{b}\right)$$

which should be compared to the Newtonian result in Eq. 10.4-9.

**10B.6 Insulation thickness for a furnace wall**

Let the regions be labeled as follows:

| | |
|---|---|
| Refractory brick | "01" |
| Insulating brick | "12" |
| Steel | "23" |

and we may used the formulas given in Eqs. 10.6-8, 9, and 10.

The minimum wall thickness will occur when $T_1 = 2000°\text{F}$. If for the sake of being on the safe side, let $T_0 = 2500°\text{F}$. Then for the region "01" the thickness must be

$$x_1 - x_0 = \frac{k_{01}(T_0 - T_1)}{q_0} = \frac{\frac{1}{2}(4.1+3.6)(2500-2000)}{5000} = 0.39\text{ft}$$

Here we have taken the thermal conductivity of the refractory brick to be the arithmetic average of the values the thermal conductivity at 2000°F and 2500°F (the latter estimated by linear extrapolation from the given data).

For the remaining two regions, we may add Eqs. 10.6-9 and 10 t get

$$T_1 - T_3 = q_0\left(\frac{x_2 - x_1}{k_{12}} + \frac{x_3 - x_2}{k_{23}}\right)$$

or, taking the steel temperature to be 100,

$$2000 - 100 = 5000\left(\frac{x_2 - x_1}{\frac{1}{2}(0.9+1.8)} + \frac{(0.25)\frac{1}{12}}{26.1}\right)$$

This gives $x_2 - x_1 = 0.51\text{ft}$.

**10B.7 Forced-convection heat transfer in flow beween parallel plates**

*a*. Since the temperature depends on both $x$ and $z$, we make an energy balance over a region of volume $W\Delta x\Delta z$, in which $W$ is the dimension of the slit in the $y$ direction. The various contributions to the energy balance are:

| | |
|---|---|
| Total energy in at $x$: | $e_x\big\|_x W\Delta z$ |
| Total energy out at $x+\Delta x$: | $e_x\big\|_{x+\Delta x} W\Delta z$ |
| Total energy in at $z$: | $e_z\big\|_z W\Delta x$ |
| Total energy out at $z+\Delta z$: | $e_z\big\|_{z+\Delta z} W\Delta x$ |
| Work done on fluid by gravity: | $\rho v_z g_z W\Delta x\Delta z$ |

When these terms are added together and divided by $W\Delta x\Delta z$, we get

$$-\frac{\partial e_x}{\partial x}-\frac{\partial e_z}{\partial z}-\rho v_z g=0$$

since gravity is acting in the -$z$ direction.

Now we use Eqs. 9.8-6 and 9.8-8 to write out the $x$ and $z$ components of the combined energy flux:

$$e_x=\tau_{xz}v_z+q_x=-\left(\mu\frac{\partial v_z}{\partial x}\right)v_z-k\frac{\partial T}{\partial x}$$

$$\begin{aligned}e_z&=\left(\tfrac{1}{2}\rho v_z^2\right)v_z+\rho\hat{H}v_z+\tau_{zz}v_z+q_z\\&=\left(\tfrac{1}{2}\rho v_z^2\right)v_z+\left(p-p^{\circ}\right)v_z+\rho\hat{C}_p\left(T-T^{\circ}\right)v_z-\left(2\mu\frac{\partial v_z}{\partial z}\right)v_z-k\frac{\partial T}{\partial z}\end{aligned}$$

Substituting these expressions into the energy balance, and making use of the fact that $v_z$ depends only on $x$ gives

$$\rho\hat{C}_p v_z \frac{\partial T}{\partial z} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right) + \mu\left(\frac{\partial v_z}{\partial x}\right)^2 + \left(-\frac{\partial p}{\partial z} + \mu\frac{\partial^2 v_z}{\partial x^2} - \rho g\right)$$

The term in the last parentheses is zero by the equation of motion, the term just before that is the viscous heating (which we neglect), and in the first parentheses we neglect the heat conduction in the z-direction.

*b*. Hence we get

$$\rho\hat{C}_p v_{z,\max}\left[1-\left(\frac{x}{B}\right)^2\right]\frac{\partial T}{\partial z} = k\frac{\partial^2 T}{\partial x^2} \qquad \text{or} \qquad \left(1-\sigma^2\right)\frac{\partial\Theta}{\partial\zeta} = \frac{\partial^2\Theta}{\partial\sigma^2}$$

with the boundary and initial conditions: at $\sigma = \pm 1$, $\pm\left(\partial\Theta/\partial\sigma\right) = 1$, and at $\zeta = 0$, $\Theta = 0$.

*c*. For large $z$ we propose the solution $\Theta(\sigma,\zeta) = C_0\zeta + \Psi(\sigma)$. Then $\Psi(\sigma)$ has to satisfy the ordinary differential equation

$$\frac{\partial^2\Psi}{\partial\sigma^2} = C_0\left(1-\sigma^2\right)$$

which is easily integrated. The expression for $\Theta(\sigma,\zeta)$ is then

$$\Theta(\sigma,\zeta) = C_0\zeta + C_0\left(\tfrac{1}{2}\sigma^2 - \tfrac{1}{12}\sigma^4\right) + C_1\sigma + C_2$$

Application of the boundary conditions at $\sigma = \pm 1$ gives $C_1 = 0$ and $C_0 = \frac{3}{2}$. The remaining constant has to be obtained from an integral condition:

$$\zeta = \int_0^1 \Theta(\sigma,\zeta)\left(1-\sigma^2\right)d\sigma$$

This gives $C_2 = -\frac{39}{280}$. Combining these results we get

$$\Theta(\sigma,\zeta) = \tfrac{3}{2}\zeta + \tfrac{3}{2}\left(\tfrac{1}{2}\sigma^2 - \tfrac{1}{12}\sigma^4\right) - \tfrac{39}{280}$$

which is in accordance with Eq. 10B.7-4.

10B.8 Electrical heating of a pipe

For the assumptions made in the problem statement we may assume that temperature is a function only of r and that

$$q_r\big|_{\kappa R} = -k\frac{\partial T}{\partial r}\bigg|_{\kappa R} = 0$$

therefore that all generated heat must leave from the surface at R by Newton's law of cooling:

$$VS_e = Ah(T_1 - T_s)$$

or

$$T\big|_R = T_1 = T_s + R(1-\kappa)S_e/2h$$

These relations provide a basis for setting up a differential equation as well as giving the boundary conditions for integrating it. The next step is a shell balance over the region between any radial position "r" and a nearby position "r + Δr":

$$2\pi rL\,q_r\big|_r + 2\pi rLS_e\Delta r = 2\pi rL\,q_r\big|_{r+\Delta r}$$

or

$$\frac{1}{r}\left[\frac{rq_r\big|_{r+\Delta r} - r\,q_r\big|_r}{\Delta r}\right] \Rightarrow \frac{1}{r}\frac{\partial}{\partial r}rq_r = S_e$$

It follows, just as for Ex. 10.2, that

$$q_r = -k\frac{\partial T}{\partial r} = \frac{rS_e}{2} + \frac{C_1}{r}$$

However, the above boundary condition now gives

$$C_1 = -\frac{(\kappa R)^2 S_e}{2}$$

and

$$\frac{\partial T}{\partial r} = -\frac{rS_e}{2k} + \frac{1}{r}\frac{(\kappa R)^2 S_e}{2k}$$

$$T = -\frac{r^2 S_e}{4k} + \ln r \cdot \frac{(\kappa R)^2 S_e}{2k} + C_2$$

Therefore

$$T = T_1 + \frac{R^2 S_e}{4k}\left\{\left[1-(r/R)^2\right] - 2\kappa^2\ln(R/r)\right\}$$

and $T_1$ is related to the surrounding temperature $T_s$ by the above equation.

**10B.9 Plug flow with forced-convection heat transfer**

*a*. We can adapt the solution in §10.8, by replacing $v_{z,\max}$ by $v_0$, and omitting the factor $[1-(r/R)^2]$ in Eq. 10.8-12. Then we can give the results corresponding to those in §10.8 as follows:

(10.8-25) $$\zeta = \int_0^1 \Theta(\xi,\zeta)\xi d\xi$$

(10.8-26) $$\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{d\Psi}{d\xi}\right) = C_0$$

(10.8-27) $$\Theta(\xi,\zeta) = C_0\zeta + \tfrac{1}{4}C_0\xi^2 + C_1\ln\xi + C_2$$

(10.8-28, 29, 30) $$C_1 = 0;\ C_0 = 2;\ C_2 = -\tfrac{1}{4}$$

With these coefficients, Eq. 10B.9-1 follows.

*b*. The solution of Problem 10B.7 can be adapted by replacing $v_{z,\max}$ by $v_0$, and omitting the factor $[1-(x/B)^2]$ in the energy equation. The results, analogous to those in part (*a*) are

(10.8-25) $$\zeta = \int_0^1 \Theta(\sigma,\zeta)\sigma d\sigma$$

(10.8-26) $$\frac{d^2\Psi}{d\sigma^2} = C_0$$

(10.8-27) $$\Theta(\sigma,\zeta) = C_0\zeta + \tfrac{1}{4}C_0\sigma^2 + C_1\sigma + C_2$$

(10.8-28, 29, 30) $$C_1 = 0;\ C_0 = 1;\ C_2 = -\tfrac{1}{6}$$

With these coefficients, one obtains Eq. 10B.9-2.

**10B.10 Free convection in an annulus of finite height**

*a*. The appropriate simplification of the energy equation is

$$0 = k\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)$$

which has the solution $T = C_1 \ln r + C_2$. When the constants are evaluated using the boundary conditions we get Eq. 10B.10-1.

*b*. We make use of the linear approximation for the density as a function of temperature:

$$\rho = \rho_1 - \rho_1\beta_1(T - T_1)$$

This leads to the equation of motion

$$0 = \mu\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) - \left(\frac{dp}{dz} + \rho_1 g\right) + \rho_1 g\beta_1(T - T_1)$$

Subtsitution of the expression for the temperature distribution from Eq. 10B.10-1 into the equation of motion, and multiplication by $R^2/\mu$ gives

$$\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{dv_z}{d\xi}\right) = \frac{R^2}{\mu}\left(\frac{dp}{dz} + \rho_1 g\right) + \left[\frac{\rho_1 g\beta_1(T_1 - T_\kappa)R^2}{\mu \ln\kappa}\right]\ln\xi$$

which is just Eq. 10B.10-2, along with the definitions of $A$ and $B$.

*c*. Integration of Eq. 10B.10.2 then yields

$$v_z = \tfrac{1}{2}A\xi^2 + \tfrac{1}{4}B\left(\xi^2\ln\xi - \xi^2\right) + C_1\ln\xi + C_2$$

The constants of integration are

$$C_2 = -\tfrac{1}{4}(A - B) \quad \text{and} \qquad C_1 = \frac{(A - B)\left(1 - \kappa^2\right) - B\kappa^2\ln\kappa}{4\ln\kappa}$$

The velocity distribution can then be written as

$$4v_z = (B-A)\left[\left(1-\xi^2\right)-\left(1-\kappa^2\right)\frac{\ln\xi}{\ln\kappa}\right]+B\left(\xi^2-\kappa^2\right)\ln\xi \qquad (*)$$

This can easily be shown to satisfy the boundary conditions for the velocity at the inner and outer walls of the annular region.

It remains to evaluate the constant $A$ (as we did in §10.9) by equating the total mass flow through a cross-section to zero:

$$\int_0^{2\pi}\int_{\kappa R}^{R}(\rho v_z)rdrd\theta = 0 \quad \text{or} \quad \int_\kappa^1\left[\rho_1-\rho_1\beta_1(T-T_1)\right]v_z\xi d\xi = 0$$

Thus we have to evaluate $A$ from

$$\int_\kappa^1\left[1+\beta_1(T_1-T_\kappa)\frac{\ln\xi}{\ln\kappa}\right]\cdot$$
$$\left\{(B-A)\left[\left(1-\xi^2\right)-\left(1-\kappa^2\right)\frac{\ln\xi}{\ln\kappa}\right]+B\left(\xi^2-\kappa^2\right)\ln\xi\right\}\xi d\xi = 0$$

First we note that $B \propto \Delta T \equiv T_1 - T_\kappa$ and therefore that $A \propto \Delta T$. To be consistent with the Boussinesq approximation, we should therefore neglect terms that are proportional to $(\Delta T)^2$. This eliminates the need for doing many of the integrations in the above equation. When the necessary integrations are performed we get, after a modest amount of algebra

$$(B-A) = \frac{B\ln\kappa}{4}\left[\frac{\left(1-\kappa^2\right)\left(1-3\kappa^2\right)-\kappa^4\ln\kappa}{\left(1-\kappa^2\right)^2+\left(1-\kappa^4\right)\ln\kappa}\right]$$

When this is substituted into (*) along with the expression for $B$ given just after Eq. 10B.10-2, we get Eq. 10B.10-3.

**10B.11 Free convection with temperature-dependent viscosity**

*a.* It is instructive and helpful to begin by going back and resolving the problem of §9.8 using the notation of the present problem. Equations 10.9-9 to 11 in dimensionless form are

$$\frac{d^2\breve{v}_z}{d\breve{y}^2} = P + \tfrac{1}{2}\mathrm{Gr}\breve{y} \qquad \text{with} \qquad \breve{v}_z(\pm 1) = 0$$

This may be integrated to give

$$\breve{v}_z = C_2 + C_1\breve{y} + \tfrac{1}{2}P\breve{y}^2 + \tfrac{1}{12}\mathrm{Gr}\breve{y}^3$$

When the boundary conditions are applied, we get $C_1 = -\frac{1}{12}\mathrm{Gr}$ and $C_2 = -\frac{1}{2}P$. Thus the velocity distribution is

$$\breve{v}_z = \tfrac{1}{12}\mathrm{Gr}\left(\breve{y}^3 - \breve{y}\right) + \tfrac{1}{2}P\left(\breve{y}^2 - 1\right)$$

which is equivalent to Eq. 10.9-12. Next we apply Eq. 10.9-13, which, in dimensionless form is

$$\int_{-1}^{+1}\left(1 - b_T\breve{y}\right)\left[\tfrac{1}{12}\mathrm{Gr}\left(\breve{y}^3 - \breve{y}\right) + \tfrac{1}{2}P\left(\breve{y}^2 - 1\right)\right]d\breve{y} = 0$$

in which $b_T = \frac{1}{2}\overline{\beta}\Delta T$; this equation has to be solved for $P$. In doing the integrations, the integrals over odd powers of $\breve{y}$ give zero, and we get

$$\mathrm{Gr}\left(-\tfrac{1}{12}\cdot\tfrac{2}{5} + \tfrac{1}{12}\cdot\tfrac{2}{3}\right)b_T + P\left(-\tfrac{1}{2}\cdot 2 + \tfrac{1}{2}\cdot\tfrac{2}{3}\right) = 0 \quad \text{from which } P = \tfrac{1}{30}\mathrm{Gr}b_T$$

When this value for $P$ is substituted into the dimensionless velocity distribution we get

$$\breve{v}_z = \tfrac{1}{12}\mathrm{Gr}\left(\breve{y}^3 - \breve{y}\right) + \tfrac{1}{60}\mathrm{Gr}b_T\left(\breve{y}^2 - 1\right)$$

On the right side, the first term contains $\Delta T$ and the second term contains $(\Delta T)^2$. In §9.8 we chose to neglect the term in $(\Delta T)^2$, which is

all right if $\Delta T$ is quite small. This led us to the result in Eq. 10.9-17, which is just the first term in the above equation.

*b*. Now we address the temperature-dependent viscosity problem in Problem 10B.11. The equation of motion is then:

$$\frac{d}{dy}\left(\mu(T)\frac{dv_z}{dy}\right)=\frac{dp}{dz}+\rho g$$

We now introduce the temperature dependence of the viscosity (Eq.10B.11-1) and density (Eq. 10.9-6) to get

$$\frac{d}{dy}\left(\frac{\overline{\mu}}{1+\overline{\beta}_\mu\left(T-\overline{T}\right)}\frac{dv_z}{dy}\right)=\frac{dp}{dz}+\overline{\rho}g-\overline{\rho}g\overline{\beta}\left(T-\overline{T}\right)$$

We now multiply this equation by $B^3\overline{\rho}/\overline{\mu}^2$ in order to get the equation in dimensionless form (using quantities defined in Eqs. 10B.11-2 and 3):

$$\frac{d}{d\breve{y}}\left(\frac{1}{1-b_\mu\breve{y}}\frac{d\breve{v}_z}{d\breve{y}}\right)=P+\tfrac{1}{2}\mathrm{Gr}\breve{y} \qquad \text{with } \breve{v}_z(\pm 1)=0$$

Integration of this equation gives

$$\breve{v}_z=\mathrm{C}_2+\mathrm{C}_1\breve{y}+\tfrac{1}{2}\left(P-\mathrm{C}_1 b_\mu\right)\breve{y}^2+\tfrac{1}{3}\left(\tfrac{1}{4}\mathrm{Gr}-Pb_\mu\right)\breve{y}^3-\tfrac{1}{16}\mathrm{Gr}b_\mu\breve{y}^4$$

Application of the boundary conditions then gives

$$\mathrm{C}_1=-\tfrac{1}{3}\left(\tfrac{1}{4}\mathrm{Gr}-\underline{Pb_\mu}\right) \qquad \text{and} \qquad \mathrm{C}_2=-\tfrac{1}{2}P+\tfrac{1}{48}\mathrm{Gr}b_\mu+\underline{\tfrac{1}{6}Pb_\mu^2}$$

In (*a*) we found that $P$ was second order in $\Delta T$ Therefore the dashed underlined terms in $\mathrm{C}_1$ and $\mathrm{C}_2$ can be expected to be of the third and fourth order respectively--thus two orders of magnitude higher than the remainder of the terms. Therefore we drop the dashed-underlined terms at this point.

Then the dimensionless velocity distribution is

$$\breve{v}_z = \left(-\tfrac{1}{2}P + \tfrac{1}{48}\mathrm{Gr}b_\mu\right) - \tfrac{1}{12}\mathrm{Gr}\breve{y} + \tfrac{1}{2}\left(P + \tfrac{1}{12}\mathrm{Gr}b_\mu\right)\breve{y}^2 + \tfrac{1}{12}\mathrm{Gr}\breve{y}^3 - \tfrac{1}{16}\mathrm{Gr}b_\mu\breve{y}^4$$

Next we deterimine $P$ from the requirement that there be no net mass movement upward in the region between the plates (i.e., Eq. 10.9-13):

$$0 = 2\left(-\tfrac{1}{2}P + \tfrac{1}{48}\mathrm{Gr}b_\mu\right) + \tfrac{2}{3}b_T\left(\tfrac{1}{12}\mathrm{Gr}\right) + \tfrac{2}{3}\cdot\tfrac{1}{2}\left(P + \tfrac{1}{12}\mathrm{Gr}b_\mu\right) - \tfrac{2}{5}b_T\left(\tfrac{1}{12}\mathrm{Gr}\right) - \tfrac{2}{5}\tfrac{1}{16}\mathrm{Gr}b_\mu$$

When this is solved for $P$ we get

$$P = \tfrac{1}{30}\mathrm{Gr}b_T + \tfrac{1}{15}\mathrm{Gr}b_\mu$$

and this is second order in $\Delta T$ as we had anticipated, and the dropping of the dashed underlined terms is fully justified. When this expression for $P$ is substituted into the dimensionless velocity distribution we finally obtain:

$$\breve{v}_z = \tfrac{1}{12}\mathrm{Gr}\left(\breve{y}^3 - \breve{y}\right) + \tfrac{1}{60}\mathrm{Gr}b_T\left(\breve{y}^2 - 1\right) - \tfrac{1}{80}\mathrm{Gr}b_\mu\left(\breve{y}^2 - 1\right)\left(5\breve{y}^2 - 1\right)$$

The first term is the "basic solution" in Eq, 10.9-17, the second term is the deviation from the basic solution when terms second order in $\Delta T$ are accounted for, and the third term is the second order term that enters in when the temperature dependence of the viscosity is taken into account. The basic solution contains only odd powers of the coordinate, whereas the second order terms contain only even powers. It is clear that this equation satisfies the boundary conditions.

**10B.12 Heat conduction with temperature-dependent thermal conductivity**

The thermal conductivity is a function of the temperature:

$$k = k_0 - (k_0 - k_\pi)\left(\frac{T_0 - T}{T_0 - T_\pi}\right) \equiv k_0 - (k_0 - k_\pi)\Theta$$

*a*. Since $T$ is a function of $\theta$ alone, we make a shell energy balance thus:

$$q_\theta\big|_\theta (r_2 - r_1)L - q_\theta\big|_{\theta+\Delta\theta}(r_2 - r_1)L = 0$$

Dividing by $(r_2 - r_1)L\Delta\theta$ and taking the limit as $\Delta\theta \to 0$ gives

$$\frac{dq_\theta}{d\theta} = 0$$

Then inserting Fourier's law with a temperature-dependent thermal conductivity, we get

$$\frac{d}{d\theta}\left[-k\frac{1}{r}\frac{dT}{d\theta}\right] = 0 \qquad \text{or} \qquad \frac{d}{d\theta}\left[-(k_0 - (k_0 - k_\pi)\Theta)\frac{1}{r}\frac{d\Theta}{d\theta}\right] = 0$$

Integrating once we get

$$(k_0 - (k_0 - k_\pi)\Theta)\frac{d\Theta}{d\theta} = C_1$$

and a second integration gives

$$k_0\Theta - \tfrac{1}{2}(k_0 - k_\pi)\Theta^2 = C_1\theta + C_2$$

When the constants are determined from the boundary conditions, $\Theta(0) = 0$ and $\Theta(\pi) = 1$, we get $C_2 = 0$ and $C_1 = (k_0 + k_\pi)/2\pi$, so that the temperature distribution in the solid is

$$\frac{k_0\Theta - \frac{1}{2}(k_0 - k_\pi)\Theta^2}{\frac{1}{2}(k_0 + k_\pi)} = \frac{\theta}{\pi}$$

Then the total heat flow through the surface at $\theta = 0$ is then

$$\begin{aligned}
Q &= \int_0^L \int_{r_1}^{r_2} \left( -k \frac{1}{r} \frac{dT}{d\theta} \right) \bigg|_{\theta=0} dr dz \\
&= k_0 (T_0 - T_\pi) \int_0^L \int_{r_1}^{r_2} \left( \frac{1}{r} \frac{d\Theta}{d\theta} \right) \bigg|_{\theta=0} dr dz \\
&= k_0 (T_0 - T_\pi) \int_0^L \int_{r_1}^{r_2} \frac{1}{r} \left( \frac{\frac{1}{2}(k_0 + k_\pi)}{\pi k_0} \right) dr dz \\
&= \left( L \ln \frac{r_2}{r_1} \right) \left( \frac{k_0 + k_\pi}{2} \right) \left( \frac{T_0 - T_\pi}{\pi} \right)
\end{aligned}$$

**10B.13 Flow reactor with exponentially temperature-dependent source**

For this problem we may take $T_0$ to be zero, since there is no particular need to do otherwise. Then the temperature is made dimensionless by dividing by the inlet temperature: $\Theta = T/T_1$. Also, the quantity $S_{c1}$ may be identified with $K$ of Eq. 10B.13-1. Therefore the quantity $S_{c1}F(\Theta)$ in Eq. 10.5-7 becomes

$$S_{c1}F(\Theta) = K\exp\left(-\frac{E}{RT}\right) = K\exp\left(-\frac{A}{\Theta}\right) \qquad \text{where} \qquad A = \frac{E}{RT_1}$$

We may then proceed to Eqs. 10.b-21 to 23:

Zone I: $\quad \Theta^{\mathrm{I}} = T^{\mathrm{I}}/T_1 = 1 \quad$ or $\quad T^{\mathrm{I}} = T_1$

Zone II: $\quad \int_{\Theta^{\mathrm{I}}}^{\Theta^{\mathrm{II}}} \frac{1}{F(\Theta)} d\Theta = \int_1^{\Theta^{\mathrm{II}}} \exp\left(+\frac{A}{\Theta}\right) d\Theta = NZ$

To do the integral we make a change of variables $A/\Theta = x$ so that $d\Theta = -Ax^{-2}dx$. Then the integral becomes

$$-A\int_1^{A/\Theta^{\mathrm{II}}} e^x x^{-2} dx = +A\frac{e^x}{x}\bigg|_1^{A/\Theta^{\mathrm{II}}} - A\int_1^{A/\Theta^{\mathrm{II}}} e^x x^{-1} dx = NZ$$

This last integral can be written as a power series. We then get:

$$A\exp\left(A/\Theta^{\mathrm{II}}\right) - Ae - A\left[\ln x + \sum_{n=1}^{\infty}\frac{x^n}{n!n}\right]_1^{A/\Theta^{\mathrm{II}}} = NZ \qquad \text{or}$$

$$\exp\left(A/\Theta^{\mathrm{II}}\right) - e - \sum_{n=1}^{\infty}\frac{1}{n!n} + \ln\left(A/\Theta^{\mathrm{II}}\right) + \sum_{n=1}^{\infty}\frac{\left(A/\Theta^{\mathrm{II}}\right)^n}{n!n} = \frac{NZ}{A}$$

for $\Theta^{II}$ as a function of $Z$. However it is easier to calculate $Z$ as a function of $\Theta^{II}$, since that does not involve any trial and error.

Zone III: $\Theta^{III} = \Theta^{II}\big|_{Z=1}$

*Another way to treat Zone II:*

If the temperature rise is not too great, we may expand $e^x x^{-2}$ in a power series about $x = 1$:

$$e^x x^{-2} = e\left[1-(x-1)+\tfrac{3}{2!}(x-1)^2-\tfrac{11}{3!}(x-1)^3+\tfrac{33}{4!}(x-1)^4+\cdots\right]$$

Then

$$\begin{aligned}
-A\int_1^{A/\Theta^{II}} e^x x^{-2}dx &= -Ae\int_1^{A/\Theta^{II}}\left[1-(1-x)+\tfrac{3}{2}(1-x)^2\cdots\right]dx \\
&= -Ae\int_0^{(A/\Theta^{II})-1}\left[1-y+\tfrac{3}{2}y^2\cdots\right]dy \\
&= -Ae\left[\left(\frac{A}{\Theta^{II}}-1\right)-\frac{1}{2}\left(\frac{A}{\Theta^{II}}-1\right)^2+\frac{3}{2}\left(\frac{A}{\Theta^{II}}-1\right)^3+\cdots\right] = NZ
\end{aligned}$$

for $\Theta^{II}$ as a function of $Z$. Also, by this method it is easier to calculate $Z$ as a function of $\Theta^{II}$, since that does not involve any trial and error.

**10B.14 Evaporation loss from an oxygen tank**

*a*. The fact that the thermal conductivity varies linearly with temperature can be written as

$$\frac{k-k_0}{k_1-k_0}=\frac{T-T_0}{T_1-T_0}\equiv\Theta$$

which also serves to define the dimensionless temperature. From an energy balance on a solid spherical shell of thickness $\Delta r$ gives

$$\left(4\pi r^2 q_r\right)\Big|_r-\left(4\pi r^2 q_r\right)\Big|_{r+\Delta r}=0$$

Division by $\Delta r$ and then taking the limit as $\Delta r$ goes to zero gives

$$\frac{d}{dr}\left(r^2 q_r\right)=0$$

Then, introduction of Fourier's law gives

$$\frac{d}{dr}\left(r^2 k\frac{dT}{dr}\right)=0 \qquad \text{or} \qquad \frac{d}{dr}\left(r^2 k\frac{d\Theta}{dr}\right)=0$$

We now integrate once with respect to $r$ to get

$$r^2 k\frac{d\Theta}{dr}=C_1 \qquad \text{or} \qquad k\frac{d\Theta}{dr}=\frac{C_1}{r^2}$$

Inserting the expression for the thermal conductivity as a function of temperature, and integrating again, we find

$$\int\left[k_0+\left(k_1-k_0\right)\Theta\right]d\Theta=-\frac{C_1}{r}+C_2$$

or

$$k_0\Theta+\tfrac{1}{2}\left(k_1-k_0\right)\Theta^2=-\frac{C_1}{r}+C_2$$

Now we apply the boundary conditions:

B. C. at $r_0$: $$0 = -\frac{C_1}{r_0} + C_2$$

B. C. at $r_1$: $$\tfrac{1}{2}(k_1 + k_0) = -\frac{C_1}{r_1} + C_2$$

Subtracting the first equation from the second eliminates $C_2$ (which we are not going to need anyway) and gives an equation for $C_1$:

$$\tfrac{1}{2}(k_1 + k_0) = C_1\left(\frac{1}{r_0} - \frac{1}{r_1}\right)$$

or

$$C_1 = \frac{\tfrac{1}{2}(k_1 + k_0)r_0 r_1}{(r_1 - r_0)}$$

The heat flow *inward* through the spherical shell at $r_0$ is now

$$Q_0 = 4\pi r_0^2\left(+k\frac{dT}{dr}\right)\bigg|_{r=r_0} = 4\pi r_0^2\left(+k_0(T_1 - T_0)\frac{d\Theta}{dr}\right)\bigg|_{r=r_0}$$

The derivative of the dimensionless temperature can be found thus:

$$k_0\frac{d\Theta}{dr} + (k_1 - k_0)\Theta\frac{d\Theta}{dr} = \frac{\tfrac{1}{2}(k_1 + k_0)r_0 r_1}{(r_1 - r_0)}\frac{1}{r^2}$$

Evaluating this expression at $r = r_0$ gives

$$k_0\frac{d\Theta}{dr}\bigg|_{r=r_0} = \frac{\tfrac{1}{2}(k_1 + k_0)r_1}{(r_1 - r_0)r_0}$$

Putting this expression into the formula for $Q_0$ gives

$$Q_0 = 4\pi r_0^2\left(+k\frac{dT}{dr}\right)\Bigg|_{r=r_0} = 4\pi r_0 r_1\left(\frac{k_0+k_1}{2}\right)\left(\frac{T_1-T_0}{r_1-r_0}\right)$$

*b* Since most of the quantities are given in the c.g.s. system, we will convert all quantities into that system. The quantity $Q_0$ may be evaluated thus:

$$Q_0 = 4\pi(36\times 2.54)(48\times 2.54)\left(\frac{(16.2\times 10^{-4})(4.136\times 10^{-3})}{2}\right)\left(\frac{183}{12\times 2.54}\right)$$
$$= 2.8182 \text{ cal/s}$$

The factor $4.136\times 10^{-3}$ needed for converting the units of the thermal conductivity is obtained from Table F.3-5. The rate of evaporation of oxygen is then:

$$R = \left(\frac{2.8182}{1636}\right)(3600) = 6.201 \text{ g-ml/hr}$$

or

$$R = \frac{(6.201)(32)}{1000} = 0.198 \text{ kg/hr}$$

**10B.15 Radial temperature gradients in an annular chemical reactor**

*a.* Consider a cylindrical shell of thickness $\Delta r$ and length $L$. We make an energy balance over this shell, by paralleling the derivation in Eqs. 10.2-2 to 6, replacing the electrical heat source by the chemical heat source $S_c$. Hence we have (cf. Eq. 10.2-6):

$$\frac{d}{dr}(rq_r) = S_c r$$

Into this we substitute Fourier's law for heat conduction in the $r$-direction to get

$$\frac{d}{dr}\left(r\left(-k_{\text{eff}}\frac{dT}{dr}\right)\right) = S_c r \qquad \text{or} \qquad k_{\text{eff}}\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = -S_c$$

provided that the effective thermal conductivity does not vary with position. The boundary conditions are:

B. C. 1: at $r = r_0$, $T = T_0$

B. C. 2: at $r = r_0$, $\dfrac{dT}{dr} = 0$

*b.* A natural choice for the dimensionless radial coordinate involves division of $r$ by either the inner or outer radius; we choose the inner radius and write $\xi = r/r_0$. Then the differential equation becomes:

$$\frac{k_{\text{eff}}}{r_0^2}\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{dT}{d\xi}\right) = -S_c \qquad \text{or} \qquad \frac{k_{\text{eff}}}{r_0^2}\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{d(T-T_0)}{d\xi}\right) = -S_c$$

From this, it is evident that $k_{\text{eff}}(T-T_0)/S_c r_0^2$ is dimensionless. By inserting a factor of 4, we get $\Theta = 4k_{\text{eff}}(T-T_0)/S_c r_0^2$. The insertion of the factor of 4 is arbitrary, but it makes the final dimensionless answer somewhat simpler. In terms of these dimensionless quantities the partial differential equation becomes:

$$\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{d\Theta}{d\xi}\right)=-4$$

with boundary conditions

B. C. 1: at $\xi=1$, $\Theta=0$
B. C. 2: at $\xi=1$, $d\Theta/d\xi=0$

*c*. Integration twice leads to

$$\Theta=-\xi^2+C_1\ln\xi+C_2$$

Application of the boundary conditions gives $C_1=2$ and $C_2=1$, so that

$$\Theta=1-\xi^2+2\ln\xi$$

*d*. The dimensionless temperature at the outer wall is then

$$\Theta(a)=1-a^2+2\ln a \qquad (a=r_1/r_0)$$

The volume averaged reduced temperature is:

$$\langle\Theta\rangle=\frac{\int_0^{2\pi}\int_1^a\left(1-\xi^2+2\ln\xi\right)\xi d\xi d\theta}{\int_0^{2\pi}\int_1^a\xi d\xi d\theta}=\frac{\left[\frac{1}{2}\xi^2-\frac{1}{4}\xi^4+\xi^2\ln\xi-\frac{1}{2}\xi^2\right]_1^a}{\left[\frac{1}{2}\xi^2\right]_1^a}$$

$$==-\tfrac{1}{2}\left(a^2+1\right)+2\frac{a^2\ln a}{a^2-1}$$

*e*. The temperature at the outer wall is

$$T=T_0+\frac{S_c r_0^2}{4k_{\text{eff}}}\left(1-a^2+2\ln a\right)$$

$$=900+\frac{\left(4800\ \frac{\text{cal}}{\text{hr}\cdot\text{cm}^3}\right)\left(3.97\times10^{-3}\ \frac{\text{Btu}}{\text{cal}}\right)\left(2.54\times12\ \frac{\text{cm}}{\text{ft}}\right)\left(\frac{0.45}{12}\right)^2}{4(0.3)}$$

$$\times\left(1-(1.11)^2+2\ln(1.11)\right)$$
$$=900+(0.681)(1-1.23+0.21)$$
$$=900+(63.3)(1-1.23+0.21)=899^\circ\text{F}$$

*f.* If the inner and outer radii were doubled, the temperature difference between the walls would be four times as great.

**10B.16 Temperature distribution in a hot-wire anemometer**

*a*. We start by writing a shell balance over a segment $\Delta z$ of the wire:

$$q_z\big|_z\left(\tfrac{1}{4}\pi D^2\right) - q_z\big|_{z+\Delta z}\left(\tfrac{1}{4}\pi D^2\right) + \frac{I^2}{k_e}\left(\tfrac{1}{4}\pi D^2\right)\Delta z - h(T - T_L)\pi D\Delta z = 0$$

Division by $\frac{1}{4}\pi D^2 \Delta z$ and taking the limit as $\Delta z \to 0$ gives

$$-\lim_{\Delta z \to 0} \frac{q_z\big|_{z+\Delta z} - q_z\big|_z}{\Delta z} + \frac{I^2}{k_e} - \frac{4h(T - T_L)}{D} = 0$$

Taking the limit and using the definition of the first derivative gives:

$$-\frac{dq_z}{dz} - \frac{4h(T - T_L)}{D} = -\frac{I^2}{k_e} \qquad \text{or} \qquad +k\frac{d^2T}{dz^2} - \frac{4h(T - T_L)}{D} = -\frac{I^2}{k_e}$$

Let us now define the dimensionless quantities

$$\zeta = \frac{z}{L} \qquad \Theta = \frac{T - T_L}{T_L} \qquad H = \frac{4hL^2}{kD} \qquad J = \frac{I^2L^2}{k_e k T_L}$$

Now when the differential equation is multiplied by $L^2/kT_L$ it becomes, in dimensionless form,

$$\frac{d^2\Theta}{d\zeta^2} - H\Theta = -J \qquad \text{with} \qquad \Theta = 0 \text{ at } \zeta = \pm 1$$

This is a nonhomogeneous, second-order differential equation, the solution of which is the sum of a complementary function and a particular integral:

$$\Theta_{\text{C.F.}} = C_1 \cosh\sqrt{H}\zeta + C_2 \sinh\sqrt{H}\zeta; \qquad \Theta_{\text{P.I.}} = J/H$$

Since the solution must be symmetric about $\zeta = 0$, the constant $C_2$ must be zero. Therefore, the complete solution has the form

$$\Theta = C_1 \cosh\sqrt{H}\zeta + \frac{J}{H} \qquad \text{where} \qquad C_1 = -\frac{J/H}{\cosh\sqrt{H}}$$

where the boundary condition that $\Theta = 0$ at $\zeta = 1$ has been used. Therefore the solution to the problem is

$$\Theta = \frac{J}{H}\left(1 - \frac{\cosh\sqrt{H}\zeta}{\cosh\sqrt{H}}\right) \qquad \text{or} \qquad T - T_L = \frac{I^2 D}{4hk_e}\left(1 - \frac{\cosh\sqrt{4h/kD}\,z}{\cosh\sqrt{4h/kD}\,L}\right)$$

*c*. First we evaluate the cosh-function in the denominator, creating dimensionless ratios with internally consistent units:

$$\cosh\sqrt{\frac{4h}{kD}}L = \cosh\sqrt{\frac{4hL^2}{kD}} = \cosh\sqrt{\left(\frac{4hL}{k}\right)\left(\frac{L}{D}\right)} =$$

$$\cosh\sqrt{\frac{4(100)((0.5)/(2.54)(12))}{(40.2)}\left(\frac{0.5}{0.0127}\right)} = \cosh 2.535 = 6.348$$

Then, at $z = 0$, the quantity in the large parentheses is

$$\left(1 - \frac{1}{\cosh\sqrt{4h/kD}\,L}\right) = 0.842$$

Then the volume rate of heat production is

$$\frac{I^2}{k_e} = \frac{(T - T_L)(4h)}{(0.842)D} = \frac{(30)(4)(100 \times 5.678)}{(0.842)(0.127 \times 0.001)} = 6.371 \times 10^8 \text{ J/m}^3\text{s}$$

$$= 6.371 \times 10^2 \text{ J/cm}^3\text{s} = 6.371 \times 10^2 \text{ amp}^2 \cdot \text{ohm/cm}^3$$

since 1 amp · volt = 1 amp$^2$ ohm = 1 J/s. Then

$$\text{current (amp)} = = \sqrt{6.371 \times 10^2 \times 10^5}\left(\tfrac{1}{4}\pi(0.0127)^2\right)$$

$$= \sqrt{63.71 \times 10^6}\left(1.267 \times 10^{-4}\right) = 1.01 \text{ amp}$$

**10B.17 Non-Newtonian flow with forced-convection heat transfer**

*a*. This problem can be solved by modifying §10.8 as follows: First replace $(1-\xi^2)$ by $(1-\xi^{s+1})$ in Eqs. 10.8-19, 25, and 26. Then the power-law analog of Eq. 10.8-27 becomes

$$\Theta(\xi,\zeta)=C_0\zeta+C_0\left(\frac{\xi^2}{4}-\frac{\xi^{s+3}}{(s+3)^2}\right)+C_1\ln\xi+C_2$$

in which the constants of integration are determined to be

$$C_1=0$$

$$C_0=2(s+3)/(2+1)$$

$$C_2=[(s+3^3)-8]/4(s+1)(s+3)(s+5)$$

With these values for the constants, we are led to Eq. 1B.17-1.

*b*. This problem may be solved by modifying Problem 10B.7, part (*b*), by replacing $(1-\sigma^2)$ by $(1-\sigma^{s+1})$ in the differential equation for $\Psi$ as well as in the integral condition. This leads to the following equation for $\Theta(\sigma,\zeta)$:

$$\Theta(\sigma,\zeta)=C_0\zeta+C_0\left(\frac{\sigma^2}{2}-\frac{\sigma^{s+3}}{(s+2)(s+3)}\right)+C_1\sigma+C_2$$

with the constants of integration given as

$$C_1=0$$

$$C_1=\frac{s+2}{s+1}$$

$$C_2=-\frac{s+2}{s+1}\left(\frac{(s+2)(s+3)(2s+5)-6}{6(s+3)(s+4)(2s+5)}\right)$$

Then we are led to Eq. 10B.17-2.

**10B.18 Reactor temperature profiles with axial heat flux**

*a*. The differential equations in Eqs. 10.5-6 to 8, and the boundary conditions in Eqs. 10.5-9 to 14 are still valid, but with a linear form for the function $F$:

$$S_c = S_{c1}\left(\frac{T - T_0}{T_1 - T_0}\right) \equiv S_{c1}\Theta$$

in which $S_{c1}$ and $T_0$ are constants describing the linear dependence of the reaction rate on the temperature. Then, if we use the dimensionless downstream coordinate $Z = z/L$, a dimensionless quantity $B = \rho\hat{C}_p v_0 L / k_{\text{eff},zz}$, and a dimensionless chemical heat source $N = S_{c1}L / \rho\hat{C}_p v_0 (T_1 - T_0)$, the differential equations for the three regions become:

$$\frac{1}{B}\frac{d^2\Theta^{\text{I}}}{dZ^2} = \frac{d\Theta^{\text{I}}}{dZ}; \quad \frac{1}{B}\frac{d^2\Theta^{\text{II}}}{dZ^2} = \frac{d\Theta^{\text{II}}}{dZ} + N\Theta^{\text{II}}; \quad \frac{1}{B}\frac{d^2\Theta^{\text{III}}}{dZ^2} = \frac{d\Theta^{\text{III}}}{dZ}$$

The boundary conditions are:

| | | |
|---|---|---|
| B.C.1: | at $Z = -\infty$, | $\Theta^{\text{I}} = 1$ |
| B.C.2: | at $Z = 0$, | $\Theta^{\text{I}} = \Theta^{\text{II}}$ |
| B.C.3: | at $Z = 0$, | $d\Theta^{\text{I}}/dZ = d\Theta^{\text{II}}/dZ$ |
| B.C.4: | at $Z = 1$, | $\Theta^{\text{II}} = \Theta^{\text{III}}$ |
| B.C.5: | at $Z = 1$, | $d\Theta^{\text{II}}/dZ = d\Theta^{\text{III}}/dZ$ |
| B.C.6: | at $Z = \infty$, | $\Theta^{\text{III}}$ = finite |

That is, we specify the condition far upstream from the reaction region, and we require that there be continuity of the temperature and the heat flux where the regions join.

The solutions of these linear, homogeneous, second-order differential equations are then

| | |
|---|---|
| Region I: | $\Theta^{\text{I}}(Z) = C_1 + C_2 e^{BZ}$ |
| Region II: | $\Theta^{\text{II}}(Z) = C_3 e^{m_+ Z} + C_4 e^{m_- Z}$ for $m_+ \neq m_-$ |
| Region III: | $\Theta^{\text{III}}(Z) = C_5 + C_6 e^{BZ}$ |

where $m_{\pm} = \frac{1}{2}B\left(1 \pm \sqrt{1-(4N/B)}\right)$.

When the integration constants are determined, we get:

Region I: $$\Theta^{\mathrm{I}}(Z) = 1 + \left(\frac{m_+ m_-(m_+ - m_-)}{m_+^2 e^{m_+} - m_-^2 e^{m_-}}\right) e^{(m_+ + m_-)Z}$$

Region II: $$\Theta^{\mathrm{II}}(Z) = \left(\frac{m_+ e^{m_+} e^{m_- Z} - m_- e^{m_-} e^{m_+ Z}}{m_+^2 e^{m_+} - m_-^2 e^{m_-}}\right)(m_+ + m_-)$$

Region III: $$\Theta^{\mathrm{III}}(Z) = \left(\frac{m_+^2 - m_-^2}{m_+^2 e^{m_+} - m_-^2 e^{m_-}}\right) e^{(m_+ + m_-)}$$

These results correspond to Eqs. 10.5-21 to 23.

*b*. In preparation for taking the limit for $B$ going to infinity, we first find the following Taylor series expansions

$$m_+ = \tfrac{1}{2}B\left(1 + (1 - \tfrac{1}{2}(4N/B) + \cdots\right) = B + N + \mathrm{O}\left(B^{-1}\right)$$
$$m_- = \tfrac{1}{2}B\left(1 - (1 - \tfrac{1}{2}(4N/B) + \cdots\right) = N + \mathrm{O}\left(B^{-1}\right)$$

It is important to note that the limiting value of $m_-$ for infinite $B$ is not zero, but $N$.

Then, the limiting expressions for the temperature profiles in the three regions are

$$\lim_{B\to\infty} \Theta^{\mathrm{I}}(Z) = \lim_{B\to\infty}\left[1 + \left(\frac{m_+ m_- e^{m_+} - \cdots}{m_+^2 e^{m_+} - \cdots}\right) e^{(m_+ + m_-)Z}\right] = 1 \quad \text{(since } Z \text{ is negative)}$$
$$\lim_{B\to\infty} \Theta^{\mathrm{I}}(Z) = \lim_{B\to\infty}\left(\frac{m_+(m_+ + \cdots)e^{m_+} e^{m_- Z}}{m_+^2 e^{m_+} - \cdots} - \frac{m_-(m_+ + \cdots)e^{m_-} e^{m_+ Z}}{m_+^2 e^{m_+} - \cdots}\right) = e^{NZ}$$
$$\lim_{B\to\infty} \Theta^{\mathrm{II}}(Z) = \lim_{B\to\infty}\left(\frac{m_+^2 - \cdots}{m_+^2 e^{m_+} - \cdots}\right) e^{(m_+ + m_-)} = e^{N}$$

These results are consistent with Eqs. 10.5-21, 22, and 23. To get the second equation above from Eq. 10.5-22, we have to substitute $F(\Theta) = \Theta$ and integrate.

**10C.1 Heating of an electric wire with temperature-dependent electrical and thermal conductivity**

*a*. Rewrite Eq. 10C.1-3 as

$$-\frac{k_0T_0}{R^2}\left[\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{k}{k_0}\frac{d\Theta}{d\xi}\right)\right]=k_{e0}\left[\frac{k_e}{k_{e0}}\left(\frac{E}{L}\right)^2\right]$$

This is easily transformed into Eq. 10C.1-4, with $B=k_{e0}R^2E^2/k_0T_0L^2$. Use of Eqs. 10C.1-1 ane 2 leads directly to Eq. 10C.1-5.

*b*. When Eq. 1C.1-8 is substituted into Eq. 1C.1-5 we get

$$\begin{aligned}
&-\frac{1}{\xi}\frac{d}{d\xi}\Big(\Big[1-\alpha_1\left\{\tfrac{1}{4}B\left(1-\xi^2\right)\left(1+B\Theta_1+\cdots\right)\right\}\\
&-\alpha_2\left\{\tfrac{1}{4}B\left(1-\xi^2\right)\left(1+B\Theta_1+\cdots\right)\right\}^2+\cdots\Big]\xi\frac{d}{d\xi}\left(\tfrac{1}{4}B\left(1-\xi^2\right)\left(1+B\Theta_1+\cdots\right)\right)\Big)\\
&\qquad=B\Big(1-\beta_1\left\{\tfrac{1}{4}B\left(1-\xi^2\right)\left(1+B\Theta_1+\cdots\right)\right\}\\
&\qquad\quad-\beta_2\left\{\tfrac{1}{4}B\left(1-\xi^2\right)\left(1+B\Theta_1+\cdots\right)\right\}^2+\cdots\Big)
\end{aligned}$$

Equating terms containing $B^1$, we get

$$-\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{d}{d\xi}\tfrac{1}{4}B\left(1-\xi^2\right)\right)=B$$

which, when the differentiations are performed, gives an identity, $B=B$, as it should. Then we equate the terms containing $B^2$, which gives

$$\begin{aligned}
&-\frac{1}{\xi}\frac{d}{d\xi}\left[\xi\frac{d}{d\xi}\tfrac{1}{4}B\left(1-\xi^2\right)B\Theta_1\right]-\frac{1}{\xi}\frac{d}{d\xi}\left[-\alpha_1\tfrac{1}{4}B\xi\frac{d}{d\xi}\tfrac{1}{4}B\left(1-\xi^2\right)\right]\\
&\qquad=-\beta_1B\cdot\tfrac{1}{4}B\left(1-\xi^2\right)
\end{aligned}$$

Division by $\frac{1}{4}B^2$ and performing the differentiation in the $\alpha_1$ term then gives the differential equation for $\Theta_1$

$$\frac{1}{\xi}\frac{d}{d\xi}\left[\xi\frac{d}{d\xi}\left(1-\xi^2\right)\Theta_1\right]=-\alpha_1\left(1-2\xi^2\right)+\beta_1\left(1-\xi^2\right)$$

Performing two integrations then gives

$$\left(1-\xi^2\right)\Theta_1=-\tfrac{1}{8}\alpha_1\left(2\xi^2-\xi^4\right)+\tfrac{1}{16}\beta_1\left(4\xi^2-\xi^4\right)+C_1\ln\xi+C_2$$

The constant $C_1$ must be zero in order to satisfy the boundary condition at the tube axis, and the application of the boundary condition at the tube wall gives

$$0=-\tfrac{1}{8}\alpha_1+\tfrac{3}{16}\beta_1+C_2$$

Taking the difference between the last two equations gives (after dividing through by $\left(1-\xi^2\right)$)

$$\Theta_1=\tfrac{1}{8}\alpha_1\left(1-\xi^2\right)-\tfrac{1}{16}\beta_1\left(3-\xi^2\right)$$

in agreement with Eq. 10C.1-9.

*c*. Rearranging Eq. 10.C-10 we have

$$\frac{k}{k_0}=\frac{k_e}{k_{e0}}\frac{T}{T_0}=\frac{k_e}{k_{e0}}(1+\Theta)$$

Into this equation we substitute Eqs. 10C.1-1 and 2, to get

$$\begin{aligned}\left(1-\alpha_1\Theta-\alpha_2\Theta^2-\cdots\right)&=\left(1-\beta_1\Theta-\beta_2\Theta^2-\cdots\right)(1+\Theta)\\&=1-\beta_1\Theta-\beta_2\Theta^2-\cdots+\Theta-\beta_1\Theta^2-\beta_2\Theta^3-\cdots\\&=1-(\beta_1-1)\Theta-(\beta_2-\beta_1)\Theta^2-\cdots\end{aligned}$$

Equating the coefficients of equal powers of $\Theta$ then gives $\alpha_1=\beta_1-1$, $\alpha_2=\beta_2-\beta_1$, and so on. Then Eq. 10C.1-12 follows directly:

$$\Theta=\tfrac{1}{4}B\left(1-\xi^2\right)\left\{1-\tfrac{1}{16}B\left[(\beta_1+2)+(\beta_1-2)\xi^2\right]+\cdots\right\}$$

**10C.2 Viscous heating with temperature-dependent viscosity and thermal conductivity**

*a*. The shell momentum and energy balances lead to

$$\frac{d}{dx}\left(\mu\frac{dv_z}{dx}\right)=0 \quad \text{and} \quad \frac{d}{dx}\left(k\frac{dT}{dx}\right)+\mu\left(\frac{dv_z}{dx}\right)^2=0$$

Mulitply the first equation by $b^2/\mu_0 v_b$ and the second by $b^2/\mu_0 v_b^2$ to get Eqs. 10C.2-3 and 4.

*b*. When Eqs. 10C.2-5, 6, 7, and 8 are used in the equation for temperature, and the coefficients of like powers of Br are equated, we get for the terms containing the first power of Br

$$\frac{d^2\Theta_1}{d\xi^2}+C_{10}^2=0$$

This equation has the solution $\Theta_1=-\frac{1}{2}C_{10}^2\xi^2+C_3\xi+C_4$. The constants of integration are determined from the boundary conditions that $\Theta_1=0$ at $\xi=0,1$. Thus the following result is obtained:

$$\Theta_1=\tfrac{1}{2}C_{10}^2\left(\xi-\xi^2\right)$$

We now turn to the $\phi$ equation (along with Eq. 10C.2-2) which is

$$\frac{d\phi}{d\xi}=C_1\left(1+\beta_1\Theta+\beta_2\Theta^2+\cdots\right)$$

The expansions in Eqs. 10C.2-6, 7, and 8 then give

$$\frac{d}{d\xi}\left(\phi_0+\mathrm{Br}\phi_1+\cdots\right)=\left(C_{10}+\mathrm{Br}C_{11}+\mathrm{Br}^2C_{12}+\cdots\right)$$
$$\left(1+\beta_1\left(\mathrm{Br}\Theta_1+\mathrm{Br}^2\Theta_2+\cdots\right)+\cdots\right)$$

Equating the coefficients of the zeroth power of Br, we get

$$\frac{d}{d\xi}\phi_0 = C_{10}$$

Integration then gives $\phi_0 = C_{10}\xi + C_2$. The boundary conditions $\phi_0(0) = 0$ and $\phi_0(1) = 1$ lead to

$$\phi_0 = \xi \qquad \text{and hence also} \qquad \Theta_1 = \tfrac{1}{2}\left(\xi - \xi^2\right)$$

These are just the results obtained in §10.4.

We next go back to the energy equation and equate the terms containing $\mathrm{Br}^2$:

$$\frac{d}{d\xi}\left(\alpha_1\Theta_1\frac{d\Theta_1}{d\xi} + \frac{d\Theta_2}{d\xi}\right) + 2C_{11} + \beta_1\Theta_1 = 0$$

Into this we insert the expression for $\Theta_1$ to obtain

$$\frac{d^2\Theta_2}{d\xi^2} = -\frac{\alpha_1}{4}\left(1 - 6\xi + 6\xi^2\right) - 2C_{11} - \frac{1}{2}\beta_1\left(\xi - \xi^2\right)$$

Integration twice gives

$$\Theta_2 = -\frac{\alpha_1}{4}\left(\frac{1}{2}\xi^2 - \xi^3 + \frac{1}{2}\xi^4\right) - C_{11}\xi^2 - \frac{1}{2}\beta_1\left(\frac{1}{6}\xi^3 - \frac{1}{12}\xi^4\right) + C_3\xi + C_4$$

Then we use the boundary conditions $\Theta_2 = 0$ at $\xi = 0,1$ leads to

$$\Theta_2 = -\frac{\alpha_1}{4}\left(\frac{1}{2}\xi^2 - \xi^3 + \frac{1}{2}\xi^4\right) - \frac{1}{24}\beta_1\left(-\xi + 2\xi^3 - \xi^4\right) + C_{11}\left(\xi - \xi^2\right)$$

The velocity distribution that goes along with this approximation is $\phi_1$. Equating the terms linear in Br gives the differential equation

$$\frac{d\phi_1}{d\xi} = C_{11} + \beta_1\Theta_1 = C_{11} + \tfrac{1}{2}\beta_1\left(\xi - \xi^2\right)$$

Integration gives

**10C.3 Viscous heating in a cone-and-plate viscometer**

*a*. First we make a table of "translation" to get from the plane Couette flow to the cone-and-plate flow:

| Plane Couette Flow (Fig. 10.4-2) | Cone-and-Plate Flow (Fig. 2B.11) |
|---|---|
| $\xi = x/b$ | $\bar{\xi} = \psi/\psi_0$ |
| $\phi = v_z/v_b$ | $\bar{\phi} = v_\phi/r\Omega$ |
| $v_b/b$ | $\Omega/\psi_0$ |
| $\mathrm{Br} = \mu v_b^2/k_0T_0$ | $\overline{\mathrm{Br}}(r/R)^2 = (\mu R^2\Omega^2/k_0T_0)(r/R)^2$ |

The torque on the cone-and-plate viscometer is given by the force times the lever arm integrated over the entire plate:

$$T_z = 2\pi\int_0^R \tau_{\theta\phi}\Big|_{\theta=\pi/2} \cdot r \cdot rdr$$

But

$$\tau_{\theta\phi} = -\mu\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\phi}{\sin\theta}\right) \approx +\mu\frac{1}{r}\frac{\partial v_\phi}{\partial\psi}$$

is a suitable approximation for the cone-plate system, with $\theta \approx \pi/2$. Therefore the torque expression becomes for a fluid at temperature $T_0$ and viscosity $\mu_0$

$$T_z = 2\pi\int_0^R \mu_0\frac{\partial v_\phi}{\partial\psi}\Big|_{\psi=0} rdr \qquad (*)$$

When there is no viscous heating, the tangential velocity is

$$v_\phi \approx r\Omega(\psi/\psi_0)$$

Combining the last two equations gives, for no viscous heating:

$$T_{z.0} = \frac{2\pi\mu_0\Omega R^3}{3\psi_0} \qquad (**)$$

When (*) is divided by (**), we get

$$\frac{T_z}{T_{z.0}} = 3\int_0^1 \left.\frac{\partial\phi}{\partial\xi}\right|_{\xi=0} \xi^2 d\xi$$

*b*. We now consider the system with viscous heating and use the cone-and-plate variables in the correspondence table. This gives

$$\frac{T_z}{T_{z.0}} = 3\int_0^1 \left(1 - \tfrac{1}{3}\overline{\mathrm{Br}}\overline{\xi}^2\beta_1 + \cdots\right)\overline{\xi}^2 d\overline{\xi}$$

where we have used the dimensionless velocity expression from part (*c* in Problem 10C.2.

When the integration is performed we get

$$\frac{T_z}{T_{z.0}} = 1 - \tfrac{1}{5}\overline{\mathrm{Br}}\beta_1 + \mathrm{O}\left(\overline{\mathrm{Br}}^2\right)$$

To get the higher-order terms shown in Eq. 10C.3-1, one would have to go back and get the higher order terms in part (*c* in Problem 10C.2.

**10D.1 Heat loss from a circular fin**

*a*. A heat balance over a ring of thickness $\Delta r$ gives

$$\left(2\pi(2B)rq_r\right)\Big|_r - \left(2\pi(2B)rq_r\right)\Big|_{r+\Delta r} - 2\cdot 2\pi r\Delta r\cdot h\left(T-T_a\right)=0$$

Divide by $2\pi(2B)\Delta r$ and take the limit as $\Delta r$ goes to zero to get

$$\lim_{r\to 0}\frac{\left(rq_r\right)\Big|_r - \left(rq_r\right)\Big|_{r+\Delta r}}{\Delta r} - \frac{rh}{B}\left(T-T_a\right)=0 \qquad \text{or}$$

from which we get the differential equation

$$-\frac{d}{dr}\left(rq_r\right)-\frac{rh}{B}\left(T-T_a\right)=0 \qquad \text{or} \qquad \frac{d}{dr}\left(r\frac{dT}{dr}\right)-\frac{rh}{Bk}\left(T-T_a\right)=0$$

In the second form, Fourier's law has been introduced. Next we rewrite the equation in terms of dimensionless variables: $\xi = r/R_0$ and $\Theta = \left(T-T_a\right)/\left(T_0-T_a\right)$:

$$\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{d\Theta}{d\xi}\right)-\left(\frac{hR_0^2}{Bk}\right)\Theta=0$$

This equation has the following solution (with $\beta^2 = hR_0^2/Bk$):

$$\Theta(\xi)=C_1I_0(\beta\xi)+C_2K_0(\beta\xi)$$

in which $I_0$ and $K_0$ are zero-order Bessel functions. The constants are determined by use of the boundary conditions: $\Theta(1)=1$ and $d\Theta/d\xi\big|_{\xi=R_1/R_0}=0$. This leads to

$$1=C_1I_0(\beta)+C_2K_0(\beta) \quad \text{and} \quad 0=C_1\beta I_1\left(\beta R_1/R_0\right)-C_2\beta K_1\left(\beta R_1/R_0\right)$$

These two equations can be solved simultaneously to give

$$C_1=\frac{\beta K_1\left(\beta R_1/R_0\right)}{I_0(\beta)\beta K_1\left(\beta R_1/R_0\right)+\beta I_1\left(\beta R_1/R_0\right)K_0(\beta)}$$

$$C_2 = \frac{\beta I_1(\beta R_1/R_0)}{I_0(\beta)\beta K_1(\beta R_1/R_0) + \beta I_1(\beta R_1/R_0)K_0(\beta)}$$

Hence the dimensionless temperature profile is

$$\Theta(\xi) = \frac{I_0(\beta\xi)K_1(\beta R_1/R_0) + I_1(\beta R_1/R_0)K_0(\beta\xi)}{I_0(\beta)K_1(\beta R_1/R_0) + I_1(\beta R_1/R_0)K_0(\beta)}$$

*b*. The total heat loss is then:

$$Q = 2\pi R_0(2B)k\left.\frac{dT}{dr}\right|_{r=R_0} = 4\pi Bk(T_0 - T_a)\left.\frac{d\Theta}{d\xi}\right|_{\xi=1}$$

$$= 4\pi Bk\beta(T_0 - T_a)\left[\frac{I_1(\beta R_1/R_0)K_1(\beta) - I_1(\beta)K_1(\beta R_1/R_0)}{I_1(\beta R_1/R_0)K_0(\beta) + I_0(\beta)K_1(\beta R_1/R_0)}\right]$$

**10D.2 Duct flow with constant wall heat flux and arbitrary velocity distribution**

*a*. The analogs of Eqs. 10.8-19 26 are

$$\phi(\xi)\frac{\partial\Theta}{\partial\zeta}=\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\Theta}{\partial\xi}\right) \qquad \text{and} \qquad \frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{d\Psi}{d\xi}\right)=C_0\phi(\xi)$$

After multiplication by $\xi$ the second of these can be integrated once to give

$$\xi\frac{d\Psi}{d\xi}=C_0\int_0^{\xi}\bar{\xi}\phi(\bar{\xi})d\bar{\xi}+C_1$$

Then division by $\xi$ and further integration gives Eq. 10D.2-1 (along with the defintion in Eq. 10D.2-2). The constant $C_1$ has to be zero, since the temperature must be finite at the tube axis (B.C. 1). Then from B.C. 2 we get $C_0 = 1/I(1)$ using the equation immediately above. To get the remaining constant, we use the integral condition analogous to Eq. 10.8-24, to get

$$\begin{aligned}\zeta &= \int_0^1\Theta(\xi,\zeta)\phi(\xi)\xi d\xi\\ &= \int_0^1\left[C_0\zeta+C_0\int_0^{\xi}\bar{\xi}^{-1}I(\bar{\xi})d\bar{\xi}+C_2\right]\phi(\xi)\xi d\xi\\ &= C_0\zeta I(1)+C_0\int_0^1\phi(\xi)\xi\left[\int_0^{\xi}\bar{\xi}^{-1}I(\bar{\xi})d\bar{\xi}\right]d\xi+C_2I(1)\\ &= \zeta+[I(1)]^{-1}\int_0^1\phi(\xi)\xi\left[\int_0^{\xi}\bar{\xi}^{-1}I(\bar{\xi})d\bar{\xi}\right]d\xi++C_2I(1)\end{aligned}$$

From the last line, Eq. 10D.2-3 follows immediately.

*b*. The wall and bulk temperatures, in dimensionless form, are given by

$$\Theta_0=\Theta(\xi=0,\zeta) \qquad \text{and} \qquad \Theta_b=\frac{\int_0^1\phi\Theta\xi d\xi}{\int_0^1\phi\xi d\xi}$$

Substitution of the result in (*a*) into these expressions gives

$$\Theta_0 = \frac{1}{I(1)}\zeta + \frac{1}{I(1)}\int_0^1 \frac{I(1)}{\xi} d\xi + C_2$$

$$\Theta_b = \frac{1}{I(1)}\zeta + \frac{1}{[I(1)]^2}\int_0^1 \phi\left[\int_0^\xi \frac{I(\bar{\xi})}{\bar{\xi}} d\bar{\xi}\right]\xi d\xi + C_2$$

Subtracting and then changing the order of integration in the second expression, we obtain

$$\Theta_0 - \Theta_b = \frac{1}{I(1)}\int_0^1 \frac{I(1)}{\xi} d\xi - \frac{1}{[I(1)]^2}\int_0^1 \frac{I(\bar{\xi})}{\bar{\xi}}\left[\int_{\bar{\xi}}^1 \phi(\xi)\xi d\xi\right] d\bar{\xi}$$

The inner integral may now be written as the difference of two integrals, the first going from 0 to 1 and the second from 0 to $\bar{\xi}$. But these two integrals are, respectively, just $I(1)$ and $I(\bar{\xi})$. Hence we get finally

$$\Theta_0 - \Theta_b = \frac{1}{I(1)}\int_0^1 \frac{I(1)}{\xi} d\xi - \frac{1}{I(1)}\int_0^1 \frac{I(1)}{\xi} d\xi + \frac{1}{[I(1)]^2}\int_0^1 \frac{[I(\bar{\xi})]^2}{\bar{\xi}} d\bar{\xi}$$

Thus, the first two terms cancel each other, and the last term gives Eq. 10D.2-4.

*c*. From the definition of the dimensionless temperature, we get

$$\Theta_0 - \Theta_b = \frac{T_0 - T_b}{q_0 R/k}$$

Taking the reciprocal, and replacing $R$ by $D/2$, we get Eq. 10D.2-5.

*d*. The quantity $I(1)$ is the ratio of the average to the maximum velocity in the tube.

$$I(1) = \int_0^1 \phi\xi d\xi = \left(1/v_{z,\max}\right)\int_0^1 v_z \xi d\xi = \langle v_z \rangle / v_{z,\max}$$

## 11A.1 Temperature in a friction bearing.

The method given in the solution of Problem 10A.3, based on Eq. 10.4-9, gives the maximum temperature rise in the lubricant as

$$\begin{aligned} T_{\max} - T_1 &= \frac{1}{8}\frac{\mu\Omega^2 R^2}{k} \\ &= \frac{1}{8}\frac{(2.0\ \text{g/cm·s})(8000\pi/60\ \text{rad/s})^2(2.54\ \text{cm})^2}{([4.0\times10^{-4}\times4.1840\times10^7]\ \text{g·cm/s}^2\text{·K})} = 16.9\text{C} \end{aligned}$$

and the maximum temperature $T_{\max}$ as 217C.

Next, we consider the analysis given in §11.4, which includes the curvature effects. Since $T_1$ and $T_\kappa$ are equal, Eq. 11.4-14 is applicable and gives

$$\xi_{\max} = \sqrt{\frac{2\ln(1/\kappa)}{((1/\kappa)^2-1)}}$$

as the location where the temperature $T_{\max}$ occurs. For this problem, $\kappa = 1/1.002$; hence,

$$\xi_{\max} = \sqrt{\frac{2\ln(1.002)}{((1.002)^2-1)}} = 0.999001165$$

which location is essentially in the middle of the gap.

To evaluate $T_{\max}$, we multiply Eq. 11.4-13 by $(T_1 - T_\kappa)$ to make each term finite; the result (after division by $\kappa^4$ in the numerator and denominator) is

$$T - T_\kappa = \frac{\mu\Omega R^2}{k}\cdot\frac{1}{((1/\kappa)^2-1)^2}\left[\left(1-\frac{1}{\xi^2}\right)-\left(1-\frac{1}{\kappa^2}\right)\frac{\ln\xi}{\ln\kappa}\right]$$

With $\xi = \xi_{\max}$, we obtain the maximum temperature rise in this system as

$$\begin{aligned} T_{\max} - T_\kappa &= \frac{(2.0\ \text{g/cm·s})(8000\pi/60\ \text{rad/s})^2(2.54\ \text{cm})^2}{([4.0\times10^{-4}\times4.1840\times10^7]\ \text{g·cm/s}^2\text{·K})}\cdot\frac{1}{(1.002^2-1)^2} \\ &\quad\cdot[-0.002000667+0.002002667] \\ &= 8.438\times10^6\cdot[2.00\times10^6] = 16.9\text{C} \end{aligned}$$

in agreement with the previous calculation. This good agreement is attributed to the narrowness of the gap relative to the cylinder radii in this problem.

**11A.2 Viscosity variation and velocity gradients in a nonisothermal film.**

*a.* We begin by determining the temperature at which the logarthmic discrepancy, $\Delta$, between the two viscosity representations in Eq. 11.4-18 is largest. The discrepancy is expressed as follows:

$$\begin{aligned}\Delta &= \ln(\text{First } \mu(x) \text{ function}) - \ln(\text{Approximation to first } \mu(x) \text{ function}) \\ &= \left[B\frac{T_0 - T_\delta}{T_0 T}\left(\frac{x}{\delta}\right)\right] - \left[B\frac{T_0 - T_\delta}{T_0 T_\delta}\left(\frac{x}{\delta}\right)\right]\end{aligned}$$

Use of Eq. 11.4-1 to express $(x/\delta)$ in terms of temperatures then gives

$$\begin{aligned}\Delta &= \left[\frac{B}{T_0 T}(T_0 - T_\delta)\left(\frac{T - T_0}{T_\delta - T_0}\right)\right] - \left[\frac{B}{T_0 T_\delta}(T_0 - T_\delta)\left(\frac{T - T_0}{T_\delta - T_0}\right)\right] \\ &= -\frac{B}{T_0}\left[\frac{T - T_0}{T} - \frac{T - T_0}{T_\delta}\right]\end{aligned}$$

The $T$-derivative of the logarithmic discrepancy is

$$\frac{d\Delta}{dT} = -\frac{B}{T_0}\left[\frac{T_0}{T^2} - \frac{1}{T_\delta}\right]$$

Setting this derivative equal to zero, we get

$$\frac{T_0}{T^2} = \frac{1}{T_\delta} \qquad \text{or} \qquad T = T_d = \sqrt{T_0 T_\delta}$$

as the temperature of maximum discrepancy between the two expressions for $\ln \mu(x)$.

*b.* For the conditions given,

$$\begin{aligned}\mu(T_0) &= \mu(80°\text{C}) = 0.3548 \times 10^{-2} \text{ g/cm·s}; \\ \mu(T_\delta) &= \mu(100°\text{C}) = 0.2821 \times 10^{-2} \text{ g/cm·s}; \\ T_d &= \sqrt{(273.15 + 80)(273.15 + 100)} = 363.01 \text{ K or } 89.86°\text{C}\end{aligned}$$

Eqs. 11.4-14 and 11.4-19 then give the viscosity at $T_d$ as

$$\begin{aligned}\mu(x_d) &= \mu_0 \exp\left[\left(\frac{\mu_\delta}{\mu_0}\right)\left(\frac{T_d - T_0}{T_\delta - T_0}\right)\right] \\ &= (0.3548 \text{ cp}) \exp\left[\left(\ln\left(\frac{0.2821}{0.3548}\right)\right)\left(\frac{89.86 - 80}{100 - 80}\right)\right] \\ &= (0.3548 \text{ cp}) \exp\left[(-0.2293)(0.493)\right] = 0.3169 \text{ cp}\end{aligned}$$

Three-point Newton interpolation of $\ln \mu$ in Table 1.1-2 gives $\mu = 0.3151$ at 89.86°C, so the largest relative discrepancy is $\Delta = -0.0057$, or $-0.6$ percent of $\mu(x)$.

**11A.3 Transpiration cooling.**

*a.* In the absence of transpiration, Eq. 11.4-1 is indeterminate, but its limiting form is obtainable by expressing the exponential functions as first-order Taylor expansions in $w_r$ (and thus in $R_0$):

$$\frac{T - T_1}{T_\kappa - T_1} = \frac{1/r - 1/R}{1/\kappa R - 1/R}$$

This profile, dcsignated as $\Theta_0$, is tabulated here for the present geometry:

| $r$, microns | 100 | 200 | 300 | 400 | 500 |
|---|---|---|---|---|---|
| $\Theta_0$ | 1.000 | 0.375 | 0.1666... | 0.0625 | 0 |

In the presence of transpiration with the given rate $w_r = 1 \times 10^{-5}$ g/s, the constant $R_0$ in Eq. 11.4-27 is

$$\begin{aligned} R_0 &= \frac{(1 \times 10^{-5}\ \text{g/s})(0.25\ \text{cal/g·C})}{(4\pi)(6.13 \times 10^{-5}\ \text{cal/cm·s·C})} \\ &= 0.003245\ \text{cm} = 32.45\ \text{microns} \end{aligned}$$

Equation 11.14-27 then gives, with $r$ in microns,

$$\frac{T - T_1}{T_\kappa - T_1} = \frac{(\exp(-32.45/r) - \exp(-32.45/500))}{((\exp(-32.45/200) - \exp(-32.45/500))}$$

A table of this function, here called $\Theta_w$, follows:

| $r$, microns | 100 | 200 | 300 | 400 | 500 |
|---|---|---|---|---|---|
| $\Theta_w$ | 1.000 | 0.406 | 0.185 | 0.070 | 0.000 |

*c.* The ratio of the heat conduction to the inner surface $r = \kappa R$ with the latter transpiration rate to that with $w_r = 0$ is, from Eq. 11.4-32,

$$\begin{aligned} \frac{Q}{Q_0} &= \frac{\phi}{\exp\phi - 1} \\ &= \frac{(R_0(1-\kappa)/\kappa R)}{\exp(R_0(1-\kappa)/\kappa R) - 1} \\ &= \frac{(32.45)(0.8)/100)}{\exp(32.45)(0.8)/100) - 1} \\ &= \frac{0.2596}{\exp(0.2596) - 1} = 0.876 \end{aligned}$$

Thus, this small rate of transpiration reduces the rate of heat conduction to the inner surface by 12.4 percent.

**11A.4 Free-convection heat loss from a vertical surface.**

The physical properties for this problem are as follows, evaluated at an average temperature $\overline{T} = (T_0 + T_1)/2 = 110°\text{F} = 43.33°\text{C} = 316.5\text{K}$:

$$\begin{aligned}
\overline{\beta} &= 1/\overline{T} = 0.00316\text{K}^{-1} \\
\overline{\rho} &= pM/R\overline{T} = 0.001154\text{g/cm}^3 \\
\mu &= 1.923 \times 10^{-4}\text{g/cm·s from Table 1.1-2} \\
\hat{C}_p &= 1.007 \text{ J/g·K from Perry's Handbook, 6th Ed., Table 3-212} \\
&= 0.2407 \text{ cal/g/K} \\
k &= 0.0276 \text{ W/m·K from Perry's Handbook, 6th Ed., Table 3-212} \\
&= 6.60 \times 10^{-5}\text{cal/s·cm·K}
\end{aligned}$$

The Prandtl and Grashof numbers are then

$$\begin{aligned}
\text{Pr} &= \frac{\hat{C}_p \mu}{k} \\
&= \frac{(0.2407\text{cal/g·K})(1.923 \times 10^{-4} \text{ g/cm·s})}{6.60 \times 10^{-5}\text{cal/s·cm·K})} = 0.701, \text{ and} \\
\text{Gr} &= \frac{\overline{\rho}^2 g \overline{\beta}(T_0 - T_1)H^2}{\mu^2} \\
&= \frac{(0.001154\text{g/cm}^3)^2(980.7\text{cm/s}^2)(0.00316\text{K}^{-1})([(150-70)/1.8]\text{K})(30\text{cm})^3}{(1.923 \times 10^{-4}\text{g/cm·s})^2} \\
&= 1.34 \times 10^8
\end{aligned}$$

The heat loss rate from one side of the plate, according to Eq. 11.4-51, is

$$Q = WHq_{\text{avg}} = WC \cdot k((T_0 - T_1)(\text{GrPr})^{1/4}$$

With Lorenz' value of $C$, this gives

$$\begin{aligned}
Q &= (50\text{cm})(0.548)(6.60 \times 10^{-5}\text{cal/s·cm·s})([80/1.8]\text{k})(1.34 \times 10^8 \times 0.701)^{1/4} \\
&= 7.9 \text{ cal/s}
\end{aligned}$$

With the value C= 0.518 recommended by Whitaker for air, the result is

$$Q = 7.5 \text{ cal/s}$$

However, this value of $C$ is based on a Prandtl number of 0.73 for air, rather than the value 0.701 found at the present conditions from more recent data. Linear extrapolation of the table on page 349 gives a C value of 0.516 at Pr= 0.701, and a corrected heat loss rate of 7.4 cal/s.

**11A.5 Velocity, temperature and pressure changes in a shock wave.**

*a.* From Eq. 11.4-7, the initial air velocity is

$$\begin{aligned} v_1 &= \mathrm{Ma}_1\sqrt{\gamma R T_1/M} \\ &= (2)\sqrt{1.4(530\ \mathrm{R})(4.9686\times 10^4\ \mathrm{lb}_m\mathrm{ft}^2/\mathrm{s}^2\cdot\ \mathrm{lb\text{-}mol\cdot R})(28.97\ \mathrm{lb}_m/\mathrm{lb\text{-}mol})} \\ &= 2256\ \mathrm{ft/s} \end{aligned}$$

*b.* The final velocity $v_\infty$ is found from Eq. 11.4-75 with $\xi \to \infty$, so that

$$\begin{aligned} \phi \to \alpha &= \frac{\gamma-1}{\gamma+1} + \frac{2}{\gamma+1}\frac{1}{\mathrm{Ma}_1^2} \\ &= \frac{0.4}{2.4} + \frac{2}{2.4}\frac{1}{4} = 0.375 \end{aligned}$$

Then Eq. 11.4-69 gives

$$v_\infty = \phi v_1 = (0.375)(2256) = 846\ \mathrm{ft/s}$$

The final temperature is obtained from the energy balance, Eq. 11.4-65:

$$\begin{aligned} T_\infty &= T_1 + \frac{1}{2}\frac{(v_1^2 - v_\infty^2)}{\hat{C}_p} \\ &= (530\ \mathrm{R}) + \frac{1}{2}\frac{(2256^2 - 846^2\ \mathrm{ft}^2/\mathrm{s}^2)/2}{(0.242\ \mathrm{Btu/lb}_m\cdot\mathrm{R}\times 2.5036\times 10^4\ \mathrm{lb}_m\mathrm{ft}^2/\mathrm{s}^2\cdot\mathrm{Btu})} \\ &= 891^\circ\mathrm{R} \end{aligned}$$

The final pressure is calculated from the foregoing results by use Eq. 11.4-61 and the ideal gas law:

$$\begin{aligned} p_\infty v_\infty/T_\infty &= p_1 v_1/T_1; \quad \text{hence} \\ p_\infty &= p_1\left(\frac{v_1}{v_\infty}\right)\left(\frac{T_\infty}{T_1}\right) \\ &= (1\ \mathrm{atm})\left(\frac{2256}{846}\right)\left(\frac{891}{530}\right) \\ &= 4.48\ \mathrm{atm} \end{aligned}$$

*c.* The change in specific internal energy is

$$\begin{aligned} \hat{U} &= \hat{C}_V\Delta T = (\hat{C}p/\gamma)\Delta T \\ &= ([0.242/1.4]\ \mathrm{Btu/lb}_m\cdot\mathrm{R})(891 - 530\ \mathrm{R}) \\ &= 62.4\ \mathrm{Btu/lb}_m, \end{aligned}$$

and the change in specific kinetic energy is

$$\begin{aligned} \Delta\hat{K} &= \frac{1}{2}\frac{(846^2 - 2256^2\ \mathrm{ft}^2/\mathrm{s}^2)}{2.5036\times 10^4\ \mathrm{lb}_m\mathrm{ft}^2/\mathrm{s}^2\cdot\mathrm{Btu})} \\ &= -87.4\ \mathrm{Btu/lb}_m \end{aligned}$$

**11A.6 Adiabatic frictionless compression of an ideal gas.**

The states encountered in such a compression satisfy Eq. 11.4-57,

$$p\rho^{-\gamma} = C_1$$

as well as

$$\frac{p}{\rho T} = R/M$$

Combining these relations, we get

$$\rho T \rho^{-\gamma} = T\rho^{1-\gamma} = C_3$$

Hence, the initial and final states in such a compression satisfy

$$\frac{T_2}{T_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma-1}$$

and the final temperature in the case considered here is

$$T_2 = (460 + 100)(10)^{1.4-1} = 1407°\text{R} = 947°\text{F}$$

**11A.7 Effect of free convection on the insulating value of a horzontal air space.**

The relevant properties of air at 1 atm and 100°C= 373.15 K are:

$$\mu = 0.02173 \text{ cp from Table 1.1-2} = 2.173 \times 10^{-4} \text{ g/cm·s}$$
$$\rho = pM/RT = 1/(82.0578 \times 373.15) = 0.0009461 \text{ g/cm}^3$$
$$\beta = 1/T = 0.002680 \text{ K}^{-1}$$
$$\hat{C}_p = 1.015 \text{ J/g·K from CRC Handbook 2000-2001, pp. 6-1, 6-2}$$
$$k = 31.40 \text{ mW/m·K from CRC Handbook 2000-2001, p. 6-185}$$
$$= 31.40 \times 10^{-5} \text{ W/cm·K}$$
$$\text{Pr} = \hat{C}_p\mu/k = \frac{(1.015 \text{ J/g·K})(2.173 \times 10^{-4} \text{ g/cm·s})}{31.40 \times 10^{-5} \text{ W/cm·K}} = 0.703$$

The no-flow state will be stable as long as the Rayleigh number, GrPr, does not exceed its critical value of 1708, given in Ref. 4 of §11.5. This gives the following restriction on the temperature difference:

$$\frac{\rho^2 g\beta(T_1 - T_0)h^3}{\mu^2}\frac{\hat{C}_p\mu}{k} \leq 1708$$

or

$$(T_1 - T_0) \leq \frac{1708\mu^2}{\rho^2 g\beta h^3 \text{ Pr}}$$
$$= \frac{1708(2.173 \times 10^{-4})^2}{(0.0009461 \text{ g/cm}^3)^2(980.665 \text{ cm/s}^2)(0.002680 \text{ K}^{-1})(2.5 \text{ cm})^3(0.703)}$$
$$= 3.1 \text{ °C}$$

If a very thin metal sheet is placed midway between the two plates, forming two cylindrical chambers of height $h/2$, then a corresponding calculation for the total temperature difference across the two chambers gives the stability condition

$$(T_1 - T_0) \leq (2)(8)(3.1) = 49.6\text{°C}$$

for absence of free convection.

**11B.1 Adiabatic frictionless processes in an ideal gas**

*a*. For adiabatic frictionless processes, the fluxes may be set equal to zero. Then the energy equation becomes

$$\rho\hat{C}_p\frac{DT}{Dt}=-\left(\frac{\partial\ln\rho}{\partial\ln T}\right)_p\frac{Dp}{Dt}\qquad\text{or}\qquad\frac{\rho\tilde{C}_p}{M}\frac{DT}{Dt}=-\left(\frac{\partial\ln\rho}{\partial\ln T}\right)_p\frac{Dp}{Dt}$$

For an ideal gas $\rho = pM/RT$, so that $(\partial\ln\rho/\partial\ln T)_p = -1$. Then

$$\frac{p}{RT}\left(a+bT+cT^2\right)\frac{DT}{Dt}=\frac{Dp}{Dt}$$

Hence for a given element of moving fluid

$$\left(\frac{a}{T}+b+cT\right)dT=R\frac{dp}{p}$$

Integration from the initial state $\rho_1, T_1$ gives answer (a) or

$$a\ln\frac{T}{T_1}+b\left(T-T_1\right)+\frac{c}{2}\left(T^2-T_1^2\right)=R\ln\frac{p}{p_1}$$

*c*. For the data given, we have

$$a\ln\frac{T_2}{T_1}=3.204\ln\frac{800}{300}=3.156\text{ cal / g - mole}\cdot\text{K}$$
$$b\left(T_2-T_1\right)=18.41\times10^{-3}(800-300)=9.205\text{ cal / g - mole}\cdot\text{K}$$
$$\frac{c}{2}\left(T_2^2-T_1^2\right)=-\frac{4.48\times10^{-6}}{2}\left(800^2-300^2\right)=-1.231\text{ cal / g - mole}\cdot\text{K}$$

Summing these results we get

$$R\ln\frac{p_2}{p_1}=11.124\quad\text{or}\qquad\frac{p_2}{p_1}=\exp\frac{11.124}{1.987}=270$$

Hence $p_2 = 270$ atm

**11B.2 Viscous heating in laminar tube flow (asymptotic solutions)**

*a*. From the energy equation we have

$$\rho\hat{C}_p v_z \frac{\partial T}{\partial z} = k\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \mu\left(\frac{\partial v_z}{\partial r}\right)^2$$

Into this we substitute the expression for the velocity distribution of a Newtonian fluid in a circular tube: $v_z = v_{z,\max}\left[1-(r/R)^2\right]$. This leads, then, to Eq. 11B.2-1.

*b*. Integration gives (for the isothermal wall) at large $z$ :

$$T = -\frac{\mu v_{z,\max}^2}{4kR^4}r^4 + C_1 \ln r + C_2$$

The constants may be evaluated using the boundary conditions given in the problem, and the result is given in Eq. 11B.2-2.

*c*. To get the $z$ dependence of the temperature we perform an average over the cross-section

$$\rho\hat{C}_p v_{z,\max}\int_0^1\left(1-\xi^2\right)\xi d\xi \cdot \frac{dT}{dz} = \frac{4\mu v_{z,\max}^2}{R^2}\int_0^1 \xi^2 \xi d\xi$$

which leads to the result

$$T - T_1 = \left(4\mu v_{z,\max}/\rho\hat{C}_p R^2\right)z$$

We then make the postulate in Eq. 11B.2-4, and substitute this postulate into Eq. 11B.2-1 to get the following equation for $f(r)$

$$\rho\hat{C}_p v_{z,\max}\left(1-\xi^2\right)\cdot\frac{4\mu v_{z,\max}}{\rho\hat{C}_p R^2} = \frac{k}{R^2}\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{df}{d\xi}\right) + \frac{4\mu v_{z,\max}^2}{R^2}\xi^2$$

This equation may be integrated to give

$$f(r) = \left(\mu v_{z,\max}^2/k\right)\left(\xi^2 - \tfrac{1}{2}\xi^4\right)$$

which leads to Eq. 11B.2-5.

**11B.3 Velocity distribution in a nonisothermal film**

*a*. When $x = \delta$, the two terms inside the large bracket are equal, but with opposite signs, and hence $v_z = 0$.

*b*. The second term in the bracket, being a constant, will not contribute to the derivative. The factors in front of the bracket are also constant. Therefore the derivative of the bracket is:

$$\frac{d}{dx}[\cdots] = \frac{0 + (1/\delta)\ln(\mu_\delta/\mu_0)}{(\mu_\delta/\mu_0)^{x/\delta}} - \frac{1 - (x/\delta)\ln(\mu_\delta/\mu_0)}{\left[(\mu_\delta/\mu_0)^{x/\delta}\right]^2}\left(\frac{\mu_\delta}{\mu_0}\right)^{x/\delta}\ln\left(\frac{\mu_\delta}{\mu_0}\right)\frac{1}{\delta}$$

When we set $x$ equal to zero in this expression we get

$$\frac{d}{dx}[\cdots]\bigg|_{x=0} = (1/\delta)\ln(\mu_\delta/\mu_0) - (1/\delta)\ln(\mu_\delta/\mu_0) = 0$$

Hence at $x = 0$, we have found that $dv_z/dx = 0$

*c*. Let $M = \ln(\mu_\delta/\mu_0)$ and $X = x/\delta$. Then Eq. 11.4-20 becomes

$$v_z = \frac{\rho g \delta^2 \cos\beta}{\mu_0}\left(\frac{1}{M^2}\right)\left(\frac{1+MX}{e^{MX}} - \frac{1+M}{e^M}\right)$$

$$= \frac{\rho g \delta^2 \cos\beta}{\mu_0}\left(\frac{1}{M^2}\right)\left(\frac{(1+MX)e^M - (1+M)e^{MX}}{e^{M(X+1)}}\right)$$

$$= \frac{\rho g \delta^2 \cos\beta}{\mu_0}\left(\frac{(1+MX)\left(1+M+\frac{1}{2}M^2+\cdots\right) - (1+M)\left(1+MX+\frac{1}{2}M^2X^2+\cdots\right)}{M^2 e^{M(X+1)}}\right)$$

$$= \frac{\rho g \cos\beta}{2\mu_0}\left(\frac{1 - X^2 + O(M)}{e^{M(X+1)}}\right)$$

When $\mu_\delta \to \mu_0$ (or $M \to 0$), the above result simplifies to

$$v_z \to \frac{\rho g \cos\beta}{2\mu_0}\left(1 - X^2\right) = \frac{\rho g \cos\beta}{2\mu_0}\left(1 - \left(\frac{x}{\delta}\right)^2\right)$$

in agreement with Eq. 2.2-18.

**11B.4 Heat conduction in a spherical shell**

In this problem, the heat flow is in the $\theta$ direction. Then Eq. B.9-3 simplifies to

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right)=0$$

The first integration leads to

$$\sin\theta\frac{dT}{d\theta}=C_1 \qquad \text{or} \qquad \frac{dT}{d\theta}=\frac{C_1}{\sin\theta}$$

The second integration gives

$$T=C_1\ln\left|\tan\tfrac{1}{2}\theta\right|+C_2=C_1\ln\tan\tfrac{1}{2}\theta+C_2$$

Since the argument of the tangent function is always less than a right angle, the absolute value sign is not needed.

The constants of integration are obtainable from the boundary conditions, which give:

$$T_1=C_1\ln\tan\tfrac{1}{2}\theta_1+C_2; \qquad T_2=C_1\ln\tan\tfrac{1}{2}(\pi-\theta_1)+C_2$$

We next form the following differences:

$$T_1-T_2=C_1\left[\ln\tan\tfrac{1}{2}\theta_1-\ln\tan\tfrac{1}{2}(\pi-\theta_1)\right]=C_1\ln\frac{\tan\frac{1}{2}\theta_1}{\tan\frac{1}{2}(\pi-\theta_1)}$$

$$T-T_2=C_1\left[\ln\tan\tfrac{1}{2}\theta-\ln\tan\tfrac{1}{2}(\pi-\theta_1)\right]=C_1\ln\frac{\tan\frac{1}{2}\theta}{\tan\frac{1}{2}(\pi-\theta_1)}$$

Finally we get for the temperature distribution in the shell:

$$\frac{T-T_2}{T_1-T_2}=\frac{\ln\left[\tan\frac{1}{2}\theta/\tan\frac{1}{2}(\pi-\theta_1)\right]}{\ln\left[\tan\frac{1}{2}\theta_1/\tan\frac{1}{2}(\pi-\theta_1)\right]}$$

This solution clearly satisfies the two boundary conditions.

**11B.5 Axial heat conduction in a wire**

*a*. This problem involves purely axial flow of heat (by conduction and convection) so that the energy equation is

$$\rho\hat{C}_p v_z \frac{dT}{dz} = k\frac{d^2T}{dz^2} \quad \text{or} \quad -\frac{\rho\hat{C}_p v}{k}\frac{dT}{dz} = \frac{d^2T}{dz^2} \qquad \text{or} \qquad -A\frac{dT}{dz} = \frac{d^2T}{dz^2}$$

in which $v_z = -v$, and $A \equiv \rho\hat{C}_p v/k$.

Integration of the differential equation gives

$$-AT = \frac{dT}{dz} + C_1$$

At $z = \infty$, we know that $T = T_\infty$ and $dT/dz = 0$; hence $C_1 = -AT_\infty$. Hence the first-order differential equation becomes

$$\frac{d\Theta}{dz} = -A\Theta \qquad \text{where} \qquad \Theta \equiv (T - T_\infty)/(T_0 - T_\infty)$$

in which (since $\Theta(0) = 1$)

$$\ln\Theta = -Az + \ln C_2 \qquad \text{or} \qquad \Theta \equiv \frac{T - T_\infty}{T_0 - T_\infty} = e^{-Az}$$

This is just Eq. 11B.5-1.

*b*. For temperature-dependent physical properties we have the following energy equation:

$$-\rho\hat{C}_{p\infty}L(\Theta)v\frac{d\Theta}{dz} = \frac{d}{dz}\left(k_\infty K(\Theta)\frac{d\Theta}{dz}\right) \text{ or } \quad -A_\infty L(\Theta)\frac{d\Theta}{dz} = \frac{d}{dz}\left(K(\Theta)\frac{d\Theta}{dz}\right)$$

in which $A_\infty = \rho\hat{C}_{p\infty}v/k_\infty$. The first integration gives

$$+A_\infty\int_z^\infty L(\Theta)\frac{d\Theta}{dz}dz = K(\Theta)\frac{d\Theta}{dz} + C_1$$

We now use the boundary conditions that at $z=\infty$, $\Theta=0$ and $d\Theta/dz=0$, to find that $C_1=0$. The above result may then be written as

$$+A_\infty\int_\Theta^0 L(\overline{\Theta})d\overline{\Theta}=K(\Theta)\frac{d\Theta}{dz} \quad \text{or} \quad -A_\infty\int_0^\Theta L(\overline{\Theta})d\overline{\Theta}=K(\Theta)\frac{d\Theta}{dz}$$

This equation may be integrated to give

$$-A_\infty\int_0^z dz=\int_1^\Theta\frac{K(\overline{\Theta})d\overline{\Theta}}{\int_0^{\overline{\Theta}}L(\overline{\overline{\Theta}})d\overline{\overline{\Theta}}}$$

This result simplifies to that in (*a*) when $K$ and $L$ are both equal to unity. Furthermore, it satisfies the boundary conditions at $z=0$ and $z=\infty$.

*c*. To show that the last equation in (*b*) satisfies the differential equation (the first equation in (*b*)), we differentiate both sides with respect to $z$ (on the right side, we differentiate with repsect to $\Theta$, using the Leibniz formula, and then multiply by $d\Theta/dz$):

$$-A_\infty=\frac{K(\Theta)}{\int_0^\Theta L(\overline{\overline{\Theta}})d\overline{\overline{\Theta}}}\cdot\frac{d\Theta}{dz} \quad \text{or} \quad -A_\infty\int_0^\Theta L(\overline{\overline{\Theta}})d\overline{\overline{\Theta}}=K(\Theta)\frac{d\Theta}{dz}$$

A second differentiation with respect to $z$ (once again using the Leibniz formula) gives

$$-A_\infty L(\Theta)\frac{d\Theta}{dz}=\frac{d}{dz}\left(K(\Theta)\frac{d\Theta}{dz}\right)$$

and this is the differential equation with which we started.

**11B.6 Transpiration cooling in a planar system**

We start with Eq. J of Table 11.4-1, and assume steady-state, negligible change in pressure with distance, and negligible viscous dissipation. Then for constant thermal conductivity this equation simplifies to

$$\rho\hat{C}_p v_y \frac{dT}{dy} = k\frac{d^2T}{dy^2} \qquad \text{or} \qquad \phi\frac{d\Theta}{d\eta} = \frac{d^2\Theta}{d\eta^2}$$

in which $\Theta = (T - T_L)/(T_0 - T_L)$, $\eta = y/L$, and $\phi = \rho\hat{C}_p v_y L/k$, a constant. This equation is to be solved with the boundary conditions that $\Theta(0) = 1$ and $\Theta(1) = 0$.

Set $p = d\Theta/d\eta$ to get the first-order separable equation

$$\phi p = \frac{dp}{d\eta} \qquad \text{or} \qquad \frac{dp}{p} = \phi d\eta$$

which may be integrated to give

$$\ln p = \phi\eta + \ln C_1 \qquad \text{or} \qquad \frac{d\Theta}{d\eta} \equiv p = C_1 e^{\phi\eta}$$

This is also a first-order separable equation and integrating it gives

$$\Theta = C_1\int e^{\phi\eta}d\eta + C_2 = \frac{C_1}{\phi}e^{\phi\eta} + C_2$$

The constants of integration are then found from the boundary conditions, and we get finally

$$\Theta = \frac{e^{\phi\eta} - e^{\phi}}{1 - e^{\phi}}$$

The heat flux at $y = 0$ is then

$$q_0 = -k\left.\frac{dT}{dy}\right|_{y=0} = -\frac{k(T_0 - T_L)}{L}\left.\frac{d\Theta}{d\eta}\right|_{\eta=0} = \frac{k(T_L - T_0)}{L}\frac{\phi}{1 - e^{\phi}}$$

**11B.7 Reduction of evaporation losses by transpiration**

*a*. Without transpiration, we have from Eq. 11.4-31

$$Q_0 = \frac{4\pi\kappa R(T_1 - T_\kappa)}{1-\kappa} == \frac{4\pi\left(\frac{1}{2}\text{ft}\right)(0.02\text{Btu/hr}\cdot\text{ft}\cdot{}^\circ\text{F})(327^\circ\text{F})}{\frac{1}{2}}$$
$$= 82\text{Btu/hr}$$

*b*. With transpiration, we get from Eq. 11.4-30

$$Q = \frac{4\pi R_0 k(T_1 - T_\kappa)}{e^{(R_0/\kappa R)(1-\kappa)} - 1} \quad \text{with} \quad R_0 = w_r \hat{C}_p / 4\pi k$$

We do not know $R_0$, but we can get it by using an energy balance in the form

$$Q = w_r \Delta\hat{H}_{\text{vap}} = \left(\frac{4\pi k R_0}{\hat{C}_p}\right)\Delta\hat{H}_{\text{vap}}$$

When this is inserted into the left side of Eq. 11.4-30, we get

$$\frac{\Delta\hat{H}_{\text{vap}}}{\hat{C}_p} = \frac{(T_1 - T_\kappa)}{e^{(R_0/\kappa R)(1-\kappa)} - 1}$$

This may be solved for $R_0$, and then the energy balance may be used to convert the result into an expression for $Q$

$$Q = \frac{4\pi k \Delta\hat{H}_{\text{vap}}}{\hat{C}_p}\left(\frac{\kappa R}{1-\kappa}\right)\ln\left(\frac{\hat{C}_p(T_1 - T_\kappa)}{\Delta\hat{H}_{\text{vap}}} + 1\right)$$
$$= \frac{4\pi(0.02)(91.7)}{0.22}\left(\frac{0.5}{0.5}\right)\ln\left(\frac{(0.22)(327)}{(91.7)} + 1\right)$$
$$= (104.76)\ln(1.7845) = (104.76)(0.5792) = 61\text{ Btu/hr}$$

**11B.8 Temperature distribution in an embedded sphere**

*a*. In both regions the partial differential equation is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)=0$$

*b*. At the surface of the embedded sphere the boundary conditions are that $T_1 = T_2$ and that $k_1(\partial T_1/\partial r) = k_0(\partial T_0/\partial r)$.

*c*. For the temperature field inside the sphere, substitution of Eq. 11B.8-1 into the terms in the above differential equation gives:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)=\left[\frac{3k_0}{k_1+2k_0}\right]2Ar^{-1}\cos\theta$$

$$\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)=-\left[\frac{3k_0}{k_1+2k_0}\right]2Ar^{-1}\cos\theta$$

Outside the sphere we get for the same two terms

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)=2Ar^{-1}\cos\theta-2\left[\frac{k_1-k_0}{k_1+2k_0}\right]\left(\frac{R}{r}\right)^3Ar^{-1}\cos\theta$$

$$\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)=-\left[1-\left[\frac{k_1-k_0}{k_1+2k_0}\right]\left(\frac{R}{r}\right)^3\right]2Ar^{-1}\cos$$

*d*. The boundary conditions are also satisfied:

$$\left[\frac{3k_0}{k_1+2k_0}\right]AR\cos\theta=\left[1-\left[\frac{k_1-k_0}{k_1+2k_0}\right]\right]AR\cos\theta$$

$$k_1\left[\frac{3k_0}{k_1+2k_0}\right]A\cos\theta=k_0\left[\frac{3k_0}{k_1+2k_0}\right]A\cos\theta+k_0\left[3\frac{k_1-k_0}{k_1+2k_0}\right]A\cos\theta$$

**11B.9 Heat flow in a solid bounded by two conical surfaces**

*a*, The temperature has to satisfy the differential equation

$$\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right)=0$$

*b*. Two successive integrations give

$$\frac{dT}{d\theta}=\frac{C_1}{\sin\theta} \qquad \text{and} \qquad T=C_1\ln\left|\tan\tfrac{1}{2}\theta\right|+C_2$$

Since in this problem, $\theta$ may go from 0 to $\frac{1}{2}\pi$, and hence the tangent will not have negative values. Therefore absolute-value signs are not needed.

*c*. The integration constants are determined from the following simultaneous equations:

$$T_1=C_1\ln\left(\tan\tfrac{1}{2}\theta_1\right)+C_2 \qquad \text{and} \qquad T_1=C_1\ln\left(\tan\tfrac{1}{2}\theta_1\right)+C_2$$

On solving them, one gets

$$C_1=\frac{T_1-T_2}{\ln\tan\frac{1}{2}\theta_1-\ln\tan\frac{1}{2}\theta_2}=\frac{T_1-T_2}{\ln\left(\tan\frac{1}{2}\theta_1/\tan\frac{1}{2}\theta_2\right)}$$

$$C_2=\frac{T_2\ln\tan\frac{1}{2}\theta_1-T_1\ln\tan\frac{1}{2}\theta_2}{\ln\tan\frac{1}{2}\theta_1-\ln\tan\frac{1}{2}\theta_2}$$

*d*. The $\theta$ component of the heat-flux vector is obtained from the first equation in (*b*) above

$$q_\theta=-k\frac{1}{r}\frac{dT}{d\theta}=-\frac{C_1k}{r\sin\theta}=-\frac{k}{r\sin\theta}\frac{T_1-T_2}{\ln\left(\tan\frac{1}{2}\theta_1/\tan\frac{1}{2}\theta_2\right)}$$

*e*. The total heat flow across the conical surface is then

$$Q=\int_0^R\int_0^{2\pi}q_\theta\big|_{\theta=\theta_1}r\sin\theta_1 d\phi dr=2\pi Rk\frac{T_1-T_2}{\ln\left(\tan\frac{1}{2}\theta_2/\tan\frac{1}{2}\theta_1\right)}$$

**11B.10 Freezing of a spherical drop**

*a*. The heat conduction equation for the solid phase is

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right)=0 \qquad (R_f \le r \le R)$$

Two integrations lead to $T=-(C_1/r)+C_2$. The constants of integration are determined from the boundary conditions

B. C. 1: At $r=R_f$, $T=T_0$; B. C. 2: At $r=R$, $-k\dfrac{dT}{dr}=h(T-T_\infty)$

This leads to the following expressions:

$$C_1=\frac{T_0-T_\infty}{(1/R)-(1/R_f)-(k/hR^2)};\ C_2=T_0+\frac{T_0-T_\infty}{\left[(1/R)-(1/R_f)-(k/hR^2)\right]R_f}$$

The total heat flow across the spherical surface at $r=R$ is then

$$Q=4\pi R^2\left(-k\frac{dT}{dr}\right)\bigg|_{r=R}=4\pi R^2\left(-\frac{k}{R^2}\right)\frac{T_0-T_\infty}{\left[(1/R)-(1/R_f)-(k/hR^2)\right]}$$

This can be rearranged to give the solution in the text.

*b*. We now have to equate the heat liberated on freezing at $r=R_f$ to the heat flowing out across the surface at $r=R$:

$$-(\rho\Delta\hat{H}_f)(4\pi R_f^2)\frac{dR_f}{dt}=\frac{h\cdot 4\pi R^2\cdot(T_0-T_\infty)}{1-(hR/k)+(hR^2/kR_f)}$$

Integration then yields

$$-(\rho\Delta\hat{H}_f)\int_R^0\left[1-(hR/k)+(hR^2/kR_f)\right]R_f^2dR_f=hR^2(T_0-T_\infty)\int_0^{t_f}dt$$

where $t_f$ is the time for the freezing of the entire droplet. Evaluation of the integrals then leads to the expression in the text.

**11B.11 Temperature rise in a catalyst pellet**

*a*. We make an energy balance over a spherical shell of thickness $\Delta r$:

$$4\pi r^2 q_r\big|_r - 4\pi(r+\Delta r)^2 q_r\big|_{r+\Delta r} + 4\pi r^2 \Delta r S_c = 0$$

Then division by $4\pi\Delta r$ gives

$$\frac{(r^2 q_r)\big|_{r+\Delta r} - (r^2 q_r)\big|_r}{\Delta r} - r^2 S_c = 0$$

When the limit is taken that $\Delta r \to 0$ and use is made of the definition of the first derivative, we get

$$\frac{d}{dr}\left(r^2 q_r\right) - r^2 S_c = 0$$

Insertion of Fourier's law then gives

$$\frac{d}{dr}\left(r^2 k \frac{dT}{dr}\right) + r^2 S_c = 0 \quad (**) \quad \text{or} \quad k\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) + r^2 S_c = 0$$

for the appropriate equation describing the heat conduction with heat generation by chemical reaction and constant $k$.

*b*. From Eq. B.9=3, with the time=derivative term set equal to zero, and all velocities set equal to zero, and all derivatives other than $r$ derivatives set equal to zero gives the heat conduction equation in spherical coordinates for a system with no chemical reaction. Therefore, we have to add a term describing the heat production per unit volume:

$$k\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) + S_c = 0$$

which is the same as the result obtained in (*a*).

*c*. The above differential equation may be integrated in a sequence of steps as follows:

$$\frac{d}{dr}\left(r^2\frac{dT}{dr}\right)=-\frac{S_c r^2}{k}; \qquad \left(r^2\frac{dT}{dr}\right)=-\frac{S_c r^3}{3k}+C_1; \qquad \frac{dT}{dr}=-\frac{S_c r}{3k}+\frac{C_1}{r^2}$$

$$T=-\frac{S_c r^2}{6k}-\frac{C_1}{r}+C_2 \qquad (***)$$

The constant $C_1$ must be zero, because neither the temperature nor its gradient are expected to be infinite. The heat loss to the surroundings provides the second boundary condition needed for getting $C_2$:

Definition of heat transfer coefficient: $q_r\big|_{r=R}=h\left(T_R-T_g\right)$

From Fourier's law: $q_r\big|_{r=R}=-k\left.\frac{dT}{dr}\right|_{r=R}=+\frac{S_c R}{3}$

Equating these expressions: $h\left(T_R-T_g\right)=\frac{S_c R}{3}$

Inserting $T_R$ from (***): $h\left(-\frac{S_c R^2}{6k}+C_2-T_g\right)=\frac{S_c R}{3}$

Solving for $C_2$: $C_2=\frac{S_c R^2}{6k}+\frac{S_c R}{3h}+T_g$

Thus we finally get the temperature profile within the catalyst pellet:

$$T-T_g=\frac{S_c R^2}{6k}\left[1-\left(\frac{r}{R}\right)^2\right]+\frac{S_c R}{3h}$$

*d*..When the heat transfer coefficient goes to infinity, the last term in the temperature distribution drops out.

*e*. The maximum temperature in the system is

$$T_{\max}-T_g=\frac{S_c R^2}{6k}+\frac{S_c R}{3h}=\frac{S_c R^2}{6k}\left(1+\frac{2k}{Rh}\right)$$

*f*. In Eq. (**) one would have to leave $k$ inside the differential operator, and insert the specific $r$ dependence of both $k$ and $S_c$ .

**11B.12 Stability of an exothermic reaction system**

*a*. For the postulated steady-state solution

$$k\frac{d^2T}{dx^2} + S_{c0}\exp\left(A(T-T_0)\right) = 0 \qquad \text{with} \qquad T = T_0 \text{ at } x = \pm B$$

*b*. Using the given dimensionless variables, we may rewrite the problem as

$$\frac{d^2\Theta}{d\xi^2} + \lambda e^{\Theta} = 0 \qquad \text{with} \qquad \Theta = 0 \text{ at } \xi = \pm 1$$

*c*. Multiply the differential equation by $2d\Theta/d\xi$ to get

$$2\frac{d\Theta}{d\xi}\frac{d^2\Theta}{d\xi^2} + 2\lambda\frac{d\Theta}{d\xi}e^{\Theta} = 0 \qquad \text{or} \qquad \frac{d}{d\xi}\left(\frac{d\Theta}{d\xi}\right)^2 + 2\lambda\frac{de^{\Theta}}{d\xi} = 0$$

Integration then gives

$$\left(\frac{d\Theta}{d\xi}\right)^2 + 2\lambda e^{\Theta} = C_1$$

We now use the fact that, from the symmetry of the problem, at $\xi = 0$ we must have $d\Theta/d\xi = 0$. Then if we let $\Theta = \Theta_0$ at $\xi = 0$, we can get an expression for $C_1$ and then write

$$\left(\frac{d\Theta}{d\xi}\right)^2 - 2\lambda\left(\exp\Theta_0 - \exp\Theta\right) = 0$$

Note that we have not "evaluated" the integration constant $C_1$, we have merely replaced it by $\Theta_0$, which has a recognizable physical meaning.

*c*. We next take the square root of both sides:

$$\frac{d\Theta}{d\xi} = \pm\sqrt{2\lambda}\sqrt{\exp\Theta_0 - \exp\Theta}$$

The minus sign has been selected, since the left side of the differential equation is inherently negative, and the quantity under the square-root sign is positive. Then we integrate over half the plate

$$\int_0^{\Theta_0} \frac{d\Theta}{\sqrt{\exp\Theta_0 - \exp\Theta}} = -\sqrt{2\lambda}\int_1^0 d\xi$$

*d*. The integral can be done analytically by making the change of variable $y^2 = \exp(\Theta - \Theta_0)$:

$$\frac{1}{\sqrt{\exp\Theta_0}}\int_0^{\Theta_0} \frac{d\Theta}{\sqrt{1-\exp(\Theta-\Theta_0)}}$$
$$= \exp\left(-\tfrac{1}{2}\Theta_0\right)\int_{\exp\left(-\frac{1}{2}\Theta_0\right)}^1 \frac{2dy}{y\sqrt{1-y^2}}$$
$$= 2\exp\left(-\tfrac{1}{2}\Theta_0\right)\operatorname{arccosh}\left(\exp\tfrac{1}{2}\Theta_0\right)$$

The integral over $y$ can be found in an integral table.

Combining these last two results we get

$$\exp\left(-\tfrac{1}{2}\Theta_0\right)\operatorname{arccosh}\left(\exp\tfrac{1}{2}\Theta_0\right) = \sqrt{\tfrac{1}{2}\lambda}$$

*e*. For $\lambda > 0.88$ no value of $\Theta_0$ can be found. This means that when $S_{c0}$ is too large, or $B$ is too large, or $k$ is too small, then the heat cannot be dissipated fast enough.

This is an important example, because it illustrates that it is not always possible to get a steady-state solution to a physical problem. To do a complete analysis of this problem, it would be necessary to solve the problem with the time-derivative term included.

**11B.13 Laminar annular flow with constant wall heat flux**

The equation analogous to Eq. 10.8-12 for the annular flow is

$$\rho\hat{C}_p v_{z,\max}\left[1-\left(\frac{r}{R}\right)^2-\frac{1-\kappa^2}{\ln(1/\kappa)}\ln\left(\frac{R}{r}\right)\right]\frac{\partial T}{\partial z}=k\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$$

where Eq. 2.4-14 has been used. We now introduce dimensionless variables: $\xi = r/R$, $\zeta = kz/\rho\hat{C}_p v_{z,\max}R^2$, and $\Theta = k(T-T_1)/q_0R$. Then the above energy equation becomes

$$\left[\left(1-\xi^2\right)-\left(1-\kappa^2\right)\frac{\ln\xi}{\ln\kappa}\right]\frac{\partial\Theta}{\partial\zeta}=\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\Theta}{\partial\xi}\right)$$

This is to be solved with the boundary conditions

$$\text{At } r=\kappa R,\quad -k\frac{\partial T}{\partial r}=q_0 \qquad \text{or} \qquad \text{at } \xi=\kappa,\quad -\frac{\partial\Theta}{\partial\xi}=1$$

$$\text{At } r=R,\quad -k\frac{\partial T}{\partial r}=0 \qquad \text{or} \qquad \text{at } \xi=1,\quad -\frac{\partial\Theta}{\partial\xi}=0$$

$$\text{At } z=0,\quad T=T_1 \qquad \text{or} \qquad \text{at } \zeta=0,\quad \Theta=0$$

We seek an asymptotic solution for large downstream distances of the form $\Theta(\xi,\zeta)=C_0\zeta+\Psi(\xi)$. The function $\Psi(\xi)$ has to satisfy

$$\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\Psi}{\partial\xi}\right)=C_0\left[\left(1-\xi^2\right)-\left(1-\kappa^2\right)\frac{\ln\xi}{\ln\kappa}\right]$$

The first and second integrations lead to (cf. Eq. 10.8-27)

$$\frac{\partial\Psi}{\partial\xi}=C_0\left[\left(\frac{\xi}{2}-\frac{\xi^3}{4}\right)-\frac{1-\kappa^2}{\ln\kappa}\left(\frac{\xi}{2}\ln\xi-\frac{\xi}{4}\right)\right]+\frac{C_1}{\xi}$$

$$\Psi=C_0\left[\left(\frac{\xi^2}{4}-\frac{\xi^4}{16}\right)-\frac{1-\kappa^2}{\ln\kappa}\left(\frac{\xi^2}{4}\ln\xi-\frac{\xi^2}{4}\right)\right]+C_1\ln\xi+C_2$$

The constants $C_0$ and $C_1$ are determined from the boundary conditions at $\xi = \kappa$ and $\xi = 1$:

$$C_0 = \frac{4\kappa}{\left(1-\kappa^4\right)+\dfrac{\left(1-\kappa^2\right)^2}{\ln\kappa}} \qquad \text{and} \quad C_1 = -\frac{\kappa\left(1+\dfrac{1-\kappa^2}{\ln\kappa}\right)}{\left(1-\kappa^4\right)+\dfrac{\left(1-\kappa^2\right)^2}{\ln\kappa}}$$

Then the equation for the radial distribution of the temperature is

$$\Psi = \frac{\kappa\xi^2\left[\left(1-\dfrac{\xi^2}{4}\right)-\dfrac{1-\kappa^2}{\ln\kappa}(\ln\xi-1)\right]-\kappa\left(1+\dfrac{1-\kappa^2}{\ln\kappa}\right)\ln\xi}{\left(1-\kappa^4\right)+\dfrac{\left(1-\kappa^2\right)^2}{\ln\kappa}}+C_2$$

The constant $C_2$ can be obtained by using an integral condition, similar to that in Eq. 10.8-24 or 25

$$2\pi\kappa Rz\cdot q_0 = \int_0^{2\pi}\int_{\kappa R}^{R}\rho\hat{C}_p(T-T_1)v_z r dr d\theta \qquad \text{or} \qquad \zeta = \frac{1}{\kappa}\int_\kappa^1 \Theta\phi\xi d\xi$$

where $\phi = v_z/v_{z,\max}$. This gives

$$\zeta = \frac{1}{\kappa}\int_\kappa^1\left\{C_0\zeta + C_0\left[\left(\frac{\xi^2}{4}-\frac{\xi^4}{16}\right)-\frac{1-\kappa^2}{\ln\kappa}\left(\frac{\xi^2}{4}\ln\xi-\frac{\xi^2}{4}\right)\right]+C_1\ln\xi+C_2\right\}\cdot \left[\left(1-\xi^2\right)-\left(1-\kappa^2\right)\frac{\ln\xi}{\ln\kappa}\right]\xi d\xi$$

or

$$0 = \frac{1}{\kappa}\int_\kappa^1\left\{C_0\left[\left(\frac{\xi^2}{4}-\frac{\xi^4}{16}\right)-\frac{1-\kappa^2}{\ln\kappa}\left(\frac{\xi^2}{4}\ln\xi-\frac{\xi^2}{4}\right)\right]+C_1\ln\xi+C_2\right\}\cdot \left[\left(1-\xi^2\right)-\left(1-\kappa^2\right)\frac{\ln\xi}{\ln\kappa}\right]\xi d\xi$$

since the terms containing $\zeta$ just cancel.

This last equation has to be solved for $C_2$. This is in principle an easy problem, but extremely time-consuming and unrewarding. It can, however, be solved by using Mathematica, and the result is*

$$C_2 = \frac{\kappa\begin{bmatrix} -9\left(1-\kappa^2\right)^3\left(11+3\kappa^2\right)-2\left(1-\kappa^2\right)^2\left(88+88\kappa^2+25\kappa^4\right)\ln\kappa \\ +3\left(-25-24\kappa^2+18\kappa^4+24\kappa^6+7\kappa^8\right)(\ln\kappa)^2-72\kappa^4(\ln\kappa)^3 \end{bmatrix}}{72\left(1-\kappa^2\right)^2\left((1-\kappa^2)+(1+\kappa^2)(\ln\kappa)\right)^2}$$

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**11B.14 Unsteady-state heating of a sphere**

*a*. The time-dependent heat conduction equation is

$$\frac{\partial T}{\partial t} = \alpha \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) \qquad \text{or} \qquad \frac{\partial \Theta}{\partial \tau} = \frac{1}{\xi^2}\frac{\partial}{\partial \xi}\left(\xi^2 \frac{\partial \Theta}{\partial \xi}\right)$$

In the second form, we have introduced the dimensionless variables: $\Theta = (T - T_0)/(T_1 - T_0)$, $\xi = r/R$, and $\tau = \alpha t/R^2$. The solution in Eq. 11B.14-1 is, in terms of the same dimensionless variables

$$\Theta = 1 + 2\sum_{n=1}^{\infty}(-1)^n \frac{\sin n\pi\xi}{n\pi\xi} e^{-n^2\pi^2\tau}$$

*b*. We do the differentiations; then it is clear that the first and last expression below are the same. Therefore Eq. 11B.14-1 satisfies the differential equation.

$$\frac{\partial \Theta}{\partial \tau} = 2\sum_{n=1}^{\infty}(-1)^n\left(-n^2\pi^2\right)\frac{\sin n\pi\xi}{n\pi\xi} e^{-n^2\pi^2\tau}$$

$$\xi^2 \frac{\partial \Theta}{\partial \xi} = 2\sum_{n=1}^{\infty}(-1)^n n\pi\left[\frac{\xi \cos n\pi\xi}{n\pi} - \frac{\sin n\pi\xi}{(n\pi)^2}\right] e^{-n^2\pi^2\tau}$$

$$\frac{\partial}{\partial \xi}\left(\xi^2 \frac{\partial \Theta}{\partial \xi}\right) = 2\sum_{n=1}^{\infty}(-1)^n n\pi\left[\frac{\cos n\pi\xi}{n\pi} - \frac{n\pi\xi \sin n\pi\xi}{n\pi} - \frac{\cos n\pi\xi}{n\pi}\right] e^{-n^2\pi^2\tau}$$

$$\frac{1}{\xi^2}\frac{\partial}{\partial \xi}\left(\xi^2 \frac{\partial \Theta}{\partial \xi}\right) = 2\sum_{n=1}^{\infty}(-1)^n n\pi\left[-\frac{n\pi \sin n\pi\xi}{n\pi\xi}\right] e^{-n^2\pi^2\tau}$$

*c*. When $\xi = 1$, we get from Eq. 11B.14-1

$$\Theta = 1 + 2\sum_{n=1}^{\infty}(-1)^n \frac{\sin n\pi}{n\pi} e^{-n^2\pi^2\tau} = 1$$

since $\sin n\pi = 0$ for all integral values of $n$.

*d*. Inasmuch as

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

the solution for $\xi = 0$ is certainly finite:

$$\Theta = 1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-n^2\pi^2\tau}$$

*e*. In dimensionless form Eq. 11B.14-2 is

$$-1 = 2\sum_{n=1}^{\infty}(-1)^n \frac{\sin n\pi\xi}{n\pi\xi}$$

Then, multiplying by $\xi \sin m\pi\xi$ and integrating gives

$$-\int_0^1 \xi \sin m\pi\xi d\xi = 2\sum_{n=1}^{\infty}(-1)^n \frac{1}{n\pi}\int_0^1 \sin m\pi\xi \sin n\pi\xi d\xi$$

The left side can be evaluated as follows:

$$-\int_0^1 \xi \sin m\pi\xi d\xi = -\frac{1}{(m\pi)^2}\int_0^{m\pi} x\sin x dx = -\frac{1}{(m\pi)^2}(\sin x - x\cos x)\Big|_0^{m\pi}$$
$$= +\frac{1}{(m\pi)^2}(m\pi\cos m\pi) = \frac{1}{m\pi}\cos m\pi = \frac{1}{m\pi}(-1)^m$$

The right side may be evaluated thus:

$$2\sum_{n=1}^{\infty}(-1)^n \frac{1}{n\pi}\int_0^1 \sin n\pi\xi \sin m\pi\xi d\xi = 2\sum_{n=1}^{\infty}(-1)^n \frac{1}{n\pi^2}\int_0^{\pi} \sin nx \sin mx dx$$
$$= 2\sum_{n=1}^{\infty}(-1)^n \frac{1}{n\pi^2}\left(\frac{\pi}{2}\delta_{mn}\right) = \frac{1}{m\pi}(-1)^m$$

Thus the two sides are equal, and it is proven that the initial condition is indeed satisfied.

**11B.15 Dimensionless variables for free convection**

*a*. When the proposed dimensionless quantities are introduced, Eqs. 11B.15-1 to 3 follow directly. Nothing more needs to be said.

*b*. We can convert Eq. 11B.15-1 into Eq. 11.4-44 if we require that

$$\text{(I)} \qquad \frac{v_{z0}y_0}{v_{y0}H} = 1$$

Then, when (I) is substituted into Eq. 11B.15.2, then the latter may be converted into Eq. 11.4-45 if we also require that

$$\text{(II)} \qquad \frac{\mu}{\rho y_0 v_{y0}} = \Pr$$

$$\text{(III)} \qquad \frac{y_0 g\beta(T_0 - T_1)}{v_{y0}v_{z0}} = \Pr$$

Next we substitute (I) into Eq. 11B.15-3 and further require that

$$\text{(IV)} \qquad \frac{k}{\rho\hat{C}_p y_0 v_{y0}} = 1$$

then Eq. 11B.15-3 becomes Eq. 11.4-35.

We thus have four equations from which to determine the three "scale factors" $y_0$, $v_{y0}$, and $v_{z0}$. However, it can be seen that Eqs. (II )and (IV) are not independent, since multiplication of Eq. (IV) by Pr just gives Eq. (II). Thus we have three independent equations from which to determine three unknowns.

We now eliminate $y_0$ by multiplying Eqs. (I) and (II) to get Eq. (V), and by dividing Eq. (III) by Eq. (I) to get Eq. (VI):

$$\text{(V)} \qquad \frac{v_{z0}\mu}{v_{y0}^2 H\rho} = \Pr$$

(VI) $$\frac{g\beta(T_0 - T_1)H}{v_{z0}^2} = \mathrm{Pr}$$

Introducing the abbreviation $B = \rho g \beta (T_0 - T_1)$, Eq. (VI) can be solved for $v_{z0}$ to give Eq. (VII), and then $v_{y0}$ is obtained from Eq. (V) to give Eq. (VIII):

(VII) $$v_{z0} = \sqrt{\frac{BH}{\rho \mathrm{Pr}}} = \sqrt{\frac{\alpha BH}{\mu}}$$

(VIII) $$v_{y0} = \sqrt{\frac{\mu}{H\rho \mathrm{Pr}}\sqrt{\frac{\alpha BH}{\mu}}} = \sqrt[4]{\frac{\alpha^3 B}{\mu H}}$$

Finally $y_0$ may be obtained from Eqs. (IV) and (VIII):

(IX) $$y_0 = \frac{k}{\rho \hat{C}_p v_{y0}} = \alpha \sqrt[4]{\frac{\mu H}{\alpha^3 B}} = \sqrt[4]{\frac{\alpha \mu H}{B}}$$

The reciprocals of these last three quantities appear in Eqs. 11.B-41 to 43.

*c*. If all the dimensionless groups were set equal to unit, then combination of Eqs. (II) and (IV) would give Pr = 1, thereby severely restricting the applicability of the results.

**11C.1 The speed of propagation of sound waves**

*a*. Equation 11C.1-1 can also be written as

$$\left(\frac{\partial p}{\partial V}\right)_S = \frac{(\partial H/\partial T)_p}{(\partial U/\partial T)_V}\left(\frac{\partial p}{\partial V}\right)_T \quad \text{or} \quad \frac{(\partial p/\partial S)_V}{(\partial V/\partial S)_p} = \frac{(\partial H/\partial T)_p}{(\partial U/\partial T)_V}\frac{(\partial p/\partial T)_V}{(\partial V/\partial T)_p}$$

This may be rearranged to give

$$\left(\frac{\partial p}{\partial S}\right)_V\left(\frac{\partial V}{\partial T}\right)_p\left(\frac{\partial U}{\partial T}\right)_V = \left(\frac{\partial p}{\partial T}\right)_V\left(\frac{\partial V}{\partial S}\right)_p\left(\frac{\partial H}{\partial T}\right)_p$$

Next we apply the Maxwell relations to the first two factors on the left and right sides, and we apply two of the four "fundamental relations" for pure fluids to modify the third factor

$$\left[-\left(\frac{\partial T}{\partial V}\right)_S\right]\left[-\left(\frac{\partial S}{\partial p}\right)_T\right]\left[T\left(\frac{\partial S}{\partial T}\right)_V\right] = \left[\left(\frac{\partial S}{\partial V}\right)_T\right]\left[\left(\frac{\partial T}{\partial p}\right)_S\right]\left[T\left(\frac{\partial S}{\partial T}\right)_p\right]$$

Then we apply the relation

$$\left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y = -1$$

to the first and third factors, to obtain

$$\frac{(\partial S/\partial p)_T}{(\partial V/\partial S)_T} = \frac{(\partial S/\partial V)_T}{(\partial p/\partial S)_T}$$

On cross-multipiclations, this gives an identity.

*b*. The equations of continuity and motion are

$$\frac{\partial}{\partial t}(\rho_0 + \rho') = -\left(\nabla\cdot(\rho_0 + \rho')(\mathbf{v}_0 + \mathbf{v}')\right);\ \frac{\partial}{\partial t}(\rho_0 + \rho')(\mathbf{v}_0 + \mathbf{v}') = -\nabla(p_0 + p')$$

Since $\rho_0$ and $p_0$ are constants and $\mathbf{v}_0$ is zero, we get (when the products of the primed quantities are neglected)

$$\frac{\partial \rho}{\partial t} = -\rho_0(\nabla \cdot \mathbf{v}) \quad \text{and} \quad \rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla p$$

*c*. Since the momentum and energy fluxes have been taken to be zero), the flow is isentropic. This enables us to rewrite the equation of motion thus:

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -(\nabla \rho)\left(\frac{\partial p}{\partial \rho}\right)_S = -(\nabla \rho)\gamma\left(\frac{\partial p}{\partial \rho}\right)_T$$

To get the second expression, Eq. 11C.1-1 has been used. Use of the definition of the speed of sound then gives Eq. 11C.1-4.

*d*. Next we take the time derivative of the equation of continuity to get

$$\frac{\partial^2 \rho}{\partial t^2} = -\frac{\partial}{\partial t}[\rho_0(\nabla \cdot \mathbf{v})] = -\left(\nabla \cdot \rho_0 \frac{\partial \mathbf{v}}{\partial t}\right) \qquad \text{or} \qquad \frac{\partial^2 \rho}{\partial t^2} = v_s^2 \nabla^2 \rho$$

To get the second form, the equation of motion of Eq. 111C.1-4 has been used.

*e*. From Eq. 11C.1-6 we get

$$\frac{\partial^2 \rho}{\partial t^2} = -\rho_0 A\left(\frac{2\pi}{\lambda}\right)^2 v_s^2 \sin\left[\frac{2\pi}{\lambda}(z - v_s t)\right]$$

$$v_s^2 \frac{\partial^2 \rho}{\partial z^2} = -v_s^2 \rho_0 A\left(\frac{2\pi}{\lambda}\right)^2 \sin\left[\frac{2\pi}{\lambda}(z - v_s t)\right]$$

Therefore, Eq. 11C.1-6 is satisfied.

**11C.2 Free convection in a slot**

*a*. The equations of change are simplified as follows:

*Continuity*:

$$\frac{\partial}{\partial t}\rho = -\left(\frac{\partial}{\partial x}\rho v_x\right) \quad \text{and at steady state and } v_x = 0 \text{ we get } 0 = 0$$

*Motion*:

$$\frac{\partial}{\partial t}\rho v_z = -\left(\frac{\partial}{\partial z}\rho v_z v_z\right) - \frac{\partial p}{\partial z} - \bar{\rho} g + \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2}\right) + \bar{\rho}\bar{\beta} g\left(T - \bar{T}\right) \text{ or}$$

$$0 = \mu\frac{\partial^2 v_z}{\partial x^2} + \bar{\rho}\bar{\beta} g\left(T - \bar{T}\right) \qquad \text{since} \qquad p \approx -\bar{\rho} g z + \text{constant}$$

*Energy*:

$$0 = k\frac{d^2 T}{dy^2} \quad \text{whence} \quad T = \bar{T} + Ay$$

*b*. The boundary conditions are;

$$v_z(\pm B, y) = 0 \qquad v_z \text{ is an odd function of } y \qquad T(x,0) - \bar{T} = 0.$$

*c*. To get the velocity profiles we rearrange the equation of motion as follows:

$$\frac{\partial^2 v_z}{\partial x^2} = -\frac{\bar{\rho}\bar{\beta} g A}{\mu} y$$

This may be integrated to give:

$$v_z(x,y) = -\frac{\bar{\rho}\bar{\beta} g A}{2\mu} x^2 y + C_1 x + C_2 \quad \text{or} \quad v_z(x,y) = \frac{\bar{\rho}\bar{\beta} g A B^2}{2\mu}\left(1 - \left(\frac{x}{B}\right)^2\right) y$$

*d*. For water at 20°C , $\bar{\rho} = 0.9982$ g/cc and $\bar{\beta} = 0.00021$ (°C)$^{-1}$. The maximum velocity will occur at $x = 0$ and $y = W$, so that

$$v_{z,\max} = \frac{\overline{\rho}\overline{\beta}gAB^2W}{2\mu}$$

Hence the corresponding temperature gradient will be

$$A = \frac{2\mu v_{z,\max}}{\overline{\rho}\overline{\beta}gB^2W}$$

$$= \frac{2\left(1.0019\times10^{-2}\ \text{g/cm}\cdot\text{s}\right)\left(\dfrac{(0.001)(2)\ \text{cm}}{3600\ \text{s}}\right)}{\left(0.99823\ \text{g/cm}^3\right)\left(0.00021\ ^\circ\text{C}^{-1}\right)\left(980.7\ \text{cm/s}^2\right)(0.01\ \text{cm})^2(0.2\ \text{cm})}$$

$$= 0.271\ ^\circ\text{C/cm}$$

**11C.3 Tangential annular flow of a highly viscous liquid**

First, rewrite Eq. 11.4-13 to eliminate $N$ in favor of the Brinkman number Br, which appears in Eq. 10.4-9:

$$\Theta = \left(1 - \frac{\ln \xi}{\ln \kappa}\right) + \mathrm{Br}\frac{\kappa^4}{\left(1-\kappa^2\right)^2}\left[\left(1 - \frac{1}{\xi^2}\right) - \left(1 - \frac{1}{\kappa^2}\right)\frac{\ln \xi}{\ln \kappa}\right]$$

To show that this reduces to Eq. 10.4-9 in the limit of a very thin annulus, begin by taking the limit of the term that does not contain Br and then we treat the term containing Br.

<u>Term without Br:</u>

We let $\kappa = 1 - \varepsilon$ and $\xi = 1 - \varepsilon(1-\eta)$, where $\eta$ is the $x/b$ of §10.4. Then using Eq. C.3-2 we find

$$1 - \frac{\ln \xi}{\ln \kappa} = 1 - \frac{-\left[\varepsilon(1-\eta) + \frac{1}{2}\varepsilon^2(1-\eta)^2 + \cdots\right]}{-\left[\varepsilon + \frac{1}{2}\varepsilon^2 + \cdots\right]} = 1 - (1-\eta) + \mathrm{O}(\varepsilon) = \eta + \mathrm{O}(\varepsilon)$$

In the limit of vanishing $\varepsilon$, this leads to the last term in Eq. 10.4-9.

<u>Term with Br:</u>

The coefficient of Br is now

$$\frac{(1-\varepsilon)^4}{\left(2\varepsilon - \varepsilon^2\right)^2}\left[\left(1 - \frac{1}{\left[1-\varepsilon(1-\eta)\right]^2}\right) - \left(1 - \frac{1}{(1-\varepsilon)^2}\right)\frac{\varepsilon(1-\eta) + \frac{1}{2}\varepsilon^2(1-\eta)^2 + \cdots}{\varepsilon + \frac{1}{2}\varepsilon^2 + \cdots}\right]$$

When everything is expanded in terms of powers of $\varepsilon$, we find

$$\frac{1}{4\varepsilon^2}\frac{1 - 4\varepsilon + 6\varepsilon^2 + \cdots}{1 - \varepsilon + \frac{1}{4}\varepsilon^2 + \cdots}\left[\left(1 - \left[1 + 2\varepsilon(1-\eta) + 3\varepsilon^2(1-\eta)^2 + \cdots\right]\right) - \left(1 - \left[1 + 2\varepsilon + 3\varepsilon^2 + \cdots\right]\right)\left(\left[1-\eta\right] + \tfrac{1}{2}\varepsilon\left(-\eta + \eta^2\right) + \cdots\right)\right]$$

Simplification of this expression leads to

$$\frac{1}{4\varepsilon^2}\left(1-3\varepsilon+\tfrac{11}{4}\varepsilon^2+\cdots\right)\Big[\left(-2\varepsilon(1-\eta)-3\varepsilon^2(1-\eta)^2\right)$$
$$-\left(-2\varepsilon-3\varepsilon^2-\cdots\right)\left([1-\eta]+\tfrac{1}{2}\varepsilon\left(-\eta+\eta^2\right)+\cdots\right)\Big]$$

Then cancellation of some terms gives

$$\frac{1}{4\varepsilon^2}(1-3\varepsilon+\cdots)\left[-3\varepsilon^2(1-\eta)^2-\left\{3\varepsilon^2(1-\eta)-\varepsilon^2\left(-\eta+\eta^2\right)+\cdots\right\}\right]$$
$$=\frac{1}{4}(1-3\varepsilon+\cdots)\left[-3\left(-\eta+\eta^2\right)+\left(-\eta+\eta^2\right)+\cdots\right]$$
$$=\frac{1}{2}\left(\eta-\eta^2\right)$$

This is in agreement with the term in Eq. 10.4-9 that is multiplied by Br.

**11C.4 Heat conduction with variable thermal conductivity**

First we obtain an expression for the gradient of $F$:

$$\begin{aligned}\nabla F &= \nabla\left(\int k dT + \text{constant}\right)\\ &= \int (\nabla k) dT\\ &= \int \frac{dk}{dT} \nabla T dT\end{aligned}$$

The $\nabla$ operator can be taken inside the integral, since it involves only differentiation with respect to position coordinates and therefore commutes with the integration over $T$. Since $k$ depends solely on $T$ (which may in turn depend on the position coordinates), we must differentiate with respect to $T$ and then perform the gradient operation on $T$, as shown above.

Next, since $\nabla T$ depends on position coordinates, and not on $T$, the $\nabla T$ may be removed from the integral sign

$$\nabla F = \nabla T \int \frac{dk}{dT} dT$$

When the integration is performed, we get

$$\nabla F = k \nabla T$$

We now form the Laplacian of $F$

$$\nabla^2 F = (\nabla \cdot \nabla F) = (\nabla \cdot k \nabla T)$$

But $(\nabla \cdot k \nabla T) = 0$ by the equation of energy. Therefore, we have finally

$$\nabla^2 F = 0$$

**11C.5 Effective thermal conductivity of a solid with spherical inclusions**

*a*. At very large distances from the region containing the inclusions, the coordinates of the various inclusions $(r,\theta)$ will not be very different from one another. Furthermore, if the density of the inclusions is small, the effect of the various inclusions will be additive. Therefore to get the temperature field far from the region containing the inclusions, we can write

$$T_0(r,\theta)-T^{\circ}=\left[1-n\frac{k_1-k_0}{k_1+2k_0}\left(\frac{R}{r}\right)^3\right]Ar\cos\theta$$

in which $n$ is the number of inclusions, each with a thermal conductivity $k_1$. This is the equation that the describes the system in Fig. 11C.5(a).

*b*. For the system in Fig. 11C.5(b), we can apply Eq. 11B.8-2 directly

$$T_0(r,\theta)-T^{\circ}=\left[1-\frac{k_{\text{eff}}-k_0}{k_{\text{eff}}+2k_0}\left(\frac{R'}{r}\right)^3\right]Ar\cos\theta$$

regarding the shaded sphere as a hypothetical material of thermal conductivity $k_{\text{eff}}$.

*c*. Next we relate the volume of the inclusions in the true system to the effective volume of the inclusions in the equivalent system: $\frac{4}{3}\pi R^3 n=\frac{4}{3}\pi R'^3\phi$. Hence $R'^3=(n/\phi)R^3$.

*d*. We can now equate the right sides of the above two equations for the temperature profiles far from the origin

$$n\frac{k_1-k_0}{k_1+2k_0}\left(\frac{R}{r}\right)^3=\frac{k_{\text{eff}}-k_0}{k_{\text{eff}}+2k_0}\left(\frac{R}{r}\right)^3\frac{n}{\phi}$$

We can now solve for the effective thermal conductivity, and express the ratio $k_{\text{eff}}/k_0$ as 1 + (deviation resulting from the inclusions). This is exactly Eq. 9.6-1.

**11C.7 Effect of surface-tension gradients on a falling film**

*a*. If we let I be the gas phase and II be the liquid phase, then the contribution from Eq. 11C.6-4 to the stress component $\tau_{xz}$ will be the *z*-component of

$$\left[\delta_x \cdot \tau^{\mathrm{II}}\right] = \nabla^s \sigma \quad \text{or} \quad \tau_{xz}^{\mathrm{II}} = \frac{\partial \sigma}{\partial z}$$

Hence, in Eq. 2.2-13 has to be replaced by

$$\tau_{xz} = (\rho g \cos\beta)x + A$$

When this is combined with Newton's law of viscosity, Eq. 2.2-14, we get

$$-\mu \frac{dv_z}{dx} = (\rho g \cos\beta)x + A$$

When this is integrated, and the no-slip boundary condition at the wall is used, we obtain the velocity profile as follows:

$$v_z = \frac{\rho g \delta^2 \cos\beta}{2\mu}\left[1 - \left(\frac{x}{\delta}\right)^2\right] + \frac{A\delta}{\mu}\left(1 - \frac{x}{\delta}\right)$$

*b*. The mass rate of flow in the film is

$$\begin{aligned} w &= \int_0^W \int_0^\delta \rho v_z dx dy = \rho W \delta \int_0^1 v_z d\xi \\ &= \rho W \delta \left[\int_0^1 \frac{\rho g \delta^2 \cos\beta}{2\mu}(1-\xi^2)d\xi + \frac{A\delta}{\mu}\int_0^1 (1-\xi)d\xi\right] \\ &= \frac{\rho^2 W g \delta^3}{3\mu} + \frac{A\rho W \delta^2}{2\mu} \end{aligned}$$

**11D.1 Equation of change for entropy**

*a*. For the volume element $\Delta x \Delta y \Delta z$ we write a balance equation (not a conservation equation) for the entropy:

$$\frac{\partial}{\partial t}\left(\rho \hat{S} \Delta x \Delta y \Delta z\right) = \left(\rho \hat{S} v_x\right)\Big|_x \Delta y \Delta z - \left(\rho \hat{S} v_x\right)\Big|_{x+\Delta x} \Delta y \Delta z + \cdots$$
$$+ s_x\big|_x \Delta y \Delta z - s_x\big|_{x+\Delta x} \Delta y \Delta z + \cdots + g_S \Delta x \Delta y \Delta z$$

Here the dots indicate the additional terms associated with the transport in the $y$ and $z$ directions. Then after dividing by $\Delta x \Delta y \Delta z$ and taking the limit as the volume element goes to zero, we get:

$$\frac{\partial}{\partial t}\rho \hat{S} = -\left(\frac{\partial}{\partial x}\rho \hat{S} v_x + \cdots\right) - \left(\frac{\partial}{\partial x} s_x + \cdots\right) + g_S$$

which is equivalent to Eq.11D.1-1.

*b* . Along a streamline we can write (using Eq. 11.2-2)

$$\frac{D\hat{U}}{Dt} = T\frac{D\hat{S}}{Dt} - p\frac{D\hat{V}}{Dt} \quad \text{or} \quad \frac{1}{\rho}\left[-(\nabla \cdot \mathbf{q}) - p(\nabla \cdot \mathbf{v}) - (\boldsymbol{\tau}:\nabla \mathbf{v})\right] = T\frac{D\hat{S}}{Dt} - p\frac{D}{Dt}\frac{1}{\rho}$$

The use of the equation of continuity and subsequent multiplication by $\rho/T$ leads to Eq. 11D.1-3.

*c*. Insertion of $\mathbf{s} = \mathbf{q}/T$ into Eq. 11D.1-3 leads to

$$\rho\frac{D\hat{S}}{Dt} = -\frac{1}{T}(\nabla \cdot T\mathbf{s}) - \frac{1}{T}(\boldsymbol{\tau}:\nabla \mathbf{v}) = -(\nabla \cdot \mathbf{s}) - \left(\frac{\mathbf{s}}{T}.\nabla T\right) - \frac{1}{T}(\boldsymbol{\tau}:\nabla \mathbf{v})$$
$$= -(\nabla \cdot \mathbf{s}) - \frac{1}{T^2}(\mathbf{q} \cdot \nabla T) - \frac{1}{T}(\boldsymbol{\tau}:\nabla \mathbf{v})$$

In the last form, the right side of the equation has the same form as Eq. 11D.1-2, so that the last two terms can be identified as the rate of entropy production (as displayed in Eq. 11D.1-4).

**11D.2 Viscous heating in laminar tube flow**

*a.* We introduce the following reduced variables:

$$\xi = \frac{r}{R} \qquad \zeta = \frac{\alpha z}{R^2 v_{\max}} \qquad \Theta = \frac{k\left(T - T^{\circ}\right)}{\mu v_{\max}^2}$$

Here $T^{\circ}$ is the wall temperature (and entry temperature) for the isothermal wall problem, and the entry temperature for the adiabatic (insulated) wall problem. Then Eq. 11B.2-1 may be written in dimensionless form as

$$\left(1-\xi^2\right)\frac{\partial\Theta}{\partial\zeta} = \frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\Theta}{\partial\xi}\right) + 4\xi^2 \qquad (*)$$

The boundary conditions are:

Isothermal wall: at $\zeta = 0$, $\Theta = 0$
at $\xi = 0$, $\partial\Theta/\partial\xi = 0$
at $\xi = 1$, $\Theta = 0$

Insulated wall: at $\zeta = 0$. $\Theta = 0$
at $\xi = 0$, $\partial\Theta/\partial\xi = 0$
at $\xi = 1$, $\partial\Theta/\partial\xi = 0$

The solution to the differential equation will be written, for both cases, as a sum of two functions: $\Theta = \Theta_1 + \Theta_2$. Here the function $\Theta_1$ is that derived in Problem 11B.2:

Isothermal wall: $\Theta_1 = \frac{1}{4}\left(1-\xi^4\right)$

Insulated wall: $\Theta_1 = 4\zeta + \left(\xi^2 - \frac{1}{2}\xi^4\right)$

When these expressions are put into the partial differential equation (*), we then get for *both* cases

Isothermal and insulated wall: $\left(1-\xi^2\right)\frac{\partial\Theta_2}{\partial\zeta} = \frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\Theta_2}{\partial\xi}\right)$

This problem is solved by the method of separation of variables, and the solution is of the form:

$$\Theta_2(\xi,\zeta) = -\sum_{i-1}^{\infty} B_i \phi_i(\xi)\psi_i(\zeta)$$

in which the $B_i$ are constants, and the functions $\phi_i(\xi)$ and $\psi_i(\zeta)$ must satisfy the following differential equations:

$$\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{d\phi_i}{d\xi}\right) + a_i\left(1-\xi^2\right)\phi_i \quad (**) \qquad\qquad \frac{d\psi_i}{d\zeta} = -a_i\psi_i$$

The $a_i$ are the eigenvalues of the problem, obtained when the solutions to the $\phi_i$ equation are required to satisfy the boundary conditions. The solution to the $\psi_i$ are proportional to $\exp(-a_i\zeta)$. The solution to the $\phi_i$ equation has to be done by a power-series expansion, and we return to that shortly. For the moment we remark that the solutions to the $\phi_i$ equation must satisfy an orthogonality relation:

$$\int_0^1 \phi_i\phi_j\left(1-\xi^2\right)\xi d\xi = 0 \qquad \text{for } i \neq j$$

The final temperature distribution expressions are then

Isothermal wall: $$\Theta = \tfrac{1}{4}\left(1-\xi^4\right) - \sum_{i=1}^{\infty} B_i\phi_i(\xi)\exp(-a_i\zeta)$$

Insulated wall: $$\Theta = 4\zeta + \left(\xi^2 - \tfrac{1}{2}\xi^4\right) - \sum_{i=1}^{\infty} B_i\phi_i(\xi)\exp(-a_i\zeta)$$

From these solutions, it can be seen that the solutions obtained in Problem 11B.2 are just the limiting solutions for large values of the dimensionless axial variable $\zeta$.

The constants $B_i$ are obtained by writing the above equations for $\zeta = 0$, then multiplying by $\phi_j\left(1-\xi^2\right)\xi$, and next integrating over $\xi$ from 0 to 1. Doing that we get:

Isothermal wall: $$B_i = \frac{1}{4}\frac{\int_0^1\left(1-\xi^4\right)\phi_j\left(1-\xi^2\right)\xi d\xi}{\int_0^1\phi_i^2\left(1-\xi^2\right)\xi d\xi}$$

Insulated wall: $$B_i = \frac{\int_0^1\left(\xi^2-\frac{1}{2}\xi^4\right)\phi_j\left(1-\xi^2\right)\xi d\xi}{\int_0^1\phi_i^2\left(1-\xi^2\right)\xi d\xi}$$

The above results were first given by H. C. Brinkman, *Appl. Sci. Res.*, **A2**, 120-124 (1951).

The eigenfunctions $\phi_i(\xi)$ that satisfy (**) can be written as a series expansion:

$$\phi_i(\xi) = \sum_{k=0}^{\infty} b_{ik}\xi^k$$

The coefficient $b_{i0}$ may be arbitrarily chosen to be unity. The next coefficient, $b_{i1}$, must be chosen to be zero in order to satisfy the boundary condition at the tube axis. Substitution of the above series solution into (**) gives the following recursion formula:

$$b_{ik} = -\frac{a_i}{k^2}\left(b_{i,k-2} - b_{i,k-5}\right)$$

Therefore, all of the $b_{ik}$ can be expressed in terms of the eigenvalues $a_i$ as follows:

| | | | | |
|---|---|---|---|---|
| $b_{i0} = 1$ | $b_{i2} = -\frac{1}{4}a_i$ | $b_{i4} = +\frac{1}{64}a_i^2$ | $b_{i6} = +\frac{1}{2304}a_i^3$ | etc. |
| $b_{i1} = 0$ | $b_{i3} = 0$ | $b_{i5} = +\frac{1}{25}a_i$ | $b_{i7} = -\frac{29}{4900}a_i^2$ | |

The eigenvalues $a_i$ are determined from the boundary conditions at $\xi = 1$, which require that:

Isothermal wall: $\sum_{k=0}^{\infty} b_{ik} = 0$

Insulated wall: $\sum_{k=0}^{\infty} k b_{ik} = 0$

The left side of each of these equations is a polynomial in $a_i$. Setting the polynomial expression equal to zero gives an algebraic equation which has to be solved for the infinite set of $a_i$. This is clearly a very tedious process. The first few eigenvalues have been calculated by Brinkman in the reference cited above. Also given there are some sample temperature profiles.

*b*. The power-law fluid can be handled by the same method used for the Newtonian fluid, given in part (*a*). The reference to Bird, cited on p. 373, gives the details and numerical results.

**11D.3 Derivation of the energy equation using integral theorems**

*a*. For an arbitrary volume $V$ fixed in space, we can write the total energy balance as

$$\frac{d}{dt}\int_V \left(\rho\hat{U} + \tfrac{1}{2}\rho v^2\right)dV = -\int_S (\mathbf{n}\cdot\mathbf{e})dS + \int_V (\mathbf{v}\cdot\mathbf{g})dV$$

$$\int_V \frac{\partial}{\partial t}\left(\rho\hat{U} + \tfrac{1}{2}\rho v^2\right)dV = -\int_V (\nabla\cdot\mathbf{e})dV + \int_V (\mathbf{v}\cdot\mathbf{g})dV$$

To get the second form, the Gauss divergence theorem was used on the right side, and on the left side the time derivative is taken inside the integral (because the volume element is fixed). Then, since the volume $V$ was chosen arbitrarily, the integral signs may be removed to get Eq. 11.1-6.

*b*. For an arbitrary moving volume element $V(t)$ of fluid, we may write the equation of conservation of energy as follows:

$$\frac{d}{dt}\int_{V(t)} \left(\rho\hat{U} + \tfrac{1}{2}\rho v^2\right)dV = -\int_{S(t)} (\mathbf{n}\cdot\mathbf{q})dS - \int_{S(t)} (\mathbf{n}\cdot[\boldsymbol{\pi}\cdot\mathbf{v}])dS + \int_{V(t)} (\mathbf{v}\cdot\mathbf{g})dV$$

This accounts for the heat transported into $V$ across the surface, and the work done on $V$ by the surface forces. There is no term accounting for convective transport across $S$, since the surface itself is moving with the fluid velocity. Next, use the Leibnitz formula on the left side to get

$$\frac{d}{dt}\int_{V(t)} \left(\rho\hat{U} + \tfrac{1}{2}\rho v^2\right)dV = \int_{V(t)} \frac{\partial}{\partial t}\left(\rho\hat{U} + \tfrac{1}{2}\rho v^2\right)dV + \int_{S(t)} \left(\rho\hat{U} + \tfrac{1}{2}\rho v^2\right)(\mathbf{n}\cdot\mathbf{v}_S)dS$$

$$\int_{V(t)} \frac{\partial}{\partial t}\left(\rho\hat{U} + \tfrac{1}{2}\rho v^2\right)dV + \int_{S(t)} \left(\mathbf{n}\cdot\left(\rho\hat{U} + \tfrac{1}{2}\rho v^2\right)\mathbf{v}_S\right)dS$$

The surface velocity $\mathbf{v}_S$ for a moving blob of fluid is identical to the fluid velocity $\mathbf{v}$. When the term containing $\mathbf{v}_S = \mathbf{v}$ is transferred to the right side and the Gauss divergence theorem is used, we get the energy balance in the following form

$$\int_{V(t)} \frac{\partial}{\partial t}\left(\rho\hat{U} + \tfrac{1}{2}\rho v^2\right)dV = -\int_{V(t)} \left(\nabla \cdot \left(\rho\hat{U} + \tfrac{1}{2}\rho v^2\right)\mathbf{v}\right)dV$$
$$- \int_{V(t)} (\nabla \cdot \mathbf{q})dV - \int_{V(t)} (\nabla \cdot [\boldsymbol{\pi} \cdot \mathbf{v}])dV + \int_{V(t)} (\mathbf{v} \cdot \mathbf{g})dV$$

Since the element of volume was completely arbitrary, we may now remove the integral signs through the entire equation and obtain Eq. 11.1-7.

**12A.1 Unsteady-state heat conduction in an iron sphere**

*a*. The thermal diffusivity of the sphere is given by Eq. 9.1-8:

$$\alpha = \frac{k}{\rho \hat{C}_p} = \frac{30}{(436)(0.12)} = 0.573 \text{ ft}^2 / \text{hr}$$

*b*. The center temperature is to be 128°F; hence

$$\frac{T_{\text{ctr}} - T_0}{T_1 - T_0} = \frac{128 - 70}{270 - 70} = 0.29$$

Then, from Fig. 12.1-3, $\alpha t / R^2 = 0.1$, and

$$t = 0.1\left(\frac{R^2}{\alpha}\right) = 0.1\left(\frac{(1/24)^2}{0.573}\right) = 3.03 \times 10^{-4} \text{ hrs} = 1.1 \text{ s}$$

*c*. By equating the dimensionless times, we get

$$\frac{\alpha_1 t_1}{R_1^2} = \frac{\alpha_2 t_2}{R_2^2}$$

or

$$\alpha_2 = \alpha_1\left(\frac{t_1}{t_2}\right) = 0.573\left(\frac{1}{2}\right) = 0.287$$

*d*. The partial differential equation from which Fig. 12.1-3 was constructed is

$$\frac{\partial T}{\partial t} = \alpha \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right)$$

**12A.2 Comparison of the two slab solutions for short times**

According to Figure 12.1-1, at $\alpha t/b^2 = 0.01$ and $y/b = 0.9$

$$\frac{T - T_0}{T_1 - T_0} \approx 0.46$$

where $y$ is the distance from the mid-plane of the slab.

Next we use Fig. 4.1-1, which can be interpreted as a plot of $(T - T_0)/(T_1 - T_0)$ vs $y'/\sqrt{4\alpha t}$, where $y' = b - y$ is the distance from the wall. We then get

$$\frac{y'}{\sqrt{4\alpha t}} = \frac{1}{2}\frac{(1-0.9)}{\sqrt{\alpha t/b^2}} = \frac{1}{2}$$

Then from Fig. 4.1-1 we find

$$\frac{T - T_0}{T_1 - T_0} \approx 0.48$$

Hence the use of the combination of variables solution introduces an error of about 4%. Smaller errors occur at smaller values of the dimensionless time $\alpha t/b^2$.

**12A.3 Bonding with a thermosetting adhesive**

The dimensionless temperature at the time of bonding is

$$\frac{T_{\text{ctr}} - T_0}{T_1 - T_0} = \frac{160 - 20}{220 - 20} = 0.70$$

This occurs very nearly at a dimensionless temperature $\alpha t/b^2 = 0.6$. Hence the time required is

$$t = (0.6)\left(\frac{b^2}{\alpha}\right) = (0.6)\left(\frac{(0.77)^2}{4.2 \times 10^{-3}}\right) = 85\text{s}$$

**12A.4 Quenching of a steel billet**

The thermal diffusivity of the steel billet is

$$\alpha = \frac{k}{\rho \hat{C}_p} = \frac{(25)\left(4.1365 \times 10^{-3}\right)}{(7.7)(0.12)} = 0.112 \text{ cm}^2/\text{s}$$

The dimensionless time is then

$$\frac{\alpha t}{R^2} = \frac{(0.112)(5 \times 60)}{(6 \times 2.54)^2} = 0.145$$

From Fig. 12.1-2, the dimensionless center-line temperature is about 0.31. Therefore

$$\frac{T_{\text{ctr}} - T_0}{T_1 - T_0} = 0.31$$

We can now solve for the centerline temperature

$$T_{\text{ctr}} = 0.31\left(T_1 - T_0\right) + T_0 = 0.31(200 - 1000) + 1000 \approx 750°\text{F}$$

**12A.5 Measurement of thermal diffusivity from amplitude of temperature oscillations**

*a*. From Eq. 12.1-40, we see that the amplitude of the temperature oscillations is the following function of $y$:

$$A(y) = \frac{q_0}{k}\sqrt{\frac{\alpha}{\omega}}\exp\left(-\sqrt{\frac{\omega}{2\alpha}}y\right)$$

Then the ratio of amplitudes at two different distances from the plane $y = 0$ is

$$\frac{A_1}{A_2} = \exp\left(-\sqrt{\frac{\omega}{2\alpha}}(y_1 - y_2)\right)$$

The logarithm of this expression is

$$\ln\left(\frac{A_1}{A_2}\right) = -\sqrt{\frac{\omega}{2\alpha}}(y_1 - y_2) = +\sqrt{\frac{\omega}{2\alpha}}(y_2 - y_1)$$

This can be solved for the thermal diffusivity to obtain

$$\alpha = \frac{\omega}{2}\left(\frac{y_2 - y_1}{\ln(A_1/A_2)}\right)^2 = \frac{2\pi\nu}{2}\left(\frac{y_2 - y_1}{\ln(A_1/A_2)}\right)^2 = \pi\nu\left(\frac{y_2 - y_1}{\ln(A_1/A_2)}\right)^2$$

where $\nu$ is the frequency in cycles per second.

*b*. Inserting the numerical values given:

$$\alpha = (3.1416)(0.0030)\left(\frac{6.15}{\ln 6.05}\right)^2 = 0.110 \ \text{cm}^2/\text{s}$$

**12A.6 Forced convection from a sphere in creeping flow**

*a*. The dimensions of the quantities in Eq. 12.4-34 can be obtained from the table on pp. 872-876, thus:

$$Q[=]\frac{ML^2}{t^3} \qquad D[=]L \qquad T[=]T \qquad k[=]\frac{ML}{t^3T}$$

$$\mathrm{Re}[=]\ \text{dimensionless} \qquad \mathrm{Pr}[=]\ \text{dimensionless}$$

To check for dimensional consistency:

$$\frac{ML^2}{t^3}[=]L\cdot T\cdot\frac{ML}{t^3T}$$

*b*. We now write Eq. 12.4-34 in terms of the Péclet number:

$$Q=\left(\pi D^2\right)\left(T_0-T_\infty\right)\left(\frac{k}{D}\right)\left[\frac{(3\pi)^{2/3}}{2^{7/3}\Gamma\left(\frac{4}{3}\right)}\right]\left(\frac{Dv_\infty\rho\hat{C}_p}{k}\right)^{1/3}$$
$$=(\pi D)\left(T_0-T_\infty\right)k\left[\frac{(3\pi)^{2/3}}{2^{7/3}\Gamma\left(\frac{4}{3}\right)}\right]\left(\frac{Dv_\infty\rho\hat{C}_p}{k}\right)^{1/3}$$

In this form, the viscosity does not appear in the expression. We now fill in the numerical values, using c.g.s. units throughout:

$$Q=\pi(0.1)(50)\left(3\times10^{-4}\right)\left[\frac{4.462}{(5.04)(0.89297)}\right]\left(\frac{(0.1)(1.0)(0.9)(0.45)}{3\times10^{-4}}\right)^{1/3}$$
$$=0.0240\ \text{cal/s}$$

**12B.1 Measurement of thermal diffusivity in an unsteady-state experiment**

We first tabulate the midplane temperatures as a function of time, and then convert them to the dimensionless midplane temperatures. Having done this, we consult Figure 12.1-1. We find the value of the dimensionless midplane temperature on the ordinate of the graph, and from this we can read off the value of $\alpha t/b^2$. This quantity then can be multiplied by $b^2$ and divided by $t$ to get the thermal diffusivity. The calculations may be summarized as follows (working to two significant figures):

| $t$ | $T_{\text{midplane}}$ | $\dfrac{T_{\text{midplane}} - T_0}{T_1 - T_0}$ | $\alpha t/b^2$ | $\alpha$ |
|---|---|---|---|---|
| 0 | 20.0 | 0 | | |
| 120 | 24.4 | 0.220 | 0.20 | $1.50\times10^{-3}$ |
| 240 | 30.5 | 0.525 | 0.40 | $1.50\times10^{-3}$ |
| 360 | 34.2 | 0.710 | 0.60 | $1.50\times10^{-3}$ |
| 480 | 36.5 | 0.825 | 0.80 | $1.50\times10^{-3}$ |
| 600 | 37.8 | 0.890 | 1.00 | $1.50\times10^{-3}$ |

Thus, the experimental data give us $\alpha = 1.50\times10^{-3}\ \text{cm}^2/\text{s}$.

Next, the thermal conductivity can be obtained as follows:

$$k = \alpha\rho\hat{C}_p = \left(1.50\times10^{-3}\ \text{cm}^2/\text{s}\right)\left(1.50\ \text{g}/\text{cm}^3\right)\left(0.365\ \text{cal}/\text{g}\cdot\text{C}\right)$$
$$= 8.21\times10^{-4}\ \text{cal}/\text{s}\cdot\text{cm}\cdot\text{C}$$

or

$$k = \left(8.21\times10^{-4}\right)\left(2.418\times10^{2}\right) = 0.20\ \text{Btu}/\text{hr}\cdot\text{ft}\cdot\text{F}$$

The conversion factor was taken from Table F.3-5.

**12B.2 Two-dimensional forced convection with a line heat source**

*a*. The energy equation simplifies to

$$\rho\hat{C}_p v_x \frac{\partial T}{\partial x} = k\frac{\partial^2 T}{\partial y^2} \qquad \text{where } v_x = v_0\text{, a constant}$$

The boundary conditions have the following meanings:

Eq. 12B.2-1: Far from the wire, the temperature of the fluid is unchanged from its value, $T_\infty$, for $x<0$

Eq. 12B.2-2: The approaching fluid is all at the temperature $T_\infty$--that is, there is no heat conduction upstream

Eq. 12B.2-3: The entering into the fluid from the wire must appear somewhere in the cross-section

In addition, we need a statement that $T(x,y)=T(x,-y)$--that is, symmetry about the plane $y=0$, which contains the wire.

*b*. The postulated solution in Eq. 12B.2-4 states that the temperature profile at any value of $x$ will be geometrically similar to the profile at any other value of $x$. When this expression for $T$ is inserted into Eq. 12B.2-3 we get:

$$\rho\hat{C}_p v_0 \int_{-\infty}^{+\infty} f(x)g(\eta)\big|_x \delta(x)d\eta = Q/L$$

When the integral on $\eta$ is evaluated, it is found that $f(x)\delta(x)=C_1$, a constant.

When the expression in Eq. 12B.2-4 is inserted into the energy equation, we get

$$\frac{v_0}{\alpha}\frac{\partial}{\partial x}[f(x)g(\eta)] = \frac{\partial^2}{\partial y^2}[f(x)g(\eta)]\text{, or}$$

$$\frac{v_0}{\alpha}\frac{\partial}{\partial x}\left[\frac{1}{\delta(x)}g\left(\frac{y}{\delta(x)}\right)\right] = \frac{\partial^2}{\partial y^2}\left[\frac{1}{\delta(x)}g\left(\frac{y}{\delta(x)}\right)\right]\text{, or}$$

$$\frac{v_0}{\alpha}\left[-\frac{1}{\delta^2}\frac{d\delta}{dx}g - \frac{\eta}{\delta^2}\frac{dg}{d\eta}\frac{d\delta}{dx}\right] = \frac{1}{\delta}\frac{d^2g}{d\eta^2}\frac{1}{\delta^2}$$

Multiplication by $\delta^3$ then gives Eq. 12B.2-5.

*c*. When the quantity in brackets in Eq. 12B.2-5 is set equal to 2, we get a first-order separable equation for $\delta(x)$--the thermal boundary-layer thickness--which is integrated from 0 to $x$ to give $\delta(x)=\sqrt{4\alpha x/v_0}$. The main reason for setting the bracketed quantity equal to 2 is that the solution to the equation for $g(\eta)$ comes out to be a little simpler.

*d*. The equation for $g(\eta)$ can be written as

$$\frac{d}{d\eta}\left(\frac{dg}{d\eta}\right)=-2\frac{d}{d\eta}(\eta g)$$

which has the solution

$$\frac{dg}{d\eta}=-2\eta g+C_2$$

Since $g(\eta)$ is symmetric about $\eta=0$, we know that $dg/d\eta=0$ at $\eta=0$; therefore, this last equation tells us that $C_2$. A further integration leads to a Gaussian function

$$g=C_3e^{-\eta^2}$$

We do not evaluate $C_3$ but instead "absorb" into the constant $C_1$.

*e*. We can now evaluate $C_1$ by substituting the postulated temperature profile into Eq. 12B.2-3 to get

$$\rho\hat{C}_p v_0 C_1\int_{-\infty}^{+\infty}g(\eta)d\eta=Q/L \qquad \text{or} \qquad \rho\hat{C}_p v_0 C_1\int_{-\infty}^{+\infty}e^{-\eta^2}d\eta=Q/L$$

which gives

$$C_1=\frac{Q/L}{\rho\hat{C}_p v_0\sqrt{\pi}}$$

Which then completes the determination of the temperature distribution in the wake of the wire.

**12B.3 Heating of a wall (constant heat flux)**

*a*. The equation to be solved for $q_y(y,t)$ is

$$\frac{\partial q_y}{\partial t} = \alpha \frac{\partial^2 q_y}{\partial y^2} \qquad \text{with } q_y(y,0)=0,\ q_y(0,t)=q_0, \text{ and } q_y(\infty,t)=0$$

This is the same mathematical problem that is solved in Example 12.1-1, so that we can write down at once

$$\frac{q_y}{q_0} = \frac{2}{\sqrt{\pi}} \int_{y/\sqrt{4\alpha t}}^{\infty} e^{-u^2} du$$

To get the temperature we use Eq. 12.1-39

$$T - T_0 = \frac{q_0}{k} \int_y^{\infty} \frac{q_y(\bar{y},t)}{q_0} d\bar{y} = \frac{q_0}{k} \frac{2}{\sqrt{\pi}} \int_y^{\infty} \int_{\bar{y}/\sqrt{4\alpha t}}^{\infty} e^{-u^2} du d\bar{y}$$

$$= \frac{q_0}{k} \frac{2}{\sqrt{\pi}} \sqrt{4\alpha t} \int_{y/\sqrt{4\alpha t}}^{\infty} \int_Y^{\infty} e^{-u^2} du dY = \frac{q_0}{k} \frac{2}{\sqrt{\pi}} \sqrt{4\alpha t} \int_{y/\sqrt{4\alpha t}}^{\infty} \int_{y/\sqrt{4\alpha t}}^{u} e^{-u^2} dY du$$

$$= \frac{q_0}{k} \frac{2}{\sqrt{\pi}} \sqrt{4\alpha t} \int_{y/\sqrt{4\alpha t}}^{\infty} e^{-u^2} \left[ u - \frac{y}{\sqrt{4\alpha t}} \right] du$$

$$= \frac{q_0}{k} \left( \sqrt{\frac{4\alpha t}{\pi}} \int_{y/\sqrt{4\alpha t}}^{\infty} e^{-u^2} u du - \frac{2y}{\sqrt{\pi}} \int_{y/\sqrt{4\alpha t}}^{\infty} e^{-u^2} du \right)$$

When the first integral is evaluated, we get Eq. 12B.3-1. In the above, to get the second line, we made a change of variables and then interchanged the order of integration; in the third line, we performed the inner integration.

*b*. Some intermediate steps in showing that the partial differential equation is satisfied: We write $k(T-T_0)/q_0 = F - G$. Then

$$\frac{\partial F}{\partial t} = \left( \sqrt{\frac{\alpha}{\pi t}} + \sqrt{\frac{1}{4\pi\alpha t}} \frac{y^2}{t} \right) e^{-y^2/4\alpha t}; \quad \alpha \frac{\partial^2 F}{\partial y^2} = \left( \sqrt{\frac{1}{4\pi\alpha t}} \frac{y^2}{t} - \sqrt{\frac{\alpha}{\pi t}} \right) e^{-y^2/4\alpha t}$$

$$\frac{\partial G}{\partial t} = \left( \sqrt{\frac{1}{4\pi\alpha t}} \frac{y^2}{t} \right) e^{-y^2/4\alpha t}; \quad \alpha \frac{\partial^2 G}{\partial y^2} = \left( -2\sqrt{\frac{\alpha}{\pi t}} + \sqrt{\frac{1}{4\pi\alpha t}} \frac{y^2}{t} \right) e^{-y^2/4\alpha t}$$

**12B.4 Heat transfer from a wall to a falling film (short contact time limit)**

*a*. From Eq. 2.2-18, we get

$$v_z = v_{z,\max}\left[1-(x/\delta)^2\right] = v_{z,\max}\left[1-\left(1-(y/\delta)\right)^2\right]$$
$$= v_{z,\max}\left[1-1+2(y/\delta)+(y/\delta)^2\right] \rightarrow 2v_{z,\max}(y/\delta)$$

this last expression is good in the vicinity of the wall, where the quadratic term can be neglected.

*b*. Equation 12B.4-2 presupposes that the heat conduction in the $z$ direction can be neglected relative to the heat convection in the $z$ direction. In addition, laminar, nonrippling flow is assumed.

*c*. The fictitious boundary condition at an infinite distance from the wall may be used instead of the boundary condition at a distance $\delta$ from the wall, since for short contact times the fluid is heated over a very short distance $y$. Therefore the inifinite boundary condition can be expected to be adequate.

*d*. Equation 12B.4-3 can be written as $y(\partial\Theta/\partial z) = \beta(\partial^2\Theta/\partial y^2)$. Next we have to convert the derivatives to derivatives with respect to the dimensionless variable $\eta$:

$$\frac{\partial\Theta}{\partial z} = \frac{d\Theta}{d\eta}\frac{\partial\eta}{\partial z} = \frac{d\Theta}{d\eta}\frac{y}{\sqrt[3]{9\beta z}}\left(-\frac{1}{3z}\right)$$

$$\frac{\partial\Theta}{\partial y} = \frac{d\Theta}{d\eta}\frac{\partial\eta}{\partial y} = \frac{d\Theta}{d\eta}\frac{1}{\sqrt[3]{9\beta z}}; \quad \frac{\partial^2\Theta}{\partial y^2} = \frac{d}{d\eta}\left(\frac{d\Theta}{d\eta}\frac{1}{\sqrt[3]{9\beta z}}\right)\frac{\partial\eta}{\partial y} = \frac{d^2\Theta}{d\eta^2}\left(\frac{1}{\sqrt[3]{9\beta z}}\right)^2$$

When these relations are substituted into the partial differential equation and use is made of the defining equation for $\eta$ we get Eq. 12B.4-7.

*e*. When we set $d\Theta/d\eta = p$, we get $dp/d\eta + 3\eta^2 p = 0$, which is first-order and separable, and the solution is given in the book. The next integration gives

$$\Theta = C_1\int_0^\eta e^{-\bar{\eta}^3}d\bar{\eta} + C_2$$

**12B.5 Temperature in a slab with heat production**

*a.* This problem is discussed on pp. 130-131 of the 2nd Edition of Carslaw and Jaeger. The solution in dimensionless form can be obtained from Eq. (7) on p. 130. We must first determine the correspondence between their symbols and ours:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| C&J | $l$ | $K$ | $v$ | $x/l$ | $\kappa$ | $\kappa t/l^2$ | $A_0$ |
| BS&L | $b$ | $k$ | $T-T_0$ | $\eta=\dfrac{y}{b}$ | $\alpha=\dfrac{k}{\rho\hat{C}_p}$ | $\tau=\dfrac{\alpha t}{b^2}$ | $S_0$ |

Therefore the temperature rise as a function of position and time is

$$\frac{k(T-T_0)}{S_0 b^2}=\frac{1}{2}\left(1-\eta^2-4\sum_{n=0}^{\infty}\frac{(-1)^n}{\left(n+\frac{1}{2}\right)^3\pi^3}\cos\left(n+\tfrac{1}{2}\right)\pi\eta\cdot e^{-\left(n+\frac{1}{2}\right)^2\pi^2\tau}\right)$$

*b,* The center-plane temperature is obtained by setting $\eta$ equal to zero. The maximum of the center-plane temperature is then

$$T_{\max}=T_0+\frac{S_0 b^2}{2k}$$

*c.* According to the figure on p. 131 in Carslaw and Jaeger, 90% of the total temperature rise occurs at about $\tau=1$, that is, at about $t=b^2/\alpha$.

**12B.6 Forced convection in slow flow across a cylinder**

*a*. First we have to determine the velocity component $v_\theta$ from Eq. 12B.6-1 (here we let $\xi = r/R$ and measure $\theta$ downstream from the stagnation locus):

$$v_\theta = \frac{\partial \psi}{\partial r} = \frac{v_\infty \sin\theta}{2S}\left[(2\ln\xi - 1) + 2 - \frac{1}{\xi^2}\right]$$

Then let $\xi = 1 + \eta$ and, for small $\eta$,, introduce the Taylor expansions $\ln\xi = (\xi - 1) + \cdots = \eta + \cdots = y/R + \cdots$ and $-1/\xi^2 = -1 + 2\eta + \cdots$. Then we may write

$$v_x = \frac{v_\infty \sin\theta}{2S}[4\eta + \cdots] \approx \frac{2v_\infty \sin\theta}{SR} y \equiv \beta y$$

Therefore, $\beta = 2v_\infty \sin\theta / SR$.

*b*. We identify the boundary-layer coordinates as $x = \theta$, $y = r - R$, and $z = z$. Then we recognize that $h_x = R$ and set $h_z = 1$. Therefore we can get the heat loss from a length $L$ of the cylinder as follows, starting from Eq. 12.4-31:

$$\begin{aligned}
Q &= \frac{3^{1/3} k(T_0 - T_\infty)L}{2\alpha^{1/3}\Gamma(\frac{4}{3})}\left(\frac{2v_\infty}{SR}\right)^{1/3}\left(\int_0^\pi \sqrt{\sin\theta}\, R d\theta\right)^{2/3} \\
&= \frac{3^{1/3} k(T_0 - T_\infty)L}{2\alpha^{1/3}\Gamma(\frac{4}{3})}\left(\frac{2v_\infty}{SR}\right)^{1/3}\left(\mathrm{B}(\tfrac{3}{4},\tfrac{1}{2})\right)^{2/3} R^{2/3} \\
&= \frac{3^{1/3}\left(\mathrm{B}(\frac{3}{4},\frac{1}{2})\right)^{2/3}}{2\pi\Gamma(\frac{4}{3})S^{1/3}}(\pi DL)(T_0 - T_\infty)\left(\frac{k}{D}\right)\left(\frac{Dv_\infty}{\nu}\cdot\frac{\nu}{\alpha}\right)^{1/3} \\
&= \frac{3^{1/3}\left(\mathrm{B}(\frac{3}{4},\frac{1}{2})\right)^{2/3}}{2\pi\Gamma(\frac{4}{3})}(\pi DL)(T_0 - T_\infty)\left(\frac{k}{D}\right)\left(\frac{\mathrm{Re}\,\mathrm{Pr}}{S}\right)^{1/3}
\end{aligned}$$

Comparison of this result with the solution to part (b) given in the text allows the constant C to be evaluated.

*c*. The boundary-layer thickness can be obtained from Eq. 12.4-29 as follows:

$$\frac{\delta_T}{R} = \frac{1}{R}\sqrt{\frac{RS}{2v_\infty \sin\theta}}\left(9\alpha\int_0^\theta \sqrt{\frac{2v_\infty \sin\theta}{RS}}R d\theta\right)^{1/3}$$
$$= \left(\frac{9S}{\mathrm{Re}\,\mathrm{Pr}}\right)^{1/3}\frac{1}{\sqrt{\sin\theta}}\left(\int_0^\theta \sqrt{\sin\theta}d\theta\right)^{1/3}$$
$$= \left(\frac{9S}{\mathrm{Re}\,\mathrm{Pr}}\right)^{1/3} f(\theta)$$

At the separation locus, $\theta = \pi$, the integral is finite (according to Eq. 12B.6-2) and the denominator in $f$ is zero, so that $f$ is infinite. Near the stagnation locus, $\theta = 0$, the numerator and denominator of $f$ may be expanded in Taylor series to give:

$$f = \frac{\left(\int_0^\theta \sqrt{\theta - \frac{1}{6}\theta^3 + \cdots}d\theta\right)^{1/3}}{\sqrt{\theta - \frac{1}{6}\theta^3 + \cdots}} = \frac{\left(\int_0^\theta \sqrt{\theta}\left(1 - \frac{1}{12}\theta^2 + \cdots\right)d\theta\right)^{1/3}}{\sqrt{\theta}\left(1 - \frac{1}{12}\theta^2 + \cdots\right)}$$
$$= \frac{1}{\sqrt{\theta}}\left(1 + \tfrac{1}{12}\theta^2 + \cdots\right)\left(\tfrac{2}{3}\theta^{3/2} - \tfrac{1}{12}\tfrac{2}{7}\theta^{7/2} + \cdots\right)^{1/3}$$

Then in the limit as $\theta \to 0$, this gives the stagnation value

$$f = \left(\tfrac{2}{3}\right)^{1/3}$$

Numerical integration gives for $\theta = \frac{1}{2}\pi$

$$f = 1.1981$$

To get the answers given in the book, we have to recognize that the theta in the problem is being measured from the stagnation point at $\theta = \pi$ to the separation point at $\theta = 0$.

Note that for this flow, just as for the flow around the sphere, the boundary layer thickness increases through finite values to an infinite value at the separation locus.

[Note: Eq. 6.2.1 of Abramowitz and Stegun (NBS, Applied Math Series, 55) is a convenient formula from which the integral from zero to $\frac{1}{2}\pi$ can be calculated in terms of gamma functions, which are tabulated.]

**12B.9 Non-Newtonian heat transfer with constant wall heat flux (asymptotic solution for small axial distances)**

For Newtonian fluids, Example 12.2-2, we wrote

$$v_z = v_{z,\max}\left[1-\left(\frac{r}{R}\right)^2\right] = v_{z,\max}\left[1-\left(1-\left(\frac{y}{R}\right)^2\right)\right] =$$

$$= v_{z,\max}\left(2\frac{y}{R}-\left(\frac{y}{R}\right)^2\right) \approx 2v_{z,\max}\frac{y}{R} = \left(\frac{(\mathcal{P}_0-\mathcal{P}_L)R^2}{2\mu L}\right)\frac{y}{R} \equiv v_0\frac{y}{R}$$

where Eqs. 2.3-18 and 19 have been used, as well as the definition $y = R - r$.

The analogous procedure for power-law fluids gives

$$v_z = v_{z,\max}\left[1-\left(\frac{r}{R}\right)^{(1/n)+1}\right] = v_{z,\max}\left[1-\left(1-\frac{y}{R}\right)^{(1/n)+1}\right]$$

$$= v_{z,\max}\left[1-1+\left(\frac{1}{n}+1\right)\left(\frac{y}{R}\right)-\frac{1}{2!}\left(\frac{1}{n}+1\right)\left(\frac{1}{n}\right)\left(\frac{y}{R}\right)^2+\cdots\right]$$

$$\approx v_{z,\max}\left(\frac{1}{n}+1\right)\left(\frac{y}{R}\right) = \left(\frac{(\mathcal{P}_0-\mathcal{P}_L)R^2}{2mL}\right)^{1/n}\frac{R}{(1/n)+1}\left(\frac{1}{n}+1\right)\left(\frac{y}{R}\right)$$

$$= \left(\frac{(\mathcal{P}_0-\mathcal{P}_L)R^2}{2mL}\right)^{1/n} R\left(\frac{y}{R}\right) \equiv v_0\frac{y}{R}$$

This gives the formula for the $v_0$ that has to be used for power-law fluids. The quantity $v_0$ , with dimensions of velocity, appears only in the dimensionless parameter $\lambda$.

The boundary condition at $\eta = 0$ gives $C_2 = 1$. The boundary condition at $\eta = \infty$, gives

$$C_1 = -\frac{1}{\int_0^\infty e^{-\bar{\eta}^3} d\bar{\eta}}$$

And the complete solution (in dimensionless form) is

$$\Theta = 1 - \frac{\int_0^\eta e^{-\bar{\eta}^3} d\bar{\eta}}{\int_0^\infty e^{-\bar{\eta}^3} d\bar{\eta}} = \frac{\int_0^\infty e^{-\bar{\eta}^3} d\bar{\eta} - \int_0^\eta e^{-\bar{\eta}^3} d\bar{\eta}}{\int_0^\infty e^{-\bar{\eta}^3} d\bar{\eta}} = \frac{\int_\eta^\infty e^{-\bar{\eta}^3} d\bar{\eta}}{\Gamma\left(\frac{4}{3}\right)}$$

The integral in the numerator cannot be integrated analytically.

*f*. The local wall heat flux (i.e., at any position $z$ down the wall) is

$$q_y\Big|_{y=0} = -k\frac{\partial T}{\partial y}\Big|_{y=0} = +k\frac{d\Theta}{d\eta}\Big|_{\eta=0}\frac{T_0 - T_1}{\sqrt[3]{9\beta z}} = \frac{k}{\Gamma\left(\frac{4}{3}\right)}\frac{T_0 - T_1}{\sqrt[3]{9\beta z}}$$

In taking the derivative, the Leibniz formula was used. Finally the average heat flux at the wall is

$$q_{y,\text{avg}}\Big|_{y=0} = \frac{1}{L}\int_0^L q_y\Big|_{y=0} dz = \frac{k}{\Gamma\left(\frac{4}{3}\right)}\frac{T_0 - T_1}{\sqrt[3]{9\beta}}\frac{1}{L}\int_0^L z^{-1/3} dz = \frac{3}{2}\frac{k}{\Gamma\left(\frac{4}{3}\right)}\frac{T_0 - T_1}{\sqrt[3]{9\beta L}}$$

This is the result that is given in Eq. 12B.4-9.

**12C.1 Product solutions for unsteady heat conduction in solids**

*a*. We begin by defining a dimensionless temperature difference by

$$\Delta = \frac{T_1 - T(x,y,z,t)}{T_1 - T_0}$$

in which $T_0$ is the initial temperature of the solid rectangular parallelepiped, and $T_1$ is the imposed temperature on the surfaces of the solid. Then the 3-dimensional heat conduction equation for the solid is

$$\frac{\partial \Delta}{\partial t} = \alpha\left(\frac{\partial^2 \Delta}{\partial x^2} + \frac{\partial^2 \Delta}{\partial y^2} + \frac{\partial^2 \Delta}{\partial z^2}\right)$$

If we now postulate a product solution

$$\Delta(x,y,z,t) = X(x,t)Y(y,t)Z(z,t)$$

then we get

$$\frac{\partial(XYZ)}{\partial t} = \alpha\left(\frac{\partial^2(XYZ)}{\partial x^2} + \frac{\partial^2(XYZ)}{\partial y^2} + \frac{\partial^2(XYZ)}{\partial z^2}\right)$$

Division by $XYZ$ then gives

$$\frac{1}{XYZ}\frac{\partial(XYZ)}{\partial t} = \alpha\left(\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}\right)$$

When the product on the left side is differentiated we get

$$\frac{1}{X}\frac{\partial X}{\partial t} + \frac{1}{Y}\frac{\partial Y}{\partial t} + \frac{1}{Z}\frac{\partial Z}{\partial t} = \alpha\left(\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}\right)$$

The first term on the left is a function of $x$ and $t$, as is the first term on the right. Similarly, the functional dependences of the second and

third terms are the same. Therefore we postulate that these pairs of terms can be equated to give:

$$\frac{\partial X}{\partial t} = \alpha \frac{\partial^2 X}{\partial x^2}; \qquad \frac{\partial Y}{\partial t} = \alpha \frac{\partial^2 Y}{\partial y^2}; \qquad \frac{\partial Z}{\partial t} = \alpha \frac{\partial^2 Z}{\partial z^2}$$

That is, we get three one-dimensional heat-conduction equations. These can be solved according to the method of separation of variables given in Ex. 12.1-2, and all of them have the same initial conditions, and the same kinds of boundary conditions. Therefore, when we combine the three solutions in the product form above we get

$$\begin{aligned}\Delta = &\left( 2\sum_{m=0}^{\infty} \frac{(-1)^m}{\left(m+\frac{1}{2}\right)\pi} e^{-\left(m+\frac{1}{2}\right)^2 \pi^2 \alpha t/a^2} \cos\left(m+\tfrac{1}{2}\right)\pi x/a \right) \\ &\cdot\left( 2\sum_{n=0}^{\infty} \frac{(-1)^n}{\left(n+\frac{1}{2}\right)\pi} e^{-\left(n+\frac{1}{2}\right)^2 \pi^2 \alpha t/b^2} \cos\left(n+\tfrac{1}{2}\right)\pi y/b \right) \\ &\cdot\left( 2\sum_{p=0}^{\infty} \frac{(-1)^p}{\left(p+\frac{1}{2}\right)\pi} e^{-\left(p+\frac{1}{2}\right)^2 \pi^2 \alpha t/c^2} \cos\left(p+\tfrac{1}{2}\right)\pi z/c \right)\end{aligned}$$

or

$$\begin{aligned}\Delta = 8\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} &\frac{(-1)^{m+n+p}}{\left(m+\frac{1}{2}\right)\left(n+\frac{1}{2}\right)\left(p+\frac{1}{2}\right)} \\ &\cdot\left[\cos\left(m+\tfrac{1}{2}\right)\pi x/a\right]\left[\cos\left(n+\tfrac{1}{2}\right)\pi y/b\right]\left[\cos\left(p+\tfrac{1}{2}\right)\pi z/c\right] \\ &\cdot e^{-\left[\left(m+\frac{1}{2}\right)^2/a^2+\left(n+\frac{1}{2}\right)^2/b^2+\left(p+\frac{1}{2}\right)^2/c^2\right]\pi^2\alpha t}\end{aligned}$$

This can be substituted into the differential equation to verify that the latter can indeed be satisfied. Also, the satisfaction of the boundary and initial conditions can be verified.

*b*. This and many other interesting and important product solutions are discussed in Carslaw and Jaeger on pp. 33, 184, and 227.

**12C.2 Heating of a semi-infinite slab with a variable thermal conductivity**

*a*. The heat conduction equation is, for variable *k*,

$$\rho\hat{C}_p\frac{\partial T}{\partial t}=\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)\quad\text{or}\quad\rho\hat{C}_p\frac{\partial\Theta}{\partial t}=k_0\frac{\partial}{\partial y}\left((1+\beta\Theta)\frac{\partial\Theta}{\partial y}\right)\quad\text{or}$$

$$\frac{\partial\Theta}{\partial t}=\alpha_0\left[\frac{\partial^2\Theta}{\partial y^2}+\beta\frac{\partial}{\partial y}\left(\Theta\frac{\partial\Theta}{\partial y}\right)\right]$$

In order to use Eq. 12C.2-2, we need to convert the derivatives:

$$\frac{\partial\Theta}{\partial t}=\frac{d\Phi}{d\eta}\frac{\partial\eta}{\partial t}=\frac{d\Phi}{d\eta}\left(-\frac{y}{\delta^2}\frac{d\delta}{dt}\right)=\frac{d\Phi}{d\eta}\left(-\frac{\eta}{\delta}\frac{d\delta}{dt}\right)$$

$$\frac{\partial\Theta}{\partial y}=\frac{d\Phi}{d\eta}\frac{\partial\eta}{\partial y}=\frac{d\Phi}{d\eta}\frac{1}{\delta}\quad\text{and}\quad\frac{\partial^2\Theta}{\partial y^2}=\frac{d^2\Phi}{d\eta^2}\frac{1}{\delta^2}$$

*b*. We now substitute the above derivatives into the heat conduction equation to get (after multiplication by $\delta^2$)

$$-\frac{d\Phi}{d\eta}\left(\eta\delta\frac{d\delta}{dt}\right)=\alpha_0\frac{d}{d\eta}\left(\frac{d\Phi}{d\eta}+\beta\Phi\frac{d\Phi}{d\eta}\right)$$

When this is integrated over $\eta$ from 0 to 1, we get

$$-\int_0^1\left[\frac{d}{d\eta}(\eta\Phi)-\Phi\right]d\eta\cdot\delta\frac{d\delta}{dt}=\alpha_0\int_0^1\frac{d}{d\eta}\left(\frac{d\Phi}{d\eta}+\beta\Phi\frac{d\Phi}{d\eta}\right)d\eta$$

Performing the integrations we finally end up with

$$-\eta\Phi\Big|_0^1+\int_0^1\Phi d\eta\cdot\delta\frac{d\delta}{dt}=\alpha_0\left(\frac{d\Phi}{d\eta}+\beta\Phi\frac{d\Phi}{d\eta}\right)\Bigg|_0^1$$

which is equivalent to Eqs. 12C.2-3 to 5.

*c.* We now use $\Phi = 1 - \frac{3}{2}\eta + \frac{1}{2}\eta^3$, which is a good choice for the approximate temperature profile, since it gives $\Phi(0) = 1$, $\Phi(1) = 0$, and $\Phi'(1) = 0$ in agreement with our intuition.

We then get $M$ and $N$ by substitution into Eqs. 12C.2-4 and 5:

$$M = \int_0^1 \Phi d\eta = \int_0^1 \left(1 - \tfrac{3}{2}\eta + \tfrac{1}{2}\eta^3\right) d\eta = \tfrac{3}{8}$$

$$N = \left(\frac{d\Phi}{d\eta} + \beta\Phi\frac{d\Phi}{d\eta}\right)\Bigg|_0^1 = \left[\left(-\tfrac{3}{2} + \tfrac{3}{2}\eta^2\right) + \beta\left(1 - \tfrac{3}{2}\eta + \tfrac{1}{2}\eta^3\right)\left(-\tfrac{3}{2} + \tfrac{3}{2}\eta^2\right)\right]\Bigg|_0^1$$
$$= \tfrac{3}{2}(1+\beta)$$

This leads to the differential equation for the boundary layer thickness:

$$\tfrac{3}{8}\delta\frac{d\delta}{dt} = \tfrac{3}{2}(1+\beta)\alpha_0$$

which when integrated gives

$$\delta = \sqrt{8(1+\beta)\alpha_0 t}$$

The time-dependent temperature distribution is then

$$\frac{T - T_0}{T_1 - T_0} = 1 - \frac{3}{8}\left(\frac{y}{\sqrt{8(1+\beta)\alpha_0 t}}\right) + \frac{1}{2}\left(\frac{y}{\sqrt{8(1+\beta)\alpha_0 t}}\right)^3$$

The heat flux at $y = 0$ is finally

$$q_y\Big|_{y=0} = -k\frac{\partial T}{\partial y}\Bigg|_{y=0} = -\frac{k(T_1 - T_0)}{\delta(t)}\frac{d\Phi}{d\eta}\Bigg|_{\eta=0} = \frac{3}{2}\frac{k(T_1 - T_0)}{\sqrt{8(1+\beta)\alpha_0 t}}$$

in which $k$ is given by Eq. 12C.2-1.

**12C.3 Heat conduction with phase change (the Neumann-Stefan problem)**

*a*. Using the definition of the dimensionless temperature differences in Eqs. 12C.3-1 and 2, we may write the heat conduction equations for the solid and liquid phases as follows:

Solid: $\dfrac{\partial \Theta_S}{\partial t} = \alpha \dfrac{\partial^2 \Theta_S}{\partial z^2}$ Liquid: $\dfrac{\partial \Theta_L}{\partial t} = \alpha \dfrac{\partial^2 \Theta_L}{\partial z^2}$

The initial and boundary conditions are:

| | | |
|---|---|---|
| I. C.: | at $t = 0$, | $\Theta_L = 1$ |
| B. C. 1: | at $z = 0$, | $\Theta_S = 0$ |
| B. C. 2: | at $z = \infty$, | $\Theta_L = 1$ |
| B. C. 3: | at $z = Z(t)$, | $\Theta_L = \Theta_S = \Theta_m$ |
| B. C. 4: | at $z = Z(t)$, | $k\dfrac{\partial \Theta_S}{\partial z} - k\dfrac{\partial \Theta_L}{\partial z} = \dfrac{\rho \Delta \hat{H}_f}{T_1 - T_0}\dfrac{dZ}{dt}$ |

*b*. The assumed forms for the solution are chosen because of their similarity to other one-dimensional heat-flow problems in a semi-infinite region:

Solid: $\Theta_S = C_1 + C_2 \mathrm{erf}\dfrac{z}{\sqrt{4\alpha t}}$

Liquid: $\Theta_L = C_3 + C_4 \mathrm{erf}\dfrac{z}{\sqrt{4\alpha t}} = (C_3 + C_4) - C_4\left(1 - \mathrm{erf}\dfrac{z}{\sqrt{4\alpha t}}\right)$

$$= (C_3 + C_4) - C_4 \mathrm{erfc}\frac{z}{\sqrt{4\alpha t}}$$

The last form for the liquid-phase temperature equation will turn out to be somewhat more convenient to use.

*c*. When we apply B. C. 1, we get $C_1 = 0$. Using B. C. 2, we get $C_3 + C_4 = 1$; we not that the initial condition is automatically satisfied. Then B. C. 3 gives

$$C_2 \mathrm{erf}\frac{Z(t)}{\sqrt{4\alpha t}} = 1 - C_4 \mathrm{erfc}\frac{Z(t)}{\sqrt{4\alpha t}} = \Theta_m$$

The only way that this equation can be satisfied, is if $Z(t) \propto \sqrt{t}$. We elect to write, then, $Z(t) = \lambda\sqrt{4\alpha t}$ where $\lambda$ is a dimensionless constant that has yet to be determined.

Finally we apply B. C. 4 to get

$$kC_2 \frac{2}{\sqrt{\pi}} e^{-Z^2/4\alpha t}\left(\frac{1}{\sqrt{4\alpha t}}\right) - k(-C_4)\frac{2}{\sqrt{\pi}} e^{-Z^2/4\alpha t}\left(\frac{-1}{\sqrt{4\alpha t}}\right)$$
$$= \frac{\rho \Delta \hat{H}_f}{T_1 - T_0}\lambda\sqrt{4\alpha}\frac{1}{2\sqrt{t}}$$

or

$$ke^{-\lambda^2}(C_2 - C_4) = \frac{\rho \Delta \hat{H}_f \lambda \alpha \sqrt{\pi}}{T_1 - T_0} \qquad (*)$$

Next we apply that part of B. C. 3 that deals with the dimensionless melting temperature:

$$C_2 \mathrm{erf}\lambda = \Theta_m \qquad \text{and} \qquad 1 - C_4 \mathrm{erfc}\lambda = \Theta_m$$

whence

$$C_2 - C_4 = \frac{\Theta_m}{\mathrm{erf}\lambda} - \frac{1-\Theta_m}{\mathrm{erfc}\lambda} \qquad (**)$$

Combining (*) and (**) and rearranging, we get

$$\frac{\Theta_m}{\mathrm{erf}\lambda} - \frac{1-\Theta_m}{\mathrm{erfc}\lambda} = \sqrt{\pi}\Lambda\lambda e^{+\lambda^2}$$

This gives $\lambda$ in terms of $\Lambda$ and $\Theta_m$. Then we get for the temperature

Solid: $\Theta_S = \Theta_m \dfrac{\mathrm{erf}(z/4\alpha t)}{\mathrm{erf}\lambda}$; Liquid: $\Theta_L = 1 - (1-\Theta_m)\dfrac{\mathrm{erfc}(z/4\alpha t)}{\mathrm{erfc}\lambda}$

Finally, we have $Z(t) = \lambda\sqrt{4\alpha t}$, where $\lambda$ is now a known function of $\Lambda$ and $\Theta_m$.

**12C.4 Viscous heating in oscillatory flow**

*a*. The starting point for this problem is Eq. 4.1-44, and the development up to and including Eq. 4.1-50 can be taken over here. B.C. 1 is also valid, but now B.C. 2 must read: at $y = b$, $v^\circ = 0$. When Eq. 4.1-50 is solved for these boundary conditions, we get

$$\frac{v^\circ(\xi)}{v_0} = \frac{\sinh c(1-\xi)}{\sinh c}$$

where $c = \sqrt{i\omega b^2/\nu} = \sqrt{\omega b^2/2\nu}(1+i) \equiv a(1+i)$ and $\xi = x/b$. The above expression for $v^\circ(\xi)$ can now be substituted into Eq. 4.1-48 to get Eq. 12C.4-1.

*b*. Next we get the dissipation function, which for this problem is $\Phi_v = \mu(\partial v_z/\partial x)^2$. The derivative of the velocity is

$$\frac{\partial v_z}{\partial x} = \Re\left\{\frac{dv^\circ}{dx}e^{i\omega t}\right\}$$

and its square is obtained by using the relation (which should be proven) $\Re\{u\}\Re\{v\} = \frac{1}{2}[\Re\{uv\} + \Re\{uv^*\}]$, where $u$ and $v$ are complex numbers and the asterisk indicates a complex conjugate:

$$\left(\frac{\partial v_z}{\partial x}\right)^2 = \frac{1}{2}\left[\Re\left\{\left(\frac{dv^\circ}{dx}\right)^2 e^{2i\omega t}\right\} + \left(\frac{dv^\circ}{dx}\right)\left(\frac{dv^\circ}{dx}\right)^*\right]$$

Next we get the time average of the above function; the first term vanishes because of the exponential (which contains sine and cosine functions) and only the second term survives:

$$\overline{\left(\frac{\partial v_z}{\partial x}\right)^2} = \frac{1}{2b^2}\left(\frac{dv^\circ}{dx}\right)\left(\frac{dv^\circ}{dx}\right)^* \quad \text{where} \quad \frac{dv^\circ}{dx} = -\frac{v_0}{b}\frac{c\cosh c(1-\xi)}{\sinh c}$$

from part (*a*). Then the time averaged dissipation function is:

$$\overline{\Phi}_v = \overline{\left(\frac{\partial v_z}{\partial x}\right)^2} = \frac{1}{2}\left(\frac{v_0}{b}\right)^2\left(\frac{c\cosh c(1-\xi)}{\sinh c}\right)\left(\frac{c\cosh c(1-\xi)}{\sinh c}\right)^2$$

$$= \frac{1}{2}\left(\frac{v_0}{b}\right)^2 \frac{(cc^*)(\cosh c(1-\xi))(\cosh c(1-\xi))^*}{(\sinh c)(\sinh c)^*}$$

$$= \frac{1}{2}\left(\frac{v_0}{b}\right)^2 \frac{\left[\cosh^2 a(1-\xi)\cos^2 a(1-\xi) + \sinh^2 a(1-\xi)\sin^2 a(1-\xi)\right]}{\sinh^2 a\cos^2 a + \cosh^2 a\sin^2 a}$$

To get the third line of the above we have used the identity

$$\cosh a(1+i)(1-\xi) = \cosh a(1-\xi)\cos a(1-\xi) + i\sinh a(1-\xi)\sin a(1-\xi)$$

which should be verified. Then, finally, use $\sin^2\theta + \cos^2\theta = 1$ and $\cosh^2\theta - \sinh^2\theta = 1$ to simplify the above expression for $\overline{\Phi}_v$, thus:

$$\overline{\Phi}_v = a^2\left(\frac{v_0}{b}\right)^2\left(\frac{\cos^2 a(1-\xi) + \sinh^2 a(1-\xi)}{\sin^2 a + \sinh^2 a}\right) \to a^2\left(\frac{v_0}{b}\right)^2 e^{-2a\xi}$$

the last expression being a limiting expression for very large frequencies.

*c.* To get the time averaged temperature distribution, we use Eq. 12C.4-4, thus

$$\frac{d^2\overline{T}}{d\xi^2} = -\frac{\mu b^2}{k}a^2\left(\frac{v_0}{b}\right)^2\left(\frac{\cos^2 a(1-\xi) + \sinh^2 a(1-\xi)}{\sin^2 a + \sinh^2 a}\right)$$

Then introducing a new variable $\eta = 1-\xi$ we write

$$\frac{d^2\overline{T}}{d(a\eta)^2} = -\frac{\mu v_0^2}{k}\left(\frac{\cos^2 a\eta + \sinh^2 a\eta}{\sin^2 a + \sinh^2 a}\right)$$

Integration once then gives

$$\frac{d\overline{T}}{d(a\eta)} = -\frac{\mu v_0^2}{4k}\left(\frac{\sin 2a\eta + \sinh 2a\eta}{\sin^2 a + \sinh^2 a}\right) + C_1$$

and a further integration then yields

$$\overline{T} = -\frac{\mu v_0^2}{4k}\left(\frac{\sin^2 a\eta + \sinh^2 a\eta}{\sin^2 a + \sinh^2 a}\right) + C_1 a\eta + C_2$$

The constants of integration may then be determined, so that we get finally:

$$\overline{T} - T_0 = \frac{\mu v_0^2}{4k}\left[(1-\xi) - \frac{\sin^2 a(1-\xi) + \sinh^2 a(1-\xi)}{\sin^2 a + \sinh^2 a}\right]$$

For the high-frequency limit, we get

$$\overline{T} - T_0 = \frac{\mu v_0^2}{4k}\left[\left(1 - e^{-2a\xi}\right) - \xi\left(1 - e^{-2a}\right)\right]$$

(*Note*: The solution given here for finding the dissipation function does not use Eq. 12C.4-1 and is hence somewhat easier than the method suggested in the text. It does, however, require somewhat more familiarity with doing manipulations with complex variables.)

**12C.5 Solar Heat Penetration**

The basis for this problem was Eq. 5-40, M. N. Özişik, Heat Conduction, 2cnd Ed. , Wiley 1993 - - - p.206. The temperature profile is re-expressed as

$$T^* \equiv \frac{T(y,t) - T(\infty)}{T_{\max}(0) - T(\infty)}$$

and should have been written as

$$T^* = \exp[-\sqrt{y^2\omega / 2\alpha}] \cdot \cos[\omega t - \sqrt{y^2\omega / 2\alpha}]$$
$$-\frac{2}{\sqrt{\pi}} \int_0^{y/\sqrt{4\alpha t}} \exp[-\eta^2] \cdot \cos[\omega t - y^2\omega / 4\alpha\eta^2] d\eta$$

The thermal properties are taken from (Rohsenow, Hartnett and Cho, Handbook of Heat Transfer, McGraw-Hill, Third Ed., p.2.68, 1998):

| Density, kg/m^3 | Thermal conductivity, W/m,K | Specific heat, J/kg,K |
|---|---|---|
| 1515 | 0.027 | 800 |

It follows that

$$\alpha = \frac{0.027}{800 \cdot 1515} \cdot \frac{m^2 W \cdot 3600 joules}{joule - W - hr} = 8.02 \cdot 10^{-5} \frac{m^2}{hr}$$

a) For the assumption of a sinusoidal input with a 24 hour period

$$T(0,t) = a \sin(n\omega\ t)]$$

Show that the mean temperature, in the absence of day-to-day variation is just $a_0$ , and for a periodicity of 24 hrs

$$\omega\ \mathrm{t} = \pi \mathrm{t}/12\mathrm{hrs}; \omega = \pi/12\mathrm{hrs}$$

The amplitude of the fluctuations at depth y relative to those at the surface, "$A_{rel}$" is just

$$A_{rel} = \exp\left(-y\sqrt{^2\omega / 2\alpha}\right) \exp\left(-40.4 \cdot y / m\right)$$

Then at a depth of 10 cm the relative amplitude

$$A_{rel} = \exp(-4.04) = 0.0176$$

as required.

b) The transient term is difficult to evaluate, but this is of no practical importance. The periodically steady oscillations are about the initial temperature of zero, and therefore all transients must be of lesser magnitude than these periodic excursions. Since the excursions are already negligible there is no need to examine this problem further.

c) Only the slowest component of the oscillations need be considered as the others damp out even faster.

**12C.6 Heat transfer in a falling non-Newtonian film**

For the non-Newtonian falling film problem, the velocity distribution has been found in Problem 8B.1:

$$v_z = \left(\frac{\rho g \delta}{m}\right)^{1/n} \frac{\delta}{(1/n)+1}\left[1-\left(\frac{x}{\delta}\right)^{(1/n)+1}\right] = v_{z,\max}\left[1-\left(\frac{x}{\delta}\right)^{(1/n)+1}\right]$$

This can be rewritten in terms of the coordinate $y$, which is the distance from the solid surface

$$v_z = v_{z,\max}\left[1-\left(1-\frac{y}{\delta}\right)^{(1/n)+1}\right]$$

When this expression is expanded in a Taylor series, we get

$$v_z = v_{z,\max}\left[1-1+\left(\frac{1}{n}+1\right)\left(\frac{y}{\delta}\right)-\frac{1}{2!}\left(\frac{1}{n}+1\right)\left(\frac{1}{n}\right)\left(\frac{y}{\delta}\right)^2+\cdots\right]$$

For positions very close to the wall, this can be approximated by

$$v_z = v_{z,\max}\left(\frac{1}{n}+1\right)\left(\frac{y}{\delta}\right)$$

Inserting the expression for the maximum velocity, we get

$$v_z = \left(\frac{\rho g \delta}{m}\right)^{1/n} y$$

which simplifies to Eq. 12B.4-1 for a Newtonian fluid.

Therefore the solution in Problem 12B.4-1 may be taken over for a power-law fluid by replacing $\beta = \mu k/\rho^2 \hat{C}_p g\delta = \left(k/\rho\hat{C}_p\right)\left(\mu/\rho g\delta\right)$ (just below Eq. 12B.4-3) by $\beta = \left(k/\rho\hat{C}_p\right)\left(m/\rho g\delta\right)^{1/n}$. The quantity $\beta$ enters into Eq. 12B.4-8 via the variable $\eta$, and explicitly in Eq. 12B.4-9.

**12D.1 Unsteady-state heating of a slab (Laplace-transform method)**

*a.* We take the Laplace transform of Eq. 12.1-14 along with the initial conditions and boundary conditions given in Eqs. 12.1-15 and 16. This gives:

$$p\overline{\Theta} - 1 = \frac{d^2\overline{\Theta}}{d\eta^2} \qquad \text{with} \qquad \overline{\Theta}(\pm 1) = 0$$

This second-order, nonhomogeneous, ordinary differential equation can be solved by getting the complementary function (from Eq. C.1-4a) and the particular integral (by trial and error); this gives

$$\overline{\Theta} = C_1 \cosh\sqrt{p}\eta + C_2 \sinh\sqrt{p}\eta + \frac{1}{p}$$

When the constants are evaluated using the boundary conditions, we find that

$$\overline{\Theta} = \frac{1}{p} - \frac{\cosh\sqrt{p}\eta}{p\cosh\sqrt{p}}$$

Taking the inverse Laplace transform, we get

$$\Theta \equiv \frac{T_1 - T}{T_1 - T_0} = 1 - \mathfrak{L}^{-1}\left\{\frac{\cosh\sqrt{p}\eta}{p\cosh\sqrt{p}}\right\}$$
$$= 2\sum_{n=0}^{\infty} \frac{(-1)^n \cos\left[\left(n+\frac{1}{2}\right)\pi\eta\right]}{\left(n+\frac{1}{2}\right)\pi} \exp\left[\left(n+\tfrac{1}{2}\right)^2 \pi^2 \tau\right]$$

The inversion was performed by using Eq. (40) on p. 259 of Erdély, W. Magnus, F. Oberhettinger, and F. G. Tricomi *Tables of Integral Transforms*, McGraw-Hill, New York (1954), with $\omega$ set equal to zero. Unfortunately, this formula has two misprints in it: the $x$ in the denominator of the first term should be $l$, and the factor $2p$ in front of the summation sign should be $2\pi$. The inversion can also be performed by using the complex inversion theorem, as explained in Carslaw and Jaeger (see reference at the bottom of p. 403).

*b*. The expression for the transformed function $\overline{\Theta}$ (in (*a*)) can also be written in terms of exponentials. Then we expand part of the denominator in a binomial expansion to get:

$$\overline{\Theta} = \frac{1}{p} - \frac{e^{\sqrt{p}\eta} + e^{-\sqrt{p}\eta}}{p\left(e^{\sqrt{p}} + e^{-\sqrt{p}}\right)} = \frac{1}{p} - \frac{1}{p}\left(e^{-\sqrt{p}(1-\eta)} + e^{-\sqrt{p}(1+\eta)}\right)\sum_{n=0}^{\infty}(-1)^n e^{-2n\sqrt{p}}$$

$$= \frac{1}{p} - \frac{1}{p}\sum_{n=0}^{\infty}(-1)^n e^{-\sqrt{p}(2n+1-\eta)} - \frac{1}{p}\sum_{n=0}^{\infty}(-1)^n e^{-\sqrt{p}(2n+1+\eta)}$$

The inverse Laplace transform is then

$$\Theta \equiv \frac{T_1 - T}{T_1 - T_0} = 1 - \sum_{n=0}^{\infty}(-1)^n \operatorname{erfc}\frac{2n+1-\eta}{\sqrt{4\tau}} - \sum_{n=0}^{\infty}(-1)^n \operatorname{erfc}\frac{2n+1+\eta}{\sqrt{4\tau}}$$

This can be rewritten in terms of the dimensionless temperature of Example 12.1-1 thus:

$$\Phi \equiv \frac{T - T_0}{T_1 - T_0} = \sum_{n=0}^{\infty}(-1)^n \operatorname{erfc}\frac{2n+1-\eta}{\sqrt{4\tau}} + \sum_{n=0}^{\infty}(-1)^n \operatorname{erfc}\frac{2n+1+\eta}{\sqrt{4\tau}}$$

Here, the dimensionless temperature $\Phi$ has been introduced to avoid confusion with the dimensionless temperature $\Theta$ used in Example 12.1-2.

*c*. To compare the results of Examples 12.1-1 and 2, we replace the position variable $y$ of Example 12.1-1 by $x$ (so that it will not be confused with the $y$ of Exammple 12.1-2). Then the first few terms of the final result in (*b*) are, written in terms of $x = b - y$, are:

$$\frac{T - T_0}{T_1 - T_0} = \operatorname{erfc}\frac{1-[1-(x/b)]}{\sqrt{4\alpha t/b^2}} - \operatorname{erfc}\frac{3-[1-(x/b)]}{\sqrt{4\alpha t/b^2}} + \cdots$$

$$+\operatorname{erfc}\frac{1+[1-(x/b)]}{\sqrt{4\alpha t/b^2}} - \operatorname{erfc}\frac{3+[1-(x/b)]}{\sqrt{4\alpha t/b^2}} + \cdots = \operatorname{erfc}\frac{x}{\sqrt{4\alpha t}} + \cdots$$

the subsequent terms being smaller.

**12D.2 The Graetz-Nusselt Problem**

*a*. The partial differential equation to be solved is

$$\rho\hat{C}_p v_{z,\max}\left[1-\left(\frac{r}{R}\right)^2\right]\frac{\partial T}{\partial z}=k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right]$$

When this is put in nondimensional form, using the dimensionless quantities defined in Eq. 12D.2-1, the following equation is obtained for the temperature distribution $\Theta(\xi,\zeta)$:

$$\phi(\xi)\frac{\partial\Theta}{\partial\zeta}=\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\Theta}{\partial\xi}\right)\text{ with }\Theta(\xi,0)=1\text{ , }\Theta(1,\zeta)=0\text{, and }\left.\frac{\partial\Theta}{\partial\xi}\right|_{\xi=0}=0$$

We now try the method of separation of variables with the dimensionless temperature given by $\Theta(\xi,\zeta)=X(\xi)Z(\zeta)$. This gives, after division by $XZ$

$$\frac{1}{Z}\frac{dZ}{d\zeta}=\frac{1}{X\phi}\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{dX}{d\xi}\right)$$

Since the left side is a function of $\zeta$ alone, and the right side is a function of $\xi$ alone, both sides have to equal a constant. We choose this constant to be $-\beta^2$. Hence we get two ordinary differential equations:

$$\frac{dZ}{d\zeta}=-\beta^2 Z \qquad \text{or} \qquad Z(\zeta)\propto\exp\left(-\beta^2\zeta\right)$$

and

$$\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{dX}{d\xi}\right)+\beta^2\phi X=0 \qquad \text{with } X(1)=0 \text{ and } X'(0)=0$$

This last equation (with boundary conditions) is a boundary-value problem of the Sturm-Liouville type. Therefore, we know that there will be an infinite number of values of eigenvalues $\beta_i^2$ and an infinite

number of eigenfunctions $X_i(\xi)$. We also know that the eigenfunctions are orthogonal on the range (0,1) with respect to the weight function $\phi\xi$. Hence the solution must have the form:

$$\Theta(\xi,\zeta)=\sum_{i=1}^{\infty}A_iX_i(\xi)\exp\left(-\beta_i^2\zeta\right) \qquad \text{with } A_i=\frac{\int_0^1 X_i\phi\xi d\xi}{\int_0^1 X_i^2\xi d\xi}$$

The expression for $A_i$ is obtained by using the requirement that $\Theta(\xi,0)=1$, and making use of the orthogonality relations for the eigenfunctions.

*b.* We write for the Newtonian fluid

$$\phi(\xi)=v_z/\langle v_z\rangle=v_{z,\max}\left(1-\xi^2\right)/\langle v_z\rangle=2\left(1-\xi^2\right)$$

so that Eq. 12D.2-3 becomes:

$$\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{dX_i}{d\xi}\right)+2\beta_i^2\left(1-\xi^2\right)X_i=0 \quad \text{with } X(1)=0 \text{ and } X'(0)=0$$

This second-order equation will have two solutions, but one of them will become infinite at $\xi=0$ and is therefore unacceptable. The remaining solution we write as a power series

$$X_i(\xi)=\sum_{j=0}^{\infty}b_{ij}\xi^j \qquad (*)$$

When this is substituted into the differential equation, we get

$$\sum_{j=0}^{\infty}j^2b_{ij}\xi^{j-2}+2\beta_i^2\left(1-\xi^2\right)\sum_{j=0}^{\infty}b_{ij}\xi^j=0$$

By collecting equal powers of $\xi$ we get the recursion relation for the coefficients:

$$b_{ij}=\left(2\beta_i^2/j^2\right)\left(b_{i,j-4}-b_{i,j-2}\right) \qquad (*)$$

The $b_{i0}$ are arbitrarily chosen to be unity, and the higher coefficients are obtained from the equation immediately above. The values of $\beta_i^2$ are then obtained from the two equations marked with (*), along with the boundary condition at $\xi = 1$, which requires that $\sum_{j=0}^{\infty} b_{ij} = 0$. This involves solving an infinite order algebraic equation by trial and error--clearly a tedious process. Methods are, however, available to get the eigenvalues $\beta_i^2$ for small and large values of $i$. For small values there is the method of Stodola and Vianello, and for large values there is the WKB (Wentzel-Kramers-Brioullin) method. The following eigenvalues have been obtained:

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $2\beta_i^2$ | 7.314 | 44.61 | 113.9 | 215.1 | 348.4 | 513.8 |

See B. C. Lyche and R. B. Bird, *Chem. Eng. Sci.*, **6**, 35-41 (1956), for the first three values both for Newtonian and power-law fluids; the higher values are obtained from the WKB method as $2\beta_i^2 = (4i - \frac{4}{3})^2$. Further discussions and literature references for non-Newtonian flow may be found in the 1st edition (Chapter 5) and the 2nd edition (Chapter 6) of R. B. Bird, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Liquids*, Vol. 1, *Fluid Mechanics* (published by Wiley-Interscience).

**12D.3 The Graetz-Nusselt problem (asymptotic solution for large z)**

*a*. Using the definition of the dimensionless temperature as given in Problem 12D.2-2, we can write

$$q_0 = q_r\big|_{r=R} = -k\frac{\partial T}{\partial r}\bigg|_{r=R} = -\frac{k(T_1 - T_0)}{R}\frac{\partial \Theta}{\partial \xi}\bigg|_{\xi=1} = -\frac{k(T_b - T_0)}{R\Theta_b}\frac{\partial \Theta}{\partial \xi}\bigg|_{\xi=1}$$

*b*. For very large z,

$$\frac{\partial \Theta}{\partial \xi}\bigg|_{\xi=1} = \sum_{i=1}^{\infty} A_i \frac{dX_i}{d\xi}\bigg|_{\xi=1} \exp\left(-\beta_i^2\zeta\right) \approx A_1 \frac{dX_1}{d\xi}\bigg|_{\xi=1} \exp\left(-\beta_1^2\zeta\right)$$

Furthermore, from the definition of the bulk temperature in Eq. 10.8-33 we find using Eqs. 12D.2-2 and 3

$$\Theta_b = \frac{\int_0^1 \phi\Theta\xi d\xi}{\int_0^1 \phi\xi d\xi} = 2\int_0^1 \phi\Theta\xi d\xi = 2\sum_{i=1}^{\infty} A_i \exp\left(-\beta_i^2\zeta\right)\int_0^1 \phi X_i \xi d\xi$$

$$= -2\sum_{i=1}^{\infty} A_i \exp\left(-\beta_i^2\zeta\right)\int_0^1 \frac{1}{\beta_i^2}\frac{d}{d\xi}\left(\xi\frac{dX_i}{d\xi}\right)d\xi$$

$$= -2\sum_{i=1}^{\infty} A_i \exp\left(-\beta_i^2\zeta\right)\frac{1}{\beta_i^2}\left(\xi\frac{dX_i}{d\xi}\right)\bigg|_0^1$$

$$= -2\sum_{i=1}^{\infty} A_i \exp\left(-\beta_i^2\zeta\right)\frac{1}{\beta_i^2}\frac{dX_i}{d\xi}\bigg|_{\xi=1} \approx -2A_1 \exp\left(-\beta_1^2\zeta\right)\frac{1}{\beta_1^2}\frac{dX_1}{d\xi}\bigg|_{\xi=1}$$

the very last expression being valid for large z. When these results are substituted into the result in (*a*), we get

$$q_0 = -\frac{k}{R}(T_b - T_0)\cdot\left(-\frac{\beta_1^2}{2}\right) = \beta_1^2\frac{k(T_b - T_0)}{D}$$

This is the same as Eq. 12D.3-3 in the text. (see also p. 217 of R. B. Bird, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Liquids*, Vol. 1, *Fluid Dynamics*, Wiley-Interscience, New York, 2nd edition (1987)).

**12D.4 The Graetz-Nusselt problem (asymptotic expression for small z)**

*a*. When Eq. 10.8-12 is written in terms of the dimensionless variables $\Theta = (T_1 - T)/(T_1 - T_0)$, $\xi = r/R$, and $\zeta = z/R$, it becomes

$$v_{z,\max}\left(1-\xi^2\right)\frac{\partial\Theta}{\partial\zeta} = \frac{\alpha}{R}\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\Theta}{\partial\xi}\right)$$

Next we introduce the three assumptions used for small $z$ in Example 12.2-2, as well as the dimensionless quantities in Eq. 12.4-1. This gives for the partial differential equation for $\Theta(\sigma,\xi)$ the following problem statement: D

$$N\sigma\frac{\partial\Theta}{\partial\zeta} = \frac{\partial^2\Theta}{\partial\sigma^2} \qquad \text{with } \Theta(\sigma,0)=0,\ \Theta(0,\zeta)=1,\ \Theta(\infty,\zeta)=0$$

We use the method of combination of independent variables, by introducing the new independent variable: $\eta = (N\sigma^3/9\zeta)^{1/3}$. Then the transformation of the derivatives proceeds as follows:

$$\left(\frac{\partial\Theta}{\partial\zeta}\right)_\sigma = \frac{d\Theta}{d\eta}\left(\frac{\partial\eta}{\partial\zeta}\right)_\sigma = \frac{d\Theta}{d\eta}\left(\frac{N\sigma^3}{9}\right)^{1/3}\left(-\frac{1}{3}\right)\zeta^{-4/3} = -\frac{\eta}{3\zeta}\frac{d\Theta}{d\eta}$$

$$\left(\frac{\partial\Theta}{\partial\sigma}\right)_\zeta = \frac{d\Theta}{d\eta}\left(\frac{\partial\eta}{\partial\sigma}\right)_\zeta = \frac{d\Theta}{d\eta}\left(\frac{N}{9\zeta}\right)^{1/3} = \frac{d\Theta}{d\eta}\frac{\eta}{\sigma} \text{ and } \left(\frac{\partial^2\Theta}{\partial\sigma^2}\right)_\zeta = \frac{d^2\Theta}{d\eta^2}\left(\frac{\eta}{\sigma}\right)^2$$

When these substitutions are made into the partial differential equation in dimensionless form, we get (after some simplification) the following ordinary differential equation and boundary conditions for $\Theta(\eta)$

$$\frac{d^2\Theta}{d\eta^2} + 3\eta^2\frac{d\Theta}{d\eta} = 0 \qquad \text{with} \qquad \Theta(0)=1 \text{ and } \Theta(\infty)=0$$

To solve this equation, we set $\Phi(\eta) = d\Theta/d\eta$, and obtain a separable first-order differential equation for $\Phi$:

$$\frac{d\Phi}{d\eta} = -3\eta^2\Phi \qquad \text{with solution} \qquad \frac{d\Theta}{d\eta} = \Phi = C_1 \exp\left(-\eta^3\right)$$

This first-order separable equation can be integrated to give

$$\Theta = C_1 \int_0^\eta \exp\left(-\overline{\eta}^3\right) d\overline{\eta} + C_2$$

The lower limit of the integral has been chosen arbitrarily. Application of the boundary conditions enables us to determine the constants of integration. The final solution is then

$$\Theta = 1 - \frac{\int_0^\eta \exp\left(-\overline{\eta}^3\right) d\overline{\eta}}{\int_0^\infty \exp\left(-\overline{\eta}^3\right) d\overline{\eta}} = \frac{\int_\eta^\infty \exp\left(-\overline{\eta}^3\right) d\overline{\eta}}{\int_0^\infty \exp\left(-\overline{\eta}^3\right) d\overline{\eta}} = \frac{1}{\Gamma\left(\frac{4}{3}\right)} \int_\eta^\infty \exp\left(-\overline{\eta}^3\right) d\overline{\eta}$$

where Eq. C.4-3 has been used to write the denominator integral in terms of a gamma function. The integral in the numerator cannot be evaluated analytically.

*b*. To get the wall heat flux, we use Fourier's law of heat conduction:

$$q_r\big|_{r=R} = -k\frac{\partial T}{\partial r}\bigg|_{r=R} = +\frac{k}{R}(T_1 - T_0)\frac{\partial \Theta}{\partial \xi}\bigg|_{\xi=1} = -\frac{k}{R}(T_1 - T_0)\frac{\partial \Theta}{\partial \sigma}\bigg|_{\sigma=0}$$
$$= -\frac{k}{R}(T_1 - T_0)\left(\frac{N}{9\zeta}\right)^{1/3}\frac{\partial \Theta}{\partial \eta}\bigg|_{\eta=0}$$

Into this expression we substitute the final result of (*a*), to obtain (with the help of the Leibniz formula for differentiating an integral, given in Eq. C.3-2)

$$q_r\big|_{r=R} = -\frac{k}{R}(T_1 - T_0)\left(\frac{N}{9\zeta}\right)^{1/3}\left(-\frac{1}{\Gamma\left(\frac{4}{3}\right)}\exp\left(-\eta^3\right)\bigg|_{\eta=0}\right)$$

$$= +\frac{k}{R}(T_1 - T_0)\frac{1}{9^{1/3}\Gamma(\frac{4}{3})}\left(\frac{4\langle v_z\rangle R}{\alpha}\cdot\frac{R}{z}\right)^{1/3}$$

$$= +\frac{k}{R}(T_1 - T_0)\frac{1}{9^{1/3}\Gamma(\frac{4}{3})}\left(\frac{D\langle v_z\rangle\rho}{\mu}\cdot\frac{\hat{C}_p\mu}{k}\cdot\frac{D}{z}\right)^{1/3}$$

$$= \frac{k}{R}(T_1 - T_0)\frac{1}{9^{1/3}\Gamma(\frac{4}{3})}\left(\mathrm{Re}\,\mathrm{Pr}\frac{D}{z}\right)^{1/3}$$

*c*. In (*a*) we can write the velocity profile for the Newtonian fluid (or any generalized Newtonian fluid) as

$$v_z = \langle v_z\rangle\phi(\xi) \qquad \text{with} \qquad \phi(1) = 0 \qquad \text{(no-slip condition)}$$

That is, for the Newtonian fluid $\phi(\xi) = 2(1-\xi^2)$. Then when we switched to the dimensionless coordinate $\sigma$ measured from the wall, the velocity profile can been written as

$$v_z = \langle v_z\rangle\psi(\sigma) \qquad \text{with} \qquad \psi(0) = 0 \qquad \text{(no-slip condition)}$$

Then, the velocity profile in the immediate vicinity of the wall is

$$v_z = \langle v_z\rangle\psi'(0)\sigma + \cdots = -\langle v_z\rangle\phi'(1)\sigma + \cdots$$

and, for the Newtonian fluid,

$$v_z = -\langle v_z\rangle[(d/d\xi)2(1-\xi^2)]\Big|_{\xi=1}\sigma\cdots = 4\langle v_z\rangle\sigma + \cdots$$

Hence, the above solution in (*a*) can be modified for the generalized Newtonian fluid by replacing $N = 4\langle v_z\rangle R/\alpha$ by

$$N = \left(-\frac{d\phi}{d\xi}\bigg|_{\xi=1}\right)\frac{\langle v_z\rangle R}{\alpha}$$

**12D.5 The Graetz problem for flow between parallel plates**

*a*. Let the flow take place in the positive $z$ direction between plates at $x = \pm B$. Then the partial differential boundary equation for $T(x,z)$ and the boundary conditions are:

$$\rho \hat{C}_p v_z(x) \frac{\partial T}{\partial z} = k \frac{\partial^2 T}{\partial x^2} \quad \text{with} \qquad T(x,0) = T_1,\ T(\pm B, z) = T_0$$

We can, if we wish, replace the boundary condition at $x = -B$ by a boundary condition $\partial T/\partial x = 0$ at $x = 0$, because of the fact that we know that the distribution of temperature about the center plane will be symmetric. Then we solve the problem for positive $x$ only, knowing that the full problem must be symmetric about the plane $x = 0$. We now introduce the dimensionless variables

$$\Theta(\xi,\zeta) = \frac{T - T_0}{T_1 - T_0} \qquad \phi(\xi) = \frac{v_z(x)}{\langle v_z \rangle} \qquad \xi = \frac{x}{B} \qquad \zeta = \frac{\alpha z}{\langle v_z \rangle B^2}$$

Then the problem may be reformulated as

$$\phi(\xi) \frac{\partial \Theta}{\partial \zeta} = \frac{\partial^2 \Theta}{\partial \xi^2} \quad \text{with} \qquad \Theta(\xi,0) = 1,\ \Theta(1,\zeta) = 0,\ \left.\frac{\partial \Theta}{\partial \xi}\right|_{\xi=0} = 0$$

We anticipate that the method of separation of variables will be appropriate, and write $\Theta(\xi,\zeta) = X(\xi)Z(\zeta)$. When this is substituted into the equation above, we get (after division by $XZ$)

$$\frac{1}{Z}\frac{dZ}{d\zeta} = \frac{1}{X\phi}\frac{d^2X}{d\xi^2} = -\beta^2$$

where we have introduced the separation constant $-\beta^2$. Hence we get two ordinary differential equations

$$\frac{dZ}{d\zeta} = -\beta^2 Z \qquad \text{whence} \qquad Z(\zeta) \propto \exp\left(-\beta^2 \zeta\right)$$

$$\frac{d^2X}{d\xi^2} + \beta^2\phi X = 0 \qquad \text{with } X(1) = 0 \text{ and } X'(0) = 0$$

Since this last equation with its boundary conditions is a boundary-value problem of the Sturm-Liouville type, there will be an infinite number of values of the eigenvalues $\beta_i^2$ and an infinite number of eigenfunctions $X_i(\xi)$. Furthermore, it is known that the eigenfunctions are orthogonal on the range (0,1) with respect to the weight function $\phi(\xi)$. Therefore, the dimensionless temperature profiles must have the form:

$$\Theta(\xi,\zeta) = \sum_{i=1}^{\infty} A_i X_i(\xi)\exp\left(-\beta_i^2\zeta\right) \quad (*) \qquad \text{with } A_i = \frac{\int_0^1 X_i(\xi)\phi(\xi)d\xi}{\int_0^1 X_i^2(\xi)d\xi}$$

The expression for the $A_i$ is obtained by using the requirement that $\Theta(\xi,0) = 1$ and using the orthogonality relations for the eigenfunctions $X_i(\xi)$.

It remains to find the eigenfunctions $X_i(\xi)$ and the eigenvalues $\beta_i^2$. This has to be done by solving the equation for $X_i(\xi)$ by a power-series technique. It is a tedious process. The results are given in the *Handbook of Heat Transfer*, by Rohsenow, Hartnett, and Cho, cited in a footnote in Table 12D.2 on p. 404.

*b*. In the limit of very large $\zeta$, only one term in the summation in (*) is needed, that is, we need only $X_1(\xi)$ and $\beta_1^2$. To get the heat flux into the wall at large distances downstream, we need to find

$$q_0 = +k\left.\frac{\partial T}{\partial x}\right|_{x=+B} = \frac{k(T_b - T_0)}{B}\frac{\left(\partial\Theta/\partial\xi\right)\big|_{\xi=1}}{\Theta_b} \quad \text{where} \qquad \Theta_b = \frac{T_b - T_0}{T_1 - T_0}$$

and $T_b$ is the bulk fluid temperature.

For very large $\zeta$,

$$\left.\frac{\partial\Theta}{\partial\xi}\right|_{\xi=1} = \sum_{i=1}^{\infty} A_i \left.\frac{dX_i}{d\xi}\right|_{\xi=1} \exp\left(-\beta_i^2\zeta\right) \approx A_1 \left.\frac{dX_1}{d\xi}\right|_{\xi=1} \exp\left(-\beta_1^2\zeta\right)$$

The bulk temperature is, for very large $\zeta$ obtained as follows:

$$\begin{aligned}
\Theta_b(\zeta) &= \frac{\int_0^1 \phi\Theta d\xi}{\int_0^1 \phi d\xi} = \sum_{i=1}^{\infty} A_i \exp\left(-\beta_i^2\zeta\right)\int_0^1 \phi X_i d\xi \\
&= -\sum_{i=1}^{\infty} A_i \exp\left(-\beta_i^2\zeta\right)\int_0^1 \frac{1}{\beta_i^2}\frac{d^2X_i}{d\xi^2} d\xi \\
&= -\sum_{i=1}^{\infty} A_i \exp\left(-\beta_i^2\zeta\right)\frac{1}{\beta_i^2}\left.\frac{dX_i}{d\xi}\right|_0^1 \\
&= -\sum_{i=1}^{\infty} A_i \exp\left(-\beta_i^2\zeta\right)\frac{1}{\beta_i^2}\left.\frac{dX_i}{d\xi}\right|_{\xi=1} \approx -A_1 \exp\left(-\beta_1^2\zeta\right)\frac{1}{\beta_1^2}\left.\frac{dX_1}{d\xi}\right|_{\xi=1}
\end{aligned}$$

Substituting the last two results into the expression for the heat flux, we find

$$q_0 = \frac{k\left(T_0 - T_b\right)}{B}\beta_1^2$$

In terms of the Nusselt number, this result may be written as

$$\mathrm{Nu} = \frac{4hB}{k} = \frac{4q_0 B}{k\left(T_0 - T_b\right)} = 4\beta_1^2 = 7.541$$

which is the result cited in Eq. (F) in Table 14.2-2.

*c*. For the limiting case of very small values of $z$, we start with the same equation used in part (*a*) for $T(x,z)$:

$$v_z(x)\frac{\partial T}{\partial z} = \alpha\frac{\partial^2 T}{\partial x^2} \qquad \text{with} \qquad T(x,0) = T_1,\ T(\pm B, z) = T_0$$

Since we are concerned only with the temperature change in the immediate vicinity of the wall, it is convenient to change to a new coordinate $y = B - x$ and to expand the velocity in powers of $y$ thus:

$$v_z = v_{z,\max}\left[1-\left(\frac{x}{B}\right)^2\right] = v_{z,\max}\left[1-\left(1-\frac{y}{B}\right)^2\right] = v_{z,\max}\left[2\left(\frac{y}{B}\right)-\left(\frac{y}{B}\right)^2\right]$$
$$\approx 2v_{z,\max}\frac{y}{B} = 3\langle v_z\rangle\frac{y}{B}$$

where we have used the result from 2B.3(b); this truncated expression is valid in the vicinity of the wall for small values of $z$. The energy equation for $T(y,z)$ can then be written as

$$3\langle v_z\rangle\frac{y}{B}\frac{\partial T}{\partial z} = \alpha\frac{\partial^2 T}{\partial y^2} \qquad \text{with } T(y,0)=T_1,\ T(0,z)=T_0,\ T(\infty,z)=T_1$$

The last boundary condition is sufficiently good for small values of $z$, since taking the center plane as being at an infinite distance from the wall will not change the temperature distribution significantly. We now introduce the following dimensionless quantities:

$$\Theta = \frac{T_1 - T}{T_1 - T_0} \qquad \eta = \frac{y}{B} \qquad \zeta = \frac{z}{B} \qquad N = \frac{3\langle v_z\rangle B}{\alpha}$$

Then the differential equation for $\Theta(\eta,\zeta)$ and the boundary conditions are:

$$N\eta\frac{\partial\Theta}{\partial\zeta} = \frac{\partial^2\Theta}{\partial\eta^2} \qquad \text{with } \Theta(\eta,0)=0,\ \Theta(0,\zeta)=1,\ \Theta(\infty,\zeta)=0$$

We now seek a solution by the method of combination of variables, using the combined variable $\chi = (N\eta^3/9\zeta)^{1/3}$. This is possible since the first and third boundary conditions both state that the dimensionless temperature is zero. We are then led to the ordinary differential equation (for the details, see the solution to Problem 12D.4)

$$\frac{d^2\Theta}{d\chi^2} + 3\chi^2\frac{d\Theta}{d\chi} = 0 \qquad \text{with} \qquad \Theta(0)=1 \text{ and } \Theta(\infty)=0$$

for which the solution is

$$\Theta = \frac{1}{\Gamma\left(\frac{4}{3}\right)} \int_{\chi}^{\infty} \exp\left(-\bar{\chi}^3\right) d\bar{\chi}$$

To get the wall heat flux, we use Fourier's law

$$q_0 = -k \left.\frac{\partial T}{\partial y}\right|_{y=0} = +\frac{k(T_1 - T_0)}{B} \left.\frac{\partial \Theta}{\partial \eta}\right|_{\eta=0} = +\frac{k(T_1 - T_0)}{B} \left(\frac{N}{9\zeta}\right)^{1/3} \left.\frac{\partial \Theta}{\partial \chi}\right|_{\chi=0}$$

$$= +\frac{k(T_1 - T_0)}{B} \left(\frac{N}{9\zeta}\right)^{1/3} \frac{1}{\Gamma\left(\frac{4}{3}\right)} (-1) \exp\left(-\chi^3\right)\Big|_{\chi=0}$$

$$= +\frac{k(T_0 - T_1)}{B} \left(\frac{N}{9\zeta}\right)^{1/3} \frac{1}{\Gamma\left(\frac{4}{3}\right)}$$

The above result can be put in the form of a Nusselt number as follows:

$$\mathrm{Nu} = \frac{4hB}{k} = \frac{4q_0 B}{k(T_0 - T_b)} \approx \frac{4q_0 B}{k(T_0 - T_1)}$$

$$= 4\left(\frac{N}{9\zeta}\right)^{1/3} \frac{1}{\Gamma\left(\frac{4}{3}\right)} = 4\left(\frac{3\langle v_z \rangle B}{9z/B}\right)^{1/3} \frac{1}{\Gamma\left(\frac{4}{3}\right)}$$

$$= \frac{4}{(3)^{1/3}\Gamma\left(\frac{4}{3}\right)} \left(\frac{\langle v_z \rangle B}{z/B}\right)^{1/3}$$

This is in agreement with Eq. (C) of Table 14.2-2. The replacement of the bulk temperature by the entrance temperature in the first line is appropriate for the entry region, inasmuch as heat has had the opportunity to penetrate only a negligible amount in that region.

**12D.6 The constant wall heat flux problem for parallel plates**

*a. Application of §10.8 to parallel-plate system*

For the laminar flow in the $z$ direction between parallel plates at $x = \pm B$, the differential equation is

$$\rho \hat{C}_p v_{\max}\left[1-\left(\frac{x}{B}\right)^2\right]\frac{\partial T}{\partial z} = k\frac{\partial^2 T}{\partial x^2}$$

We now introduce the dimensionless variables

$$\xi = \frac{x}{B} \qquad \zeta = \frac{kz}{\rho \hat{C}_p v_{\max} B^2} \qquad \Theta(\xi,\zeta) = \frac{k(T-T_1)}{q_0 B}$$

When the differential equation is multiplied by $B/q_0$, the above differential equation can be rewritten as

$$\left(1-\xi^2\right)\frac{\partial \Theta}{\partial \zeta} = \frac{\partial^2 \Theta}{\partial \xi^2} \qquad (**)$$

with the boundary conditions:

| | | |
|---|---|---|
| B. C. 1: | at $\xi = 0$, | $\dfrac{\partial \Theta}{\partial \xi} = 0$ |
| B. C. 2: | at $\xi = +1$, | $\dfrac{\partial \Theta}{\partial \xi} = 1$ |
| B. C. 3: | at $\zeta = 0$, | $\Theta = 0$ |

We now seek a solution valid for large downstream distance, and try a solution of the form

$$\Theta(\xi,\zeta) = C_0\zeta + \Psi(\xi) \qquad (*)$$

Since this function cannot satisfy B. C. 3, we replace the latter by B. C. 4:

Condition 4: at any plane $z$, $\quad 2zWq_0 = \int_0^W \int_{-B}^{+B} \rho \hat{C}_p (T-T_1) v_z dx dy$

or, in dimensionless variables

$$\text{at any plane } \zeta, \qquad \zeta = \int_0^1 \Theta(\xi,\zeta)\left(1-\xi^2\right)d\xi$$

When the trial function in (*) is substituted into (**), we get

$$\left(1-\xi^2\right)C_0 = \frac{d^2\Psi}{d\xi^2}$$

Integration of this equation gives

$$\frac{d\Psi}{d\xi} = C_0\left(\xi - \tfrac{1}{3}\xi^3\right) + C_1 \qquad (***)$$

Further integration gives

$$\Psi = C_0\left(\tfrac{1}{2}\xi^2 - \tfrac{1}{12}\xi^4\right) + C_1\xi + C_2$$

so that

$$\Theta(\xi,\zeta) = C_0\zeta + C_0\left(\tfrac{1}{2}\xi^2 - \tfrac{1}{12}\xi^4\right) + C_1\xi + C_2$$

B. C. 1 leads to $C_1 = 0$. Furthermore, B. C. 2, with the help of (***), gives $C_0 = \frac{3}{2}$. Finally Condition 4 leads to:

$$\zeta = \int_0^1 \left[\tfrac{3}{2}\zeta + \tfrac{3}{2}\left(\tfrac{1}{2}\xi^2 - \tfrac{1}{12}\xi^4\right) + C_2\right]\left(1-\xi^2\right)d\xi$$

from which we get $C_2 = -\frac{39}{280}$. Therefore, finally

$$\Theta(\xi,\zeta) = \tfrac{3}{2}\zeta + \tfrac{3}{4}\left(\xi^2 - \tfrac{1}{6}\xi^4\right) - \tfrac{39}{280}$$

Then the bulk temperature as a function of downstream distance is

$$T_b - T_1 = \frac{\int_0^B [T(x,z) - T_1] v_z(x) dx}{\int_0^B v_z(x) dx} = \frac{q_0 B}{k} \frac{\int_0^1 \Theta(\xi,\zeta)(1-\xi^2) d\xi}{\int_0^1 (1-\xi^2) d\xi} = \frac{q_0 B}{k}\left(\tfrac{3}{2}\zeta\right)$$

and the wall temperature as a function of downstream distance is

$$T_0 - T_1 = \frac{q_0 B}{k}\left(\tfrac{3}{2}\zeta + \tfrac{3}{4}(1-\tfrac{1}{6}) - \tfrac{39}{280}\right) = \frac{q_0 B}{k}\left(\tfrac{3}{2}\zeta + \tfrac{17}{35}\right)$$

Therefore

$$T_0 - T_b = \frac{17}{35}\frac{q_0 B}{k} \qquad \text{or} \qquad \mathrm{Nu} = \frac{4 q_0 B}{k(T_0 - T_1)} = \frac{140}{17}$$

in agreement with Eq. (L) of Table 14.2-2 (note the definition of Nu in the table title!).

*b*. For very small values of $z$ the velocity distribution the linear portion very near the wall is the only part of the distribution that is important; if we let $y$ be the distance from the wall at $x = B$, we have

$$v_z = v_{\max}\left[1 - \left(\frac{x}{B}\right)^2\right] = v_{\max}\left[1 - \left(1 - \left(\frac{y}{B}\right)\right)^2\right]$$
$$= v_{\max}\left(2\frac{y}{B} - \left(\frac{y}{B}\right)^2\right) \approx 2 v_{\max}\frac{y}{B} = 3\langle v_z\rangle \frac{y}{B} \equiv v_0 \frac{y}{B}$$

which serves to define $v_0$. Then the first equation in (*a*) becomes

$$v_0 \frac{y}{B}\frac{\partial T}{\partial z} = \alpha \frac{\partial^2 T}{\partial y^2} \qquad \text{or} \qquad v_0 \frac{1}{B}\frac{\partial T}{\partial z} = \alpha \frac{1}{y}\frac{\partial^2 T}{\partial y^2}$$

Aside from the replacement of $R$ by $B$, these equations are identical to Eqs. 12.2-13 and 14, and they have the same boundary conditions. Therefore, the remainder of Example 12.2-2 is valid for the slit flow (with the replacement of $R$ by $B$).

When $\eta = 0$, we also have $\chi = 0$, and Eq. 12.2-24 gives for the difference between the wall and bulk temperatures

$$\frac{T_0 - T_1}{(q_0 B/k)} = \frac{\sqrt[3]{9\lambda}}{\Gamma(\frac{2}{3})} = \frac{\sqrt[3]{9}}{\Gamma(\frac{2}{3})}\sqrt[3]{\frac{\alpha z}{v_0 B^2}} = \frac{\sqrt[3]{3}}{\Gamma(\frac{2}{3})}\sqrt[3]{\frac{\alpha z}{\langle v_z \rangle B^2}}$$

inasmuch as the bulk temperature is virtually the same as the entrance temperature, the heat having penetrated to only a very small distance into the fluid. Then the Nusselt number is

$$\mathrm{Nu} = \frac{4hB}{k} = \frac{4q_0 B}{k(T_0 - T_1)} = \frac{4\Gamma(\frac{2}{3})}{\sqrt[3]{3}}\sqrt[3]{\frac{\langle v_z \rangle B^2}{\alpha z}}$$

in agreement with Eq. (H) of Table 14.2-2.

*c.* The slit analogs of Eqs. 10.8-12 and 19 are

$$\rho \hat{C}_p v_{z,\max}\left[1 - \left(\frac{x}{B}\right)^2\right]\frac{\partial T}{\partial z} = k\frac{\partial^2 T}{\partial x^2} \qquad \text{or} \qquad (1 - \xi^2)\frac{\partial \Theta}{\partial \zeta} = \frac{\partial^2 \Theta}{\partial \xi^2}$$

In the second equation we have introduced the dimensionless variables

$$\Theta = \frac{T - T_1}{q_0 B/k} \qquad \xi = \frac{x}{B} \qquad \zeta = \frac{z}{\rho \hat{C}_p v_{z,\max} B^2/k}$$

From §10.8 we know that the asymptotic solution for large downstream distances is

$$\Theta_\infty(\xi,\zeta) = \tfrac{3}{2}\zeta + \tfrac{3}{4}\left(\xi^2 - \tfrac{1}{6}\xi^4\right) - \tfrac{39}{280}$$

The complete solution is assumed to be of the following form

$$\Theta(\xi,\zeta) = \Theta_\infty(\xi,\zeta) - \Theta_d(\xi,\zeta)$$

Substituting this into the partial differential equation above tells us that the time-decaying function $\Theta_d(\xi,\zeta)$ must satisfy the equation

$$\left(1-\xi^2\right)\frac{\partial\Theta_d}{\partial\zeta}=\frac{\partial^2\Theta_d}{\partial\xi^2}$$

with the following boundary conditions:

B. C. 1: at $\xi=0$, $\dfrac{\partial\Theta_d}{\partial\xi}=0$

B. C. 2: at $\xi=1$, $\dfrac{\partial\Theta_d}{\partial\xi}=0$

B. C. 3: at $\zeta=0$, $\Theta_d=\Theta_\infty(\xi,0)$

We try a solution by the method of separation of variables, letting $\Theta_d(\xi,\zeta)=X(\xi)Z(\zeta)$. The functions $X$ and $Z$ must then satisfy the following ordinary differential equations

$$\frac{dZ}{d\zeta}=-c^2Z \qquad \frac{d^2X}{d\xi^2}+c^2\left(1-\xi^2\right)X=0 \qquad (****)$$

where $-c^2$ is the separation constant. We then expect to have a complete solution of the form

$$\Theta(\xi,\zeta)=\Theta_\infty(\xi,\zeta)-\sum_{k=1}^{\infty}B_k\exp\left(-c_k^2\zeta\right)X_k(\xi)$$

where

$$B_k=\frac{\int_0^1\Theta_\infty(\xi,0)\left[X_k(\xi)\right]\left(1-\xi^2\right)d\xi}{\int_0^1\left[X_k(\xi)\right]^2\left(1-\xi^2\right)d\xi}$$

Therefore, the problem is reduced to one of finding the eigenfunctions $X_k(\xi)$ of (****), and then getting the eigenvalues $c_k$, by applying the boundary condition at the wall $\xi=1$.

The Nusselt number is shown in Fig. 14.2-1 over the entire range, including the limiting cases given in parts (*a*) and (*b*).

**12D.7 Asymptotic solution for small $z$ for laminar tube flow with constant heat flux**

Exchange the order of integration in Eq. 12D.7-1, recognizing that the area of integration is a triangular area bounded by the lines $\bar{\chi} = \chi$, $\bar{\chi} = \bar{\bar{\chi}}$, and $\bar{\bar{\chi}} = \infty$. Then the inner integration can be performed analytically:

$$\Theta(\eta,\lambda) = \sqrt[3]{9\lambda}\int_{\chi}^{\infty} \psi d\bar{\chi} = \sqrt[3]{9\lambda}\int_{\chi}^{\infty}\left[\frac{3}{\Gamma\left(\frac{2}{3}\right)}\int_{\bar{\chi}}^{\infty}\bar{\bar{\chi}}e^{-\bar{\bar{\chi}}^3}d\bar{\bar{\chi}}\right]d\bar{\chi}$$

$$= \sqrt[3]{9\lambda}\,\frac{3}{\Gamma\left(\frac{2}{3}\right)}\int_{\chi}^{\infty}\bar{\bar{\chi}}e^{-\bar{\bar{\chi}}^3}\left[\int_{\chi}^{\bar{\bar{\chi}}}d\bar{\chi}\right]d\bar{\bar{\chi}}$$

$$= \sqrt[3]{9\lambda}\,\frac{3}{\Gamma\left(\frac{2}{3}\right)}\int_{\chi}^{\infty}\bar{\bar{\chi}}e^{-\bar{\bar{\chi}}^3}\left[\bar{\bar{\chi}} - \chi\right]d\bar{\bar{\chi}}$$

Next rewrite the result as

$$\Theta(\eta,\lambda) = = \frac{\sqrt[3]{9\lambda}}{\Gamma\left(\frac{2}{3}\right)}\left[3\int_{\chi}^{\infty}\bar{\bar{\chi}}^2 e^{-\bar{\bar{\chi}}^3}d\bar{\bar{\chi}} - 3\chi\left(\int_0^{\infty}\bar{\bar{\chi}}e^{-\bar{\bar{\chi}}^3}d\bar{\bar{\chi}} - \int_0^{\chi}\bar{\bar{\chi}}e^{-\bar{\bar{\chi}}^3}d\bar{\bar{\chi}}\right)\right]$$

The first integration can be performed analytically, the second integral can be put in the form of a (complete) gamma function, and the third integral can be written as an incomplete gamma function. In the second integral, we set $t = \bar{\bar{\chi}}^3$, so that $dt = 3\bar{\bar{\chi}}^2 d\bar{\bar{\chi}}$ and $d\bar{\bar{\chi}} = \frac{1}{3}t^{-2/3}dt$. Then

$$\int_0^{\infty}\bar{\bar{\chi}}e^{-\bar{\bar{\chi}}^3}d\bar{\bar{\chi}} = \int_0^{\infty}t^{1/3}e^{-t}\left(\tfrac{1}{3}t^{-2/3}\right)dt = \tfrac{1}{3}\int_0^{\infty}t^{(2/3)-1}e^{-t}dt = \tfrac{1}{3}\Gamma\left(\tfrac{2}{3}\right)$$

A similar derivation can be performed for the indefinite integral. Then we are led directly to Eq. 12.2-24.

**12D.8 Forced conduction heat transfer from a flat plate (thermal boundary layer extends beyond the momentum boundary layer)**

Evaluate the left side of Eq. 12.4-5 by using Eq. 12.4-8:

$$k\frac{\partial T}{\partial y}\bigg|_{y=0} = \frac{2k(T_\infty - T_0)}{\delta_T}$$

Next, evaluate the the integral on the right side of Eq. 12.4-5:

$$\int_0^\infty \rho\hat{C}_p v_x (T_\infty - T)dy = \rho\hat{C}_p v_\infty (T_\infty - T_0)\delta_T \int_0^1 \left(\frac{v_x}{v_\infty}\right)\left(1 - \frac{T - T_0}{T_\infty - T_0}\right)d\eta_T$$

The dimensionless integral on the right side can now be split into two parts to perform the integration, since the equation for the velocity profile changes before we get to the upper limit:

$$\begin{aligned}
&\int_0^{1/\Delta}\left(2\eta_T\Delta - 2\eta_T^3\Delta^3 + \eta_T^4\Delta^3\right)\left(1 - 2\eta_T + 2\eta_T^3 - \eta_T^4\right)d\eta_T \\
&\qquad + \int_{1/\Delta}^1 (1)\left(1 - 2\eta_T + 2\eta_T^3 - \eta_T^4\right)d\eta_T \\
&= \left(\tfrac{7}{10}\Delta^{-1} - \tfrac{13}{15}\Delta^{-2} + \tfrac{67}{140}\Delta^{-4} - \tfrac{7}{36}\Delta^{-5}\right) + \left(\tfrac{3}{10} - \Delta^{-1} + \Delta^{-2} - \tfrac{1}{2}\Delta^{-4} + \tfrac{1}{5}\Delta^{-5}\right) \\
&= \tfrac{3}{10} - \tfrac{3}{10}\Delta^{-1} + \tfrac{2}{15}\Delta^{-2} - \tfrac{3}{140}\Delta^{-4} + \tfrac{1}{180}\Delta^{-5} \equiv F(\Delta)
\end{aligned}$$

We can now write Eq. 12.4-5 as

$$\frac{2k(T_\infty - T_0)}{\delta_T} = \frac{d}{dx}\rho\hat{C}_p v_\infty (T_\infty - T_0)\delta_T F(\Delta) \quad \text{or} \quad \frac{2\alpha}{v_\infty} = F(\Delta)\delta_T\frac{d\delta_T}{dx}$$

Integration of this equation then gives $\delta_T = \sqrt{4(\alpha x/v_\infty)/F(\Delta)}$. The result in Eq. 12.4-12 is valid for $\Delta \geq 1$, so that the ratio of the two boundary-layer thicknesses is

$$\Delta = \frac{\delta_T}{\delta} = \sqrt{\frac{4(\alpha x/v_\infty)}{F(\Delta)}\cdot\frac{37}{1260}\left(\frac{v_\infty}{\nu x}\right)} = \sqrt{\frac{37}{315\,\mathrm{Pr}\,F(\Delta)}}$$

When this is squared, we get Eq. 12D.8-1.

**13B.1 Wall heat flux for turbulent flow in tubes (approximate)**

The integral in Eq. 13.3-6 can be evaluated by making a change of variable. Let

$$\mathrm{Pr}^{1/3}\left(\frac{yv_*}{14.5\nu}\right) = x$$

Then Eq. 13.3-6 becomes (after setting the upper limit equal to infinity):

$$q_0\,\mathrm{Pr}^{-1/3}\left(\frac{v_*}{14.5\nu}\right)^{-1}\int_0^\infty \frac{dx}{1+x^3} = k\left(T_0 - \overline{T}_R\right)$$

The integral may be found in a table of integrals, where we find

$$\int_0^\infty \frac{dx}{1+x^3} = \frac{\pi}{3\sin\frac{1}{3}\pi} = \frac{\pi}{3\left(\frac{1}{2}\sqrt{3}\right)} = \frac{2\pi}{3\sqrt{3}}$$

Hence Eq. 13.3-6 becomes

$$q_0\,\mathrm{Pr}^{-1/3}\left(\frac{v_*}{14.5\nu}\right)^{-1}\frac{2\pi}{3\sqrt{3}} = k\left(T_0 - \overline{T}_R\right)$$

Then the dimensionless wall heat flux is

$$\frac{q_0 D}{k\left(T_0 - \overline{T}_R\right)} = \frac{3\sqrt{3}}{2\pi}\left(\frac{v_*}{14.5\nu}\right)\mathrm{Pr}^{1/3}\,D$$

This now has to be written in terms of dimensionless groups:

$$\frac{q_0 D}{k\left(T_0 - \overline{T}_R\right)} = \frac{3\sqrt{3}}{2\pi(14.5)}\left(\frac{v_*}{\langle v_z\rangle}\right)\left(\frac{D\langle v_z\rangle\rho}{\mu}\right)\mathrm{Pr}^{1/3} = \frac{3\sqrt{3}}{2\pi(14.5)}\sqrt{\frac{f}{2}}\,\mathrm{Re}\,\mathrm{Pr}^{1/3}$$

where Eq. 6.1-4a has been used in order to introduce the friction factor. Thus we have obtained Eq. 13.3-7.

**13B.2 Wall heat flux for turbulent flow in tubes**

*a*. The assumptions are, for the asymptotic solution discussed here (the asymptotic form of the solution is introduced in Eq. 13.4-10):

i. Fully developed turbulent flow
ii. Axial heat conduction is presumed to be negligible with respect to axial heat convection
iii. The turbulent Prandtl number can be taken to be unity (in developing Eq. 13.4-20)
iv. The modified van Driest equaion of Eq. 5.4-7 is used to describe the turbulent velocity profile (in development of Eq. 13.4-20)

*b*. To get the constant $C_0$ in Eq. 13.4-10, we use B. C. 2. First we have to get the derivative $d\Theta/d\xi$ using the Leibniz formula

$$\frac{d\Theta}{d\xi} = 0 + C_0 \frac{I(\xi)}{\xi\left[1+\left(\alpha^{(t)}/\alpha\right)\right]}$$

Then at $\xi = 1$ (the tube wall), the turbulent thermal diffusivity vanishes, and the dimensionless temperature gradient is unity, so that we get the result in Eq. 13.4-14:

$$1 = C_0 \frac{I(1)}{1[1+0]} \qquad \text{or} \qquad C_0 = [I(1)]^{-1}$$

The dimensionless wall temperature is obtained from setting $\xi = 1$ in Eq. 13.4-12 (and setting $C_1 = 0$)

$$\Theta_0 = C_0\zeta + C_0\int_0^1 \frac{I(\xi)}{\xi\left[1+\left(\alpha^{(t)}/\alpha\right)\right]}d\xi + C_2$$

The dimensionless bulk temperature is obtained using the definition in Eq. 10.8-33, starting with Eq. 13.4-12 (and setting $C_1 = 0$):

$$\Theta_b = \frac{\int_0^1 \phi\Theta\xi d\xi}{\int_0^1 \phi\xi d\xi} = C_0\zeta + C_0\frac{1}{I(1)}\int_0^1 \phi\left[\int_0^\xi \frac{I(\bar{\xi})}{\bar{\xi}\left[1+\left(\alpha^{(t)}/\alpha\right)\right]}d\bar{\xi}\right]\xi d\xi + C_2$$

Then we get the dimensionless temperature difference:

$$\Theta_0 - \Theta_b = C_0 \int_0^1 \frac{I(\xi)}{\xi\left[1+\left(\alpha^{(t)}/\alpha\right)\right]} d\xi - C_0 \frac{1}{I(1)} \int_0^1 \phi \left[ \int_0^{\xi} \frac{I(\bar{\xi})}{\bar{\xi}\left[1+\left(\alpha^{(t)}/\alpha\right)\right]} d\bar{\xi} \right] \xi d\xi$$

Then we exchange the order of integration to get the second line of Eq. 13.4-15:

$$\begin{aligned}
\Theta_0 - \Theta_b &= C_0 \int_0^1 \frac{I(\xi)}{\xi\left[1+\left(\alpha^{(t)}/\alpha\right)\right]} d\xi \\
&\qquad - \frac{C_0}{I(1)} \int_0^1 \frac{I(\bar{\xi})}{\bar{\xi}\left[1+\left(\alpha^{(t)}/\alpha\right)\right]} \left[ \int_0^1 \phi \xi d\xi - \int_0^{\bar{\xi}} \phi \xi d\xi \right] d\bar{\xi} \\
&= C_0 \int_0^1 \frac{I(\xi)}{\xi\left[1+\left(\alpha^{(t)}/\alpha\right)\right]} d\xi - \frac{C_0}{I(1)} \int_0^1 \frac{I(\bar{\xi})}{\bar{\xi}\left[1+\left(\alpha^{(t)}/\alpha\right)\right]} \left[ I(1) - I(\bar{\xi}) \right] d\bar{\xi} \\
&= C_0 \int_0^1 \frac{I(\xi)}{\xi\left[1+\left(\alpha^{(t)}/\alpha\right)\right]} d\xi - C_0 \int_0^1 \frac{I(\xi)}{\xi\left[1+\left(\alpha^{(t)}/\alpha\right)\right]} d\xi \\
&\qquad + \frac{C_0}{I(1)} \int_0^1 \frac{\left[I(\bar{\xi})\right]^2}{\bar{\xi}\left[1+\left(\alpha^{(t)}/\alpha\right)\right]} d\bar{\xi}
\end{aligned}$$

The first two terms cancel, and the third term gives Eq. 13.4-16, when the expression for $C_0$ in Eq. 13.4-14 is used.

*c*. It is not necessary to find the constant $C_2$ in this problem because we are interested only in the dimensionless temperature *difference*. In obtaining this difference, the integration constant $C_2$ cancels out. If, however, we want to find the complete temperature profile, then we need an expression for $C_2$.

**13C.1 Constant wall heat flux for turbulent flow between two parallel walls**

*a*. For slit flow, the analog of Eq. 13.4-6 is

$$\phi\frac{\partial\Theta}{\partial\zeta}=\frac{\partial}{\partial\xi}\left[\left(1+\frac{\alpha^{(t)}}{\alpha}\right)\frac{\partial\Theta}{\partial\xi}\right]$$

with the boundary conditions $\partial\Theta/\partial\xi=0$ at $\xi=0$, $\partial\Theta/\partial\xi=1$ at $\xi=1$, and $\Theta=0$ at $\zeta=0$. Here the dimensionless variables are defined as:

$$\Theta=\frac{\overline{T}-T_1}{q_0B/k};\qquad \phi=\frac{\overline{v}_z}{\overline{v}_{z,\max}}\qquad \xi=\frac{x}{b}\qquad \zeta=\frac{z}{\rho\hat{C}_p\overline{v}_{z,\max}B^2/k}$$

We now try a solution of the form $\Theta(\xi,\zeta)=C_0\zeta+\Psi(\xi)$, which should be asymptotically correct at distances far down the tube. This leads then directly to the following differential equation

$$C_0\phi=\frac{d}{d\xi}\left[\left(1+\frac{\alpha^{(t)}}{\alpha}\right)\frac{d\Psi}{d\xi}\right]$$

A first integration then gives

$$\left(1+\frac{\alpha^{(t)}}{\alpha}\right)\frac{d\Psi}{d\xi}=C_0\int_0^{\xi}\phi\left(\overline{\xi}\right)d\overline{\xi}+C_1=C_0J(\xi)$$

where we have introduced the abbreviation $J(\xi)=\int_0^{\xi}\phi\left(\overline{\xi}\right)d\overline{\xi}$ and the constant of integration has been set equal to zero since $\partial\Theta/\partial\xi=0$ at t he tube axis. Then, since $\partial\Theta/\partial\xi=1$ and $\alpha^{(t)}=0$ at the tube wall, we find that $C_0=1/J(1)$. Thus the dimensionless temperature profile is

$$\Theta(\xi,\zeta)=C_0\zeta+C_0\int_0^{\xi}\frac{J\left(\overline{\xi}\right)}{1+\left(\alpha^{(t)}/\alpha\right)}d\overline{\xi}+C_2$$

in which it is understood that $\alpha^{(t)}$ is a function of $\overline{\xi}$. Next we form

$$\Theta_0 - \Theta_b = C_0 \int_0^1 \frac{J(\bar{\xi})}{1+\left(\alpha^{(t)}/\alpha\right)} d\bar{\xi} - C_0 \frac{\int_0^1 \phi(\xi)\left[\int_0^\xi \frac{J(\bar{\xi})}{1+\left(\alpha^{(t)}/\alpha\right)} d\bar{\xi}\right] d\xi}{\int_0^1 \phi(\xi) d\xi}$$

Next we exchange the order of integration in the numerator and make use of the definition of $J$ in the denominator to find

$$\Theta_0 - \Theta_b = C_0 \int_0^1 \frac{J(\bar{\xi})}{1+\left(\alpha^{(t)}/\alpha\right)} d\bar{\xi} - \frac{C_0}{J(1)} \int_0^1 \frac{J(\bar{\xi})}{1+\left(\alpha^{(t)}/\alpha\right)} \left[\int_{\bar{\xi}}^1 \phi(\xi) d\xi\right] d\bar{\xi}$$

The inner integral of the second term may be written as $J(1) - J(\bar{\xi})$, and the $J(1)$ contribution to the second term of $\Theta_0 - \Theta_b$ exactly cancels the first term. The final expression for $\Theta_0 - \Theta_b$ is thus

$$\Theta_0 - \Theta_b = \frac{1}{[J(1)]^2} \int_0^1 \frac{[J(\xi)]^2}{1+\left(\alpha^{(t)}/\alpha\right)} d\xi$$

Then, since $q_0 B/k(T_0 - T_b) = 1/(\Theta_0 - \Theta_b)$, Eq. 13C.1-1 follows.

*b*. For laminar Newtonian flow, $\phi = 1 - \xi^2$ and $\alpha^{(t)} = 0$; then

$J(\xi) = \int_0^\xi \left(1 - \bar{\xi}^2\right) d\bar{\xi} = \xi - \frac{1}{3}\xi^3$ and $J(1) = \frac{2}{3}$. Furthermore

$$\int_0^1 [J(\xi)]^2 d\xi = \int_0^1 \left(\xi^2 - \tfrac{2}{3}\xi^4 + \tfrac{1}{9}\xi^6\right) d\xi = \tfrac{68}{315}$$

Hence, finally, for laminar Newtonian flow (using the mean hydraulic radius in the expression for the Nusselt number)

$$\mathrm{Nu} = \frac{q_0(4B)}{k(T_0 - T_b)} = \frac{140}{17} \qquad \text{(see the entry in Table 14.2-2)}$$

For plug flow, $J(\xi) = \int_0^\xi d\bar{\xi} = \xi$, $J(1) = 1$, and $\int_0^1 [J(\xi)]^2 d\xi = \int_0^1 \xi^2 d\xi = \frac{1}{3}$. Hence we finally get Nu = 12, which is the entry in Table 14.2-2.

**13D.1 The temperature profile for turbulent flow in a tube**

*a*. Condition #4 is $\int_0^1 \Theta(\xi,\zeta)\phi(\xi)\xi d\xi = \zeta$. Into this, we now substitute the expression for dimensionless temperature of Eq. 13.4-12, in order to get $C_2$. We do this term by term:

1st term: $\int_0^1 C_0\zeta\phi(\xi)\xi d\xi = C_0\zeta\int_0^1\phi(\xi)\xi d\xi = \zeta$ (Eq. 13.4-14 was used)

4th term: $\int_0^1 C_2\phi\xi d\xi = C_2 I(1)$

2nd term:
$$\int_0^1 C_0\left[\int_0^{\xi}\frac{I(\bar{\xi})}{\bar{\xi}\left[1+\left(\alpha^{(t)}/\alpha\right)\right]}d\bar{\xi}\right]\phi(\xi)\xi d\xi$$
$$= C_0\int_0^1\frac{I(\bar{\xi})}{\bar{\xi}\left[1+\left(\alpha^{(t)}/\alpha\right)\right]}\left[\int_{\bar{\xi}}^1\phi(\xi)\xi d\xi\right]d\bar{\xi}$$
$$= C_0\int_0^1\frac{I(\bar{\xi})}{\bar{\xi}\left[1+\left(\alpha^{(t)}/\alpha\right)\right]}\left[\int_0^1\phi(\xi)\xi d\xi - \int_0^{\bar{\xi}}\phi(\xi)\xi d\xi\right]d\bar{\xi}$$
$$= \int_0^1\frac{I(\bar{\xi})}{\bar{\xi}\left[1+\left(\alpha^{(t)}/\alpha\right)\right]}d\bar{\xi} - \frac{1}{I(1)}\int_0^1\frac{\left[I(\bar{\xi})\right]^2}{\bar{\xi}\left[1+\left(\alpha^{(t)}/\alpha\right)\right]}d\bar{\xi}$$

Substitution into the 4th condition above and solving for $C_2$ gives

$$C_2 = \int_0^1\frac{\left[I(\bar{\xi})/I(1)\right]^2}{\bar{\xi}\left[1+\left(\alpha^{(t)}/\alpha\right)\right]}d\bar{\xi} - \int_0^1\frac{\left[I(\bar{\xi})/I(1)\right]}{\bar{\xi}\left[1+\left(\alpha^{(t)}/\alpha\right)\right]}d\bar{\xi}$$

which is the same as Eq. 13D.1-1.

*b*. For a Newtonian fluid $I(\xi) = \int_0^{\xi}\left(1-\bar{\xi}^2\right)\bar{\xi}d\bar{\xi} = \frac{1}{2}\xi^2 - \frac{1}{4}\xi^4$ and $I(1) = \frac{1}{4}$. Then $\left[I(\xi)/I(1)\right] = 2\xi^2 - \xi^4$, $\left[I(\xi)/I(1)\right]^2 = 4\xi^4 - 4\xi^6 + \xi^8$, and

$$C_2 = \int_0^1\left(4\xi^3 - 4\xi^5 + \xi^7\right)d\xi - \int_0^1\left(2\xi - \xi^3\right)d\xi = -\tfrac{7}{24}$$

**14A.1 Average heat transfer coefficients.**

The total heat transfer rate is:

$$\begin{aligned} Q &= w\hat{C}_p(T_{b2} - T_{b1}) \\ &= (10,000 \text{ lb}_m/\text{hr})(0.6 \text{ Btu/lb}_m\cdot\text{F})(200 - 100 \text{ F}) \\ &= 600,000 \text{ Btu/hr} \end{aligned}$$

The total inside surface area of the tubes is:

$$A = \pi D L_{\text{tot}} = (\pi)(1.00 - 2 \times 0.065 \text{ in})(1/12 \text{ ft/in})(300 \text{ ft}) = 68.3 \text{ ft}^2$$

The various temperature differences between the inner tube surfaces and the oil are:

$$\begin{aligned} (T_0 - T_b)_1 &= 213 - 100 = 113 \text{ F} \\ (T_0 - T_b)_a &= (113 + 13)/2 = 63 \text{ F} \\ (T_0 - T_b)_{\ln} &= (113 - 13)/\ln(113/13) = 46.2 \text{ F} \end{aligned}$$

Insertion of these values into Eqs. 14.1-2,3,4 then gives the heat transfer coefficients:

$$\begin{aligned} h_1 &= (600,000)/(68.3 \times 113) = 78 \text{ Btu/hr}\cdot\text{ft}^2\cdot\text{F} \\ h_a &= (600,000)/(68.3 \times 63) = 139 \text{ Btu/hr}\cdot\text{ft}^2\cdot\text{F} \\ h_{\ln} &= (600,000)/(68.3 \times 46.2) = 190 \text{ Btu/hr}\cdot\text{ft}^2\cdot\text{F} \end{aligned}$$

**14A.2 Heat transfer in laminar tube flow.**

**(a)** The Prandtl number, based on the property values given, is

$$\mathrm{Pr} = \frac{\hat{C}_p \mu}{k} = \frac{(0.49\ \mathrm{Btu/lb}_m\cdot\mathrm{F})(1.42\ \mathrm{lb}_m/\mathrm{ft}\cdot\mathrm{hr})}{(0.0825\ \mathrm{Btu/hr\cdot ft\cdot F})} = 8.43$$

**(b)** The Reynolds umber is

$$\begin{aligned}\mathrm{Re} &= \frac{D\langle v\rangle\rho}{\mu} = \frac{4w}{\pi D\mu} \\ &= \frac{(4)(100\ \mathrm{lb}_m/\mathrm{hr})}{(\pi)(1/12\ \mathrm{ft})(1.42\ \mathrm{lb}_m/\mathrm{ft}\cdot\mathrm{hr})} \\ &= 1076\end{aligned}$$

**(c)** From Fig. 14.3-2, at $L/D = (20\ \mathrm{ft})/(1/12\ \mathrm{ft}) = 240$, we read

$$\frac{(T_{b2} - T_{b1})}{(T_0 - T_b)_{\ln}} \cdot \left(\frac{D}{4L}\right) \cdot \left(\frac{\hat{C}_p \mu}{k}\right)^{2/3} \left(\frac{\mu_b}{\mu_0}\right)^{-0.14} = 0.0028$$

Now, for uniform $T_0$,

$$\frac{(T_{b2} - T_{b1})}{(T_0 - T_b)_{\ln}} = \ln\left(\frac{T_0 - T_{b1}}{T_0 - T_{b2}}\right)$$

Hence, for this problem,

$$\ln\left(\frac{T_0 - T_{b1}}{T_0 - T_{b2}}\right) = (0.0028)(4 \times 240)(8.43)^{-2/3}(1.0) = 0.649,$$

giving

$$\frac{T_0 - T_{b2}}{T_0 - T_{b1}} = \exp(-0.649) = 0.523$$

Insertion of $T_0 = 215°\mathrm{F}$ and $T_{b1} = 100°\mathrm{F}$ gives

$$T_{b2} = 215 - 0.523(215 - 100) = 155°\mathrm{F}$$

**14A.3 Effect of flow rate on exit temperature from a heat exchanger.**

(a) From the solution of Problem 14A.2 we find that Re= $10.76w$ and that

$$\frac{T_0 - T_{b2}}{T_0 - T_{b1}} = \exp\left(\frac{-0.649Y}{0.0028}\right) = \exp(-232Y)$$

in which $w$ is the mass flow rate in $\text{lb}_m/\text{hr}$ and $Y$ is the ordinate of Fig. 14.3-2 at the prevailing Reynolds number. The exit bulk temperature is then

$$T_{b2} = T_{b1} + (T_0 - T_{b1})\left[1 - \exp(-232Y)\right]$$

(b) The total heat flow through the tube wall is

$$Q = w\hat{C}_p(T_{b2} - T_{b1})$$

Calculations for (a) and (c) are summarized below:

| $w$, $\text{lb}_m/\text{hr}$ | Re | $Y$ | $1 - e^{-232Y}$ | $T_{b2} - T_{b1}$, °F | $T_{b2}$, °F | $Q$, Btu/hr |
|---|---|---|---|---|---|---|
| 100 | 1076 | 0.0028 | 0.478 | 54.9 | 154.9 | 2690 |
| 200 | 2152 | 0.00185 | 0.349 | 40.1 | 140.1 | 3930 |
| 400 | 4304 | 0.0036 | 0.566 | 65.1 | 165.1 | 12760 |
| 800 | 8608 | 0.0040 | 0.605 | 69.5 | 169.5 | 27260 |
| 1600 | 17216 | 0.0037 | 0.576 | 66.3 | 166.3 | 51950 |
| 3200 | 34432 | 0.0033 | 0.535 | 61.5 | 161.5 | 96460 |

**14A.4 Local heat transfer coefficient for turbulent forced convection in a tube.**

Fig. 14.3-2 requires the viscosity values $\mu_b$ at $T_b$ and $\mu_0$ at $T_0$. Interpolation in Table 1.1-1, or on page 6-3 of *CRC Handbook of Chemistry and Physics,* 81st Edition (2000-2001) gives $\mu_b = 1.13$ mPa·S at $T_b = 60°\text{F} = 15.56°\text{C}$, and $\mu_0 = 0.398$ mPa·s at $T_0 = 160°\text{F} = 71.11°\text{C}$, whence $(\mu_b/\mu_0) = 2.84$. The other physical properties in Fig. 14.3-2, including $\mu$ in the Reynolds and Prandtl numbers, are evaluated at the "film temperature" $T_f = (T_0 + T_b)/2 = 110°\text{F} = 43.33°\text{C}$, giving $\mu = 1.489$ lb$_m$/hr·ft, $\hat{C}_p = 4.1792$ J/g·K $= 0.99885$ Btu/lb$_m$·F, and $k = 0.6348$ W/m·K $= 0.36679$ Btu/hr·ft·F. Then the Prandtl number at $T_f$ is

$$\frac{\hat{C}_p\mu}{k} = \frac{(0.99885\ \text{Btu/lb}_m\cdot\text{F})(1.489\ \text{Btu/hr·ft·F})}{0.36679\ \text{Btu/hr·ft·F}} = 4.05$$

and the Reynolds number calculated at $T_f$ is

$$\begin{aligned}\text{Re} &= \frac{DG}{\mu} = \frac{4w}{\pi D\mu} \\ &= \frac{4 \times (15{,}000\ \text{lb}_m/\text{hr})}{\pi(2/12\ \text{ft})(1.489\ \text{lb}_m/\text{hr·ft})} = 7.70 \times 10^4\end{aligned}$$

From Fig. 14.3-2 at this value of Re, we get the ordinate expression

$$\frac{h_{\text{ln}}}{\hat{C}_pG}\left(\frac{\hat{C}_p\mu}{k}\right)^{2/3}\left(\frac{\mu_b}{\mu_0}\right)^{-0.14} = 0.0028 \qquad (14\text{A}.4-1)$$

in which $h_{\text{ln}}$ can be regarded also as $h_{\text{loc}}$ according to the analysis in Problem 14B.5, and $G = 4w/\pi D^2 = (4\pi)(15{,}000\ \text{lb}_m/\text{hr})/(2/12\ \text{ft})^2 = 6.88 \times 10^5$ lb$_m$/hr·ft$^2$.

Insertion of the foregoing results into Eq. 14A.4-1 then gives

$$\begin{aligned}h_{\text{loc}} &= 0.0028\hat{C}_pG\left(\frac{\hat{C}_p\mu}{k}\right)^{-2/3}\left(\frac{\mu_b}{\mu_0}\right)^{+0.14} \\ &= 0.0028(0.99885\ \text{Btu/lb}_m\cdot\text{F})(6.88 \times 10^5\ \text{lb}_m/\text{hr·ft}^2)(4.05)^{-2/3}(2.84)^{+0.14} \\ &= 7.8 \times 10^2\ \text{Btu/hr·ft}^2\cdot\text{F}\end{aligned}$$

as the asymptotic value of the local heat transfer coefficient, and

$$\begin{aligned}q_r|_{r=R} = -q_0 &= h_{\text{loc}}(T_b - T_0) \\ &= (7.8 \times 10^2)(60 - 160) = -7.8 \times 10^4\end{aligned}$$

as the radial heat flux at the inner wall of the pipe.

**14A.5 Heat transfer from condensing vapors.**

**(a)** The boundaries of the condensate layer are at $T_0 = 190°\text{F}$ and $T_d = 212°\text{F}$; thus the film temperature $T_f$ is $(190 + 212)/2 = 201°\text{F}$. The physical properties at this temperature are well approximated by the values at 200°F, given in Ex. 14.7-1:

$$\begin{aligned} \Delta\hat{H}_{\text{vap}} &= 978 \text{ Btu/lb}_m \\ k &= 0.393 \text{ Btu/hr·ft·F} \\ \rho &= 60.1 \text{ lb}_m/\text{ft}^3 \\ \mu &= 0.738 \text{ lb}_m/\text{hr·ft} \end{aligned}$$

The resulting abscissa for Fig. 14.7-2 is:

$$\begin{aligned} &\frac{k\rho^{2/3}g^{1/3}(T_d - T_0)L}{\mu^{5/3}\Delta\hat{H}_{\text{vap}}} \\ &\quad = \frac{(0.393 \text{ Btu/hr·ft·F})(60.1 \text{ lb}_m/\text{ft}^3)^{2/3}(4.17 \times 10^8 \text{ ft/hr}^2)^{1/3}(22 \text{ F})(1.0 \text{ ft})}{(0.738 \text{ lb}_m/\text{ft·hr})^{5/3}(978 \text{ Btu/lb}_m)} \\ &\quad = 168 \text{ Btu}^{1-1}\text{hr}^{-1-2/3+5/3}\text{ft}^{1-2+1/3+1+5/3}\text{F}^{-1+1}, \text{ dimensionless} \end{aligned}$$

This value falls in the laminar region of Fig. 14.7-2. Extrapolation of the laminar line with a slope of 3/4, consistent with Eq. 14.7-5, gives

$$\Gamma/\mu = 170(0.168)^{3/4} = 45$$

The heat transfer rate, neglecting subcooling, is

$$\begin{aligned} Q &= \pi D\Gamma\Delta\hat{H}_{\text{vap}} = \pi(1/12 \text{ ft})(45 \times 0.738 \text{ lb}_m/\text{ft·hr})(978 \text{ Btu/lb}_m) \\ &= 8400 \text{ Btu/hr.} \end{aligned}$$

A similar result is obtainable from Eq. 14.7-5, once the flow is known to be laminar.

**(b)** Comparison of Eqs. 14.7-5 and 6 gives, for laminar condensate flow:

$$\frac{Q_{\text{hor.}}}{Q_{\text{vert.}}} = \frac{0.725}{0.943}\left(\frac{L}{D}\right)^{1/4}$$

Hence, if the tube were horizontal the heat transfer rate would be:

$$Q_{\text{hor.}} = (8400)(0.725/0.943)(12)^{1/4} = 12{,}000 \text{ Btu/hr}$$

The assumption of laminar condensate flow on the horizontal tube is clearly reasonable, given the result of **(a)** and the still smaller value of $\Gamma/\mu$ in **(b)**.

**14A.6 Forced-convection heat transfer from an isolated sphere.**

**(a)** The physical properties of air at 1 atm and $T_f = \frac{1}{2}(T_0 + T_\infty) = 150°\text{F} = 65.56°\text{C} = 338.7\text{K}$ are:

$\rho = pM/RT_f = 1.042 \times 10^{-3}$ g/cm$^3$

$\mu = 0.02023$ cp $= 2.023 \times 10^{-4}$ g/cm·s, from Table 1.1-2

$\hat{C}_p = 1.008$ J/g·K $= 1.008$ W·s/g·K from CRC Handbook 2000-2001, pp. 6-1,6-2.

$k = 26.9 \times 10^{-3}$ W/m·K from CRC Handbook 2000-2001, p. 6-185

The Reynolds and Prandtl numbers are

$$\begin{aligned}
\text{Re} &= \frac{Dv_\infty \rho}{\mu} \\
&= \frac{(2.54\text{ cm})(100 \times 12 \times 2.54\text{ cm/s})(1.042 \times 10^{-3}\text{ g/cm}^3)}{2.023 \times 10^{-4}\text{ g/cm·s}} \\
&= 3.99 \times 10^4 \\
\text{Pr} &= \frac{\hat{C}_p \mu}{k} \\
&= \frac{(1.008\text{ W·s/g·K})(2.023 \times 10^{-4}\text{ g/cm·s})}{26.9 \times 10^{-5}\text{ W/cm·K}} \\
&= 0.703
\end{aligned}$$

Substitution of these values into Eq. 14.4-5 gives

$$\begin{aligned}
\text{Nu}_m &= 2 + 0.60(3.99 \times 10^4)^{1/2}(0.703)^{1/3} \\
&= 108.6
\end{aligned}$$

Hence,

$$\begin{aligned}
h_m &= 108.6k/D = 108.6(26.9 \times 10^{-5}\text{ W/cm·K})/(2.54\text{ cm}) \\
&= 0.01150\text{ W/cm}^2\text{·K}
\end{aligned}$$

and the convective heat loss rate is

$$\begin{aligned}
Q &= \pi D^2 h_m (T_0 - T_\infty) \\
&= \pi(2.54^2\text{ cm}^2)(0.01150\text{ W/cm}^2\text{·K})([100/1.8]\text{ K}) \\
&= 12.9\text{ W} = 3.1\text{ cal/s}
\end{aligned}$$

according to Eq. 14.4-5. The radiative loss is about 1.0 W for a perfectly black sphere in a large enclosure with walls at 100°F, and can be estimated by the methods of §16.5.

**(b)** For Eq. 14.4-6, we need $\mu_0 = 0.02144$ cp at $T_0 = 200°\text{F} = 93.3°\text{C}$ and the following property values at $T_\infty = 100°\text{F} = 37.8°\text{C} = 310.9$ K:

$$\begin{aligned}
\mu_\infty &= 0.01898 \text{ cp} = 1.898 \times 10^{-4} \text{ g/cm·s} \\
\rho &= pM/RT_\infty = 1.136 \times 10^{-3} \text{ g/cm}^3 \\
\hat{C}_p &= 1.007 \text{ J/g·K} = 1.007 \text{ W·s/g·K} \\
k &= 27.0 \times 10^{-3} \text{ mW/m·K} = 27.0 \times 10^{-5} \text{ W/cm·K}
\end{aligned}$$

The resulting values of Re and Pr calculated at the upstream state are

$$\begin{aligned}
\text{Re} &= \frac{(2.54 \text{ cm})(100 \times 12 \times 2.54 \text{ cm/s})(1.136 \times 10^{-3} \text{ g/cm}^3)}{1.898 \times 10^{-4} \text{ g/cm·s}} \\
&= 4.63 \times 10^4 \\
\text{Pr} &= \frac{(1.007 \text{ W·s/g·K})(1.898 \times 10^{-4} \text{ g/cm·s})}{27.0 \times 10^{-5} \text{ W/cm·K}} \\
&= 0.708
\end{aligned}$$

Substitution of these values into Eq. 14.4-6 gives

$$\begin{aligned}
\text{Nu}_m &= 2 + (0.4 \text{ Re}^{1/2} + 0.06 \text{ Re}^{2/3}) \text{ Pr}^{0.4} \left(\frac{\mu_\infty}{\mu_0}\right)^{1/4} \\
&= 2 + 138.1 = 140.1
\end{aligned}$$

whence

$$\begin{aligned}
h_m &= 140.1 k_\infty/D = 140.1(27.0 \times 10^{-5} \text{ W/cm·K})/(2.54 \text{ cm}) \\
&= 0.0149 \text{ W/cm}^2\text{·K}
\end{aligned}$$

and the convective heat loss rate is

$$\begin{aligned}
Q &= \pi D^2 h_m (T_0 - T_\infty) \\
&= \pi(2.54 \text{ cm})^2(0.0149 \text{ W/cm}^2\text{·K})([100/1.8] \text{ K}) \\
&= 16.8 \text{ W} = 4.0 \text{ cal/s}
\end{aligned}$$

according to Eq. 14.4-6. This result is believed to be more accurate than that found in **(a)**.

**14A.7 Free-convection heat transfer from an isolated sphere.**

For the conditions of this problem, the thermal expansion coefficient $\beta = 1/T_f$ is (1/338.7 K), and the other physical properties are the same as in part **(a)** of Problem 14A.6. (Note that, for the correlations in §14.6, $\beta$ and $\rho$ are evaluated at $T_f$ rather than $\overline{T}$ for calculation of Gr.) Then

$$\begin{aligned}
\mathrm{GrPr} &= \left(\frac{D^3\rho^2 g\beta\Delta T}{\mu^2}\right)\left(\frac{\hat{C}_p\mu}{k}\right) \\
&= \frac{(2.54\ \mathrm{cm})^3(0.001042\ \mathrm{g/cm{\cdot}s})^2(980.7\ \mathrm{cm/s^2})(100/[1.8\times 338.7])}{(2.023\times 10^{-4}\ \mathrm{g/cm{\cdot}s})^2}(0.703) \\
&= 4.92\times 10^4
\end{aligned}$$

Eq. 14.6-4 gives

$$\begin{aligned}
\mathrm{Nu}_m^{\mathrm{lam}} &= \frac{0.878\times 0.671}{[1+(0.492/\ \mathrm{Pr})^{9/16}]^{4/9}}\,(\ \mathrm{GrPr})^{1/4} \\
&= \frac{(0.878)(0.671)}{[1+(0.492/0.703)^{9/16}]^{4/9}}\,(4.92\times 10^4)^{1/4} \\
&= 6.73
\end{aligned}$$

Hence,

$$\begin{aligned}
h_m &= 6.73k/D = (6.73)(26.9\times 10^{-5}\ \mathrm{W/cm{\cdot}K})/(2.54\ \mathrm{cm}) \\
&= 0.000712\ \mathrm{W/cm^2{\cdot}K}
\end{aligned}$$

and the convective heat loss rate is

$$\begin{aligned}
Q &= \pi D^2 h_m(T_0 - T_\infty) \\
&= \pi(2.54\ \mathrm{cm})^2(0.000712\ \mathrm{W/cm^2{\cdot}K})([100/1.8]\ \mathrm{K}) \\
&= 0.80\ \mathrm{W} = 0.20\ \mathrm{cal/s}
\end{aligned}$$

By the methods of §16.5, one can calculate that the rate of heat loss by radiation is of comparable magnitude: 1.0 W for a perfectly black sphere in a large enclosure with walls at 100°F.

**14A.8 Heat loss by free convection from a horizontal pipe immersed in a liquid.**

From the data provided, we find the following values at $T_f = 32.3°\text{C}$:

$$\beta = \frac{-1}{\rho}\frac{d\rho}{dT} = -\frac{0.99463 - 0.99528}{0.99496(33.3 - 31.3)}$$
$$= 3.27 \times 10^{-4}\ \text{K}^{-1} = 1.815 \times 10^{-4}\ \text{F}^{-1}$$
$$\rho = 0.99496\ \text{g/cm}^3)(12 \times 2.54\ \text{cm/ft})^3/(453.59\ \text{g/lb}_m) = 62.11\ \text{lb}_m/\text{ft}^3$$
$$\hat{C}_p = 0.9986\ \text{cal/g·C} \approx 0.9986\ \text{Btu/lb}_m\text{·F}$$
$$\mu = 0.7632\ \text{cp} = 1.8463\ \text{lb}_m/\text{hr·ft}$$
$$k = 0.363\ \text{Btu/hr·ft·F}$$
$$\frac{\hat{C}_p\mu}{k} = \frac{(0.9986\ \text{Btu/lb}_m\text{·F})(1.8463\ \text{lb}_m/\text{hr·ft})}{0.363\ \text{Btu/hr·ft·F}} = 5.08$$

Hence,

$$\text{GrPr} = \frac{(0.5\ \text{ft})^3(62.11\ \text{lb}_m/\text{ft}^3)^2(4.17 \times 10^8\ \text{ft/hr}^2)(1.815 \times 10^{-4} \times 20)}{(1.8463\ \text{lb}_m/\text{hr·ft})^2}(5.08)$$
$$= 1.088 \times 10^9$$

Then from Eqs. 14.6-1 to 3 and Table 14.6-1 we get

$$\text{Nu}_m = 0.772\left(\frac{0.671}{[1 + (0.492/5.08)^{9/16}]^{4/9}}\right)(1.088 \times 10^9)^{1/4}$$
$$= 0.772\left(\frac{0.671}{1.112}\right)(181.6) = 84.6$$

The heat transfer coefficient is then

$$h_m = \text{Nu}\frac{k}{D} = 84.6\left(\frac{0.363}{0.5}\right) = 61.4\ \text{Btu/hr·ft}^2\text{·F}$$

and the rate of convective heat loss per unit length of the pipe is

$$\frac{Q}{L} = \frac{h_m A \Delta T}{L} = h_m \pi D \Delta T$$
$$= (61.4\ \text{Btu/hr·ft}^2\text{·F})(3.1415)(0.5\ \text{ft})(20°\text{F}) = 1930\ \text{Btu/hr·ft}$$

**14B .1 Limiting local Nusselt numbers for plug flow with constant heat flux**

(*Note: Problem 10B.9-1 should be worked prior do doing this problem*)

*a*. For circular tubes with plug flow, the dimensionless temperature distribution, the dimensionless wall temperature, and the dimensionless bulk temperature are obtainable from Eq. 10B.9-1 on p. 325:

$$\Theta = \frac{k(T - T_1)}{q_0 R} = 2\zeta + \tfrac{1}{2}\xi^2 - \tfrac{1}{4} \qquad \text{(where } \xi = \frac{r}{R} \text{ and } \zeta = \frac{\alpha z}{v_0 R^2}\text{)}$$

$$\Theta_0 = \Theta\big|_{\xi=1} = 2\zeta + \tfrac{1}{4}$$

$$\Theta_b = \frac{\int_0^1 \Theta v_0 \xi d\xi}{\int_0^1 v_0 \xi d\xi} = 2\int_0^1 \Theta \xi d\xi = 2\int_0^1 \left(2\zeta + \tfrac{1}{2}\xi^2 - \tfrac{1}{4}\right)\xi d\xi = 2\zeta$$

Then the difference between the wall temperature and the bulk temperature is

$$\Theta_0 - \Theta_b = \frac{k(T_0 - T_b)}{q_0 R} = \frac{1}{4}$$

and the Nusselt number is

$$\mathrm{Nu} = \frac{hD}{k} = \frac{q_0(2R)}{k(T_0 - T_b)} = 4 \cdot 2 = 8$$

in agreement with Eq. (J) on p. 430. Note that, by convention, the Nusselt number for tubes is defined using the diameter rather than the radius, and this definition introduces the factor of 2.

*b*. For the plug flow in a slit of width $2B$, we have for the dimensionless temperature, wall temperature, and bulk temperature, all obtainable from the results of part (*b*) of Problem 10B.9:

$$\Theta = \frac{k(T - T_1)}{q_0 B} = \zeta + \tfrac{1}{2}\sigma^2 - \tfrac{1}{6} \qquad \text{(where } \sigma = \frac{x}{B} \text{ and } \zeta = \frac{\alpha z}{v_0 B^2}\text{)}$$

$$\Theta_0 = \Theta\big|_{\sigma=1} = \zeta + \tfrac{1}{3}$$

$$\Theta_b = \frac{\int_0^1 \Theta v_0 d\sigma}{\int_0^1 v_0 d\sigma} = \int_0^1 \Theta d\sigma = \int_0^1 \left(\zeta + \tfrac{1}{2}\sigma^2 - \tfrac{1}{6}\right) d\sigma = \zeta$$

Then the difference between the wall temperature and the bulk temperature is

$$\Theta_0 - \Theta_b = \frac{k(T_0 - T_b)}{q_0 B} = \frac{1}{3}$$

The Nusselt number is then

$$\mathrm{Nu} = \frac{h(4B)}{k} = \frac{q_0(4B)}{k(T_0 - T_b)} = 3 \cdot 4 = 12$$

in agreement with Eq. (J) on p. 431. Note that the Nusselt number for slits is defined in terms of $4B$, and this is the origin of the factor of 4 which appears here (see heading of Tables 14.2-1 and 2, as well as the caption for Figure 14.2-1 on p. 429.

**14B.2 Local overall heat transfer coefficient.**

Let 0 and 1 denote the inner and outer surfaces of the tube, and $h_0$ and $h_1$ denote the local heat transfer coefficients on those surfaces at the cross-section where the oil bulk temperature is 150°F. According to the development in §9.6, the temperature drops within a cross-section have the same ratio as the corresponding resistance terms that sum to $1/(r_0 U_0)$:

$$\frac{T_1 - 150}{213 - T_1} = \left[\frac{1}{r_0 h_0} + \frac{\ln(r_1/r_0)}{k_{01}}\right] / \left[\frac{1}{r_1 h_1}\right]$$

The numerator on the right is

$$\frac{1}{(0.435/12)(190)} + \frac{\ln(0.5/0.435)}{220}$$
$$= 0.1452 + 0.0006 = 0.1458 \text{ hr·ft·F/Btu}$$

in which a thernal conductivity of $k_{01} = 220$ Btu/hr·ft·F has been used for copper at $T \approx 190$°F, based on Tables 9.1-5 and F.3-5. To calculate the denominator, we use Eq. 14.7-3 for the heat transfer coefficient for filmwise condensation on horizontal tubes. Iteration is required, since the temperature difference across the condensate film is unknown. As a first approximation, we choose $T_1 = 190$°F, and use the physical properties at 200°F from Example 14.7-1:

$$\Delta\hat{H}_{\text{vap}} = 978 \text{ Btu/lb}_m$$
$$k = 0.393 \text{ Btu/hr·ft·F}$$
$$\rho = 60.1 \text{ lb}_m/\text{ft}^3$$
$$\mu = 0.738 \text{ lb}_m/\text{hr·ft}$$

Then Eq. 14.7-3 gives

$$h_1 = h_m = 0.725 \left[\frac{k^3 \rho^2 g \Delta\hat{H}_{\text{vap}}}{\mu D (T_d - T_1)}\right]^{1/4}$$
$$= 0.725 \left[\frac{0.393^3 \rho^2 (4.17 \times 10^8)(978.8)}{(0.738)(0.5/12)(T_d - T_1)}\right]^{1/4}$$
$$= 12,500(213 - T_1)^{-1/4}$$

Equating the heat flow through the numerator and denominator resistances gives

$$(T_1 - 150)/0.1458 = (213 - T1) r_1 h_1 = (213 - T_1)^{3/4}(0.5/12)(12,500)$$

or

$$213 - T_1 = 0.0000183(T_1 - 150)^{4/3}$$

Successive substitutions of $T_1$ in the right-hand term give a rapidly converging sequence of left-hand values: to the solution: $T_1 = 212.9975, 212.9954, 212.9954, \ldots$. Thus, the outer-surface temperature of the tubes at this cross-section is 212.9954° F. The temperature drop through the tube wall is $0.0006/0.1458(212.995 - 150) = 0.25$F. Thus, the thermal resitances of the tube wall and condensate film are unimportant here, as assumed in Problem 14A.1.

**14B.3 The hot-wire anemometer**

*a*. The physical properties of interest at $p$ = 1 atm and a film temperature of 335°F are:

$$\rho = 0.0499 \text{ lb}_m/\text{ft}^3$$

$$\hat{C}_p = 0.242 \text{ Btu/lb}_m\cdot{}^\circ\text{F}$$

$$\mu = 0.0594 \text{ lb}_m/\text{ft}\cdot\text{hr} = 1.64\times10^{-5} \text{ lb}_m/\text{ft}\cdot\text{s} \quad \text{(from Eq. 1.4.14)}$$

$$k = \left(0.242 + \frac{5}{4}\left(\frac{1.986}{29}\right)\right)\left(1.64\times10^{-5}\right) = 5.373\times10^{-6} \text{ Btu/ft}\cdot\text{s}^\circ\text{F}$$

(from Eq. 9.3-15)

Pr = 0.74 (from Eq. 9.3-16)

Also the Reynolds number is $\text{Re} = \dfrac{(0.01/12)(100)(0.0499)}{\left(1.64\times10^{-5}\right)} = 254$

Then Eq. 14.4-8 gives

$$\text{Nu}_m = (5.99 + 2.29)(0.905) + 0.92(-3.54 + 1062)^{-1/3}(6.33)(0.905) = 8.010$$

Then we get the heat transfer coefficient from

$$h_m = \frac{\text{Nu}_m k}{D} = \frac{(8.010)\left(5.373\times10^{-6}\right)}{(0.01/12)} = 0.0516 \text{ Btu/ft}^2\cdot\text{s}^\circ\text{F}$$

$$= 186 \text{ Btu/ft}^2\cdot\text{hr}^\circ\text{F}$$

Finally, the heat loss from the wire is

$$Q = h_m A\Delta T = h_m\cdot\pi DL\cdot\left(T_0 - T_\infty\right)$$

$$= (186)\left(\pi\frac{0.01}{12}\frac{0.5}{12}\right)(600-70)\text{Btu/hr}$$

$$= 10.75 \text{ Btu/hr} = 3.15 \text{ W} = 10.75 \text{ W}$$

*b*. For an approach velocity of 300 ft/s, Re = 762. Equation 14.4-8 gives $\text{Nu}_m = 14.20$, and $\text{Q}(300)/\text{Q}(100) = 14.20/8.010 = 1.77$. This is very close to $\sqrt{3} = 1.73$ from King's relation.

**14B.4 Dimensional analysis.**

**(a)** The left-hand member of Eq. 14B.4-1 is expressible in terms of integrals of the dimensionless product function $\breve{v}_z \breve{T}$ as follows:

$$\frac{T_{b2} - T_{b1}}{T_{b0} - T_{b1}} = \left[ \langle \overline{\breve{v}_z \breve{T}}|_{\breve{z}=0} \rangle - \langle \overline{\breve{v}_z \breve{T}}|_{\breve{z}=L/D} \rangle \right] \tag{A}$$

Here the angle brackets denote cross-sectional averages as in Eqs. 10.8-32 and 33, whereas overlines denote long-term time averages. Thus the averages on the right depend only on Re, Pr, and L/D, when viscous dissipation and radiant energy absorption and emission are neglected in the energy equation. With these assumptions, and the further neglect of axial heat conduction, the quotient $(T_{b2} - T_{b1})/(T_0 - T_{b1})$ is equal to the time-average of $Q/(w\hat{C}_p(T_0 - T_{b1})$.

**(b)** The heat transfer coefficients $h_a$ and $h_{\ln}$ each differ from $h_1$ only by the ratio of the corresponding $\Delta T$ definitions given in §14.1. But $\Delta T_a$ and $\Delta T_{\ln}$ are expressible in terms of $\Delta T_1$ and $\Delta T_2$, so their ratios depend only on Re, Pr, and L/D according to the result in **(a)**. Hence, $\mathrm{Nu}_a$ and $\mathrm{Nu}_{\ln}$ are functions of the same arguments as $\mathrm{Nu}_1$, confirming Eqs. 14.3-12 and 13.

Equation 14.3-14 requires extension of Eq. A to a variable upper limit $\breve{z}$, giving

$$\frac{T_b(z) - T_{b1}}{T_0 - T_{b1}} = \text{ a function of } (\mathrm{Re}, \mathrm{Pr}, \breve{z}) \tag{B}$$

Then, according to Eq. 14.1-18,

$$\begin{aligned} \frac{h_{\mathrm{loc}} D}{k} &= \frac{w\hat{C}_p}{\pi D k} \frac{d \ln(T_0 - T_b)}{d(z/D)} \\ &= \left( \frac{w}{\pi D \mu} \right) \left( \frac{\hat{C}_p \mu}{k} \right) \frac{d \ln(T_0 - T_b)}{d\breve{z}} \\ &= 4 \ \mathrm{RePr} \frac{d \ln[(T_0 - T_b)/(T_0 - T_{b1})]}{d\breve{z}} \end{aligned}$$

Combining this result with Eq. B, we get (for this special case of uniform wall temperature in the heat-exchange section),

$$\mathrm{Nu}_{\mathrm{loc}} = \mathrm{Nu}_{\mathrm{loc}} \left( \mathrm{Re}, \mathrm{Pr}, \left( \frac{z}{D} \right) \right)$$

in agreement with Eq. 14.3-14.

**14B.5 Relation between $h_{\text{loc}}$ and $h_{\text{ln}}$**

*a*. We relate the rate of heat transfer across the increment of surface $\pi D dz$ to the decrease in the internal energy within the volume element $\frac{1}{4}\pi D^2$:

$$h_{\text{loc}}(\pi D dz)(T_b - T_0) = -\left(\tfrac{1}{4}\pi D^2\right)\rho\hat{C}_p\langle v\rangle dT$$

This is really an application of the *d*-form of the energy balance discussed in §15.5, and given specifically in Eq. 15.4-4. It is clear from this equation that the kinetic and potential energy changes are being neglected, and that Eq. 15.4-4 has been multiplied through by $w$.

The above equation may now be integrated over a length $L$ of the tube to get, with $T_0 = \alpha + \beta T_b$,

$$\begin{aligned}
\int_0^L h_{\text{loc}} dz &= -\tfrac{1}{4}D\rho\hat{C}_p\langle v\rangle\int_{T_b(0)}^{T_b(L)}\frac{dT_b}{T_b - T_0} = -\tfrac{1}{4}D\rho\hat{C}_p\langle v\rangle\int_{T_b(0)}^{T_b(L)}\frac{dT_b}{T_b(1-\beta)-\alpha} \\
&= -\frac{\frac{1}{4}D\rho\hat{C}_p\langle v\rangle}{1-\beta}\ln\left[T_b(1-\beta)-\alpha\right]\Big|_{T_b(0)}^{T_b(L)} \\
&= -\frac{\frac{1}{4}D\rho\hat{C}_p\langle v\rangle}{1-\beta}\left\{\ln\left[T_b(L)(1-\beta)-\alpha\right]-\ln\left[T_b(0)(1-\beta)-\alpha\right]\right\} \\
&= -\frac{\frac{1}{4}D\rho\hat{C}_p\langle v\rangle}{1-\beta}\left\{\ln\left[T_b(L)-T_0(L)\right]-\ln\left[T_b(0)-T_0(0)\right]\right\} \\
&= -\frac{\frac{1}{4}D\rho\hat{C}_p\langle v\rangle}{1-\beta}\frac{\left[T_b(L)-T_0(L)\right]-\left[T_b(0)-T_0(0)\right]}{(T_b-T_0)_{\text{ln}}} \\
&= -\tfrac{1}{4}D\rho\hat{C}_p\langle v\rangle\frac{\left[T_b(L)-(\alpha/(1-\beta))\right]-\left[T_b(0)-(\alpha/(1-\beta))\right]}{(T_b-T_0)_{\text{ln}}} \\
&= +\tfrac{1}{4}D\rho\hat{C}_p\langle v\rangle\frac{T_b(L)-T_b(0)}{(T_0-T_b)_{\text{ln}}}
\end{aligned}$$

which agrees with Eq. 15B.5-2.

*b*. Equation 14.1-14 can be written as

$$h_{\ln} = \frac{\frac{1}{4} D\rho \hat{C}_p \langle v \rangle}{L} \frac{[T_b(L) - T_b(0)]}{(T_0 - T_b)_{\ln}}$$

from which Eq. 14B.5-3 follows at once:

$$h_{\ln} = \frac{1}{L} \int_0^L h_{\text{loc}} dz$$

Then, differentiating the integral in Eq. 14B.5-3 using the Leibniz formula in Eq. C.3-2, we get

$$\frac{dh_{\ln}}{dL} = -\frac{1}{L^2}(Lh_{\ln}) + \frac{1}{L} h_{\text{loc}} \Big|_{z=L}$$

or

$$h_{\text{loc}} \big|_{z=L} = h_{\ln} + L \frac{dh_{\ln}}{dL}$$

which is Eq. 14B.5-4

**14B.6 Heat loss by free convection from a pipe**

The properties of air at 1 atm and a film temperature of 190°F (or 650° R) are

$$\mu = 0.216 \text{ cp}$$
$$k = 0.0173 \text{ Btu/hr}\cdot\text{ft}\cdot{}^\circ\text{F}$$
$$\beta = 1/650 \ (^\circ\text{R})^{-1}$$
$$\hat{C}_p = 0.242 \text{ Btu/lb}_m\cdot{}^\circ\text{F}$$
$$\text{Pr} = 0.727$$

Since the temperature difference is the same in both the original problem (of Example 14.6-1) and the new problem (Problem 14B.6), it suffices to determine the ratio $h'_m/h_m$ , where the accent indicates the result for the "new" problem.

Next we calculate the ratio of the heat-transfer coefficients:

$$\frac{h'_m}{h_m} = \left(\frac{k'}{k}\right)^{3/4}\left(\frac{\beta'}{\beta}\cdot\frac{\hat{C}'_p}{\hat{C}_p}\cdot\frac{\mu}{\mu'}\right)^{1/4}\left(\frac{\left[1+\left(0.492/\text{Pr}\right)^{9/16}\right]^{4/9}}{\left[1+\left(0.492/\text{Pr}'\right)^{9/16}\right]^{4/9}}\right)$$
$$= \left(\frac{0.0173}{0.0152}\right)^{3/4}\left(\frac{550}{650}\cdot\frac{0.242}{0.241}\cdot\frac{0.0190}{0.0216}\right)^{1/4}\left(\frac{1.299}{1.299}\right)$$
$$= (1.102)(0.930)(1.000) = 1.025$$

Thus, in the "new" problem, the heat-transfer rate is only slightly greater than in the original problem.

If the thermal conductivity and viscosity had been assumed to vary with temperature by the simple power-law suggested by the simplified kinetic theory of Chapters 9 and 1, then almost no change would have been predicted.

**14D.1 Heat transfer from an oblate ellipsoid of revolution**

*a*. From Eq. A.7-13 we get (with $\Sigma = \sinh\xi$, $\mathrm{K} = \cosh\xi$, $S = \sin\eta$, $C = \cos\eta$, $s = \sin\psi$, $c = \cos\psi$)

$$h_\xi^2 = \sum_{i=1}^{3}\left(\frac{\partial x_i}{\partial \xi}\right)^2 = a^2\left[(\Sigma Sc)^2 + (\Sigma Ss)^2 + (\mathrm{K}C)^2\right] = a^2\left[(\Sigma S)^2 + (\mathrm{K}C)^2\right]$$
$$= a^2\left[\left(\mathrm{K}^2 - 1\right)S^2 + \mathrm{K}^2\left(1 - S^2\right)\right] = a^2\left(\mathrm{K}^2 - S^2\right) \quad \therefore h_\xi = a\sqrt{\mathrm{K}^2 - S^2}$$

$$h_\eta^2 = \sum_{i=1}^{3}\left(\frac{\partial x_i}{\partial \eta}\right)^2 = a^2\left[(\mathrm{K}Cc)^2 + (\mathrm{K}Cs)^2 + (\Sigma S)^2\right] = a^2\left[(\mathrm{K}C)^2 + (\Sigma S)^2\right]$$
$$= a^2\left[\mathrm{K}^2\left(1 - S^2\right) + \left(\mathrm{K}^2 - 1\right)S^2\right] = a^2\left(\mathrm{K}^2 - S^2\right) \quad \therefore h_\eta = a\sqrt{\mathrm{K}^2 - S^2}$$

$$h_\psi^2 = \sum_{i=1}^{3}\left(\frac{\partial x_i}{\partial \psi}\right)^2 = a^2\left[(\mathrm{K}Ss)^2 + (\mathrm{K}Sc)^2\right] = a^2\mathrm{K}^2S^2 \quad \therefore h_\psi = a\mathrm{K}S$$

These results may be used with the expression for surface elements after Eq. A.7-18 to get

$$S_{\eta\psi} = a^2\sqrt{\cosh^2\xi - \sin^2\eta}\,(\cosh\xi\sin\eta)\,d\eta\, d\psi$$

*b*. From Eq. A.7-17, we get Laplace's equation as

$$\nabla^2\Theta = \frac{1}{h_\xi h_\eta h_\psi}\left(\frac{\partial}{\partial\xi}\left(\frac{h_\eta h_\psi}{h_\xi}\frac{\partial\Theta}{\partial\xi}\right) + \cdots\right) = 0 \quad \text{or} \quad \frac{d}{d\xi}\left(\cosh\xi\frac{d\Theta}{d\xi}\right) = 0$$

since heat is flowing in the $\xi$ direction only. This equation may be integrated to give

$$\Theta = K_1\arctan(\sinh\xi) + K_2$$

with the boundary conditions: $\Theta(\xi_0) = 0$ and $\Theta(\infty) = 1$. This leads to Eq. 14D.1-7.

*c*. In the limit as $\xi_0 \to 0$ (a two-sided disk of radius a = *R*), the result of part (*b*), namely Eq. 14D.1-7, simplifies to

$$\Theta = 1 - \frac{\frac{1}{2}\pi - \arctan(\sinh\xi)}{\frac{1}{2}\pi - 0} = \frac{2}{\pi}\arctan(\sinh\xi)$$

The normal dimensionless temperature gradient at the surface is then obtained by using Eq. A.7-15, thus:

$$\begin{aligned}
\frac{1}{h_\xi}\frac{d\Theta}{d\xi}\bigg|_{\xi=0} &= \frac{1}{R\sqrt{1-\sin^2\eta}}\frac{d}{d\xi}\left(\frac{2}{\pi}\arctan(\sinh\xi)\right)\bigg|_{\xi=0} \\
&= \frac{1}{R\sqrt{1-\sin^2\eta}}\left(\frac{2}{\pi}\right)\left(\frac{1}{1+\sinh^2\xi}\cdot\cosh\xi\right)\bigg|_{\xi=0} \\
&= \frac{1}{R\sqrt{1-\sin^2\eta}}\left(\frac{2}{\pi}\right)(\cosh 0) = \left(\frac{2}{\pi}\right)\frac{1}{R\sqrt{1-\sin^2\eta}}
\end{aligned}$$

Then the total heat loss through both sides of the disk is

$$\begin{aligned}
Q &= 2\int(\mathbf{n}\cdot\mathbf{q})dS = -2k\int(\mathbf{n}\cdot\nabla T)dS \\
&= 2k(T_0 - T_\infty)\cdot\frac{2}{\pi R}\int_0^{2\pi}\int_0^{\pi/2}\frac{1}{\cos\eta}R^2\cos\eta\sin\eta d\eta d\psi \\
&= 2k(T_0 - T_\infty)\cdot\frac{2}{\pi R}\cdot 2\pi R^2\int_0^{\pi}\sin\eta d\eta = 8kR(T_0 - T_\infty)
\end{aligned}$$

The heat transfer coefficient is then

$$h_m = \frac{Q}{A\Delta T} = \frac{8kR(T_0 - T_\infty)}{(\pi R^2)(T_0 - T_\infty)} = \frac{8k}{\pi R}$$

and the Nusselt number is

$$\mathrm{Nu}_m = \frac{h_m(2R)}{k} = \frac{8k}{\pi R}\cdot\frac{2R}{k} = \frac{16}{\pi} = 5.09$$

**15A.1 Heat transfer in double-pipe heat exchangers.**

**(a)** In the absence of heat loss to the surroundings, Eqs. 15.4-7,8 give

$$w_c\hat{C}_{pc}(T_{c2} - T_{c1}) = -w_h\hat{C}_{ph}(T_{h2} - T_{h1})$$

with each flow rate $w$ expressed from plane 1 toward plane 2. Insertion of the data then gives

$$Q_c = (5000)(1.00)(T_{c2} - 60) = -(-10,000)(0.60)(200 - 100) = 600,000 \text{ Btu/hr}$$

whence

$$T_{c2} = 60 + 120 = 180°\text{F}$$

The log-mean temperature difference is

$$(\Delta T)_{\ln} = (20 - 40)/\ln(20/40) = 28.85\text{F}$$

and the required heat exchange area, from Eq. 15.4-15, is

$$A_0 = \frac{Q_c}{U_0(\Delta T)_{\ln}} = \frac{(600,000 \text{ Btu/hr})}{(200 \text{ Btu/hr·ft}^2\text{·F})(28.85\text{F})} = 104 \text{ ft}^2$$

**(b)** Eq. 15B.1-2 gives

$$\begin{aligned}\frac{Q}{A} &= \frac{U_1\Delta T_2 - U_2\Delta T_1}{\ln(U_1\Delta T_2/U_2\Delta T_1)} \\ &= \frac{(50 \times 20) - (350 \times 40)}{\ln((50 \times 20)/350 \times 40))} \\ &= 4926 \text{ Btu/hr·ft}^2\end{aligned}$$

The required heat exchange area is then

$$A = \frac{Q}{Q/A} = \frac{600,000}{4926} = 122 \text{ ft}^2$$

**(c)** The minimum usable flow rate of water to cool the oil to 100°F in counterflow is

$$w_c = \frac{(10,000)(0.60)(200 - 100)}{(1.00)(200 - 60)} = 4286 \text{ lb}_m\text{/hr}$$

whereas the minimum usable flow rate of water in parallel flow is

$$w_c = \frac{(10,000)(0.60)(200 - 100)}{(1.00)(100 - 60)} = 15,000 \text{ lb}_m\text{/hr}$$

**(d)** If parallel flow is used, with $w_c$ =15,500 $\text{lb}_m$/hr of water, the outlet water temperature will be

$$T_{c2} = 60 + (10,000)(0.60)(200 - 100)/(15,500)(1.00) = 98.71°\text{F}$$

Then $(\Delta T)_{\ln} = (140 - 1.29)/\ln(140/1.29) = 29.6°\text{F}$ and the required heat exchange area is

$$A = \frac{Q}{U(\Delta T)_{\ln}} = \frac{(10,000)(0.60)(200 - 100)}{(200)(29.6)} = 101 \text{ ft}^2$$

**15A.2 Adiabatic flow of natural gas in a pipeline.**

**(a)** The density and mass flux at plane 1 are

$$\rho_1 = \frac{p_1 M}{RT} = \frac{(100 \times 144 \text{ lb}_f/\text{ft}^2)(16.04 \text{ lb}_m/\text{lb-mol})}{(1545 \times 530 \text{ ft}\cdot\text{lb}_f/\text{lb-mol})} = 0.2821 \text{ lb}_m/\text{ft}^3$$

$$G := \rho_1 v_1 = (0.2821 \text{ lb}_m/\text{ft}^3)(40 \text{ ft/s}) = 11.28 \text{ lb}_m/\text{ft}^2\cdot\text{s})$$

Re and $f$ at plane 1 are the same here as in Example 15.4-2, and $f$ is approximated as constant at 0.0025 along the pipeline, giving

$$e_v = 4fL/D = (4)(0.0025)(52,800/2) = 264.$$

Furthermore,

$$\begin{aligned} p_1\rho_1 &= (100 \times 144 \text{ lb}_f/\text{ft}^2)(32.2 \text{ lb}_m\text{ft}/\text{lb}_f\cdot\text{s}^2)(0.2821 \text{ lb}_m/\text{ft}^3) \\ &= 1.308 \times 10^5 \text{ lb}_m^2/\text{ft}^4\cdot\text{s}^2. \end{aligned}$$

Eq. 15B3-7 may be rearranged to give

$$\frac{e_v - [(\gamma+1)/2\gamma]\ln s}{1-s} - \frac{\gamma-1}{2\gamma} = \frac{\rho_1 p_1}{G^2}$$

or, for this problem with $\gamma = 1.3$ for methane,

$$\frac{264 - [2.3/2.6]\ln s}{1-s} - \frac{0.3}{2.6} = \frac{1.308 \times 10^5}{(11.28)^2} = 1026$$

This equation has the solution $s = 0.74$, corresponding to $\rho_2/\rho_1 = \sqrt{0.74} = 0.86$ and $\rho_2 = 0.86 \times 0.2821 = 0.243 \text{ lb}_m/\text{ft}^3$.

**(b)** Eq. 15B.3-8 and the result of **(a)** give

$$\begin{aligned} p_2 &= p_1 \frac{\rho_2}{\rho_1}\left[1 + \frac{[1-(\rho_1/\rho_2)^2]G^2}{\rho_1 p_1}\left(\frac{\gamma-1}{2\gamma}\right)\right] \\ &= (100 \text{ psia})(0.86)\left[1 + \frac{[1-0.86^{-2}](11.28 \text{ lb}_m/\text{ft}^2\cdot\text{s})^2}{(1.308 \times 10^5 \text{ lb}_m^2/\text{ft}^4\cdot\text{s}^2)}\left(\frac{0.3}{2.6}\right)\right] \\ &= 86 \text{ psia} \end{aligned}$$

**(c)** The temperature at the compressor inlet is

$$T_2 = T_1(p_2/p_1)(\rho_1/\rho_2) = 530(86/100)/(0.86) = 530°\text{R}$$

Eq. 15.4-22 then gives

$$\begin{aligned}
\hat{W}_m &= \frac{v_1^2}{2}\left[1-\left(\frac{\rho_1}{\rho_2}\right)^2\right] \\
&\quad + \frac{RT_2}{M}\frac{\gamma}{\gamma-1}\left[\left(\frac{p_1}{p_2}\right)^{(\gamma-1)/\gamma}-1\right] \\
&= \frac{(40)^2}{2}\left[1-\left(\frac{1}{0.86}\right)^2\right] \\
&\quad + \frac{(49686\times 530\ \mathrm{lb}_m\mathrm{ft}^2/\mathrm{s}^2\cdot\mathrm{lb\text{-}mol})}{16.04\ \mathrm{lb}_m/\mathrm{lb\text{-}mol})}\frac{1.3}{0.3}\left[(\frac{100}{86})^{0.3/1.3}-1\right] \\
&= (-282+2.520\times 10^5\ \mathrm{ft}^2/\mathrm{s}^2)/(32.2\ \mathrm{lb}_m\mathrm{ft}/\mathrm{s}^2\cdot\mathrm{lb}_f = 7817\ \mathrm{ft}\cdot\mathrm{lb}_f/\mathrm{lb}_m
\end{aligned}$$

and the required compressor power output is

$$\begin{aligned}
W_m &= w\hat{W}_m = \left(\frac{\pi D^2}{4}\right)G\hat{W}_m \\
&= \left(\frac{\pi(2\ \mathrm{ft})^2)}{4}\right)(11.28\ \mathrm{lb}_m/\mathrm{ft}^2\cdot\mathrm{s})(7817\ \mathrm{ft}\cdot\mathrm{lb}_f/\mathrm{lb}_m) = 2.770\times 10^5\ \mathrm{ft}\cdot\mathrm{lb}_f/\mathrm{s} \\
&= 504\ \mathrm{hp}
\end{aligned}$$

**15A.3 Mixing of two ideal-gas streams.**

(a) The right sides of Eqs. 15.3-6, 7 and 10 are

$$\begin{aligned} w &= w_{1a} + w_{1b} = 1000 + 10,000 \\ &= 11,000 \text{ lb}_m/\text{hr} = 3.0556 \text{ lb}_m/\text{s} \end{aligned}$$

$$\begin{aligned} P &= v_{1a}w_{1a} + v_{1b}w_{1b} + p_{1a}S_{1a} + p_{1b}S_{1b} \\ &= v_{1a}w_{1a} + v_{1b}w_{1b} + \frac{RT_{1a}}{M}\frac{w_{1a}}{v_{1a}} + \frac{RT_{1b}}{M}\frac{w_{1b}}{v_{1b}} \\ &= [(1000)(1000/3600) + (100)(10,000/3600) \text{ lb}_m\cdot\text{ft/s}^2] \\ &\quad + \left(\frac{49686 \times 540}{28.97} \text{ ft}^2/\text{s}^2\right)\left[\frac{1000/3600}{1000} + \frac{10,000/3600}{100} \text{ lb}_m/\text{ft}\right] \\ &= 555.6 + 25983.5 = 26539.1 \text{ lb}_m\cdot\text{ft/s}^2] \end{aligned}$$

With $\widetilde{C}_p = 6.97$ Btu/lb-mole·F, hence $\hat{C}_p = 6.97/28.97$ Btu/lb$_m$·F, the right side of Eq. 15.3-10 becomes

$$\begin{aligned} E &= \hat{C}_p\,[w_{1a}T_{1a} + w_{1b}T_{1b}] + \frac{1}{2}\left[w_{1a}v_{1a}^2 + w_{1b}v_{1b}^2\right] \\ &= (6.97 \times 540/28.97 \text{ Btu/lb}_m)(25036 \text{ lb}_m\text{ft}^2/\text{s}^2\cdot \text{ Btu})\left[\frac{1000}{3600} + \frac{10,000}{3600} \text{ lb}_m/\text{s}\right] \\ &\quad + \frac{1}{2}\left[\frac{1000}{3600}(1000)^2 + \frac{10,000}{3600}(100)^2 \text{ lb}_m\text{ft}^2/\text{s}^3\right] \\ &= 9.939 \times 10^6 + 1.390 \times 10^5 = 1.0078 \times 10^7 \text{ lb}_m\text{ft}^2/\text{s}^3 \end{aligned}$$

and $\gamma = \widetilde{C}_p/(\widetilde{C}_p - R) = 6.97/(6.97 - 1.9872) = 1.399$.

Then Eq. 15.3-13 yields the solutions

$$\begin{aligned} v_2 &= \left(\frac{1.399}{2.399}\right)\frac{26539.1}{3.05556}\left[1 \pm \sqrt{1 - 2\left(\frac{1.399^2 - 1}{1.399^2}\right)\frac{3.05556 \times 1.0078 \times 10^7}{(26539.1)^2}}\right] \\ &= 5065[1 \pm 0.97838] = 109.5 \text{ ft/s and } 10020 \text{ ft/s} \end{aligned}$$

the smaller one of which is stable.

Then Eq. 15.3-10 gives the temperature of the mixed stream:

$$\begin{aligned} T_2 &= [E/w - \frac{1}{2}v_2^2]/\hat{C}_p \\ &= \left[\frac{1.0078 \times 10^7}{3.05556} - \frac{1}{2}(109.5)^2 \text{ ft}^2/\text{s}^2\right]\frac{28.97}{6.97 \times 25036 \text{ °R}\cdot\text{s}^2/\text{ft}^2} \\ &= 546.5\text{°R} = 86.5\text{°F} \end{aligned}$$

The cross-sectional area of the mixed stream is

$$
\begin{aligned}
S_2 = S_{1a} + S_{1b} &= \frac{w_{1a}}{\rho_{1a} v_{1a}} + \frac{w_{1b}}{\rho_{1b} v_{1b}} = \frac{RT_1}{p_1 M}\left[\frac{w_{1a}}{v_{1a}} + \frac{w_{1b}}{v_{1b}}\right] \\
&= \left(\frac{49686 \times 540}{68086 \times 28.97}\ \mathrm{ft^3/lb}_m\right)\left[\frac{1000/3600}{1000} + \frac{10,000/3600}{100}\ \mathrm{lb}_m/\mathrm{ft}\right] \\
&= 0.3816\ \mathrm{ft}^2
\end{aligned}
$$

and the pressure $p_2$ of the mixed stream is obtained from Eq. 15.3-7:

$$
\begin{aligned}
p_2 &= \frac{P - v_2 w_2}{S_2} \\
&= \left[\frac{26539.1 - 109.5 \times 11,000/3600}{0.3816}\right] = 6.86 \times 10^4\ \mathrm{lb}_m/\mathrm{ft{\cdot}s^2} = 1.00\ \mathrm{atm}
\end{aligned}
$$

**(b)** If the fluid density were treated as constant, Eq. 15.3-6 would give

$$(S_{1a} + S_{1b}) v_2 = S_{1a} v_{1a} + S_{1b} v_{1b}$$

whence

$$
\begin{aligned}
v_2 &= \frac{S_{1a} v_{1a} + S_{1b} v_{1b}}{S_{1a} + S_{1b}} \\
&= \frac{w_{1a}/\rho + w_{1b}\rho}{w_{1a}/\rho v_{1a} + w_{1b}/\rho v_{1b}} \\
&= \frac{w_{1a} + w_{1b}}{w_{1a}/v_{1a} + w_{1b}/v_{1b}} \\
&= \frac{11,000}{1 + 100} = 108.9\ \mathrm{ft/s}
\end{aligned}
$$

**(c)** Eq. D of Table 15.3-1 gives, for the conditions of **(b)**,

$$
\begin{aligned}
\hat{E}_v &= (w_{1a}/w_2)\frac{v_{1a}^2}{2} + (w_{1b}/w_2)\frac{v_{1b}^2}{2} - \frac{v_2^2}{2} \\
&= (1000/11000)\frac{(1000^2}{2} + (10,000/11000)\frac{(100)^2}{2} - \frac{(109.5)^2}{2} \\
&= 45454.5 + 4545.5 - 5995.1\ \mathrm{ft^2/s^2} \\
&= 44004.8\ \mathrm{ft^2/s^2} \times 3.1081 \times 10^{-2}\ \mathrm{ft{\cdot}lb}_f/(\mathrm{lb}_m \mathrm{ft^2/s^2} = 1.368 \times 10^3\ \mathrm{ft{\cdot}lb}_f/\mathrm{lb}_m
\end{aligned}
$$

**15A.4 Flow through a Venturi tube.**

**(a)** Eq. 15.5-34 requires the calculated values

$$S_0 = (\pi/4)(D_0)^2 = (\pi/4)(3/12 \text{ ft})^2 = 0.04909 \text{ ft}^2,$$

$$\rho_2 = (\frac{p_1 M}{RT_1})(p_2/p_1)^{1/\gamma} = \frac{(14.696 \times 144 \text{ lb}_f/\text{ft}^2)(28.97 \text{ lb}_m/\text{lb-mol})}{(1544.3 \times 529.7 \text{ ft}\cdot\text{lb}_f/\text{lb-mol})}(0.75)^{1/1.4}$$

$$= 0.0610 \text{ lb}_m/\text{ft}^3,$$

$$\frac{p_1}{\rho_1} = \frac{RT_1}{M} = \left(\frac{49686 \times 529.7 \text{ lb}_m\cdot\text{ft}^2/\text{s}^2\cdot\text{lb-mol}}{28.97 \text{ lb}_m/\text{lb-mol}}\right) = 9.085 \times 10^5 \text{ ft}^2/\text{s}^2,$$

$$1 - (S_0/S_1)^2(p_2/p_1)^{2/\gamma} = 1 - (D_0/D_1)^4(p_2/p_1)^{2/\gamma} = (3/12)^4(0.75)^{2/1.4} = 0.99990.$$

Substitution of these and known values into Eq. 15.5-34 gives

$$w = (0.98)(0.0610)(0.04909)\sqrt{\frac{2(9.085 \times 10^5)[1.4/0.4][1 - 0.75^{0.4/1.4}]}{0.99990}}$$

$$= 2.08 \text{ lb}_m/\text{s}$$

**(b)** For isothermal flow,

$$\rho_2 = \frac{p_2 M}{RT_1} = \frac{(0.75 \times 14.696 \times 144 \text{ lb}_f/\text{ft}^2)(28.97 \text{ lb}_m/\text{lb-mol})}{(1544.3 \times 529.7 \text{ ft}\cdot\text{lb}_f/\text{lb-mol})} = 0.0562 \text{ lb}_m/\text{ft}^3,$$

$$1 - (\rho_2 S_0/\rho_1 S_1)^2 = 1 - (p_2/p_1)^2(D_2/D_1)^4 = 1 - (0.75)^2((3/12)^4 = 0.9978,$$

and

$$-\int_1^2 (1/\rho)dp = \frac{RT_1}{M}\ln(p_1/p_2) = \frac{(49686 \times 529.7}{28.97}\ln(1/0.75) = 2.614 \times 10^5 \text{ ft}^2/\text{s}^2$$

Then Eq. 15.5-33 gives

$$w = (0.98)(0.0562)(0.04909)\sqrt{\frac{2 \times 2.614 \times 10^5}{0.9978}} = 1.96 \text{ lb}_m/\text{s}$$

**(c)** For constant density at the entering value,

$$\rho = \frac{p_1 M}{RT_1} = \frac{(14.696 \times 144 \text{ lb}_f/\text{ft}^2)(28.97 \text{ lb}_m/\text{lb-mol})}{(1544.3 \times 529.7 \text{ ft}\cdot\text{lb}_f/\text{lb-mol})} = 0.0750 \text{ lb}_m/\text{ft}^3,$$

$$1 - (\rho_2 S_0/\rho_1 S_1)^2 = 1 - (D_0/D_1)^4 = 0.9961$$

$$-\int_1^2 (1/\rho)dp = \frac{RT_1}{Mp_1}(p_1 - p_2) = \frac{RT_1}{Mp_1}(p1 - 0.75p_1)$$

$$= \frac{0.25RT_1}{M} = \frac{0.25 \times 49686 \times 529.7}{28.97} = 2.271 \times 10^5 \text{ ft}^2/\text{s}^2$$

Then Eq. 15.5-33 gives

$$w = (0.98)(0.0750)(0.04909)\sqrt{\frac{2 \times 2.271 \times 10^5}{0.9961}} = 2.44 \text{ lb}_m/\text{s}$$

**15A.5 Free batch expansion of a compressible fluid.**

**(a)** Just after the start of the discharge, $p_1$ is equal to $p_0$ and $p_2/p_1$ attains the critical value given in Eq 15.5-43. Hence,

$$p_2 = p_0 \left(\frac{2}{1.4+1}\right)^{1.4/(1.4-1)} = 52.8 \text{ atm}$$

The expansion from $p_1$ to $p_2$ satisfies $p/\rho T = R/M$, hence

$$\frac{T_2}{T_0} = \frac{p_2}{p_0}\frac{\rho_0}{\rho_2}$$

Inserting $\rho_2/\rho_0 = (p_2/p_0)^{1/\gamma}$ according to Eq. 15.5-39, we get

$$\begin{aligned} T_2 &= T_0(p_2/p_0)^{(\gamma-1)/\gamma} \\ &= T_0\left[\left(\frac{2}{\gamma+1}\right)^{\gamma/(\gamma-1)}\right]^{(\gamma-1)/\gamma} \\ &= T_0\left(\frac{2}{\gamma+1}\right) = 300\left(\frac{2}{2.4}\right) = 250 \text{ K} \end{aligned}$$

**(b)** When $p_2 = p_a = 1$ atm, $p_1 = 1/0.528 = 1.893$ atm. Then the temperature within the main part of the tank is

$$T_1 = T_0\left(\frac{p_1}{p_0}\right)^{(\gamma-1)/\gamma} = 300\left(\frac{1.893}{100}\right)^{0.4/1.4} = 300 \times 0.5078 = 152 \text{ K}$$

**(c)** At the state described in **(b)**, $\rho_1/\rho_0 = (1.893/100)^{1/1.4} = 0.0588$. The time required to reach this state is computable from Eq. 15.5-46:

$$\begin{aligned} t &= \frac{(V/S_2)}{\sqrt{(\gamma RT_0/M)(2/(\gamma+1))^{(\gamma+1)/(\gamma-1)}}}\left(\frac{2}{\gamma-1}\right)\left[\left(\frac{\rho_1}{\rho_0}\right)^{(1-\gamma)/2} - 1\right] \\ &= \frac{(10/0.1 \text{ ft})}{\sqrt{(1.4 \times 49686 \times 300 \times 1.8/28.97 \text{ ft}^2/\text{s}^2)(2/2.4)^{2.4/0.4}}} \\ &\quad \cdot \left(\frac{2}{0.4}\right)\left[(0.0588)^{-0.4/2} - 1\right] \\ &= 0.58 \text{ s} \end{aligned}$$

**15A.6 Heating of air in a tube.**

For the air contained in the 20-ft length of tube, we write Eq. 15.1-3 with $\hat{W}_m = 0$ (since this system has no moving surfaces); also we use the approximation $\Delta(v^2/2)$ of the kinetic energy change from inlet to outlet. This gives the simplified steady-state energy balance

$$\Delta\left(\hat{H} + \frac{1}{2}v^2\right) = \hat{Q}$$

for horizontal flow, or

$$\frac{1}{M}\int_{T_1}^{T_2} \tilde{C}_p dT + \frac{1}{2}v_1^2\left[\left(\frac{p_1 T_2}{p_2 T_1}\right)^2 - 1\right] = \hat{Q}$$

for horizontal flow of an ideal gas in a tube. Proceeding as in Example 15.3-1, we get the algebraic equation

$$\frac{1}{28.97}[6.39(T_2 - T_1) + \frac{9.8}{2} \times 10^{-4}(T_2^2 - T_1^2) - \frac{8.18}{3} \times 10^{-8}(T_2^3 - T_1^3)\ \text{Btu/lb}_m]$$
$$+ \frac{1}{2}(75\ \text{ft/s})^2 \times 3.9942 \times 10^{-5}\ \text{Btu/(lb}_m\ \text{ft}^2/\text{s}^2))\left[\left(\frac{40T_2}{15T_1}\right)^2 - 1\right]$$
$$= (800 \times 20/185 = 86.4865\ \text{Btu/lb}_m)$$

Inserting $T_1 = 5°\text{F}$, and an initial guess of 800°R for $T_2$, we get the following Newton iteration sequence, converging to 864°R≈ 354°F:

| Old $T_2$, °R | LHS, Btu/lb$_m$ | $f$ =LHS −RHS | $f' = df/dT_2$, Btu/lb$_m$·R | $\Delta T_2 = -f/f'$ | New $T_2$, °R |
|---|---|---|---|---|---|
| 800 | 82.9245 | -3.5620 | 0.2458 | 14.49 | 814.49 |
| 814.49 | 86.5761 | +0.0896 | 0.2474 | -0.36 | 814.12 |

**15A.7 Operation of a simple double-pipe heat exchanger.**

**(a)** The two exchangers will work most effectively if connected so as to simulate a single double-pipe exchanger operating in countercurrent flow. That is, the stream to be heated should enter exchanger A through its inner pipe at plane 1, next to the outlet of the twice-cooled hot water stream, and should exit the inner pipe of exchanger B at plane 2, next to the original inlet of the hot-water stream.

**(b)** The Reynolds and Prandtl numbers for the stream being heated are:

$$\mathrm{Re} = \left(\frac{4w}{\pi D\mu}\right) = \left(\frac{(4)(5400\ \mathrm{lb}_m/\mathrm{hr}}{\pi(0.0875\ \mathrm{ft})(1.09\ \mathrm{lb}_m/\mathrm{hr{\cdot}ft})}\right) = 7.21 \times 10^4$$

$$\mathrm{Pr} = \left(\frac{\hat{C}_p\mu}{k}\right) = \left(\frac{1.00\ \mathrm{Btu/lb}_m{\cdot}\mathrm{F})(1.09\ \mathrm{lb}_m/\mathrm{hr{\cdot}ft})}{0.376\ \mathrm{Btu/hr{\cdot}ft{\cdot}F}}\right) = 2.90$$

At these conditions, with $\mu_0$ and $\mu_b$ not distinguished, Fig. 14.3-2 gives

$$\frac{h_{\ln}D}{k}(\mathrm{Re})^{-1}(\mathrm{Pr})^{-1/3} = 0.0028$$

and the insensitivity of $h_{\ln}$ to $L/D$ at Re above 8000 gives $h_{\mathrm{loc}} = h_{\ln}$ according to Eq. 14B.5-4. Hence, the local heat transfer coefficient for the inner pipe wall is

$$\begin{aligned} h_i &= 0.0028\frac{k}{D_i}(\mathrm{Re})(\mathrm{Pr})^{1/3} \\ &= 0.0028\frac{(0.376\ \mathrm{Btu/hr{\cdot}ft{\cdot}F})}{(0.0875\ \mathrm{ft})}(7.21 \times 10^4)(2.90)^{1/3} \\ &= 1237\ \mathrm{Btu/hr{\cdot}ft^2{\cdot}F} \end{aligned}$$

and the local overall coefficient $U_i$ based on $D_i$ satisfies

$$\begin{aligned} \frac{1}{D_iU_i} &= \frac{1}{D_ih_i} + \frac{1}{D_ih_{\mathrm{annulus}}} + \text{Wall and deposit resistance based on } D_i \\ &= \frac{1}{(0.0875 \times 1237)} + \frac{1}{(0.0875 \times 2000)} + \frac{0.001}{0.0875}\ \mathrm{hr{\cdot}ft{\cdot}F/Btu} \end{aligned}$$

or

$$\begin{aligned} \frac{1}{U_i} &= \frac{1}{1237} + \frac{1}{2000} + 0.001 \\ &= 0.000808 + 0.0005 + 0.001 = 0.002308\ \mathrm{hr{\cdot}ft^2{\cdot}F/Btu} \end{aligned}$$

Hence, $U_i = (1/0.002308)$ Btu/hr·ft²·F. Equation 15.4-13 then gives

$$\ln\left(\frac{T_{h1} - T_{c1}}{T_{h2} - T_{c2}}\right) = U_i\left(\frac{1}{w_h\hat{C}_{ph}} + \frac{1}{w_c\hat{C}_{pc}}\right)(\pi D_iL) \qquad (15A.7-1)$$

or

$$\ln\left(\frac{T_{h1}-70}{200-T_{c2}}\right) = \frac{1}{0.002308}\left(\frac{1}{(5400)(1.00)} + \frac{1}{(-8100)(1.00)}\right)(\pi \times 0.0875 \times 40)$$
$$= 0.2941$$

in which $L$, $w_h$ and $w_c$ are measured in the direction of the cold water flow.

The computation of $T_{c2}$ is summarized below. For each trial value of $T_{c2}$, an energy balance with heat capacities of 1.00 Btu/lb$_m$·F gives

$$5400(1.00)(T_{c2}-70) = 8100(1.00)(200-T_{h1}) \qquad (15A.7-2)$$

from which the outlet temperature $T_{h1}$ of the cooled hot water can be calculated for testing against Eq. 15A.7-1. The final $T_{h1}$ is very close to 136°F.

| $T_{c2}$ | $T_{h1} = 200 - \frac{2}{3}(T_{c2}-70)$ | $T_{h1}-70$ | $200-T_{c2}$ | $\ln\frac{T_{h1}-70}{200-T_{c2}}$ |
|---|---|---|---|---|
| 130 | 160 | 90 | 70 | 0.2513 |
| 132 | 158.667 | 88.667 | 68 | 0.2654 |
| 135 | 156.667 | 86.667 | 65 | 0.2877 |
| 136 | 156 | 86 | 64 | 0.2955 |
| 135.8 | 156.133 | 86.133 | 64.2 | 0.2939 |

**15B.1 Performance of a double-pipe heat exchanger with variable overall heat-transfer coefficient**

*a*. We start with Eq. 15.4-12

$$-\frac{d(T_h - T_c)}{(T_h - T_c)} = U\left(\frac{1}{w_h\hat{C}_{ph}} + \frac{1}{w_c\hat{C}_{pc}}\right)(2\pi r_0)dl$$

Since $U$ is a linear function of $T_h - T_c$, we write

$$U = \alpha + \beta(T_h - T_c)$$

and we choose $\alpha$ and $\beta$ by requiring that the conditions at planes 1 and 2 are satisfied:

At plane 1: $U_1 = \alpha + \beta(T_{h1} - T_{c1})$

At plane 2: $U_2 = \alpha + \beta(T_{h2} - T_{c2})$

This gives us two equations and for finding the two constants $\alpha$ and $\beta$:

$$\alpha = U_1 - \frac{(U_2 - U_1)(T_{h1} - T_{c1})}{(T_{h2} - T_{c2}) - (T_{h1} - T_{c1})}$$

$$\beta = \frac{(U_2 - U_1)}{(T_{h2} - T_{c2}) - (T_{h1} - T_{c1})}$$

Then the overall heat-transfer coefficient can be written as:

$$\frac{U - U_1}{U_2 - U_1} = \frac{(T_h - T_c) - (T_{h1} - T_{c1})}{(T_{h2} - T_{c2}) - (T_{h1} - T_{c1})} \equiv \frac{\Delta T - \Delta T_1}{\Delta T_2 - \Delta T_1}$$

*b*. The first equation in (*a*) may now be written as:

$$-\frac{d\Delta T}{\Delta T} = (\alpha + \beta\Delta T)\left(\frac{1}{w_h\hat{C}_{ph}} + \frac{1}{w_c\hat{C}_{pc}}\right)(2\pi r_0)dl$$

Part of the right side may be rewritten by using Eqs. 15.4-7 and 8:

$$\left(\frac{1}{w_h\hat{C}_{ph}}+\frac{1}{w_c\hat{C}_{pc}}\right)=\left(\frac{T_{h2}-T_{h1}}{-Q_c}+\frac{T_{c2}-T_{c1}}{Q_c}\right)$$
$$=\frac{1}{Q_c}\left((T_{h1}-T_{c1})-(T_{h2}-T_{c2})\right)=\frac{1}{Q_c}(\Delta T_1-\Delta T_2)$$

The differential equation now becomes

$$-\frac{d\Delta T}{\Delta T(\alpha+\beta\Delta T)}=\frac{1}{Q_c}(\Delta T_1-\Delta T_2)(2\pi r_0)dl$$

Integration then gives:

$$-\int_{\Delta T_1}^{\Delta T_2}\frac{d\Delta T}{\Delta T(\alpha+\beta\Delta T)}=\frac{1}{Q_c}(\Delta T_1-\Delta T_2)(2\pi r_0)\int_0^L dl$$

or

$$+\frac{1}{\alpha}\ln\left|\frac{\alpha+\beta\Delta T}{\Delta T}\right|_{\Delta T_1}^{\Delta T_2}=\frac{A}{Q_c}(\Delta T_1-\Delta T_2)$$

where $A$ is the area of the heat-transfer surface. Then, introducing the expression for $\alpha$ developed in (*a*), we get

$$Q_c\ln\left(\frac{U_2}{U_1}\cdot\frac{\Delta T_1}{\Delta T_2}\right)=-A\left[U_1(\Delta T_2-\Delta T_1)-(U_2-U_1)\Delta T_1\right]$$

Finally, solving for $Q_c$ we find

$$Q_c=A\frac{U_1\Delta T_2-U_2\Delta T_1}{\ln(U_1\Delta T_2)-\ln(U_2\Delta T_1)}$$

This is the same as Eq. 15B.1-2.

**15B.2 Pressure drop in turbulent flow in a slightly converging tube**

*a*. We begin with the *d*-form of the mechanical energy balance as given in Eq. 15.4-2

$$vdv + gdh + \frac{1}{\rho}dp = d\hat{W} - \tfrac{1}{2}v^2 \frac{f}{R_h}dz$$

When this is integrated from $z=0$ to $z=L$ for flow of an incompressible fluid in a horizontal tube and no mechanical work term, we get (after using Eq. 6.2-16)

$$\tfrac{1}{2}\left(v_2^2 - v_1^2\right) + \frac{1}{\rho}\left(p_2 - p_1\right) = -2\int_0^L \frac{v^2 f}{D}dz$$

*b*. Since $w = \rho v S = \rho v(\pi D^2)$ is constant for each cross section, we have $\rho_1 v_1 D_1^2 = \rho v D^2$ so that, for an incompressible fluid

$$\frac{v}{v_1} = \left(\frac{D_1}{D}\right)^2$$

Furthermore, this result can be used to rewrite the local friction factor thus

$$f = \frac{0.0791}{(Dv/\nu)^{1/4}} = \frac{0.0791}{(D/\nu)^{1/4}(v_1)^{1/4}}\left(\frac{D}{D_1}\right)^{1/2}$$
$$= \frac{0.0791}{(D/\nu)^{1/4}(v_1)^{1/4}(D_1/D)^{1/4}}\left(\frac{D}{D_1}\right)^{1/4} = \frac{0.0791}{(D_1 v_1/\nu)^{1/4}}\left(\frac{D}{D_1}\right)^{1/4}$$

*c*. Use Eq. 15B.2-1 to make a change of variable in the integral in Eq. 15B.2-3:

$$\int_0^L \frac{v^2 f}{D}dz = \int_{D_1}^{D_2} \frac{v^2 f}{D}\left(\frac{dz}{dD}\right)dD = \frac{L}{D_2 - D_1}\int_{D_1}^{D_2} \frac{v^2 f}{D}dD$$

The limits on the integral over $D$ are fixed by using Eq. 15B.2-1.

*d*. When the expressions for the velocity and friction factor in (*b*) are substituted into the integral in (*c*), the integration can be performed to get

$$\int_0^L \frac{v^2 f}{D} dz = \frac{0.0791 L v_1^2}{(D_2 - D_1)(D_1 v_1/\nu)^{1/4}} \int_{D_1}^{D_2} \frac{(D_1/D)^4 (D/D_1)^{1/4}}{D} dD$$

$$= \frac{0.0791 L v_1^2}{(D_2 - D_1)(D_1 v_1/\nu)^{1/4}} \int_1^{D_2/D_1} x^{-19/4} dx$$

$$= \frac{\left(\frac{4}{15}\right) 0.0791 L v_1^2}{(D_1 - D_2)(D_1 v_1/\nu)^{1/4}} \left[ \left( \frac{D_1}{D_2} \right)^{15/4} - 1 \right]$$

Then Eq. 15B.2-3 becomes

$$\frac{1}{\rho}(p_1 - p_2) = \frac{1}{2} v_1^2 \left[ \left( \frac{D_1}{D_2} \right)^4 - 1 \right] + \frac{\left(\frac{8}{15}\right) 0.0791 L v_1^2}{(D_1 - D_2)(D_1 v_1/\nu)^{1/4}} \left[ \left( \frac{D_1}{D_2} \right)^{15/4} - 1 \right]$$

*e*. From Eqs. 6.1-4 and 15B.2-2, we get for $D_1 = D_2 = D$ and $v_1 = v_2 = v$

$$\frac{1}{\rho}(p_1 - p_2) = \frac{2Lv^2}{D} f = \frac{2Lv^2}{D} \frac{0.0791}{(Dv\rho/\mu)^{1/4}}$$

We now have to show that the result in (*d*) simplifies to this result when $D_1 = D_2 = D$ and $v_1 = v_2 = v$. Clearly the first term on the right side of Eq. 15B.2-6 vanishes when $D_1 = D_2 = D$. The next term gives 0/0, and hence we have to apply L'Hôpital's rule:

$$\lim_{D_2 \to D_1} \frac{\left[ (D_1/D_2)^{15/4} - 1 \right]}{(D_1/D_2) - 1} = \lim_{x \to 1} \frac{x^{15/4} - 1}{x - 1} = \lim_{x \to 1} \frac{\frac{15}{4} x^{11/4}}{1} = \frac{15}{4}$$

This, along with the statements that $D_1 = D_2 = D$ and $v_1 = v_2 = v$ completes the proof of equivalence.

**15B.3 Steady flow of ideal gases in ducts of constant cross section**

*a*, In the absence of work terms associated with moving mechanical parts and for a duct which is horizontal, the second term on the left side of 15.4-2 and the first term on the right side can be omitted. For a circular tube, 4 times the mean hydraulic radius equals the tube diameter, we get for a differential length $dL$ of the tube

$$vdv + \frac{1}{\rho}dp + 2v^2\frac{f}{D}dL = 0 \qquad \text{or} \qquad vdv + \frac{1}{\rho}dp + \tfrac{1}{2}v^2 de_v = 0$$

*b*. Use of the product rule for differentiation,

$$d\left(\frac{p}{\rho}\right) = \frac{1}{\rho}dp - \left(\frac{p}{\rho^2}\right)d\rho$$

leads directly from Eq. 15B.3-1 to Eq. 15B.3-2. The *d*-form of the mass balance is

$$dw = 0 \qquad \text{or} \qquad d(\rho v S) = 0 \ \text{ or } \qquad \rho dv + v d\rho = 0 \qquad \text{or} \qquad d\rho = \rho\frac{dv}{v}$$

For an ideal gas, $p\tilde{V} = RT$ or $pM\hat{V} = RT$ or $pM = \rho RT$. Hence for the isothermal flow of an ideal gas, the second term in Eq. 15B.3-2 is zero, and that equation becomes, after multiplication by $2/v^2$

$$de_v = -\left(\frac{2}{v^2}\right)vdv - \left(\frac{2}{v^2}\right)\left(\frac{\rho RT}{M}\right)\left(\frac{1}{\rho^2}\right)d\rho$$

Then, combining the last two equations, we get Eq. 15B.3-3.

*c*. When the result in Eq. 15B.3-3 is integrated from "1" to "2" encompassing a length $L$ of the pipe, we get

$$e_v = \frac{2RT}{M}\left(\frac{v_2^{-2}}{-2} - \frac{v_1^{-2}}{-2}\right) - 2(\ln v_2 - \ln v_1) = \frac{RT}{Mv_1^2}\left(1 - \frac{v_1^2}{v_2^2}\right) + \ln\left(\frac{v_1}{v_2}\right)^2$$

$$= \frac{RT\rho_1^2}{M(\rho_1 v_1)^2}\left[1 - \left(\frac{p_2}{p_1}\right)^2\right] + \ln\left(\frac{p_2}{p_1}\right)^2 = \frac{p_1\rho_1}{G^2}(1-r) + \ln r$$

This may be solved for the "mass velocity" $G$ to give

$$G = \sqrt{\frac{p_1\rho_1(1-r)}{e_v - \ln r}}$$

Then

$$\frac{dG^2}{dr} = p_1\rho_1\left(\frac{-1}{e_v - \ln r} - \frac{(-1/r)}{(e_v - \ln r)^2}\right) = 0$$

so that the critical value of $r$ is given by the solution of the equation

$$e_v - \ln r_c = \frac{1 - r_c}{r_c}$$

*d*. From Eq. 15.4-4, we find that the right side is zero (for an adiabatic system with no work done by moving parts), and that the ideal gas law makes the fourth term on the left be zero. In addition, since the tube is horizontal, the second term on the left is also zero. Hence we are left with

$$vdv + \hat{C}_p dT = 0 \quad \text{or} \quad vdv + \frac{\tilde{C}_p}{M} dT = 0 \quad \text{or} \quad vdv + \frac{R}{M}\frac{\gamma}{\gamma - 1} dT = 0$$

The third expression comes from the use of the ideal-gas expression $\tilde{C}_p - \tilde{C}_V = R$ and the definition $\gamma = \tilde{C}_p / \tilde{C}_V$, which give

$$\frac{\tilde{C}_p}{R} = \frac{\tilde{C}_p}{\tilde{C}_p - \tilde{C}_V} = \frac{(\tilde{C}_p/\tilde{C}_V)}{(\tilde{C}_p/\tilde{C}_V) - 1} = \frac{\gamma}{\gamma - 1}$$

Next the energy equation can be integrated to give

$$\tfrac{1}{2}v^2 + \left(\frac{\gamma}{\gamma - 1}\right)\frac{RT}{M} = \text{const} \quad \text{or} \quad \tfrac{1}{2}v^2 + \left(\frac{\gamma}{\gamma - 1}\right)\frac{p}{\rho} = \tfrac{1}{2}v_1^2 + \left(\frac{\gamma}{\gamma - 1}\right)\frac{p_1}{\rho_1}$$

where the ideal gas law has been used to get the second form. Next, after solving the integrated form of the energy equation for $p/\rho$, we substitute this expression for $p/\rho$ into the second and third terms of Eq. 15B.3-2 to get (after multiplying the latter by $2/v^2$)

$$2\frac{dv}{v} - 2\left(\frac{\gamma-1}{\gamma}\right)\frac{dv}{v} + \frac{2}{v^2}\left(\frac{p_1}{\rho_1} + \frac{\gamma-1}{\gamma}\tfrac{1}{2}v_1^2 - \frac{\gamma-1}{\gamma}\tfrac{1}{2}v^2\right)\frac{d\rho}{\rho} = -de_v$$

Then we use the $d$-form of the mass balance ($\rho dv + v d\rho = 0$) to get

$$2\left(\frac{1}{\gamma}\right)\frac{dv}{v} - \frac{2}{v^2}\left(\frac{p_1}{\rho_1} + \frac{\gamma-1}{\gamma}\tfrac{1}{2}v_1^2 - \frac{\gamma-1}{\gamma}\tfrac{1}{2}v^2\right)\frac{dv}{v} = -de_v$$

or, when some terms are combined,

$$\left(\frac{\gamma+1}{\gamma}\right)\frac{dv}{v} - \frac{2}{v^3}\left(\frac{p_1}{\rho_1} + \frac{\gamma-1}{\gamma}\tfrac{1}{2}v_1^2\right)dv = -de_v$$

This equation can be integrated to give

$$\left(\frac{\gamma+1}{\gamma}\right)\ln\frac{v_2}{v_1} + \left(\frac{p_1}{\rho_1} + \frac{\gamma-1}{\gamma}\tfrac{1}{2}v_1^2\right)\left(\frac{1}{v_2^2} - \frac{1}{v_1^2}\right) = -e_v$$

Using the macroscopic balance, $v_2/v_1 = \rho_1/\rho_2$, we get

$$\left(\frac{\gamma+1}{\gamma}\right)\ln\frac{\rho_1}{\rho_2} + \frac{p_1\rho_1}{(\rho_1 v_1)^2}\left(\frac{v_1^2}{v_2^2} - 1\right) + \frac{1}{2}\left(\frac{\gamma-1}{\gamma}\right)\left(\frac{v_1^2}{v_2^2} - 1\right) = -e_v$$

We can now solve for $G = \rho_1 v_1$ (using $s = (\rho_2/\rho_1)^2$) to get

$$G = \rho_1 v_1 = \sqrt{\frac{p_1\rho_1}{\dfrac{e_v - [(\gamma+1)/\gamma]\ln s}{1-s} - \dfrac{\gamma-1}{2\gamma}}}$$

*e*. The macroscopic energy balance is

$$\hat{C}_p(T_2 - T_1) + \tfrac{1}{2}\left(v_2^2 - v_1^2\right) = 0$$

Then using the ideal gas law in the first term and the macroscopic mass balance on the second term gives

$$\frac{M\hat{C}_p}{R}\left(\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1}\right) + \tfrac{1}{2}G^2\left(\frac{1}{\rho_2^2} - \frac{1}{\rho_1^2}\right) = 0$$

Then, since $M\hat{C}_p/R = \gamma/(1-\gamma)$, this equation can be rearranged thus

$$\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1} = \left(\frac{1-\gamma}{2\gamma}\right)\frac{G^2}{\rho_1^2}\left[1 - \left(\frac{\rho_1}{\rho_2}\right)^2\right]$$

And this expression can be rearranged further to give Eq. 15B.3-8. Let the right side of the above equation be designated by $X$ and proceed as follows:

$$\frac{p_2}{p_1}\frac{p_1}{\rho_2} - \frac{p_1}{\rho_1} = X \quad \text{or} \quad \frac{p_2}{p_1} - \frac{\rho_2}{\rho_1} = \frac{\rho_2}{p_1}X \quad \text{or} \quad \frac{p_2}{p_1} = \frac{\rho_2}{\rho_1}\left(1 + \frac{\rho_1}{p_1}X\right)$$

Inserting the expression for $X$ will then lead directly to Eq. 15B.3-8.

**15B.4 The Mach number in the mixing of two fluid streams**

$s$. Requiring that the radicand in Eq. 15.3-13 be zero is equivalent to the statement that

$$\frac{1}{2}=\left(\frac{\gamma^2-1}{\gamma^2}\right)\left(\frac{w}{P}\right)^2\left(\frac{E}{w}\right)$$

The factor $(w/P)$ can be obtained from Eq. 15.3-11, and the factor $(E/w)$ can be obtained from Eq. 15.3-10. This gives

$$\frac{1}{2}=\left(\frac{\gamma^2-1}{\gamma^2}\right)\left(v_2+\frac{RT_2}{Mv_2}\right)^{-2}\left(\hat{C}_pT_2+\tfrac{1}{2}v_2^2\right)$$

We now move the second factor on the right side to the left side, multiply the entire equation by $v_2^2$, and replace $\hat{C}_p$ by $(R/M)[\gamma/(\gamma-1)]$ to get

$$\frac{1}{2}\left(v_2^2+\frac{RT_2}{M}\right)^2=\left(\frac{\gamma^2-1}{\gamma^2}\right)\left[\frac{R}{M}\left(\frac{\gamma}{\gamma-1}\right)T_2+\tfrac{1}{2}v_2^2\right]v_2^2$$

We next multiply by 2 and rewrite the equation thus:

$$\left[v_2^4+\frac{2RT_2}{M}v_2^2+\left(\frac{RT_2}{M}\right)^2\right]=\left(\frac{\gamma+1}{\gamma}\right)\frac{2RT_2}{M}v_2^2+\left(\frac{\gamma^2-1}{\gamma^2}\right)v_2^4$$

Then we collect terms in the same powers of the velocity at plane "2" to get

$$\left(\frac{1}{\gamma^2}\right)v_2^4-\frac{2RT_2}{M}\left(\frac{1}{\gamma}\right)v_2^2+\left(\frac{RT_2}{M}\right)^2=0 \qquad \text{or} \qquad \left(\frac{1}{\gamma}v_2^2-\frac{RT_2}{M}\right)^2=0$$

This equation may be solved for the velocity at plane "2" to give

$$v_2=\sqrt{\gamma\frac{RT_2}{M}}$$

This is exactly the speed of sound, given in Problem 11C.1(*c*).

*b*. To describe the behavior of a gas passing through a sudden enlargement, we can set

$$w_{a1} = w_1 \quad \text{and} \quad w_{b1} = 0$$

and also

$$p_{1a} = p_1 \qquad T_{1a} = T_1 \qquad S_{a1} = S_1 \qquad S_{b1} = 0$$

Then when $w$, $P$, and $E$ are defined analogously to the quantities in (*a*), we find that Eqs. 15.3-11, 12, and 13 remain unchanged.

**15B.5 Limiting discharge rates for Venturi meters**

*a*. First, we take the square of Eq. 15.5-34

$$w^2 = C_d^2 \rho_1^2 S_0^2 \left( \frac{2p_1}{\rho_1} \right) \left( \frac{\gamma}{\gamma - 1} \right) \left[ \frac{(\rho_2/\rho_1)^2 \left(1 - r^{(\gamma-1)/\gamma}\right)}{1 - \beta r^{2/\gamma}} \right]$$

in which $\beta = (S_0/S_1)^2$ and $r = p_2/p_1$. Next we set $dw^2/dr = 0$ to get

$$\frac{d}{dr} \left[ \frac{r^{2/\gamma}\left(1 - r^{(\gamma-1)/\gamma}\right)}{1 - \beta r^{2/\gamma}} \right] = 0$$

and, performing the differentiation, we have

$$\frac{(2/\gamma) r^{(2/\gamma)-1} - \left((1/\gamma)+1\right) r^{1/\gamma}}{1 - \beta r^{2/\gamma}} - \frac{r^{2/\gamma}\left(1 - r^{(\gamma-1)/\gamma}\right) - \beta (2/\gamma) r^{(2/\gamma)-1}}{\left(1 - \beta r^{2/\gamma}\right)^2} = 0 \quad \text{or}$$

$$\left[(2/\gamma) r^{(2/\gamma)-1} - \left((1/\gamma)+1\right) r^{1/\gamma}\right]\left(1 - \beta r^{2/\gamma}\right) - r^{2/\gamma}\left(1 - r^{(\gamma-1)/\gamma}\right)\left[-\beta (2/\gamma) r^{(2/\gamma)-1}\right] = 0$$

When this is multiplied out, two terms cancel, and the remaining terms are:

$$\frac{2}{\gamma} r^{(2/\gamma)-1} + \beta \left( \frac{1}{\gamma} + 1 \right) r^{3/\gamma} - \left( \frac{1}{\gamma} + 1 \right) r^{1/\gamma} - \beta \frac{2}{\gamma} r^{3/\gamma} = 0$$

Multiplication by $\gamma r^{-3/\gamma}$ finally yields

$$\beta(\gamma - 1) + \frac{2}{r^{(1/\gamma)+1}} - \frac{(1+\gamma)}{r^{2/\gamma}} = 0$$

This is equivalent to Eq. 15B.5-1.

*b*. First we solve Eq. 15B.5-1 for $\beta = 0$ and get

$$r = \left(\frac{2}{\gamma+1}\right)^{\gamma/(\gamma-1)}$$

We substitute this expression for $r$ into the first equation above in (*a*) in for $\beta = 0$ and then rearrange it as follows:

$$\begin{aligned}
w_{\max} &= C_d p_1 S_0 \sqrt{2\frac{\rho_1}{p_1}\left(\frac{\gamma}{\gamma-1}\right) r^{2/\gamma}\left(1 - r^{(\gamma-1)/\gamma}\right)} \\
&= C_d p_1 S_0 \sqrt{2\frac{M}{RT_1}\left(\frac{\gamma}{\gamma-1}\right)\left(\frac{2}{\gamma+1}\right)^{2/(\gamma-1)}\left(1 - \frac{2}{\gamma+1}\right)} \\
&= C_d p_1 S_0 \sqrt{2\frac{M}{RT_1}\left(\frac{\gamma}{\gamma+1}\right)\left(\frac{2}{\gamma+1}\right)^{2/(\gamma-1)}} \\
&= C_d p_1 S_0 \sqrt{\frac{\gamma M}{RT_1}\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}}
\end{aligned}$$

in agreement with Eq. 15B.5-2.

*c*. For isothermal flow, we get from Eq. 15.5-33

$$\begin{aligned}
w &= C_d \rho_2 S_0 \sqrt{\frac{-2\int_{p_1}^{p_2}(RT/pM)dp}{1-(p_2 S_0/p_1 S_1)^2}} = C_d \rho_2 S_0 \sqrt{\frac{(2RT/M)\ln(p_1/p_2)}{1-(p_2 S_0/p_1 S_1)^2}} \\
&= C_d \rho_1 S_0 r \sqrt{\frac{(2RT/M)\ln(1/r)}{1-\beta r^2}}
\end{aligned}$$

in which $r = p_2/p_1$ and $\beta = (S_0/S_1)^2$.

Now for negligibly small $\beta$, we can find the maximum discharge rate by setting $dw^2/dr = 0$ which gives

$$\frac{d}{dr}\left(r^2 \ln\frac{1}{r}\right) = 2r\ln\frac{1}{r} - 1 = 0$$

This gives the value of $r$ for the maximum discharge rate as

$$r = \frac{p_2}{p_1} = \frac{1}{\sqrt{e}}$$

Then

$$w_{\max} = C_d \rho_1 S_0 \frac{1}{\sqrt{e}} \sqrt{\frac{2RT}{M} \cdot \frac{1}{2}} = C_d \rho_1 S_0 \sqrt{\frac{RT}{eM}} = C_d p_1 S_0 \sqrt{\frac{M}{eRT}}$$

It is interesting to compare the results for air ($\gamma = 1.4$) for the two cases. For adiabatic flow, we get

$$\frac{w_{\max}}{C_d p_1 S_0 \sqrt{M/RT_1}} = \sqrt{\gamma \left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}} = \sqrt{1.4\left(\frac{2}{2.4}\right)^{2.4/0.4}} = \sqrt{1.4}(0.8333)^3$$
$$= (1.183)(0.5787) = 0.6846$$

For isothermal flow, we have

$$\frac{w_{\max}}{C_d p_1 S_0 \sqrt{M/RT}} = \frac{1}{\sqrt{e}} = \frac{1}{\sqrt{2.718}} = 0.6065$$

The ratio of these two results is 1.13.

We estimate that this ratio will amost never be greater than 25%. Thus, the simpler isothermal expressions are frequently useful for preliminary estimates of flow behavior

**15B.6 Flow of a compressible fluid through a convergent-divergent nozzle**

*a*. From Eq. 15.2-2 with no work done by moving parts and with viscous heating neglected, we get for the assumption of flat velocity profiles:

$$\tfrac{1}{2}\left(v_2^2 - v_1^2\right) + \int_1^2 \frac{1}{\rho} dp = 0$$

Then, we set $v_1 = 0$ and, assuming adiabatic flow, we can use Eq. 15.2-5 to evaluate the integral. This results in

$$\tfrac{1}{2}v_2^2 + \frac{p_1}{\rho_1}\frac{\gamma}{\gamma-1}\left[\left(\frac{p_2}{p_1}\right)^{(\gamma-1)/\gamma} - 1\right] = 0$$

Then the ideal gas law may be used to rewrite this expression as

$$\tfrac{1}{2}v_2^2 = \frac{RT_1}{M}\frac{\gamma}{\gamma-1}\left[1 - \left(\frac{p_2}{p_1}\right)^{(\gamma-1)/\gamma}\right]$$

*b*. We may start with Eq. 15.5-34 with $S_1$ considered to be very large, and $S_0$ replaced by $S_2$

$$w = C_d \rho_2 S_2 \sqrt{2\left(\frac{p_1}{\rho_1}\right)\left(\frac{\gamma}{\gamma-1}\right)\left[1 - \left(\frac{p_2}{p_1}\right)^{(\gamma-1)/\gamma}\right]}$$

We next replace $\rho_2$ by $\rho_1(p_2/p_1)^{1/\gamma}$ and solve for $S_2$, assuming that $C_d = 1$. This gives, with $r = p_2/p_1$

$$S_2 = \frac{w}{\rho_1\sqrt{2\left(\frac{p_1}{\rho_1}\right)\left(\frac{\gamma}{\gamma-1}\right)\left(r^{2/\gamma} - r^{(\gamma+1)/\gamma}\right)}}$$

The minimum cross-section for given values of $w$, $p_1$, and $\rho_1$ will occur when

$$\frac{dS_2^2}{dr} = 0 = \frac{d}{dr}\left(r^{2/\gamma} - r^{(\gamma+1)/\gamma}\right)^{-1} = \frac{(-1)\left[(2/\gamma)r^{2/\gamma} - ((\gamma+1)/\gamma)r^{(\gamma+1)/\gamma}\right]}{r\left(r^{2/\gamma} - r^{(\gamma+1)/\gamma}\right)^2}$$

When this equation is solved for $r$, we get

$$r = \frac{p_2}{p_1} = \left(\frac{2}{\gamma+1}\right)^{\gamma/(\gamma-1)}$$

which is Eq. 15B.6-2.

*c*. Combining the results from (*a*) and (*b*) we get

$$\tfrac{1}{2}v_2^2 = \frac{RT_1}{M}\frac{\gamma}{\gamma-1}\left[1 - \frac{2}{\gamma+1}\right] = \frac{RT_1}{M}\frac{\gamma}{\gamma-1}\left(\frac{\gamma-1}{\gamma+1}\right) = \frac{RT_1}{M}\frac{\gamma}{\gamma+1}$$

Then, from Eq. 11.4-56 and Eq. 15B.6-2, we find

$$T_1 = T_2\left(\frac{p_1}{p_2}\right)^{(\gamma-1)/\gamma} = T_2\left(\frac{\gamma+1}{2}\right)$$

From the last two equations we get

$$\tfrac{1}{2}v_2^2 = \frac{RT_2}{M}\left(\frac{\gamma}{2}\right) \quad \text{or} \quad v_2 = \sqrt{\frac{\gamma RT_2}{M}}$$

Therefore, the velocity at surface "2" is sonic (cf. Eq. 9.4-4). It is clear that this problem is very similar to that in Problem 15B.5. Here, however, we are considering the effect of varying $S_2$, whereas in Problem 15B.5 the reverse procedure was used.

*d*. Since we now want to get the velocity as a function of the *local* pressure, we have to replace the first equation in (*a*) by

$$\tfrac{1}{2}\left(v^2 - v_1^2\right) + \int_{p_1}^{p} \frac{1}{\rho}dp = 0$$

We then use the fact that $v_1 = 0$, and perform the integration to get the velocity at any value of the local pressure for an ideal gas in adiabatic flow

$$\begin{aligned}
\tfrac{1}{2}v^2 &= -\int_{p_1}^{p} \frac{1}{\rho} dp = -\int_{p_1}^{p} \left( \frac{p_1^{1/\gamma}}{\rho_1} \right) p^{-1/\gamma} dp = -\left( \frac{p_1^{1/\gamma}}{\rho_1} \right) \left( \frac{\gamma}{\gamma - 1} \right) \left( p^{(\gamma-1)/\gamma} - p_1^{(\gamma-1)/\gamma} \right) \\
&= +\left( \frac{p_1^{1/\gamma}}{\rho_1} \right) \left( \frac{\gamma}{\gamma - 1} \right) p_1^{(\gamma-1)/\gamma} \left[ 1 - \left( \frac{p}{p_1} \right)^{(\gamma-1)/\gamma} \right] \\
&= \left( \frac{p_1}{\rho_1} \right) \left( \frac{\gamma}{\gamma - 1} \right) \left( 1 - r^{(\gamma-1)/\gamma} \right)
\end{aligned}$$

where $r = p/p_1$ (!). Then taking the square root and using the ideal gas law, we get

$$v = \sqrt{\frac{2RT_1}{M} \left( \frac{\gamma}{\gamma - 1} \right) \left( 1 - r^{(\gamma-1)/\gamma} \right)}$$

The cross-section at any value of the local pressure can then be obtained from (with $r = p/p_1$):

$$\begin{aligned}
w &= \rho S \sqrt{2 \left( \frac{p_1}{\rho_1} \right) \left( \frac{\gamma}{\gamma - 1} \right) \left[ 1 - \left( \frac{p}{p_1} \right)^{(\gamma-1)/\gamma} \right]} \\
&= \rho_1 S r^{1/\gamma} \sqrt{2 \left( \frac{RT_1}{M} \right) \left( \frac{\gamma}{\gamma - 1} \right) \left( 1 - r^{(\gamma-1)/\gamma} \right)}
\end{aligned}$$

Then, solving for the cross-section $S$, we get

$$S = \frac{w/\rho_1}{r^{1/\gamma} \sqrt{2 \left( \frac{RT_1}{M} \right) \left( \frac{\gamma}{\gamma - 1} \right) \left( 1 - r^{(\gamma-1)/\gamma} \right)}}$$

Next, for the conditions of the problem given in part (*d*) of the problem, we have

$$R = (1544)(32.17) = \text{ft}^2\text{lb}_m/\text{lb-mole}\cdot\text{s}^2\cdot\text{R}$$
$$RT_1/M = (1544)(32.17)(560)/29 = 9.592\times10^5\ \text{ft}^2/\text{s}^2$$
$$\sqrt{\frac{2RT_1}{M}\left(\frac{\gamma}{\gamma-1}\right)} = \sqrt{\left(9.592\times10^5\right)\frac{2(1.4)}{0.4}} = 2591\ \text{ft/s}$$
$$w = (10\ \text{lb-mol/s})(29\ \text{lb}_m/\text{lb-mol}$$
$$\rho_1 = \frac{p_1M}{RT_1} = \frac{(10\ \text{atm})\left(6.8087\times10^4\ (\text{lb}_m/\text{ft}\cdot\text{s}^2)\ \text{per atm.}\right)}{\left(4.9686\times10^4\ \text{lb}_m\text{ft}^2/\text{s}^2\cdot\text{lb-mol}\cdot{}^\circ\text{K}\right)(560^\circ\text{K})}$$
$$= 0.710\ \text{lb}_m/\text{ft}^3$$

We may now summarize the calculations of $v$, $T$, and $S$ thus:

$$v = 2591\sqrt{1-r^{0.286}}\,; \qquad T = 560r^{0.286}; \qquad S = \frac{0.157r^{-0.714}}{\sqrt{1-r^{0.286}}}$$

| $p$ (atm) | $r$ | $r^{0.286}$ | $v$ | $T$ | $S$ |
|---|---|---|---|---|---|
| 10 | 1.0 | 1.000 | 0000 | 560 | ∞ |
| 9 | 0.9 | 0.970 | 449 | 543 | 0.977 |
| 8 | 0.8 | 0.938 | 645 | 525 | 0.739 |
| 7 | 0.7 | 0.903 | 807 | 506 | 0.650 |
| 6 | 0.6 | 0.864 | 956 | 484 | 0.613 |
| 5.28* | 0.528 | 0.833 | 1058 | 466 | 0.606 |
| 5 | 0.5 | 0.820 | 1099 | 459 | 0.607 |
| 4 | 0.4 | 0.769 | 1245 | 431 | 0.628 |
| 3 | 0.3 | 0.709 | 1398 | 397 | 0.688 |
| 2 | 0.2 | 0.631 | 1574 | 353 | 0.816 |
| 1 | 0.1 | 0.518 | 1798 | 290 | 1.171 |
| 0 | 0.0 | 0.000 | 2591 | 0 | ∞ |

*Pressure at the minimum cross-section

**15B.7 Transient thermal behavor of a chromatographic device**

We start with the energy equation of Eq. 15.1-2, which we apply to the gas inside the chromatographic column. We are concerned only with the change in time (and not distance) within the column. Since there are no moving mechanical parts, the equation simplifies to

$$\frac{d}{dt}U_{\text{tot}} = Q = UA\left(T_g - T\right)$$

in which the total kinetic and potential energy in the system are considered constant, and $U$ is the overall heat transfer coefficient between the external gas stream and the gas within the chromatographic device.

From thermodynamics we can write

$$d\hat{U} = \left(\frac{\partial \hat{U}}{\partial T}\right)_{\hat{V}} dT + \left(\frac{\partial \hat{U}}{\partial \hat{V}}\right)_T d\hat{V} = \hat{C}_V dT + \left[-p + T\left(\frac{\partial p}{\partial T}\right)_{\hat{V}}\right] d\hat{V}$$

The quantity inside the brackets is zero for an ideal gas and is neglected here. Then $\hat{U} - \hat{U}^\circ = \hat{C}_V(T - T^\circ)$, where the superscripts $^\circ$ stand for the reference state, and

$$U_{\text{tot}} = \int \rho \hat{U} dV = \rho \hat{U}^\circ V_{\text{tot}} + \rho \hat{C}_V (T - T^\circ) V_{\text{tot}}$$

where the density and heat capacity are taken to be constants. The energy balance becomes

$$\rho \hat{C}_V V_{\text{tot}} \frac{dT}{dt} = UA\left(T_g - T\right) \qquad \text{or} \qquad \frac{dT}{dt} = \frac{UA}{\rho \hat{C}_V V_{\text{tot}}}\left[T_{g0}\left(1 + \frac{t}{t_0}\right) - T\right]$$

We now introduce the dimensionless temperature $\Theta = T/T_{g0}$ and the dimensionless time $\tau = t/t_0$ and rewrite the energy balance as

$$\frac{d\Theta}{d\tau} = B\left[(1+\tau) - \Theta\right] \qquad \text{with} \qquad B = UAt_0/\rho \hat{C}_V V_{\text{tot}}$$

This differential equation can be put into the form

$$\frac{d\Theta}{d\tau} + B\Theta = B(1+\tau) \qquad \text{with} \qquad \Theta = 1 \text{ at } \tau = 0$$

This is a linear first-order differential equation (see Eq. C.1-2), which with the indicated initial condition has the solution

$$\Theta = \left(1 + \tau - \frac{1}{B}\right) + \frac{1}{B}e^{-B\tau}$$

This is probably the simplest solution that can be offered for this problem. We are now in a position to answer the questions in the problem:

*a*. The difference between the external gas temperature and the temperature within the chromatographic device is

$$T_g - T = T_{g0}(1-\tau) - T_{g0}\left(1 + \tau - \frac{1}{B} + \frac{1}{B}e^{-B\tau}\right) = T_{g0}\left(\frac{1}{B} - \frac{1}{B}e^{-B\tau}\right)$$

*b*. The limiting value of the temperature difference will be

$$(\Delta T)_\infty = T_{g0}\frac{1}{B}$$

*c*. The time required for the temperature difference to be within 1% of the limiting value is given by the solution of the equation $e^{-B\tau} = 0.01$ or $\tau = (2.303)(2)/B$.

*d*. Constant physical properties; no dependence of temperature as a function of distance down the column; the neglecting of potential energy and kinetic energy effects.

*e*. All the quantities that are contained in $B$.

**15B.8 Continuous heating of a slurry in an agitated tank**

*a.* The starting point is Eq. 15.1-1 or 2. On the left side the potential energy term $\Phi_{tot}$ is omitted because for a full tank the $\Phi_{tot}$ is a constant. Similarly $K_{tot}$ is time-independent and can be omitted. The term $U_{tot} = \int_V \rho \hat{U} dV$ will be retained. Since $\hat{U} = \hat{H} - (p/\rho)$, and $p$ and $\rho$ are time-independent, we can replace the internal energy by enthalpy, which is given by Eq. 9.8-8, the second term on the right side of that equation being negligible. Furthermore, since the heat capacity is considered constant, Eq. 9.8-8 becomes simply

$$\hat{H} = \hat{H}_i + \hat{C}_p (T - T_i)$$

where the inlet slurry temperature $T_i$ has been chosen to the datum plane for the enthalpy.

On the right side of Eq. 15.1-1, the mechanical work term and the kinetic energy and potential energy terms can be disregarded. But the heat addition term $Q$ and the outlet enthalpy term must be retained. Therefore, Eq. 15.1-1 becomes

$$\frac{d}{dt} U_{tot} = -\hat{H}_2 w + Q$$

or

$$\rho \hat{C}_p V \frac{dT}{dt} = -w\hat{C}_p (T - T_i) + UA(T_s - T)$$

which is Eq. 15B.8-1.

*b.* When the system has attained steady state operation, the left side of the above equation becomes zero, and the slurry temperature has leveled off to $T_\infty$. This quantity is then defined by the equation

$$0 = -w\hat{C}_p (T_\infty - T_i) + UA(T_s - T_\infty)$$

or (with the definition that $UA/w\hat{C}_p = \Omega$):

$$T_\infty = \frac{T_i + \left(UA/w\hat{C}_p\right)T_s}{1+\left(UA/w\hat{C}_p\right)} = \frac{T_i + \Omega T_s}{1+\Omega}$$

From the last equation we may also get the relation

$$\frac{T_s - T_\infty}{T_i - T_\infty} = -\frac{w\hat{C}_p}{UA} = -\frac{1}{\Omega}$$

which is needed presently.

Now the differential equation can be rewritten in terms of the dimensionless variables given in Eqs. 15B.8-2,3:

$$\begin{aligned}\frac{d}{d\tau}\left(\frac{T-T_\infty}{T_i-T_\infty}\right) &= \Omega\left(\frac{T_s-T}{T_i-T_\infty}\right) - \left(\frac{T-T_i}{T_i-T_\infty}\right) \\ &= \Omega\left(\frac{(T_s-T_\infty)-(T-T_\infty)}{T_i-T_\infty}\right) - \left(\frac{(T-T_\infty)-(T_i-T_\infty)}{T_i-T_\infty}\right) \\ &= \Omega\left(-\frac{1}{\Omega}-\Theta\right) - (\Theta-1) = -\Theta(1+\Omega)\end{aligned}$$

Hence the differential equation in dimensionless form is

$$\frac{d\Theta}{d\tau} = -\Theta(1+\Omega)$$

*c*. This equation is solved to give

$$\int_1^\Theta \frac{d\Theta}{\Theta} = -(1+\Omega)\int_0^\tau d\tau \qquad \text{or} \qquad \Theta = e^{-(1+\Omega)\tau}$$

This is just the dimensionless form of the solution given in the text.

*d*. When the solution is written in the dimensionless form, it is easy to see that it satisfies the differential equation and initial condition. It is also evident that $\Theta \to 0$ (or $T \to T_\infty$), which is to be expected.

**15C.1 Parallel-counterflow heat exchangers**

*a.* From Eq. 15.1-3, with only the enthalpy terms contributing, and Eq. 9.3-8 with only the first term being important (and the heat capacity constant), we get for the region between *a* and *b*

$$w_B\hat{C}_{pB}(T_B - T_{B2}) - w_A\hat{C}_{pA}\left(T_A^{\mathrm{II}} - T_A^{\mathrm{I}}\right) = 0$$

from which Eq. 15C.1-1 follows.

*b.* Then the application of Eq. 15.4-4 gives, for the region between the dashed lines in Fig. 15C.1, three differential energy balances over the heat-transfer surface $dA$:

$$w_A\hat{C}_{pA}dT_A^{\mathrm{I}} = \frac{1}{2}U\left(T_B - T_A^{\mathrm{I}}\right)dA$$

$$w_A\hat{C}_{pA}dT_A^{\mathrm{II}} = -\frac{1}{2}U\left(T_B - T_A^{\mathrm{II}}\right)dA$$

$$w_B\hat{C}_{pB}dT_B = \left[\frac{1}{2}U\left(T_A^{\mathrm{I}} - T_B\right) + \frac{1}{2}U\left(T_A^{\mathrm{II}} - T_B\right)\right]dA$$

Then introducing the ratio $R$, and the dimensionless differential area $d\alpha$ (defined in the problem statement) leads directly to Eqs. 15C.1-2, 3, and 4.

*c.* First we differentiate Eq. 15C.1-4 with respect to $\alpha$ to get

$$\frac{1}{R}\frac{d^2T_B}{d\alpha^2} + \frac{dT_B}{d\alpha} - \frac{1}{2}\left(\frac{dT_A^{\mathrm{I}}}{d\alpha} + \frac{dT_A^{\mathrm{II}}}{d\alpha}\right) = 0$$

Then using Eqs. 15C.1-2 and 3 this becomes

$$\frac{1}{R}\frac{d^2T_B}{d\alpha^2} + \frac{dT_B}{d\alpha} - \frac{1}{4}\left(T_B - T_A^{\mathrm{I}}\right) - \frac{1}{4}\left(T_A^{\mathrm{II}} - T_B\right) = 0$$

Next use Eq. 15C.1-1 to rewrite this as

$$\frac{d^2T_B}{d\alpha^2} + R\frac{dT_B}{d\alpha} - \frac{1}{4}(T_B - T_{B2}) = 0 \qquad \text{or} \qquad \frac{d^2\Theta}{d\alpha^2} + R\frac{d\Theta}{d\alpha} - \frac{1}{4}\Theta = 0$$

In the second equation we have used the dimensionless temperature. This differential equation may be solved (see Eq. C.1-7a) as follows:

$$\Theta = C_1 \exp\left[-\tfrac{1}{2}\left(R - \sqrt{R^2-1}\right)\alpha\right] + C_2 \exp\left[-\tfrac{1}{2}\left(R + \sqrt{R^2-1}\right)\alpha\right]$$
$$\equiv C_1 e^{m_-\alpha} + C_2 e^{m_+\alpha}$$

Application of the boundary conditions gives two equations for the integration constants:

$$1 = C_1 + C_2$$

$$0 = C_1 e^{m_-\alpha_T} + C_2 e^{m_+\alpha_T}$$

The integration constants are therefore:

$$C_1 = -\frac{e^{m_+\alpha_T}}{e^{m_-\alpha_T} - e^{m_+\alpha_T}} = \frac{1}{1 - e^{(m_- - m_+)\alpha_T}} = \frac{1}{1 - e^{\sqrt{R^2-1}\,\alpha_T}}$$

$$C_2 = +\frac{e^{m_-\alpha_T}}{e^{m_-\alpha_T} - e^{m_+\alpha_T}} = \frac{1}{1 - e^{(m_+ - m_-)\alpha_T}} = \frac{1}{1 - e^{-\sqrt{R^2-1}\,\alpha_T}}$$

*d*. Next we obtain $d\Theta/d\alpha$:

$$\frac{d\Theta}{d\alpha} = \frac{m_- e^{m_-\alpha}}{1 - e^{\sqrt{R^2-1}\,\alpha_T}} + \frac{m_+ e^{m_+\alpha}}{1 - e^{-\sqrt{R^2-1}\,\alpha_T}}$$

Then evaluate this at $\alpha = 0$ to get

$$\frac{1}{T_{B1} - T_{B2}} \frac{dT_B}{d\alpha}\bigg|_{\alpha=0} = \frac{m_-}{1 - e^{\sqrt{R^2-1}\,\alpha_T}} + \frac{m_+}{1 - e^{-\sqrt{R^2-1}\,\alpha_T}}$$

Then eliminate the derivative by use of Eq. 15C.1-4 (also evaluated at $\alpha = 0$) to obtain:

$$\frac{R}{T_{B1}-T_{B2}}\left[-T_{B1}+\tfrac{1}{2}\left(T_{A1}+T_{A2}\right)\right]=\frac{m_-}{1-e^{\sqrt{R^2-1}\,\alpha_T}}+\frac{m_+}{1-e^{-\sqrt{R^2-1}\,\alpha_T}} \qquad (*)$$

We now replace the denominator on the left side by using Eq. 15C.1-1, and then we have to manipulate the left side in such a way as to obtain the ratio $\Psi$ defined after Eq. 15C.1-8; thus the left side becomes:

$$\frac{\tfrac{1}{2}\left(T_{A1}-T_{B1}\right)+\tfrac{1}{2}\left(T_{A2}-T_{B1}\right)}{T_{A2}-T_{A1}}=\frac{\left(T_{A1}-T_{B1}\right)+\tfrac{1}{2}\left(T_{A2}-T_{B1}-T_{A1}+T_{B1}\right)}{T_{A2}-T_{A1}}$$
$$=-\frac{T_{B1}-T_{A1}}{T_{A2}-T_{A1}}+\frac{1}{2}=-\frac{1}{\Psi}+\frac{1}{2}\equiv C$$

With this substitution Eq. (*) gives $\Psi$ as a function of $\alpha_T$. But we would like to have $\alpha_T$ as a function of $\Psi$. That is, we have to solved the following equation for $x$:

$$C=\frac{A}{1-x}+\frac{B}{1-(1/x)} \qquad \text{with } A=m_-, \qquad B=m_+. \qquad x=e^{\sqrt{R^2-1}\,\alpha_T}$$

But this is just a quadratic equation, which can be solved by the standard method, which yields two solutions:

$$x=1 \quad \text{and} \quad x=\frac{C-A}{C-B}$$

The solution $x=1$ is physically uninteresting. The other solution gives

$$e^{\sqrt{R^2-1}\,\alpha_T}=\frac{\left(-(1/\Psi)+\tfrac{1}{2}\right)-m_-}{\left(-(1/\Psi)+\tfrac{1}{2}\right)-m_+}=\frac{(-2+\Psi)-2\Psi m_-}{(-2+\Psi)-2\Psi m_+}=\frac{2-\Psi(1-2m_-)}{2-\Psi(1-2m_+)} \quad \text{or}$$

$$e^{\sqrt{R^2-1}\,\alpha_T}=\frac{2-\Psi\left(1-1+R-\sqrt{R^2+1}\right)}{2-\Psi\left(1-1+R+\sqrt{R^2+1}\right)}$$

Taking the logarithm of this equation gives Eq. 15C.1-8.

**15D.1 The macroscopic entropy balance**

*a*. First rewrite Eq. 11D.1-3 as

$$\frac{\partial}{\partial t}\rho\hat{S} = -\left(\nabla\cdot\rho\mathbf{v}\hat{S}\right) - \frac{1}{T}(\nabla\cdot\mathbf{q}) - \frac{1}{T}(\boldsymbol{\tau}:\nabla\mathbf{v})$$
$$= -\left(\nabla\cdot\rho\mathbf{v}\hat{S}\right) - \left(\nabla\cdot\frac{1}{T}\mathbf{q}\right) - \frac{1}{T^2}(\mathbf{q}\cdot\nabla T) - \frac{1}{T}(\boldsymbol{\tau}:\nabla\mathbf{v})$$

We now integrate this over the entire macroscopic flow system, which is presumed to have some moving parts in it so that the volume and surface of the system are time dependent:

$$\int_{V(t)}\frac{\partial}{\partial t}\rho\hat{S}dV = -\int_{V(t)}\left(\nabla\cdot\rho\mathbf{v}\hat{S}\right)dV - \int_{V(t)}\left(\nabla\cdot\frac{1}{T}\mathbf{q}\right)dV$$
$$-\int_{V(t)}\frac{1}{T}\left[(\mathbf{q}\cdot\nabla\ln T) + (\boldsymbol{\tau}:\nabla\mathbf{v})\right]dV$$

Since the last term is the integral of the entropy production terms in Eq. 11D.1-4, we can label this term $g_{S,\text{tot}}$. We may now use the Leibniz rule to move the time-derivative operator to the left of the integral sign and the Gauss divergence theorem to transform the volume integral into a surface integral:

$$\int_{V(t)}\frac{\partial}{\partial t}\rho\hat{S}dV = \frac{d}{dt}\int_{V(t)}\rho\hat{S}dV + \int_{S(t)}\left(n\cdot\rho\mathbf{v}_S\hat{S}\right)dS = \frac{d}{dt}S_{\text{tot}} + \int_{S(t)}\left(\mathbf{n}\cdot\rho\mathbf{v}_S\hat{S}\right)dS$$

When this is inserted into the preceding equation and the Gauss theorem applied to the integrals containing divergences, we get:

$$\frac{d}{dt}S_{\text{tot}} = -\int_{S(t)}\left(\mathbf{n}\cdot\rho(\mathbf{v}-\mathbf{v}_S)\hat{S}\right)dS - \int_{S(t)}\left(\mathbf{n}\cdot\frac{1}{T}\mathbf{q}\right)dS + g_{S,\text{tot}}$$

We now divide the surface into four parts: $S(t) = S_1 + S_2 + S_f + S_m$, the cross-section of the entrance (1), the cross-section of the exit (2), the fixed solid surfaces ($f$, and the moving surfaces ($m$). The first surface integral above will contribute only at 1 and 2:

$$-\int_{S(t)}\left(\mathbf{n}\cdot\rho(\mathbf{v}-\mathbf{v}_S)\hat{S}\right)dS=\rho_1 v_1\hat{S}_1 S_1-\rho_2 v_2\hat{S}_2 S_2=w_1\hat{S}_1-w_2\hat{S}_2$$

The second surface integral will, however, contribute at all four surfaces:

$$-\int_{S(t)}\left(\mathbf{n}\cdot\frac{1}{T}\mathbf{q}\right)dS=\frac{q_1 S_1}{T_1}-\frac{q_2 S_2}{T_2}-\int_{S_f+S_m}\left(\mathbf{n}\cdot\frac{1}{T}\mathbf{q}\right)dS=\frac{w_1 q_1}{\rho_1 v_1 T_1}-\frac{w_2 q_2}{\rho_2 v_2 T_2}+Q_S$$

Hence the macroscopic entropy balance becomes:

$$\frac{d}{dt}S_{\text{tot}}=\left(\hat{S}_1+\frac{q_1}{\rho_1 v_1 T_1}\right)w_1-\left(\hat{S}_2+\frac{q_2}{\rho_2 v_2 T_2}\right)w_2+Q_S+g_{S,\text{tot}}$$

which is Eq. 15D.1-1.

*b*. The macroscopic energy balances states that the increase in the total entropy of the system results from (1) the convection of entropy in and out of the system $(w_1\hat{S}_1-w_2\hat{S}_2)$; (2) the entropy transport at the inlet and outlet of the system by heat conduction $((q_1 S_1/T_1)-(q_2 S_2/T_2))$--presumably very small compared with the entropy convection terms; (3) the entropy transport through the walls of the system, and (4) the entropy production within the system, which consists of a term involving the heat flux and a term involving the momentum flux. Entropy is thus produced by the dissipative effects of heat conduction and viscous flow. These terms are positive for linear flux laws.

*c*. The term in the entropy production term involving the stress tensor can be written

$$-\int_{V(t)}\frac{1}{T}(\boldsymbol{\tau}:\nabla\mathbf{v})dV\doteq-\frac{1}{T_{\text{avg}}}\int_{V(t)}(\boldsymbol{\tau}:\nabla\mathbf{v})dV=\frac{E_v}{T_{\text{avg}}}$$

if the temperature variation throughout the system is not too large with respect to the absolute temperature. Then the integral is indeed the energy dissipation by viscous heating divided by the average temperature.

**15D.2 Derivation of the macroscopic energy balance**

When Eq. (N) of Table 11.4-1 is integrated over the entire volume of the flow system we get:

$$\int_{V(t)} \frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho v^2 + \rho\hat{U} + \rho\hat{\Phi}\right)dV = -\int_{V(t)} \left(\nabla\cdot\left(\tfrac{1}{2}\rho v^2 + \rho\hat{U} + \rho\hat{\Phi}\right)\mathbf{v}\right)dV$$
$$- \int_{V(t)} (\nabla\cdot\mathbf{q})dV - \int_{V(t)} (\nabla\cdot p\mathbf{v})dV - \int_{V(t)} (\nabla\cdot[\boldsymbol{\tau}\cdot\mathbf{v}])dV$$

We now apply the Leibniz formula to the integral on the left side of the equation to get

$$\int_{V(t)} \frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho v^2 + \rho\hat{U} + \rho\hat{\Phi}\right)dV$$
$$= \frac{d}{dt}\int_{V(t)} \left(\tfrac{1}{2}\rho v^2 + \rho\hat{U} + \rho\hat{\Phi}\right)dV + \int_{S(t)} \left(\mathbf{n}\cdot\left(\tfrac{1}{2}\rho v^2 + \rho\hat{U} + \rho\hat{\Phi}\right)\mathbf{v}_S\right)dS$$
$$= \frac{d}{dt}\left(K_{\text{tot}} + U_{\text{tot}} + \Phi_{\text{tot}}\right) + \int_{S(t)} \left(\mathbf{n}\cdot\left(\tfrac{1}{2}\rho v^2 + \rho\hat{U} + \rho\hat{\Phi}\right)\mathbf{v}_S\right)dS$$

This is now substituted into the preceding equation (in which the volume integrals have been converted into surface integrals by using the Gauss divergence theorem) to get:

$$\frac{d}{dt}\left(K_{\text{tot}} + U_{\text{tot}} + \Phi_{\text{tot}}\right) = -\int_{S(t)} \left(\mathbf{n}\cdot\left(\tfrac{1}{2}\rho v^2 + \rho\hat{U} + \rho\hat{\Phi}\right)(\mathbf{v} - \mathbf{v}_S)\right)dS$$
$$- \int_{S(t)} (n\cdot\mathbf{q})dS - \int_{S(t)} (\mathbf{n}\cdot p\mathbf{v})dS - \int_{S(t)} (\mathbf{n}\cdot[\boldsymbol{\tau}\cdot\mathbf{v}])dS$$

We now divide the surface into four parts: $S(t) = S_1 + S_2 + S_f + S_m$, the cross-section of the entrance (1), the cross-section of the exit (2), the fixed solid surfaces ($f$, and the moving surfaces ($m$). The first surface integral above will contribute only at 1 and 2:

$$-\int_{S(t)} \left(\mathbf{n}\cdot\left(\tfrac{1}{2}\rho v^2 + \rho\hat{U} + \rho\hat{\Phi}\right)(\mathbf{v} - \mathbf{v}_S)\right)dS$$
$$= \left(\tfrac{1}{2}\rho_1\langle v_1^3\rangle + \rho_1\hat{U}_1\langle v_1\rangle + \rho_1\hat{\Phi}_1\langle v_1\rangle\right)S_1 - \left(\tfrac{1}{2}\rho_2\langle v_2^3\rangle + \rho_2\hat{U}_2\langle v_2\rangle + \rho_2\hat{\Phi}_2\langle v_2\rangle\right)S_2$$

The contribution of the second surface integral at 1 and 2 is presumed to be negligible with respect to the convective energy transport, but will in general have a nonzero value on the surfaces; we call this $Q$. The third integral will contribute both at "1" and "2" and on the moving surfaces:

$$-\int_{S(t)}(\mathbf{n}\cdot p\mathbf{v})dS = +p_1\langle v_1\rangle S_1 - p_2\langle v_2\rangle S_2 + W_m^{(p)}$$

where $W_m^{(p)}$ is the pressure contribution to the work done on the system by the via the moving surfaces. The fourth integral will be:

$$-\int_{S(t)}(\mathbf{n}\cdot[\boldsymbol{\tau}\cdot\mathbf{v}])dS + W_m^{(\tau)}$$

which is the stress contribution to the work done on the system by the via the moving surfaces. Of course there are also contributions at 1 and 2, but these are considered to be negligible to the pressure forces. When all of the above contributions are assembled, we get

$$\frac{d}{dt}(K_{\text{tot}} + U_{\text{tot}} + \Phi_{\text{tot}}) =$$
$$\left(\tfrac{1}{2}\rho_1\langle v_1^3\rangle + \rho_1\hat{U}_1\langle v_1\rangle + \rho_1\hat{\Phi}_1\langle v_1\rangle\right)S_1 - \left(\tfrac{1}{2}\rho_2\langle v_2^3\rangle + \rho_2\hat{U}_2\langle v_2\rangle + \rho_2\hat{\Phi}_2\langle v_2\rangle\right)S_2$$
$$+ Q + p_1\langle v_1\rangle S_1 - p_2\langle v_2\rangle S_2 + W_m^{(p)} + W_m^{(\tau)}$$

which is the macroscopic energy balance given in Eq. 15.1-1.

**15D.5 The classical Bernoulli equation**

For an isentropic flow,

$$d\hat{U} = -pd\hat{V} = -pd\frac{1}{\rho} \qquad \text{or} \qquad \nabla\hat{U} = -p\nabla\hat{V} = -p\nabla\frac{1}{\rho}$$

Then Eq. (E) of Table 11.4-1, with the heat and momentum fluxes set equal to zero, becomes for a steady-state system

$$\rho\left(\mathbf{v}\cdot\nabla\tfrac{1}{2}v^2\right) + \rho\left(\mathbf{v}\cdot\left(-p\nabla\rho^{-1}\right)\right) = -\left(\nabla\cdot p\mathbf{v}\right) + \rho\left(\mathbf{v}\cdot\mathbf{g}\right)$$

Using $\mathbf{g} = -g\nabla h$ and some standard vector identities

$$\rho\left(\mathbf{v}\cdot\nabla\tfrac{1}{2}v^2\right) + p\rho^{-1}\left(\mathbf{v}\cdot\nabla\rho\right) = -p\left(\nabla\cdot\mathbf{v}\right) - \left(\mathbf{v}\cdot\nabla p\right) - \rho\left(\mathbf{v}\cdot\nabla h\right)$$

The second term on the left and the second term on the right can be shown to be equal, using the equation of continuity (appropriately simplified for steady-state flow). Next, consider that the del operators are acting only along a streamline, so that $\nabla = (\mathbf{v}/v)(d/ds)$, and hence we get

$$\rho v\frac{d}{ds}\left(\tfrac{1}{2}v^2\right) = -\frac{\rho v}{\rho}\frac{d}{ds}p - \rho v g\frac{d}{ds}h$$

When this equation is divided through by $\rho v$, we get Eq. 3.5-11. Then Eq. 3.5-12 follows at once.

**16A.1 Approximation of a black body by a hole in a sphere**
Use Eq. 16.2-12

$$e_{\text{hole}} = \frac{e}{e + f(1-e)}$$

or

$$f = \frac{e(1-e_{\text{hole}})}{e_{\text{hole}}(1-e)}$$

Into this we insert the values $e = 0.57$ and $e_{\text{hole}} = 0.99$ and get

$$f = \frac{0.57(1-0.99)}{0.99(1-0.57)} = \frac{0.57(0.01)}{0.99(0.43)} = \frac{0.0057}{0.4257} = 0.01339$$

Then using the definition of $f$ we write

$$f = 0.01339 = \frac{\pi r_{\text{hole}}^2}{4\pi(3^2)}$$

Then solving for the hole radius we get

$$r_{\text{hole}} = 2\sqrt{0.01339}(3) = 0.69 \text{ in.}$$

**16A.2 Efficiency of a solar engine**

The area of the mirror is $(\pi R^2) = 25\pi\ \text{ft}^2$. Since the solar constant (heat flux entering the earth's atmosphere) is, according to Example 16.4-1, 430 $\text{Btu/hr}\cdot\text{ft}^2$, the energy input to the solar device is

$$\left(430\,\text{Btu/hr}\cdot\text{ft}^2\right)\left(25\pi\ \text{ft}^2\right)\left(3.93\times10^{-4}\ \text{hp/Btu}\cdot\text{hr}^{-1}\right) = 13.3\ \text{hp}$$

Therefore the efficiency of the solar device is

$$\text{Efficiency} = \frac{2}{13.3} = 0.150 = 15\%$$

## 16A.3 Radiant heating requirements

The heat required is the sum of the radiant heat-transfer rates betwen the floor and each of the other surfaces. Since no pertinent data are supplied for its estimation, the convective heat transfer will be neglected. It can be expected to be appreciable, however.

The total radiant heat-transfer rate is then

$$Q_{\text{rad}} = \sigma A_{\text{floor}}\left(T_{\text{floor}}^4 - T_{\text{walls}}^4\right)\sum F_i$$

Floor to ceiling: $F_1 = 0.49$

Floor to large walls: $F_2 = F_3 = 0.17$

Floor to small walls: $F_4 = F_5 = 0.075$

Summing the contributions, we get:

$$\sum F_i = 0.49 + 2(0.17) + 2(0.075) = 0.49 + 0.34 + 0.15 = 0.98$$

Alternatively, we may consider the floor to be completely surrounded by black surfaces. From that point of view we know that the sum of the $F_i$ should be 1.00. A cumulative error of 2% has thus resulted from considering each of the cold surfaces separately. Then we get

$$\begin{aligned} Q_{\text{rad}} &= (1.712\times10^{-9}\ \text{Btu/hr}\cdot\text{ft}^2\text{R}^4)(450\text{ft}^2) \\ &\qquad \cdot\left[(75+460)^4 - (-10+460)^4\right]\text{R}^4 \\ &= \left(1.712\times10^{-9}\right)(450)\left(8.19\times10^{10} - 4.10\times10^{10}\right) \\ &= 3.15\times10^4\ \text{Btu/hr} \end{aligned}$$

Here we have used the value of the Stefan-Boltzmann constant given on p. 867.

## 16A.4 Steady-state temperature of a roof

Since June 21 is (conveniently) very close to the summer solstice, the angle of incidence of the sun's rays on a flat roof may be calculated quite simply. We know that the earth's axis is tilted at an angle of about 23.5 degrees. Thus the angle of incidence of a flat roof at 45 degrees north latitude will be about $45 - 23.5 = 21.5$ degrees, and the heat received by the roof will be given by the solar constant multiplied by the cosine of the angle of incidence and then further multiplied by the absorptivity of the surface: (430)(cos 21.5°)*a*. = (430)(0.9304)*a*., in units of Btu/hr.ft$^2$. We now equate the radiant energy received from the sun by the roof to the radiant energy emitted by the roof plus the heat lost by convective heat transfer for the two cases given in parts (a) and (b):

$$(430)(0.9304)a = h\left(T_{\text{roof}} - T_{\text{air}}\right) + \sigma e T_{\text{roof}}^4$$

*a*. For a perfectly black roof, we have

$$(430)(0.9304)(1.00) = 2.0\left(T_{\text{roof}} - 560\right) + \left(1.712 \times 10^{-9}\right)(1.00)T_{\text{roof}}^4$$

This equation may be solved by trial and error to get about 625°R or 165°F.

*b*. For the flat roof with $a = 0.3$ and $e = 0.07$ we get

$$(430)(0.9304)(0.3) = 2.0\left(T_{\text{roof}} - 560\right) + \left(1.712 \times 10^{-9}\right)(0.07)T_{\text{roof}}^4$$

which may be solved by trial and error to give 610°R or 150°F.

## 16A.5 Radiation errors in temperature measurements

Assume that the thermocouple behaves as a gray body in a large black enclosure, and equat the net radiation loss to the convective heat input:

$$e\sigma\left(T_{\text{th}}^4 - T_{\text{wall}}^4\right) = h\left(T_{\text{gas}} - T_{\text{th}}\right)$$

in which $T_{\text{th}}$ is the thermocouple junction temperature.

For the conditions of this problem

$$(0.8)\left(1.712\times10^{-9}\right)\left[(960)^4 - (760)^4\right] = (50)\left(T_{\text{gas}} - 960\right)$$

Solving this for $T_{\text{gas}}$ we get

$$T_{\text{gas}} = 960 + \frac{705}{50} = 971°\text{R} = 514°\text{F}$$

There is thus a 14°F difference between the calculated gas temperature and the thermocouple reading.

**16A.6 Surface temperatures on Earth's moon.**

**(a)** A quasi-steady-state energy balance on a lunar surface element that directly faces the sun gives

$$I_0 a = e\sigma T_{s,\max}^4$$

in which $I_0$ is the solar constant, $T_{s,\max}$ is the temperature of that surface element, and $a$ and $e$ are its total absorptivity and emissivity. Setting $a = e$ for a gray surface, and using the value calculated in Example 16.4-1 for the solar constant, we obtain the quasi-steady-state estimate

$$T_{s,\max} = (I_0\sigma)^{1/4} = (430/1.7124 \times 10^{-9})^{1/4} = 708°\text{R}$$

of the maximum temperature on the moon.

**(b)** For a spherical lunar surface, receiving radiation from the sun only, the local intensity of incident radiation is

$$I(\theta) = \begin{cases} I_0 \cos\theta, & \text{for } 0 \le \theta \le \pi/2; \\ 0 & \text{for } \pi/2 \le \theta \le \pi, \end{cases}$$

when the sun is treated as an infinitely distant point radiator. Here $\theta$ is the angular displacement from the surface location nearest the sun. Replacement of $I_0$ by $I(\theta)$ in the result of **(a)** gives the quasi-steady-state temperature prediction

$$T(\theta) = \begin{cases} T_{s,\max} \cos^{1/4}(\theta) & \text{for } 0 \le \theta \le \pi/2; \\ 0 & \text{for } \pi/2 \le \theta \le \pi \end{cases}$$

This prediction becomes less accurate in the partially shadowed region (penumbra), given by $|\theta - \pi| \approx 0.0046$ for a spherical lunar surface and the solar dimensions shown in Fig. 16.4-1. Transient energy transport and shadows cast by the rugged lunar topography also become important in this region, and radiation emitted and reflected from Earth and the other planets become significant on the dark side of the moon.

**16B.1 Reference temperature for effective emissivity**

By the assumption of linear variation of emissivity with temperature we write

$$e = a + bT$$

where $a$ and $b$ are constants. Then

$$e_1 - e^\circ = b(T_1 - T^\circ) \qquad \text{and} \qquad e_2 - e^\circ = b(T_2 - T^\circ)$$

and

$$(e_1 - e^\circ)T_1^4 = b(T_1 - T^\circ)T_1^4$$

$$(e_2 - e^\circ)T_2^4 = b(T_2 - T^\circ)T_2^4$$

However

$$(e_1 - e^\circ)T_1^4 = (e_2 - e^\circ)T_2^4$$

and therefore

$$T_1^5 - T_1^4 T^\circ = T_2^5 - T_2^4 T^\circ$$

This may be solved for the reference temperature to give

$$T^\circ = \frac{T_1^5 - T_2^5}{T_1^4 - T_2^4}$$

This is the same as Eq. 16B.1-2.

**16B.2 Radiation across an annular gap**

The system in this problem is very similar that depicted in Fig. 16.5-1 and described in Eq. 16.5-9. Since all the radiation leaving the inner surface ("1") of the annulus is directly intercepted by the outer surface ("2"), we know from the interpretation of $F_{12}$ after Eq. 16.4-11 that $F_{12} = 1$. Then

$$Q_{12} = \frac{\sigma\left(T_1^4 - T_2^4\right)}{\dfrac{1-e_1}{e_1 A_1} + \dfrac{1}{A_1} + \dfrac{1-e_2}{e_2 A_2}}$$

$$= \frac{\sigma\left(T_1^4 - T_2^4\right)}{\dfrac{1}{e_1 A_1} - \dfrac{1}{A_1} + \dfrac{1}{A_1} + \dfrac{1}{e_2 A_2} - \dfrac{1}{A_2}}$$

$$= \frac{\sigma\left(T_1^4 - T_2^4\right)}{\dfrac{1}{e_1 A_1} + \dfrac{1}{A_2}\left(\dfrac{1}{e_2} - 1\right)}$$

Note that if, in Eq. 16.5-9 we had replaced $A_1 F_{12}$ by $A_2 F_{21}$, then we cannot set $F_{21} = 0$ in this problem, inasmuch as light leaving surface "2" is not all intercepted by surface "1". In fact, it is not difficult to show that $F_{21} = A_1/A_2$. Then if the inner cylindrical surface is only slightly smaller than the outer cylindrical surface, then $F_{21}$ will be just slightly smaller than 1, and if the inner cylinder shrinks to a wire, then $F_{21}$ will approach zero.

**16B.3 Multiple radiation shields.**

**(a)** Equation 16.5-1 gives the radiant heat flow between successive planes in the series as

$$Q_{i,i+1} = \frac{\sigma(T_i^4 - T_{i+1}^4)}{[1/e_i + 1/e_{i+1} - 1]} = \frac{\sigma(T_i^4 - T_{i+1}^4)}{R_{i,i+1}}$$

in which

$$R_{i,i+1} = [1/e_i + 1/e_{i+1} - 1]/A_i$$

Summing the thermal resistances $R_{i,i+1}$ from $i = 1$ to $i = n - 1$, and equating the heat flows through all of them, gives

$$Q \sum_{i=1}^{i=n-1} R_{i,i+1} = \sigma(T_1^4 - T_n^4)$$

whence

$$Q = \frac{\sigma(T_1^4 - T_n^4)}{\sum_{i=1}^{i=n-1} R_{i,i+1}}$$

**(b)** The ratio of the radiant heat flow through $n$ identical sheets to that between two is

$$\frac{Q(n \text{ sheets})}{Q(2 \text{ sheets})} = \frac{R_{1,2}}{(n-1)R_{1,2}} = \frac{1}{n-1}$$

**(c)** The ratio of the heat flow with three non-identical sheets to the heat flow without the middle sheet is

$$\frac{R_{1,3}}{R_{1,2} + R_{2,3}} = \frac{[1/e_1 + 1/e_3 - 1]}{[1/e_1 + 1/e_2 - 1] + [1/e_2 + 1/e_3 - 1]}$$

in agreement with the result of Example 16.5-1.

**16B.4 Radiation and conduction through absorbing media**

*a*. We begin by combining Eqs. 16.6-5 and 6 to get

$$0 = -\frac{d}{dz} q_z^{(r)} - m_a q_z^{(r)}$$

This may be integrated to give

$$\ln q_z^{(r)} = -m_a z + C$$

The constant of integration may be obtained from the boundary condition at $z = 0$, so that

$$q_z^{(r)} = q_0^{(r)} e^{-m_a z} \qquad \text{and (from Eq. 16.6-6)} \qquad \mathcal{Q} = m_a q_0^{(r)} e^{-m_a z}$$

Next, use Eq. 16.6-4 to get

$$0 = k\frac{d^2T}{dz^2} + m_a q_0^{(r)} e^{-m_a z}$$

Integration twice with respect to z gives

$$T(z) = -\frac{q_0^{(r)}}{m_a k} e^{-m_a z} + C_1 z + C_2$$

Next we apply the boundary conditions that $T(0) = T_0$ and $T(\delta) = T_\delta$ to get

$$T(z) - T_0 = \frac{q_0^{(r)}}{m_a k}\left(1 - e^{-m_a z}\right) + \left[\left(T_\delta - T_0\right) - \frac{q_0^{(r)}}{m_a k}\left(1 - e^{-m_a \delta}\right)\right]\frac{z}{\delta}$$

*b*. The conductive heat flux is given by Fourier's law:

$$q_z = -k\frac{dT}{dz} = -q_0^{(r)} e^{-m_a \delta} + k\frac{T_\delta - T_0}{\delta} - \frac{q_0^{(r)}}{m_a \delta}\left(1 - e^{-m_a \delta}\right)$$

For very large values of $m_a$, the first and third terms become negligibly small, and we are left with

Very large $m_a$: $$q_z \approx -k\frac{T_0 - T_\delta}{\delta}$$

For very small values of $m_a$ we can expand the first and third terms in a Taylor series and get:

Very small $m_a$:
$$\begin{aligned} q_z &= -q_0^{(r)}(1 - m_a z + \cdots) - k\frac{T_0 - T_\delta}{\delta} \\ &\quad - \frac{q_0^{(r)}}{m_a \delta}(1 - 1 + m_a \delta - \cdots) \\ &\approx -k\frac{T_0 - T_\delta}{\delta} \end{aligned}$$

Thus in both limits, the conductive heat flux is virtually unaffected by the radiation.

**16B.5 Cooling of a black body in vacuo**

An energy balance over the black body in the enclosure is

$$\frac{d}{dt}U_{\text{tot}} = Q \qquad \text{or} \qquad \rho\hat{C}_p V\frac{dT}{dt} = Aq_b^{(e)}$$

When the Stefan-Boltzmann law is inserted, we get

$$\rho\hat{C}_p V\frac{dT}{dt} = A\sigma\left(T_2^4 - T^4\right)$$

This separable, first-order equation may be integrated as follows

$$\int_{T_1}^{T}\frac{dT}{T_2^4 - T^4} = \frac{A\sigma}{\rho\hat{C}_p V}\int_0^t dt$$

The integral on the left side may be integrated with the help of an integral table to give

$$\left(\frac{1}{4T_2^3}\ln\left|\frac{T_2+T}{T_2-T}\right| + \frac{1}{2T_2^3}\arctan\frac{T}{T_2}\right)\Bigg|_{T_1}^{T} = \frac{A\sigma t}{\rho\hat{C}_p V}$$

Therefore the temperature-time relation is

$$\ln\left|\frac{T_2+T}{T_2-T}\right| - \ln\left|\frac{T_2+T_1}{T_2-T_1}\right| + 2\arctan\frac{T}{T_2} - 2\arctan\frac{T_1}{T_2} = \frac{4A\sigma T_2^3 t}{\rho\hat{C}_p V}$$

The left side gives $T = T_2$ as $t$ goes to infinity, and $T = T_1$ as $t$ goes to zero.

The left side is dimensionless, and the right side is also, as can be seen by using the table of notation given on pp. 872 et seq.

$$\frac{\left(L^2\right)\left(\frac{M}{t^3T^4}\right)\left(T^3\right)(t)}{\left(\frac{M}{L^3}\right)\left(\frac{L^2}{t^2T}\right)\left(L^3\right)} = \text{dimensionless}$$

**16B.6 Heat loss from an insulated pipe.**

**(a)** Eq. 10.6-29 gives for this problem,

$$Q_0^{(\text{cond})}/L = \frac{2\pi(T_0 - T_2)}{\frac{\ln(r_1/r_0)}{k_{01}} + \frac{\ln(r_2/r_1)}{k_{12}}}$$

with $r_0 = 1.0335$ in., $r_1 = 1.0335 + 0.154 = 1.1875$ in.,and $r_2 = 1.1875 + 2 = 3.1875$ in. With the given thermal conductivity values, we then obtain

$$\begin{aligned} Q_0^{(\text{cond})}/L &= \frac{2\pi(T_0 - T_a)}{\frac{\ln(1.1975/1.0335)}{26} + \frac{\ln(3.1875/1.1875)}{0.35}} \\ &= \frac{2\pi(250 - T_a)}{0.0053 + 2.8} \\ &= \begin{cases} 334 \text{ Btu/hr·ft} & \text{if } T_0 = 100°\text{F}; \\ 0 & \text{if } T_2 = 250°\text{F} \end{cases} \end{aligned}$$

**(b)** The net radiative heat loss is given by Eq. 16.5-3. Setting $e = a = 0.05$ for the aluminum foil, we get

$$\begin{aligned} Q^{(\text{rad})}/L &= \sigma\pi D_2(0.05)(T_0^4 - T_a^4) \\ &= 0.1712 \times 10^{-8}\pi(2 \times 3.1875)(0.05)(T_0^4 - 540^4) \\ &= \begin{cases} 23 \text{ Btu/hr·ft} & \text{if } T_0 = 100°\text{F} = 560°\text{R}; \\ 290 \text{ Btu/hr·ft} & \text{if } T_0 = 250°\text{F} = 710°\text{R} \end{cases} \end{aligned}$$

The free-convective heat loss is predictable as in §14.6. For $T_a = 100°$F, Example 14.6-1 gives

$$Q^{(\text{conv})}/L = 18 \text{ Btu/hr·ft}$$

For $T_a = 250°$F, Eqs. 14.6-4,5 and Tables 14.6-1,2 give

$$Nu_m^{\text{lam}} = 0.772(0.515)(\text{GrPr})^{1/4} = 0.398(\text{GrPr})^{1/4}$$

The needed properties of air at $T_f = (80 + 250)/2 = 165°$F$= 74°$C are obtained from the ideal gas law, from Table 1.1-2, and from *CRC Handbook of Chemistry and Physics,* 81st Ed., 2001-2002, pp. 6-1, 6-2, and 6-185.

$$\begin{aligned} \mu &= 0.0206 \text{ mPa·s} = 0.0498 \text{ lb}_m\text{/ft·hr} \\ \rho &= pM/RT = 1.017 \times 10^{-3} \text{ g/cm}^3 = 0.0634 \text{ lb}_m\text{/ft}^3 \\ \hat{C}_p &= 1.012 \text{ J/g/K} = 0.2420 \text{ Btu/lb}_m/cdot\text{R} \\ k &= 29.5\text{mW/m·K} = 0.0170 \text{ Btu/hr·ft·F} \\ \beta &= 1/T_f = (1/625)\text{R}^{-1} \end{aligned}$$

Hence,

$$\begin{aligned}\mathrm{GrPr} &= \left(\frac{(2\times 3.1875/12)^3(0.0634)^2(4.17\times 10^8)(170/625)}{(0.0498)^2}\right)\left(\frac{(0.2420)(0.0498)}{(0.0170)}\right)\\ &= (2.76\times 10^7)(0.709) = 1.96\times 10^7\end{aligned}$$

and

$$\begin{aligned}\mathrm{Nu}_m &= (0.398)(\mathrm{GrPr})^{1/4}\\ &= (0.398)(1.96\times 10^7)^{1/4} = 26.5\end{aligned}$$

Therefore,

$$\begin{aligned}Q^{(\mathrm{conv})}/L &= h_m\pi D(T_0 - T_a) = (\mathrm{Nu}_m k/D)(\pi D)(T_0 - T_a)\\ &= \mathrm{Nu}_m \pi k(T_0 - T_a)\\ &= (26.5)\pi(0.0170)(250 - 80) = 241 \text{ Btu/hr·ft}\end{aligned}$$

giving

$$\frac{Q^{(\mathrm{cond})} - Q^{(\mathrm{rad})} - Q^{(\mathrm{conv})}}{L} = \begin{cases} 334 - 23 - 18 = 293 \text{ Btu/hr·ft} & \text{if } T_0 = 100°\mathrm{F};\\ 0 - 290 - 241 = -531 \text{ Btu/hr·ft} & \text{if } T_0 = 250°\mathrm{F}.\end{cases}$$

**(c)** Linear interpolation to zero heat accumulation at the outer surface gives the steady-state values

$$T_0 = 100 + 293/824(250 - 100) = 153°\mathrm{F}$$

$$Q^{(\mathrm{cond})}/L = Q^{(\mathrm{rad})+(\mathrm{conv})}/L = 334 + (293/824)(0 - 334) = 215 \text{ Btu/hr·ft}$$

**17A.1 Prediction of a low-density binary diffusivity.**

**(a)** We begin by looking up the needed properties of the species from Table E.1:

| Species | $M$, g/g-mol | $T_c$, K | $p_c$, atm |
|---|---|---|---|
| A: $CH_4$ | 16.04 | 191.1 | 45.8 |
| B: $C_2H_6$ | 30.07 | 305.4 | 48.2 |

Equation 17.2-1 then gives the following prediction of $\mathcal{D}_{AB}$ for methane-ethane (treated here as a nonpolar gas-pair) at $p = 1$ atm and $T = 293$K:

$$\begin{aligned}
\mathcal{D}_{AB} &= a\left(\frac{T}{\sqrt{T_{cA}T_{cB}}}\right)^b (p_{cA}p_{cB})^{1/3}(T_{cA}T_{cB})^{5/12}(1/M_A + 1/M_B)^{1/2}/p \\
&= 2.745 \times 10^{-4}\left(\frac{293}{\sqrt{191.1 \times 305.4}}\right)^{1.823} \\
&\quad \cdot (45.8 \times 48.2)^{1/3}(191.1 \times 305.4)^{5/12}(1/16.04 + 1/30.07)^{1/2}/1 \text{ atm} \\
&= 0.152 \text{ cm}^2/\text{s}
\end{aligned}$$

**(b)** Equation 17.2-3 gives

$$\begin{aligned}
(c\mathcal{D}_{AB})_c &= 2.96 \times 10^{-6}\left(\frac{1}{M_A} + \frac{1}{M_B}\right)^{1/2} \frac{(p_{cA}p_{cB})^{1/3}}{(T_{cA}T_{cB})^{1/12}} \\
&= 2.96 \times 10^{-6}\left(\frac{1}{16.04} + \frac{1}{30.07}\right)^{1/2} \frac{(45.8 \times 48.2)^{1/3}}{(191.1 \times 305.4)^{1/12}} \\
&= 4.78 \times 10^{-6} \text{ g-mol/cm·s}
\end{aligned}$$

The reduced conditions for Fig. 17.2-1 for this problem, calculated as described on page 522, are

$$T_r = \frac{T}{\sqrt{T_{cA}T_{cB}}} = \frac{293}{\sqrt{191.1 \times 305.4}} = 1.21$$

$$p_r = \frac{p}{\sqrt{p_{cA}p_{cB}}} = \frac{1.0}{\sqrt{45.8 \times 48.2}} = 0.021$$

At this reduced state, Fig. 17.2-1 gives $(c\mathcal{D}_{AB})_r = 1.20$. Hence, the predicted value of $c\mathcal{D}_{AB}$ is $1.20(c\mathcal{D}_{AB})_c = 5.74 \times 10^{-6}$. Dividing this result by the ideal-gas prediction $c = p/RT = 4.16 \times 10^{-5}$ g-mol/cm$^{-3}$ at this low-density condition gives $\mathcal{D}_{AB} = 0.138$ cm$^2$/s.

**(c)** Equations 17.3-14 and 15 give the binary interaction parameters

$$\sigma_{AB} = (3.780 + 4.388)/2 = 4.084\,\text{Å} \quad \text{and} \quad \varepsilon_{AB}/K = \sqrt{154 \times 232} = 189 \text{ K}$$

when the Lennard-Jones parameters of Table E.1 are used for the individual species. Then, at $\kappa T/\varepsilon_{AB} = 293/189 = 1.550$, Table E.2 gives $\Omega_{\mathcal{D},AB} = 1.183$, and Eq. 17.3-12 gives the prediction

$$\begin{aligned}
\mathcal{D}_{AB} &= 0.0018583\sqrt{T^3\left(\frac{1}{M_A}+\frac{1}{M_B}\right)}\frac{1}{p\sigma_{AB}^2\Omega_{calD,AB}} \\
&= 0.0018583\sqrt{(293)^3\left(\frac{1}{16.04}+\frac{1}{30.07}\right)}\frac{1}{(1)(4.084)^2(1.183)} \\
&= 0.146\ \mathrm{cm}^2/\mathrm{s}
\end{aligned}$$

**(d)** Use of Eqs. 1.4-11a,c with the combining rules of Eqs. 17.3-14,15 gives the estimates

$$\varepsilon_{AB}/\kappa = 0.77\sqrt{191.1 \times 305.4} = 186.0\ \mathrm{K}$$

$$\sigma_{AB} = \frac{2.44}{2}\left[\left(\frac{T_{cA}}{p_{cA}}\right)^{1/3} + \left(\frac{T_{cA}}{p_{cB}}\right)^{1/3}\right] = 4.222\,\mathrm{\AA}$$

Then at $\kappa T/\varepsilon_{AB} = 293/186.0 = 1.575$, Table E.2 gives $\Omega_{\mathcal{D},AB} = 1.1755$, whereupon Eq. 17.3-12 gives

$$\begin{aligned}
\mathcal{D}_{AB} &= 0.0018583\sqrt{(293)^3\left(\frac{1}{16.04}+\frac{1}{30.07}\right)}\frac{1}{(1)(4.222)^2(1.1755)} \\
&= 0.138\ \mathrm{cm}^2/\mathrm{s}
\end{aligned}$$

**17A.2 Extrapolation of binary diffusivity to a very high temperature.**

**(a)** Equation 17.2-1 in the nonpolar form gives

$$\mathcal{D}_{AB}\big|_{1500\text{K}} = \mathcal{D}_{AB}\big|_{293\text{K}}(1500/293)^{1.823} = 2.96 \text{ cm}^2/\text{s}$$

**(b)** Equation 17.3-10, with $n = p/\kappa T$, predicts $\mathcal{D}_{AB} \propto T^{3/2}$, thus giving

$$\mathcal{D}_{AB}\big|_{1500\text{K}} = \mathcal{D}_{AB}\big|_{293\text{K}}(1500/293)^{1.5} = 1.75 \text{ cm}^2/\text{s}$$

**(c)** Equation 17.3-12 and Table E.2, with $\varepsilon_{AB} = 135.8$ from Table E.1 and Eq. 17.3-15, gives

$$\mathcal{D}_{AB}\big|_{1500\text{K}} = \mathcal{D}_{AB}\big|_{293\text{K}}(1500/293)^{3/2}\left(\frac{\Omega_{\mathcal{D},AB}\big|_{293\kappa/\varepsilon_{AB}}}{\Omega_{\mathcal{D},AB}\big|_{1500\kappa/\varepsilon_{AB}}}\right) = 2.51 \text{ cm}^2/\text{s}$$

The superior agreement of Method (c) with the experimental value of 2.45 $\text{cm}^2/\text{s}$ illustrates the wider range provided by the Chapman-Enskog theory when combined with the Lennard-Jones potential-energy model.

**17A.3 Self-diffusion in liquid mercury.**

Equation 17.4-5 gives

$$\mathcal{D}_{AA^*} = \frac{\kappa T}{2\pi\mu_A}\left(\frac{\widetilde{N}_A}{\widetilde{V}_A}\right)^{1/3}$$

or in cgs units,

$$\mathcal{D}_{AA^*} = \frac{(1.38066\times 10^{-16}\text{erg/K})(T,\text{K})}{2\pi(\mu_A,\text{g/cm·s})}\left(\frac{6.02214\times 10^{23}\text{molecules/g-mol}}{(200.61\text{g/g-mol})(\hat{V}_A,\text{cm}^3/\text{g})}\right)^{1/3}$$

Insertion of the values tabulated for this problem, with $\mu$(cp) divided by 100 to get $\mu_A$, g/cm·s, gives the following results:

| $T(K)$ | $\mathcal{D}_{AA^*}$, obs. | $\mathcal{D}_{AA^*}$, pred. | Ratio, pred./obs. |
|---|---|---|---|
| 275.7 | $1.52\times 10^{-5}$ | $1.24\times 10^{-5}$ | 0.82 |
| 289.6 | $1.68\times 10^{-5}$ | $1.40\times 10^{-5}$ | 0.84 |
| 364.2 | $2.57\times 10^{-5}$ | $2.16\times 10^{-5}$ | 0.84 |

The predicted self-diffusivities are about 5/6 of the measured values.

**17A.4 Schmidt numbers for binary gas mixtures at low density.**

We begin by tabulating the needed molecular parameters for species $A$ and $B$ from Table E.1, and estimating the binary parameters $\sigma_{AB}$ and $\varepsilon_{AB}/\kappa$ from Eqs. 17.3-14 and 15:

| Species | $M$, g/g-mol | $\sigma$, Å | $\varepsilon/\kappa$, K |
|---|---|---|---|
| A: $H_2$ | 2.016 | 2.915 | 38.0 |
| B: $CCl_2F_2$ | 120.92 | 5.116 | 280. |
| AB: | | 4.0155 | 103.15 |

Equation 17.3-11 and Table E.2 then give the following prediction of $c\mathcal{D}_{AB}$ for binary mixtures of $H_2$ and Freon-12 at $T = 25°C$=298.15 K:

$$
\begin{aligned}
c\mathcal{D}_{AB} &= 2.2646 \times 10^{-5} \sqrt{T\left(\frac{1}{M_A} + \frac{1}{M_B}\right)} \frac{1}{\sigma_{AB}^2 \Omega_{\mathcal{D},AB}} \\
&= 2.2646 \times 10^{-5} \sqrt{298.15\left(\frac{1}{2.016} + \frac{1}{120.92}\right)} \frac{1}{(4.0155)^2(0.9597)} \\
&= 1.794 \times 10^{-5} \text{ g-mol/cm}\cdot\text{/s}
\end{aligned}
$$

With this prediction of $c\mathcal{D}_{AB}$ and the viscosity data of Problem 1A.4, the Schmidt number can then be calculated as

$$
\text{Sc} = \frac{\mu}{\rho \mathcal{D}_{AB}} = \frac{\mu}{M c\mathcal{D}_{AB}} = \frac{\mu}{(x_A M_A + x_B M_B) c\mathcal{D}_{AB}}
$$

in accordance with Eqs. G and L of Table 17.7-1. Results are as follows:

| $x_A = x_{H_2}$ | 0.00 | 0.25 | 0.50 | 0.75 | 1.00 |
|---|---|---|---|---|---|
| $\mu$, g/cm·s | $124.0 \times 10^{-6}$ | $128.1 \times 10^{-6}$ | $131.9 \times 10^{-6}$ | $135.1 \times 10^{-6}$ | $88.4 \times 10^{-6}$ |
| $M$, g/g-mol | 120.92 | 91.194 | 61.468 | 31.742 | 2.016 |
| Sc | 0.057 | 0.078 | 0.120 | 0.237 | 2.44 |

We see that the Schmidt number depends strongly on the composition when $M_A$ and $M_B$ differ greatly. This fact is also illustrated in Table 17.3-1.

**17A.5 Estimation of diffusivity for a binary mixture at high density.**

The following properties of $N_2$ and $C_2H_6$ for this problem are obtained from Table E.1:

| Species | $M$, g/g-mol | $T_c$, K | $p_c$, atm |
|---|---|---|---|
| A: $N_2$ | 28.01 | 126.2 | 33.5 |
| B: $C_2H_6$ | 30.07 | 305.4 | 48.2 |

**(a)** The measured value $\mathcal{D}_{AB} = 0.148$ cm$^2$/s at $T = 298.2$ K permits calculation of an experimentally-based value of $(c\mathcal{D}_{AB})_c$. The reduced conditions for this measurement are

$$T_r = \frac{T}{\sqrt{T_{cA}T_{cB}}} = \frac{298.2}{\sqrt{126.2 \times 305.4}} = 1.52;$$
$$p_r = \frac{p}{\sqrt{p_{cA}p_{cB}}} = \frac{1.0}{\sqrt{33.5 \times 48.2}} = 0.025$$

On Fig. 17.2-1, this state lies essentially at the low-pressure limit, with $(c\mathcal{D}_{AB})_r = 1.47$. Accordingly expressing $c$ by the ideal gas law, we find

$$\begin{aligned} c\mathcal{D}_{AB} &= \frac{p}{RT}\mathcal{D}_{AB} \\ &= \frac{1 \text{ atm}}{(82.06 \times 298.2 \text{ cm}^3\cdot\text{atm/g-mol}}(0.148 \text{ cm}^2/\text{s}) \\ &= 6.05 \times 10^{-6} \text{ g-mol/cm·s} \end{aligned}$$

Hence, the critical $c\mathcal{D}_{AB}$ value is

$$(c\mathcal{D}_{AB})_c = c\mathcal{D}_{AB}/(c\mathcal{D}_{AB})_r = 6.05 \times 10^{-6}/1.47 = 4.12 \times 10^{-6} \text{ g-mol/cm·s}$$

Now, the reduced conditions for the desired prediction are

$$T_r = \frac{T}{\sqrt{T_{cA}T_{cB}}} = \frac{288.2}{\sqrt{126.2 \times 305.4}} = 1.47;$$
$$p_r = \frac{p}{\sqrt{p_{cA}p_{cB}}} = \frac{40}{\sqrt{33.5 \times 48.2}} = 0.995$$

at which state Fig. 17.2-1 gives $(c\mathcal{D}_{AB})_r = 1.42$. The resulting prediction is then

$$\begin{aligned} c\mathcal{D}_{AB} &= (c\mathcal{D}_{AB})_c(c\mathcal{D}_{AB})_r \\ &= (4.12 \times 10^{-6})(1.42) = 5.8 \times 10^{-6} \text{ g-mol/cm·s.} \end{aligned}$$

**(b)** Equation 17.2-3 gives the predicted critical value

$$\begin{aligned} (c\mathcal{D}_{AB})_c &= 2.96 \times 10^{-6}\left(\frac{1}{28.01} + \frac{1}{30.07}\right)^{1/2} \frac{(33.5 \times 48.2)^{1/3}}{(126.2 \times 305.4)^{1/12}} \\ &= 3.78 \times 10^{-6} \text{ g-mol/cm·s.} \end{aligned}$$

Multiplication by $(c\mathcal{D}_{AB})_r = 1.42$ as in **(a)** gives

$$c\mathcal{D}_{AB} = (3.78 \times 10^{-6})(1.42) = 5.4 \times 10^{-6} \text{ g-mol/cm·s.}$$

**17A.6 Diffusivity and Schmidt number for chlorine-air mixtures.**

**(a)** We begin by tabulating molecular parameters for chlorine and air from Table E.1, and estimating the binary parameters $\sigma_{AB}$ and $\varepsilon_{AB}/\kappa$ from Eqs. 17.3-14 and 15:

| Species | $M$, g/g-mol | $\sigma$, Å | $\varepsilon/\kappa$, K |
|---|---|---|---|
| A: $Cl_2$ | 70.91 | 4.115 | 357. |
| B: Air | 28.97 | 3.617 | 97.0 |
| AB: | | 3.866 | 186.1 |

Equation 17.3-12 and Table E.2 then give the following prediction of $\mathcal{D}_{AB}$ for chlorine-air mixtures at $T = 75°\text{F} = 23.89°\text{C} = 297.04$ K:

$$\begin{aligned}\mathcal{D}_{AB} &= 0.0018583\sqrt{T^3\left(\frac{1}{M_A}+\frac{1}{M_B}\right)}\frac{1}{p\sigma_{AB}^2\Omega_{\mathcal{D},AB}}\\ &= 0.0018583\sqrt{(297.04)^3\left(\frac{1}{28.97}+\frac{1}{70.91}\right)}\frac{1}{(1)(3.866)^2(1.169)}\\ &= 0.120\ \text{cm}^2/\text{s}\end{aligned}$$

**(b)** Equation 17.2-1 needs the following values from Table E.1:

| Component | $M$, g/g-mol | $T_c$, K | $p_c$, atm |
|---|---|---|---|
| A: $Cl_2$ | 70.91 | 417. | 76.1 |
| B: Air | 28.97 | 132. | 36.4 |

The nonpolar version of Eq. 17.2-1 then gives the prediction

$$\begin{aligned}\mathcal{D}_{AB} &= 2.745\times10^{-4}\left(\frac{297.04}{\sqrt{417.\times132.}}\right)^{1.823}\\ &\quad\cdot(76.1\times36.4)^{1/3}(417.\times132.)^{5/12}(1/70.91+1/28.97)^{1/2}/1\ \text{atm}\\ &= 0.123\ \text{cm}^2/\text{s}\end{aligned}$$

**(c)** The result of **(a)**, and the ideal gas expression for $c$, give

$$c\mathcal{D}_{AB} = \frac{p}{RT}\mathcal{D}_{AB} = 4.92\times10^{-6}\ \text{g-mol/cm·s}$$

With this prediction of $c\mathcal{D}_{AB}$ and the viscosity predictions of Problem 1A.4, the Schmidt number can be calculated as

$$\text{Sc} = \frac{\mu}{\rho\mathcal{D}_{AB}} = \frac{\mu}{Mc\mathcal{D}_{AB}} = \frac{\mu}{(x_AM_A+x_BM_B)c\mathcal{D}_{AB}}$$

in accordance with Eqs. G and L of Table 17.7-1. Results are as follows:

| | | | | | |
|---|---|---|---|---|---|
| $x_{Cl_2}$ | 0.00 | 0.25 | 0.50 | 0.75 | 1.00 |
| $\mu$, g/cm·s | 0.000183 | 0.000164 | 0.000150 | 0.000139 | 0.000131 |
| $M$, g/g-mol | 28.97 | 39.455 | 49.94 | 60.425 | 70.91 |
| Sc | 1.28 | 0.84 | 0.61 | 0.47 | 0.375 |

We see that the Schmidt number depends strongly on the composition when $M_A$ and $M_B$ differ greatly. This fact is also illustrated in Table 17.3-1 and in Problem 17A.4.

**17A.7 The Schmidt number for self-diffusion**

**(a)** Equation 1.3-1b, written for non-tracer species A, gives

$$\mu_c = 7.70 \times 10^{-6} M_A^{1/2} p_{cA}^{2/3} T_{cA}^{-1/6}$$

for the critical viscosity in g/cm·s. Here $M_A$ is in g/g-mol, $p_{cA}$ in atm and $T_{cA}$ in K. Eq. 17.2-2 gives

$$(c\mathcal{D}_{AA*})_c = 2.96 \times 10^{-6} \left( \frac{1}{M_A} + \frac{1}{M_{A*}} \right)^{1/2} p_{cA}^{1/3} T_{cA}^{-1/6}$$

The resulting critical Schmidt number for self-diffusion with $M_A \approx M_{A*}$ is

$$\left( \frac{\mu}{M_A c\mathcal{D}_{AA*}} \right)_c = \frac{7.70}{2.96\sqrt{2}} = 1.84$$

**(b)** Figs. 1.3-1 and 17.2-1, with the result in **(a)** for $M_A \approx M_{A*}$, give Sc for self-diffusion as the following function of $T_r$ and $p_r$:

$$\mathrm{Sc}_{AA*} = 1.84 \frac{\mu_r(T_r, p_r)}{(c\mathcal{D}_{AA*})_r(T_r, p_r)}$$

Calculations from this formula are summarized below:

| Phase→ | Gas | Gas | Gas | Liquid | Gas | Gas |
|---|---|---|---|---|---|---|
| $T_r$ | 0.7 | 1.0 | 5.0 | 0.7 | 1.0 | 2.0 |
| $p_r$ | 0.0 | 0.0 | 0.0 | sat. | 1.0 | 1.0 |
| $\mu_r$ | 0.32 | 0.45 | 1.62 | 7.8 | 1.00 | 0.94 |
| $(c\mathcal{D}_{AA*})_r$ | 0.72 | 1.01 | 3.65 | 0.37 | 1.03 | 1.83 |
| $\mathrm{Sc}_{AA*}$ | 0.82 | 0.82 | 0.82 | 39 | 1.79 | 0.94 |

**17A.8 Correction of high-density diffusivity for temperature.**

The following properties of $CH_4$ and $C_2H_6$ for this problem are obtained from Table E.1:

| Species | $M$, g/g-mol | $T_c$, K | $p_c$, atm |
|---|---|---|---|
| A: $CH_4$ | 16.04 | 191.1 | 45.8 |
| B: $C_2H_6$ | 30.07 | 305.4 | 48.2 |

The reduced conditions (State 1) for the given $c\mathcal{D}_{AB}$ value, calculated as described on page 522, are

$$T_r = \frac{313}{\sqrt{191.1 \times 305.4}} = 1.30; \qquad p_r = \frac{136}{\sqrt{45.8 \times 48.2}} = 2.89$$

and Fig. 17.2-1 gives $(c\mathcal{D}_{AB})_r = 1.27$ at this state. The reduced conditions (State 2) for the desired prediction are

$$T_r = \frac{351}{\sqrt{191.1 \times 305.4}} = 1.45; \qquad p_r = \frac{136}{\sqrt{45.8 \times 48.2}} = 2.89$$

and Fig. 17.2-1 gives $(c\mathcal{D}_{AB})_r = 1.40$ at this state. The prediction of $c\mathcal{D}_{AB}$ is then obtained as follows:

$$\begin{aligned}(c\mathcal{D}_{AB})\big|_{\text{State 2}} &= (c\mathcal{D}_{AB})\big|_{\text{State 1}} \frac{(c\mathcal{D})_r\big|_{\text{State 2}}}{(c\mathcal{D})_r\big|_{\text{State 1}}} \\ &= 6.0 \times 10^{-6} \frac{1.40}{1.27} = 6.6 \times 10^{-6} \text{ g-mol/c·s}\end{aligned}$$

The observed $c\mathcal{D}_{AB}$ value at State 2 is $6.3 \times 10^{-6}$ g-mol/cm·s, in fair agreement with this prediction.

**17A.9 Prediction of critical $c\mathcal{D}_{AB}$ values.**

**(a)** Equation 17A.9-1 gives $\kappa T/\varepsilon_{AA*} = 1/0.77 = 1.2987$ at $T = T_{cA}$, and Table E.2 gives $\Omega_{\mathcal{D},AA*} = 1.2746$ at this argument value. Insertion of this result, along with Eq. 17A.9-2, into Eq. 17A.9-1 gives

$$(c\mathcal{D}_{AA*})_c = \frac{2.2646 \times 10^{-5}}{1.01}\sqrt{T_{cA}\left(\frac{1}{M_A} + \frac{1}{M_{A*}}\right)}\frac{1}{(2.44(T_{cA}/p_{cA})^{1/3})^2(1.2746)}$$
$$= 2.955 \times 10^{-6}\left(\frac{1}{M_A} + \frac{1}{M_{A*}}\right)^{1/2}\frac{p_{cA}^{2/3}}{T_{cA}^{1/6}}$$

which verifies Eq. 17.2-2, within the uncertainty of the coefficient 1.01 determined from low-density self-diffusion data.

**(b)** Evaluation of the component parameters in Eqs. 17A.9-4,5 according to Eqs. 1.4-11a,c gives

$$\sigma_{AB} = 2.44\sqrt{\left(\frac{T_{cA}}{p_{cA}}\right)^{1/3}\left(\frac{T_{cB}}{p_{cB}}\right)^{1/3}} = 2.44\left(\frac{T_{cA}T_{cB}}{p_{cA}p_{cB}}\right)^{1/6}$$

and

$$\frac{\varepsilon_{AB}}{\kappa} = 0.77\sqrt{T_{cA}T_{cB}}$$

Replacement of $A*$ by $B$ and $T_{cA}$ by $\sqrt{T_{cA}T_{cB}}$ in Eq. 17A.9-1 gives

$$(c\mathcal{D}_{AB})_c = \frac{2.2646 \times 10^{-6}}{1.01}\sqrt{T_{cA}T_{cB}\left(\frac{1}{M_A} + \frac{1}{M_B}\right)}\frac{1}{\sigma_{AB}^2\Omega_{\mathcal{D},AB}}$$

Evaluation of $\sigma_{AB}$ and $\varepsilon_{AB}/\kappa$ according to Eqs. 17A.9-6,7 then gives

$$(c\mathcal{D}_{AB})_c = 2.955 \times 10^{-6}\left(\frac{1}{M_A} + \frac{1}{M_B}\right)^{1/2}\frac{(p_{cA}p_{cB})^{1/3}}{(T_{cA}T_{cB})^{1/6}}$$

by a procedure analogous to that given in part **(a)**. This result reproduces Eq. 17.2-3 within the uncertainty of the empirical coefficient 1.01.

**17A.10 Estimation of liquid diffusivities.**

**(a)** We begin by evaluating the solvent properties $\psi_B = 2.6$, $M_B = 18.016$, and the dilute-solution viscosity $\mu = 1.22$ cp at 12.5°C from a plot of the data for liquid water in Table 1.1-2. The solute, acetic acid, has a molecular weight $M_A = 60.052$, and its molar volume $\widetilde{V}_A = M_A/\rho$ at its normal boiling point is $60.052/0.937 = 64.1$ cm$^3$/g-mole. The Wilke-Chang formula, Eq. 17.4-8, then gives the following prediction at 12.5°C:

$$\begin{aligned}
\mathcal{D}_{AB} &= 7.4 \times 10^{-8} \frac{\sqrt{\psi_B M_B}\, T}{\mu \widetilde{V}_A^{0.6}} \\
&= 7.4 \times 10^{-8} \frac{\sqrt{(2.6)(18.016)}(273.15 + 12.5)}{(1.22)(64.1)^{0.6}} \\
&= 9.8 \times 10^{-6}
\end{aligned}$$

**(b)** For a given composition, Eq. 17.4-8 predicts $\mathcal{D}_{AB}$ to be proportional to $T/\mu$. Let $T_1 = 15°\text{C} = 288.15\text{K}$, and $T_2 = 100°\text{C} = 373.15\text{K}$. Then, the predicted viscosity for a dilute aqueous solution of methanol at $T_2$ is

$$\begin{aligned}
\mathcal{D}_{AB}\big|_{T_2} &= \frac{(\mathcal{D}_{AB}\mu)\big|_{T_1}}{\mu\big|_{T_2}} \frac{T_2}{T_1} \\
&= (1.28 \times 10^{-5}\ \text{cm}^2/\text{s}) \left(\frac{1.14\ \text{cp}}{0.2821\ \text{cp}}\right) \left(\frac{373.15\text{K}}{288.15\text{K}}\right) \\
&= 6.7 \times 10^{-6}\ \text{cm}^2/\text{s}
\end{aligned}$$

**17B.1 Interrelation of composition variables in mixtures**

*a.*. To get Eq. (H), use Eqs. (A) and (D)

$$c_\alpha = \frac{\text{moles of } \alpha}{\text{volume}} = \frac{\rho_\alpha}{M_\alpha} = \frac{(\text{mass of } \alpha)\,/\,\text{volume}}{(\text{mass of } \alpha)\,/\,(\text{moles of } \alpha)}$$

and Eq. (I) follows immediately.

To get Eq. (J), use Eqs. (F) and (E)

$$\sum_{\alpha=1}^{N} x_\alpha = \frac{1}{c}\sum_{\alpha=1}^{N} c_\alpha = \frac{c}{c} = 1$$

Equation (K) may be obtained similarly.

To get Eq. (L), use Eqs. (F), (J), and (B) thus

$$\sum_{\alpha=1}^{N} x_\alpha M_\alpha = \frac{1}{c}\sum_{\alpha=1}^{N} c_\alpha M_\alpha = \frac{1}{c}\sum_{\alpha=1}^{N} \rho_\alpha = \frac{\rho}{c} = M$$

and Eq. (M) may be obtained in a similar way.

To derive Eq (N), we use Eqs. (F), (H), and (C) to get

$$x_\alpha = \frac{c_\alpha}{c} = \frac{(\rho_\alpha/M_\alpha)}{\sum_{\beta=1}^{N}(\rho_\beta/M_\beta)} = \frac{(\rho_\alpha/M_\alpha)(1/\rho)}{\sum_{\beta=1}^{N}(\rho_\beta/M_\beta)(1/\rho)}\,\frac{(\omega_\alpha/M_\alpha)}{\sum_{\beta=1}^{N}(\omega_\beta/M_\beta)}$$

The derivation of Eq. (O) is done in like manner.

*b.* To get Eq. (P') from Eq. (P),

$$\begin{aligned}
\nabla x_A &= -\frac{M^2}{M_A}\left[\frac{1}{M} + \omega_A\left(\frac{1}{M_B} - \frac{1}{M_A}\right)\right]\nabla\omega_B \\
&= \frac{M^2}{M_A}\left[\left(\frac{\omega_A}{M_A} + \frac{\omega_B}{M_B}\right) + \left(\frac{\omega_A}{M_B} - \frac{\omega_A}{M_A}\right)\right]\nabla\omega_A = \frac{M^2}{M_A M_B}\nabla\omega_A \\
&= \frac{\dfrac{1}{M_A M_B}}{\dfrac{1}{M^2}}\nabla\omega_A = \frac{\dfrac{1}{M_A M_B}}{\left(\dfrac{\omega_A}{M_A} + \dfrac{\omega_B}{M_B}\right)^2}\nabla\omega_A
\end{aligned}$$

In going from line 1 to line 2, we used Eq. (M) and then Eq. Eq. (K). To get line 3, we used Eq. (K) again. The same method is used to get Eq. (Q') fro Eq. (Q).

*c*. First we must write the binary equivalent of Eq. (N) in such a way that only one variable appears on the right side (we have to keep in mind that the two mass fractions cannot be varied independently):

$$x_A = \frac{\dfrac{\omega_A}{M_A}}{\dfrac{\omega_A}{M_A} + \dfrac{\omega_B}{M_B}} = \frac{\dfrac{\omega_A}{M_A}}{\dfrac{\omega_A}{M_A} + \dfrac{1-\omega_A}{M_B}}$$

Then to get Eq. (P'), we differentiate

$$\begin{aligned}
\nabla x_A &= \frac{\dfrac{1}{M_A}}{\dfrac{\omega_A}{M_A} + \dfrac{1-\omega_A}{M_B}} \nabla\omega_A - \frac{\dfrac{\omega_A}{M_A}}{\left(\dfrac{\omega_A}{M_A} + \dfrac{1-\omega_A}{M_B}\right)^2}\left(\frac{1}{M_A} - \frac{1}{M_B}\right)\nabla\omega_A \\
&= \frac{\dfrac{1}{M_A}\left(\dfrac{\omega_A}{M_A} + \dfrac{1-\omega_A}{M_B}\right) - \dfrac{\omega_A}{M_A}\left(\dfrac{1}{M_A} - \dfrac{1}{M_B}\right)}{\left(\dfrac{\omega_A}{M_A} + \dfrac{1-\omega_A}{M_B}\right)^2}\nabla\omega_A \\
&= \frac{\dfrac{1}{M_A}\left(\dfrac{1-\omega_A}{M_B}\right) + \dfrac{\omega_A}{M_A}\left(\dfrac{1}{M_B}\right)}{\left(\dfrac{\omega_A}{M_A} + \dfrac{1-\omega_A}{M_B}\right)^2}\nabla\omega_A \\
&= \frac{\dfrac{\omega_A + \omega_B}{M_A M_B}}{\left(\dfrac{\omega_A}{M_A} + \dfrac{\omega_B}{M_B}\right)^2}\nabla\omega_A = \frac{\dfrac{1}{M_A M_B}}{\left(\dfrac{\omega_A}{M_A} + \dfrac{\omega_B}{M_B}\right)^2}\nabla\omega_A
\end{aligned}$$

The derivation of Eq. (Q') from Eq. (O) proceeds similarly.

**17B.2 Relations among the fluxes in multicomponent systems**

*a*. To verify Eq. (K) of Table 17.8-1, we proceed as follows:

$$\begin{aligned}\sum_{\alpha=1}^{N} \mathbf{j}_\alpha &= \sum_{\alpha=1}^{N} \rho_\alpha(\mathbf{v}_\alpha - \mathbf{v}) = \sum_{\alpha=1}^{N} \rho_\alpha \mathbf{v}_\alpha - \sum_{\alpha=1}^{N} \rho_\alpha \mathbf{v} \\ &= \rho \sum_{\alpha=1}^{N} \omega_\alpha \mathbf{v} - \mathbf{v} \sum_{\alpha=1}^{N} \rho_\alpha \\ &= \rho\mathbf{v} - \mathbf{v}\rho = 0\end{aligned}$$

In the second line, we have used Eq. (C) of Table 17.7-1, and to get the third line we used Eq. (B) of the same table, as well as Eq. (B) of Table 17.2-2.

The proof of Eq. (O) of Table 17.8-1 proceeds analogously.

*b*. The verification of Eq. (T) of Table 17.8-1 follows directly from the definitions of the fluxes:

$$\mathbf{j}_\alpha = \rho_\alpha(\mathbf{v}_\alpha - \mathbf{v}) \qquad \text{and} \qquad \mathbf{n}_\alpha = \rho_\alpha \mathbf{v}_\alpha$$

Substitution of these definitions into Eq. (T) gives

$$\rho_\alpha(\mathbf{v}_\alpha - \mathbf{v}) = \rho_\alpha \mathbf{v}_\alpha - \omega_\alpha \sum_{\beta=1}^{N} \rho_\beta \mathbf{v}_\beta$$

This can be rewritten as

$$\rho_\alpha \mathbf{v}_\alpha - \rho_\alpha \mathbf{v} = \rho_\alpha \mathbf{v}_\alpha - \rho\omega_\alpha \sum_{\beta=1}^{N} (\rho_\beta/\rho)\mathbf{v}_\beta$$

or, making use of Eq. (C) of Table 17.7-1,

$$-\rho_\alpha \mathbf{v} = -\rho_\alpha \sum_{\beta=1}^{N} \omega_\beta \mathbf{v}_\beta$$

Then, use of Eq. (B) of Table 17.7-2 completes the verification.

The verification of Eq. (X) of Table 17.8-1 may be done analogously.

**17B.3 Relations between the fluxes in binary systems**

The expressions for the fluxes are

$$\mathbf{j}_A = \rho_A(\mathbf{v}_A - \mathbf{v})$$
$$\mathbf{J}_A^* = c_A(\mathbf{v}_A - \mathbf{v}^*) \qquad \text{or} \qquad M_A\mathbf{J}_A^* = \rho_A(\mathbf{v}_A - \mathbf{v}^*)$$

Next form the difference between the above expressions, and then repeat for species $B$

$$\mathbf{j}_A - M_A\mathbf{J}_A^* = \rho_A(\mathbf{v} - \mathbf{v}^*) \qquad \text{and} \qquad \mathbf{j}_B - M_B\mathbf{J}_B^* = \rho_B(\mathbf{v} - \mathbf{v}^*)$$

Then eliminate $\mathbf{v} - \mathbf{v}^*$ between these two equations to get

$$\frac{\mathbf{j}_A}{\rho_A} - \frac{M_A\mathbf{J}_A^*}{\rho_A} = \frac{\mathbf{j}_B}{\rho_B} - \frac{M_B\mathbf{J}_B^*}{\rho_B}$$

Then use Eqs. (K) and (O) of Table 17.8-1 to eliminat the fluxes with subscripts $B$ and then rearrange to get

$$\mathbf{j}_A\left(\frac{1}{\rho_A} + \frac{1}{\rho_B}\right) = \mathbf{J}_A^*\left(\frac{M_A}{\rho_A} + \frac{M_B}{\rho_B}\right) \qquad \text{or} \qquad \mathbf{j}_A\left(\frac{1}{\rho_A} + \frac{1}{\rho_B}\right) = \mathbf{J}_A^*\left(\frac{1}{c_A} + \frac{1}{c_B}\right)$$

whence

$$\mathbf{j}_A\left(\frac{\rho_A + \rho_B}{\rho_A\rho_B}\right) = \mathbf{J}_A^*\left(\frac{c_A + c_B}{c_Ac_B}\right)$$

where Eq. (H) of Table 17.7-1 has been used. Next use Eqs. (B), (C), (E), and (F) to get Eq. 17B.3-1.

**17B.4 Equivalence of various forms of Fick's law for binary mixtures**
(*Note*: Problem 17B.3 should be worked prior to Problem 17B.4)

*a*. To get Eq. (B ) of Table 17.8-2, we start by rewriting Eq. (A) by using Eq. 17B.3-1 and Eqs. (Q') and (L) of Table 17.7-1:

$$\mathbf{J}_A^* \frac{\rho\omega_A\omega_B}{cx_Ax_B} = -\rho\mathcal{D}_{AB}\frac{M_AM_B}{M^2}\nabla x_A$$

Next, use of Eq. (G) of Table 17.7-1 allows us to rewrite this as

$$\mathbf{J}_A^* = -\frac{cx_Ax_B}{\rho\omega_A\omega_B}\rho\mathcal{D}_{AB}\frac{M_AM_B}{(\rho/c)^2}\nabla x_A$$

Then we note that Eqs. (C), (F), and (H) of Table 17.7-1 give

$$\frac{x_A}{\omega_A} = \frac{c_A}{c}\frac{\rho}{\rho_A} = \frac{\rho}{c}\frac{1}{M_A}$$

so that

$$\mathbf{J}_A^* = -\frac{c}{\rho}\left(\frac{\rho}{c}\frac{1}{M_A}\cdot\frac{\rho}{c}\frac{1}{M_B}\right)\rho\mathcal{D}_{AB}\frac{M_AM_B}{(\rho/c)^2}\nabla x_A = -c\mathcal{D}_{AB}\nabla x_A$$

Equation (D) of Table 17.8-2 follows at once from Eq. (B) of Table 17.8-2 and and Eq. (V) of Table 17.8-1:

$$\mathbf{N}_A = c_A\mathbf{v}^* + \mathbf{J}_A^* = c_A\mathbf{v}^* - c\mathcal{D}_{AB}\nabla x_A$$

Equation (F) of Table 17.8-2 is obtained by writing Eq. (C) of the same table for both species $A$ and $B$ after using Eq. (I) of Table 17.8-1:

$$c_A(\mathbf{v}_A - \mathbf{v}^*) = -c\mathcal{D}_{AB}\nabla x_A \text{ and } c_B(\mathbf{v}_B - \mathbf{v}^*) = -c\mathcal{D}_{AB}\nabla x_B = +c\mathcal{D}_{AB}\nabla x_A$$

These equations may be rewritten as

$$(\mathbf{v}_A - \mathbf{v}^*) = -\frac{1}{x_A}\mathcal{D}_{AB}\nabla x_A \qquad \text{and} \qquad (\mathbf{v}_B - \mathbf{v}^*) = +\frac{1}{x_B}\mathcal{D}_{AB}\nabla x_A$$

Subtraction of these equations eliminates the molar average velocity, and we get

$$\mathbf{v}_A - \mathbf{v}_B = -\left(\frac{1}{x_A} + \frac{1}{x_B}\right)\mathcal{D}_{AB}\nabla x_A = -\frac{\mathcal{D}_{AB}}{x_A x_B}\nabla x_A$$

*b*. To get the first equation, we use is Eq. (Q') of Table 17.7-1

$$\mathbf{j}_\alpha = -\rho\mathcal{D}_{AB}\nabla\rho_A = -\rho\mathcal{D}_{AB}\frac{M_A M_B}{M^2}\nabla x_A$$

From this equation, we conclude that the expressions written in a mixture of molar and mass quantities are more complex than those written entirely in molar quantities alone or mass quantities alone.

To get the second equation, we start with Eq. (B) of Table 17.8-2 (which we have seen in (*a*) can be obtained from Eq. (A)), written for species *A* and *B*

$$\mathbf{J}_A^* = -c\mathcal{D}_{AB}\nabla x_A$$

$$\mathbf{J}_B^* = -c\mathcal{D}_{AB}\nabla x_B = +c\mathcal{D}_{AB}\nabla x_A$$

Then we rewrite these equations in terms of the combined molar fluxes (using Eq. (V) of Table 17.8-1)

$$\mathbf{N}_A - c_A\mathbf{v}^* = -c\mathcal{D}_{AB}\nabla x_A$$

$$\mathbf{N}_B - c_B\mathbf{v}^* = +c\mathcal{D}_{AB}\nabla x_A$$

We now multiply the first equation by $x_B$ and the second by $x_A$ to get

$$x_B\mathbf{N}_A - cx_A x_B\mathbf{v}^* = -x_B c\mathcal{D}_{AB}\nabla x_A$$

$$x_A\mathbf{N}_B - cx_A x_B\mathbf{v}^* = +x_A c\mathcal{D}_{AB}\nabla x_A$$

Subtracting the second equation from the first then gives Eq. 17B,4-2 after use has been made of Eq. (J) of Table 17.7-1

$$x_A \mathbf{N}_B - x_B \mathbf{N}_A = c\mathcal{D}_{AB} \nabla x_A$$

This equation contains no reference to the mass average velocity or the molar average velocity. That is true also of Eq. (E) of Table 17.8-2, which can be derived in analogous fashion from Eq. (A) of the same table.

*c*. We start by rewriting Eq. (F) in the following form

$$\mathbf{v}_A - \mathbf{v}_B = -\mathcal{D}_{AB} \frac{1}{x_A x_B} \nabla x_A = -\mathcal{D}_{AB} \left( \frac{x_B + x_A}{x_A x_B} \right) \nabla x_A$$

This may also be written as

$$\mathbf{v}_A - \mathbf{v}_B = -\mathcal{D}_{AB} \left( \frac{1}{x_A} + \frac{1}{x_B} \right) \nabla x_A = -\mathcal{D}_{AB} \left( \frac{\nabla x_A}{x_A} - \frac{\nabla x_B}{x_B} \right)$$

Then, finally

$$\mathbf{v}_A - \mathbf{v}_B = -\mathcal{D}_{AB} \left( \nabla \ln x_A - \nabla \ln x_B \right) = -\mathcal{D}_{AB} \nabla \ln \frac{x_A}{x_B}$$

**17C.1 Mass flux with respect to the volume average velocity**

*a*. Using the definitions of $\mathbf{j}_A^{\blacksquare}$ and $\mathbf{j}_A$ we get

$$\mathbf{v}^{\blacksquare} = c_A \overline{V}_A \mathbf{v}_A + c_B \overline{V}_B \mathbf{v}_B = c_A \overline{V}_A \left( \frac{\mathbf{j}_A}{\rho} + \mathbf{v} \right) + c_B \overline{V}_B \left( \frac{\mathbf{j}_B}{\rho} + \mathbf{v} \right)$$

Using the relation just after Eq. 17C.1-3 the above result gives the relation between the volume average velocity and the mass average velocity

$$\mathbf{v}^{\blacksquare} - \mathbf{v} = \left( \frac{\overline{V}_A}{M_A} - \frac{\overline{V}_B}{M_B} \right) \mathbf{j}_A$$

If we now subtract Eq. (E) of Table 17.8-1 from Eq. 17C.1-2 we get

$$\mathbf{j}_A^{\blacksquare} - \mathbf{j}_A = -\rho_A \left( \mathbf{v}^{\blacksquare} - \mathbf{v} \right) = -\rho_A \left( \frac{\overline{V}_A}{M_A} - \frac{\overline{V}_B}{M_B} \right) \mathbf{j}_A$$

Then, using the relation just after Eq. 17C.1-3 we get

$$\mathbf{j}_A^{\blacksquare} = \mathbf{j}_A \left( 1 - \rho_A \frac{\overline{V}_A}{M_A} + \rho_A \frac{\overline{V}_B}{M_B} \right) = \mathbf{j}_A \left( \rho_A \frac{\overline{V}_A}{M_A} + \rho_B \frac{\overline{V}_B}{M_B} - \rho_A \frac{\overline{V}_A}{M_A} + \rho_A \frac{\overline{V}_B}{M_B} \right)$$

and this leads to Eq. 17C.1-3

$$\mathbf{j}_A^{\blacksquare} = \mathbf{j}_A \left( \rho_B \frac{\overline{V}_B}{M_B} + \rho_A \frac{\overline{V}_B}{M_B} \right) = \mathbf{j}_A \rho \frac{\overline{V}_B}{M_B}$$

*b*. Starting from Eq. 17C.1-3 we have

$$\mathbf{j}_A^{\blacksquare} = \mathbf{j}_A \rho \frac{\overline{V}_B}{M_B} = \left( -\rho \mathcal{D}_{AB} \nabla \omega_A \right) \rho \left( \frac{c_B \overline{V}_B}{c_B M_B} \right)$$

$$= -\rho \mathcal{D}_{AB} \left( \frac{1}{\rho} \nabla \rho_A + \rho_A \nabla \frac{1}{\rho} \right) \rho \left( \frac{1 - c_A \overline{V}_A}{c_B M_B} \right)$$

$$= -\rho\mathcal{D}_{AB}\left(\frac{1}{\rho}\nabla\rho_A - \frac{\rho_A}{\rho^2}\nabla\rho\right)\left(\frac{\rho_A}{M_B} + \frac{\rho_B}{M_B}\right)\left(\frac{1 - c_A\overline{V}_A}{c_B}\right)$$

$$= -\mathcal{D}_{AB}\nabla\rho_A - \left(\mathcal{D}_{AB}\nabla\rho_A\right)\left(\frac{\rho_A}{M_B}\frac{c_B\overline{V}_B}{c_B} - \frac{\rho_B}{M_B}\frac{c_A\overline{V}_A}{c_B}\right)$$

$$+\left(\frac{\rho_A}{\rho}\mathcal{D}_{AB}\nabla\rho\right)\left(\frac{\rho}{M_B}\right)\left(\frac{c_B\overline{V}_B}{c_B}\right)$$

$$= -\mathcal{D}_{AB}\nabla\rho_A - \left(\mathcal{D}_{AB}\nabla\rho_A\right)\left(\frac{\rho_A}{\rho_B}c_B\overline{V}_B - c_A\overline{V}_A\right)$$

$$+\left(\frac{\rho_A}{\rho_B}\mathcal{D}_{AB}\nabla\left(\rho_A + \rho_B\right)\right)c_B\overline{V}_B$$

$$= -\mathcal{D}_{AB}\nabla\rho_A + c_A\overline{V}_A\mathcal{D}_{AB}\nabla\rho_A + c_B\overline{V}_B\frac{\rho_A}{\rho_B}\mathcal{D}_{AB}\nabla\rho_B$$

$$= -\mathcal{D}_{AB}\nabla\rho_A + \rho_A\mathcal{D}_{AB}\left(\overline{V}_A\nabla c_A + \overline{V}_B\nabla c_B\right)$$

Since the second term on the right side is zero, we get Eq. 17C.1-4.

To get the equation just after Eq. 17C.1-4, we proceed thus. The definitions of the partial molar volumes enable us to write

$$dV = \overline{V}_A dn_A + \overline{V}_B dn_B$$

Since the volume is a homogeneous function of order 1, we get from Euler's theorem:

$V = n_A\overline{V}_A + n_B\overline{V}_B$, or by dividing by $V$: $\qquad 1 = c_A\overline{V}_A + c_B\overline{V}_B \quad (\#)$

Forming the differential of $V$ from this equation gives

$$dV = n_A d\overline{V}_A + \overline{V}_A dn_A + n_B d\overline{V}_B + \overline{V}_B dn_B$$

Comparing this with the first equation gives

$n_A d\overline{V}_A + n_B d\overline{V}_B = 0$, or by dividing by $V$: $\qquad c_A d\overline{V}_A + c_B d\overline{V}_B = 0 \quad (*)$

Differentiating (#) and subtracting (*) gives the desired result.

**17C.2 Mass flux with respect to the solvent velocity**

*a*. First we rewrite Eq. 17C.2-1 as

$$\mathbf{j}_\alpha^N = \rho_\alpha(\mathbf{v}_\alpha - \mathbf{v}) - \rho_\alpha(\mathbf{v}_N - \mathbf{v})$$

Then we modify the terms on the right side as follows

$$\mathbf{j}_\alpha^N = \rho_\alpha(\mathbf{v}_\alpha - \mathbf{v}) - \left(\frac{\rho_\alpha}{\rho_N}\right)\rho_N(\mathbf{v}_N - \mathbf{v}) = \mathbf{j}_\alpha - \left(\frac{\rho_\alpha}{\rho_N}\right)\mathbf{j}_N$$

Here we have used the definition in Eq. H of Table 17.8-1.

*b*. Application of Eq. 17C.2-2 to a binary system with $N = B$ gives

$$\mathbf{j}_A^N = \mathbf{j}_A - \left(\frac{\rho_A}{\rho_B}\right)\mathbf{j}_B$$

Then we make use of Eq. (K) of Table 17.8-1 and Eq. (A) of Table 17.8-2 to get

$$\mathbf{j}_A^B = \mathbf{j}_A - \left(\frac{\rho_A}{\rho_B}\right)(-\mathbf{j}_A) = \mathbf{j}_A\left(1 + \frac{\rho_A}{\rho_B}\right)$$

$$= \mathbf{j}_A\left(\frac{\rho_A + \rho_B}{\rho_B}\right) = \mathbf{j}_A\left(\frac{\rho}{\rho_B}\right) = -\left(\frac{\rho}{\rho_B}\right)\rho\mathcal{D}_{AB}\nabla\omega_A$$

*c*. For a very dilute solution of $A$ in $B$, we know that $\rho_B \approx \rho$, so that

$$\mathbf{j}_A^B = -\rho\mathcal{D}_{AB}\nabla\omega_A$$

**17C.3 Determination of Lennard-Jones parameters from diffusivity data for a binary mixture.**

(a) Computational software is preferable to graphical methods for treatment of this statistical problem. The results shown below were obtained by Dr. Mike Caracotsios, using Athena Visual Workbench, a Windows package for formulation and solution of chemical process models. The problem formulation and results are displayed fully at www.AthenaVisual.com /transportphenomena, a website provided by Stewart & Associates Engineering Software, Inc.

In the following table, PAR(1) (the first adjustable parameter), denotes $\sigma_{AB}$, and PAR(2) (the second adjustable parameter), denotes $\varepsilon_{AB}/K$. The values in the "Last PAR(I)" column, with status "Estimated", are the recommended parameter values in the sense of maximum posterior probability density based on the data provided. The individual uncertainty of each parameter is reported as "95% Marginal HPD Interval" (the interval of highest posterior density containing 95% of the posterior probability distribution for that parameter; we prefer this Bayesian interpretation to the notion of a "confidence interval", which invokes a hypothetical population of alternative sets of data). The two columns labeled DYDPAR(1:NPAR) are the parametric sensitivities of the predicted diffusivities to the individual parameters. Further documentation of this problem and the modeling capabilities of Athena Visual Workbench are provided at the website named above.

NONLINEAR SINGLE-RESPONSE ESTIMATION STATISTICS SUMMARY

SUM OF SQUARES OF RESIDUALS..... 3.15638D-05
RESIDUAL DEGREES OF FREEDOM..... 6
MODEL PARAMETERS ESTIMATED..... 2

RMS VALUE OF WEIGHTED RESIDUALS..... 2.29361D-03
T(0.025)-STUDENT DISTRIBUTION VALUE..... 2.447

| ADJ.PARS | Last PAR(I) | Status | 95% Marginal HPD Interval |
|---|---|---|---|
| PAR( 1) | 3.313569E+00 | Estimated | 3.313569E+00 +- 4.080E-02 |
| PAR( 2) | 8.024473E+01 | Estimated | 8.024473E+01 +- 1.205E+01 |

| EVENT | OBSERVED | PREDICTED | RESIDUAL | ERROR(%) | DYDPAR(1:NPAR)-> | |
|---|---|---|---|---|---|---|
| 1 | 4.7000E-01 | 4.7273E-01 | -2.7282E-03 | -0.58 | -2.8533E-01 | -1.2242E-03 |
| 2 | 6.9000E-01 | 6.9072E-01 | -7.2080E-04 | -0.10 | -4.1690E-01 | -1.6487E-03 |
| 3 | 9.4000E-01 | 9.3925E-01 | 7.5217E-04 | 0.08 | -5.6691E-01 | -2.1090E-03 |
| 4 | 1.2200E+00 | 1.2161E+00 | 3.8625E-03 | 0.32 | -7.3404E-01 | -2.6078E-03 |
| 5 | 1.5200E+00 | 1.5198E+00 | 2.2434E-04 | 0.01 | -9.1731E-01 | -3.1524E-03 |
| 6 | 1.8500E+00 | 1.8489E+00 | 1.0664E-03 | 0.06 | -1.1160E+00 | -3.7483E-03 |
| 7 | 2.2000E+00 | 2.2026E+00 | -2.6318E-03 | -0.12 | -1.3295E+00 | -4.3978E-03 |
| 8 | 2.5800E+00 | 2.5801E+00 | -5.1116E-05 | 0.00 | -1.5573E+00 | -5.1009E-03 |

NUMBER OF MODEL CALLS..... 13
NUMBER OF ITERATIONS..... 4

**18A.1 Evaporation rate.**

Let $A$ denote chloropicrin and $B$ denote air; then Eq. 18.2-14 gives, for constant total pressure $p$ and ideal gas behavior,

$$N_{Az}\Big|_{z=z_1} = \frac{c\mathcal{D}_{AB}}{(z_2 - z_1)} \ln\left(\frac{x_{B2}}{x_{B1}}\right)$$
$$= \frac{p}{RT}\frac{\mathcal{D}_{AB}}{(z_2 - z_1)} \ln\left(\frac{p_{B2}}{p_{B1}}\right)$$

Inserting the values $T = 15°\text{C} = 298.15\text{K}$, $p_{B2} = p = 770$ mm Hg and $p_{B1} = 770 - 23.81 = 746.19$ mm Hg, we get

$$N_{Az}\Big|_{z=z_1} = \frac{(770/760)\text{ atm}}{(82.06 \times 298.15)\text{ cm}^3\text{atm/g-mol}} \frac{(0.088\text{ cm}^2/\text{s})}{(11.14\text{ cm})} \ln\left(\frac{770}{746.19}\right)$$
$$= 1.03 \times 10^{-8}\text{ g-mol/cm}^2\cdot\text{s}$$

Finally, the evaporation rate in g/hr is

$$W_A = N_{Az}M_AS$$
$$= (1.03 \times 10^{-8}\text{ g-mol/cm}^2\cdot\text{s})(164.4\text{ g/g-mol})(2.29\text{ cm}^2)(3600\text{ s/hr})$$
$$= 0.0139\text{ g/hr}$$

**18A.2 Sublimation of small iodine spheres in still air.**

**(a)** From Table E.1 and Eqs. 17.3-14,15, we get the following values for the system $I_2$-air:

| Species | $M$ | $\sigma$, Å | $\varepsilon/\kappa$, K |
|---|---|---|---|
| A: $I_2$ | 253.81 | 4.982 | 550. |
| B: air | 28.97 | 3.617 | 97.0 |
| AB | | 4.2995 | 231.0 |

Thus, at $T = 40°\text{C} = 313.15\text{K}$, we get the argument value

$$\kappa T/\varepsilon_{AB} = 313.15/231.0 = 1.356,$$

at which Table E.2 gives $\Omega_{\mathcal{D},AB} = 1.251$. Equation 17.3-12 then gives

$$\begin{aligned}
\mathcal{D}_{AB} &= 0.0018583\sqrt{T^3\left(\frac{1}{M_A}+\frac{1}{M_B}\right)}\frac{1}{p\sigma^2\Omega_{\mathcal{D},AB}} \\
&= 0.0018583\sqrt{(313.15)^3\left(\frac{1}{253.81}+\frac{1}{28.97}\right)}\frac{1}{(747/760)(4.2995)^2(1.251)} \\
&= 0.0888\ \text{cm}^2/\text{s}
\end{aligned}$$

**(b)** Equation 18.2-27, with $r_2 \to \infty$, gives

$$\begin{aligned}
W_A &= 4\pi r_1 c\mathcal{D}_{AB}\ln\left(\frac{1-x_{A2}}{1-x_{A1}}\right) \\
&= 4\pi r_1\frac{p\mathcal{D}_{AB}}{RT}\ln\left(\frac{p}{p-p_{A,\text{vap}}}\right) \\
&= 4\pi(0.5\ \text{cm})\frac{(747/760\ \text{atm})(0.0888\ \text{cm}^2/\text{s})}{(82.06\times 313.15\ \text{cm}^3\text{atm/g-mol})}\ln\left(\frac{747}{747-1.03}\right) \\
&= 2.95\times 10^{-8}\ \text{g-mol/s}\times 3600\ \text{s/hr} \\
&= 1.06\times 10^{-4}\ \text{g-mol/hr}
\end{aligned}$$

**18A.3 Estimating the error in calculating the absorption rate.**

Equation 18.5-18 gives

$$W_A = Kc_{A0}\sqrt{\mathcal{D}_{AB}}$$

in which $K$ is a product of known quantities. Then the error in $W_A$ resulting from small errors $\Delta c_{A0}$ and $\Delta \mathcal{D}_{AB}$ is

$$\begin{aligned}\Delta W_A &= \left(\frac{\partial W_A}{\partial c_{A0}}\right)\Delta c_{A0} + \left(\frac{\partial W_A}{\partial \mathcal{D}_{AB}}\right)\Delta \mathcal{D}_{AB} \\ &= K\sqrt{\mathcal{D}_{AB}}\Delta c_{A0} + \frac{Kc_{A0}}{2\sqrt{\mathcal{D}_{AB}}}\Delta \mathcal{D}_{AB}\end{aligned}$$

Division by $W_A$ then gives the fractional error expression

$$\frac{\Delta W_A}{W_A} \approx \frac{\Delta c_{A0}}{c_{A0}} + \frac{1}{2}\frac{\Delta \mathcal{D}_{AB}}{\mathcal{D}_{AB}}$$

Hence, the maximum absolute percentage error in the calculation of $W_A$ under the given conditions is

$$\left|100\frac{\Delta W_A}{W_A}\right|_{\max} = 5\% + \frac{1}{2}(10\%) = 10\%$$

**18A.4 Chlorine absorption in a falling film.**

The absorption rate is predicted by Eq. 18.5-18, which may be rewritten in terms of the average film velocity by use of Eq. 2.2-20:

$$W_A = 2\pi R L c_{A0} \sqrt{\frac{6\mathcal{D}_{AB}\langle v_z \rangle}{\pi L}}$$

The solubility is

$$\begin{aligned} c_{A0} &= \rho\omega_{A0}/M_A \\ &\approx (0.998 \text{ g soln/cm}^3)(0.00823 \text{ g Cl}_2\text{/g soln})/(70.91 \text{ g Cl}_2\text{/g-mol Cl}_2) \\ &= 1.16 \times 10^{-4} \text{ g-mol Cl}_2\text{/cm}^3 \end{aligned}$$

Then the predicted absorption rate is

$$\begin{aligned} W_A &= 2\pi(1.4 \times 13 \text{ cm}^2)(1.16 \times 10^{-4} \text{ g-mol Cl}_2\text{/cm}^3) \\ &\quad \times \sqrt{\frac{6(1.26 \times 10^{-5} \text{ cm}^2\text{/s})(17.7 \text{ cm/s})}{(\pi)(13 \text{ cm})}} \\ &= 7.58 \times 10^{-5} \text{ g-mol/s} = 0.273 \text{ g-mol/hr} \end{aligned}$$

**18A.5 Measurement of diffusivity by the point-source method.**

**(a)** Directly downstream of the source, the distance $s$ from the source reduces to $z$; hence, Eq. 18C.1-3 takes the form

$$c_A = \frac{W_A}{4\pi \mathcal{D}_{AB} z}$$

Therefore, the injection rate $W_A$ required to produce a mole fraction $x_A \approx 0.01$ at $p = 1$ atm and $T = 800°\text{C} = 1073\text{K}$ at a point 1 cm downstream of the source is

$$\begin{aligned} W_A &= 4\pi \mathcal{D}_{AB}(0.01c)z \\ &= 4\pi \mathcal{D}_{AB}(0.01p/RT)z \\ &= 4\pi(5\ \text{cm}^2/\text{s})(0.01)[(1\ \text{atm})/(82.06 \times 1073\ \text{cm}^3\text{atm/g-mol})](1\ \text{cm}) \\ &= 7.1 \times 10^{-6}\ \text{g-mol/s} \end{aligned}$$

**(b)** Expansion of $s$ in powers of $r^2$ at constant $z$ gives, to second order in $r$,

$$s = \sqrt{z^2 + r^2} = z\sqrt{1 + \frac{r^2}{z^2}} = z\left[1 + \frac{1}{2}\frac{r^2}{z^2} - \ldots\right]$$

and

$$s - z = z\left[\frac{1}{2}\frac{r^2}{z^2} - \ldots\right]$$

and

$$\frac{z}{s} = \left[1 - \frac{1}{2}\frac{r^2}{z^2} + \ldots\right]$$

Equation 18C.1-3 then yields the following expansion for constant $z$, complete through order $r^2$:

$$\begin{aligned} c_A(r,z) &= \frac{W_A}{4\pi \mathcal{D}_{AB} z}\frac{z}{s}\exp[-(v_0/2\mathcal{D}_{AB})(s-z)] \\ &= c_A(0,z)\left[1 - \frac{1}{2}\frac{r^2}{z^2} + \ldots\right]\left[1 - (v_0/2\mathcal{D}_{AB})z\left[\frac{1}{2}\frac{r^2}{z^2} - \ldots\right] + \ldots\right] \\ &= c_A(0,z)\left[1 - \frac{1}{2}\frac{r^2}{z^2}(1 + (v_0 z/2\mathcal{D}_{AB})) + O\left(\frac{r^4}{z^4}\right)\right] \end{aligned}$$

Thus, $c_A(r,z)$ and $x_A(r,z)$ will be within 1% of their centerline values as long as the second-order term within the square brackets does not exceed 0.01. The deviation $r_s$ of the sample collector from the axis, at the given conditions, therefore must satisfy

$$\begin{aligned} r_s^2 &\le (0.01)\frac{2z^2}{1 + v_0 z/2\mathcal{D}_{AB}} \\ &= \frac{(0.02)(1\ \text{cm}^2)}{1 + (50\ \text{cm/s})(1\ \text{cm})/(2 \times 5\ \text{cm}^2/\text{s})} \\ &= 0.00333\ \text{cm}^2 \end{aligned}$$

Hence,

$$r_s \le \sqrt{0.00333\ \text{cm}^2} = 0.058\ \text{cm}$$

**18A.6 Determination of diffusivity for ether-air system.**

**(a)** Equation 18.2-17 gives

$$\begin{aligned}\mathcal{D}_{AB} &= -\frac{\rho^{(A)}}{M_A}\frac{\Delta z_1}{\Delta t}\cdot\frac{(z_2-z_1)(x_B)_{\ln}}{c(x_{A1}-x_{A2})}\\ &= -\frac{\rho^{(A)}}{M_A}\frac{\Delta z_1}{\Delta t}\cdot\frac{RT}{p}\frac{(z_2-z_1)}{\ln(x_{B2}/x_{B1})}\\ &= -\frac{\rho^{(A)}}{M_A}\frac{\Delta z_1}{\Delta t}\cdot\frac{RT}{p}\frac{(z_2-z_1)}{\ln(p/(p-p_{A,\text{vap}}))}\end{aligned}$$

for each finite increment $\Delta t$ of time and $-\Delta z_1$ of decrease in liquid height. Insertion of the given constants gives

$$\begin{aligned}\mathcal{D}_{AB},\ \text{cm}^2/\text{s} &= +\frac{(0.712\ \text{g/cm}^3)}{(74.12\ \text{g/g-mol})}\frac{(0.2\ \text{cm})}{(\Delta t,\text{s})}\frac{82.06\times 295.15\ \text{cm}^3\text{atm/g-mol})}{(747/760\ \text{atm})}\\ &\quad\cdot\frac{(z_2-z_1,\ \text{cm})}{\ln(747/(747-480))}\\ &= \frac{(24642\ \text{cm}^3/\text{g-mol gas})}{(104.1\ \text{cm}^3/\text{g-mol liquid A})}\frac{(0.2\ \text{cm})}{(\Delta t,\ \text{s})}\frac{(z_2-z_1,\ \text{cm})}{\ln(747/267)}\\ &= 46.015\frac{(z_2-z_1,\ \text{cm})}{(\Delta t,\ \text{s for 0.2 cm level decrease})}\end{aligned}$$

and the following calculated values for the six time intervals:

| | | | | | | |
|---|---|---|---|---|---|---|
| $(z_2-z_1)$, av., cm | 1.0 | 1.5 | 2.0 | 2.5 | 3.5 | 4.5 |
| $\Delta t$ for $\Delta z_1=-0.2$ cm | 590 | 895 | 1185 | 1480 | 2055 | 2655 |
| $\mathcal{D}_{AB}$, cm$^2$/s | 0.0780 | 0.0771 | 0.0779 | 0.0777 | 0.0784 | 0.0780 |

The average of these values is 0.07785 cm$^2$/s.

**(b)** Conversion of the above result to 760 mm Hg and 0°C, using Eq. 17.2-1 in its nonpolar form, gives

$$\mathcal{D}_{AB} = 0.07785\times(747/760)\times(273.15/295.15)^{1.823} = 0.0664\ \text{cm}^2/\text{s}$$

This result appears preferable to the one reported by Jost.

**18A.7 Mass flux from a circulating bubble.**

**(a)** With the data provided, Eq. 18.5-20 gives the surface-average mass flux

$$
\begin{aligned}
(N_A)_{\text{avg}} &= \sqrt{\frac{4\mathcal{D}_{AB}v_t}{\pi D}}c_{A0} \\
&= \sqrt{\frac{(4)(1.46\times10^{-5}\ \text{cm}^2/\text{s})(22\ \text{cm/s})}{(\pi)(0.5\ \text{cm})}}(0.041\times10^{-3}\ \text{g-mol/cm}^3) \\
&= 1.17\times10^{-6}\ \text{g-mol/cm}^2\cdot\text{s}
\end{aligned}
$$

**(b)** Equation 18.1-2, with the surface-averaged $k_c$ obtained by Hamerton and Garner, gives

$$
\begin{aligned}
(N_A)_{\text{avg}} &= k_c(c_{A0} - c_{A\infty}) \\
&= (117\ \text{cm/hr})(1\ \text{hr}/3600\ \text{s})(0.041\times10^{-3}\ \text{g-mol/cm}^3) \\
&= 1.14\times10^{-6}\ \text{g-mol/cm}^2\cdot\text{s}
\end{aligned}
$$

**18B.1 Diffusion through a stagnant film--alternate derivation**

From Equation 18.2-1 we get

$$\frac{1}{1-x_A}\frac{dx_A}{dz} = -\frac{N_{Az}}{c\mathcal{D}_{AB}}$$

or

$$-\frac{1}{x_B}\frac{dx_B}{dz} = -\frac{N_{Az}}{c\mathcal{D}_{AB}}$$

Integration gives

$$\int_{x_{B1}}^{x_{B2}} \frac{dx_B}{x_B} = \frac{N_{Az}}{c\mathcal{D}_{AB}}\int_{z_1}^{z_2} dz$$

Hence

$$\ln\frac{x_{B2}}{x_{B1}} = \frac{N_{Az}}{c\mathcal{D}_{AB}}(z_2 - z_1)$$

or

$$N_{Az} = \frac{c\mathcal{D}_{AB}}{z_2 - z_1}\ln\frac{x_{B2}}{x_{B1}}$$

Which is in agreement with Eq. 18.2-14

**18B.2 Error in neglecting the convection term in evaporation**

*a*. Without the convection term, Eq 18.2-1 and 4 become

$$N_{Az} = -c\mathcal{D}_{AB}\frac{dx_A}{dz} \qquad \text{and} \qquad \frac{d^2x_A}{dz^2} = 0$$

Integration of the latter equation twice then gives

$$x_A = C_1 z + C_2$$

Then application of the boundary conditions gives two equations:

$$x_{A1} = C_1 z_1 + C_2 \quad \text{and} \quad x_{A2} = C_1 z_2 + C_2$$

These equation may be solved simultaneously to give

$$C_1 = -\frac{x_{A1} - x_{A2}}{z_2 - z_1} \quad \text{and} \quad C_2 = x_{A1} + \frac{x_{A1} - x_{A2}}{z_2 - z_1} z_1$$

Therefore, in the approximation being considered here, the mole-fraction profile in the system is given by

$$\frac{x_{A1} - x_A}{x_{A1} - x_{A2}} = \frac{z - z_1}{z_2 - z_1}$$

*b*. To get the result in (*a*) from Eq. 18.2-14, we can expand the latter in a Taylor series, as was done in getting Eq. 18.2-16.

*c*. To get the solution of Example 18.2-2 by using the result in (*a*), we make the following calculation

$$\mathcal{D}_{AB} = \frac{N_{Az}RT}{p(-dx_A/dz)} = \frac{N_{Az}RT(z_2 - z_1)}{p(x_{A1} - x_{A2})} = \frac{(7.26\times10^{-9})(82.06)(273)(17.1)}{(755/760)[(33/755) - 0]}$$

$$= 0.0641 \text{ cm}^2/\text{s}$$

Hence the error is $\dfrac{0.0641 - 0.0636}{0.0636} \times 100 = 0.79\%$

**18B.3 Effect of mass transfer rate on the concentration profiles**

*a*. Rewrite Eq. 18.2-14 as

$$N_{Az} = \frac{c\mathcal{D}_{AB}}{z_2 - z_1} \ln \frac{1 - x_{A2}}{1 - x_{A1}} \qquad \text{or} \qquad \frac{1 - x_{A2}}{1 - x_{A1}} = \exp \frac{N_{Az}(z_2 - z_1)}{c\mathcal{D}_{AB}}$$

When this is substituted into the right side of Eq. 18.2-11, we get

$$\frac{1 - x_A}{1 - x_{A1}} = \left( \exp \frac{N_{Az}(z_2 - z_1)}{c\mathcal{D}_{AB}} \right)^{(z - z_1)/(z_2 - z_1)} = \left( \exp \frac{N_{Az}(z - z_1)}{c\mathcal{D}_{AB}} \right)$$

*b*. Starting with Eq. 18.2-1 and integrating directly we get

$$\frac{1}{1 - x_A} \frac{d}{dz} = -\frac{N_{Az}}{c\mathcal{D}_{AB}} \qquad \text{and} \qquad \int_{x_{A1}}^{x_A} \frac{1}{1 - x_A} dx_A = -\frac{N_{Az}}{c\mathcal{D}_{AB}} \int_{z_1}^{z} dz$$

When the integrals are evaluated we get

$$-\ln \frac{1 - x_A}{1 - x_{A1}} = -\frac{N_{Az}}{c\mathcal{D}_{AB}}(z - z_1)$$

Changing signs and taking the antilogarithm of both sides then gives the result in (*a*).

*c*. Expanding the right side of Eq. 18B.3-1 in a Taylor series in the argument of the exponent, we get

$$\frac{1 - x_A}{1 - x_{A1}} = 1 + \frac{N_{Az}}{c\mathcal{D}_{AB}}(z - z_1) + \cdots$$

If we retain just two terms in the Taylor series, and bring the "1" on the right side over to the left side, we get

$$x_A = x_{A1} - \frac{N_{Az}(1 - x_{A1})}{c\mathcal{D}_{AB}}(z - z_1)$$

which is of the form $x_A = mz + b$, that is, a straight line function.

**18B.4 Absorption with chemical reaction**

*a*. Equation 18.4-8 remains valid, but now the boundary conditions are: at $\zeta = 1$, $\Gamma = 1$; and at $\zeta = 0$, $d\Gamma/d\zeta = 0$. Hence the boundary conditions lead to a pair of simultaneous equations:

$$1 = C_1 \cosh\phi + C_2 \sinh\phi \qquad \text{and} \qquad 0 = 0 + C_2\phi$$

from which it follows that $C_2 = 0$ and $C_1 = 1/\cosh\phi$. Then the analog of Eq. 18.4-9 is

$$\Gamma = \cosh\phi\zeta/\cosh\phi$$

The mass flux at the liquid gas interface at $z = L$ is then:

$$N_{Az}\Big|_{z=L} = +\mathcal{D}_{AB}\frac{dc_A}{dz}\bigg|_{z=L} = \frac{c_{A0}\mathcal{D}_{AB}}{L}\frac{d\Gamma}{d\zeta}\bigg|_{\zeta=1} = \frac{c_{A0}\mathcal{D}_{AB}}{L}\frac{\phi\sinh\phi\zeta}{\cosh\phi}\bigg|_{\zeta=1}$$

which leads directly to the result in Eq. 18.4-12.

*b*. We start with Eq. 18.4-7, the solution of which can be written in the form of a superposition of exponentials

$$\Gamma = C_1 e^{\phi\zeta} + C_2 e^{-\phi\zeta}$$

which is to be solved with the boundary conditions given just above Eq. 18.4-8. This leads to the following equations for the constants of integration:

$$1 = C_1 + C_2 \qquad \text{and} \qquad 0 = C_1\phi e^{\phi} - C_2\phi e^{-\phi}$$

These two simultaneous equations can be solved to give

$$C_1 = \frac{1}{1+e^{2\phi}} = \frac{e^{-\phi}}{e^{\phi}+e^{-\phi}} \qquad \text{and} \qquad C_2 = \frac{e^{2\phi}}{1+e^{2\phi}} = \frac{e^{\phi}}{e^{\phi}+e^{-\phi}}$$

Therefore , the dimensionless concentration profile is

$$\Gamma = \frac{e^{-\phi}e^{\phi\zeta}}{e^{\phi}+e^{-\phi}} + \frac{e^{\phi}e^{-\phi\zeta}}{e^{\phi}+e^{-\phi}} = \frac{\frac{1}{2}\left[e^{\phi(1-\zeta)}+e^{-\phi(1-\zeta)}\right]}{\frac{1}{2}\left[e^{\phi}+e^{-\phi}\right]} = \frac{\cosh\left[\phi(1-\zeta)\right]}{\cosh\phi}$$

Thus we are led to the same result that we obtained in Eq. 18.4-9.

*c*. Equation 18.4-12 can be written thus

$$N_{Az}\big|_{z=0} = \left(\frac{c_{A0}\mathfrak{D}_{AB}}{L}\right)\sqrt{\frac{k_1'''L^2}{\mathfrak{D}_{AB}}}\tanh\sqrt{\frac{k_1'''L^2}{\mathfrak{D}_{AB}}}$$

Therefore, for very large $L$ we get (using an expansion appropriate for large values of the argument)

$$N_{Az}\big|_{z=0} = c_{A0}\sqrt{\mathfrak{D}_{AB}k_1'''}\left(1-2\exp\left(-2k_1'''L^2/\mathfrak{D}_{AB}\right)+\cdots\right) \xrightarrow{L\to\infty} c_{A0}\sqrt{\mathfrak{D}_{AB}k_1'''}$$

For very small values of $L$ we get (using an expansion appropriate for very small values of the argument)

$$N_{Az}\big|_{z=0} = c_{A0}\sqrt{\mathfrak{D}_{AB}k_1'''}\sqrt{\frac{k_1'''L^2}{\mathfrak{D}_{AB}}}\left(1-\frac{1}{3}\frac{k_1'''L^2}{\mathfrak{D}_{AB}}+\cdots\right)$$

$$= c_{A0}k_1'''L\left(1-\frac{1}{3}\frac{k_1'''L^2}{\mathfrak{D}_{AB}}+\cdots\right) \xrightarrow{L\to 0} 0$$

Similarly, Eq. 18.4-10 can be written as

$$\frac{c_A}{c_{A0}} = \cosh\sqrt{\frac{k_1'''}{\mathfrak{D}_{AB}}}z - \tanh\sqrt{\frac{k_1'''L^2}{\mathfrak{D}_{AB}}}\sinh\sqrt{\frac{k_1'''}{\mathfrak{D}_{AB}}}z$$

As $L$ becomes infinite, this becomes

$$\frac{c_A}{c_{A0}} = \cosh\sqrt{\frac{k_1'''}{\mathfrak{D}_{AB}}}z - \sinh\sqrt{\frac{k_1'''}{\mathfrak{D}_{AB}}}z = \exp\left(-\sqrt{\frac{k_1'''}{\mathfrak{D}_{AB}}}z\right)$$

As $L$ becomes zero, the dimensionless concentration becomes unity.

**18B.5 Absorption of chlorine by cyclohexene**

*a*. For a second-order reaction, Eq. 18.4-4 has to be replaced by

$$-\mathcal{D}_{AB}\frac{d^2c_A}{dz^2}+k_2''c_A^2=0$$

with the same boundary conditions as before. Introduce the dimensionless variables $\Gamma=c_A/c_{A0}$ and $\zeta=\sqrt{k_2''c_{A0}/6\mathcal{D}_{AB}}z$. Then the differential equation becomes

$$\frac{d^2\Gamma}{d\zeta^2}-6\Gamma^2=0$$

with boundary conditions $\Gamma(0)=1$ and $\Gamma(\infty)=0$. We now let $d\Gamma/d\zeta=p(\Gamma)$, so that $d^2\Gamma/d\zeta^2=dp/d\zeta=(dp/d\Gamma)(d\Gamma/d\zeta)=p(dp/d\Gamma)$. Then we obtain a differential equation that is first order and separable

$$p\frac{dp}{d\Gamma}=6\Gamma^2$$

This may be inegrated, and we use the boundary condition that at $\zeta=\infty$, $\Gamma=0$ and *also* that $d\Gamma/d\zeta=p(\Gamma)=0$:

$$\int_0^p p\,dp=\int_0^\Gamma 6\Gamma^2 d\Gamma$$

from which

$$\tfrac{1}{2}p^2=2\Gamma^3 \qquad \text{or} \qquad \frac{d\Gamma}{d\zeta}=\pm 2\Gamma^{3/2}$$

Here we must choose the minus sign, since the slope of the concentration vs. distance curve is negative. Then using the first boundary condition, we can integrate this equation to get:

$$\int_1^\Gamma \Gamma^{-3/2}d\Gamma=-\int_0^\zeta d\zeta \qquad \text{or} \qquad \Gamma^{-1}=(1+\zeta)^2$$

Hence the final expression for the concentration profile is

$$\frac{c_{A0}}{c_A} = \left(1 + \sqrt{\frac{k_2''' c_{A0}}{6\mathfrak{D}_{AB}}} z\right)^2$$

*b*. From the result of (*a*) we get the absorption rate at the liquid gas interface:

$$N_{Az}\big|_{z=0} = -\mathfrak{D}_{AB}\frac{dc_A}{dz}\bigg|_{z=0} = \sqrt{\tfrac{2}{3}k_2''' c_A^3 \mathfrak{D}_{AB}}$$

*c*. The equation to be solved is

$$-\mathfrak{D}_{AB}\frac{d^2c_A}{dz^2} + f(c_A) = 0 \qquad \text{or} \qquad p\frac{dp}{dc_A} = \frac{f(c_A)}{\mathfrak{D}_{AB}}$$

where we have introduced the variable $p$ as before. The resulting equation is integrated, as before, to give

$$\int_0^p \bar{p}\,d\bar{p} = \frac{1}{\mathfrak{D}_{AB}}\int_0^{c_A} f(\bar{c}_A)d\bar{c}_A \qquad \text{or} \qquad p = \frac{dc_A}{dz} = -\sqrt{\frac{2}{\mathfrak{D}_{AB}}\int_0^{c_A} f(\bar{c}_A)d\bar{c}_A}$$

A second integration yields

$$\int_{c_{A0}}^{c_A} \frac{d\bar{c}_A}{\sqrt{(2/\mathfrak{D}_{AB})\int_0^{\bar{c}_A} f(\bar{\bar{c}}_A)d\bar{\bar{c}}_A}} = -\int_0^1 dz = -z$$

Then we differentiate both sides with respect to $z$ to get

$$\frac{1}{\sqrt{(2/\mathfrak{D}_{AB})\int_0^{c_A} f(\bar{c}_A)d\bar{c}_A}}\frac{dc_A}{dz} = -1 \text{ and } \frac{dc_A}{dz}\bigg|_{z=0} = -\sqrt{(2/\mathfrak{D}_{AB})\int_0^{c_{A0}} f(c_A)dc_A}$$

This together with $N_{Az}\big|_{z=0} = -\mathfrak{D}_{AB}(dc_A/dz)\big|_{z=0}$ gives Eq. 18B.5-2.

**18B.6 Two-bulb experiment for measuring gas diffusivity--quasi-steady-state analysis**

*a*. The molar shell balance on $\Delta z$ gives for species $A$

$$SN_{Az}\big|_z - SN_{Az}\big|_{z+\Delta z} = 0$$

Division by $\Delta z$ and taking the limit as $\Delta z$ goes to zero gives $dN_{Az}/dz = 0$ or $N_{Az}$= constant.

*b*. Equation 18.0-1, for this problem, may be simplified thus:

$$N_{Az} = -c\mathcal{D}_{AB}(dx_A/dz) + x_A(N_{Az} + N_{Bz}) = -c\mathcal{D}_{AB}(dx_A/dz)$$

since $N_{Az} = -N_{Bz}$; this is true, because in a system at constant $c$ for every molecule of $A$ that moves to the right, a molecule of $B$ must move to the left.

*c*. The equation in (*b*), with $N_{Az}$= constant, then becomes $dx_A/dz = -(N_{Az}/c\mathcal{D}_{AB})$, which when integrated becomes

$$x_A = -\frac{N_{Az}}{c\mathcal{D}_{AB}}z + C_1$$

*d*. At $z = L$, $x_A = x_A^+$, so that

$$x_A^+ = -\frac{N_{Az}}{c\mathcal{D}_{AB}}L + C_1$$

Subtracting the last two equations gives then

$$x_A - x_A^+ = \frac{N_{Az}}{c\mathcal{D}_{AB}}(L - z)$$

*e*. At $z = -L$, we know that $x_A = x_A^- = 1 - x_A^+$, so that

$$\left(1 - x_A^+\right) - x_A^+ = \frac{N_{Az}}{c\mathcal{D}_{AB}}\left(L - (-L)\right)$$

from which Eq. 18B.6-2 follows directly.

*f*. A mass balance over the right bulb states that the time rate of change of moles within $V$ must exactly equal the rate at which moles enter $V$ by diffusion at the end of the tube. That is

$$\frac{d}{dt}\left(Vcx_A^+\right) = SN_{Az} \qquad \text{or} \qquad Vc\frac{dx_A^+}{dt} = S\left(\tfrac{1}{2} - x_A^+\right)\frac{c\mathcal{D}_{AB}}{L}$$

in which $S$ is the cross-section of the connecting tube.

*g*. The equation in (*f*) may be integrated

$$\int \frac{dx_A^+}{\frac{1}{2} - x_A^+} = \int \frac{S\mathcal{D}_{AB}}{LV}dt + C_2 \qquad \text{or} \qquad -\ln\left(\tfrac{1}{2} - x_A^+\right) = \frac{S\mathcal{D}_{AB}}{LV}t + C_2$$

The integration constant may be obtained by using the fact that at $t$ = 0, we know that the mole fraction of $A$ in the right bulb will be zero, or $C_2 = -\ln\left(\frac{1}{2} - 0\right)$. Therefore

$$\ln\left(\tfrac{1}{2} - x_A^+\right) - \ln\tfrac{1}{2} = -\frac{S\mathcal{D}_{AB}}{LV}t \qquad \text{or} \qquad \frac{\frac{1}{2} - x_A^+}{\frac{1}{2}} = \exp\left(-\frac{S\mathcal{D}_{AB}}{LV}t\right)$$

*h*. If we plot

$$-\frac{LV}{S}\ln\left(\frac{\frac{1}{2} - x_A^+}{\frac{1}{2}}\right) \text{ vs. } t$$

the slope will be the diffusivity, $\mathcal{D}_{AB}$.

**18B.7 Diffusion from a suspended droplet**

*a*. A mass balance on $A$ over a spherical shell of thickness $\Delta r$ is (in molar units)

$$4\pi r^2 \cdot N_{Ar}\big|_r - 4\pi(r+\Delta r)^2 \cdot N_{Ar}\big|_{r+\Delta r} = 0$$

or, equivalently

$$(4\pi r^2 N_{Ar})\big|_r - (4\pi r^2 N_{Ar})\big|_{r+\Delta r} = 0$$

Now divide by $4\pi\Delta r$ and take the limit as $\Delta r$ goes to zero to get

$$\frac{d}{dr}(r^2 N_{Ar}) = 0$$

This may be integrated to give $r^2 N_{Ar} = C_1$. We may use the boundary condition that $N_{Ar} = N_{Ar1}$ at $r = r_1$ (the gas liquid interface) to evaluate the constant and obtain $r^2 N_{Ar} = r_1^2 N_{Ar1}$.

*b* . Equation 18.0-1, written for the radial component in spherical coordinates, is

$$N_{Ar} = -c\mathcal{D}_{AB}\frac{dx_A}{dr} + x_A\left(N_{Ar} + N_{Br}\right)$$

If gas $B$ is not moving, then $N_{Br}$ may be set equal to zero and the equation may be solved for the molar flux of $A$:

$$N_{Ar} = -\frac{c\mathcal{D}_{AB}}{1-x_A}\frac{dx_A}{dr}$$

Multiplying by $r^2$ and using the result obtained in (*a*), we get Eq. 19B.7-1:

$$r_1^2 N_{Ar1} = -\frac{c\mathcal{D}_{AB}}{1-x_A} r^2 \frac{dx_A}{dr}$$

*c*. Equation 19B.7-1 can be rearranged to give

$$r_1^2 N_{Ar1} \frac{dr}{r^2} = -c\mathcal{D}_{AB} \frac{dx_A}{1-x_A}$$

Integration then gives

$$r_1^2 N_{Ar1} \left( -\frac{1}{r} \right) \Bigg|_{r_1}^{r_2} = -c\mathcal{D}_{AB} \left[ -\ln(1-x_A) \right] \Big|_{x_{A1}}^{x_{A2}}$$

or

$$r_1^2 N_{Ar1} \left( \frac{1}{r_1} - \frac{1}{r_2} \right) = +c\mathcal{D}_{AB} \left[ \ln(1-x_{A2}) - \ln(1-x_{A1}) \right]$$

This may be rearranged to give

$$r_1^2 N_{Ar1} \left( \frac{r_2 - r_1}{r_1 r_2} \right) = c\mathcal{D}_{AB} \ln \frac{x_{B2}}{x_{B1}}$$

or, when solved for the molar flux of $A$

$$N_{Ar1} = \frac{c\mathcal{D}_{AB}}{r_2 - r_1} \left( \frac{r_2}{r_1} \right) \ln \frac{x_{B2}}{x_{B1}}$$

When $r_2 \to \infty$ (and presumably also $x_{A2} \to 0$), this last result gives

$$N_{Ar1} = \frac{c\mathcal{D}_{AB}}{r_1} \ln \frac{1}{x_{B1}}$$

**18B.8 Method for separating helium from natural gas**

Let $A$ be helium and $B$ be pyrex. Then a shell mass balance gives the following equation

$$\frac{d}{dr}(rN_{Ar}) = 0$$

Insertion of Eq. 18.0-1. with $N_{Br} = 0$ and $x_A << 1$, gives for constant diffusivity

$$\frac{d}{dr}\left(r\frac{dc_A}{dr}\right) = 0$$

Integrating twice we get

$$c_A = C_1 \ln r + C_2$$

The boundary conditions are: at $r = R_1$, $c_A = c_{A1}$ and at $r = R_2$, $c_A = c_{A2}$. Evaluation of the constants of integration gives for the concentration profile:

$$\frac{c_A - c_{A2}}{c_{A1} - c_{A2}} = \frac{\ln(R_2/r)}{\ln(R_2/R_1)}$$

Then the molar rate of diffusion through the wall is

$$N_{Ar} = -\mathcal{D}_{AB}\frac{dc_A}{dr} = \frac{+\mathcal{D}_{AB}(c_{A1} - c_{A2})}{r\ln(R_2/R_1)}$$

and

$$W_A = 2\pi rL \cdot N_{Ar} = \frac{2\pi L\mathcal{D}_{AB}(c_{A1} - c_{A2})}{\ln(R_2/R_1)}$$

is the molar flow rate of the helium through the pyrex tube.

**18B.9 Rate of leaching**

*a*. The molar balance for substance $A$ over a thin slab of thickness $\Delta z$ is

$$N_{Az}\big|_z S - N_{Az}\big|_{z+\Delta z} S = 0$$

Division by $S\Delta z$ and letting the slab thickness $\Delta z$ go to zero yields

$$\frac{dN_{Az}}{dz} = 0$$

Then inserting an approximate version of Fick's law gives

$$N_{Az} = -\mathcal{D}_{AB}\frac{dc_A}{dz}$$

which is good for a dilute solution of $A$ in $B$. Thus the diffusion equation becomes

$$\frac{d^2c_A}{dz^2} = 0$$

*b*. The above differential equation may be integrated to give

$$c_A = C_1 z + C_2$$

The constants are determined from the boundary conditions that $c_A = c_{A0}$ at $z = 0$, and $c_A = c_{A\delta}$ at $z = \delta$. The final expression is then

$$\frac{c_A - c_{A0}}{c_{A\delta} - c_{A0}} = \frac{z}{\delta} \qquad \text{or} \qquad \frac{c_A - c_{A\delta}}{c_{A0} - c_{A\delta}} = 1 - \frac{z}{\delta}$$

*c*. The rate of leaching (per unit area) is then

$$N_{Az}\big|_{z=0} = -\mathcal{D}_{AB}\frac{dc_A}{dz}\bigg|_{z=0} = -\mathcal{D}_{AB}\left(c_{A0} - c_{A\delta}\right)\left(-\frac{1}{\delta}\right) = \frac{\mathcal{D}_{AB}\left(c_{A0} - c_{A\delta}\right)}{\delta}$$

**18B.10 Constant-evaporating mixtures.**

**(a)** For this one-dimensional, steady-state, nonreactive system, the species conservation equations take the form

$$dN_{\alpha z}/dz = 0,$$

and give

$$N_{\alpha z} = \text{const. for each species.}$$

The coefficients $c\mathcal{D}_{\alpha\beta}$ depend only on $T$ in low-density systems, according to Eq. 17.3-16 and the corresponding formula of Mason and Monchick for polar gas mixtures; thus, each of these coefficients is predicted to be constant over this isothermal system.

Assuming insolubility of nitrogen (3) in the liquid mixture, we obtain $N_{3z} = 0$; then Eq. 17.9-1 gives

$$\frac{dy_3}{dz} = \frac{N_{1z}}{c\mathcal{D}_{13}} + \frac{N_{2z}}{c\mathcal{D}_{23}}$$

as the Maxwell-Stefan equation for $y_3$. Integration and use of the boundary condition at $z = L$ give

$$\ln y_3 = \left[\frac{N_{1z}}{c\mathcal{D}_{13}} + \frac{N_{2z}}{c\mathcal{D}_{23}}\right](z - L) = A(z - L)$$

whence

$$y_3 = \exp[A(z - L)] \qquad (18B.10-1)$$

Equation 17.9-1 and the condition $y_1 + y_2 + y_3 = 1$ then give

$$\begin{aligned}
-\frac{dy_1}{dz} &= \frac{y_2 N_{1z} - y_1 N_{2z}}{c\mathcal{D}_{12}} + \frac{y_3 N_{1z}}{c\mathcal{D}_{13}} \\
&= \frac{(1 - y_1 - y_3)N_{1z} - y_1 N_{2z}}{c\mathcal{D}_{12}} + \frac{y_3 N_{1z}}{c\mathcal{D}_{13}} \\
&= \frac{N_{1z}}{c\mathcal{D}_{12}} - y_1 \frac{N_{1z} + N_{2z}}{c\mathcal{D}_{12}} + y_3\left(\frac{-N_{1z}}{c\mathcal{D}_{12}} + \frac{N_{1z}}{c\mathcal{D}_{13}}\right)
\end{aligned}$$

as the differential equation for the toluene mole fraction profile $y_1(z)$. Insertion of the notations $B$, $C$ and $D$, and the solution for $y_3$, gives

$$-\frac{dy_1}{dz} = C - y_1 B - D\exp[A(z - L)]$$

Rearrangement of this result gives

$$\frac{dy_1}{dz} - By_1 = -C + D\exp[A(z - L)]$$

which has the form of Eq. C.1-2, with $f(z) = -B$ and $g(z) = -C + D\exp[A(z-L)]$. Using the solution indicated there, we get

$$
\begin{aligned}
y_1 &= \exp(Bz)\left[\int \exp(-Bz)\{-C + D\exp[A(z-L]\}dz + K\right] \\
&= Ke^{Bz} - Ce^{Bz}\int e^{-Bz}dz + De^{Bz}\int e^{[-Bz+Az-AL]}dz \\
&= Ke^{Bz} - Ce^{Bz}\frac{e^{-Bz}}{-B} + De^{[Bz-AL]}\int e^{(A-B)z}dz \\
&= Ke^{Bz} + \frac{C}{B} + \frac{D}{A-B}e^{[A(z-L)]}
\end{aligned}
$$

The boundary condition $y_1 = 0$ at $z = L$ gives

$$0 = Ke^{BL} + \frac{C}{B} + \frac{D}{A-B}$$

The resulting integration constant is

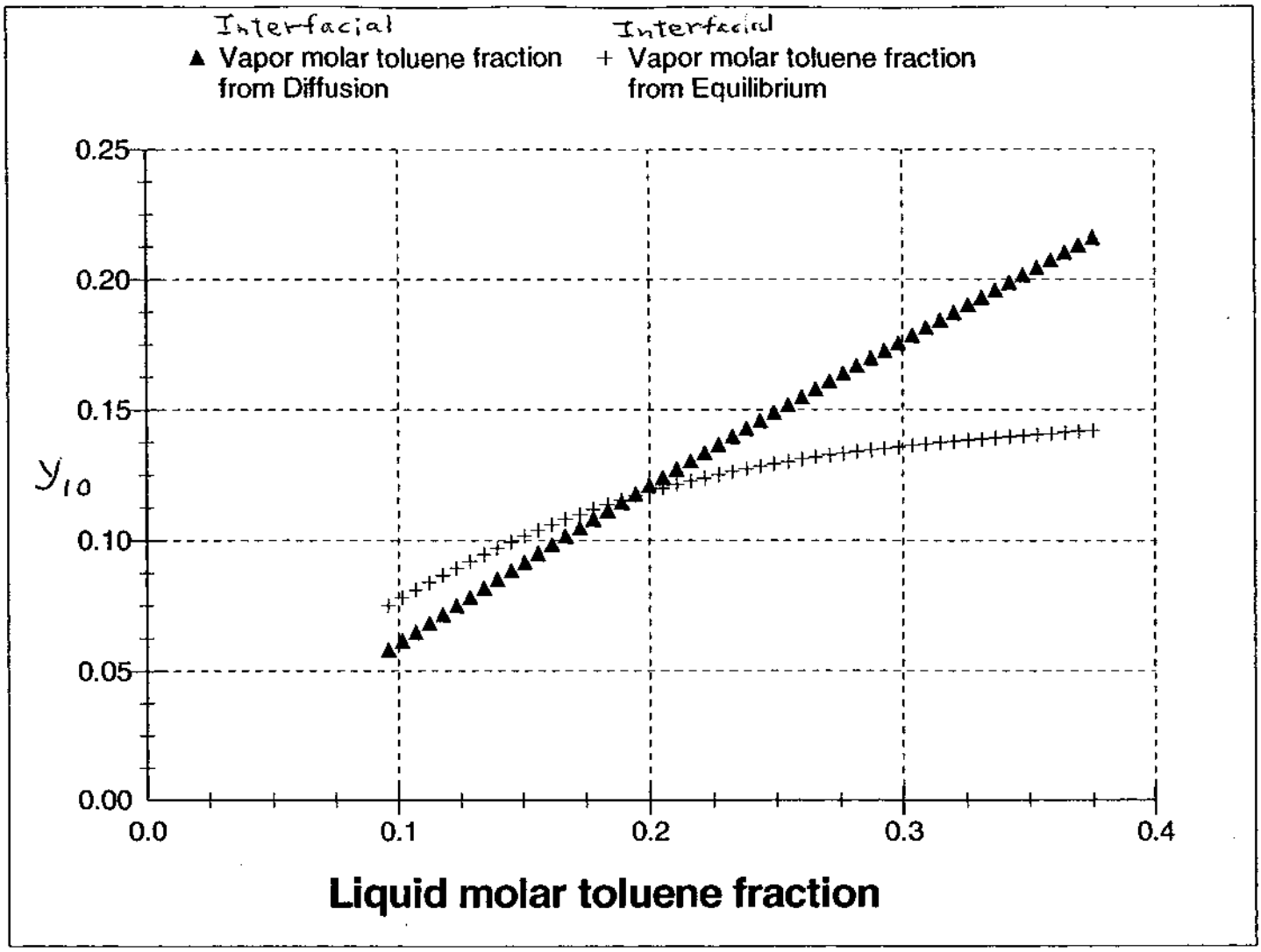
$$K = \left[\frac{-C}{B} - \frac{D}{A-B}\right]e^{-BL}$$

Thus, the toluene mole-fraction profile is

$$y_1(z) = \frac{D}{A-B}e^{[A(z-L)]} - \left[\frac{C}{B} + \frac{D}{A-B}\right]e^{-BL+Bz} + \frac{C}{B}$$

in agreement with Eq. 18B.10-1.

**(b)** Numerical results of the suggested calculation procedure are shown in the following graph, prepared by Mike Caracotsios. Cubic spline interpolation was used to get equilibrium vapor compositions over the tabulated range of liquid compositions. The interfacial vapor composition $y_{10}$ calculated from the equilibrium data becomes equal to that calculated in step (iv) at a liquid composition $x_{10} = 0.192$.


Interfacial
▲ Vapor molar toluene fraction from Diffusion
Interfacial
+ Vapor molar toluene fraction from Equilibrium
0.25
0.20
0.15
0.10
0.05
0.00
$y_{10}$
0.0
0.1
0.2
0.3
0.4
Liquid molar toluene fraction

**18B.11 Diffusion with fast second-order reaction**

*a*. A shell balance on species $A$ plus Fick's first law of diffusion leads to

$$\frac{d^2x_A}{dz^2}=0 \quad \text{which has the solution} \qquad x_A = C_1 z + C_2$$

When the constants of integration are determined from the boundary conditions that $x_A(0)=x_{A0}$ and $x_A(\delta)=0$, the mole-fraction profile becomes

$$x_A = x_{A0}\left(1-\frac{z}{\delta}\right)$$

Then the rate of dissolution of $A$ at the solid-liquid interface is

$$N_{Az}\Big|_{z=0} = -c\mathcal{D}_{AS}\frac{dx_A}{dz}\Big|_{z=0} = \frac{c\mathcal{D}_{AS}x_{A0}}{\delta}$$

*b*. For the system pictured in Fig. 18B.11, we have the following differential equations

$$\frac{d^2x_A}{dz^2}=0 \quad (\text{for } 0\le z\le \kappa\delta) \quad \text{and} \quad \frac{d^2x_B}{dz^2}=0 \quad (\text{for } \kappa\delta\le z\le\delta)$$

where $\kappa$ is as yet unknown. These equations have the solutions

$$x_A = C_1 z + C_2 \qquad \text{and} \qquad x_B = C_3 z + C_4$$

The boundary conditions are now: $x_A(0)=x_{A0}$, $x_A(\kappa\delta)=0$, $x_B(\kappa\delta)=0$, and $x_B(\delta)=x_{B\infty}$. When these are used to get the four constants of integration, we get the following linear profiles:

$$x_A = x_{A0}\left(1-\frac{z}{\kappa\delta}\right) \qquad \text{and} \qquad x_B = x_{B\infty}\left(1-\frac{1}{1-\kappa}+\frac{1}{(1-\kappa)\delta}z\right)$$

valid in their respective regions.

Up to this point $\delta$ is unknown. This quantity can be determined from the statement that the rate of diffusion of $A$ into the reaction plane must be exactly equal to the rate of diffusion of $B$ into the reaction plane at $z = \delta$. This is expressed mathematically as:

$$-c\mathcal{D}_{AS}\frac{dx_A}{dz}\bigg|_{z=\kappa\delta} = +c\mathcal{D}_{BS}\frac{dx_B}{dz}\bigg|_{z=\kappa\delta}$$

When the differentiations are performed, we get

$$-c\mathcal{D}_{AS}\left(-\frac{x_{A0}}{\kappa\delta}\right) = +c\mathcal{D}_{BS}\left(\frac{x_{B\infty}}{(1-\kappa)\delta}\right)$$

or

$$\frac{\kappa}{\mathcal{D}_{AS}x_{A0}} = \frac{1-\kappa}{\mathcal{D}_{BS}x_{B\infty}}$$

From this we get

$$\frac{1}{\kappa} = \left(1 + \frac{\mathcal{D}_{BS}x_{B\infty}}{\mathcal{D}_{AS}x_{A0}}\right)$$

Then the rate of dissolution of $A$ at the solid-liquid interface is

$$N_{Az}\big|_{z=0} = -c\mathcal{D}_{AS}\frac{dx_A}{dz}\bigg|_{z=0} = \frac{c\mathcal{D}_{AS}x_{A0}}{\kappa\delta} = \frac{c\mathcal{D}_{AS}x_{A0}}{\delta}\left(1 + \frac{\mathcal{D}_{BS}x_{B\infty}}{\mathcal{D}_{AS}x_{A0}}\right)$$

**18B.12 A sectioned-cell experiment for measuring gas-phase diffusivity**

*a*. Equation 18.2-1 can be rewritten as

$$N_{Az} = +\frac{c\mathcal{D}_{AB}}{x_B}\frac{dx_B}{dz}$$

Integration from $z$ to the top of the diffusion tube ($z_2$) gives (since $N_{Az}$ is constant according to Eq. 18.2-3)

$$\int_{x_B}^{x_{B2}} \frac{dx_B}{x_B} = \frac{N_{Az}}{c\mathcal{D}_{AB}}\int_z^{z_2} dz \qquad \text{or} \qquad \ln\frac{x_{B2}}{x_B} = \frac{N_{Az}(z_2 - z)}{c\mathcal{D}_{AB}}$$

This may now be rewritten as

$$\ln x_B = \left(\ln x_{B2} - \frac{N_{Az}z_2}{c\mathcal{D}_{AB}}\right) + \left(\frac{N_{Az}}{c\mathcal{D}_{AB}}\right)z$$

This suggests that when $\ln x_B$ ($or$ $\log_{10} x_B$) is plotted against $z$, the slope of the resulting straight line will be $N_{Az}/c\mathcal{D}_{AB}$ ($or$ $(N_{Az}/c\mathcal{D}_{AB})$ divided by 2.303).

*b*. The plot of the experimental data of $\log_{10} x_B$ vs $z$ gives a straight line of slope 0.171. The intercept at $z = z_1 = 0$ is the logarithm of the mole fraction of $B$ at the interface. The intercept at $\log_{10} x_B = 0$ is the total length of the diffusion path, where the mole fraction of $A$ is maintained at zero.

*c*. We can tabulate, for the four sections of the diffusion tube, the average distance, the average mole fraction of $B$ , and the logarithm (to the base 10) of the mole fraction of $B$, thus:

| $z$ (avg) | $x_B$ | $\log_{10} x_B$ |
|---|---|---|
| 0.60 | 0.243 | -0.6144 |
| 1.60 | 0.359 | -0.4444 |
| 2.60 | 0.531 | -0.2749 |
| 3.60 | 0.785 | -0.1051 |

We may now use the equation in (*a*) to get the diffusivity. We convert all data to c.g.s. units for the numerical calculation. We use the result from (*b*) that the slope of straight line of $\log_{10} x_B$ vs $z$ (from graph) is 0.171. Then

$$0.171 = \frac{N_{Az}}{2.303c\mathcal{D}_{AB}}$$

The molar flux of $A$ is obtained from the evaporation rate across a circular region of diameter 2 cm as follows:

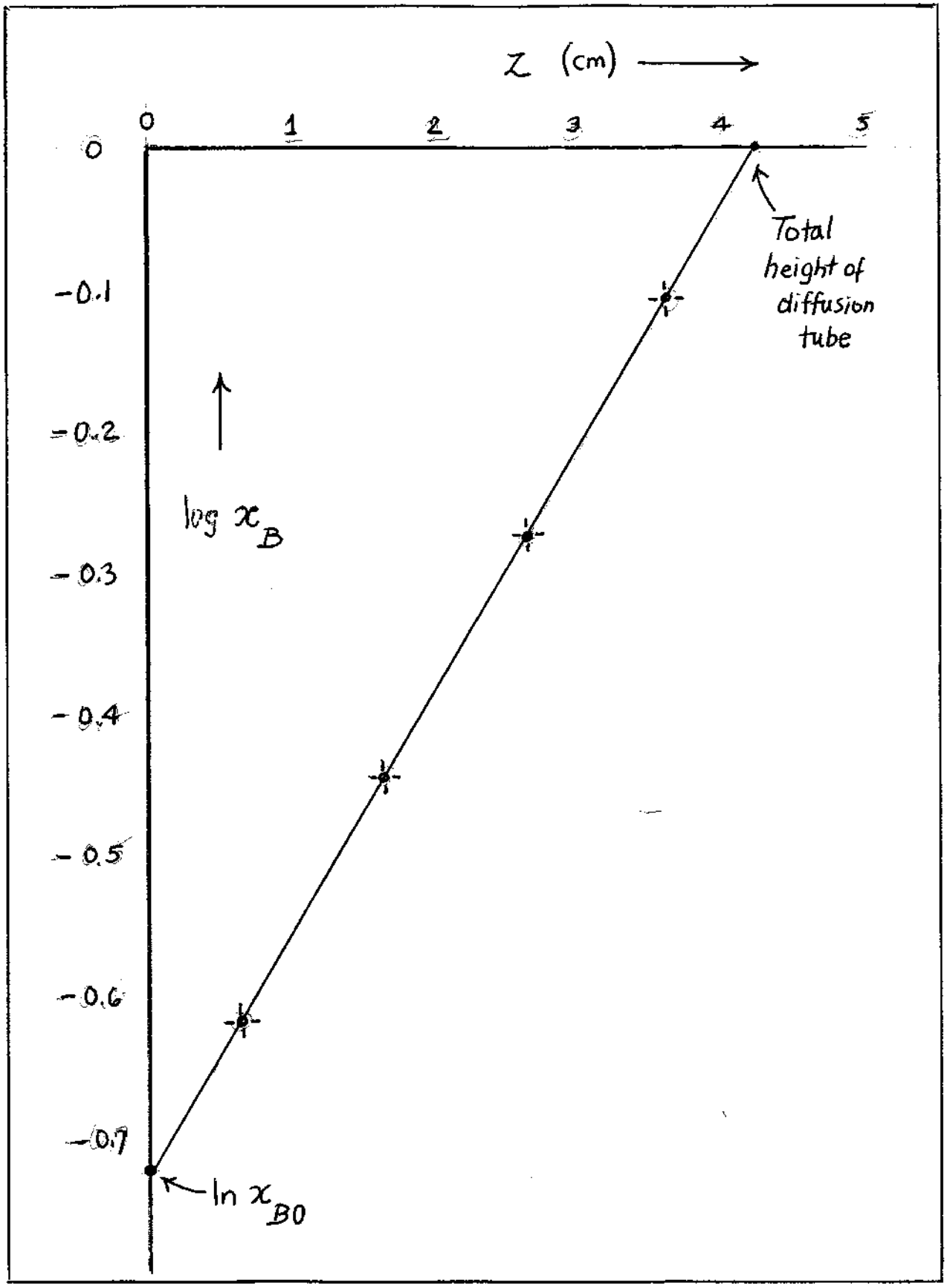
$$N_{Az} = \frac{(0.0274 \text{ g-moles/hr})}{(\pi \cdot 1^2 \text{ cm}^2)(3600 \text{ s/hr})} = 2.423 \times 10^{-6} \text{ g-moles/cm}^2\text{s}$$

The total molar concentration is obtained from the ideal gas law, $p = cRT$:

$$c = \frac{(741/760 \text{ atm})}{(82.06 \text{ cm}^3\text{atm/g-mole} \cdot \text{K})(25 + 273.2 \text{ K})}$$
$$= 3.984 \times 10^{-5} \text{ g-mole/atm}$$

Hence, the diffusivity is

$$\mathcal{D}_{AB} = \frac{N_{Az}}{(0.171)(2.303)c} = \frac{2.423 \times 10^{-6}}{(0.171)(2.303)(3.984 \times 10^{-5})} = 0.154 \text{ cm}^2/\text{s}$$


z (cm)
0
1
2
3
4
5
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
log $x_B$
Total height of diffusion tube
ln $x_{B0}$

**18B.13 Tarnishing of metal surfaces**

Fick's first law as applied to the diffusion of oxygen gas through the oxide film is

$$N_{O_2} = -\mathcal{D}_{O_2,MO_x} \frac{dc_{O_2}}{dz}$$

This is the form of Fick's law for a dilute solution of oxygen in virtually stationary metal oxide. This equation may be integrated (for a *quasi-steady-state* situation) with the boundary conditions that $c_{O_2} = c_0$ atz = 0, and $c_{O_2} = 0$ at $z = z_f(t)$ , where $z_f(t)$ is the slowly varying oxide film thickness at time $t$. The integration gives

$$N_{O_2} = \mathcal{D}_{O_2,MO_x} \frac{c_0}{z_f}$$

Next we write an *unsteady-state* mass balance for the region between $z_f(t)$ and $z_f(t+\Delta t)$ to account for the increase in oxygen content between the two planes and the oxygen addition by diffusion

$$\left(z_f(t+\Delta t) - z_f(t)\right)c_f x = N_{O_2}\Delta t$$

where $x$ is the stoichiometric coefficient in the oxidation equation, and $c_f$ is the molar density of the oxide film, $MO_x$. Dividing both sides by $\Delta t$ and letting $\Delta t$ go to zero then gives the differential equation for the location of the (slowly moving) oxide front

$$\frac{dz_f}{dt} = \frac{N_{O_2}}{c_f x} = \frac{1}{c_f x}\left(\mathcal{D}_{O_2,MO_x} \frac{c_0}{z_f}\right)$$

This equation may be integrated with respect to time, using the initial condition that $z_f = 0$ at $t = 0$. This gives Eq. 18B.13.1 as follows:

$$\frac{1}{2}z_f^2 = \frac{c_0}{c_f x}\mathcal{D}_{O_2,MO_x} t \qquad \text{or} \qquad z_f = \sqrt{\frac{2c_0}{c_f x}\mathcal{D}_{O_2,MO_x} t}$$

**18B.14 Effectiveness factors for thin disks**

If we make the assumption that the catalyst can be treated as a continuum, then a shell balance on species $A$ gives:

$$\frac{dN_{Az}}{dz} = R_A$$

Then, inserting the expression for the molar flux analogous to Eq. 18.7-4, we get for an irreversible, first-order chemical reaction

$$\frac{d}{dz}\left(-\mathcal{D}_A \frac{dc_A}{dz}\right) = -k_1''ac_A \qquad \text{or} \qquad \mathcal{D}_A \frac{d^2c_A}{dz^2} - k_1''ac_A = 0$$

In the second form, it has been assumed that the effective diffusivity is constant. This differential equation may be solved for the boundary conditions that $c_A(\pm b) = c_{As}$; the result is

$$\frac{c_A}{c_{As}} = \frac{\cosh\sqrt{k_1''a/\mathcal{D}_A}\,z}{\cosh\sqrt{k_1''a/\mathcal{D}_A}\,b} \equiv \frac{\cosh\lambda z}{\cosh\lambda b}$$

The total molar flow (or the effective reaction rate in moles/time) is

$$|W_A| = 2\cdot\pi R^2|N_{Az}|_{z=b} = 2\cdot\pi R^2\left|-\mathcal{D}_A\frac{dc_A}{dz}\right|_{z=b} = 2\cdot\pi R^2\mathcal{D}_A c_{As}\lambda\tanh\lambda b$$

For $n$ disks of thickness $b/n$, this result may be modified thus:

$|W_A^{(n)}| = n\left(2\pi R^2\mathcal{D}_A c_{As}\lambda\tanh(\lambda b/n)\right)$, and letting $n\to\infty$

$$\lim_{n\to\infty}|W_A^{(n)}| = \lim_{n\to\infty} n\left(2\pi R^2\mathcal{D}_A c_{As}\lambda\tanh(\lambda b/n)\right)$$
$$= \lim_{n\to\infty} n\left(2\pi R^2\mathcal{D}_A c_{As}\lambda\left[(\lambda b/n)+\cdots\right]\right) = 2\pi R^2\mathcal{D}_A c_{As}\lambda^2 b = 2\pi R^2 bk_1''ac_{As}$$

Then the effectiveness factor is

$$\eta_A = \lim_{n\to\infty}\left(W_A/W_A^{(n)}\right) = \frac{2\pi R^2\mathcal{D}_A c_{As}\lambda\tanh\lambda b}{2\pi R^2\mathcal{D}_A c_{As}\lambda^2 b} = \frac{\tanh\lambda b}{\lambda b}$$

**18B.15 Diffusion and heterogeneous reaction in a slender cylindrical tube with a closed end**

*a.*. The mass balance over a small segment of the tube is

$$n_{Az}\big|_z S - n_{Az}\big|_{z+\Delta z} S - P\Delta z f(\omega_{A0}) = 0$$

Division by $S\Delta z$ and letting the segment thickness go to zero gives

$$-\frac{dn_{Az}}{dz} - \frac{P}{S} f(\omega_{A0}) = 0$$

*b*. The mass-average velocity is given by

$$v_z = \omega_A v_{Az} + \omega_B v_{Bz}$$

At steady state, $B$ is not moving so that $v_{Bz}$ is zero; if substance $A$ is present only in very small concentrations, then is quite small. Hence $v_z \approx 0$.

*c*. Because of the result in (*b*), we can write the mass flux as

$$n_{Az} = -\rho \mathcal{D}_{AB} \frac{d\omega_A}{dz}$$

Then the diffusion equation becomes

$$\rho \mathcal{D}_{AB} \frac{d^2\omega_A}{dz^2} - \frac{P}{S} f(\omega_A) = 0$$

or

$$\frac{d^2\omega_A}{d\zeta^2} - \frac{PL^2}{S\rho \mathcal{D}_{AB}} k_1'' \omega_A = 0$$

Here we have set the wall mass fraction $\omega_{A0}$ equal to that in the main stream $\omega_A$ and we have introduced explicitly the first order reaction kinetics. We have also switched over to a dimensionless coordinate $\zeta = z/L$.

The differential equation is to be solved with the boundary conditions:

B. C. 1: at $\zeta = 0$, $\omega_A = \omega_{Ai}$

B. C. 2: at $\zeta = 1$, $d\omega_A/dz = 0$

The general solution is

$$\omega_A = C_1 \cosh N\zeta + C_2 \sinh N\zeta$$

in which $N = \sqrt{PL^2 k_1''/S\rho\mathcal{D}_{AB}}$. When the constants are determined from the boundary conditions we get

$$\begin{aligned}\frac{\omega_A}{\omega_{Ai}} &= \cosh N\zeta - \frac{\sinh N}{\cosh N}\sinh N\zeta \\ &= \frac{\cosh N\zeta \cosh N - \sinh N \sinh N\zeta}{\cosh N} \\ &= \frac{\cosh N(1-\zeta)}{\cosh N}.\end{aligned}$$

*d*. The mass flow rate of *A* into the capillary is

$$\begin{aligned}w_A\big|_{z=0} &= -\rho\mathcal{D}_{AB}S\frac{d\omega_A}{dz}\bigg|_{z=0} \\ &= -\frac{\rho\mathcal{D}_{AB}S}{L}\frac{d\omega_A}{d\zeta}\bigg|_{\zeta=0} \\ &= -\frac{\rho\mathcal{D}_{AB}S\omega_{Ai}}{L}\frac{(-N)\sinh N(1-\zeta)}{\cosh N}\bigg|_{\zeta=0} \\ &= \frac{\rho\mathcal{D}_{AB}S\omega_{Ai}}{L}(N\tanh N)\end{aligned}$$

**18B.16 Effects of temperature and pressure on evaporation rate**

*a*. For ideal gas behavior, $x_{A1} = p_{\text{vap},1}/p$, as pointed out several paragraphs before Eq. 18.2-1. Since the vapor pressure increases with temperature, according to the Clausius-Clapeyron equation, $x_{A1}$ will also increase. When the total pressure increases, $x_{A1}$ will decrease.

*b*. The evaporation rate is given by Eqs 18.2-14 and 15. Since, at constant temperature, $c \propto p$ and $\mathcal{D}_{AB} \propto 1/p$, the evaporation rate should be nearly independent of the imposed pressure.

*c*. The temperature dependence of the evaporation rate can be estimated as follows: $c \propto 1/T$ and (for the simple kinetic theory of rigid spheres) $\mathcal{D}_{AB} \propto T^{3/2}$. This suggests that

$$(\text{Evaporation rate at } T' : \text{Evaporation rate at } T) = (T'/T)^{1/2}$$

However, it is known that the kinetic theory of rigid spheres underestimates the temperature dependence. If one reads off the slope of the curve in Fig. 17.2-1, we estimate that

$$(\text{Evaporation rate at } T' : \text{Evaporation rate at } T) \approx (T'/T)^{3/4}$$

which is a more believable result.

**18B.17 Reaction rates in large and small particles**

*a*. Equation 18.7-11 for small $R$ becomes, on using the expansion of the hyperbolic cotangent for small arguments

$$W_{AR} = 4\pi R\mathcal{D}_A c_{AR}\left[1-\left(1+\tfrac{1}{3}(k_1''a/\mathcal{D}_A)R^2+\cdots\right)\right]$$
$$\approx -\left(\tfrac{4}{3}\pi R^3\right)(k_1''a)c_{AR}$$

where we have retained only the leading term.

For large $R$, we use the expansion of the hyperbolic cotangent for large arguments (which may be found in mathematics handbooks) to get

$$W_{AR} = 4\pi R\mathcal{D}_A c_{AR}\left[1-\left(\lambda R+2\lambda R\exp(-2\lambda R)+\cdots\right)\right]$$
$$\approx -\left(4\pi R^2\right)\sqrt{k_1''a\mathcal{D}_A}\,c_{AR}$$

where $\lambda = \sqrt{k_1''a/\mathcal{D}_A}$, and only the leading term has been retained.

Note that for small $R$ the rate of disappearance of $A$ is proportional to the volume of the catalyst, whereas for large $R$ the rate is proportional to the surface area.

*b*. Equation 18B.14-2 for small $R$ becomes, using the expansion for the hyperbolic tangent for small arguments

$$|W_A| = 2\cdot\pi R^2 c_{As}\mathcal{D}_A\lambda\left(\lambda b-\tfrac{1}{3}\lambda^3 b^3+\cdots\right)$$
$$\approx \left(2b\pi R^2\right)(k_1''a)c_{As}$$

Here, again, we let $\lambda = \sqrt{k_1''a/\mathcal{D}_A}$, and we have kept only the leading term.

For large $R$ we use the expansion for the hyperbolic tangent for large arguments to get

$$|W_A| = 2\cdot\pi R^2 c_{As}\mathcal{D}_A\lambda\left(1-2\exp(-2\lambda b)+\cdots\right)$$
$$\approx \left(2\pi R^2\right)\sqrt{k_1''a\mathcal{D}_A}\,c_{As}$$

Here again, $|W_A| \propto$ volume for small $R$, and $|W_A| \propto$ surface area for large $R$.

**18B.18 Evaporation rate for small mole fraction of the volatile liquid**

According to Eq. 18.2-13

$$\frac{1}{(x_B)_{\ln}} = \frac{\ln(x_{B2}/x_{B1})}{x_{B2}-x_{B1}} = \frac{1}{x_{A1}-x_{A2}}\ln\frac{1-x_{A2}}{1-x_{A1}} = \frac{\ln(1-x_{A2})-\ln(1-x_{A1})}{x_{A1}-x_{A2}}$$

We now wish to expand the logarithms for the situation where the mole fractions are much less than unity. To do this we may use Eq. C.2-3. This gives

$$\begin{aligned}\frac{1}{(x_B)_{\ln}} &= \frac{-\left(x_{A2}+\frac{1}{2}x_{A2}^2+\frac{1}{3}x_{A2}^3+\cdots\right)+\left(x_{A1}+\frac{1}{2}x_{A1}^2+\frac{1}{3}x_{A1}^3+\cdots\right)}{x_{A1}-x_{A2}} \\ &= \frac{(x_{A1}-x_{A2})+\frac{1}{2}\left(x_{A1}^2-x_{A2}^2\right)+\frac{1}{3}\left(x_{A1}^3-x_{A2}^3\right)+\cdots}{x_{A1}-x_{A2}} \\ &= 1+\tfrac{1}{2}(x_{A1}+x_{A2})+\tfrac{1}{3}\left(x_{A1}^2+x_{A1}x_{A2}+x_{A2}^2\right)+\cdots\end{aligned}$$

This is the expression in brackets in Eq. 18.2-16.

**18B.19 Oxygen uptake by a bacterial aggregate**

*a*. A mass balance over a thin spherical shell gives

$$\left(4\pi r^2 n_{Ar}\right)\Big|_r - \left(4\pi r^2 n_{Ar}\right)\Big|_{r+\Delta r} + \left(4\pi r^2 \Delta r\right) r_A = 0$$

in which $A$ is stands for oxygen. Dividing by $4\pi\Delta r$ and taking the limit as the spherical-shell thickness goes to zero, we get

$$-\frac{d}{dr}\left(r^2 n_{Ar}\right) = -r^2 r_A$$

Then inserting Fick's first law of diffusion and introducing the zero-order chemical reaction rate constant to describe the disappearance of the oxygen, we have

$$\mathcal{D}_{AB}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\rho_A}{dr}\right) = -r_A = k_0'''$$

The $B$ in $\mathcal{D}_{AB}$ stands for the medium through which the oxygen is diffusing. We then multiply the equation by $R^2/\rho_0\mathcal{D}_{AB}$, and define the dimensionless quantities: $\xi = r/R$, $\chi = \rho_A/\rho_0$, and $N = R^2 k_0'''/\rho_0\mathcal{D}_{AB}$. Then the diffusion equation becomes, in dimensionless form

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\chi}{d\xi}\right) = N$$

*b*. If we assume the presence of an anoxic (oxygen-free) core, then we have the following boundary conditions: at $\xi = 1$, $\chi = 1$; at $\xi = 0$, $\chi =$ finite; and at $\xi = \xi_0$, $\chi = 0$ and $d\chi/d\xi = 0$. The physical significance of the last boundary condition is that there must be no diffusion across the boundary between the anoxic core and the outer region of the aggregate. This may occur if all the material that enters across the outer surface of the aggregate is completely used up by the time it reaches the spherical surface at $\xi = \xi_0$.

*c*. Two integrations of the above equation give:

$$\xi^2 \frac{d\chi}{d\xi} = \tfrac{1}{3} N\xi^3 + C_1 \qquad \text{and} \quad \chi = \tfrac{1}{6} N\xi^2 - C_1\xi^{-1} + C_2$$

The constant $C_2$ can be found from the boundary condition at $\xi = 1$, so that

$$\chi = \tfrac{1}{6} N\left(\xi^2 - 1\right) - C_1\left(\xi^{-1} - 1\right) + 1$$

We now have to apply the two boundary conditions at $\xi = \xi_0$, in order to determine the two quantities $C_1$ and $\xi_0$ (which as yet is unspecified):

$d\chi/d\xi = 0$ at $\xi = \xi_0$: $\qquad 0 = \tfrac{1}{3} N\xi_0 + \dfrac{C_1}{\xi_0^2} \quad$ or $\quad C_1 = -\tfrac{1}{3} N\xi_0^3$

$\chi = 0$ at $\xi = \xi_0$: $\qquad 0 = \tfrac{1}{6} N\left(\xi_0^2 - 1\right) - C_1\left(\xi_0^{-1} - 1\right) + 1$

Elimination of $C_1$ between the last results gives, after rearranging

$$\xi_0^3 - \frac{3}{2}\xi_0^2 + \left(\frac{1}{2} - \frac{3}{N}\right) = 0 \qquad (***)$$

If $N$ is very large, the cubic relation has the roots 1, 1, and $-\frac{1}{2}$. The last of these is physically impossible, and the other two roots indicate that if the reaction rate is exceedingly large, there the anoxic region will occupy the entire sphere. If we set $\xi_0$ equal to zero, we find the minimum value of $N$, namely $N = 6$. Below this value, there will be no anoxic region. Thus the final expression for the concentration profile is:

$$\chi = 1 - \tfrac{1}{6} N\left(1 - \xi^2\right) + \tfrac{1}{3} N\xi_0^3\left(\xi^{-1} - 1\right) \qquad \text{for} \quad 1 \geq \xi \geq \xi_0 \geq 0$$

and $\chi = 0$ for $0 \leq \xi \leq \xi_0$. Equation (***) above gives the boundary of the anoxic region in terms of $N$.

**18C.1 Diffusion from a point source in a moving stream**

*a*. Make a mass balance (in molar units) over the ring-shaped area element shown in Fig. 18C.1:

$$\left(2\pi r\Delta r N_{Az}\right)\Big|_{z} - \left(2\pi r\Delta r N_{Az}\right)\Big|_{z+\Delta z} + \left(2\pi\Delta z r N_{Ar}\right)\Big|_{r} - + \left(2\pi\Delta z r N_{Ar}\right)\Big|_{r+r} = 0$$

Divide by $2\pi\Delta z\Delta r$ and take the limit as the dimensions of the ring-shaped element go to zero:

$$\lim_{\Delta z\to 0}\frac{\left(rN_{Az}\right)\Big|_{z+\Delta z} - \left(rN_{Az}\right)\Big|_{z}}{\Delta z} + \lim_{\Delta r\to 0}\frac{\left(rN_{Ar}\right)\Big|_{r+\Delta r} - \left(rN_{Ar}\right)\Big|_{r}}{\Delta r} = 0$$

Then using the definition of the partial derivative, we get

$$r\frac{\partial N_{Az}}{\partial z} + \frac{\partial}{\partial r}\left(r\frac{\partial N_{Ar}}{\partial r}\right) = 0 \qquad \text{or} \qquad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial N_{Ar}}{\partial r}\right) + \frac{\partial N_{Az}}{\partial z} = 0$$

We now use the expression for the molar flux given in Eq. (D) of Table 17.8-2:

$$N_{Ar} = -c\mathcal{D}_{AB}\frac{\partial x_A}{\partial r} + c_A v_r^* \approx -\mathcal{D}_{AB}\frac{\partial c_A}{\partial r}$$
$$N_{Az} = -c\mathcal{D}_{AB}\frac{\partial x_A}{\partial z} + c_A v_z^* \approx -\mathcal{D}_{AB}\frac{\partial c_A}{\partial z} + c_A v_0$$

In getting the approximate expressions, we have assumed that there is negligible convective diffusion in the $r$ direction, and that the distinction between molar and mass average velocity is unimportant in this system. These assumptions would be valid if the concentration of $A$ in the mixture is small. When these expressions are inserted into the equation for the molar flux, we get Eq. 18C.1-1.

*b*. To make the change of variable, we use $c_A(r,z) = c_A(s,z)$, and apply the chain rule of partial differentiation as follows:

$$\left(\frac{\partial c_A}{\partial z}\right)_r = \left(\frac{\partial c_A}{\partial s}\right)_z\left(\frac{\partial s}{\partial z}\right)_r + \left(\frac{\partial c_A}{\partial z}\right)_s = \left(\frac{\partial c_A}{\partial s}\right)_z\frac{z}{s} + \left(\frac{\partial c_A}{\partial z}\right)_s$$

$$\left(\frac{\partial^2 c_A}{\partial z^2}\right)_r = \frac{\partial}{\partial z}\left(\frac{\partial c_A}{\partial z}\right)_r = \left(\frac{\partial}{\partial s}\left(\frac{\partial c_A}{\partial z}\right)_r\right)_z \frac{z}{s} + \left(\frac{\partial}{\partial z}\left(\frac{\partial c_A}{\partial z}\right)_r\right)_s$$

$$= \left(\frac{\partial^2 c_A}{\partial s^2}\frac{z}{s} - \frac{\partial c_A}{\partial s}\frac{z}{s^2} + \frac{\partial^2 c_A}{\partial s \partial z}\right)\frac{z}{s} + \left(\frac{\partial^2 c_A}{\partial s \partial z}\frac{z}{s} + \frac{1}{s}\frac{\partial c_A}{\partial s} + \frac{\partial^2 c_A}{\partial z^2}\right)$$

$$\left(\frac{\partial c_A}{\partial r}\right)_z = \left(\frac{\partial c_A}{\partial s}\right)_z \left(\frac{\partial s}{\partial r}\right)_z = \left(\frac{\partial c_A}{\partial s}\right)_z \frac{r}{s}$$

$$r\left(\frac{\partial c_A}{\partial r}\right)_z = \left(\frac{\partial c_A}{\partial s}\right)_z \frac{r^2}{s} = \left(\frac{\partial c_A}{\partial s}\right)_z \frac{s^2 - z^2}{s}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial c_A}{\partial r}\right)_z\right] = \frac{1}{s}\left(\frac{\partial}{\partial s}\left[r\left(\frac{\partial c_A}{\partial r}\right)_z\right]\right)_z = \left(\frac{\partial^2 c_A}{\partial s^2}\right)\frac{s^2 - z^2}{s^2} + \frac{1}{s}\left(\frac{\partial c_A}{\partial s}\right)_z \frac{s^2 + z^2}{s^2}$$

When these are substituted into Eq. 18C.1-1, we then get Eq. 18C.1-2.

*c*. Let

$$f(s,z) = c_A \cdot \frac{4\pi \mathcal{D}_{AB}}{W_A} = \frac{1}{s}\exp\left(-\frac{v_0}{2\mathcal{D}_{AB}}(s-z)\right) \equiv \frac{1}{s}\exp(-\alpha(s-z))$$

Then

$$\left(\frac{\partial f}{\partial z}\right) = \alpha f; \qquad \left(\frac{\partial f}{\partial s}\right) = -\frac{f}{s} - \alpha f; \qquad \frac{z}{s}\left(\frac{\partial f}{\partial s}\right) = -\frac{zf}{s^2} - \frac{z}{s}\alpha f$$

$$\left(\frac{\partial^2 f}{\partial z^2}\right) = \alpha^2 f; \qquad 2\frac{z}{s}\left(\frac{\partial^2 f}{\partial s \partial z}\right) = \frac{z}{s}\left(-2\alpha\frac{f}{s} - 2\alpha^2 f\right)$$

$$\frac{1}{s^2}\frac{\partial}{\partial s}\left(s^2 \frac{\partial f}{\partial s}\right) = \alpha^2 f$$

When these expressions are substituted into Eq. 18C.1-2, an identity is obtained.

*c*. B. C. 1 is clearly satisfied. To examine B. C. 2, we first have to calculate the derivative

$$\frac{\partial c_A}{\partial s} = \frac{W_A}{4\pi \mathcal{D}_{AB}}\left(-\frac{1}{s^2}e^{-\alpha(s-z)} - \frac{\alpha}{s}e^{-\alpha(s-z)}\right)$$

Then

$$-4\pi s^2 \mathcal{D}_{AB}\frac{\partial c_A}{\partial s} = W_A e^{-\alpha(s-z)}(1-\alpha s)$$

When $s$ is made to go to zero (which implies that $z$ goes to zero as well), the right side of the above equation goes to $W_A$, and therefore the second boundary condition is satisfied.

To examine B. C. 3 we have to calculate the derivative with respect to $r$:

$$\left(\frac{\partial c_A}{\partial r}\right)_z = \left(\frac{\partial c_A}{\partial s}\right)_z \frac{r}{s} = \frac{W_A}{4\pi \mathcal{D}_{AB}} e^{-\alpha(s-z)}(1-\alpha s)\frac{r}{s}$$

where $s = \sqrt{r^2 + z^2}$. When $r \to 0$, $s$ goes to $z$, and B. C. 3 is satisfied.

The meanings of the three boundary conditions are:

B. C. 1: The concentration of $A$ on a spherical surface at infinite distance from the injection point must be zero (since $A$ is diffusing in all directions)

B. C. 2: This is a statement that $W_A$ is the injection rate of $A$.

B. C. 3: This means that the maximum in the concentration must be on the $z$ axis.

*e*. To determine the diffusivity, one can write Eq. 18C.1-3 as

$$\ln(c_A s) = \ln\left(W_A/4\pi \mathcal{D}_{AB}\right) - \left(v_0/2\mathcal{D}_{AB}\right)(s-z)$$

Hence, if $c_A s$ is plotted vs. $(s-z)$ on semi-logarithmic paper, the slope will be $-\left(v_0/2\mathcal{D}_{AB}\right)$ and the intercept will be $\left(W_A/4\pi \mathcal{D}_{AB}\right)$.

For more on this subject, see T. K. Sherwood and R. L. Pigford, *Absorption and Extraction*, McGraw-Hill, New York (1952), pp. 42-43; H. S. Carslaw and J. C. Jaeger, *Heat Conduction in Solids*, Oxford University Press, 2d ed. (1959), Eq. 2; H. A. Wilson, *Proc. Camb. Phil. Soc.*, **12**, 406 (1904).

**18C.2 Diffusion and reaction in a partially impregnated catalyst**

Use the notation of §18.7 as well as the following dimensionless quantities: $\xi = r/R$, $\Gamma = c_A/c_{AR}$, and $\phi = \sqrt{k_0'' a R^2/\mathcal{D}_A}$. Superscripts I and II indicate the two regions.

*a*. In Region I, the diffusion equation and its solution are

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\Gamma^{(\mathrm{I})}}{d\xi}\right) = 0 \quad \text{and} \quad \Gamma^{(\mathrm{I})} = -\frac{C_1^{\mathrm{I}}}{\xi} + C_2^{\mathrm{I}}$$

In Regiion II, the diffusion equation and its solution are (cf. Eqs. 18.7-6 and 9)

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\Gamma^{(\mathrm{II})}}{d\xi}\right) = \phi^2\Gamma^{(\mathrm{II})} \quad \text{and} \quad \Gamma^{(\mathrm{II})} = \frac{C_1^{\mathrm{II}}}{\xi}\cosh\phi\xi + \frac{C_2^{\mathrm{II}}}{\xi}\sinh\phi\xi$$

The boundary conditions that have to be satisfied are:

B. C. 1: at $\xi = 0$, $\Gamma^{(\mathrm{I})}$ is finite (whence $C_1^{\mathrm{I}} = 0$)

B. C. 2: at $\xi = \kappa$, $\Gamma^{(\mathrm{I})} = \Gamma^{(\mathrm{II})}$

B. C. 3: at $\xi = \kappa$, $d\Gamma^{(\mathrm{I})}/d\xi = d\Gamma^{(\mathrm{II})}/d\xi$

B. C. 4: at $\xi = 1$, $\Gamma^{(\mathrm{II})} = 1$

Boundary conditions 4 and 3 give us the following two equations:

$$1 = C_1^{\mathrm{II}}\cosh\phi + C_2^{\mathrm{II}}\sinh\phi$$

$$0 = -\frac{C_1^{\mathrm{II}}}{\kappa^2}\cosh\phi\kappa + \frac{C_1^{\mathrm{II}}}{\kappa}\phi\sinh\phi\kappa - \frac{C_2^{\mathrm{II}}}{\kappa^2}\sinh\phi\kappa + \frac{C_2^{\mathrm{II}}}{\kappa}\phi\cosh\phi\kappa$$

These two equation may be solved simultaneously to get

$$C_1^{\mathrm{II}} = \frac{-\sinh\phi\kappa + \phi\kappa\cosh\phi\kappa}{\sinh\phi(1-\kappa) - \phi\kappa\cosh\phi(1+\kappa)}$$

$$C_2^{\mathrm{II}} = \frac{\cosh\phi\kappa - \phi\kappa\sinh\phi\kappa}{\sinh\phi(1-\kappa) - \phi\kappa\cosh\phi(1+\kappa)}$$

Therefore the concentration profiles in the two regions are:

$$\Gamma^{(\mathrm{II})} = \frac{1}{\xi}\frac{\begin{bmatrix}(-\sinh\phi\kappa + \phi\kappa\cosh\phi\kappa)\cosh\phi\xi \\ +(\cosh\phi\kappa - \phi\kappa\sinh\phi\kappa)\sinh\phi\xi\end{bmatrix}}{\sinh\phi(1-\kappa) - \phi\kappa\cosh\phi(1+\kappa)}$$

$$\Gamma^{(\mathrm{I})} = \frac{1}{\xi}\frac{\phi\kappa}{\sinh\phi(1-\kappa) - \cosh\phi(1+\kappa)}$$

To get the concentration profile for region I, we used B. C. 2.

*b*. To get the molar flux at the outer surface, we need

$$W_A = -4\pi R^2 \mathcal{D}_A \frac{dc_A^{\mathrm{II}}}{dr}\bigg|_{r=R} = -4\pi R \mathcal{D}_A c_{AR} \frac{d\Gamma^{(\mathrm{II})}}{d\xi}\bigg|_{\xi=1}$$

After evaluating the derivative at the surface, we get finally

$$W_A = 4\pi R \mathcal{D}_A c_{AR}\left[1 - \phi\left(\frac{\cosh\phi(1-\kappa) + \phi\kappa\sinh\phi(1+\kappa)}{\sinh\phi(1-\kappa) - \phi\kappa\cosh\phi(1+\kappa)}\right)\right]$$

In the limit that $\kappa \to 0$, this result simplifies to Eq. 18.7-11.

**18C.3 Absorption rate in a falling film**

*a*. The total moles of $A$ transferred per unit time across the gas-liquid interface is $W_A$. This has to be equated to the amount if $A$ that is leaving in the film of finite thickness $\delta$:

$$W_A = W\int_0^\delta c_A(x,z)\big|_{z=L} v_z(x)dx$$

When it is assumed that $A$ diffuses only a very short distance into the film, then $v_z(x)$ may be set equal to the fluid velocity at the gas-liquid interface, $v_{z,\max}$, and taken outside the integral. Furthermore, since $c_A(x,z)$ is virtually zero beyond a distance small compared to $\delta$, the integration can be extended to infinity. This reasoning leads to Eq. 18C.3-1.

*b*. Inserting $c_A(x,z)$ into Eq. 18C.3-1 and changing to the variable $u$ requires no further explanation.

*c*. Changing the order of integration requires specifying the region of integration. In this case it is a triangular region extending from $u=0$ to $u=\infty$,and from the diagonal line $\xi = u$ across to $\xi = \infty$. When the order of integration is reversed, it is necessary to integrate over exactly the same region, but this time from $\xi = 0$ to $\xi = \infty$ and from $u=0$ up to the diagonal $u=\xi$. This leads to Eq. 18C.3-3.

Having done this, the inside integral can be performed analytically to give

$$W_A = WLc_{A0}\sqrt{\frac{4\mathcal{D}_{AB}v_{z,\max}}{\pi L}}\cdot 2\int_0^\infty \xi\exp(-\xi^2)d\xi$$

Now the remaining integral can be done also to give

$$= WLc_{A0}\sqrt{\frac{4\mathcal{D}_{AB}v_{z,\max}}{\pi L}}\left(-\exp(-\xi^2)\right)\Big|_0^\infty = WLc_{A0}\sqrt{\frac{4\mathcal{D}_{AB}v_{z,\max}}{\pi L}}$$

which is in agreement with Eq. 18.5-18.

**18C.4 Estimation of the required length of an isothermal reactor**

*a*. The steady state mass balance over a length $\Delta l$ of the reactor is

$$w\omega_{A0}\big|_{l} - w\omega_{A0}\big|_{l+\Delta l} - (S\Delta l)an_A = 0$$

Dividing by $w\Delta l$ and letting $\Delta l$ go to zero gives

$$\frac{d\omega_{A0}}{dl} = -\frac{San_A}{w}$$

or

$$\frac{d\omega_{A0}}{dl} = \frac{SaN_A M_A}{w}$$

*b*. Next we want to use the result of Eq. 18.3-9 to write

$$\frac{M_A M_B (dx_{A0}/dl)}{(M_A x_{A0} + M_B x_{B0})^2} = \frac{SaM_A}{w}\left(\frac{2c\mathcal{D}_{AB}}{\delta}\ln\frac{1}{1-\frac{1}{2}x_{A0}}\right)$$

Then integration gives

$$\left(\frac{M_B w\delta}{2Sac\mathcal{D}_{AB}}\right)\int_{x_A(0)}^{x_A(L)}\frac{dx_{A0}}{\left[M_A x_{A0} + M_B(1-x_{A0})\right]^2 \ln\left(1-\frac{1}{2}x_{A0}\right)} = L$$

**18C.5 Steady-state evaporation**

*a*, We start with the second form of Eq. 17.9-1, written for diffusion of a 3-component mixture in the $z$ direction. For convenience, we omit the index $z$ on the species molar fluxes:

$$\frac{dx_\alpha}{dz} = \sum_{\beta=1}^{3} \frac{1}{c\mathfrak{D}_{\alpha\beta}} \left( x_\alpha N_\beta - x_\beta N_\alpha \right)$$

Consider now the concentration gradient of species 3, taking into account that $N_1$ and $N_2$ are constants, and that $N_3$ is zero:

$$\frac{dx_3}{dz} = \frac{1}{c\mathfrak{D}_{13}} (x_3 N_1) + \frac{1}{c\mathfrak{D}_{23}} (x_3 N_2) = \left( \frac{N_1}{c\mathfrak{D}_{13}} + \frac{N_2}{c\mathfrak{D}_{23}} \right) x_3$$

This first-order, separable differential equation is easily integrated to give the result in Eq. 18C.5-1:

$$x_3 = x_{30} \exp(\nu_{113} + \nu_{223})\zeta \equiv x_{30} \exp A\zeta$$

in which the notation $\nu_{\alpha\beta\gamma}$ is a symbol defined in the textbooi.

*b* . Next we consider the concentration gradient for species 2:

$$\frac{dx_2}{dz} = \frac{1}{c\mathfrak{D}_{12}} (x_2 N_1 - x_1 N_2) + \frac{1}{c\mathfrak{D}_{23}} (x_2 N_3 - x_3 N_2)$$

or, in dimensionless form,

$$\begin{aligned} \frac{dx_2}{d\zeta} &= (\nu_{112} x_2 - \nu_{212} x_1) - \nu_{223} x_3 = \nu_{112} x_2 - \nu_{212}(1 - x_2 - x_3) - \nu_{223} x_3 \\ &= (\nu_{112} + \nu_{212}) x_2 + (\nu_{212} - \nu_{223}) x_3 - \nu_{212} \end{aligned}$$

When we use the abbreviations given in the text, this equation can be recast in the form

$$\frac{dx_2}{d\zeta} - (\nu_{112} + \nu_{212}) x_2 = (\nu_{212} - \nu_{223}) x_{30} \exp(\nu_{113} + \nu_{223})\zeta - \nu_{212} \qquad \text{or}$$

$$\frac{dx_2}{d\zeta} - Bx_2 = Cx_{30} \exp A\zeta - v_{212}$$

This is a first-order linear equation, which is readily solved

$$x_2 = e^{B\zeta}\left[\int_0^{\zeta} e^{-B\zeta}\left(Cx_{30}e^{A\zeta} - v_{212}\right)\right]d\zeta + C_1 \qquad (C_1 = x_{20} \text{ from B. C.})$$

$$= e^{B\zeta}\left[Cx_{30}\frac{e^{(A-B)\zeta} - 1}{(A-B)} - v_{212}\frac{e^{-B\zeta} - 1}{-B} + x_{20}\right]$$

$$= \frac{Cx_{30}}{A-B}e^{A\zeta} + \left(x_{20} - \frac{Cx_{30}}{A-B} - \frac{v_{212}}{B}\right)e^{B\zeta} + \frac{v_{212}}{B}$$

in which

$$\frac{v_{212}}{B} = \frac{N_2}{N_1 + N_2}; \quad e^{B\zeta} = \exp\left[\frac{(N_1 + N_2)z}{c\mathfrak{D}_{12}}\right]; \quad e^{A\zeta} = \exp\left[\left(\frac{N_1}{c\mathfrak{D}_{13}} + \frac{N_2}{c\mathfrak{D}_{23}}\right)z\right]$$

$$\frac{C}{A-B} = \frac{N_2\left(\dfrac{\mathfrak{D}_{23} - \mathfrak{D}_{12}}{\mathfrak{D}_{23}}\right)}{N_1\left(\dfrac{\mathfrak{D}_{12} - \mathfrak{D}_{13}}{\mathfrak{D}_{13}}\right) + N_2\left(\dfrac{\mathfrak{D}_{12} - \mathfrak{D}_{23}}{\mathfrak{D}_{23}}\right)}$$

*c*. The results in (*a*) and (*b*) above are in agreement with Eqs. (2) and (3) of the article by Carty and Schrodt, cited in the text.

*d*. Figure 3 in the article by Carty and Schrodt indicates good agreement between the above equations and the experimental data.

**18D.1 Effectiveness factors for long cylinders**

Instead of Eq. 18.7-6 (for spheres) we have now (for cylinders):

$$\mathfrak{D}_A \frac{1}{r}\frac{d}{dr}\left(r\frac{dc_A}{dr}\right) = k_1'' a c_A \qquad \text{with } c_A(R) = c_{AR} \text{ and } c_A(0) = \text{finite}$$

The differential equation may also be put into the form

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dc_A}{dr}\right) - \left(\frac{k_1'' a}{\mathfrak{D}_A}\right)c_A = 0$$

This may be recognized as the equation for a first-order modified Bessel function. Therefore, the solution is

$$c_A = C_1 I_0\left(\sqrt{\frac{k_1'' a}{\mathfrak{D}_A}}\,r\right) + C_2 K_0\left(\sqrt{\frac{k_1'' a}{\mathfrak{D}_A}}\,r\right)$$

Inasmuch as $K_0$ cannot satisfy the boundary condition at $r = 0$, we must set $C_2$ equal to zero. The boundary condition at $r = R$ then gives the constant $C_1$. The concentration profile is therefore

$$\frac{c_A}{c_{AR}} = \frac{I_0\left(\sqrt{k_1'' a/\mathfrak{D}_A}\,r\right)}{I_0\left(\sqrt{k_1'' a/\mathfrak{D}_A}\,R\right)}$$

This is the cylindrical analog of Eq. 18.7-9.

The molar flow of $A$ at the surface is then

$$W_{AR} = 2\pi RL\left(-\mathfrak{D}_A \frac{dc_A}{dr}\right)\Bigg|_{r=R}$$
$$= -\frac{(2\pi RL)\mathfrak{D}_A c_{AR}}{I_0\left(\sqrt{k_1'' a/\mathfrak{D}_A}\,R\right)} I_1\left(\sqrt{k_1'' a/\mathfrak{D}_A}\,R\right)\cdot\sqrt{k_1'' a/\mathfrak{D}_A}$$

which is the cylindrical analog of Eq. 18.7-11. The analog of Eq. 18.7-12 is then

$$W_{AR,0} = \left(\pi R^2 L\right)(a)\left(-k_1'' c_A\right)$$

Next we evaluate the effectiveness factor

$$\eta_A = \frac{W_{AR}}{W_{AR,0}} = \frac{2I_1\left(\sqrt{k_1'' a/\mathfrak{D}_A}\,R\right)}{\left(\sqrt{k_1'' a/\mathfrak{D}_A}\,R\right)I_0\left(\sqrt{k_1'' a/\mathfrak{D}_A}\,R\right)}$$

The generalized modulus for the cylinder is

$$\Lambda = \sqrt{\frac{k_1'' a}{\mathfrak{D}_A}}\frac{V_P}{S_P} = \sqrt{\frac{k_1'' a}{\mathfrak{D}_A}}\frac{\pi R^2 L}{2\pi R L} = \sqrt{\frac{k_1'' a}{\mathfrak{D}_A}}\frac{R}{2}$$

Therefore the argument of the Bessel functions is

$$\sqrt{k_1'' a/\mathfrak{D}_A}\,R = 2\Lambda$$

We may now express the effectiveness factor in terms of the generalized modulus thus:

$$\eta_A = \frac{I_1(2\Lambda)}{\Lambda I_0(2\Lambda)}$$

which is the desired result.

**18D.2 Gas absorption in a falling film with chemical reaction**

The problem to be solved is

$$v_{\max}\frac{\partial c_A}{\partial z} = \mathcal{D}_{AB}\frac{\partial^2 c_A}{\partial x^2} - k_1'' c_A$$

with boundary conditions $c_A(x,0)=0$, $c_A(0,z)=c_{A0}$, and $c_A(\infty,z)=0$, or, in dimensionless notation

$$\frac{\partial c}{\partial \zeta} = \frac{\partial^2 c}{\partial \xi^2} - ac \qquad \text{with } c(\xi,0)=0,\ c(0,\zeta)=1,\ c(\infty,\zeta)=0$$

in which $c = c_A/c_{A0}$, $\zeta = z/L$, $\xi = x\sqrt{v_{\max}/\mathcal{D}_{AB}L}$, and $a = k_1''L/v_{\max}$.

We now take the Laplace transform the partial differential equation and the boundary conditions to get

$$s\bar{c} - c(\xi,0) = \frac{d^2\bar{c}}{d\xi^2} - a\bar{c} \qquad \text{with} \qquad \bar{c}(0)=1/s,\ \bar{c}(\infty)=0$$

Because of the boundary condition at $\zeta = 0$, $c(\xi,0) = 0$. The solution of the ordinary differential equation for $\bar{c}$ with its boundary conditions is

$$\bar{c} = \frac{1}{s}\exp\left(-\sqrt{(s+a)}\xi\right)$$

We will not invert this, since all we need is the molar flux at the wall,

$$N_{Ax}(z)\big|_{x=0} = -\mathcal{D}_{AB}\frac{\partial c_A}{\partial x}\bigg|_{x=0} = -\mathcal{D}_{AB}c_{A0}\sqrt{\frac{v_{\max}}{\mathcal{D}_{AB}L}}\frac{dc}{d\xi}\bigg|_{\xi=0}$$

The Laplace transform of this expression is

$$\overline{N_{Ax}(z)\big|_{x=0}} = -\mathcal{D}_{AB}c_{A0}\sqrt{\frac{v_{\max}}{\mathcal{D}_{AB}L}}\frac{d\bar{c}}{d\xi}\bigg|_{\xi=0} = +\mathcal{D}_{AB}c_{A0}\sqrt{\frac{v_{\max}}{\mathcal{D}_{AB}L}}\frac{\sqrt{s+a}}{s}$$

Hence the molar flux at the wall is

$$
\begin{aligned}
N_{Ax}(z)\big|_{x=0} &= +\mathfrak{D}_{AB}c_{A0}\sqrt{\frac{v_{\max}}{\mathfrak{D}_{AB}L}}\,\mathfrak{L}^{-1}\left\{\frac{\sqrt{s+a}}{s}\right\} \\
&= +\mathfrak{D}_{AB}c_{A0}\sqrt{\frac{v_{\max}}{\mathfrak{D}_{AB}L}}\,\mathfrak{L}^{-1}\left\{\frac{s+a}{s\sqrt{s+a}}\right\} \\
&= +\mathfrak{D}_{AB}c_{A0}\sqrt{\frac{v_{\max}}{\mathfrak{D}_{AB}L}}\left[\mathfrak{L}^{-1}\left\{\frac{1}{\sqrt{s+a}}\right\}+a\mathfrak{L}^{-1}\left\{\frac{1}{s}\frac{1}{\sqrt{s+a}}\right\}\right] \\
&= +\mathfrak{D}_{AB}c_{A0}\sqrt{\frac{v_{\max}}{\mathfrak{D}_{AB}L}}\left(\frac{\exp(-a\zeta)}{\sqrt{\pi\zeta}}+\int_0^{\zeta}\frac{\exp\left(-a\bar{\zeta}\right)}{\sqrt{\pi\bar{\zeta}}}d\bar{\zeta}\right)
\end{aligned}
$$

The first transform was obtained from a table of transforms, and the second by using the convolution theorem.

The total molar flow rate through the interface is then

$$
\begin{aligned}
W_A &= W\int_0^L N_{Ax}(z)\big|_{x=0}dz \\
&= WL\mathfrak{D}_{AB}c_{A0}\sqrt{\frac{v_{\max}}{\mathfrak{D}_{AB}L}}\int_0^1\left(\frac{\exp(-a\zeta)}{\sqrt{\pi\zeta}}+\int_0^{\zeta}\frac{\exp\left(-a\bar{\zeta}\right)}{\sqrt{\pi\bar{\zeta}}}d\bar{\zeta}\right)d\zeta \\
&= WL\mathfrak{D}_{AB}c_{A0}\sqrt{\frac{v_{\max}}{\mathfrak{D}_{AB}L}}\left[\frac{\operatorname{erf}\sqrt{a}}{\sqrt{a}}+a\left(\frac{\operatorname{erf}\sqrt{a}}{\sqrt{a}}+\frac{\exp(-a)}{\sqrt{\pi a}}-\frac{\operatorname{erf}\sqrt{a}}{2a\sqrt{a}}\right)\right] \\
&= Wc_{A0}v_{\max}\sqrt{\mathfrak{D}_{AB}/k_1''}\left[\left(\tfrac{1}{2}+a\right)\operatorname{erf}\sqrt{a}+\sqrt{a/\pi}\exp(-a)\right]
\end{aligned}
$$

The double integral in the second line is evaluated by exchanging the order of integration and then performing the inner integral. In going from the third to the fourth line, $\sqrt{a}$ was factored out of the denominator.

**19A.1 Dehumidification of air.**

**(a)** Let A denote $H_2O$ and B denote air, as in Example 19.4-1. The interfacial mole fraction of A is then estimated as

$$x_{A0} = \frac{p_{H_2O,\text{vap}}}{p} = \frac{0.178 \text{ psia}}{14.696 \text{ psia}} = 0.0121$$

The "film temperature" $T_f = \frac{1}{2}(T_0 + T_\delta)$ is 65°F= 291.48 K. The gas-phase properties at $T_f$ and $x_B \to 1.0$ are:

$$\mathcal{D}_{AB} = 0.246 \text{ cm}^2/\text{s from Eq. 17.2-1}$$
$$c = p/RT = 4.18 \times 10^{-5} \text{ g-mol/cm}^3$$
$$k = 25.5 \times 10^{-5} \text{ W/cm·K from } \textit{CRC Handbook 2000-2001}\text{, p. 6-185}$$

The molar heat capacity of the transferred vapor at $T_f$ is

$$\widetilde{C}_{pA} = 8.00 \text{ cal/g-mol·K} = 33.47 \text{ J/g-mol·K}$$

from O. A. Hougen and K. M. Watson, CPP Charts, Wiley, New York (1943), Fig. 26.

Substituting into Eq. 19.4-5 we get:

$$\frac{N_{Ay}\widetilde{C}_{pA}\delta}{k} = \left(\frac{N_{Ay}\delta}{c\mathcal{D}_{AB}}\right)\left(\frac{c\mathcal{D}_{AB}\widetilde{C}_{pA}}{k}\right) = \left(\ln\frac{1-x_{A\delta}}{1-x_{A0}}\right)\left(\frac{c\mathcal{D}_{AB}\widetilde{C}_{pA}}{k}\right)$$
$$= \left(\frac{1-0.0180}{1-0.0121}\right)$$
$$\cdot\left(\frac{0.0000418 \times 0.246 \text{ g-mol/cm·s})(33.47 \text{ J/g-mol·K})}{25.5 \times 10^{-5} \text{ W/cm·K}}\right)$$
$$= (-0.00599)(1.350) = -0.0081$$

Then the right-hand member of Eq. 19.4-9 takes the value

$$\frac{-(N_{Ay}\widetilde{C}_{pA}/k)\delta}{1-\exp(N_{Ay}\widetilde{C}_{pA}/k)} = \frac{0.0081}{1-\exp(-0.0081)} = \frac{0.0081}{1-\left[1-0.0081+\frac{1}{2}(0.0081)^2+\ldots\right]}$$
$$= \frac{1}{1-\frac{1}{2}(0.0081)+\ldots} = 1.004\ldots$$

**(b)** From Eq. 19.4-9 we see that the quotient just calculated is the ratio of the interfacial conduction fluxes calculated with and without allowance for diffusive energy flux. The diffusive energy flux is evidently unimportant in this problem.

**19B.1 Steady-state evaporation**

*a*. From Eq. (M) of Table 17.8-1 and the fact that $B$ is stagnant leads to $N_{Az} = cv_z^*$. Equation (D) of Table 17.8-2 states that $N_{Az} = c_A v_z^* - c\mathcal{D}_{AB}(dx_A/dz)$. Equating the right sides of these two expressions and dividing by $c$ gives

$$v_z^* = x_A v_z^* - \mathcal{D}_{AB}\frac{dx_A}{dz} \quad \text{or} \quad v_z^* = -\frac{\mathcal{D}_{AB}}{1-x_A}\frac{dx_A}{dz}$$

*b*. When this expression for the molar average velocity is combined with Eq. 19.1-17 (when simplified for steady state and unidirectional diffusion with no chemical reactions), we get

$$-\frac{\mathcal{D}_{AB}}{1-x_A}\frac{dx_A}{dz}\cdot\frac{dx_A}{dz} = \mathcal{D}_{AB}\frac{d^2x_A}{dz^2} \qquad \text{or} \qquad \frac{d^2x_A}{dz^2} + \frac{1}{1-x_A}\left(\frac{dx_A}{dz}\right)^2 = 0$$

which is just Eq. 19B.1-1.

*c*. Equation 19B.1-1 can also be written as (cf. Eq. 18.2-5)

$$(1-x_A)\frac{d}{dz}\left(\frac{1}{1-x_A}\frac{dx_A}{dz}\right) = 0$$

One integration of this equation gives (cf. Eq. 18.2-6)

$$\frac{1}{1-x_A}\frac{dx_A}{dz} = C_1$$

and the second integration gives (cf. Eqs. 18.2-7 and 8)

$$-\ln(1-x_A) = C_1 z + C_2$$

Then one can follow the text in §18.2 until Eq. 18.2-11 is obtained.

**19B.2 Gas absorption with chemical reaction**

Equation 19.1-16 is

$$\rho\left(\frac{\partial \omega_A}{\partial t} + \mathbf{v}\cdot\nabla\omega_A\right) = \rho\mathcal{D}_{AB}\nabla^2\omega_A + r_A$$

For constant mass density, $\rho$ can be taken inside the time and space derivatives. This gives:

$$\left(\frac{\partial \rho_A}{\partial t} + \mathbf{v}\cdot\nabla\rho_A\right) = \mathcal{D}_{AB}\nabla^2\rho_A + r_A$$

We now divide by the molecular weight of species $A$ to get

$$\left(\frac{\partial c_A}{\partial t} + \mathbf{v}\cdot\nabla c_A\right) = \mathcal{D}_{AB}\nabla^2 c_A + R_A$$

For steady-state diffusion, the time-derivative term can be omitted. Since the bulk motion in the direction of diffusion is presumed to be small, the $\mathbf{v}$ term can be omitted. Then we further simplify the equation to diffusion in the z direction. This gives

$$0 = \mathcal{D}_{AB}\frac{d^2 c_A}{dz^2} + R_A$$

where $R_A$ is the molar rate of production of species $A$ per unit volume. If $A$ is disappearing by an irreversible, first-order reaction, then $R_A = -k_1''' c_A$, so that we arrive at

$$0 = \mathcal{D}_{AB}\frac{d^2 c_A}{dz^2} - k_1''' c_A$$

which is just Eq. 18.4-4.

**19B.3 Concentration-dependent diffusivity**

*a.*. For this problem the diffusion equation simplifies to

$$0 = \frac{d}{dz}\left( \mathcal{D}_{AB}(c_A)\frac{dc_A}{dz} \right)$$

*b*. One integration of this differential equation gives

$$\mathcal{D}_{AB}(c_A)\frac{dc_A}{dz} = C_1$$

and a second integration gives

$$\int_{c_{A0}}^{c_A} \mathcal{D}_{AB}(c'_A)dc'_A = C_1\int_0^z dz$$

The integration constant may be obtained by applying the boundary condition at $z = b$:

$$\int_{c_{A0}}^{c_{Ab}} \mathcal{D}_{AB}(c'_A)dc'_A = C_1\int_0^b dz$$

Taking the quotient of these two equations gives

$$\frac{\int_{c_{A0}}^{c_A} \mathcal{D}_{AB}(c'_A)dc'_A}{\int_{c_{A0}}^{c_{Ab}} \mathcal{D}_{AB}(c'_A)dc'_A} = \frac{z}{b}$$

which is equivalent to Eq. 19B.3-1.

*c*. The molar flux at the solid-liquid interface is then

$$N_{Az}\big|_{z=0} = -\mathcal{D}_{AB}\frac{dc_A}{dz}\bigg|_{z=0} = -\mathcal{D}_{AB}\left( \frac{1}{b\mathcal{D}_{AB}}\int_{c_{A0}}^{c_{Ab}} \mathcal{D}_{AB}(c'_A)dc'_A \right)$$

which is equivalent to Eq. 19B.3-2. The quantity in parentheses is the concentration gradient, obtained by differentiating the result in (*b*) with respect to z using the Leibniz formula.

*d*. When Eq. 19B.3-3 is inserted into Eq. 19B.3-2, we get

$$N_{Az}\big|_{z=0} = \frac{\overline{\mathfrak{D}}_{AB}}{b}\int_{c_{Ab}}^{c_{A0}}\left[1+\beta_1\left(c_A-\bar{c}_A\right)+\beta_2\left(c_A-\bar{c}_A\right)^2+\cdots\right]dc_A$$
$$= \frac{\overline{\mathfrak{D}}_{AB}}{b}\left[\left(c_{A0}-c_{Ab}\right)+\tfrac{1}{2}\beta_1\left(c_A-\bar{c}_A\right)^2\Big|_{c_{Ab}}^{c_{A0}}+\tfrac{1}{3}\beta_2\left(c_A-\bar{c}_A\right)^3\Big|_{c_{Ab}}^{c_{A0}}+\cdots\right]$$

In the second term on the right side we have

$$\left(c_A-\bar{c}_A\right)^2\Big|_{c_{Ab}}^{c_{A0}} = \left(c_{A0}^2-2c_{A0}\bar{c}_A+\bar{c}_A^2\right)-\left(c_{Ab}^2-2c_{Ab}\bar{c}_A+\bar{c}_A^2\right)$$
$$=\left(c_{A0}^2-c_{Ab}^2\right)-2\left(c_{A0}-c_{Ab}\right)\bar{c}_A$$
$$=\left(c_{A0}^2-c_{Ab}^2\right)-2\left(c_{A0}-c_{Ab}\right)\tfrac{1}{2}\left(c_{A0}+c_{Ab}\right)=0$$

In the third term on the right side we have

$$\left(c_A-\bar{c}_A\right)^3\Big|_{c_{Ab}}^{c_{A0}} = \left(c_{A0}^3-3c_{A0}^2\bar{c}_A+3c_{A0}\bar{c}_A^2-\bar{c}_A^3\right)$$
$$-\left(c_{Ab}^3-3c_{Ab}^2\bar{c}_A+3c_{Ab}\bar{c}_A^2-\bar{c}_A^3\right)$$
$$=\left(c_{A0}^3-c_{Ab}^3\right)+3\left(c_{A0}^2-c_{Ab}^2\right)\bar{c}_A+3\left(c_{A0}-c_{Ab}\right)\bar{c}_A^2$$
$$=\left(c_{A0}-c_{Ab}\right)\left[\left(c_{A0}^2+c_{A0}c_{Ab}+c_{Ab}^2\right)-3\left(c_{A0}+c_{Ab}\right)\tfrac{1}{2}\left(c_{A0}+c_{Ab}\right)\right.$$
$$\left.+3\left(\tfrac{1}{2}\right)^2\left(c_{A0}+c_{Ab}\right)^2\right]$$
$$=\left(c_{A0}-c_{Ab}\right)\left[\tfrac{1}{4}c_{A0}^2-\tfrac{1}{2}c_{A0}c_{Ab}+\tfrac{1}{4}c_{Ab}^2\right]$$
$$=\tfrac{1}{4}\left(c_{A0}-c_{Ab}\right)\left(c_{A0}-c_{Ab}\right)^2$$

When this is substituted into the molar-flux expression above, we get the result in Eq. 19B.3-4.

*e*. If the diffusivity is linear in the concentration, so that the terms in Eq. 19B.3-3 containing terms higher than the quadratic term may be omitted, then the result in Eq. 19.3-4 is valid, but the expression in brackets is just unity. This means that one gets a valid expression for the mass flux by using the formula for constant diffusivity, but using the diffusivity at the average concentration.

**19B.4 Oxidation of silicon**

*a*. Solution of Fick's second law (Eq. 19.1-18) for unidirectional diffusion with the quasi-steady-state assumption gives $c_A(z) = C_1 z + C_2$. The integration constants are determined from the boundary conditions that $c_A(0) = c_{A0}$ and $c_A(\delta) = c_{A\delta}$. This gives Eq. 19B.4-1.

*b*. The unsteady molar balance on oxygen gives

$$\frac{d}{dt}\int_{V(t)} c_A(z,t)dV = S\left(-\mathcal{D}_{AB}\frac{dc_A}{dz}\bigg|_{z=0} - k_1'' c_{A\delta}\right)$$

The left side is the time rate of change of total number of moles of oxygen within the region. The first term on the right side is the rate at which oxygen enters an area $S$ of the gas-solid interface, and the second term is the rate at which oxygen leaves the system through an area of the silicon-silicon-dioxide interface, by virtue of surface reaction. The left side can be rewritten (using the Leibniz formula) as

$$\int_{V(t)} \frac{\partial c_A}{\partial t}dV + \int_{S(t)} (\mathbf{n}\cdot\mathbf{v}_S)c_A dS \approx c_{A\delta} S\frac{d\delta}{dt}$$

The first term on the left can be neglected, because of the quasi-steady-state assumption. When the right side of this expression and the right side of the previous equation are equated (and divided through by $S$) we getEq. 19B.4-2.

*c*. The unsteady molar balance on the silicon dioxide gives

$$\frac{d}{dt}\int_{V(t)} c_B(z,t)dV = S(k_1'' c_{A\delta})$$

This states that the mass of silicon oxide increases because of the surface reaction of oxygen with silicon and also because of the change in volume of the silicon oxide region. Once again the Leibniz formula is used to rewrite the left side as

$$\int_{V(t)} \frac{\partial c_B}{\partial t}dV + \int_{S(t)} (\mathbf{n}\cdot\mathbf{v}_S)c_B dS \approx c_{B\delta} S\frac{d\delta}{dt} \approx \frac{1}{\tilde{V}_B} S\frac{d\delta}{dt}$$

where $\tilde{V}_B$ is the molar volume of the silicon dioxide. Equating the right sides of the last two equations gives Eq. 19B.4-3.

*d*. Following the directions given in the text we get

$$\tilde{V}_B k_1'' c_{A\delta}^2 = -\mathfrak{D}_{AB}\left(-\frac{c_{A0} - c_{A\delta}}{\delta}\right) - k_1'' c_{A\delta}$$

and this may be rearranged to give Eq. 19B.4-4.

*e*. The differential equation for the movement of the boundary is

$$k_1''\left[\frac{c_{A0}}{1 + \left(k_1''\delta/\mathfrak{D}_{AB}\right)}\right] = \frac{1}{\tilde{V}_B}\frac{d\delta}{dt}$$

or

$$\tilde{V}_B c_{A0} k_1'' = \left(1 + \frac{k_1''\delta}{\mathfrak{D}_{AB}}\right)\frac{d\delta}{dt}$$

This is a separable first-order differential equation, which may be solved with the initial condition that $\delta(0) = 0$:

$$\tilde{V}_B c_{A0} k_1'' \int_0^t dt = \int_0^\delta \left(1 + \frac{k_1''\delta}{\mathfrak{D}_{AB}}\right) d\delta$$

and this yields

$$\tilde{V}_B c_{A0} k_1'' t = \delta + \frac{k_1''\delta^2}{2\mathfrak{D}_{AB}}$$

which may be rearranged to give Eq. 19B.4-5.

**19B.5 The Maxwell-Stefan equations for multicomponent gas mixtures**

*a*. If we start with the first form of the Maxwell-Stefan equations, we have

$$\nabla x_A = -\frac{x_A x_b}{\mathcal{D}_{AB}}(\mathbf{v}_A - \mathbf{v}_B)$$

From this, we get, by rearranging

$$c(\mathbf{v}_A - \mathbf{v}_B) = -\frac{c\mathcal{D}_{AB}}{x_A x_b}\nabla x_A$$

which is just Eq. (F) of Table 17.8-2.

*b*. If we start with the second form of the Maxwell-Stefan equation we have

$$\nabla x_A = -\frac{1}{c\mathcal{D}_{AB}}(x_B\mathbf{N}_A - x_A\mathbf{N}_B) \qquad \text{or} \qquad x_B\mathbf{N}_A - x_A\mathbf{N}_B = -c\mathcal{D}_{AB}\nabla x_A$$

If now we add and subtract $x_A\mathbf{N}_A$ on the left side, we get

$$(x_A + x_B)\mathbf{N}_A - x_A(\mathbf{N}_A + \mathbf{N}_B) = -c\mathcal{D}_{AB}\nabla x_A$$

or

$$\mathbf{N}_A = x_A(\mathbf{N}_A + \mathbf{N}_B) - c\mathcal{D}_{AB}\nabla x_A$$

which is just Eq. (D) of Table 17.8-2.

**19B.6 Diffusion and chemical reaction in a liquid**

*a*. The differential equation for the steady-state diffusion from a sphere is

$$\mathcal{D}_{AB}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dc_A}{dr}\right)-k_1'''c_A=0 \qquad \text{or} \qquad \frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\Gamma}{d\xi}\right)-b^2\Gamma=0$$

According to Eq. C.1-6b, the solution to this equation is

$$\Gamma=\frac{C_1}{\xi}e^{-b\xi}+\frac{C_2}{\xi}e^{b\xi}$$

Application of the boundary conditions that $\Gamma(1)=1$ and $\Gamma(\infty)=0$ determines the constants, and the final result is (cf. Eq. 19B.6-1)

$$\Gamma=\frac{1}{\xi}\frac{e^{-b\xi}}{e^{-b}}$$

The mass flux at the sphere surface is then

$$N_{Ar}\big|_{r=R}=-\mathcal{D}_{AB}\frac{dc_A}{dr}\bigg|_{r=R}=-\frac{\mathcal{D}_{AB}c_{A0}}{R}\frac{d\Gamma}{d\xi}\bigg|_{\xi=1}=+\frac{\mathcal{D}_{AB}c_{A0}}{R}(1+b)$$

and the total loss of $A$ from the sphere in moles per unit time is

$$W_A=4\pi R^2\left(\frac{\mathcal{D}_{AB}c_{A0}}{R}\right)\left(1+\sqrt{\frac{k_1'''R^2}{\mathcal{D}_{AB}}}\right)$$

*b*. The unsteady-state mass balance on the dissolving sphere of species $A$ is

$$-\frac{d}{dt}\left(\tfrac{4}{3}\pi R^3\rho_{\text{sph}}\right)=4\pi R^2\left(\frac{\mathcal{D}_{AB}c_{A0}M_A}{R}\right)\left(1+\sqrt{\frac{k_1'''R^2}{\mathcal{D}_{AB}}}\right)$$

Since we have used here the steady-state expression for the molar flux at the surface of the sphere, this is a quasi-steady-state treatment.

The integration of this equation can be accomplished as follows: First we divide through by $4\pi\rho_{\text{sph}}$ to get

$$-R^2\frac{dR}{dt}=\frac{\mathcal{D}_{AB}c_{A0}M_A}{\rho_{\text{sph}}}R\left(1+\sqrt{\frac{k_1'''R^2}{\mathcal{D}_{AB}}}\right)$$

Next, put all the factors containing $R$ on the left side

$$-\frac{R}{1+\left(k_1'''R^2/\mathcal{D}_{AB}\right)}\frac{dR}{dt}=\frac{\mathcal{D}_{AB}c_{A0}M_A}{\rho_{\text{sph}}}$$

Next we introduce a new dimensionless variable $Y=R\sqrt{k_1'''/\mathcal{D}_{AB}}$ and write

$$-\frac{Y}{1+Y}\frac{dY}{dt}=\frac{k_1'''c_{A0}M_A}{\rho_{\text{sph}}}$$

Then we integrate

$$-\int_{Y_0}^{Y}\frac{Y}{1+Y}dY=\frac{k_1'''c_{A0}M_A}{\rho_{\text{sph}}}\int_{t_0}^{t}dt$$

and finally

$$(Y-Y_0)-\ln\frac{1+Y}{1+Y_0}=-\frac{k_1'''c_{A0}M_A}{\rho_{\text{sph}}}(t-t_0)\quad\text{or}$$

$$\sqrt{\frac{k_1'''}{\mathcal{D}_{AB}}}(R-R_0)-\ln\frac{1+\sqrt{k_1'''/\mathcal{D}_{AB}}R}{1+\sqrt{k_1'''/\mathcal{D}_{AB}}R_0}=-\frac{k_1'''c_{A0}M_A}{\rho_{\text{sph}}}(t-t_0)$$

From this one can get the dependence of the sphere radius on the time.

**19B.7 Various forms of the species continuity equation**

*a*. To get Eq. (A) of Table 19.2-1 from Eq. 19.1-7, use Eq. (C) of Table 17.7-1, in the form $\rho_\alpha = \rho\omega_\alpha$ on the left side of Eq. 19.1-7, and Eq. (S) of Table 17.8-1 on the right side.

To get Eq. (b) of Table 19.2-3 from Eq. 19.1-7, move the term $-(\nabla \cdot \rho_\alpha \mathbf{v})$ to the left side of the equation, and then use Eq. 3.5-4 (with *f* identified as $\omega_\alpha$).

*b*. Rearrange Eq. 19.1-11 to get

$$\frac{\partial c_\alpha}{\partial t} + (\nabla \cdot c_\alpha \mathbf{v}^*) = -(\nabla \cdot \mathbf{J}_\alpha^*) + R_\alpha$$

Then rewrite the divergence term on the left side by differentiating the product

$$\frac{\partial c_\alpha}{\partial t} + c_\alpha(\nabla \cdot \mathbf{v}^*) + (\mathbf{v}^* \cdot \nabla c_\alpha) = -(\nabla \cdot \mathbf{J}_\alpha^*) + R_\alpha$$

Then write $c_\alpha = c x_\alpha$ and once again differentiate the products, thus:

$$c\frac{\partial x_\alpha}{\partial t} + x_\alpha \frac{\partial c}{\partial t} + x_\alpha c(\nabla \cdot \mathbf{v}^*) + c(\mathbf{v}^* \cdot \nabla x_\alpha) + x_\alpha(\mathbf{v}^* \cdot \nabla c) = -(\nabla \cdot \mathbf{J}_\alpha^*) + R_\alpha$$

or, on rearranging

$$c\left(\frac{\partial x_\alpha}{\partial t} + (\mathbf{v}^* \cdot \nabla x_\alpha)\right) + x_\alpha\left(\frac{\partial c}{\partial t} + (\nabla \cdot c\mathbf{v}^*)\right) = -(\nabla \cdot \mathbf{J}_\alpha^*) + R_\alpha$$

Next we may make use of the overall equation of continuity in Eq. 19.1-12 to modify the second term on the left (with the prefactor $x_\alpha$)

$$c\left(\frac{\partial x_\alpha}{\partial t} + (\mathbf{v}^* \cdot \nabla x_\alpha)\right) + x_\alpha \sum_{\beta=1}^{N} R_\beta = -(\nabla \cdot \mathbf{J}_\alpha^*) + R_\alpha$$

This equation can be put into the form given in Eq. 19.1-15. No assumptions have been made in getting to this result from Eq. 19.1-11.

## 19C.1 Alternate form of the binary diffusion equation

Equation 19.1-17, in the absence of chemical reactions, is

$$\frac{\partial x_A}{\partial t} + \left(\mathbf{v}^* \cdot \nabla x_A\right) = \mathscr{D}_{AB} \nabla^2 x_A$$

Equation 19C.1-1 can be written as

$$\frac{1}{M}\frac{\partial M}{\partial t} + \frac{1}{M}(\mathbf{v} \cdot \nabla M) = \mathscr{D}_{AB}\left(\nabla \cdot \frac{1}{M}\nabla M\right)$$
$$= \mathscr{D}_{AB}\left[\frac{1}{M}\nabla^2 M - \frac{1}{M^2}(\nabla M \cdot \nabla M)\right]$$

To compare this with Eq. 19.1-17, we multiply through by $M$ and then replace $M$ by its definition: $M = x_A M_A + x_B M_B = x_A M_A + (1 - x_A) M_B = x_A (M_A - M_B) + M_B$. This leads to

$$\frac{\partial x_A}{\partial t} + \left(\mathbf{v} \cdot \nabla x_A\right) = \mathscr{D}_{AB} \nabla^2 x_A + \mathscr{D}_{AB} \frac{(M_A - M_B)(\nabla x_A \cdot \nabla x_A)}{M}$$

To get this result, we have also divided through by the factor $M_A - M_B$ (which is never exactly equal to zero).

When this last equation is compared to the first equation above, we see that the two results are the same, if

$$\mathbf{v}^* = \mathbf{v} + (1/M)(M_A - M_B)\mathscr{D}_{AB} \nabla x_A$$

To show this we introduce the definitions of $\mathbf{v}$ and $\mathbf{v}^*$, and also Fick's first law in the form of Eq. (F) of Table 17.8-2. This yields

$$x_A \mathbf{v}_A + x_B \mathbf{v}_B = \frac{1}{M}(M_A x_A \mathbf{v}_A + M_B x_B \mathbf{v}_B) - \frac{M_A - M_B}{M} x_A x_B (\mathbf{v}_A - \mathbf{v}_B)$$
$$= \frac{1}{M}\left[(M_A - M_A x_B + M_B x_B) x_A \mathbf{v}_A + (M_B - M_B x_A + M_A x_A) x_B \mathbf{v}_B\right]$$
$$= (1/M)\left[(M_A x_A + M_B x_B) x_A \mathbf{v}_A + (M_B x_B + M_A x_A) x_B \mathbf{v}_B\right]$$

Since this is an identity, the proof is completed.

**19D.1 Derivation of the equation of continuity**

*a*. The equation of continuity for species $A$ states that for an arbitrary fixed volume $V$ the time rate of change of the total mass of species $A$ must equal the net rate of addition of $A$ over the bounding surface $S$ (by convection and diffusion) plus the rate of production of species $A$ by chemical reactions. This statement in integral form is given by Eq. 19D.1-1. The surface integral may be transformed into a volume integral by using the Gauss divergence theorem to give

$$\int_V \frac{\partial}{\partial t}\rho_A dV = -\int_V (\nabla \cdot \mathbf{n}_A) dV + \int_V r_A dV$$

Here we have also moved the time-derivative operator inside the integral, inasmuch as the volume is not changing with time. Since the volume is completely arbitrary, we may remove the integral signs to get the equation of motion in the form of Eq. 19.1-6:

$$\frac{\partial \rho_A}{\partial t} = -(\nabla \cdot \mathbf{n}_A) + r_A$$

*b*. For an arbitrarily chosen "blob" of fluid, contained within the volume $V(t)$, whose boundaries are moving with the mass-average velocity, we may write the mass conservation statement as follows: the time rate of change of $A$ within the volume equals the rate of addition of $A$ by diffusion across the surface $S(t)$, plus the rate of production of $A$ by chemical reaction within $V(t)$ :

$$\frac{d}{dt}\int_{V(t)} \rho_A dV = -\int_{S(t)} (\mathbf{n} \cdot \mathbf{j}_A) dS + \int_{V(t)} r_A dV$$

Next we apply the Leibniz formula to the left side to get

$$\int_{V(t)} \frac{\partial}{\partial t}\rho_A dV + \int_{S(t)} \rho_A (\mathbf{n} \cdot \mathbf{v}_S) dS = -\int_{S(t)} (\mathbf{n} \cdot \mathbf{j}_A) dS + \int_{V(t)} r_A dV$$

This may be rearranged to give

$$\int_{V(t)} \frac{\partial}{\partial t}\rho_A dV = -\int_{S(t)} \left(\mathbf{n}\cdot\left(\mathbf{j}_A + \rho_A \mathbf{v}\right)\right) dS + \int_{V(t)} r_A dV$$

where use has been made of the fact that the surface of the blob is everywhere moving with the mass-average velocity, i.e., $\mathbf{v}_S = \mathbf{v}$. Then, we make use of the Gauss divergence theorem for the surface integral, to obtain

$$\int_{V(t)} \frac{\partial}{\partial t}\rho_A dV = -\int_{V(t)} \left(\nabla\cdot\left(\mathbf{j}_A + \rho_A \mathbf{v}\right)\right) dV + \int_{V(t)} r_A dV$$

Since the volume element $V(t)$ was chosen arbitrarily, we may remove the integrals to get the species equation of continuity for $A$ in the form of Eq. 19.1-7:

$$\frac{\partial \rho_A}{\partial t} = -\left(\nabla\cdot\left(\mathbf{j}_A + \rho_A \mathbf{v}\right)\right) + r_A$$

**19D.2 Derivation of the eqution of change for temperature for a multicomponent system**

*a*. Applying the chain rule for the functions related in Eq. 19D.2-1, we get

$$\left(\frac{\partial H}{\partial m_\alpha}\right)_{m_\gamma} = \sum_{\beta=1}^{N-1}\left(\frac{\partial(m\hat{H})}{\partial \omega_\beta}\right)_{\omega_\gamma,m}\left(\frac{\partial \omega_\beta}{\partial m_\alpha}\right)_{m_\gamma} + \left(\frac{\partial(m\hat{H})}{\partial m}\right)_{\omega_\gamma}\left(\frac{\partial m}{\partial m_\alpha}\right)_{m_\gamma}$$

$$= m\sum_{\beta=1}^{N-1}\left(\frac{\partial \hat{H}}{\partial \omega_\beta}\right)_{\omega_\gamma}\left(\frac{\delta_{\alpha\beta}}{\Sigma_\gamma m_\gamma} - \frac{m_\beta}{\left(\Sigma_\gamma m_\gamma\right)^2}\right) + \hat{H}$$

$$= \sum_{\beta=1}^{N-1}\left(\frac{\partial \hat{H}}{\partial \omega_\beta}\right)_{\omega_\gamma}\left(\delta_{\alpha\beta} - \frac{m_\beta}{m}\right) + \hat{H} \qquad \text{(for } \alpha \neq N\text{)}$$

$$\left(\frac{\partial H}{\partial m_N}\right)_{m_\gamma} = \sum_{\beta=1}^{N-1}\left(\frac{\partial(m\hat{H})}{\partial \omega_\beta}\right)_{\omega_\gamma,m}\left(\frac{\partial \omega_\beta}{\partial m_N}\right)_{m_\gamma} + \left(\frac{\partial(m\hat{H})}{\partial m}\right)_{\omega_\gamma}\left(\frac{\partial m}{\partial m_N}\right)_{m_\gamma}$$

$$= m\sum_{\beta=1}^{N-1}\left(\frac{\partial \hat{H}}{\partial \omega_\beta}\right)\left(-\frac{m_\beta}{m^2}\right) + \hat{H} = \sum_{\beta=1}^{N-1}\left(\frac{\partial \hat{H}}{\partial \omega_\beta}\right)\left(-\frac{m_\beta}{m}\right) + \hat{H} \quad \text{(for } \alpha = N\text{)}$$

Subtraction of these two results gives Eq. 19D.2-4.

*b*. Regard $H$ as a function of $p$, $T$, and the first ($N$ - 1) mass fractions, so that

$$d\hat{H} = \left(\frac{\partial \hat{H}}{\partial p}\right)_{T,\omega_\gamma} dp + \left(\frac{\partial \hat{H}}{\partial T}\right)_{p,\omega_\gamma} dT + \sum_{\alpha=1}^{N-1}\left(\frac{\partial \hat{H}}{\partial \omega_\alpha}\right)_{p,T,\omega_\gamma} d\omega_\alpha$$

When this is applied to a fluid element moving with the fluid, Eq. 19D.2-5 results. Equations 19D.2-6 and 7 are standard thermodynamic results.

*c*. Because of the relation $m_\alpha = n_\alpha M_\alpha$, the differential quotients in the last term in the last equation can be rewritten, with the help of Eq. 19D.2-4, in terms of partial molar quantities:

$$\left(\frac{\partial \hat{H}}{\partial \omega_\alpha}\right)_{\omega_\gamma} = \left(\frac{\partial H}{\partial m_\alpha}\right)_{m_\gamma} - \left(\frac{\partial H}{\partial m_N}\right)_{m_\gamma}$$
$$= \frac{1}{M_\alpha}\left(\frac{\partial H}{\partial n_\alpha}\right)_{m_\gamma} - \frac{1}{M_N}\left(\frac{\partial H}{\partial n_N}\right)_{m_\gamma} = \frac{\overline{H}_\alpha}{M_\alpha} - \frac{\overline{H}_N}{M_N}$$

In Eq. 19D.2-5, we use Eq. (E) of Table 19.2-4 for the substantial derivative of the enthalpy, and we use Eq. 19.1-14 for the substantial derivative of the mole fractions. We also use Eqs. 19D.2-6 and 7 for the other differential quotients. Then Eq. 19D.2-5 becomes

$$-(\nabla\cdot\mathbf{q}) - (\boldsymbol{\tau}:\nabla\mathbf{v}) + \frac{Dp}{Dt} = \left[1 - \left(\frac{\partial \ln \hat{V}}{\partial \ln T}\right)_{p,\omega_\gamma}\right]\frac{Dp}{Dt} + \rho\hat{C}_p\frac{DT}{Dt}$$
$$+ \sum_{\alpha=1}^{N-1}\left(\frac{\overline{H}_\alpha}{M_\alpha} - \frac{\overline{H}_N}{M_N}\right)\left(-(\nabla\cdot\mathbf{j}_\alpha) + r_\alpha\right)$$

*d*. Rearranging we get

$$\rho\hat{C}_p\frac{DT}{Dt} = -(\nabla\cdot\mathbf{q}) - (\boldsymbol{\tau}:\nabla\mathbf{v}) + \left(\frac{\partial \ln \hat{V}}{\partial \ln T}\right)_{p,\omega_\gamma}\frac{Dp}{Dt}$$
$$+ \sum_{\alpha=1}^{N-1}\left(\frac{\overline{H}_\alpha}{M_\alpha} - \frac{\overline{H}_N}{M_N}\right)\left((\nabla\cdot\mathbf{j}_\alpha) - r_\alpha\right)$$

The last term in this equation can be simplified as follows:

$$+\sum_{\alpha=1}^{N-1}\frac{\overline{H}_\alpha}{M_\alpha}\left((\nabla\cdot\mathbf{j}_\alpha) - r_\alpha\right) - \left(\sum_{\alpha=1}^{N-1}\left((\nabla\cdot\mathbf{j}_\alpha) - r_\alpha\right)\right)\frac{\overline{H}_N}{M_N}$$
$$= \sum_{\alpha=1}^{N-1}\overline{H}_\alpha\left((\nabla\cdot\mathbf{J}_\alpha) - R_\alpha\right) + \overline{H}_N\left((\nabla\cdot\mathbf{J}_N) - R_N\right)$$
$$= \sum_{\alpha=1}^{N}\overline{H}_\alpha\left((\nabla\cdot\mathbf{J}_\alpha) - R_\alpha\right)$$

This, then, leads to Eq. (F) of Table 19.2-4.

**19D.3 Gas separation by atmolysis or "sweep diffusion"**

*a*. The Maxwell-Stefan equation for a three-component mixture are

$$\frac{dx_A}{dz} = \frac{1}{c\mathcal{D}_{AB}}(x_A N_{Bz} - x_B N_{Az}) + \frac{1}{c\mathcal{D}_{AC}}(x_A N_{Cz} - x_C N_{Az})$$

$$\frac{dx_B}{dz} = \frac{1}{c\mathcal{D}_{AB}}(x_B N_{Az} - x_A N_{Bz}) + \frac{1}{c\mathcal{D}_{BC}}(x_B N_{Cz} - x_C N_{Bz})$$

We now make the substitution $x_C = 1 - x_A - x_B$ in order to have just two independent variables. Then by introducing the indicated dimensionless variables, we get Eqs. 19D.3-1 and 2.

*b*. Next we take the Laplace transform of Eqs. 19D.3-1 and 2:

$$p\bar{x}_A - x_{A1} = Y_{AA}\bar{x}_A + Y_{AB}\bar{x}_B + p^{-1}Y_A$$

$$p\bar{x}_B - x_{B1} = Y_{BA}\bar{x}_A + Y_{BB}\bar{x}_B + p^{-1}Y_B$$

This set of equations can be solved simultaneously to give for $\bar{x}_A$

$$\bar{x}_A = \frac{X_A(x_{A1}, x_{B1}; p)}{p(p - p_+)(p - p_-)}$$

in which $p_+$ and $p_-$ are given by Eq. 19D.3-4, and $X_A(x_{A1}, x_{B1}; p)$ by Eq. 19D.3-5. Similar equations can be given for $\bar{x}_B$ by interchanging the subscripts *A* and *B*.

The transformed expression $\bar{x}_A$ can be inverted by using the Heaviside partial fractions expansion theorem if we exclude the relatively unimportant cases $p_+ = p_-$ and $p_+ (\text{or } p_-) = 0$. Thus we get

$$x_A(\zeta) = \frac{X(x_{A1}, x_{B1}; 0)}{p_+ p_-} + \frac{X(x_{A1}, x_{B1}; p_+)e^{p_+\zeta}}{p_+(p_+ - p_-)} + \frac{X(x_{A1}, x_{B1}; p_-)e^{p_-\zeta}}{p_-(p_- - p_+)}$$

The expression for $x_B$ is obtained by interchanging all indices *A* and *B*.

*c*. Finally, at $\zeta = 1$, $x_A = x_{A2}$, so that

$$x_{A2} = \frac{X(x_{A1}, x_{B1}; 0)}{p_+ p_-} + \frac{X(x_{A1}, x_{B1}; p_+) e^{p_+ \zeta}}{p_+ (p_+ - p_-)} + \frac{X(x_{A1}, x_{B1}; p_-) e^{p_- \zeta}}{p_- (p_- - p_+)}$$

and a similar expression for $x_B$. We hence have a pair of equations giving the relations among $x_{A1}$, $x_{A2}$, $x_{B1}$, and $x_{B2}$, the dimensionless fluxes $\nu_\alpha$, and the dimensionless diffusivity ratios. Keys and Pigford go further and give a plot of the separation factor

$$\alpha = \frac{x_{A2}/x_{B2}}{x_{A1}/x_{B1}}$$

for the special case of very small mole fractions.

Note that the choice of notation in this problem has the advantage that there is symmetry between *A* and *B*, which makes it easy to give the results for both species after the result for one species has been worked out.

**19D.4 Steady-state diffusion from a rotating disk**

*a.* Equation 19.1-16 for steady-state diffusion in the absence of chemical reactions is

$$(\mathbf{v}\cdot\nabla\rho_A) = \mathscr{D}_{AB}\nabla^2\rho_A(z)$$

For the special case that $\rho_A$ depends on $z$ alone, the diffusion equation simplifies further to

$$v_z\frac{d\rho_A}{dz} = \mathscr{D}_{AB}\frac{d^2\rho_A}{dz^2} \qquad \text{or} \qquad H(\zeta)\frac{d\rho_A}{d\zeta} = \frac{1}{\mathrm{Sc}}\frac{d^2\rho_A}{d\zeta^2}$$

in which the dimensionless coordinate $\zeta = z\sqrt{\Omega/\nu}$ has been introduced.

*b.* The differential equation is solved by setting $d\rho_A/d\zeta = p$, so that we get the first-order separable equation $dp/d\zeta = \mathrm{Sc}H(\zeta)p$, which has the solution

$$\ln p = \mathrm{Sc}\int_0^\zeta H(\zeta)d\zeta + \ln C_1 \qquad \text{or} \qquad \frac{d\rho_A}{d\zeta} = p = C_1\exp\left(\mathrm{Sc}\int_0^\zeta H(\bar\zeta)d\bar\zeta\right)$$

A further integration gives

$$\rho_A = C_1\int_0^\zeta \exp\left(\mathrm{Sc}\int_0^{\bar\zeta} H(\bar{\bar\zeta})d\bar{\bar\zeta}\right)d\bar\zeta + C_2$$

The boundary conditions that $\rho_A(0) = \rho_{A0}$ and $\rho_A(\infty) = 0$ then give

$$\frac{\rho_A}{\rho_{A0}} = 1 - \frac{\int_0^\zeta \exp\left(\mathrm{Sc}\int_0^{\bar\zeta} H(\bar{\bar\zeta})d\bar{\bar\zeta}\right)d\bar\zeta}{\int_0^0 \exp\left(\mathrm{Sc}\int_0^{\bar\zeta} H(\bar{\bar\zeta})d\bar{\bar\zeta}\right)d\bar\zeta} \to 1 - \frac{\int_0^\zeta \exp\left(\frac{1}{3}\mathrm{Sc}a\bar\zeta^3\right)d\bar\zeta}{\int_0^\infty \exp\left(\frac{1}{3}\mathrm{Sc}a\bar\zeta^3\right)d\bar\zeta}$$

the second expression being the high Schmidt number limit, i.e., taking only the first term in the expansion for $H$. The integral in the denominator can be evaluated analytically

$$\int_0^\infty \exp\left(\tfrac{1}{3}\mathrm{Sc}a\bar{\zeta}^3\right)d\bar{\zeta} = \frac{\int_0^\infty \exp\left(-u^3\right)du}{\sqrt[3]{\tfrac{1}{3}\mathrm{Sc}a}} = \frac{\tfrac{1}{3}\Gamma\left(\tfrac{1}{3}\right)}{\sqrt[3]{\tfrac{1}{3}\mathrm{Sc}a}}$$

Hence the final expression for the concentration profile is

$$\frac{\rho_A}{\rho_{A0}} = 1 - \frac{\sqrt[3]{\tfrac{1}{3}\mathrm{Sc}a}}{\Gamma\left(\tfrac{4}{3}\right)}\int_0^\zeta \exp\left(-\tfrac{1}{3}\mathrm{Sc}a\bar{\zeta}^3\right)d\bar{\zeta}$$

*c.* The mass flux in the $z$ direction is then

$$j_{Az} = -\mathcal{D}_{AB}\frac{d\rho_A}{dz} = +\rho_{A0}\mathcal{D}_{AB}\frac{\sqrt[3]{\tfrac{1}{3}\mathrm{Sc}a}}{\Gamma\left(\tfrac{4}{3}\right)}\exp\left(-\tfrac{1}{3}\mathrm{Sc}a\zeta^3\right)\frac{d\zeta}{dz}$$

in which $d\zeta/dz = \sqrt{\Omega/\nu}$. At the surface of the disk the mass flux is

$$\begin{aligned} j_{Az}\big|_{z=0} &= \rho_{A0}\mathcal{D}_{AB}\frac{\sqrt[3]{\tfrac{1}{3}\mathrm{Sc}a}}{\Gamma\left(\tfrac{4}{3}\right)}\sqrt{\frac{\Omega}{\nu}} = \frac{\sqrt[3]{\tfrac{1}{3}(0.510)}}{\Gamma\left(\tfrac{4}{3}\right)}\rho_{A0}\mathcal{D}_{AB}\mathrm{Sc}^{1/3}\Omega^{1/2}\nu^{-1/2} \\ &= 0.620\rho_{A0}\mathcal{D}_{AB}\left(\nu/\mathcal{D}_{AB}\right)^{1/3}\Omega^{1/2}\nu^{-1/2} \end{aligned}$$

which is the result in Eq. 19D.4-7.

**20A.1 Measurement of diffusivity by unsteady-state evaporation.**

Assuming ideal gas behavior, and insolubility of species "B" in the liquid, we estimate the mole fraction of ethyl propionate in the interfacial vapor as

$$x_{A0} = \frac{p_{A,\text{vap}}}{p} = \frac{41.5}{761.2} = 0.0545$$

Eq. 20.1-1-22 gives

$$\Delta V(t) = S x_{A0} \psi \sqrt{\frac{4\mathcal{D}_{AB}}{\pi}} [\sqrt{t} - \sqrt{240}]$$

Linear interpolation to $x_{A0} = 0.0545$ in Table 20.1-1 gives $\psi = 1.0235$, whence

$$\begin{aligned} \mathcal{D}_{AB} \text{ cm}^2/\text{s} &= \frac{\pi}{4(S x_{A0} \psi)^2} \left[ \frac{\Delta V(t)}{\sqrt{t} - \sqrt{240}} \right]^2 \\ &= 16.48 \left[ \frac{\Delta V(t)}{\sqrt{t} - \sqrt{240}} \right]^2 \end{aligned}$$

with $\Delta V$ in cm$^3$ and $t$ in s. Application of this formula to the tabulated data gives the following results:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $\sqrt{t}$ | 15.5 | 19.4 | 23.4 | 26.9 | 30.5 | 34.0 | 37.5 | 41.5 |
| $\mathcal{D}_{AB}$ | 1.2 | 0.0281 | 0.0278 | 0.0272 | 0.0273 | 0.0270 | 0.0273 | 0.0269 |

The average of the last seven determinations of $\mathcal{D}_{AB}$ is 0.0274 cm$^2/s$.

**20A.2 Absorption of oxygen from a growing bubble.**

The interfacial molar flux of oxygen into the liquid is given by Eq. 20.1-75 as

$$N_{A0}(t) = c_{A0}\sqrt{\frac{(2n+1)\mathcal{D}_{AB}}{\pi t}}$$

for a bubble with interfacial area $S(t) = at^n$ and interfacial liquid concentration $c_{A0}$, when $a$, $n$ and $c_{A0}$ are constants. The solubility $\omega_{A0}$ corresponds to a molar concentration

$$\begin{aligned} c_{A0} &= \rho\omega_{A0}/M_A \\ &= (1.0 \text{ g soln/cm}^3)(7.78 \times 10^{-4} \text{ g O}_2\text{/g soln})/(32 \text{ g O}_2\text{/g-mol O}_2) \\ &= 2.43 \times 10^{-5} \text{ g-mol O}_2\text{/cm}^3 \end{aligned}$$

which is used here as the interfacial concentration of dissolved $O_2$.

**(a)** For constant growth rate of the bubble volume, $r_s^3 \propto t$, so that $r_s \propto t^{1/3}$ and $S(t) \propto r_s^2 \propto t^{2/3}$, giving $n = 2/3$ in Eq. 20.1-75. Then at $t = 2$ s, the interfacial molar flux of $O_2$ into the liquid is

$$\begin{aligned} N_{A0}(t) &= c_{A0}\sqrt{\frac{(\frac{4}{3}+1)\mathcal{D}_{AB}}{\pi t}} \\ &= (2.43 \times 10^{-5} \text{ g-mol/cm}^3)\sqrt{\frac{(7/3)(2.60 \times 10^{-5} \text{ cm}^2\text{/s})}{(3.1416)(2 \text{ s})}} \\ &= 7.55 \times 10^{-8} \text{ g-mol/cm}^2\text{·s} \end{aligned}$$

The total absorption rate in g/s is then

$$\begin{aligned} w_A(t) &= 4\pi r_s^2(t) N_{A0}(t) M_A \\ &= 4\pi(0.1/2 \text{ cm})^2)(7.55 \times 10^{-8} \text{ g-mol/cm}^2\text{·s})(32 \text{ g/g-mol}) \\ &= 7.6 \times 10^{-8} \text{ g/s} \end{aligned}$$

**(b)** For constant radial growth rate, $r_s \propto t$ and $S(t) \propto t^2$, giving $n = 2$ in Eq. 20.1-75. Then the interfacial flux at time $t = 2$ s is

$$\begin{aligned} N_{A0} &= c_{A0}\sqrt{\frac{(4+1)\mathcal{D}_{AB}}{\pi t}} \\ &= (2.43 \times 10^{-5} \text{ g-mol/cm}^3)\sqrt{\frac{(5)(2.60 \times 10^{-5} \text{ cm}^2\text{/s})}{\pi(2 \text{ s})}} \\ &= 1.11 \times 10^{-7} \text{ g-mol/cm}^2\text{·s} \end{aligned}$$

and the total absorption rate in g/s is

$$\begin{aligned} w_A &= 4\pi r_s^2(t) N_{A0}(t) M_A \\ &= 4\pi(0.1/2 \text{ cm})^2(1.11 \times 10^{-7} \text{ g-mol/cm}^2\text{·s})(32 \text{ g/g-mol}) \\ &= 1.11 \times 10^{-7} \text{ g/s} \end{aligned}$$

**20A.3 Rate of evaporation of $n$-octane.**

Table E-1 gives the following Lennard-Jones parameters:

| Species | M | $\sigma$, Å | $\varepsilon$, K |
|---|---|---|---|
| A: $n$-$C_8H_{18}$ | 114.23 | 7.035 | 361. |
| B: $N_2$ | 28.013 | 3.667 | 99.8 |

Eqs. 17.3-14,15 then give the interaction parameters $\sigma_{AB} = 5.351$ Å and $\varepsilon_{AB} =$ 189.8 K, and Eq. 17.3-10 gives

$$\mathcal{D}_{AB},\ \text{cm}^2/\text{s} = 0.0018583\sqrt{(293.15)^3\left(\frac{1}{114.23}+\frac{1}{28.013}\right)}\frac{1}{(p,\ \text{atm})(5.351)^2(1.185)}$$
$$= 0.0580/(p,\ \text{atm})$$

**(a)** If $p = 1$ atm, then $\mathcal{D}_{AB} = 0.0580$ cm$^2$/s and $x_{A0} = 10.45/760 = 0.01375$. Interpolation in Table 20.1-1 gives $\varphi = 0.00859$, and Eq. 20.1-20 gives the volume of vapor produced in 24.5 hr as

$$V_A = S\varphi\sqrt{4\mathcal{D}_{AB}t}$$
$$= (1.29\ \text{cm}^2)(0.00859)\sqrt{4(0.0580\ \text{cm}^2/\text{s})(24.5\times 3600\ \text{s})}$$
$$= 1.585\ \text{cm}^3$$

The mass of vapor produced in 24.5 hr is

$$m_A = \frac{pV_AM_A}{RT}$$
$$= \frac{(1\ \text{atm})(1.585\ \text{cm}^3)(114.23\ \text{g/g-mol})}{82.0578\times 293.15\ \text{cm}^3\ \text{atm/g-mol}}$$
$$= 0.0075\ \text{g}$$

**(b)** If $p = 2$ atm, then $\mathcal{D}_{AB} = 0.0290$ cm$^2$/s and $x_{A0} = 0.01375/2 = 0.0688$. Table 20.1-1 then gives $\varphi = 0.00430$, whence

$$V_A = S\varphi\sqrt{4\mathcal{D}_{AB}t}$$
$$= (1.29\ \text{cm}^2)(0.00438)\sqrt{4(0.00290\ \text{cm}^2/\text{s})(24.5\times 3600\ \text{s})}$$
$$= 0.561\ \text{cm}^3$$

and the mass of vapor produced in 24.5 hr is

$$m_A = \frac{pV_AM_A}{RT}$$
$$= \frac{(2\ \text{atm})(0.561\ \text{cm}^3)(114.23\ \text{g/g-mol})}{82.0578\times 293.15\ \text{cm}^3\ \text{atm/g-mol}}$$
$$= 0.0053\ \text{g}$$

**20A.4 Effect of bubble size on interfacial composition.**

Equations 20A.4-1 and 2 give

$$\omega_A = H\left[p_\infty + \frac{2\sigma}{r_s}\right]$$

Assuming $H$ to be independent of $r_s$, the ratio of $\omega_A$ to its value for a very large bubble is

$$\frac{\omega_{A0}(r)}{\omega_{A0}(\infty)} = \frac{p_\infty + 2\sigma/r_s}{p_\infty} = 1 + \frac{2\sigma}{p_\infty r_s}$$

Thus, the bubble radius corresponding to a 10% increase of $\omega_{A0}$ over its value for a very large bubble is given by

$$\frac{2\sigma}{p_\infty r_s} = 0.1 \qquad \text{or} \qquad r_s = \frac{2\sigma}{p_\infty}$$

For a gas bubble in water at 25°C, with $p_\infty =$ 1 atm, this gives the required bubble radius as

$$r_s = \frac{(2)(72 \text{ dynes/cm})}{(0.1)(1.0133 \times 10^6 \text{ dyne/cm}^2)}$$
$$= 0.00142 \text{ cm} = 14 \text{ microns}$$

The (normally minor) dependence of liquid-phase free energy on total pressure has been neglected here. To include this effect, one would need partial molar volume data for the particular system.

**20A.5 Absorption with rapid second-order reaction**

*a*. The first thing one has to do is to determine the parameter $\gamma$ from Eq. 20.1-37, using the concentrations and diffusivities given in Fig. 20.1-2. By a trial-and-error procedure we have found that $\gamma = 4.9\times10^{-5}$ mm$^2$/s. We now verify this by substituting into Eq. 20.1-37. We first have to convert the diffusivities in ft$^2$/hr into units of mm$^2$/s:

$$\mathcal{D}_{AS} = \left(3.9\times10^{-5}\ \text{ft}^2/\text{hr}\right)\frac{\left(12\cdot2.54\cdot10\ \text{mm/ft}\right)^2}{3600\ \text{s/hr}} = 1.006\times10^{-3}\ \text{mm}^2/\text{s}$$

$$\mathcal{D}_{BS} = 1.95\times10^{-5}\ \text{ft}^2/\text{hr} = 0.503\times10^{-3}\ \text{mm}^2/\text{s}$$

Substituting into Eq. 20.1-37 now gives:

$$1-\text{erf}\sqrt{\frac{4.9\times10^{-5}}{0.503\times10^{-3}}}$$
$$=\left(\frac{4}{1}\right)\sqrt{\frac{0.503\times10^{-3}}{1.006\times10^{-3}}}\text{erf}\sqrt{\frac{4.9\times10^{-5}}{1.006\times10^{-3}}}\exp\left(\frac{4.9\times10^{-5}}{1.006\times10^{-3}}-\frac{4.9\times10^{-5}}{0.503\times10^{-3}}\right)$$

or

$$1-\text{erf}\ 0.3121 = (2.828)(\text{erf}\ 0.2207)(0.9525)$$

When the error functions are evaluated, using a table, we get

$$1-0.341 = (2.828)(0.245)(0.9525)$$

This gives 0.66 = 0.66. Therefore, $z_R(t)=\sqrt{4\gamma t}$ gives the location of the reaction zone as a function of $t$. Thus we get the following table of results that can be compared with Fig. 19.1-2:

| $t$ | $z_R(t)=\sqrt{4\gamma t}$ |
|---|---|
| 0.625 s | 0.011 mm |
| 2.5 | 0.022 |
| 10.0 | 0.044 |

This is not in very good agreement with the graph in Fig. 19.1-2. The reason for this may be that Eq. 415 on p. 336 of *Absorption and Extraction* by Sherwood and Pigford (which corresponds to our Eq. 20.1-37), contains two errors: the $r$ on the left side should be $\sqrt{r}$ and the $D_A$ in the argument of the second error function should be $D_B$.

*b*. To get $N_{A0}$ at 2.5 seconds, we use Eq. 20.1-38, as follows:

$$N_{A0} = \frac{c_{A0}}{\operatorname{erf}\sqrt{\gamma/\mathcal{D}_{AS}}}\sqrt{\frac{\mathcal{D}_{AS}}{\pi t}}$$

$$= \frac{\left(1\ \text{gmol/dm}^3\right)\left(10\ \text{cm/dm}\right)^3}{\operatorname{erf}\sqrt{\left(4.9\times10^{-5}\right)/\left(1.006\times10^{-3}\right)}}\sqrt{\frac{1.006\times10^{-3}\ \text{mm}^2/\text{s}}{(3.14159)(2.5\ \text{s})\left(100\ \text{mm}^2/\text{cm}^2\right)}}$$

$$= \frac{1000}{0.245}\left(1.13\times10^{-3}\right) = 4.61\ \text{gmol/cm}^2\text{s}$$

**20A.6 Rapid forced-convection mass transfer into a laminar boundary layer.**

Equation (20.2-51) gives, for the conditions of this problem,

$$R_\omega = \frac{(\omega_0 - \omega_\infty)}{n_{A0}/(n_{A0} + n_{B0}) - \omega_{A0}}$$
$$= \frac{(0.9 - 0.1)}{1 - 0.9} = 8.0$$

Fig. 22.8-5 gives a dimensionless mass flux $\phi$ of 1.55 for mass transfer at $R_\omega = 8.0$ and Sc = 2.0. With this result, Eq. 22.8-21 and Table 20.2-1 give

$$1.55 = \frac{K\Lambda}{\Pi'(0, \Lambda, 0)} = \frac{K \times 2.0}{0.5972}$$

so that

$$K = 1.55 \times 0.5972/2 = 0.463$$

The definition of $K$ in Eq. 20.2-48 then gives the total interfacial mass flux

$$n_{A0}(x) + n_{B0}(x) = \rho_0 v_0(x) = K\rho v_\infty \sqrt{\frac{\nu}{2v_\infty x}}$$
$$= 0.33\sqrt{\rho v_\infty \mu / x}$$

In this problem $n_{B0}(x)$ is stated to be zero, so the previous result reduces to

$$n_{A0}(x) = 0.33\sqrt{\rho v_\infty \mu / x}$$

**20A.7 Slow forced-convection mass transfer into a laminar boundary layer.**

**(a)** Equation (20.2-51) gives, for the conditions of this problem, the binary mass flux ratio

$$R_\omega = \frac{(\omega_0 - \omega_\infty)}{n_{A0}/(n_{A0} + n_{B0}) - \omega_{A0}}$$
$$= \frac{(0.05 - 0.01)}{1 - 0.05} = 0.0421$$

Eq. (20.2-55) and Table 20.2-2, with Sc= 0.6, then give

$$K = a\Lambda^{-2/3}\frac{R_\omega}{1 + bR_\omega}$$
$$= 0.4642 \times (0.6^{-2/3})\frac{0.0421}{1 + 0.768 \times 0.0421} = 0.0266$$

Equation 20.2-48, with the specification $n_{B0}(x) = 0$, then gives the evaporative mass flux

$$n_{A0}(x) = \rho_0 v_0(x) = K\rho v_\infty\sqrt{\frac{\nu}{2v_\infty x}}$$
$$= 0.0266\sqrt{\rho v_\infty \mu/2x}$$
$$= 0.0188\sqrt{\rho v_\infty \mu/x}$$

**(b)** Eq. 20.2-57 gives

$$\frac{n_{A0} - \omega_{A0}(n_{A0} + n_{B0})}{\rho v_\infty}(\omega_{A0} - \omega_{A\infty})\mathrm{Sc}^{2/3} = 0.332\sqrt{\frac{\nu}{v_\infty x}}$$

which gives, since $n_{B0} = 0$,

$$n_{A0} = 0.332\mathrm{Sc}^{-2/3}\frac{\omega_{A0} - \omega_{A\infty}}{1 - \omega_{A0}}\rho v_\infty\sqrt{\frac{\nu}{v_\infty x}}$$
$$= 0.332(0.6)^{-2/3}\frac{0.05 - 0.01}{1 - 0.05}\sqrt{\rho v_\infty \mu/x}$$
$$= 0.0196\sqrt{\rho v_\infty \mu/x}$$

**(c)** At the value $K = 0.0266$ found in (a), Table 20.2-1 gives the interpolated interfacial gradient $\Pi'(0, \mathrm{Sc}, K) = 0.3797$. Then Eq. 20.2-47, with $n_{B0} = 0$, gives the reference solution

$$n_{A0} = \frac{\omega_0 - \omega_\infty}{1 - \omega_0}\frac{\Pi'(0, \mathrm{Sc}, K)}{\mathrm{Sc}}\rho v_\infty\sqrt{\frac{\nu}{2v_\infty x}}$$
$$= \frac{0.05 - 0.01}{1 - 0.05}\frac{0.3797}{0.6}\sqrt{\rho v_\infty \mu/2x}$$
$$= 0.0188\sqrt{\rho v_\infty \mu/x}$$

in excellent agreement with the truncated expansion used in part (a).

**20B.1 Extension of the Arnold problem to account for interphase transfer of both species**

Equations 20.1-1, 2, and 3 are still valid if both species are crossing the interface. In Eq.20.1-3 we now eliminate $c$ by using Eq. (D) of Table 17.8-2 (evaluated at the interface) to get

$$v_z^* = -\frac{(N_{Az0}+N_{Bz0})}{N_{Az0}-x_{A0}(N_{Az0}+N_{Bz0})}\frac{\partial x_A}{\partial z}\bigg|_{z=0} = -\frac{(1+r)}{1-x_{A0}(1+r)}\frac{\partial x_A}{\partial z}\bigg|_{z=0}$$

When $N_{Bz0}=0$ (or the ratio $r=N_{Bz0}/N_{Az0}$ goes to zero), this equation simplifies to Eq. 20.1-4.

Equation 20.1-5 is then replaced by

$$\frac{\partial x_A}{\partial t}-\left(\frac{(1+r)}{1-x_{A0}(1+r)}\frac{\partial x_A}{\partial z}\bigg|_{z=0}\right)\frac{\partial x_A}{\partial z}=\mathcal{D}_{AB}\frac{\partial^2 x_A}{\partial z^2}$$

We now introduce the dimensionless variables $\Pi$ (defined in Eq. 20.1-23) and $Z$ (defined just below Eq. 20.1-8); note that when $x_{A\infty}=0$, $\Pi$ is the same as $-(X-1)$. Then the combinarion of variables method gives (cf. 20.1-9) the ordinary differential equation

$$\frac{d^2\Pi}{dZ^2}+2(Z-\varphi)\frac{d\Pi}{dZ}=0$$

along with

$$\varphi(x_{A0},r)=+\frac{1}{2}\frac{(x_{A0}-x_{A\infty})(1+r)}{1-x_{A0}(1+r)}\frac{d\Pi}{dZ}\bigg|_{Z=0}$$

This is Eq. 20.1-24. If $r=0$ and $x_{A\infty}=0$, it reduces to Eq. 20.1-10. The sign discrepancy comes about because (in the limit that $x_{A\infty}$ is zero, $\Pi$ is the same as $-(X-1)$. The solution of the ordinary differential equation for $\Pi$ proceeds exactly as that for $X$ in the text, and the final result is

$$\Pi=\frac{\operatorname{erf}(Z-\varphi)+\operatorname{erf}\varphi}{1+\operatorname{erf}\varphi}$$

for the concentration profile. If $r=0$ and $x_{A\infty}=0$, this reduces exactly to Eq. 20.1-16. When this concentration profile is used to evaluate $d\Pi/dZ$ at $Z=0$, then we get an expression for $\varphi(x_{A0},r)$:

$$\varphi(x_{A0},r)=+\frac{1}{2}\frac{(x_{A0}-x_{A\infty})(1+r)}{1-x_{A0}(1+r)}\left(\frac{2}{\sqrt{\pi}}\frac{\exp(-\varphi^2)}{1+\operatorname{erf}\varphi}\right)$$

This may be rearranged to give

$$\frac{(x_{A0}-x_{A\infty})(1+r)}{1-x_{A0}(1+r)}=\sqrt{\pi}(1+\operatorname{erf}\varphi)\varphi\exp(+\varphi^2)$$

in agreement with Eq. 20.1-25.

**20B.2 Extension of the Arnold problem to nonisothermal diffusion**

*a*. Equation (M) of Table 19.2-4 without the last two terms is:

$$\frac{\partial}{\partial t}\sum_{\alpha=1}^{N} c_\alpha \overline{H}_\alpha + \left(\nabla \cdot \sum_{\alpha=1}^{N} \mathbf{N}_\alpha \overline{H}_\alpha\right) = (\nabla \cdot k\nabla T)$$

Replacing the partial molar quantities by quantities per mole (see comment after Eq. 19.3-6), and assuming that $k$ is constant then gives

$$\frac{\partial}{\partial t}\sum_{\alpha=1}^{N} c_\alpha \tilde{H}_\alpha + \left(\nabla \cdot \sum_{\alpha=1}^{N} \mathbf{N}_\alpha \tilde{H}_\alpha\right) = k\nabla^2 T$$

Differentiating the products in the first and second terms allows us to rewrite this as

$$\sum_{\alpha=1}^{N} c_\alpha \frac{\partial \tilde{H}_\alpha}{\partial t} + \sum_{\alpha=1}^{N} \frac{\partial c_\alpha}{\partial t}\tilde{H}_\alpha + \sum_{\alpha=1}^{N}(\nabla \cdot \mathbf{N}_\alpha)\tilde{H}_\alpha + \sum_{\alpha=1}^{N}\left(\mathbf{N}_\alpha \cdot \nabla \tilde{H}_\alpha\right) = k\nabla^2 T$$

In the second term of this equation, we replace the derivative of the concentration by using Eq. 19.1-10 (omitting the reaction-rate term), and then we see that the second and third terms just exactly cancel. Next, we replace the enthalpy by the heat capacity multiplied by a temperature difference in accordance with Eq. 9.8-8, to get

$$\sum_{\alpha=1}^{N} c_\alpha \frac{\partial}{\partial t}\tilde{C}_{p\alpha}\left(T - T^{\circ}\right) + \sum_{\alpha=1}^{N}\left(\mathbf{N}_\alpha \cdot \nabla \tilde{C}_{p\alpha}\left(T - T^{\circ}\right)\right) = k\nabla^2 T$$

For constant heat capacities this then becomes:

$$\sum_{\alpha=1}^{N} c_\alpha \tilde{C}_{p\alpha} \frac{\partial T}{\partial t} + \sum_{\alpha=1}^{N} \tilde{C}_{p\alpha}\left(\mathbf{N}_\alpha \cdot \nabla T\right) = k\nabla^2 T$$

If all the heat capacities are alike, then they can be taken outside the summation to get

$$c\tilde{C}_p \frac{\partial T}{\partial t} + \tilde{C}_p\left(\sum_{\alpha=1}^{N} \mathbf{N}_\alpha \cdot \nabla T\right) = k\nabla^2 T$$

Next we make use of $c\tilde{C}_p = \rho\hat{C}_p$, and then divide the entire equation by $\rho\hat{C}_p$; this gives, with the help of Eq. (M) of Table 17.8-1

$$\frac{\partial T}{\partial t} + \left(\left(\frac{1}{c}\sum_{\alpha=1}^{N}\mathbf{N}_\alpha\right)\cdot\nabla T\right) = \alpha\nabla^2 T \qquad \text{or} \qquad \frac{\partial T}{\partial t} + (\mathbf{v}^*\cdot\nabla T) = \alpha\nabla^2 T$$

which is just the 3-dimensional version of Eq. 20B.2-1.

*b*. When Eq. 20B.2-1 is rewritten in terms of the dimensionless temperature, we get

$$\frac{\partial \Pi_T}{\partial t} + v_z^*\frac{\partial \Pi_T}{\partial z} = \alpha\frac{\partial^2 \Pi_T}{\partial z^2}$$

We now postulate that the dimensionless temperature is a function only of the variable $Z_T = z/\sqrt{4\alpha t}$. This leads us to the following ordinary differential equation:

$$\frac{d^2\Pi_T}{dZ_T^2} + 2(Z_T - \varphi_T)\frac{d\Pi_T}{dZ_T} = 0 \qquad \text{where} \qquad \varphi_T = v_z^*\sqrt{\frac{t}{\alpha}} = \varphi\sqrt{\frac{\mathcal{D}_{AB}}{\alpha}}$$

It was shown in Eq. 21.1-24 that $\varphi$ is a function of the terminal mole fractions and the interfacial molar-fluxes, but not of time. Therefore, since $\varphi_T$ is just $\varphi$ multiplied by the square root of the Lewis number, it may be treated as a constant in the above ordinary differential equation. This equation may be solved by the same technique used in §20.1 to give

$$\Pi_T = 1 - \frac{1 - \operatorname{erf}(Z_T - \varphi_T)}{1 + \operatorname{erf}\varphi_T} = \frac{\operatorname{erf}(Z_T - \varphi_T) + \operatorname{erf}\varphi_T}{1 + \operatorname{erf}\varphi_T}$$

*c*. To get the interfacial heat flux, we use Fourier's law:

$$q_0 = -k\left.\frac{\partial T}{\partial z}\right|_{z=0} = -\frac{k(T_\infty - T_0)}{1 + \operatorname{erf}\varphi_T}\frac{d}{dz}\operatorname{erf}(Z_T - \varphi_T)\Big|_{z=0}$$

$$
\begin{aligned}
&= -\frac{k(T_\infty - T_0)}{1+\operatorname{erf}\varphi_T}\frac{2}{\sqrt{\pi}}\exp\left[-(Z_T-\varphi_T)^2\right]\Big|_{z=0}\frac{d}{dz}(Z_T-\varphi_T) \\
&= -\frac{k(T_\infty - T_0)}{1+\operatorname{erf}\varphi_T}\frac{2}{\sqrt{\pi}}\exp\left(-\varphi_T^2\right)\frac{1}{\sqrt{4\alpha t}} \\
&= \frac{k(T_0 - T_\infty)}{\sqrt{\pi}(1+\operatorname{erf}\varphi_T)\exp\left(+\varphi_T^2\right)\sqrt{\alpha t}}
\end{aligned}
$$

Next we calculate the ratio in Eq. 20B.2-4:

$$
\begin{aligned}
\frac{N_{A0}+N_{B0}}{q_0/\tilde{C}_p(T_0-T_\infty)} &= \frac{cv_z^*}{q_0/\tilde{C}_p(T_0-T_\infty)} = \frac{c\varphi_T\sqrt{\alpha/t}}{q_0/\tilde{C}_p(T_0-T_\infty)} \\
&= \frac{c\tilde{C}_p\varphi_T\sqrt{\alpha/t}}{k}\left[\sqrt{\pi}(1+\operatorname{erf}\varphi_T)\exp\left(+\varphi_T^2\right)\sqrt{\alpha t}\right] \\
&= \frac{\alpha\varphi_T}{k/\rho\hat{C}_p}\left[\sqrt{\pi}(1+\operatorname{erf}\varphi_T)\exp\left(+\varphi_T^2\right)\right] \\
&= \sqrt{\pi}(1+\operatorname{erf}\varphi_T)\varphi_T\exp\left(+\varphi_T^2\right)
\end{aligned}
$$

In the first step, we used Eq. (M) of Table 17.8-1, and later we used the relation $c\tilde{C}_p = \rho\hat{C}_p$.

**20B.3 Stoichiometric boundary condition for rapid irreversible reaction**

We begin by rewriting Eq. 20B.3-1 as

$$\frac{1}{a}c_A\left[\left(v_{Az}-v_z^*\right)+\left(v_z^*-v_R\right)\right]=-\frac{1}{b}c_A\left[\left(v_{Bz}-v_z^*\right)+\left(v_z^*-v_R\right)\right]$$

Then, using Fick's first law in the form of Eq. (B) of Table 17.8-2 (along with Eq. (I) of Table 17.8-1, we get

$$-\frac{1}{a}c\mathfrak{D}_{AS}\frac{\partial x_A}{\partial z}+\frac{1}{a}c_A\left(v_z^*-v_R\right)=+\frac{1}{b}c\mathfrak{D}_{BS}\frac{\partial x_B}{\partial z}-\frac{1}{b}c_B\left(v_z^*-v_R\right)$$

We now make use of the fact that the system has constant $c$ so that

$$-\frac{1}{a}\mathfrak{D}_{AS}\frac{\partial c_A}{\partial z}=+\frac{1}{b}\mathfrak{D}_{BS}\frac{\partial c_B}{\partial z}-\left(v_z^*-v_R\right)\left(\frac{1}{a}c_A+\frac{1}{b}c_B\right)$$

At the plane $z=z_R$, there is no $A$ or $B$ present, so that the last term on the right side is zero. Therefore, we get

$$-\frac{1}{a}\mathfrak{D}_{AS}\frac{\partial c_A}{\partial z}=+\frac{1}{b}\mathfrak{D}_{BS}\frac{\partial c_B}{\partial z}$$

for one of the boundary conditions at $z=z_R$ (cf. 20.1-31).

**20B.4 Taylor dispersion in slit flow**

This problem can be solved by paralleling the treatment in §20.5 for circular tubes. We give here the intermediate results along with the corresponding equation numbers in the text. Some results from Problem 2B.3 are needed here.

$$\frac{\partial \omega_A}{\partial t} + v_{z,\max}\left[1-\left(\frac{x}{B}\right)^2\right]\frac{\partial \omega_A}{\partial z} = \mathcal{D}_{AB}\frac{\partial^2 \omega_A}{\partial x^2} \qquad (20.5\text{-}1)$$

$$\langle \omega_A \rangle = \frac{1}{B}\int_0^B \omega_A dx \qquad (20.5\text{-}3)$$

$$\frac{\partial^2 \omega_A}{\partial \xi^2} = \frac{v_{z,\max}B^2}{\mathcal{D}_{AB}}\left(\tfrac{1}{3}-\xi^2\right)\frac{\partial \omega_A}{\partial \bar{z}} \qquad \text{where } \xi = \frac{x}{B} \qquad (20.5\text{-}7)$$

$$\omega_A = \frac{v_{z,\max}B^2}{6\mathcal{D}_{AB}}\left(\xi^2 - \tfrac{1}{2}\xi^4\right)\frac{\partial \langle \omega_A \rangle}{\partial \bar{z}} + \omega_A(0,\bar{z}) \qquad (20.5\text{-}10)$$

$$\langle \omega_A \rangle = \frac{v_{z,\max}B^2}{6\mathcal{D}_{AB}}\left(\frac{7}{30}\right)\frac{\partial \langle \omega_A \rangle}{\partial \bar{z}} + \omega_A(0,\bar{z}) \qquad (20.5\text{-}11)$$

$$\omega_A - \langle \omega_A \rangle = \frac{\frac{3}{2}\langle v_z \rangle B^2}{6\mathcal{D}_{AB}}\frac{\partial \langle \omega_A \rangle}{\partial \bar{z}}\left(-\tfrac{7}{30} + \xi^2 - \tfrac{1}{2}\xi^4\right) \qquad (20.5\text{-}12)$$

$$\begin{aligned}
2BW\rho\langle \omega_A(v_z - \langle v_z \rangle)\rangle &= 2BW\rho\langle v_z \rangle\left\langle \omega_A\left(\frac{v_z}{\langle v_z \rangle} - 1\right)\right\rangle \\
&= 2BW\rho\langle v_z \rangle\left[\frac{\frac{3}{2}\langle v_z \rangle B^2}{6\mathcal{D}_{AB}}\frac{\partial \langle \omega_A \rangle}{\partial \bar{z}}\right]\int_0^1\left(-\tfrac{7}{30} + \xi^2 - \tfrac{1}{2}\xi^4\right)\tfrac{3}{2}\left(\tfrac{1}{3} - \xi^2\right)d\xi \\
&= = 2BW\rho\langle v_z \rangle\left[\frac{\frac{3}{2}\langle v_z \rangle B^2}{6\mathcal{D}_{AB}}\frac{\partial \langle \omega_A \rangle}{\partial \bar{z}}\right]\left(-\frac{8}{105}\right) \\
&= -2BW\rho\frac{\partial \langle \omega_A \rangle}{\partial \bar{z}}\left[\frac{2B^2\langle v_z \rangle^2}{105\mathcal{D}_{AB}}\right] \qquad (20.5\text{-}13)
\end{aligned}$$

The quantity in brackets is the Taylor dispersion coefficient.

**20B.5 Diffusion from an instantaneous point source**

*a*. The relevant diffusion equation is

$$\frac{\partial \rho_A}{\partial t} = \mathcal{D}_{AB}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \rho_A}{\partial r}\right)$$

The time derivative of Eq. 20B.5-1 is

$$\frac{\partial \rho_A}{\partial t} = \rho_A\left(-\frac{3}{2t}+\frac{r^2}{4\mathcal{D}_{AB}t^2}\right)$$

Next we build up the quantity on the right side of the diffusion equation by differentiating Eq. 20B.5-1, thus:

$$\frac{\partial \rho_A}{\partial r} = \rho_A\left(-\frac{r}{2\mathcal{D}_{AB}t}\right) \quad \text{and} \quad r^2\frac{\partial \rho_A}{\partial r} = \rho_A\left(-\frac{r^3}{2\mathcal{D}_{AB}t}\right)$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial \rho_A}{\partial r}\right) = \rho_A\left[\left(-\frac{r}{2\mathcal{D}_{AB}t}\right)\left(-\frac{r^3}{2\mathcal{D}_{AB}t}\right)-\frac{3r^2}{2\mathcal{D}_{AB}t}\right]$$

$$\mathcal{D}_{AB}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \rho_A}{\partial r}\right) = \rho_A\left(\frac{r^2}{4\mathcal{D}_{AB}t}-\frac{3}{2t}\right)$$

*b*. As $r \to \infty$, we find that $\rho_A \to 0$, as it should.

*c*. Integrating over all space in spherical coordinates, we get

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\frac{m_A}{(4\pi\mathcal{D}_{AB}t)^{3/2}}\exp\left(-r^2/4\mathcal{D}_{AB}t\right)r^2 dr\sin\theta d\theta d\phi \\
&\quad = 4\pi\frac{m_A}{(4\pi\mathcal{D}_{AB}t)^{3/2}}\int_0^{\infty}\exp\left(-u^2\right)u^2 du\cdot\left(4\mathcal{D}_{AB}t\right)^{3/2} \\
&\quad = 4\pi\frac{m_A}{(4\pi\mathcal{D}_{AB}t)^{3/2}}\cdot\frac{\sqrt{\pi}}{4}\cdot\left(4\mathcal{D}_{AB}t\right)^{3/2} = m_A
\end{aligned}$$

In going from the first to the second line, we have made a change of variable: $r/\sqrt{4\mathcal{D}_{AB}t} = u$.

*d*. Let $n = (1/4\mathcal{D}_{AB}t)$. Then Eq. 20B.5-1 becomes

$$\rho_A = m_A (n/\pi)^{3/2} \exp(-nr^2)$$

The limit of this function as $n \to \infty$ is $\rho_A = m_A \delta(x)\delta(y)\delta(z)$. That is we get a delta function in the three spatial variables. In other words, all the material is "piled up" at the origin. [For more on this, see R. B. Bird, C. F. Curtiss, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Liquids*, Vol. 2, *Kinetic Theory*, Wiley-Interscience, New York (1987), §E.4, p. 405.]

20B.6 Unsteady diffusion with first-order chemical reaction

(a) the thin disk catalyst particle

The reference problem is

$$\frac{\partial \Theta}{\partial \tau} = \frac{\partial^2 \Theta}{\partial \eta^2}; \Theta(0,\eta) = 1; \Theta(\pm 1) = 0$$

The problem of interest here is

$$\frac{\partial \omega}{\partial \tau} = \frac{\partial^2 \omega}{\partial \eta^2} - \kappa\omega; \omega(0,\eta) = 1; \omega(\pm 1) = 0$$

Attempt a solution of the form

$$\omega = \Theta\, exp(-\kappa\tau)$$

Putting this trial solution into the differential equation for ω gives the original equation for Θ, and the new equation also satisfies the initial an boundary conditions. It is our desired solution.

(b) diffusion and reaction from a point source

Again we may write the solution by inspection as

$$p = ge^{kt}$$

Here g is the normalized probability of finding a diffusing particle, originally at the coordinate origin, at any point and time.

**20B.7 Simultaneous momentum, heat, and mass transfer: alternate boundary conditions.**

**(a)** With $n_{B0} = 0$, Eq. 20.2-51 gives the mass flux ratio

$$R_\omega = \frac{\omega_{A0} - \omega_{A\infty}}{1 - \omega_{A0}}$$

Then with $\Lambda = \mathrm{Sc}$ for binary diffusion, Eq. 20.2-52 gives the following implicit equation for the dimensionless mass flux,

$$K = \frac{1}{\mathrm{Sc}} \frac{\omega_{A0} - \omega_{A\infty}}{1 - \omega_{A0}} \Pi'(0, \mathrm{Sc}, K)$$

which can be solved conveniently with Fig. 22.8-5, Eq. 22.8-21, and Table 20.2-1, in the manner of the solution given for Problem 20A.6.

**(b)** With $\omega_A = 0$ and $n_{B0} = -2n_{A0}$, Eq. 20.2-51 gives the mass flux ratio

$$R_\omega = \frac{0 - \omega_{A\infty}}{-1 - 0} = \omega_{A\infty}$$

Then with $\Lambda = \mathrm{Sc}$ for diffusion in the binary gas phase, Eq. 20.2-52 gives the following implicit equation for the dimensionless mass flux,

$$K = \frac{1}{\mathrm{Sc}} \omega_{A\infty} \Pi'(0, \mathrm{Sc}, K)$$

which can be solved directly with the aid of Fig. 22.8-5, as noted in (a).

**(c)** A steady-state energy balance from the inner to the outer boundary of the wall gives, for the region of laminar boundary layer,

$$\rho_0 v_0 \hat{C}_p T_a = \rho v_0 \hat{C}_p T_0 + q_0$$

whence

$$q_0 = \rho_0 \hat{C}_p (T_a - T_0) = n_{A0} \hat{C}_p (T_a - T_0)$$

for this one-component system. With this substitution, Eq. 20.2-50 gives the energy flux ratio

$$R_T = \frac{n_{A0} \hat{C}_p (T_0 - T_\infty)}{n_{A0} \hat{C}_p (T_a - T_0)} = \frac{T_0 - T_\infty}{T_a - T_0}$$

Then with $\Lambda = \mathrm{Pr}$, Eq. 20.2-52 gives the following implicit equation for the dimensionless mass flux,

$$K = \frac{1}{\mathrm{Pr}} \frac{T_0 - T_\infty}{T_a - T_0} \Pi'(0, \mathrm{Pr}, K)$$

which can be solved directly with the aid of Fig. 22.8-5, as noted in (a) and (b).

**20B.8 Absorption from a pulsating bubble**

*a*. The essential point here is the calculation of the integral in Eq. 20.1-72. Let the surface of the bubble oscillate between $S_1 = 4\pi R_1^2$ and $S_2 = 4\pi R_2^2$, with $S_2/S_1 \geq 1$, so that

$$S(t) = S_1 \qquad 2n \leq \omega t \leq 2n+1 \qquad \text{for } n = 0,1,2,\cdots$$
$$S(t) = S_2 \qquad 2n+1 \leq \omega t \leq 2n+2 \qquad \text{for } n = 0,1,2,\cdots$$

We further let $(S_2/S_1)^2 = r \geq 1$. Then in each time region, we can calculate the value of $1/t$ times the integral in Eq. 20.1-72 as follows:

$0 \leq \omega t \leq 1$:
$$\frac{1}{t}\int_0^t \left(\frac{S_1}{S_1}\right)^2 d\bar{t} = 1$$

$1 \leq \omega t \leq 2$:
$$\frac{1}{t}\int_0^{1/\omega} \left(\frac{S_1}{S_2}\right)^2 d\bar{t} + \frac{1}{t}\int_{1/\omega}^{t} \left(\frac{S_2}{S_2}\right)^2 d\bar{t} = \frac{1}{\omega t}\left(\frac{1}{r} - 1\right) + 1$$

$2 \leq \omega t \leq 3$:
$$\frac{1}{t}\int_0^{1/\omega} \left(\frac{S_1}{S_1}\right)^2 d\bar{t} + \frac{1}{t}\int_{1/\omega}^{2/\omega} \left(\frac{S_2}{S_1}\right)^2 d\bar{t} + \frac{1}{t}\int_{2/\omega}^{t} \left(\frac{S_1}{S_1}\right)^2 d\bar{t}$$
$$= \frac{1}{\omega t}(r-1) + 1$$

$3 \leq \omega t \leq 4$:
$$\frac{1}{t}\int_0^{1/\omega} \left(\frac{S_1}{S_2}\right)^2 d\bar{t} + \frac{1}{t}\int_{1/\omega}^{2/\omega} \left(\frac{S_2}{S_2}\right)^2 d\bar{t} + \frac{1}{t}\int_{2/\omega}^{3/\omega} \left(\frac{S_1}{S_2}\right)^2 d\bar{t}$$
$$+ \frac{1}{t}\int_{3/\omega}^{t} \left(\frac{S_1}{S_2}\right)^2 d\bar{t} = \frac{2}{\omega t}\left(\frac{1}{r} - 1\right) + 1$$

$4 \leq \omega t \leq 5$:
$$\frac{2}{\omega t}(r-1) + 1$$

$5 \leq \omega t \leq 6$
$$\frac{3}{\omega t}\left(\frac{1}{r} - 1\right) + 1$$

We have now gone sufficiently far to be able to see what the pattern is, and we can summarize the results as follows:

| | $j$ even | $j$ odd |
|---|---|---|
| $j \le \omega t \le j+1$ <br> $(j = 0,1,2,...)$ | $\frac{j}{2\omega t}(r-1)+1$ | $\frac{(j+1)}{2\omega t}\left(\frac{1}{r}-1\right)+1$ |

In for large values of the time, these expressions become very nearly

| | | |
|---|---|---|
| $j \le \omega t \le j+1$ <br> $(j = 0,1,2,...)$ | $\frac{1}{2}(1+r)$ | $\frac{1}{2}\left(1+\frac{1}{r}\right)$ |

Thus, the dimensionless molar transfer rate at the interface becomes (according to Eq. 20.1-72) at large values of the time:

| | | | |
|---|---|---|---|
| $j \le \omega t \le j+1$ <br> $(j = 0,1,2,...)$ | $\frac{N_{A0}(t)}{c_{A0}\sqrt{\mathcal{D}_{AB}/\pi t}} =$ | $\sqrt{\frac{2}{1+r}}$ | $\sqrt{\frac{2}{1+(1/r)}}$ |

If $r = 1$, then there is no droplet oscillation and the dimensionless molar transfer rate is 1. For $r = 2$, say, the dimensionless molar transfer rate has a square-wave form, oscillating between $\sqrt{\frac{2}{3}}$ and $\sqrt{\frac{4}{3}}$.

*b*. A more interesting problem might be to get the total moles of $A$ transferred at time $t$, using Eq. 20.1-73; for large values of the time, we get:

| | | | |
|---|---|---|---|
| $j \le \omega t \le j+1$ <br> $(j = 0,1,2,...)$ | $\frac{M_A(t)/S(t)}{c_{A0}\sqrt{4\mathcal{D}_{AB}t/\pi}} =$ | $\sqrt{\frac{1}{2}(1+r)}$ | $\sqrt{\frac{1}{2}\left(1+\frac{1}{r}\right)}$ |

Now, if $S(t) = \frac{1}{2}(S_1 + S_2)$ (a constant),

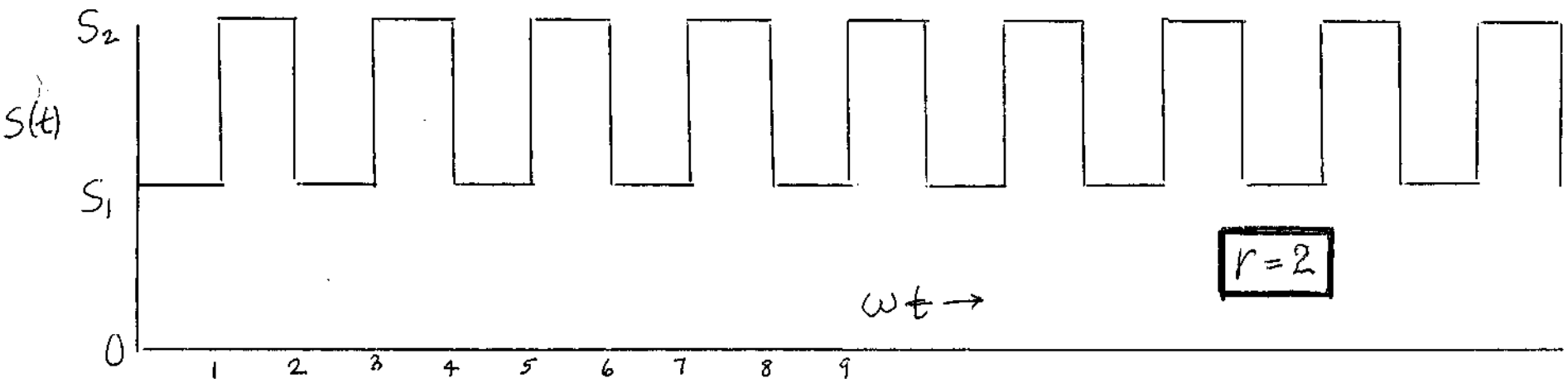
$$M_A(t) = \tfrac{1}{2}S_1(1+r)c_{A0}\sqrt{4\mathcal{D}_{AB}t/\pi}$$

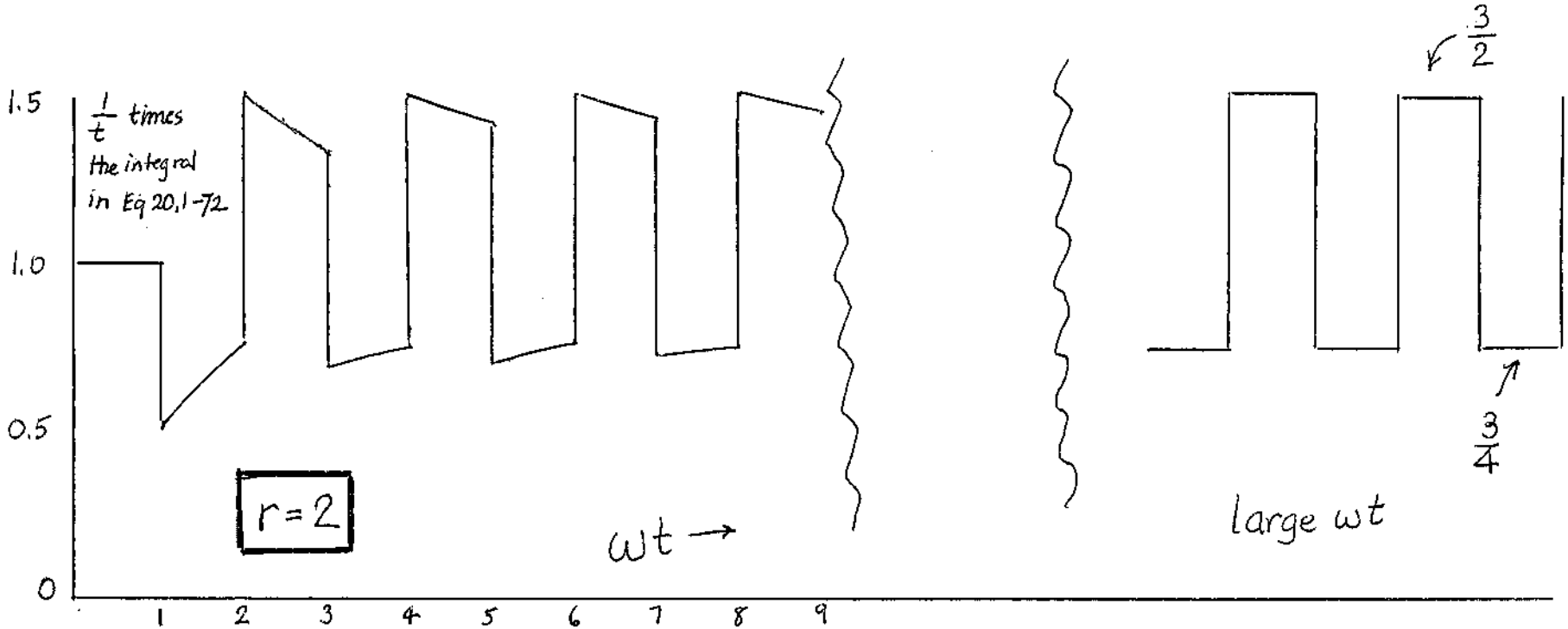
If, on the other hand, $S(t)$ is oscillating between $S_1$ and $S_2$,

$$M_A(t) = \left[ \tfrac{1}{2} S_1 \sqrt{\frac{1}{2}(1+r)} + \tfrac{1}{2} S_2 \sqrt{\frac{1}{2}\left(1+\frac{1}{r}\right)} \right] c_{A0} \sqrt{4\mathcal{D}_{AB} t/\pi}$$

$$= \frac{1}{2\sqrt{2}} S_1 \left[ \sqrt{(1+r)} + r\sqrt{\left(1+\frac{1}{r}\right)} \right] c_{A0} \sqrt{4\mathcal{D}_{AB} t/\pi}$$

For $r = 2$, the first expression (constant surface area) gives a numerical constant of 1.50, whereas the second (oscillating area) gives 1.47. Keep in mind that the second expression is approximate (for large $t$ and that this may account for the discrepancy between the two values.


$S_2$
$S(t)$
$S_1$
0
1 2 3 4 5 6 7 8 9
$\omega t \rightarrow$
$r = 2$


1.5
1.0
0.5
0
$\frac{1}{t}$ times
the integral
in Eq 20.1-72
$r = 2$
1 2 3 4 5 6 7 8 9
$\omega t \rightarrow$
large $\omega t$
$\frac{3}{2}$
$\frac{3}{4}$

20B.9 Verification of the solution of the Taylor-dispersion equation

(a) Begin by writing

$$u \equiv z - < v > t$$

$$dq = \left(\frac{\partial q}{\partial t}\right)_z dt + \left(\frac{\partial q}{\partial z}\right)_t dz$$

$$\left(\frac{\partial q}{\partial t}\right)_u = \left(\frac{\partial q}{\partial t}\right)_z \cdot 1 + \left(\frac{\partial q}{\partial z}\right)_t \left(\frac{\partial z}{\partial t}\right)_u$$

$$= \left(\frac{\partial q}{\partial t}\right)_z + \left(\frac{\partial q}{\partial z}\right)_t (< v >)$$

$$\left(\frac{\partial q}{\partial u}\right)_t = \left(\frac{\partial q}{\partial z}\right)_t ; \left(\frac{\partial^2 q}{\partial u^2}\right) = \left(\frac{\partial^2 q}{\partial z^2}\right)$$

where q is any scalar, here $< \rho >$. Putting these results into Eq. 20.5-17 gives

$$\left(\frac{\partial < \rho >}{\partial t}\right)_u = K \left(\frac{\partial^2 < \delta >}{\partial u^2}\right)_t$$

which is the desired result.

(b) Our differential equation is of the form $\dfrac{\partial q}{\partial t} = K \dfrac{\partial^2 q}{\partial z^2}$

where q is <ρ>and the proposed solution can be written as

$$q = \frac{A}{\sqrt{t}} exp\left(-\frac{u^2}{4Kt}\right)$$

Now

$$\partial q / \partial t = -\frac{q}{2t} + \frac{u^2}{4Kt^2} q$$

$$\frac{\partial q}{\partial u} = -\frac{u}{2Kt} q$$

$$\frac{\partial^2 q}{\partial u^2} = -\frac{q}{2Kt} + \frac{u^2 q}{4K^2 t^2}$$

$$K \frac{\partial^2 q}{\partial u^2} = -\frac{q}{2t} + \frac{u^2 q}{4Kt^2}$$

The differential equation is satisfied.

(c) Simple inspection shows that these conditions are met.

< end >

20C.1 Order-of-magnitude analysis of gas absorption from a growing bubble

(a) The volumetric flow rate for this system is constant at a value

$$Q = \frac{d}{dt}\frac{4}{3}\pi r_s^3 = 4\pi r_s^2 \frac{dr_s}{dt}$$

$$v_r\Big|_{r=r_s} = (Q/4\pi)\frac{1}{r_s^2}$$

The same volumetric flow must occur at any distance, and this constraint requires the suggested result.

(b) This is obtained simply by putting the above result into the spherically symmetric continuity equation.

(c) The expression

$$\frac{r_s^2}{(y-r_s)} = \frac{1}{(1-y/r_s)^2}$$

The desired expression can be obtained simply by division, and it is also available in many standard tables, for example, H. B. Dwight, Tables of Integrals and Other Mathematical Data, Macmillan, #9.06.

(d) Terms (5), (6) and (7) are second, third and forth order respectively.

< end >

**20C.3 Absorption with chemical reaction in a semi-infinite medium**

*a*. First, introduce the following dimensionless variables: $\Gamma = c_A/c_{A0}$, $\xi = \sqrt{k_1'''/\mathcal{D}_{AB}}\,x$, and $\tau = k_1''t$; also, define the combinations $m = \left[\left(\xi/2\sqrt{\tau}\right) - \sqrt{\tau}\right]$ and $p = \left[\left(\xi/2\sqrt{\tau}\right) + \sqrt{\tau}\right]$. Then Eqs. 20C.3-1 and 2 are

$$\frac{\partial \Gamma}{\partial \tau} = \frac{\partial^2 \Gamma}{\partial \xi^2} - \Gamma \qquad \text{and} \qquad 2\Gamma = e^{-\xi}\operatorname{erfc} m + e^{\xi}\operatorname{erfc} p$$

Then we calculate the derivatives as follows:

$$2\frac{\partial \Gamma}{\partial \tau} = \frac{e^{-\xi}e^{-m^2}}{\sqrt{\pi}}\left(\frac{\xi}{2\tau^{3/2}} + \frac{1}{\sqrt{\tau}}\right) + \frac{e^{\xi}e^{-p^2}}{\sqrt{\pi}}\left(\frac{\xi}{2\tau^{3/2}} - \frac{1}{\sqrt{\tau}}\right)$$

$$2\frac{\partial \Gamma}{\partial \xi} = -e^{-\xi}\operatorname{erfc} m - \frac{e^{-\xi}e^{-m^2}}{\sqrt{\pi\tau}} + e^{\xi}\operatorname{erfc} p - \frac{e^{\xi}e^{-p^2}}{\sqrt{\pi\tau}}$$

$$2\frac{\partial^2 \Gamma}{\partial \xi^2} = e^{-\xi}\operatorname{erfc} m + \frac{e^{-\xi}e^{-m^2}}{\sqrt{\pi\tau}} + \frac{e^{-\xi}e^{-m^2}}{\sqrt{\pi\tau}} + \frac{e^{-\xi}e^{-m^2}}{\sqrt{\pi\tau}}\left(\frac{\xi}{2\sqrt{\tau}} - \sqrt{\tau}\right)$$
$$+ e^{\xi}\operatorname{erfc} p - \frac{e^{\xi}e^{-p^2}}{\sqrt{\pi\tau}} - \frac{e^{\xi}e^{-p^2}}{\sqrt{\pi\tau}} + \frac{e^{\xi}e^{-p^2}}{\sqrt{\pi\tau}}\left(\frac{\xi}{2\sqrt{\tau}} + \sqrt{\tau}\right)$$

When these contributions are substituted into the diffusion equation, it may be seen that the equation is satisfied.

At $t = 0$, we find that $m = \infty$ and $p = \infty$, so that erfc $m$ and erfc $p$ are both equal to zero, and hence the initial condition is satsified.

At $x = 0$, $\Gamma = \frac{1}{2}\left[\operatorname{erfc}(-k_1''t) + \operatorname{erfc}(+k_1''t)\right]$ which may also be written as $\frac{1}{2}\left[1 - \operatorname{erf}(-k_1''t) + 1 - \operatorname{erf}(+k_1''t)\right]$; however $\operatorname{erf}(-u) = -\operatorname{erf}(u)$, so that we are left with $\frac{1}{2}\cdot 2 = 1$, which indicates that the boundary condition at $x = 0$ is satisfied.

At $x = \infty$, erfc$m$ and erfc$p$ both go to zero, and $e^{-\xi}$ also. Hence we have to prove that $e^{\xi}\operatorname{erfc} p$ goes to zero:

$$\lim_{\xi\to\infty}\frac{\operatorname{erfc}\left(\xi/2\sqrt{t}\right)}{e^{-\xi}}=\lim_{\xi\to\infty}\frac{\left(2/\sqrt{\pi}\right)\left(\exp(-\xi^2/4t)\right)\left(1/2\sqrt{t}\right)}{-e^{-\xi}}=0$$

which shows that the boundary condition at $x=\infty$ is satisfied.

*b*. To get the molar flux at the wall, we once again make use of dimensionless variables:

$$\begin{aligned}
N_{Ax}\big|_{x=0} &= -\mathcal{D}_{AB}c_{A0}\frac{\partial\Gamma}{\partial x}\bigg|_{x=0} = -\mathcal{D}_{AB}c_{A0}\sqrt{\frac{k_1'''}{\mathcal{D}_{AB}}}\frac{\partial\Gamma}{\partial\xi}\bigg|_{\xi=0} \\
&= -\frac{1}{2}c_{A0}\sqrt{\mathcal{D}_{AB}k_1'''}\left[-e^{-\xi}\operatorname{erfc}m-\frac{e^{-\xi}e^{-m^2}}{\sqrt{\pi\tau}}+e^{\xi}\operatorname{erfc}p-\frac{e^{\xi}e^{-p^2}}{\sqrt{\pi\tau}}\right]\bigg|_{\xi=0} \\
&= -\frac{1}{2}c_{A0}\sqrt{\mathcal{D}_{AB}k_1'''}\left[-\operatorname{erfc}\left(-\sqrt{\tau}\right)-\frac{e^{-\tau}}{\sqrt{\pi\tau}}+\operatorname{erfc}\left(\sqrt{\tau}\right)-\frac{e^{-\tau}}{\sqrt{\pi\tau}}\right] \\
&= -\frac{1}{2}c_{A0}\sqrt{\mathcal{D}_{AB}k_1'''}\left[-2\operatorname{erf}\sqrt{\tau}-\frac{2e^{-\tau}}{\sqrt{\pi\tau}}\right]
\end{aligned}$$

which is equivalent to Eq. 20C.3-3. Actually a neater form of the result is:

$$N_{Ax}\big|_{x=0}=c_{A0}\sqrt{\frac{\mathcal{D}_{AB}}{\pi t}}\left[\sqrt{\pi k_1'''t}\operatorname{erf}\sqrt{k_1'''t}+e^{-k_1'''t}\right]$$

In this form, the quantity in the brackets is the correction factor for the absorption rate because of the chemical reaction that is occurring.

*c*. The total moles absorbed through an area $A$ is

$$\begin{aligned}
M_A &= A\int_0^{k_1'''t}N_{Ax}\big|_{x=0}dt=\frac{A}{k_1'''}\int_0^{k_1'''t}N_{Ax}\big|_{x=0}d\tau \\
&= \frac{A}{k_1'''}c_{A0}\sqrt{\mathcal{D}_{AB}k_1'''}\int_0^{k_1'''t}\left[\operatorname{erf}\sqrt{\tau}+\frac{e^{-\tau}}{\sqrt{\pi\tau}}\right]d\tau
\end{aligned}$$

We now perform the two integrations; in both of them we make the change of variable $\sqrt{\tau}=y$:

$$\int_0^{k_1''t} \frac{e^{-\tau}}{\sqrt{\pi\tau}} d\tau = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{k_1''t}} e^{-y^2} dy = \operatorname{erf}\sqrt{k_1''t}$$

$$\begin{aligned}
\int_0^{k_1''t} \operatorname{erf}\sqrt{\tau} d\tau &= \int_0^{\sqrt{k_1''t}} (\operatorname{erf} y) 2y dy = \int_0^{\sqrt{k_1''t}} \left( \frac{2}{\sqrt{\pi}} \int_0^y e^{-s^2} ds \right) 2y dy \\
&= \frac{4}{\sqrt{\pi}} \int_0^{\sqrt{k_1''t}} \left( \int_s^{\sqrt{k_1''t}} y dy \right) e^{-s^2} ds = \frac{4}{\sqrt{\pi}} \int_0^{\sqrt{k_1''t}} \left[ \tfrac{1}{2}\left(k_1''t - s^2\right) \right] e^{-s^2} ds \\
&= k_1''t \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{k_1''t}} e^{-s^2} ds - \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{k_1''t}} s^2 e^{-s^2} ds \\
&= k_1''t \operatorname{erf}\sqrt{k_1''t} - \frac{2}{\sqrt{\pi}} \left[ -\tfrac{1}{2} s e^{-s^2} \Big|_0^{\sqrt{k_1''t}} + \tfrac{1}{2} \int_0^{\sqrt{k_1''t}} e^{-s^2} ds \right] \\
&= k_1''t \operatorname{erf}\sqrt{k_1''t} + \frac{1}{\sqrt{\pi}} \sqrt{k_1''t} e^{-k_1''t} - \tfrac{1}{2} \operatorname{erf}\sqrt{k_1''t}
\end{aligned}$$

When these integrals are substituted into the expression for $M_A$, we end up with Eq. 20C.3-4.

*d*. For large values of $k_1''t$, the exponential in the last term goes to zero, the error function goes to unity, and the only remaining terms are those in the large parentheses in Eq. 20C.3-4. In this way, Eq. 20C.3-5 is obtained.

20C.4 Design of fluid control circuits

The response to a pulse input to a "long" tube can be expressed as

$$\rho_i(L,t) = \frac{m_i / A}{\sqrt{4\pi Et}} exp\left[\frac{(t - L/v)^2}{4(Et/v^2)}\right]$$

Here $m_i$ is the mass of solute "i" fed to a tube of length L and cross-sectional area A. This is just 20.5-18, modified to be explicit in time, and using a generalized dispersion coefficient E.

For a short pulse most of the solute leaves near the mean residence time

$$\bar{t} \equiv L / v$$

where $v$ is the flow average velocity, <v>. The term "E" is the sum of Taylor dispersion and molecular diffusivity:

$$E = K + \mathcal{D}_{im}$$

If we then approximate $t$ as $\bar{t}$ the exit composition has the form of a normal probability distribution with variance

$$\sigma^2 = 2E\bar{t} / v^2$$

and $\sigma$ is the standard deviation of the distribution.

A reasonable set of design criteria is then that

$$\sqrt{2E\bar{t}} / v \leq 0.05t_0$$

$$L >> 2vR^2 / (3.8)^2 \mathcal{D}_{im} \text{ (Eq. 20.5-4)}$$

In practice the ">>" requirement need not be very large. It is really only necessary that there not be much tailing. A discussion of this point is beyond the scope of the present discussion.

In any event for most "long" ducts we may approximate "E" as the "K" of Eq. 20.5-15:

$$K = R^2 v^2 / 48\mathcal{D}_{im}; 2Rv / \nu < 2,100$$

For higher Re turbulence results in much smaller dispersion coefficients

$$K \approx 10Rv^*$$

We will not explore this turbulent region here.

We thus find

$$\frac{R^2 v^2}{48\mathcal{D}_{im}} < 0.05^2 t_0^2 \frac{v^3}{L}$$

$$t_0^2 > 8.333\frac{LR^2}{v\mathcal{D}_{im}}; t_0 > 2.89\sqrt{\frac{LR^2}{v\mathcal{D}_{im}}}$$

with the constraint on length introduced above. This in turn means that

$$t_0 >> \frac{2.89\cdot\sqrt{2}}{3.8}\sqrt{\frac{R^2}{\mathcal{D}_{im}}} = 1.07R^2 / \mathcal{D}_{im}$$

$$= 2.7\cdot 10^{-3} cm^2 / \mathcal{D}_{im}$$

Then for typical liquids, with diffusivities on the order of $10^{-5}$ $cm^2/s$,

$$t_0 >> 2.7\cdot 10^2 s = 4.5\, min$$

whereas for gases with diffusivities of, let us say 0.27,

$$t_0 >> 0.01s$$

< end >

**20C.5 Dissociation of a gas caused by a temperature gradient**
[The small Roman numerals after the equations correspond to the equations given by Dirac in his paper.]

*a. The equation of state*

It if assumed that the diffusing mixture is at constant temperature and that the *ideal gas law* is applicable, then we have

$$p = (n_1 + n_2)\kappa T \qquad \text{or} \qquad (n_1 + n_2)T = p/\kappa = \text{constant}$$

Here $n_1$ and $n_2$ are the number densities of the single and double species at any point in the tube. Then if $n_1^0$ and $n_2^0$ are the values of $n_1$ and $n_2$ at the same pressure and temperature in a system with no temperature gradient, then we have also

$$\left(n_1^0 + n_2^0\right)T = p/\kappa = \text{constant}$$

Then Dirac defines a quantity $\sigma$ by the following:

$$n_1 - n_1^0 = \sigma \qquad \text{and} \qquad n_2 - n_2^0 = -\sigma \qquad \text{(i)}$$

Thus $\sigma$ gives the deviation from the equilibrium state, resulting from the temperature gradient.

*b. The equation of continuity for the mixture*

Equation 3.1-4 (or Eq. (D) of Table 19.2-1) gives, for steady state,

$$\frac{d}{dz}\rho v_z = 0 \qquad \text{or} \qquad \rho v_z = \text{constant}$$

where $v_z$ is the mass-average velocity. Therefore, if we make use of Eq. (B) of Table 17.7-2, and if we make use of the fact that there is no net movement of mass across any plane z = constant, then we have

$$\rho_1 v_1 + \rho_2 v_2 = 0 \qquad \text{or} \qquad n_1 m_1 v_1 + n_2 m_2 v_2 = 0$$

where $m_1$ and $m_2$ are the masses of single and double molecules, respectively. Then

$$n_1 m_1 v_1 + n_2 (2m_1) v_2 = 0 \qquad \text{or} \qquad n_1 v_1 + 2n_2 v_2 = 0 \qquad \text{(ii)}$$

*c. The energy equation for the mixture*

For a steady-state system with all species subject to the same external force, the one-dimensional energy equation of Eq. (C) gives $e_z = E$, a constant. Then we use the approximate expression for the energy flux (also used by Dirac) given in Eq. 19.3-6

$$-k\frac{dT}{dz} + \left(N_1\overline{H}_1 + N_2\overline{H}_2\right) = E \quad \text{or} \quad -k\frac{dT}{dz} + c_1 v_1 \tilde{H}_1 + c_2 v_2 \tilde{H}_2 = E$$

where the partial molar enthalpies have been replaced by the enthalpies per mole, as is normally done for ideal-gas systems. Then switching to quantities per molecule, we may write

$$-k\frac{dT}{dz} + n_1 v_1 h_1 + n_2 v_2 h_2 = E \quad \text{or} \quad -k\frac{dT}{dz} + \tfrac{1}{2} n_1 v_1 \left(2h_1 - h_2\right) = E$$

Then if we define the heat of dissociation by $\Delta h_{\text{dissoc}} = 2h_1 - h_2$, the energy equation becomes

$$-k\frac{dT}{dz} + \tfrac{1}{2} n_1 v_1 \Delta h_{\text{dissoc}} = E \qquad \text{(iii)}$$

*d. The expression for the molar flux*

If we write Eq. 24.2-8 for species "1" in a 2-component mixture, it becomes:

$$\frac{x_1 x_2}{\mathfrak{D}_{12}}\left(v_1 - v_2\right) = -\frac{dx_1}{dz} - \frac{x_1 x_2}{\mathfrak{D}_{12}}\left(\frac{D_1^T}{\rho_1} - \frac{D_2^T}{\rho_2}\right)\frac{d\ln T}{dz}$$

This equation is a generalization of Eq. (F) of Table 17.8-2, which takes into account the extra term needed to describe thermal diffusion. Since $D_1^T + D_2^T = 0$ (see paragraph just above Eq. 24.2-3), the above equation may be written as

$$v_1 - v_2 = -\frac{\mathfrak{D}_{12}}{x_1 x_2}\frac{dx_1}{dz} - \frac{D_1^T}{\rho}\frac{d\ln T}{dz} \qquad \text{(iv)}$$

This is exactly Dirac's equation (iv) if $-D_1^T/\rho$ is relabeled as $D'$.

*e. The equation of continuity for species "1"*

Equation 19.1-10 at steady state is

$$0 = -\frac{d}{dz}c_1 v_1 + R_1 \qquad \text{or} \qquad 0 = -\frac{d}{dz}n_1 v_1 + R_1 \tilde{N}$$

the second form being in molecular, rather than molar, quantities. Dirac then postulates that, for systems not too far removed from equilibrium, first order kinetics can be assumed so that

$$\frac{d}{dz}n_1 v_1 = -k_1'''\sigma \qquad \text{(v)}$$

The above set of equations (i to v) are the equations that Dirac derived, and these have to be solved simultaneously for $n_1$, $n_2$, $v_1$, $v_2$, and $T$. for temperature dependent thermal conductivity, diffusivity, and thermal diffusion coefficient.

First he shows how to reduce the set of five equations to a set of two equations. Then he resorts to numerical methods to complete the calculation for some sample values of the constants in the system.

**20D.1 Two-bulb experiment for measuring gas diffusivities--analytical solution**

The diffusion equation to be solved is

$$\frac{\partial c_A}{\partial t} = \mathfrak{D}_{AB} \frac{\partial^2 c_A}{\partial z^2}$$

with the initial conditions that $c_A(z,0) = c_A^+$ for $0 < z < L$ and that $c_A(z,0) = c_A^-$ for $-L < z < 0$, and the following boundary conditions:

$$\text{At } z = 0, \quad c_A = \tfrac{1}{2}\left(c_A^+ + c_A^-\right)$$

$$\text{At } z = \pm L, \quad \mp \mathfrak{D}_{AB} \frac{\partial c_A}{\partial z} = V \frac{\partial c_A}{\partial t}$$

Next, we introduce dimensionless quantities as follows: $\zeta = z/L$, $\tau = \mathfrak{D}_{AB} t/L^2$, $\Gamma = \left[c_A - \tfrac{1}{2}\left(c_A^+ + c_A^-\right)\right] \Big/ \tfrac{1}{2}\left(c_A^+ - c_A^-\right)$, and $N = SL/V$. Then the problem may be restated as

$$\frac{\partial \Gamma}{\partial t} = \frac{\partial^2 \Gamma}{\partial \zeta^2}$$

with initial condition that $\Gamma(\zeta,0) = 1$ for $0 < \zeta < 1$ and the boundary conditions

$$\text{At } \zeta = 0, \quad \Gamma = 0; \qquad \text{and at } \Gamma = 1, \qquad -\frac{\partial \Gamma}{\partial \zeta} = \frac{1}{N} \frac{\partial \Gamma}{\partial \tau}$$

That is, we solve only the right half of the problem, with the understanding that the left side can be obtained from symmetry conditions.

We assume a product solution $\Gamma(\zeta,\tau) = Z(\zeta)T(\tau)$. Then when this is substituted into the partial differential equation, we get the following two ordinary differential equations

$$\frac{dT}{d\tau} = -\gamma^2 T \qquad \text{and} \qquad \frac{d^2 Z}{d\zeta^2} + \gamma^2 Z = 0$$

in which $\gamma^2$ is the separation constant. These equations have the solutions

$$T = A_0 \exp\left(-\gamma^2 \tau\right) \qquad \text{and} \qquad Z = A_1 \sin\gamma\zeta + A_2 \cos\gamma\zeta$$

The first bounday condition gives $A_2 = 0$. The second boundary condition gives $\gamma \tan\gamma = N$, which has an infinite number of solutions, which we call $\gamma_n$. Hence the solution to the partial differential equation is of the form

$$\Gamma = \sum_{n=1}^{\infty} D_n \exp\left(-\gamma_n^2 \tau\right) \sin\gamma_n \zeta$$

The $D_n$ are to be determined from the initial condition that

$$1 = \sum_{n=1}^{\infty} D_n \sin\gamma_n \zeta \qquad (\#)$$

However, the $\sin\gamma_n\zeta$ are not orthogonal for $0 < \zeta < 1$, as may be seen from

$$\int_0^1 \sin\gamma_m\zeta \sin\gamma_n\zeta d\zeta = \frac{1}{2}\left[\frac{\sin(\gamma_n - \gamma_m)}{\gamma_n - \gamma_m} - \frac{\sin(\gamma_n + \gamma_m)}{\gamma_n + \gamma_m}\right] \qquad (m \neq n)$$

If we use the fact that $\gamma_n \tan\gamma_n = N$, we can show that

$$\sin\gamma_n = \frac{N(-1)^{n+1}}{\sqrt{\gamma_n^2 + N^2}} \qquad \text{and} \qquad \cos\gamma_n = \frac{\gamma_n(-1)^{n+1}}{\sqrt{\gamma_n^2 + N^2}}$$

Then we get

$$\int_0^1 \sin\gamma_m\zeta \sin\gamma_n\zeta d\zeta = -N(-1)^{m+n}\frac{1}{\sqrt{\gamma_n^2+N^2}}\frac{1}{\sqrt{\gamma_m^2+N^2}} \neq 0 \quad (m \neq n)$$

Although the sine functions are not orthogonal, the cosines are:

$$\int_0^1 \cos\gamma_m\zeta \cos\gamma_n\zeta d\zeta = \frac{1}{2}\left(\frac{\gamma_n^2+N^2+N}{\gamma_n^2+N^2}\right)\delta_{mn}$$

Next we use this last bit of information to get the $D_n$. The late Professor R. J. Buehler (Department of Statistics, University of Minnesota) suggested the following procedure: We first rewrite the initial condition (#) as

$$\lim_{B\to 0}\left(1-e^{-\zeta/B}\right) = \sum_{n=1}^{\infty} D_n \sin\gamma_n\zeta$$

Then we consider the same condition, but without taking the limit:

$$\left(1-e^{-\zeta/B}\right) = \sum_{n=1}^{\infty} D_n(B)\sin\gamma_n\zeta$$

This may be differentiated to give

$$\frac{1}{B}e^{-\zeta/B} = \sum_{n=1}^{\infty} D_n(B)\gamma_n \cos\gamma_n\zeta$$

Now we may multiply by $\cos\gamma_m\zeta$ and integrate from 0 to 1

$$\frac{1}{B}\int_0^1 e^{-\zeta/B}\cos\gamma_m\zeta d\zeta = \sum_{n=1}^{\infty} D_n(B)\gamma_n \int_0^1 \cos\gamma_n\zeta \cos\gamma_m\zeta d\zeta$$

The integrations are now performed, and use is made of the orthoginality relation for the cosine functions. This result is

$$\frac{1}{1+\gamma_m^2B^2} - \frac{e^{-1/B}}{1+\gamma_m^2B^2}\left(\cos\gamma_m + \gamma_m\sin\gamma_m\right) = D_m(B)\frac{\gamma_m}{2}\left(\frac{\gamma_m^2+N^2+N}{\gamma_m^2+N^2}\right)$$

Now take the limit as $B$ goes to zero, so that the left side becomes unity, and

$$D_m = \frac{2}{\gamma_m}\left(\frac{\gamma_m^2 + N^2}{\gamma_m^2 + N^2 + N}\right)$$

Hence the final solution for the concentration in the two-bulb system is

$$\Gamma(\zeta,\tau) = \sum_{n=1}^{\infty}\frac{2}{\gamma_n}\left(\frac{\gamma_n^2 + N^2}{\gamma_n^2 + N^2 + N}\right)\exp(-\gamma_n^2\tau)\sin\gamma_n\zeta$$

The concentration in the reservoir at $\zeta = 1$ is then

$$\begin{aligned}\Gamma(1,\tau) &= \sum_{n=1}^{\infty}\frac{2}{\gamma_n}\left(\frac{\gamma_n^2 + N^2}{\gamma_n^2 + N^2 + N}\right)\exp(-\gamma_n^2\tau)\sin\gamma_n \\ &= \sum_{n=1}^{\infty}\frac{2}{\gamma_n}\left(\frac{\gamma_n^2 + N^2}{\gamma_n^2 + N^2 + N}\right)\exp(-\gamma_n^2\tau)\frac{(-1)^{n+1}N}{\sqrt{\gamma_n^2 + N^2}} \\ &= \sum_{n=1}^{\infty}\frac{2N(-1)^{n+1}}{\gamma_n}\left(\frac{\sqrt{\gamma_n^2 + N^2}}{\gamma_n^2 + N^2 + N}\right)\exp(-\gamma_n^2\tau)\end{aligned}$$

**20D.2 Unsteady-state interphase diffusion**

*a.* Take the Laplace transform of Eq. 20C.2-1 and 2:

$$p\bar{c}_{\mathrm{I}} - c_{\mathrm{I}}(0) = \mathscr{D}_{\mathrm{I}} \frac{d^2\bar{c}_{\mathrm{I}}}{dz^2} \quad \text{and} \quad p\bar{c}_{\mathrm{II}} - c_{\mathrm{II}}(0) = \mathscr{D}_{\mathrm{II}} \frac{d^2\bar{c}_{\mathrm{II}}}{dz^2}$$

Then using the initial conditions and rearranging, we get

$$\mathscr{D}_{\mathrm{I}} \frac{d^2\bar{c}_{\mathrm{I}}}{dz^2} - p\bar{c}_{\mathrm{I}} = -c_{\mathrm{I}}^{\mathrm{o}} \quad \text{and} \quad \mathscr{D}_{\mathrm{II}} \frac{d^2\bar{c}_{\mathrm{II}}}{dz^2} - p\bar{c}_{\mathrm{II}} = -c_{\mathrm{II}}^{\mathrm{o}}$$

These ordinary differential equations can be solved to give

$$\bar{c}_{\mathrm{I}} = C_1 \exp\left(\sqrt{p/\mathscr{D}_{\mathrm{I}}}\, z\right) + C_2 \exp\left(-\sqrt{p/\mathscr{D}_{\mathrm{I}}}\, z\right) + \frac{c_{\mathrm{I}}^{\mathrm{o}}}{p}$$

$$\bar{c}_{\mathrm{II}} = C_3 \exp\left(\sqrt{p/\mathscr{D}_{\mathrm{II}}}\, z\right) + C_4 \exp\left(-\sqrt{p/\mathscr{D}_{\mathrm{II}}}\, z\right) + \frac{c_{\mathrm{II}}^{\mathrm{o}}}{p}$$

The boundary conditions at plus and minus infinity require that $C_2$ and $C_3$ be zero. The boundary conditions at the interface ($z = 0$) yield:

$$\left(C_1 + \frac{c_{\mathrm{I}}^{\mathrm{o}}}{p}\right) m = \left(C_4 + \frac{c_{\mathrm{II}}^{\mathrm{o}}}{p}\right) \quad \text{and} \quad C_4 = -\sqrt{\frac{\mathscr{D}_{\mathrm{I}}}{\mathscr{D}_{\mathrm{II}}}}\, C_1$$

From these two equations we get the integration constants

$$C_1 = \frac{1}{p} \frac{c_{\mathrm{II}}^{\mathrm{o}} - m c_{\mathrm{I}}^{\mathrm{o}}}{m + \sqrt{\mathscr{D}_{\mathrm{I}}/\mathscr{D}_{\mathrm{II}}}} \quad \text{and} \quad C_4 = -\sqrt{\frac{\mathscr{D}_{\mathrm{I}}}{\mathscr{D}_{\mathrm{II}}}}\, C_1 C_4 = -\frac{1}{p} \frac{c_{\mathrm{II}}^{\mathrm{o}} - m c_{\mathrm{I}}^{\mathrm{o}}}{m\sqrt{\mathscr{D}_{\mathrm{II}}/\mathscr{D}_{\mathrm{I}}}}$$

Then the expressions for the transformed concentrations are

$$\bar{c}_{\mathrm{I}} = \frac{1}{p} \frac{c_{\mathrm{II}}^{\mathrm{o}} - m c_{\mathrm{I}}^{\mathrm{o}}}{m + \sqrt{\mathscr{D}_{\mathrm{I}}/\mathscr{D}_{\mathrm{II}}}} \exp\left(\sqrt{p/\mathscr{D}_{\mathrm{I}}}\, z\right) + \frac{c_{\mathrm{I}}^{\mathrm{o}}}{p}$$

$$\bar{c}_{\mathrm{II}} = -\frac{1}{p}\frac{c_{\mathrm{II}}^{\circ} - mc_{\mathrm{I}}^{\circ}}{m\sqrt{\mathcal{D}_{\mathrm{II}}/\mathcal{D}_{\mathrm{I}}}}\exp\left(-\sqrt{p/\mathcal{D}_{\mathrm{II}}}z\right) + \frac{c_{\mathrm{II}}^{\circ}}{p}$$

Then, taking the inverse transform (using a table of transforms), we get

$$c_{\mathrm{I}} = \left(\frac{c_{\mathrm{II}}^{\circ} - mc_{\mathrm{I}}^{\circ}}{m + \sqrt{\mathcal{D}_{\mathrm{I}}/\mathcal{D}_{\mathrm{II}}}}\right)\mathrm{erfc}\left(-\frac{z}{\sqrt{4\mathcal{D}_{\mathrm{I}}t}}\right) + c_{\mathrm{I}}^{\circ}$$

$$c_{\mathrm{II}} = -\frac{c_{\mathrm{II}}^{\circ} - mc_{\mathrm{I}}^{\circ}}{m\sqrt{\mathcal{D}_{\mathrm{II}}/\mathcal{D}_{\mathrm{I}}}}\mathrm{erfc}\left(\frac{z}{\sqrt{4\mathcal{D}_{\mathrm{II}}t}}\right) + c_{\mathrm{II}}^{\circ}$$

Then using the fact that $\mathrm{erfc}\, x = 1 - \mathrm{erf}\, x$, and that $-\mathrm{erf}\,(-x) = \mathrm{erf}\, x$, we finally get

$$c_{\mathrm{II}} = -\frac{1}{p}\frac{c_{\mathrm{II}}^{\circ} - mc_{\mathrm{I}}^{\circ}}{m\sqrt{\mathcal{D}_{\mathrm{II}}/\mathcal{D}_{\mathrm{I}}} + 1}\mathrm{erfc}\left(\frac{z}{\sqrt{4\mathcal{D}_{\mathrm{II}}t}}\right) + c_{\mathrm{II}}^{\circ}$$

$$\frac{c_{\mathrm{II}} - c_{\mathrm{II}}^{\circ}}{c_{\mathrm{I}}^{\circ} - (1/m)c_{\mathrm{II}}^{\circ}} = \frac{1 - \mathrm{erf}\left(z/\sqrt{4\mathcal{D}_{\mathrm{II}}t}\right)}{(1/m) + \sqrt{\mathcal{D}_{\mathrm{II}}/\mathcal{D}_{\mathrm{I}}}}$$

*b*. To get the flux at the interface, we can differentiate the concentration profiles:

$$\begin{aligned}
N_{Az}\big|_{z=0} &= -\mathcal{D}_{\mathrm{I}}\frac{\partial c_{\mathrm{I}}}{\partial z}\bigg|_{z=0} \\
&= -\mathcal{D}_{\mathrm{I}}\left(\frac{c_{\mathrm{II}}^{\circ} - mc_{\mathrm{I}}^{\circ}}{m + \sqrt{\mathcal{D}_{\mathrm{I}}/\mathcal{D}_{\mathrm{II}}}}\right)\frac{2}{\sqrt{\pi}}\exp\left(-z^2/4\mathcal{D}_{\mathrm{I}}t\right)\left(\frac{1}{\sqrt{4\mathcal{D}_{\mathrm{I}}t}}\right)\bigg|_{z=0} \\
&= -\left(\frac{c_{\mathrm{II}}^{\circ} - mc_{\mathrm{I}}^{\circ}}{m + \sqrt{\mathcal{D}_{\mathrm{I}}/\mathcal{D}_{\mathrm{II}}}}\right)\sqrt{\frac{\mathcal{D}_{\mathrm{I}}}{\pi t}}
\end{aligned}$$

Alternatively one can take the transform of the molar flux expression, differentiate $\bar{c}_{\mathrm{I}}$ with respect to $z$, then take the inverse transform to get the molar flux expression just found.

20D.3 Critical size of an autocatalytic system

(a) The general non-reactive transient solution for cylinders is

$$c_i = \sum_1^\infty A_n J_0(\alpha_n r) exp\left(-\alpha_n^2 \mathcal{D} t / R^2\right)$$

and the reactive solution is just

$$c_{i,reactive} = \sum_1^\infty A_n J_0(\alpha_n r) exp\left(k_1''' t - \alpha_n^2 t \mathcal{D} / R^2\right)$$

(b) Initial concentrations will decrease whenever all exponentials are negative, and the first term will be the most likely to be positive. Critical mass is then reached when

$$R = \alpha_1 \sqrt{\mathcal{D} / k_1'''}$$

(c) This answer is given in the problem statement.

20D.4 Dispersion of a broad pulse in steady laminar axial flow in a tube

(a) The approach here is to take advantage of the problem linearity and to add weighted pulse response solutions, or transfer functions, which are called here h(z,t). For our situation the transfer function is

$$h(L,t) = \frac{1}{\sqrt{4\pi Kt}} exp\left[-\frac{(L- < v > t)^2}{4Kt}\right]$$

It only remains now to weight individual differential pulses, by f(t'), each introduced at a time t', and add them up by integration from time zero to observer time t.

The response to an input of mass $m_i$ added over a cross-sectional area of column A, in a short time period is just

$$(m_i / A)h(L,t)$$

The mass of solute per unit area can in turn be written as

$$dm_i / A = \rho_A\Big|_{0,t'} < v > dt' = fdt'$$

for uniform concentration over the inlet cross-section. Summing, or more specifically integrating over the time interval of interest then leads directly to the solution suggested

$$< \rho_A >|_{L,t} = \frac{1}{\sqrt{4\pi K}} \int_{-\infty}^{t} f(t') \frac{exp\left[-\frac{(l- < v > t')^2}{4Kt'}\right]}{\sqrt{t'}} dt'$$

(b) For the square pulse one need only replace $f$ by $f_0 = \rho_{A0} < v >$ and change the lower limit of integration to "0".

< end >

**20D.5 Velocity divergence in interfacially embedded coordinates.**

**(a)** Integration of Eq. 20.4-7 over the boundary $S_D$ of any closed domain $D(u, w, y)$ in the boundary layer gives

$$\int_{S_D} (\boldsymbol{V} \bullet d\boldsymbol{S}_D) = \int_{S_D} (\boldsymbol{v} \bullet d\boldsymbol{S}_D) + \int_{S_D} \left( \frac{\partial \boldsymbol{r}(u, w, y, t)}{\partial t} \bullet d\boldsymbol{S}_D \right) \qquad (20D.5-1)$$

**(b)** Since the third dot product is the outward normal component of the velocity of the boundary element $d\boldsymbol{S}_D$ relative to stationary coordinates, its integral is the rate of change of the volume $V_D$ of the domain $D(u, w, y)$. With the aid of Eq. 20.4-3, we can rewrite this term as follows,

$$\int_{S_D} \left( \frac{\partial \boldsymbol{r}(u, w, y, t)}{\partial t} \bullet d\boldsymbol{S}_D \right) = \frac{d}{dt} \int_D \sqrt{g(u, w, y, t)} du dw dy$$
$$= \int_D \frac{\partial \sqrt{g(u, w, y, t)}}{\partial t} du dw dy \quad (20D.5-2)$$

using Leibniz' rule to differentiate the second integral.

**(c)** Application of Eq. A.5-1 (in which $\boldsymbol{v}$ is any differentiable vector function) to Eq. 20D.5-1, after use of Eq. 20D.5-2, gives the following equation in terms of volume integrals:

$$\int_{V_D} (\nabla \bullet \boldsymbol{V}) dV_D = \int_{V_D} (\nabla \bullet \boldsymbol{v}) dV_D + \int_D \frac{\partial \sqrt{g(u, w, y, t)}}{\partial t} du dw dy$$

Expressing each integral in the coordinates of Eq. 20.4-3, we then obtain

$$\int_D \left[ (\nabla \bullet \boldsymbol{V}) - \nabla \bullet \boldsymbol{v}) - \frac{1}{\sqrt{g(u, w, y, t)}} \frac{d\sqrt{g(u, w, y, t)}}{dt} \right] \sqrt{g(u, w, y, t)} du dw dy = 0$$

In order that this equation hold for every domain $D$ of nonzero $g(u, w, y, t)$, the quantity in square brackets must vanish identically in such regions, giving

$$(\nabla \bullet \boldsymbol{V}) = (\nabla \bullet \boldsymbol{v}) + \frac{\partial \ln \sqrt{g(u, w, y, t)}}{\partial t}$$

there in accordance with Eq. 20.4-8.

21A.1 Determination of eddy diffusivity

$$\ln(sc_A) = \ln(W_A / 4\pi D_{AB}) - \left(v_0 / 2D_{AB}\right)(s - z)$$

Assume that

$$W_{CO_2} = 0.001 \bullet W_{Air}$$

(a) Begin by examining the behavior at r = 0, and note that

$$W_{air} = c_{tot} v \left( \frac{\pi D^2}{4} \right)$$

so that the volume fraction (equal to the mole fraction) of carbon dioxide

$$\phi_{CO_2} = \frac{1}{16} \bullet \frac{v_0 D^2}{z K_{ACO_2}} \bullet 0.001$$

where $K_{ACO_2}$ is the effective diffusivity of carbon dioxide in air. Here we have assumed the ideal gas law to hold. We must now estimate the volume fractions, and here they will be taken as 0.0105 and 0.007 for distances of 112.5 and 152.7 cm respectively. Solving the above equation for the effective diffusivity then gives:

$$K_{A-CO_2} = 18.6,\ 21.9\ cm^2 / s$$

respectively.

21A.2 Heat and mass transfer analogy

Begin by defining the parameters

$$\Theta_{AB} = \frac{c_A - c_{A1}}{j_{A0} R / D_{AB}} \quad \xi = \frac{z}{v_{\max} R^2 / D_{AB}}$$

The desired solution then follows directly

$$\frac{D_{AB}(c_0 - c_b)}{j_{A0} D} = 2\left(\frac{v_{\max}}{< v_z >}\right)^2 \int_0^1 \frac{[I(\xi)]^2}{\xi[1 + (\nu^{(t)} / \nu)(Sc / Sc^{(t)})]} d\xi$$

**21B.1 Wall mass flux for turbulent flow with no chemical reactions.**

*a.* The Blasius formula for turbulent flow in a tube may be found in Eq. 6.2-12:

$$f = \frac{0.0791}{\mathrm{Re}^{1/4}}$$

The mass-transfer analog of Eq. 13.3-7 is

$$\mathrm{Sh} = \frac{1}{17.5}\sqrt{\frac{f}{2}}\,\mathrm{Re}\,\mathrm{Sc}^{1/3}$$

When these are combined, we get

$$\mathrm{Sh} = 0.0114\,\mathrm{Re}^{7/8}\,\mathrm{Sc}^{1/3}$$

*b.* If on the other hand, we use the mass-transfer analog of Eq. 13.4-20,

$$\mathrm{Sh} = \frac{1}{12.48}\sqrt{\frac{f}{2}}\,\mathrm{Re}\,\mathrm{Sc}^{1/3} \qquad \text{(large Sc)}$$

then we get

$$\mathrm{Sh} = 0.0160\,\mathrm{Re}^{7/8}\,\mathrm{Sc}^{1/3}$$

which is in better agreement with experimental data.

21B.2 Alternate expressions for the turbulent mass flux.

Seek an asymptotic expression for the turbulent mass flux for long circular tubes and a boundary condition of constant wall mass flux. Assume net mass transfer across the wall is negligible.

a) Parallel the approach to laminar flow heat transfer in §10.8 to write

$$\Pi(\xi,\zeta) = C_1\zeta + \Pi_\infty(\xi) + C_2$$

and show that

$$C_1 = 4$$

Here $\Pi = -(\omega_A - \omega_{A0})/(j_{A0}D/\rho D_{AW})$; $\xi = r/D$; $\zeta = (z/D)/\mathrm{Re}\,Sc$ and the subscript "0" indicates conditions at the wall. This is given in Section 10.8

b) Put these results into the species continuity equation to obtain

$$-4\frac{v_z}{<v_z>} = \frac{1}{\xi}\frac{d}{d\xi}\left[\left(1+\frac{Sc}{Sc^{(t)}}\frac{\mu^{(t)}}{\mu}\right)\xi\frac{d\Pi_\infty}{d\xi}\right]$$

This equation is to be integrated with the boundary conditions

At $\xi = 1/2$ $\quad \Pi_\infty = 0$ and $d\Pi_\infty/d\xi = -1$

c) Integrate once with respect to $\xi$ to obtain:

$$-\frac{d\Pi_\infty}{d\xi} = \frac{\frac{1}{2} - 4\int_\xi^{1/2}\frac{v_z}{<v_z>}\xi d\xi}{\xi\left[1+\frac{Sc}{Sc^{(t)}}\frac{\mu^{(t)}}{\mu}\right]}$$

Here $Sc^{(t)} = \mu^{(t)}/\rho D_{AW}^{(t)}$.

21B.3 An asymptotic expression for the turbulent mass flux.

Start with Eq. 21B.3-1 in the form

$$-\frac{1}{D}\frac{d\Pi_\infty}{d\xi}=+\frac{d\Pi}{dy}=\frac{1}{D}\frac{1}{\left[1+Sc(yv^*/14.5\nu)^3\right]}$$

Making the suggested substitution yields

$$\frac{d\Pi}{dy}=\frac{d\Pi}{d\eta}\frac{Sc^{1/3}v^*}{14.5\nu}$$

so that

$$\frac{d\Pi}{d\eta}=\frac{14.5\nu}{Dv_*Sc^{1/3}}\frac{1}{1+\eta^3}$$

and

$$\Pi(\infty)-\Pi(0)=\frac{14.5\nu}{Dv_*Sc^{1/3}}\int_0^\infty\frac{1}{1+\eta^3}$$

The integral is equal to

$$I=\frac{\pi/3}{\sin(\pi/3)}=\frac{\pi}{3}\frac{2}{\sqrt{3}}=1.209$$

It follows that

$$Sh=\frac{1}{\Pi(\infty)-\Pi(0)}=0.057\,\mathrm{Re}\,Sc^{1/3}\sqrt{f/2}$$

If the Blasius expression is used for the friction factor we obtain

$$Sh=0.0108\,\mathrm{Re}^{7/8}\,Sc^{1/3}$$

which is within about 5% of the above value. This is good agreement.

21B.4: Deposition of silver from a turbulent stream.

(a) We may write immediately from the results of the previous problem that

$$\frac{d\Pi}{d\eta} = \frac{14.5}{\mathrm{Re}\,Sc^{1/3}\sqrt{f/2}}\frac{1}{1+\eta^3}$$

where η is defined as in the last example. In our situation. However, for our initial purposes it is most convenient to write

$$(D_{eq}v_*/\nu) = \mathrm{Re}\sqrt{f/2} = \frac{2.54cm \bullet 11.4cm/s}{0.0101cm^2/s} = 2{,}863$$

The corresponding Schmidt number is 952 as stated in Fig.21B.3(b). The concentration gradient is then defined by

$$\frac{d\Pi}{d\eta} = \frac{1}{1942}\int_o^\eta \frac{d\eta}{1+\eta^3}$$

and we are now able to calculate the concentration gradient explicitly in terms of η. We may now calculate the concentration gradient, and the results are shown in Fig. 21B.3(b). However, the abscissa is written in terms of

$$s^+ = sv_*/\nu$$

where s is distance measured into the fluid from the surface under consideration. This is done to facilitate comparison with velocity profiles. We can calculate the concentration profile by integration of the concentration gradient with respect to $s^+$, but this result is not shown on the figure. It rises so rapidly to a limiting value

$$\Pi(\infty) - \Pi(0) = 1.209/1942 = 6.23 \bullet 10^{-4}$$

that the initial slope appears infinite on the scale of the figure, and the ordinate is 0.000623 for all readable positions.

$Sh = \frac{2864 \bullet 9.84}{1.209 \bullet 14.5} = 1606$ Under these conditions the cathode is completely polarized, i.e. the silver ion concentration there is essentially. This is also the situation of maximum silver flux and the corresponding maximum, or *limiting* current.

The corresponding Sherwood Number is

and the mass transfer coefficient for $Ag^+$ is

$$k_c = \left( \frac{D_{Ag-W}}{D} \right) Sh = \frac{1.06 \bullet 10^{-5}}{2.54} \frac{cm}{s} 1606 = 6.7 \bullet 10^{-3} cm/s$$

The corresponding silver ion flux

$$N_{Ag^+} = 6.7 \bullet 10^{-3} \frac{cm}{s} \bullet 10^{-9} \frac{g-eqts}{cm^3} = 6.7 \bullet 10^{-12} \frac{g-eqts}{cm^2 s}$$

**21B.5 Mixing-length expression for the velocity profile.**

**(a)** For steadily driven turbulent flow in a tube, Eq. 5.5-3 gives

$$\frac{d}{dr}(r\overline{\tau}_{rz}) = \frac{r(\mathcal{P}_0 - \mathcal{P}_L)}{L}$$

in which $\overline{\tau} = \overline{\tau}^{(v)} + \overline{\tau}^{(t)}$. Integration gives

$$r\overline{\tau}_{rz} = \frac{r^2}{2}\frac{(\mathcal{P}_0 - \mathcal{P}_L)}{L} + C_0$$

in which the integration constant $C_0$ is zero because the other terms vanish at $r = 0$. Hence,

$$\overline{\tau}_{rz} = \frac{r}{2}\frac{(\mathcal{P}_0 - \mathcal{P}_L)}{L}$$

and the wall shear stress is

$$\tau_0 = \frac{R}{2}\frac{(\mathcal{P}_0 - \mathcal{P}_L)}{L}$$

Combining the last two equations, we get

$$\overline{\tau}_{rz} = \tau_0\frac{r}{R} = \tau_0\frac{R-y}{R} = \tau_0\left(1 - \frac{y}{R}\right) \text{ for } 0 \le y \le R \qquad (21B.5-1)$$

**(b)** Inserting the viscous momentum-flux expression

$$\overline{\tau}_{rz}^{(v)} = \mu\frac{d\overline{v}_z}{dy}$$

and the mixing-length model

$$\overline{\tau}_{rz}^{(t)} = \rho\ell^2\left(\frac{d\overline{v}_z}{dy}\right)^2$$

into the result of **(a)**, we get the differential equation

$$\rho\ell^2\left(\frac{d\overline{v}_z}{dy}\right)^2 + \mu\frac{d\overline{v}_z}{dy} = \tau_0\left(1 - \frac{y}{R}\right) \text{ for } 0 \le y \le R \qquad (21B.5-2)$$

for the time-smoothed velocity distribution $\overline{v}_z(r)$.

**(c)** Setting $v_* = \sqrt{\tau_0/\rho}$, $\overline{v}_z = v_* v^+$, $y = \nu y^+/v_*$, and $\ell = \nu\ell^+/v_*$ in Eq. 21B.5-2 gives

$$\rho\left(\frac{\nu\ell^+}{v_*}\right)^2\left(\frac{d(v_*v^+)}{d(\nu y^+/v_*)}\right)^2 + \mu\frac{d(v_*v^+)}{d(\nu y^+/v_*)} = \tau_0\left(1 - \frac{y^+}{R^+}\right) \text{ for } 0 \le y^+ \le R^+$$

Division of the terms of this equation by their common factor $\tau_0 = \rho v_*^2 = (\mu/\nu)v_*^2$ gives

$$(\ell^+)^2\left(\frac{dv^+}{dy^+}\right)^2 + \frac{dv^+}{dy^+} = \left(1 - \frac{y^+}{R^+}\right) \text{ for } 0 \le y^+ \le R^+$$

in agreement with Eq. 21.4-13.

22A.1 Prediction of mass transfer coefficients in closed channels

a) Neglect the velocity of the water film surface.

$$\mathrm{Re} = \frac{D(v\rho)}{\mu} = \frac{D(4W/\pi D^2)}{\mu} = \frac{4}{\pi}\frac{W}{\mu D}$$

To a presently acceptable approximation the viscosity of air (in cp) at low density is given by

$$\mu = 0.0716 + 4.94 \bullet 10^{-5} T - 3.71 \bullet 10^{-8} T^2$$

where T is degrees centigrade. It follows that at 25 C

$$\mu = 0.0184 cp = 0.000184 g/cm - s$$

$$\mathrm{Re} = \frac{4}{\pi}\frac{1570 \bullet 453.6 \bullet 0.032808}{0.000184 \bullet 0.5 \bullet 3600} = 89{,}820$$

At 2 atmospheres the kinematic viscosity is given approximately by

$$\nu = (1/2)[0.13269 + 8.73 \bullet 10^{-4} T + 9.8 \bullet 10^{-7} T^2 \text{ in cm^2/s}$$

or $\nu = 0.0775 cm^2/s$

Then

$$Sc = \nu / D_{AW} = 0.60$$

It follows that the Sherwood number

$$Sh = 0.023\left[89{,}820^{0.83} \bullet 0.6^{0.44}\right] = 237$$

and

$$k_c = 2.02 cm/s$$

b) Here we must go back to Section 14.5 and the mass transfer analogs to the heat transfer expressions of Table 22.2-1.

We begin by writing

$$j_D = j_H = 2.19\mathrm{Re}^{-2/3} + 0.78\mathrm{Re}^{-0.381}$$

with

$$\mathrm{Re} = \frac{6G_0}{a\mu} = 89{,}820 \bullet \frac{6}{(100/ft)(0.5ft)} = 10{,}778$$

Then

$$j_D = \frac{2.19}{10{,}788^{2/3}} + \frac{0.78}{\mathrm{Re}^{0.381}} = 0.0272$$

We now turn to Table. 22.2-1 to write

$$j_D = \frac{k_c}{v_0} Sc^{2/3} = (0.6)^{2/3} \frac{k_c}{v_0};\ k_c = \frac{0.0272}{0.6} v_0 = 0.0382 v_0$$

and it only remains to find the superficial velocity

$$v_0 = \frac{1570 lb/hr}{(\pi/4)(1/4 ft^2)\rho_{air}}$$

with air density given by the ideal gas law as

$$\rho_{air} = \frac{M_{air} p}{RT} = \frac{(28.97 lbs/lb-mol)(2atm)}{(538R)(0.73atm - ft^3/lb-mol, R)} = 0.1475 lbs/ft^3$$

Then

$$v_0 = \frac{1570 lb/hr}{(\pi/4)(1/4 ft^2)(0.1475 lbs/ft^3)} \bullet \frac{hr}{3600s} = 15.1 ft/s$$

or $\quad v_0 = 460 cm/s$

Then $k_c = 17.6 cm/s$

22A.2 Calculation of gas composition from psychrometric data.

Solution is conveniently started with Eq. 22.3-39, with n equal to 0.44 (see Prob. 22A.1 statement). The molar heat capacity at the mean "film" temperature of 105 F is 6.98 Btu/lb-F (see Ex. 22.3-2), and the heat of vaporization at 80 F is 18,851 Btu/lb-mol. The vapor pressure of water at 80 F, 26.7 C, is 26.0 mm Hg. [***Erratum***: the coefficient of $T^2$ in Fig. 22B.7 should be $-7.98 \bullet 10^{-5}$]. It follows that the water vapor mole fraction at the water surface

$$x_{wo} = 26/800 = 0.0325$$

We may then write:

$$\frac{0.0325 - x_{w\infty}}{(130-80)(1-0.0325)} = \frac{6.98}{18{,}851}\left(\frac{0.58}{0.74}\right)^{0.56} = 3.23 \bullet 10^{-4}$$

It then follows that $x_{w\infty} = 0.0158$

This differs from the answer given in the text primarily because I used a value of 0.56 for n, not 2/3. This corresponds to assuming Sh $\alpha$ $Sc^{0.44}$, Nu $\alpha$ $Pr^{0.44}$.

**22A.3 Calculating the inlet air temperature for drying in a fixed bed.**

For water at 60°F= 15.56°C, the fitted equation in Fig. 22B.7 gives $p_{A,\text{vap}}$ = 13.15 mm Hg. The resulting mole fraction of $H_2O$ in the interfacial air, assuming local equilibrium and ideal gas behavior, is $x_{A0} = (p_{A,\text{vap}}/p) = 13.15/(1.1 \times 760) = 0.0157$. The film composition in the shallow bed is then $x_{Af} = 0.5(0 + 0.0157) = 0.0079$. The molar enthalpy of vaporization of water at this temperature, from Lange's Handbook (1999), is 19,044 Btu/lb-mol.

At the estimated film temperature $T_f$ of 90°F, the molar heat capacities of dry air and of water vapor are found as 6.96 and 8.01, respectively, in Btu/lb-mol·F in Fig. 60 of Hougen Watson and Ragatz, *Chemical Process Principles Charts*, 2nd Ed., Wiley, New York (1960). Thus, the humid air at film conditions has a heat capacity $\widetilde{C}_p$ of $0.0079 \times 8.01 + .9921 \times 6.96 = 6.97$ btu/lb-mol·F. The Prandtl number and Schmidt number at the film condition are taken as 0.74 and 0.58, respectively, as in Example 22.3-1.

The development in Example 22.3-2 can then be paralleled, with these comments:

(i) For a shallow bed, $x_{Ab}$ and $T_b$ reduce to the inlet conditions $x_{A1}$ and $T_1$.

(ii) The exponent $n = 1/3$ in Eq. 14.5-3, and in the analogous function $j_D$, is supported for this geometry by the data analysis reported in §14.5. The exponent $1 - n = 0.56$, reported by Bedingfield and Drew and cited on page 684, is for flow across a single cylinder in a large duct.

Therefore, subject to the indicated neglect of radiation, Eq. 22.3-38 can be taken over to calculate the inlet temperature as follows, with physical properties evaluated at $(T_f, x_{Af})$:

$$
\begin{aligned}
T_1 &= T_0 + \left(\frac{x_{A0} - x_{A1}}{1 - x_{A0}}\right)\left(\frac{\text{Sc}}{\text{Pr}}\right)^{-2/3} \frac{\Delta\widehat{H}_{\text{vap}}}{\widehat{C}_p} \\
&= 60 + \left(\frac{0.0157 - 0}{1 - 0.0157}\right)\left(\frac{0.74}{0.58}\right)^{2/3}\left(\frac{19,091}{6.97}\right) \\
&= 111.4°\text{F}
\end{aligned}
$$

**22A.4 Rate of drying of granular solids in a fixed bed.**

The calculations made in the solution of Problem 22A.3 are applicable here, along with the comments (i) and (II). Note also that the flux ratio $R_x$ of Table 22.8-1 has the value

$$R_x = \frac{x_{A0} - x_{A\infty}}{1 - x_{A0}} = \frac{0.0157 - 0}{1 - 0.0157} = 0.0160$$

here, when the solubility of species B (air) in the interstitial liquid water is neglected. At this small value of $R_x$, the mass-transfer coefficient $k_{x,\text{loc}}$ is negligibly affected by the net molar transfer, as shown in Fig. 22.8-1 on page 710.

Then the binary analog of Eq. 14.5-1 gives the rate of water removal per unit volume of bed as

$$\frac{dW_{A0}}{Sdz} = (k_{x,\text{loc}}a)\frac{(x_{A0} - x_{Ab})}{(1 - x_{A0})}$$

and the Chilton-Colburn analogy $j_H = j_D$ in Eq. 22.3-23, when applied to Eqs. 14.5-2,3,4, gives the following binary mass-transfer formula in molar units (see also Table 22.2-1):

$$j_D = \frac{k_{x,\text{loc}}}{cv_0}(\text{Sc})^{2/3} = 2.19\text{Re}^{-2/3} + 0.78\text{Re}^{-0.381}$$

in which

$$\text{Re} = \frac{6\rho v_0}{a\mu\psi} = \frac{6v_0}{a\nu\psi}$$

Insertion of numerical values for this problem (including $\nu = 0.162$ cm$^2$/s=0.000174 ft$^2$/s from Table 1.2.1 and §F.3, $\psi = 0.92$ for cylinders from page 441, and $c = p/RT = 0.00275$ lb-mol/ft$^3$) gives

$$\text{Re} = \frac{(6)(15)}{(180)(0.000174)(0.92)} = 3.123 \times 10^3$$

$$j_D = (2.19)(3123)^{-2/3} + (0.78)(3123)^{-0.381} = 0.0466$$

$$\begin{aligned} k_{x,\text{loc}} &= j_D c v_0 (\text{Sc})^{-2/3} \\ &= (0.0466)(0.00275)(15)(0.58)^{-2/3} = 0.00276 \text{ lb-mol/s}\cdot/\text{ft}^2 \end{aligned}$$

and finally, the initial rate of water removal per unit volume of bed is

$$\begin{aligned} \frac{dW_{A0}}{Sdz} &= (0.00276 \text{ lb-mol/s}\cdot\text{ft}^2)(180 \text{ ft}^2/\text{ft}^3)\frac{0.0157 - 0}{1 - 0.0157} \\ &= 7.9 \times 10^{-3} \text{ lb-mol/s}\cdot\text{ft}^3 \\ &= 28.5 \text{ lb-mol/hr}\cdot\text{ft}^3 \end{aligned}$$

22B.1 Evaporation of a freely falling drop

Begin by noting that acceptable approximations for all needed physical properties have been provided in the above solutions except water density which is 0.998 g/cm$^3$.

(a) From Eq. 6.1-7 we may write

$$\mathrm{Re}^2 f = \frac{4}{3}\frac{D^3 g \Delta\rho}{\mu}$$

and for this problem

D = 0.1 cm, g = 980 cm/s$^2$ , $\Delta\rho$ = (0.998-0.0014)g/cm$^3$, $\mu$=1.84•10$^{-4}$ g/cm,s

so that

$$\mathrm{Re}\sqrt{f} = 212.5$$

A plot of this f(Re) relation on Fig. 6.3-1 is line of slope minus 2 which intersects the friction factor-Re curve at a Reynolds number of 250. It follows that

$$v = 250\mu / D\rho \approx 390 cm/s$$

(b) To the accuracy of this problem we may use the simple approximation of Eq. 14.4-5 and its mass transfer counterpart

$$Nu_m = 2 + 0.6\mathrm{Re}^{1/2}\mathrm{Pr}^{1/3} = 10.6$$

$$Sh_m = 2 + 0.6\mathrm{Re}^{1/2} Sc^{1/3} = 10$$

It follows that to a reasonable approximation

$$\frac{h}{k_x} = \frac{10.6}{10} C_p \left(\frac{Sc}{\mathrm{Pr}}\right) \approx 6$$

cal/g-mole, C or Btu/lb-mole, F. We may then write approximately

$$\frac{x_{w0} - 0}{(100 - T_0)(1 - x_{wo})} \approx \frac{6}{19{,}100} \approx 3.2 \bullet 10^{-4} F^{-1}$$

This equation is to be solved simultaneously with the equilibrium relation

$$x_{wo} = p_w^{\circ} / 1 atm$$

with the vapor pressure of water $\overset{\circ}{p}_w$ obtained for example from Fig. 22B.6. This process yields a surface temperature

$$T_0 \approx 54F \text{ and } x_{w0} \approx 0.0141$$

(c) A mass balance over the sphere gives

$$-\frac{d}{dt}\left(\frac{4}{3}\pi R^3 c_L\right) = 4\pi R^2 N_{wr0}$$

This equation is easily rewritten as

$$-\frac{dD}{dt} = \frac{2k_{xm}}{c_L}\frac{x_{w0} - x_{w\infty}}{1 - x_{w0}}$$

We now assume pseudo-steady condition such that sphere surface temperature is that given by the above analysis, and we use the following parameter values

$$c_G = p/RT = 1/(82.05 \bullet 285) = 4.28 \bullet 10^{-5}\, g = moles/cm^3$$

$$\nu_G = 0.13269 + 0.0008729 + 9.82 \bullet 10^{-7}$$

$$= 0.1433 cm^2/s$$

$$D_{WA} = \nu_G / Sc = (0.1433 cm^2/s)/0.625$$

$$= 0.229 cm^2/s$$

$$k_{xm} = 10\frac{(cD_{wa})_G}{D} = \frac{10 \bullet (4.06 \bullet 10^{-5}) \bullet 0.229}{0.1}$$

$$= 9.3 \bullet 10^{-4}\, g - mols/cm^2, s$$

Then

$$-\frac{dD}{dt} = \frac{2 \bullet 9.3 \bullet 10^{-4}\, g - mols/cm^2}{0.0555 g - mols/cm^3} \bullet \frac{0.0141}{0.9859} = 4.79 \bullet 10^{-4}\, cm/s$$

22B.2 Effect of radiation on psychrometric measurements

(a) Only convective and radiant heat transfer occur for the dry-bulb thermometer, and we may write

$$h_{db}(T_\infty - T_{db}) + \sigma\left[a_{db}T_s^4 - e_{db}T_{db}^4\right]$$

Here the subscripts db, ∞ and s refer to the mean dry bulb temperature, the air temperature outside the dry-bulb boundary layer and the effective radiation temperature of the surroundings, respectively.

This equation permits determining the bulk air temperature, $T_\infty$, from the dry-bulb reading and $T_s$, presumed known.

(b) Here we must consider convective heat transfer and write

$$N_{w0}\Delta\tilde{H}_{w,vap} = k_{xwm}\frac{x_{w0} - x_{w\infty}}{1 - x_{w0}}\Delta\tilde{H}_{w,vap}$$

$$= h_{wb}(T_\infty - T_{wb}) + \sigma\left[a_{wb}T_s^4 - e_{wb}T_{wb}^4\right]$$

(c) Rearranging the above expression yields

$$\frac{(x_{w0} - x_{w\infty})}{(T_\infty - T_0)(1 - x_{w0})} = \frac{h_{wb}}{k_{xwb}\Delta\tilde{H}_{w,evap}} \bullet (1 + C)$$

where C is a radiation correction factor for the wet-bulb thermometer defined by

$$C = \frac{\sigma\left[a_{wb}T_s^4 - e_{wb}T_{wb}^4\right]}{h_{wb}(T_\infty - T_{wb})}$$

The first job is to find the true entering air temperature, and for this we need $h_{db}$. We obtain this from the formula

$$h = j_H(\mathrm{Re})c_{air}\tilde{C}_{p,air}v_\infty / \mathrm{Pr}^{2/3}$$

The contributions to this equation are estimated as follows:

$$\mathrm{Re} = \frac{0.254cm \bullet 457.2cm/s}{0.187} = 621$$

and from Fig. 14.4-2

$$j_H \approx 0.57/\sqrt{\mathrm{Re}} = 0.57/\sqrt{621} = 0.023$$

Here kinematic air viscosity is calculated for 60 C (140 F) from the equation

$$\nu_{air} = 0.1327 + 0.00087T + 9.82 \bullet 10^{-7} T^2$$

where ν is in cp, T is in C, and the pressure is one atmosphere. It then follows from the gas law and the other data given that

$$h_{db} \approx 24 Btu/hr, ft^3, ^o F$$

Then $T_\infty \approx 140 + \dfrac{0.1712 \bullet 0.93}{24}\left[\left(\dfrac{600}{100}\right)^4 - \left(\dfrac{590}{100}\right)^4\right] = 140.6F$

This is a quite minor correction.

Following the same approach we find that

$$h_{wb} \approx 20 Btu/hr, ft^2\, ^o F$$

and $$\frac{h_{wb}}{k_{xw}\Delta\tilde{H}} = \frac{\tilde{C}_{p,w}}{\Delta\tilde{H}}\left[\frac{Sc}{\mathrm{Pr}}\right]^{2/3} = 3.14 \bullet 10^{-4} / {}^0F$$

Putting this result into the first equation of (c) yields

$$x_{w\infty} = 0.0021$$

which is significantly smaller than the result in Ex. 22.3-2.

22B.3 Film theory with variable transport properties

Begin with Eq. 19.4-10 and rearrange to

$$cD_{AB}\frac{dx_A}{dy} = x_A(N_{A0} + N_{B0}) - N_{A0}$$

and separate variables to obtain

$$\frac{dy}{cD_{AB}} = \frac{dx_A}{x_A(N_{A0} + N_{B0}) - N_{A0}}$$
$$= \frac{d\ln[x_A(N_{A0} + N_{B0}) - N_{A0}]}{(N_{A0} + N_{B0})}$$

$$I \equiv (N_{A0} + N_{A0})\int_0^y \frac{dy}{cD_{AB}} = \ln[x_A(N_{A0} + N_{A0}) - N_{A0}\Big|_{x_{A0}}^{x_A}$$

or

$$e^I = \frac{x_A(N_{A0} + N_{B0}) - N_{A0}}{x_{A0}(N_{A0} + N_{B0}) - N_{A0}}$$

Then

$$I = \frac{N_{A0} - x_A(N_{A0} + N_{B0})}{N_{A0} - x_{A0}(N_{A0} + N_{B0})}$$

Now subtract

$$\frac{N_{A0} - x_{A0}(N_{A0} + N_{B0})}{N_{A0} - x_{A0}(N_{A0} + N_{B0})}$$

from both sides to obtain the desired result:

$$\frac{(N_{A0} + N_{B0})(x_A - x_{A0})}{N_{A0} - x_{A0}(N_{A0} + N_{B0})} = 1 - \exp\left[(N_{A0} + N_{B0})\int_0^y \frac{dy}{cD_{AB}}\right]$$

The heat transfer analog can be obtained by a similar procedure. Parts (b) and (c) are left to the reader.

22B.4 An evaporative ice maker

Here we are only interested in a preliminary study so will assume steady behavior. We will also accept the boundary layer simplifications introduced in this chapter and the simplified boundary conditions that go with it. We begin by neglecting the differences in Sc and Pr so that we can use the additivity of Grashof numbers introduced in Ex. 22.6-1, and we apply this concept to the horizontal plate with flow inward from the surroundings. For illustrative processes we use the laminar-flow expression for a horizontal plate. Now the flow of fluid will be toward the plate and out at the periphery, and this is the situation of a cooled plate: the surface is cooler than the surroundings. We thus begin with Eq. 14.6-7:

$$Nu_m = \frac{0.527}{[1+(1.9/\mathrm{Pr})^{9/10}]^{2/9}}(Gr\,\mathrm{Pr})^{1/5}$$

For our present purposes Gr is replace by ($Gr+Gr_{AB}$), but in this situation $Gr_{AB}$ is quite small and can be neglected. Moreover, it should be noted that the denominator has the magnitude

$$[1+(1.9/0.74)^{9/10}]^{2/9} = 1.30$$

and the corresponding expression for Sc of 0.58 is 1.35, which is not significantly different for our present purposes.

We may this write

$$Nu_m \approx \frac{0.57}{1.3}0.74^{1/5}Gr^{1/5} = 0.412Gr^{1/5}$$

$$Sh_m \approx \frac{0.57}{1.35}0.58^{1/5} = 0.379Gr^{1/5}$$

and

$$h/k_{xw} = \frac{0.412}{0.379}\frac{Sc}{\mathrm{Pr}}\tilde{C}_p \approx 0.85\tilde{C}_p$$

It follows that

$$\frac{(x_{w0}-x_{w\infty})}{(T_\infty - T_0)(1-x_{w0})} = 0.85\frac{\tilde{C}_p}{\Delta\tilde{H}_{vap}}$$

Heat of vaporization should be calculated at the freezing point and is about

$$\Delta\tilde{H}_{vap} = 19{,}313 Btu/lb-mol = 10{,}735 cal/g-mol$$

The molar heat capacity should be evaluated at a somewhat higher but unknown temperature. Since its temperature sensitivity is small we shall use a value of

$$\tilde{C}_p \approx 8 Btu/lb-mole, F = 8 cal/g-mole, F$$

which corresponds to about 40 F or approximately 5 C.

Using the vapor pressure data of Fig. 22B.6 we find the relation etween surrounding mole fraction and temperature is

$$\frac{0.00626 - x_{w\infty}}{T_\infty(1-0.00626)} = 0.85\frac{8}{10{,}735}$$

or

$$x_{w\infty} = 0.00676 - 6.29\bullet 10^{-4}T \; x_w = 0.00626 - 6.29\bullet 10^{-4}T$$

This equation is plotted on the attached psychrometric chart. It may be seen here that the maximum possible temperature is about 5.5 C 0r about 42 F, and that even for extremely dry air, the maximum is a bit below 10 C or 50 F.

We must now see how effective radiation to the night sky can be, and we start with a result already obtained in Prob. 22B.2:

$$\frac{0.00626 - x_{w\infty}}{T_\infty(1-0.00626)} = 0.85\frac{8}{10{,}735}\left\{1+\left[\frac{\sigma e(T_s^4 - T_0^4}{hT_\infty}\right]\right\}$$

Note though that radiation now opposes convective heat transfer, and that the radiation flux is independent of local ambient temperature. For the given conditions and a grey surface, with absorptivity equal to emissivity of 0.93, as in Prob. 22B.2, we find

$$q_{rad} = 1.355\bullet 0.93\left(0.15^4 - 0.273^4\right) = -5.04\bullet 10^{-3}\frac{cal}{cm^2, s}$$

We must now calculate the free convection heat transfer coefficient from

$$h_m = 0.412\frac{k}{D}Gr^{0.2}$$

with $Gr = \dfrac{D^3 g\beta\Delta T}{\nu^2}$

$k \approx 5.9\bullet10^{-5}$ cal/cm,s,°K (Bejan), D = 50 cm, g = 980 cm/s$^2$, $\beta = 1/T \approx 1/273 = 0.00366$, $\nu \approx 0.133$ cm$^2$/s

$$Gr \approx \frac{50^3\bullet980\bullet\Delta T}{273\bullet(0.133)^2} = 2.54\bullet10^7\,\Delta T$$

and $$Gr^{1/5} = 30.2(\Delta T)^{0.2}(^oC)^{-0.2}$$

Then $h = 0.412\dfrac{6\bullet10^{-5}}{50}30.2(\Delta T)^{0.2} = 1.46\bullet10^{-5}\dfrac{cal}{cm^2,s,(^0C)^{0.2}}$

and the convective heat transfer rate is

$$q_{conv} = 1.46\bullet10^{-5}(\Delta T)^{1.2}\,cal/cm^2,s,(^0C)^{1.2}$$

We may now calculate the local ambient temperature for which no net evaporation takes place: where radiation out just balances convection in to the water surface.

From the left side of the wet-bulb expression this is the temperature at which 30% saturated air has a water vapor mole fraction of 0.00626. From our psychrometric chart this is about 20 C. However, at this temperature

$$q_{conv} = 4.2\bullet10^{-4}\,cal/cm^2,s$$

which is insufficient to supply the radiative loss. Water vapor must now condense to supply the radiative loss, and local ambient temperature must be well above 20 C.

We could go further, but already it is clear that radiation is very powerful, and that the prediction is greatly at variance with observation. Ice is never found at ambient temperatures even of 20 C. There are many possible reasons for this discrepancy between this simple theory and actual observation. They include

conduction from the ground, forced convection in the form of wind and turbulence in the boundary layer. This last is unlikely as flows about a cold plate facing upwards are very stable (Rohsenow et al, p. 4.17). However, one of the most important is fog formation (see Ex. 19.5-2),very often seen on still nights. Fog very effectively inhibits radiation. Ice can form at very high ambient temperatures in a desert.

Finally we note that the efficiency of our device can be significantly improved by the presence of a small lip as this adds a convective heat transfer resistance without appreciably affecting radiation. A stagnant film has a thermal resistance which can be calculated from

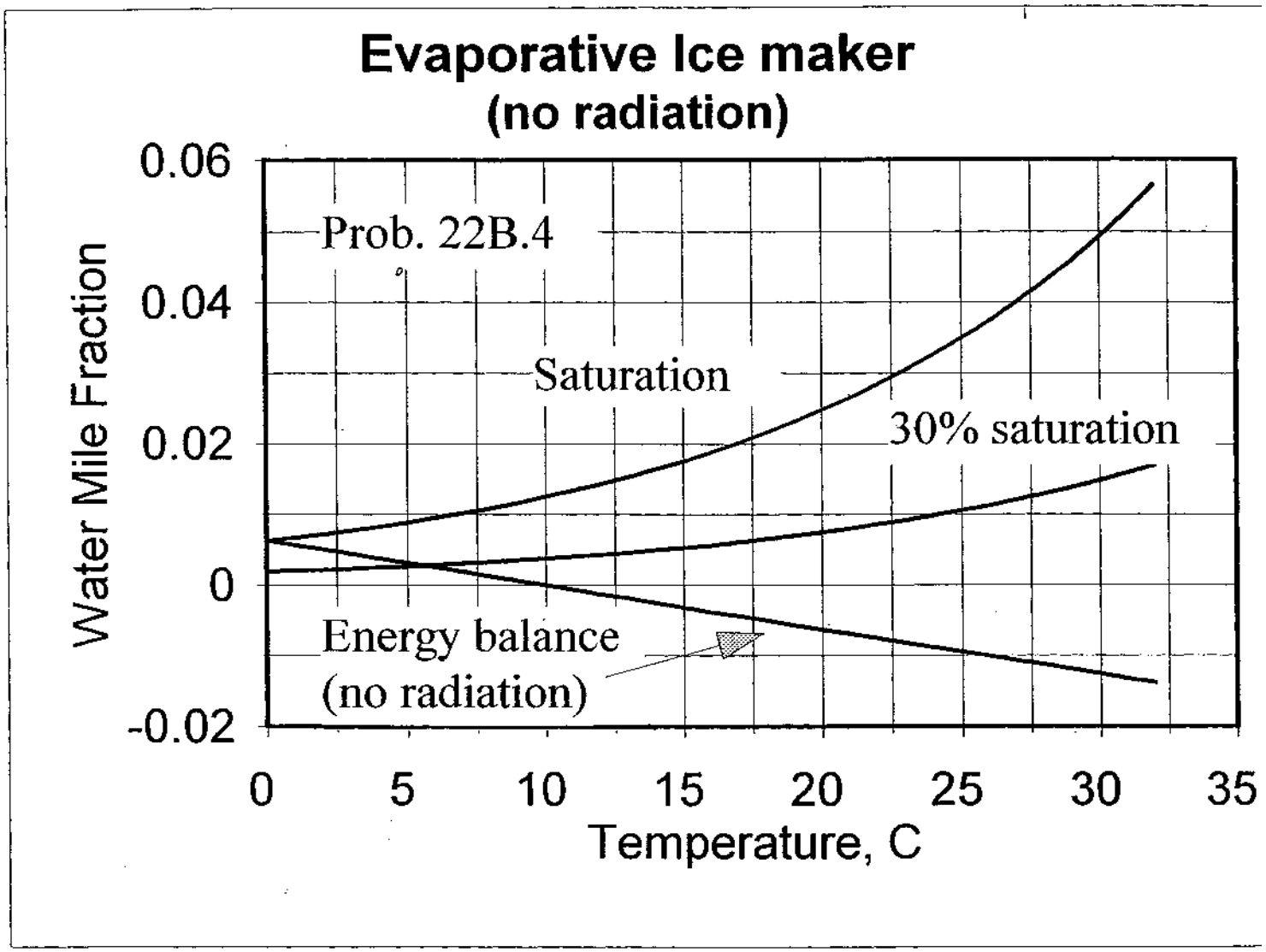
$$Nu = hD/k = D/\delta$$

where δ is the film thickness. Then for 2 mm film

$$h = k/\delta = (5.9 \bullet 10^{-5} cal/cm,s.^{o}C)/0.2cm$$

or $$h_{film} = 2.95 \bullet 10^{-5} cal/cm^{2},s,^{o}C$$

and its resistance is the reciprocal of this quantity.


Evaporative Ice maker
(no radiation)
Prob. 22B.4
Saturation
30% saturation
Energy balance
(no radiation)
Water Mile Fraction
0.06
0.04
0.02
0
-0.02
0
5
10
15
20
25
30
35
Temperature, C

22B.5 Oxygen stripping

Once again we must begin by determining terminal velocity, and we begin here with the fundamental momentum balance

$$F = \frac{4}{3}\pi R^3 \Delta\rho g = \frac{1}{2}\pi R^2 \rho_w v^2 f$$

and rearranging as before we find

$$\mathrm{Re}^2 f = \frac{4}{3}\frac{D^3 \rho_w \Delta\rho g}{\mu_w^2}$$

Looking ahead to the inaccuracies involved in using the graphical form of the f(Re) relation shown in Fig. 6.3-1 we make the approximations

$$\Delta\rho \approx \rho_w ;\ \mu_w^2 / \rho_w \Delta\rho \approx v_w^2 \approx 10^{-4} cm^4 / s^2$$

It follows that

$$\mathrm{Re}^2 f = \frac{4}{3}\frac{10^{-3} \bullet 980}{10^{-4}} = 1.3 \bullet 10^4$$

Sketching this f-Re relation on Fig.6.3-1 shows that the Reynolds number is of the order of 10, where

$$f \approx 18.5 / \mathrm{Re}^{3/5}$$

Eliminating the friction factor between these two expressions yields

$$\mathrm{Re}^{7/5} = 1.3 \bullet 10^4 / 18.5 \ ;\ \mathrm{Re} \approx 108$$

We are now ready to estimate ass transfer rates, and we begin by anticipating the results of Prob. 22B.8 by assuming that essentially all mass transfer resistance occurs in the water phase.

The best available correlation is then that

$$Sh_m \approx 2 + 0.6 \mathrm{Re}^{1/2} Sc^{1/3} = k_{cm} D / D_{AW}$$

and it remains to determine magnitudes of key system properties. Here we shall assume oxygen solubility to be given by

$$c_{O_2} = [2.17 - 0.0507T + 5.604 \bullet 10^{-4} T^2](p / 1atm)$$

where concentration is mmols/L, and temperature is in degrees centigrade [regression of data in J. E. Bailey and D. F. Ollis , Biochemical Engineering Fundamentals, 2cnd Ed., McGraw-Hill, 1986 -- - p. 463]. Then for our circumstances, the entering oxygen concentration is about

$$c_{O_2} \approx 1.38 mmols / L = 1.38 \bullet 10^{-6} g - mols / cm^3$$

For oxygen-water diffusivity we shall use

$$D_{O_2W} \approx (1.04 + 0.053T) \bullet 10^{-5} cm^2 / s$$

[This is a regression of data at 37 C by E. E. Spaeth and S. K. Friedlander, Biophys. J. 1967, p.827 and data at 25 C reported in T. K. Sherwood, et al, Mass transfer 1975, Wiley] It follows that

$$D_{O_2W} \approx 2.1 \bullet 10^{-5} cm^2 / s$$

Then the Schmidt number

$$Sc \approx 0.0100137 / [2.1 \bullet 10^{-5}] = 476$$

and

$$k_c = \frac{2.1 \bullet 10^{-5} cm^2 / s}{0.1 cm} \left[ 2 + 0.6 \bullet 108^{1/2} \bullet 476^{1/3} \right] \approx 0.0106 cm / s$$

The inward directly oxygen flux at the bubble surface is then

$$-N_{O_2 r}\Big|_{r=R} = k_c \Delta c_{O_2} = 0.010106 \bullet 1.38 \bullet 10^{-8} gmoles / cm^2, s$$

22B.6 Controlling diffusional resistances

No first of al because there is no diffusional resistance in pure oxygen. However, the same answer would be true even if the gas phase were any nitrogen-oxygen mixture. The gas phase contribution is truly negligible, so it is only necessary to recognize that the gas phase diffusivity is in the range of

$$0.1 < D_{GM} < 1.0 cm^2 / s$$

where G is "any" gas in a gas containing "M" and "G" for these conditions. To show this begin by writing

$$N_{O_2 0} = k_{cL}(c_{Lb} - c_{L0}) = k_{cG}(c_{G0} - c_{Gb}) = K_{cL}(c_{Lb} - c_{Le})$$

Now define

$$(c_{G0} - c_{Gb}) = \frac{1}{m}(c_{L0} - c_{Le})$$

to write

$$\frac{1}{k_{cL}} + \frac{m}{k_{cG}} = \frac{1}{K_{cL}} \text{ or } \frac{k_{cL}}{K_{cL}} = 1 + m\frac{k_{cL}}{k_{cG}}$$

Noting that

$$Sh = Dk_c / D_{im}$$

we may write

$$\frac{k_{cL}}{K_{cL}} = 1 + m\frac{D_{O_2W}}{D_{O_2A}}\frac{Sh_L}{Sh_A}$$

and to an order of magnitude approximation

$$(Sh_L / Sh_G) << (D_{O_2A} / D_{2OW})^n$$

where "n" is 1/3 for the situation being considered here.

Note that for our conditions

$$c_G = p / RT = \frac{1 atm, g = mole}{82.05 cm^3 \bullet 293.16} \bullet 10^6 \frac{mmols, cm^3}{g - mole, L} = 41.6 mmols / L$$

so that $$m \approx 0.03$$

22B.7 Determination of diffusivity

The basic problem here is to calculate the quantity ψ. this may be shown as follows:

For the actual situation,

$$\frac{dV}{dt} = S\phi\sqrt{D_{WA}/t}$$

whereas in the absence of convection

$$\frac{dV}{dt} = Sx_{W0}\sqrt{D_{WA}/\pi t}$$

The ratio of these two quantities is then just ψ.

Determination of ψ is difficult, as stated in Ex. 20.1-1, but the inverse is straightforward. Examples in the supplement attached show that significant corrections are needed even for mole fractions as low as 0.03. These calculations were made using the approximation of the error function

$$erf(x) \approx 1 - (a_1 t + a_2 t^2 + a_3 t^3)e^{-x^2}$$

$$t = 1/1 + 0.47047x$$

with

$$a_1 = 0.34802$$

$$a_2 = -0.09587$$

$$a_3 = 0.74785$$

(Abramowitz and Stegun, 7.1.25.

Mole fractions can be related to temperature using Fig. 22B.6.

Now just looking at Table 20.1-1 we see that ψ begins to increase very rapidly with interfacial water vapor mole fraction, and this means that heat supply can become a problem.

Now the oxygen-air diffusivity is to the approximations of Chap. 17 equal to oxygen-air diffusivity, and from Table 17.1-1 this is 0.181 $cm^2/s$. From Eq. 17.2-1

$$D_{O_2A} = (0.18)(293.2/273.2)^{1.82} = 0.205 cm^2/s$$

We then find that

$$\frac{k_{cL}}{K_{cL}} << 0.03\left(\frac{2.1\bullet 10^{-5}}{2.05\bullet 10^{-1}}\right)^{2/3} = 6.6\bullet 10^{-5}$$

which is truly negligible. Similar remarks hold or almost all systems of sparingly soluble gases.

22B.8 Marangoni effects in condensation of vapors

We begin by calculating film thickness in the absence of surface tension gradients as

$$\delta = k / h = (0.73/5{,}000)m$$
$$= 1.46 \bullet 10^{-4} m$$

From Eq.2.2-19

$$< v >= g\delta^2 / 3\nu = 24.0 cm/s$$

and the mass flow rate per unit width

$$\rho < v >= 24.0 \bullet 0.96 = 23.1 g/cm,s$$

We may now parallel the development of Ex. 2.2 and write

$$\tau_{xz} = \rho g x + C_1$$

but now the integration constant is to be evaluated from the new BC so that

$$\tau_{xz} = \rho g x + \tau_s = -\mu dv_z / dx$$

Integrating this equation with respect to x, and using the BC

$$v_z\big|_{x=\delta} = 0$$

yields

$$\rho < v >= \left[\frac{\rho^2 g \delta^2}{3\mu}\right]\left(1 + \frac{3}{2}\frac{\tau_s}{\rho g \delta}\right)$$

The term in $\tau_s$ represents the effect of surface tension gradients, and where this term is small its denominator will be about that for no gradient. Now for our circumstances $\rho g \delta \doteq 14.3 dyne/cm^2$. Surface tension effects will thus be small for systems like ours where surface tension increases downwards. In the reverse case however, even small gradients can cause hydrodynamic instabilities and thus have major effects.

22B.9 and 22B.10 Film models for spheres and cylinders.

For all three of our basic coordinate systems we may write

$$N_A - x_A\left(N_A + N_B\right) = -cD_{AB}\frac{dx_A}{dr}$$

where r represents distance, y for rectangular coordinates and r for cylindrical and spherical. A subscript "r" is understood for each of the molar fluxes.

It is necessary for this development that the left side of the equation become independent of r, so we rewrite it as

$$r^n\left[N_A - x_A\left(N_A + N_B\right)\right] = -cD_{Ab}r^n\frac{dx_A}{dr}$$

where n = 0 for slabs, 2 for spheres and 1 for cylinders. Now for all three

$$r^n N_i = R^n N_i(R)$$

a constant, and R is any convenient value of r. We may then write

$$R^n\left[N_{AR} - x_A\left(N_{AR} + N_{BR}\right)\right]$$
$$= \frac{cD_{AB}}{R^n\left[N_{AR} + N_{BR}\right]}r^n\frac{d\ln}{dr}\left[R^n\left(N_{AR} + N_{BR}\right)\right]$$

with boundary conditions

$$x_A = x_{A0} \text{ at } r = R_0$$
$$x_A = x_{A\delta} \text{ at } r = R_\delta$$

Here the subscript "R" denotes evaluation at r = R.

Separation of variables and integration gives

$$\frac{R^n\left[N_{AR} - x_{A\delta}(N_{AR} + N_{BR})\right]}{R^n\left[N_{AR} - x_{A0}(N_{AR} + N_{AR})\right]} = \exp\left[\frac{R^n(N_{AR} + N_{AR})}{cD_B}\int_{R_0}^{R_\delta}\frac{dr}{r^n}\right] \equiv E$$

$$E = \frac{(N_{A0} + N_{B0})}{cD_{Ab}}\frac{R_0}{R_\delta}(R_\delta - R_0)$$

Note now that R cancels out on the left side of this equation. We can this eliminate it and, just for convenience, replace the "R" subscripts by "0", referring to conditions at $R_0$. Moreover we can add 1 + -1 to the left side in the form of

$$1-\frac{N_{A0}-x_{A0}(N_{A0}+N_{B0})}{N_{A0}-x_{A0}(N_{A0}+N_{B0})}$$

We thus obtain

$$1+\frac{(N_{A0}+N_{B0})(x_{A0}-x_{A\delta})}{N_{A0}-x_{A0}(N_{A0}+N_{B0})}=E$$

as required. Moreover, E can be expressed as

$$E=\exp\left[(N_{A0}+N_{B0})/k_{x,loc}\right]$$

to complete the development. Remember that we are using the mass transfer coefficient for negligible convection, so that

$$x_{A0}(N_{A0}+N_{B0})<<N_{A0}$$

To show this for spheres, start with a spherical shell and write

$$\nabla^2 x_A=0=\frac{1}{r^2}\frac{d}{dr}r^2\frac{dx_A}{dr}$$

For the above boundary conditions this yields

$$\frac{x_A-x_{A0}}{x_{A\delta}-x_{A0}}=\frac{R_\delta-R_0}{R_\delta}\left(1-\frac{R_0}{r}\right)$$

$$N_{A0}=\left[cD_{AB}\left(\frac{R_\delta}{R_0}\frac{1}{(R_\delta-R_0)}\right)\right](x_{A0}-x_{A\delta})=k_{x,loc}(c_0-c_\delta)$$

The cylindrical result is obtained similarly, and the heat transfer results can be obtained directly by analogy:

$$N_A \to N_A \tilde{C}_{pA}$$
$$cD_{AB} \to k$$
$$x_A \to T$$
$$J^*_{A0} \to q_0$$

Moreover, one can extend this result to variable properties by including the transport properties inside the integral.

22C.1 Calculation of ultrafiltration rates

The calculational technique is described in detail on pages 714 and 715, and the results are shown in Fig. 22.8-9.

Prob. 23A.1 Expansion of a gas mixture: very slow reaction rate

We first the gas composition from Eqs. 23.5-56 through 59:

$$K_H = \frac{(x_{H_2O})(x_{CO})}{(x_{H_2})(x_{CO_2})} = \frac{(x_{CO})^2}{(x_{CO_2})^2} = \left(\frac{x_{CO}}{0.5 - x_{CO_2}}\right)^2$$

$$x_{CO} = \frac{0.5\sqrt{K}}{1+\sqrt{K}};\ \sqrt{K} = 10^{-0.15/2} = 0.841;$$

$$x_{CO} = x_{H_2O} = \frac{0.5 \cdot 0.841}{1+0.841} = 0.228;$$

$$x_{H_2} = x_{CO_2} = 0.5 - 0.228 = 0.272$$

It follows that

$$\sum_i x_i M_i = 0.228(28.01+18.02) + 0.272(2.016+44.01) = 23.01$$

$$\sum_i x_i \tilde{C}_{pi} = 0.228(7.93+9.86) + 0.272(7.217+12.995)$$

$$= 9.554 cal/g-mol,^o K$$

Substitution of this last result in Eq. 23.5-54 yields

$$T_2 = T_1(p_2/p_1)^{R/\Sigma x_i \tilde{C}_{pi}} = 1000(1/1.5)^{(1.987/9.55)} = 920K$$

Substitution of the above values in Eq. 23.5-55 , using Table F3.3, yields

$$v_2 = \left\{2(T_1 - T_2)\frac{\sum x_i \tilde{C}_{pi}}{\sum x_i M_i}\right\}^{1/2}$$

$$= \left[2(1000-920)\frac{9.554 \cdot 4.1840 \cdot 10^7}{23.01}\right]^{1/2}$$

$$= 2.78 \cdot 10^9 cm/s = 1726 ft/s$$

The speed of sound can be determined from Eq. 9.4-4 as

$$v_s = \sqrt{\gamma R T_2 / M} = \sqrt{\frac{\tilde{C}_p}{\tilde{C}_p - R} \cdot \frac{R T_2}{M}}$$

$$= \sqrt{\frac{9.554}{9.554 - 1.987} \cdot \frac{1.987 \cdot 4.184 \cdot 10^7 \cdot 920}{23.01}}$$

$$= 6.48 \cdot 10^9 \, cm/s = 2126 \, ft/s$$

The flow is therefore subsonic, and the assumption that $p_2$ is equal to the ambient pressure is justified.

23A.2 Height of a packed-tower absorber

Begin by completing the mass balance as shown on the accompanying figure. The cyclohexane free gas stream rate in lb-moles/hr

$$W_G = (363\,ft^3/\min)(60\min/hr)\left(1.05\cdot\frac{273}{303}\frac{lb-mole}{359\,ft^3}\right)\cdot 0.99$$

$$= 56.8 lb-moles/hr$$

and

$$Y_{H2} = 0.01/0.99 = 0.0101$$

The corresponding exit mole ratio is

$$Y_{H1} = 0.0101\cdot 0.1 = 0.00101$$

The exit liquid mole ratio is now obtained from the macroscopic mass balance

$$X_{H2} = X_{H1} - (-56.8/19.94)(0.00909)$$

$$= 0.0289$$

More generally we may write

$$Y = 0.00101 + (19.94/56.8)(X - 0.003)$$

$$= 0.351X - 0.00043$$

This is the answer to part (a).

We now begin with Raoult's law in the form

$$y_0 = \frac{p_{vap}}{p_{tot}}x_0 = \frac{121}{1.05\cdot 760}x_0 = 0.1516x_0$$

This result can be used along with the definition of the mole ratio to obtain an equilibrium expression in terms of mole ratio:

$$Y = y/(1-y) = \alpha x/[1-\alpha x];\ x = X/(1+X)$$

where α equals 0.1516. This expression is also drawn on the accompanying figure, and it may be seen to effectively straight, because all solutions are so dilute. This is the solution to part (b).

We may now determine the interface conditions from the relation

$$\frac{Y - Y_0}{X - X_0} = -\left(\frac{0.32}{14.2}\right) = -0.0225$$

Since both the equilibrium and operating lines are linear to the accuracy of the technique we can simplify this expression to

$$Y_0 = 0.15X_0;\ Y = 0.351X - 0.00043$$

We thus find

$$X_0 \approx \frac{Y + 0.0225X}{0.15 + 0.0225}$$

$$Y_0 \approx 0.15X_0$$

These results are tabulated on the attached spread sheet, and they include the answer to part (c).

We may next calculate the tower height from the expression

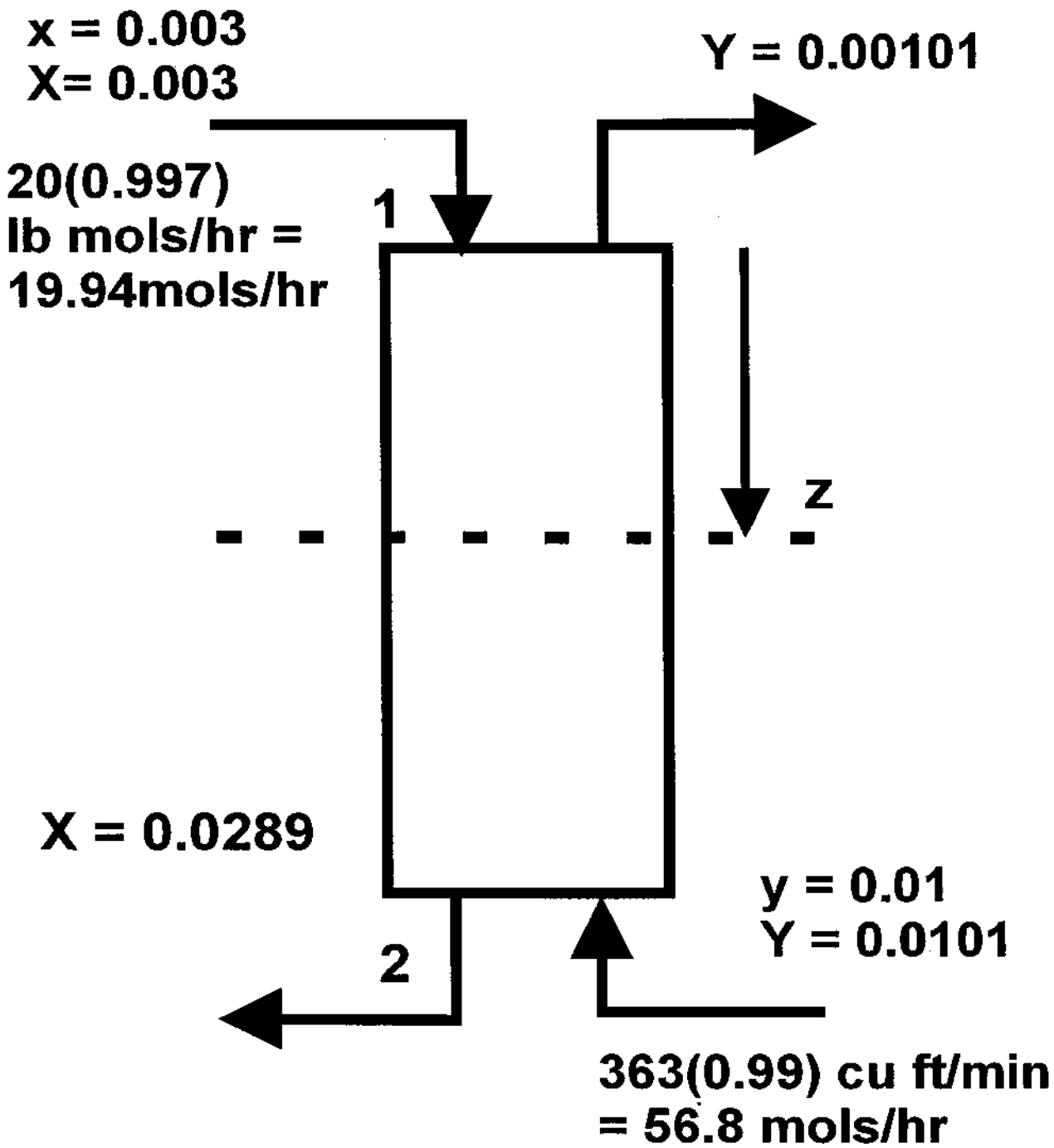
$$Z = -\frac{W_G}{S(k_y a)} \int_{Y_1}^{Y_2} \frac{dY}{(Y - Y_0)} = -\left(\frac{56.8}{2 \cdot 14.2}\right) I$$

Numerical integration gives about 62 ft, which is the answer to part (d). Use of the log mean approximation yields

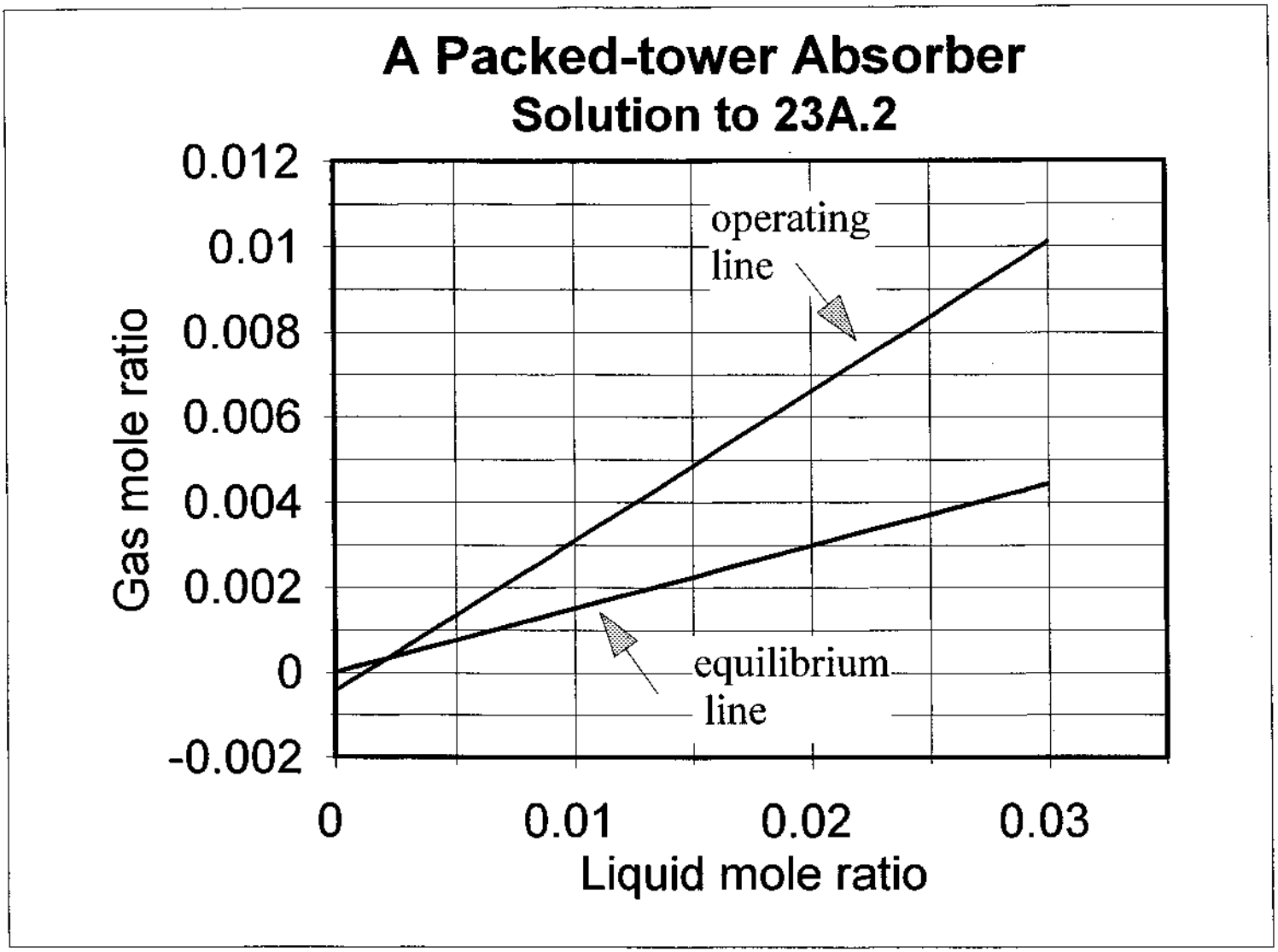
$$I \approx \frac{(Y_2 - Y_1)}{(Y - Y_0)_{\ln}}$$

with $\quad Y_2 - Y_1 = 0.00909;\ (Y - Y_0)_{\ln} = 0.0003$

and a packed height of 60 ft.




## 23A.2: Mass balances


A Packed-tower Absorber
Solution to 23A.2
0.012
0.01
0.008
0.006
0.004
0.002
0
-0.002
Gas mole ratio
operating line
equilibrium line
0
0.01
0.02
0.03
Liquid mole ratio

23A.2 Graphical constructions

| X | Y | x | y* | Y* | X/0 | Y/0 |
|---|---|---|---|---|---|---|
| 0 | -0.00043 | 0 | 0 | 0 | 0 | |
| 0.001 | -7.9E-05 | 0.000999 | 0.0001514 | 0.000151 | 0 | |
| 0.002 | 0.000272 | 0.001996 | 0.0003026 | 0.000303 | 0 | |
| 0.003 | 0.000623 | 0.002991 | 0.0004534 | 0.000454 | 0.004003 | 0.0006 |
| 0.004 | 0.000974 | 0.003984 | 0.000604 | 0.000604 | 0.006168 | 0.000925 |
| 0.005 | 0.001325 | 0.004975 | 0.0007542 | 0.000755 | 0.008333 | 0.00125 |
| 0.006 | 0.001676 | 0.005964 | 0.0009042 | 0.000905 | 0.010499 | 0.001575 |
| 0.007 | 0.002027 | 0.006951 | 0.0010538 | 0.001055 | 0.012664 | 0.0019 |
| 0.008 | 0.002378 | 0.007937 | 0.0012032 | 0.001205 | 0.014829 | 0.002224 |
| 0.009 | 0.002729 | 0.00892 | 0.0013522 | 0.001354 | 0.016994 | 0.002549 |
| 0.01 | 0.00308 | 0.009901 | 0.001501 | 0.001503 | 0.019159 | 0.002874 |
| 0.011 | 0.003431 | 0.01088 | 0.0016495 | 0.001652 | 0.021325 | 0.003199 |
| 0.012 | 0.003782 | 0.011858 | 0.0017976 | 0.001801 | 0.02349 | 0.003523 |
| 0.013 | 0.004133 | 0.012833 | 0.0019455 | 0.001949 | 0.025655 | 0.003848 |
| 0.014 | 0.004484 | 0.013807 | 0.0020931 | 0.002097 | 0.02782 | 0.004173 |
| 0.015 | 0.004835 | 0.014778 | 0.0022404 | 0.002245 | 0.029986 | 0.004498 |
| 0.016 | 0.005186 | 0.015748 | 0.0023874 | 0.002393 | 0.032151 | 0.004823 |
| 0.017 | 0.005537 | 0.016716 | 0.0025341 | 0.002541 | 0.034316 | 0.005147 |
| 0.018 | 0.005888 | 0.017682 | 0.0026806 | 0.002688 | 0.036481 | 0.005472 |
| 0.019 | 0.006239 | 0.018646 | 0.0028267 | 0.002835 | 0.038646 | 0.005797 |
| 0.02 | 0.00659 | 0.019608 | 0.0029725 | 0.002981 | 0.040812 | 0.006122 |
| 0.021 | 0.006941 | 0.020568 | 0.0031181 | 0.003128 | 0.042977 | 0.006447 |
| 0.022 | 0.007292 | 0.021526 | 0.0032634 | 0.003274 | 0.045142 | 0.006771 |
| 0.023 | 0.007643 | 0.022483 | 0.0034084 | 0.00342 | 0.047307 | 0.007096 |
| 0.024 | 0.007994 | 0.023438 | 0.0035531 | 0.003566 | 0.049472 | 0.007421 |
| 0.025 | 0.008345 | 0.02439 | 0.0036976 | 0.003711 | 0.051638 | 0.007746 |
| 0.026 | 0.008696 | 0.025341 | 0.0038417 | 0.003857 | 0.053803 | 0.00807 |
| 0.027 | 0.009047 | 0.02629 | 0.0039856 | 0.004002 | 0.055968 | 0.008395 |
| 0.028 | 0.009398 | 0.027237 | 0.0041292 | 0.004146 | 0.058133 | 0.00872 |
| 0.029 | 0.009749 | 0.028183 | 0.0042725 | 0.004291 | 0.060299 | 0.009045 |
| 0.03 | 0.0101 | 0.029126 | 0.0044155 | 0.004435 | 0.062464 | 0.00937 |

23B.1: Effective average driving forces in a gas absorber.

This problem has really been solved in the above discussion of Prob. 23A.2, but perhaps a more detailed analysis is in order. (a) We therefore begin by writing

$$Y_e = m_1 X + b_1$$
$$Y = m_2 X + b_2$$

It follows that $$Y_e - Y = (m_2 - m_1)X + (b_2 - b_1)$$

which is the required linear relation. It follows directly that

$$Y - Y_e = C_1 Y + C_2$$

(b) We now write directly from Eq. 23.5-21

$$Y - Y_0 = m_3 (X - X_0)$$

and $$Y_0 = m_1 X_0 + b_1$$

It follows that

$$Y - Y_0 = m_3 \left[ \frac{Y - b_2}{m_2} - \frac{Y_0 - b_1}{m_1} \right]$$

which is the required linear relation between Y and $Y_0$. It follows that there is also a linear relation between $(Y-Y_0)$ and Y so that

$$Y - Y_0 = C_3 Y + C_4$$

where $C_3$ and $C_4$ are constants, obtainable from the above relations.

(c) We now begin with the relation

$$Z = -\frac{W_G}{S(k_y a)} \int_{Y_1}^{Y_2} \frac{dY}{C_3 Y + C_4}$$

For $C_3$ non-zero and a finite denominator

$$Z = -\frac{1}{C_3} \frac{W_G}{S(k_y a)} \ln \left[ \frac{C_3 Y_2 + C_4}{C_3 Y_1 + C_4} \right]$$

$$C_3 Y_2 + C_4 = (Y - Y_0)_2$$

Now

$$C_3 Y_1 + C_4 = (Y - Y_0)_1$$

$$C_3 = \frac{(Y - Y_0)_2 - (Y - Y_0)_1}{(Y_2 - Y_1)}$$

and

$$Z = -\frac{W_G}{S(k_y a)} \frac{(Y_2 - Y_1)}{(Y - Y_0)_2 - (Y - Y_0)_1} \ln\left[\frac{(Y - Y_0)_2}{(Y - Y_0)_1}\right]$$

or

$$Z = -\frac{W_G}{S(k_y a)} \frac{(Y_2 - Y_1)}{(Y - Y_0)_{\ln}}$$

This result also holds for the degenerate of $C_3$ equal to zero where $(Y\text{-}Y_0)_{\ln}$ becomes equal to the driving force at any position in the column.

The corresponding expression in terms of $(K_y a)$ follows by analogy.

23B.2 Expansion of a gas mixture: very fast reaction rate.

Begin by computing the temperature distribution of $\tilde{H}$ as follows:

| | 900 K | 950 K | 1,000 K |
|---|---|---|---|
| $\sqrt{K}$ | 0.676 | 0.754 | 0.841 |
| $x_{H_2} = x_{CO_2} = 0.5/\sqrt{K}$ | 0.298 | 0.285 | 0.272 |
| $x_{CO} = x_{H_2O} = 0.5 - x_{H_2}$ | 0.202 | 0.215 | 0.228 |
| $\tilde{H}_{H_2} = 6340 + 7.217(T - 900)$ | 6340 | 6701 | 7062 |
| $\tilde{H}_{CO_2} = -83,242 + 12.995(T - 900)$ | -83,242 | -82,592 | -81,942 |
| $\tilde{H}_{CO} = -16,636 + 7.932(T - 900)$ | -16,636 | -16,239 | -15,843 |
| $\tilde{H}_{H_2O} = -49,378 + 9.861(T - 900)$ | -49,378 | -48,885 | -48,392 |
| $\tilde{H} = \sum x_i \tilde{H}_i, cal/g-mole$ | -36,252 | -35,631 | -35,013 |
| $\Delta\tilde{H}/\Delta T$ | 12.42 | 12.36 | |

The proposed value of 12.4 is clearly adequate. This is the answer to part (a).

(b) From Eq. 23.5-53, which holds for constant mean molecular weight we get

$$\ln\left(\frac{T_2}{T_1}\right) - \frac{1.987}{12.40}\ln\left(\frac{1}{1.5}\right) = -0.065$$

so that $$T_2 = 1,000 \cdot e^{-0.056} = 937K$$

(c) For consistency with (b) we do not interpolate for $\tilde{H}_2$. Rather we set

$$\tilde{H}_2 - \tilde{H}_1 \approx 12.4(937 - 1,000) = -781 cal/g-mole$$

Then $$\hat{H}_1 - \hat{H}_2 = \frac{\tilde{H}_1 - \tilde{H}_2}{\sum x_i M_i} = \frac{781 \cdot 4.184 \cdot 10^7}{23.01}$$

$$= 1.42 \cdot 10^9 \, erg/g = 1.42 \cdot 10^9 \, cm^2/s^2$$

and $$v_2 = \sqrt{2(\hat{H}_1 - \hat{H}_2)} = 5.33 \cdot 10^4 \, cm/s = 1748 \, ft/s$$

23B.3 Startup of a chemical reactor

It is convenient to start by scaling moles of B by defining

$$u \equiv M_{B,tot} \cdot \frac{\rho k_{1c}'''}{w c_{A0}}$$

and to write $$u \equiv u_{tr} + u_{\infty}$$

where the subscripts "tr" and "∞" refer to transient and steady contributions, respectively. We then find that

$$\frac{d^2u}{dt^2} + \Sigma \frac{du}{dt} + Pu = 1$$

where $\Sigma = (k_{1A}''' + k_{1B}''' + k_{1C}'')$ and $P = k_{1B}''' k_{1C}''$. It also follows that

$$u_{\infty} = 1/P;\ \frac{d^2u}{dt^2} + \Sigma \frac{du}{dt} = 0$$

with the initial conditions that at time zero

$$u_{tr} = -1/P;\ du_{tr}/dt = 0$$

We now assume solutions of the form

$$u_{tr} = a_{\pm} e^{s_{\pm} t}$$

Putting this trial solution into the differential equation gives

$$s^2 + s\Sigma + P = 0$$

Solving for s using the quadratic formula gives Eq. 23.6-9, and use of the boundary conditions yields Eq. 23.6-28, with $u_{\infty}$ included, thus completing the formal solution to part (a).

Using the identity in the hint shows that the argument of the radical is indeed real, and smaller than Σ. Both of the exponents are thus negative as required for part (b). We also have the explicit expression for B required in part (c) and need only that for A. This latter can be obtained directly from Eq. 23.6-3 as

$$M_{A,tot} = \frac{1}{k_{1B}''} \left[ \frac{dM_{B,tot}}{dt} + (k_{1A}''' + k_{1C}''') M_{B,tot} \right]$$

with

$$\frac{dM_{B,tot}}{dt} = \frac{wc_{A0}}{\rho k_{1C}^{'''}} \cdot \frac{s_+ s_-}{(s_+ - s_-)} \left[ \exp(s_+ t) - \exp(s_t) \right]$$

which completes the solution.

**23B.4 Irreversible first-order reaction in a continuous reactor**

The unsteady state mass balance on substance $A$ over the reactor is (see Eq. 21.1-3)

$$\frac{d}{dt}M_{A,\text{tot}} = W_{A1} - W_{A2} - W_{A,0} + R_{A,\text{tot}}$$

Into this we substitute $M_{A,\text{tot}} = Vc_A$, $W_{A1} = (w/\rho)c_{A0}$, $W_{A2} = (w/\rho)c_A$, and $R_{A,\text{tot}} = -(k_1'''c_A)V$. Then division by $V$ gives

$$\frac{dc_A}{dt} = \left(\frac{w}{\rho V}\right)c_{A0} - \left(\frac{w}{\rho V} + k_1'''\right)c_A$$

Designate the coefficient of the last term by $t_0^{-1}$, where $t_0$ has dimensions of time. Then the separable first-order equation may be integrated to give

$$\int_{c_{A0}}^{c_A} \frac{dc_A}{\left(t_0^{-1} - k_1'''\right)c_{A0} - t_0^{-1}c_A} = \int_0^t dt$$

Performing the integration gives

$$-t_0 \ln \frac{\left(t_0^{-1} - k_1'''\right)c_{A0} - t_0^{-1}c_A}{-k_1'''c_{A0}} = t \qquad \text{or} \qquad \ln \frac{c_A - \left(1 - k_1'''t_0\right)c_{A0}}{k_1'''t_0 c_{A0}} = -\frac{t}{t_0}$$

Taking the antilogarithm of both sides then yields

$$\frac{c_A}{c_{A0}} = \left(1 - k_1'''t_0\right) + k_1'''t_0 \exp\left(-\frac{t}{t_0}\right)$$

Then, transforming back to the original variables, we get

$$\frac{c_A}{c_{A0}} = \frac{w/\rho V}{(w/\rho V) + k_1'''} + \left(1 - \frac{w/\rho V}{(w/\rho V) + k_1'''}\right)\exp\left[-\left((w/\rho V) + k_1'''\right)t\right]$$

which is the same as Eq. 23B.4-1.

23B.4 Irreversible first-order reaction in a continuous reactor (Alternative Solution)

A balance on species "A" in the reactor gives

$$\frac{d}{dt}M_{A,tot} = W_{A1} - W_{A2} - W_{A0} + R_{A,tot}$$

For t > 0, this equation is to be integrated as

$$V\int_{c_{A0}}^{c_A} \frac{dc_A}{Qc_{A0} - (Q + k_1'''V)c_A} = t$$

from which we may write

$$\left(c_A - \frac{Qc_{A0}}{Q + k_1'''V}\right) = \left(c_{A0} - \frac{Qc_{A0}}{Q + k_1'''V}\right) \cdot \exp\left[-\left(\frac{Q + k_1'''V}{V}\right)t\right]$$

or

$$\frac{c_A}{c_{A0}} = \frac{Q}{Q + k_1'''V} + \left(1 - \frac{Q}{Q + k_1'''V}\right)\exp\left[-\left(\frac{Q + k_1'''V}{V}\right)t\right]$$

23B.5 Math and enthalpy balances in an adiabatic splitter

The conservation relations for this systems are:

Total mass balance: $F = P + W = 100$

Mass of ammonia: $zF = yP + xW = 40$

Enthalpy: $HF = HP + hW = 42{,}000$

Equilibrium: $Y = 10^{1.4} X^{1.53} = 25.12 X^{1.53}$

We now write explicit expressions for the conditions of the problem, beginning by using the overall balance in the ammonia balance to write:

$$y\Theta + x(1-\Theta) = 0.4$$

or

$$\Theta = \frac{0.4 - x}{y - x}$$

where $\Theta$ = P/F is the product (fractional) *cut*. We next eliminate P and W in the enthalpy balance by writing

$$(1{,}210 - 465y - 115y^{12})\Theta$$
$$+(330 - 950x + 740x^2)(1-\Theta) = 420$$

or

$$(2.88 - 1.107y - 0.2378y^{12})\Theta$$
$$+(0.7857 - 2.26x + 1.762x^2)(1-\Theta) = 1$$

or

$$A\Theta + B(1-\Theta) = 1$$

which can be rearranged to $\Theta = \dfrac{1-B}{A-B}$

Clearly a numerical solution is required, and the following approach is used here:

1) assume a value of x, and compute the corresponding values of X and Y.
2) calculate the cut corresponding to these choices from the species mass balance, and plot $\Theta(x)$, and

3) do the same for the enthalpy balance.

4) The desired solution occurs when these two equations give the same cut.

This process is shown on the attached spread sheet, where it I found that

$$\mathbf{x = 0.22;\ y = 0.713;\ \Theta = 0.365}$$

Enthalpies can then be obtained from the thermal data just given:

$$\hat{h} = 330 - 950 \cdot 0.22 + 740 \cdot (0.22^2) = 157\,Btu/lb$$

$$\hat{H} = 1210 - 465y - 115y\ \hat{H} = 1210 - 465y - 115y^{12} = 877\,Btu/lb$$

These results vary significantly from the answers given in the problem statement.

Mass and energy balances for Prob. 23B.5

| x | X | Y | y | Theta-x | A | B | Theta-h | H-E |
|---|---|---|---|---|---|---|---|---|
| 0.1 | 0.111111 | 0.741344 | 0.425731 | 0.9210058 | 2.408708 | 0.57732 | 0.230798 | -0.69021 |
| 0.11 | 0.123596 | 0.85773 | 0.461709 | 0.8245463 | 2.368866 | 0.55842 | 0.243907 | -0.58064 |
| 0.12 | 0.136364 | 0.979867 | 0.494915 | 0.746835 | 2.332077 | 0.539873 | 0.256738 | -0.4901 |
| 0.13 | 0.149425 | 1.107524 | 0.525509 | 0.6826638 | 2.298156 | 0.521678 | 0.269253 | -0.41341 |
| 0.14 | 0.162791 | 1.240496 | 0.55367 | 0.6285198 | 2.26689 | 0.503835 | 0.281423 | -0.3471 |
| 0.15 | 0.176471 | 1.378603 | 0.579585 | 0.5819568 | 2.238058 | 0.486345 | 0.29323 | -0.28873 |
| 0.16 | 0.190476 | 1.52168 | 0.603439 | 0.5412245 | 2.211439 | 0.469207 | 0.304663 | -0.23656 |
| 0.17 | 0.204819 | 1.669577 | 0.625409 | 0.5050407 | 2.186821 | 0.452422 | 0.315716 | -0.18932 |
| 0.18 | 0.219512 | 1.82216 | 0.645662 | 0.4724462 | 2.164005 | 0.435989 | 0.326392 | -0.14605 |
| 0.19 | 0.234568 | 1.979305 | 0.664351 | 0.4427099 | 2.142805 | 0.419908 | 0.336696 | -0.10601 |
| 0.2 | 0.25 | 2.140896 | 0.681619 | 0.4152656 | 2.123055 | 0.40418 | 0.346634 | -0.06863 |
| 0.21 | 0.265823 | 2.306828 | 0.697595 | 0.3896674 | 2.104604 | 0.388804 | 0.356216 | -0.03345 |
| 0.22 | 0.282051 | 2.477002 | 0.712396 | 0.3655595 | 2.087315 | 0.373781 | 0.365455 | -0.0001 |
| 0.23 | 0.298701 | 2.651327 | 0.726127 | 0.3426542 | 2.071068 | 0.35911 | 0.374361 | 0.031707 |
| 0.24 | 0.315789 | 2.829716 | 0.738884 | 0.3207158 | 2.055758 | 0.344791 | 0.382946 | 0.062231 |
| 0.25 | 0.333333 | 3.012089 | 0.750753 | 0.2995487 | 2.041292 | 0.330825 | 0.391224 | 0.091675 |
| 0.26 | 0.351351 | 3.198371 | 0.761812 | 0.2789887 | 2.027587 | 0.317211 | 0.399204 | 0.120215 |
| 0.27 | 0.369863 | 3.38849 | 0.772131 | 0.2588965 | 2.014572 | 0.30395 | 0.406899 | 0.148002 |
| 0.28 | 0.388889 | 3.582378 | 0.781773 | 0.2391521 | 2.002185 | 0.291041 | 0.414319 | 0.175167 |
| 0.29 | 0.408451 | 3.779972 | 0.790794 | 0.2196513 | 1.990369 | 0.278484 | 0.421475 | 0.201823 |
| 0.3 | 0.428571 | 3.981211 | 0.799246 | 0.2003022 | 1.979078 | 0.26628 | 0.428375 | 0.228073 |
| 0.31 | 0.449275 | 4.186037 | 0.807175 | 0.181023 | 1.968268 | 0.254428 | 0.43503 | 0.254007 |
| 0.32 | 0.470588 | 4.394395 | 0.814622 | 0.1617395 | 1.957904 | 0.242929 | 0.441447 | 0.279708 |
| 0.33 | 0.492537 | 4.606234 | 0.821627 | 0.1423843 | 1.947952 | 0.231782 | 0.447635 | 0.305251 |
| 0.34 | 0.515152 | 4.821502 | 0.828223 | 0.1228947 | 1.938384 | 0.220987 | 0.453601 | 0.330706 |
| 0.35 | 0.538462 | 5.040153 | 0.834441 | 0.1032117 | 1.929174 | 0.210545 | 0.459352 | 0.35614 |
| 0.36 | 0.5625 | 5.262141 | 0.84031 | 0.0832795 | 1.920299 | 0.200455 | 0.464894 | 0.381614 |
| 0.37 | 0.587302 | 5.487421 | 0.845856 | 0.0630443 | 1.91174 | 0.190718 | 0.470234 | 0.407189 |
| 0.38 | 0.612903 | 5.715952 | 0.851101 | 0.0424538 | 1.903477 | 0.181333 | 0.475377 | 0.432923 |
| 0.39 | 0.639344 | 5.947692 | 0.856067 | 0.0214561 | 1.895495 | 0.1723 | 0.480329 | 0.458873 |
| 0.4 | 0.666667 | 6.182604 | 0.860775 | -4.82E-16 | 1.887778 | 0.16362 | 0.485095 | 0.485095 |
| 0.41 | 0.694915 | 6.420649 | 0.865241 | -0.0219664 | 1.880313 | 0.155292 | 0.48968 | 0.511646 |
| 0.42 | 0.724138 | 6.661792 | 0.869482 | -0.0444956 | 1.873088 | 0.147317 | 0.494088 | 0.538584 |

23B.6 Flow Distribution in an ideal cascade

Begin with a mole fraction of $x_W = x_0 = 0.1$ in liquid downflow. Then

$$X_0 = x_0/(1-x_0) = 0.1/0.9 = 0.1111$$

and a relative volatility of 2.5. Then

$$X_{n+1} = \sqrt{2.5}X_n = 1.581X_n$$

Values through $X_{11}$ are given on the attached spread sheet. These in turn may be expressed as mole fractions. Thus

Plate "0" (reboiler): $x_0 = 0.1;\ y_0 = 0.217391$

because the upflow stream leaving the reboiler must be in equilibrium with the downflowing stream, which is the product. Compositions are shown in the attached spread sheet. It may be seen there that the mole ratio for stage 9, the tenth stage is 17.1428, corresponding to a mole fraction of 0.9488. [The large numbers of digits being carried is to ensure agreement with ideal stage theory. Note that 13 stages are provided for in the spread sheet. There is no upper limit at this stage of the calculations.]

These compositions must be correct for any ideal cascade with a bottoms mole fraction of 0.1 and constant relative volatility of 2.5. it remains however, to make mass balances for this system, and for convenience we shall assume saturated liquid feed of mole fraction 0.523249, the predicted composition of $x_5$.

We can then make mass balances for the column as a whole, using as basis a feed rate of unity:

$$1 = P + W$$

$$0.5232 = 0.9449P + 0.1W$$

We thus find that

$$P = 0.501$$

$$W = 0.499$$

$$F = 1.000$$

and we are now ready to calculate intermediate stream rates. This done via overall and solute mass balances. Thus for the section below the feed plate, the "exhausting" section, we begin by writing mass balances or plate "0":

$$D_1 - U_0 = W$$

$$x_1 D_1 - y_0 U_0 = x_0 W$$

or

$$D_1 = W\left(\frac{y_0 - x_0}{y_0 - x_1}\right)$$

More generally

$$D_{n+1} = W\left(\frac{x_w - y_n}{x_{n+1} - y_n}\right)$$

The results are shown in the attached spread sheet.

At the top of the column, plate 12, part of the upflow is removed as product and the remainder converted to a downflow. The mass balances are then

$$P = U_n - D_{n+1}$$

$$y_p P = y_n U_n - x_{n+1} D_{n+1}$$

$$D_n = P\left(\frac{y_p - y_{n-1}}{y_{n-1} - x_n}\right)$$

These results are also shown in the spread sheet.

Prob. 23B.6 solutions

| n | Xn | Yn | xn | yn | D | -f | U |
|---|---|---|---|---|---|---|---|
| 0 | 0.111111 | 0.277778 | 0.1 | 0.217391 | 0.499 | -0.499 | |
| 1 | 0.175666 | 0.439166 | 0.149419 | 0.305153 | 0.861792 | -0.86179 | |
| 2 | 0.277729 | 0.694322 | 0.217361 | 0.409793 | 1.166069 | -1.16607 | |
| 3 | 0.439089 | 1.097723 | 0.305116 | 0.523293 | 1.476794 | -1.47679 | |
| 4 | 0.6942 | 1.7355 | 0.409751 | 0.634436 | 1.860312 | -1.86031 | |
| 5 | 1.09753 | 2.743825 | 0.523249 | 0.732894 | 2.398508 | -2.39851 | |
| 6 | 1.735195 | 4.337987 | 0.634395 | 0.812663 | 1.078348 | -1.07835 | |
| 7 | 2.743343 | 6.858358 | 0.732859 | 0.872747 | 0.830161 | -0.83016 | |
| 8 | 4.337226 | 10.84306 | 0.812637 | 0.915562 | 0.601373 | -0.60137 | |
| 9 | 6.857154 | 17.14288 | 0.872727 | 0.944882 | 0.343134 | -0.34313 | |
| 10 | 10.84116 | 27.1029 | 0.915549 | 0.964416 | | | |
| 11 | 17.13987 | 42.84969 | 0.944873 | 0.977195 | | | |
| 12 | 27.09814 | 67.74535 | 0.96441 | 0.985454 | | | |
| 13 | | | 0.985454 | | | | |

Note: P = 0.501 and W = 0.499

23B.7 Isotope fractionation and the value function

The fundamental equation needed for this comparison are

$$\Delta / F = \Theta V(y) + (1-\Theta)V(x) - V(z) \qquad (1)$$

$$z = y\Theta + x(1-\Theta) \qquad (2)$$

$$V(w) = (2w-1)\ln[w/(1-w)] \qquad (3)$$

We must first calculate the material balances for the two processes using Eq. 2:

Process 1:

$$0.1 = 0.9\Theta + 0.01(1-\Theta);\ \Theta = 0.101$$

Process 2:

$$0.1 = 0.95\Theta + 0.02(1-\Theta);\ \Theta = 0.86$$

We next need values for each of these compositions:

| | | | | | |
|---|---|---|---|---|---|
| w | 0.01 | 0.02 | 0.1 | 0.9 | 0.95 |
| V | 4.503 | 3.736 | 1.758 | 1.758 | 2.650 |

We may then calculate the separative work done by the two processes as:

Process 1:

$$\Delta / F = 0.101 \cdot 1.758 + 0.899 \cdot 4.503 - 1.758 = 2.47$$

Process 2:

$$\Delta / F = 0.86 \cdot 2.650 + 0.14 \cdot 3.736 - 1.758 = 1.04$$

The first process is the more effective.

< end >

**23C.1 Irreversible second-order reaction in an agitated tank**

*a*. The unsteady state mass balance on substance $A$ over the reactor is (see Eq. 21.1-3)

$$\frac{d}{dt}M_{A,\text{tot}} = W_{A1} - W_{A2} - W_{A,0} + R_{A,\text{tot}}$$

Into this we substitute $M_{A,\text{tot}} = Vc_A$, $W_{A1} = (w/\rho)c_{A0}$, $W_{A2} = (w/\rho)c_A$, and $R_{A,\text{tot}} = -(k_2'''c_A^2)V$. Then division by $V$ gives

$$\frac{dc_A}{dt} = \left(\frac{w}{\rho V}\right)c_{A0} - \left(\frac{w}{\rho V}\right)c_A - k_2'''c_A^2$$

*b*. We now make the change of variable indicated in the problem statement:

$$c_A = u - \frac{1}{2k_2'''}\left(\frac{w}{\rho V}\right)(1+Z) \qquad \text{where} \qquad Z = \sqrt{1 + 4k_2'''c_{A0}\left(\frac{\rho V}{w}\right)}$$

This gives

$$\frac{du}{dt} = \underbrace{\left(\frac{w}{\rho V}\right)c_{A0}}_{==} - \left(\frac{w}{\rho V}\right)u + \frac{1}{2k_2'''}\left(\frac{w}{\rho V}\right)^2\underbrace{(1+Z)}_{==\ ---}$$

$$-k_2'''u^2 + \left(\frac{w}{\rho V}\right)(1+Z)u - \frac{1}{4k_2'''}\left(\frac{w}{\rho V}\right)^2\underbrace{\left(1+2Z+Z^2\right)}_{==\ ---\ \ ==}$$

The --- terms cancel, as do the == terms, so that the following equation results:

$$\frac{du}{dt} - \left(\frac{w}{\rho V}\right)Zu = -k_2'''u^2 \qquad \text{(Bernoulli equation)}$$

The accompanying initial condition is

At $t = 0$: $\quad u = c_{A0} + \frac{1}{2k_2'''}\left(\frac{w}{\rho V}\right)(1+Z)$

*c*. We now make the change of variable $v = 1/u$, which converts the Bernoulli equation into a linear first-order differential equation:

$$\frac{dv}{dt} + \left(\frac{w}{\rho V}\right)Zv = k'''$$

This can be solved at once to give

$$v = C\exp\left(-\left(\frac{w}{\rho V}\right)Zt\right) + k_2'''\left(\frac{\rho V}{wZ}\right)$$

The constant of integration is obtained by using the initial condition given in (*b*)

$$\frac{1}{c_{A0} + (1/2k_2''')(w/\rho V)(1+Z)} = C + k_2'''\left(\frac{\rho V}{wZ}\right)$$

When we insert this constant into the solution and then revert to the original variable, we get

$$\frac{1}{c_A + (1/2k_2''')(w/\rho V)(1+Z)}$$
$$= \left(\frac{1}{c_{A0} + (1/2k_2''')(w/\rho V)(1+Z)} - k_2'''\left(\frac{\rho V}{wZ}\right)\right)\exp\left(-\left(\frac{w}{\rho V}\right)Zt\right) + k_2'''\left(\frac{\rho V}{wZ}\right)$$

When the expression for $Z$ is inserted, we then have the final concentration vs. time expression in terms of the original variables.

**Prob. 23C.2 Protein Purification**

(a) $Y = N_1 / N_2 = cvS_1x_1 / cvS_2 / x_2 = (S_1 / S_2)X_1$

(b) Begin with writing material balances over the column:

$$0.99P = 0.95 \cdot 0.9F;\ P = 0.8636F$$
$$W = (1 - 0.8636)F = 0.1364F$$
$$x_W = 0.05 \cdot 0.9F / 0.1364F = 0.3300$$

We thus find that

$$Y_P = 0.99/0.01 = 99;\ X_W = 0.33/0.67 = 0.4925$$

Then, counting from the bottom, so that stage 1 produces the product W, we find that

$$Y_1 = \alpha\ X_W = 0.4925\alpha;\ Y_N = Y_P = 99$$
$$Y_N / Y_1 = (99/0.4925) = 201 = \left(\sqrt{\alpha}\right)^N = \alpha^{N/2}$$

or

$$N = 2 \cdot \ln 201 / \ln \alpha$$

The relation between N and α is calculated on the attached spread sheet and shown on the attached graph.

(c) For a 3-stage ideal cascade, the down flow from the top stage and the up flow from the bottom must both have the composition of the feed: a mole fraction of 0.9 and a mole ratio of 9. The product mole ratio must then be

$$Y_P = 40 \cdot 9 = 360;\ y_P = 360/361 = 0.997$$

and the bottoms must be

$$X_W = 9/40 = 0.225;\ x_W = 0.225/1.225 = 0.184$$

The over up and down flows from the feed stage in turn are

$$Y_F = Y_P / \sqrt{40} = 56.92;\ y_F = 56.92/57.92 = 0.983$$
$$X_F = Y_F / 40 = 1.423;\ \mathrm{x_F} = 1.423/1.423 = 0.590$$

which completes the determination of stage compositions.

Calculation of stage stream flows is calculated as follows:

$$P,W: \begin{array}{l} F = P + W \\ 0.9F = 0.997P + 0.184W \end{array}$$

It follows that

$$P = 0.8807F;\ W = 0.1193F$$

$$U_F, D_P:$$

$$U_F = P + D_P$$

$$0.983U_F = 0.997P + 0.9D_P$$

Thus

$$U_F = 1.1687P = 1.029F$$

$$D_P = 0.1687P = 0.1485F$$

Similarly

$$D_F = W + U_W$$

$$x_F D_F = x_W W + y_W U_W$$

so that

$$D_F = 2.31W = 0.275F$$

$$U_W = 1.31W = 0.156F$$

These results can be summarized as follows:

| Stream | to product | to top stage | to feed stage | to bottom stage | to bottoms |
|---|---|---|---|---|---|
| Rate/F | 0.8807 | 1.029 | 1.30 | 0.275 | 0.1193 |

The sum of flows into the three stages is then

## Sum of stage inflows = 2.60 F

which determines the total membrane requirement. The yield is

$$Yield = 0.997 \cdot 0.8807 / 0.9 = 0.9756$$

(d) For a simple single stage operation the defining equations are

$$F = P + W$$

$$zF = yP + xW$$

$$Y = 40X$$

For P = 0.8807 F we may write

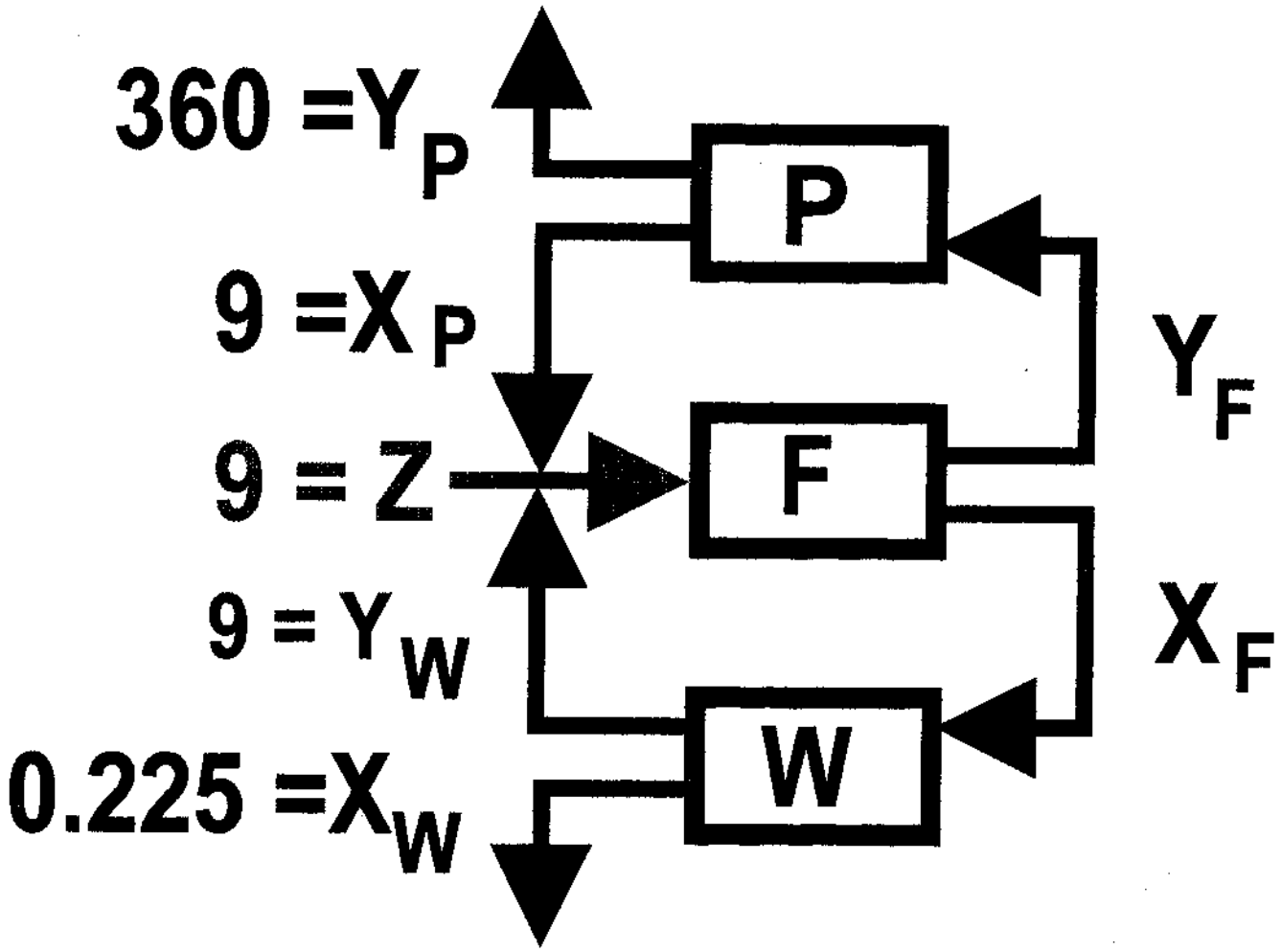
$$y = \frac{0.9 - 0.1193x}{0.8807}; x = \frac{y}{40 - 39y}$$

and therefore since y must be less than unity,

0.974

$$Yield = 0.974 \cdot 0.8807 / 0.9 = 0.953$$

Both results are quite good, but product purity is substantially higher for the 3-stage cascade.




$y_p = 0.997$

$z = 0.9$

$x_w = 0.184$

$\alpha = 40$

# A Three-stage Protein Fractionator

(X, Y, Z = mole ratios of desired to undesired component; x, y, z = correspoinding mole fractions)

23C.3 Physical significance of the zeroth and first moments

There is no end to the systems that can be examined, but one must limit behavior to a single inlet and single outlet with negligible diffusion across both, and effectively uniform concentration across both inlet and outlet. Here we merely summarize the key findings. These two moments enable calculation of the *effective* volume of an otherwise ill defined system and the flow rate of fluid through it.

For the zeroth moment one can relax all constraints except those of time independent flow and both uniform concentration and negligible diffusion across the outlet. The tracer must also be introduced over a finite time interval. Then

$$\int_{-\infty}^{\infty} Qcdt = Q\int_{-\infty}^{\infty} cdt = QM_0 = m$$

over any flow cross-section meeting the above requirements. It follows that

$$Q = m / M_0$$

with what are often minimal reservations.

We may generalize the treatment of the first moment to

$$\frac{M_1\big|_{out} - M_1\big|_{in}}{M_0} = t_{res} \equiv \frac{V_{eff}}{Q}$$

where the *effective volume*

$$V_{eff} = \int_V K_D dV$$

where $K_D$ is the equilibrium ratio of local tracer concentration to that in the stream flowing into, through and out of the system under investigation. For tissue permeated by blood vessels for example

$$K_D = \frac{c_{tissue}}{c_{blood}}$$

This generalization is very important in pharmacokinetics for example.

23C.4 Analogy between the unsteady operation of an adsorption column and a cross-flow heat exchanger.

There are three basic cross flow exchanger morphologies: both streams mixed, only one mixed and neither mixed. The analog sought here is the last of these, and it was first described by Nusselt (W. Nusselt, Z. Ver. deut. Ing., 1911, **55**, 2021; Tech. Mech. Thermodynam., 1930, **1**, 417; Jakob, Max, "Heat Transfer", Wiley, 1957, pp. 217-227 and 253). It is difficult to find more than graphical representations of the solution because of numerical difficulties. Moreover, even Jakob's detailed analysis is cumbersome and difficult to follow. Now however many available programs can handle functions like the integral in Eq. 25 quite simply and effectively.

The analogs of Eq. 23.6-10 and 11 are

$$v_{x0}\rho_s\hat{C}_{pS}\frac{\partial T_S}{\partial x} = (Ua)(T_M - T_S)$$

$$v_{z0}\rho_m\hat{C}_{pM}\frac{\partial T_M}{\partial z} = -(Ua)(T_M - T_S)$$

Here the subscript "S" refers to the analog of the stationary phase, and x replaces t. The terms $v_{x0}$ and $v_{z0}$ are the superficial velocities in the x and x directions respectively, and they are defined by

$$v_{x0} = v_x/(1-\varepsilon);\ \ v_{z0} = v_z/\varepsilon$$

where the unadorned velocities are the mean velocities in the "x" and "z" channels, which occupy fractions ε and (1-ε) of the total fluid space respectively. These differ from their mass transfer analogs primarily in that there are convective terms in both regions, and no transient terms in either. However, they have the same basic mathematical forms as Eqs. 23.6-15 and 16 respectively. They are to be solved with the boundary\t conditions

$$\text{At x} = 0,\ T_S = T_{S0};\ \ \text{At z} = 0,\ T_M = T_{M0}$$

We may thus write the description in the form

$$\frac{\partial\Theta_S}{\partial\xi} = (\Theta_M - \Theta_S);\ \ \frac{\partial\Theta_M}{\partial\varsigma} = -(\Theta_M - \Theta_S)$$

with $\Theta_S(\zeta,0) = 0 \qquad \Theta_M(0,\xi) = 1$

where

$$\Theta = (T - T_{S0})/(T_{M0} - T_{S0})$$

$$\xi = x(Ua)(T_{M0} - T_{S0}) / v_{x0}\rho_S \hat{C}_{pS}$$

$$\varsigma = z(Ua)(T_{M0} - T_{S0}) / v_{z0}\rho_M \hat{C}_{pM}$$

The solution to the $T_M$ profile is then given directly by Eq. 23.6-25 with $\tau$ replaced by $\xi$.

However, the outlet temperature varies with position, and the average outlet temperature is given by

$$\bar{\Theta}_M(\Xi) = \frac{1}{\Xi}\int_0^{\Xi}\left[1 - \int_0^{Z}\exp\left(-(\xi + Z)J_0\left(i\sqrt{4\zeta\xi}\right)d\zeta\right)\right]d\xi$$

where $\Xi$ is the width of the exchanger (in the $\xi$ direction).

23D.1 Unsteady-state operation of a packed column

The suggested solution is in effect just reverse engineering, and it seems more useful to show how the result can be obtained. Here we follow A. Anzelius (of Stockholm, Z. angew. Math u. Mech. 1926, **6**, #4, 291-293) who provides a clear and elegant proof, as well as the first of many. In our notation he starts with Eqs. 23.6-23 and 24:

$$\frac{\partial X}{\partial \varsigma} = Y - X; \quad \frac{\partial Y}{\partial \tau} = X - Y$$

The key to Anzelius' solution is to define a function

$$G(\zeta, \tau) = e^{\zeta + \tau}(X - Y)$$

which satisfies the relation

$$\frac{\partial^2 G}{\partial \tau \partial \zeta} = G$$

as can be shown by differentiation with the aid of Eqs. 23.6-23 and 24. To show this note that

$$\frac{\partial^2 Y}{\partial \zeta \partial \tau} = \frac{\partial}{\partial \zeta}(X - Y)$$

and similarly for the mixed second derivative of X.

We now look again at Eq. 23.6-23 and consider initial conditions at the column entrance:

$$\left.\frac{\partial X}{\partial \zeta}\right|_{\zeta,0} = Y(\zeta, 0) - X(\zeta, 0) = -X(\zeta, 0)$$

or $$X(\zeta, 0) = e^{-\zeta}$$

Similarly $$Y(0, \tau) = 1 - e^{-\tau}$$

It follows that the boundary conditions on G are

$$G(\zeta, 0) = G(0, \tau) = 1$$

Anzelius obtained G by an iterative process in series form

$$G = G_0 + G_1 + G_2 + \cdots$$

for which $$\frac{\partial^2 G_n}{\partial\tau\partial\zeta} = G_{n-1} \text{ and } G_0 = 1$$

and $$G_n = \int_0^\zeta \cdots \bullet \int_0^\zeta d\zeta^n \bullet \int_0^\tau \cdots \bullet \int_0^\tau d\tau^n = \frac{(\zeta\tau)^n}{(n!)^2}$$

This series converges to $$G = \sum_{n=0}^{\infty} \frac{(\zeta\tau)^n}{(n!)^2} = J_0\left(2i\sqrt{\zeta\tau}\right)$$

and it follows that $$X - Y = e^{-(\tau+\zeta)} J_0\left(2i\sqrt{\zeta\tau}\right)$$

The desired concentration profiles can now be obtained directly from Eqs. 23.6-23 and 24:

$$X = 1 - \int_0^\zeta e^{-(\tau+s)} J_0\left(2i\sqrt{\tau s}\right) ds$$

$$Y = \int_0^\tau e^{-(\zeta+u)} J_o\left(2i\sqrt{\zeta u}\right) du$$

Note that this elegant and concise solution requires no special skill beyond recognizing the infinite series as a well characterized function.

23D.2 Additivity of the lower moments

[For background see Bracewell, Ron, "The Fourier Transform and its Applications", McGraw-Hill, 3rd Ed, 1999]

(i) It was already shown in Prob. 23C.3 that the zeroth moment

$$M_0 = m / Q$$

where m is the mass of tracer fed, and Q is the solvent flow rate. This quantity is thus the same for all input and output streams in a series of units, which need not be identical.

(ii) We now begin with the first absolute moment for one system

$$M_1 = \frac{h'}{h}$$

For two systems in series

$$M_1 = \frac{F'(0)}{F(0)} = \frac{h_1 h_2' + h_1' h_2}{h_1 h_2} = \frac{h_2'}{h_2} + \frac{h_1'}{h_1} = (M_1)_1 + (M_1)_2$$

as required. For the second central moment we may write for one system

$$\mu_2 = \frac{h''}{h} - \left(\frac{h'}{h}\right)^2$$

while for two systems in series $\mu_2 = \left[\frac{h''}{h} - \left(\frac{h'}{h}\right)^2\right]_1 + \left[\frac{h''}{h} - \left(\frac{h'}{h}\right)^2\right]_2$

as required. The second central moment with respect to time is the variance of the distribution and thus the measure of dispersion about the mean. The third central moment is given by

$$\mu_3 = -\frac{h'''}{h} + 3\frac{h'h''}{h^2} - 2\left(\frac{h'}{h}\right)^3$$

Again we find

$$\mu_3 = (\mu_3)_1 + (\mu_3)_2$$

This third central moment is the primary measure of skewness.

Additivity does not, however, occur four the fourth and higher central moments. As a practical matter this is not very important because these moments are usually not obtainable from experimental data with useful accuracy.

It should be noted that additivity is not restricted to two systems in series but holds for any number.

**23D.3 Startup of a chemical reactor**

Taking the Laplace transform of Eqs. 23.6-2 and 3, we get

$$p\overline{M}_{A,\text{tot}} - M_{A,\text{tot}}(0) = \frac{wc_{A0}/\rho}{p} - k_{1B}'''\overline{M}_{A,\text{tot}} + k_{1A}'''\overline{M}_{B,\text{tot}}$$

$$p\overline{M}_{B,\text{tot}} - M_{B,\text{tot}}(0) = -\left(k_{1A}''' + k_{1C}'''\right)\overline{M}_{B,\text{tot}} + k_{1B}'''\overline{M}_{A,\text{tot}}$$

Both $M_{A,\text{tot}}$ and $M_{B,\text{tot}}$ are zero at $t = 0$. The above pair of algebraic equations can be rearranged thus:

$$\left(p + k_{1B}'''\right)\overline{M}_{A,\text{tot}} - k_{1A}'''\overline{M}_{B,\text{tot}} = \frac{wc_{A0}/\rho}{p}$$

$$-k_{1B}'''\overline{M}_{A,\text{tot}} + \left(p + k_{1A}''' + k_{1C}'''\right)\overline{M}_{B,\text{tot}} = 0$$

These equations can be solved for $\overline{M}_{A,\text{tot}}$ and $\overline{M}_{B,\text{tot}}$. We are interested only in $\overline{M}_{B,\text{tot}}$. which can be written down immediately:

$$\overline{M}_{B,\text{tot}} = \frac{\begin{vmatrix} p & \dfrac{wc_{A0}/\rho}{p} \\ -k_{1B}''' & 0 \end{vmatrix}}{\begin{vmatrix} p + k_{1B}''' & -k_{1A}''' \\ -k_{1B}''' & p + k_{1A}''' + k_{1C}''' \end{vmatrix}} = \frac{k_{1B}'''\dfrac{wc_{A0}/\rho}{p}}{\left(p + k_{1B}'''\right)\left(p + k_{1A}''' + k_{1C}'''\right) - k_{1A}'''k_{1B}'''}$$

$$= \left(k_{1B}'''\frac{wc_{A0}}{\rho}\right)\left(\frac{1}{p(p - p_+)(p - p_-)}\right)$$

The second expression gives the form of the result, and $p_+$ and $p_-$ may be found by equating the first and second expressions:

$$2p_\pm = -\left(k_{1A}''' + k_{1B}''' + k_{1C}'''\right) \pm \sqrt{\left(k_{1A}''' + k_{1B}''' + k_{1C}'''\right)^2 - 4k_{1B}'''k_{1C}'''}$$

which is Eq. 23.6-9.

Next we recognize that the above expression for $\overline{M}_{B,\text{tot}}$ is of the form of a "numerator function" $N(p)$ divided by a "denominator

function" $D(p)$, which can be expanded in partial fractions (the "Heaviside partial fractions expansion theorem), thus:

$$\frac{N(p)}{D(p)} = \sum_k \frac{N(a_k)}{D'(a_k)} \frac{1}{p - a_k}$$

Then the inverse transform is taken:

$$\mathfrak{L}^{-1}\left\{\frac{N(p)}{D(p)}\right\} = \sum_k \frac{N(a_k)}{D'(a_k)} \exp(a_k t)$$

Therefore

$$M_{B,\text{tot}} = \left(k_{1B}''' \frac{wc_{A0}}{\rho}\right)\left(\frac{1}{(0-p_+)(0-p_-)} + \frac{1}{p_+(p_+ - p_-)} \exp p_+ t + \frac{1}{p_-(p_- - p_+)} \exp p_- t\right)$$

Then, since $p_+ p_- = k_{1B}''' k_{1C}'''$, the above expression may be rewritten thus:

$$M_{B,\text{tot}} = \left(\frac{wc_{A0}}{\rho k_{1C}'''}\right)\left(1 + \frac{p_- \exp p_+ t - p_+ \exp p_- t}{(p_+ - p_-)}\right)$$

which is Eq. 23.6-8.

23D.4 Transient behavior of N reactors in series

Begin by redefining subscripts to facilitate generalization and to follow Acrivos and Amundson:

$$A \Rightarrow 1;\ B \Rightarrow 2;\ C \Rightarrow 3$$

$$k'''_{1AB} \equiv k_1;\ k'''_{1BA} \equiv k'_1;\ k'''_{1BC} \equiv k_2;\ k'''_{1CB} \equiv k'_2$$

We may then write a continuity equation for species 1 (A) in tank 1 as:

$$\frac{dc_1(1)}{dt} = -(k_1 + h)c_1(1) + k'_1 c_2(1) + hc_1(0)$$

where $c_1(0)$ is the concentration at which solute 1 (A) enters the first tank. One can write similar equations for the two remaining solutes, but it is simpler to write a single matrix equation

$$dc(1)/dt = -\bar{B}c(1) + hc(0)$$

where

$$c(1) = \begin{pmatrix} c_1(1) \\ c_2(1) \\ c_3(1) \end{pmatrix} \qquad c(0) = \begin{pmatrix} c_1(0) \\ c_2(0) \\ c_3(0) \end{pmatrix}$$

$$\bar{B} = \begin{pmatrix} (k_1 + h) & -k'_1 & 0 \\ -k_1 & (k'_1 + k_2 + h) & -k'_2 \\ 0 & -k_2 & (k'_2 + h) \end{pmatrix}$$

The solution to this set of equations for tank 1 is

$$c(1) = h\left(I - e^{-\bar{B}t}\right)\bar{B}^{-1}c(0)$$

Note that this is slightly more general than the solution to Prob. 23D.4 in that it permits all three solutes to enter the first tank. However, it is also more convenient in providing similarities with the behavior of subsequent tanks. Thus for the second tank

$$dc(2)/dt = hc(1) - \bar{B}c(2)$$

and

$$c(2) = h^2\left(I - e^{-\overline{B}t} - \overline{B}te^{-\overline{B}t}\right)\overline{B}^{-2}c(0)$$

and for any tank n

$$c(n) = h^n\left(I - e^{-\overline{B}t}\sum_{i=0}^{n-1}\frac{(\overline{B}t)^i}{i!}\right)\overline{B}^{-n}c(0)$$

This expression may be written out explicitly using Sylvester's theorem as

$$c(n) = h^n\sum_{k=1}^{3}\left(\frac{1}{\lambda_k}\right)^n\left\{\left(1 - e^{-(\lambda_k t)}\right)\sum_{i=1}^{n-1}\frac{(\lambda_k t)^i}{i!}\right\}\frac{adj}{\pi\ i \neq k}\frac{(\lambda_k I - \overline{B})}{(\lambda_k - \lambda_i)}c(0)$$

where the $\lambda_k, (1 \leq k \leq 3)$, are the characteristic roots of $\overline{B}$.

Acrivos and Amundson provide the following numerical example to illustrate use of this approach:

$$h = 1; k_1 = 0.2; k_1' = 0.05; k_2 = 0.10; k_2' = 0.05$$

$$c_1(0) = 1; c_2(0) = 0; c_3(0) = 0$$

Then

$$\overline{B} = \begin{pmatrix} 1.2 & -0.05 & 0 \\ -0.2 & 1.15 & -0.05 \\ 0 & -0.1 & 1.05 \end{pmatrix}$$

and the characteristic equation for $\overline{B}$ is

$$|\lambda I - B| = \lambda^3 - 3.4000\lambda^2 + 3.8325\lambda - 1.4325 = 0$$

The roots of this equation are then

$$\lambda_1 = 1.2866; \lambda_2 = 1.1133; \lambda_3 = 1.0000$$

Therefore

$$c(n) = (0.7772)^n \left[ 1 - e^{-1.2866t} \sum_{i=0}^{n-1} \frac{(1.2866)^i}{i!} \right] \begin{pmatrix} 0.5501 \\ -0.9527 \\ 0.4027 \end{pmatrix}$$

$$+(0.8982)^n \left[ 1 - e^{-1.1133t} \sum_{i=0}^{n-1} \frac{(1.1133t)^i}{i!} \right] \begin{pmatrix} 0.3730 \\ 0.6448 \\ -1.0186 \end{pmatrix}$$

$$+\left[ 1 + e^{-t} \sum_{i=0}^{n-1} \frac{t^1}{i!} \right] \begin{pmatrix} 0.0769 \\ 0.3080 \\ 0.6159 \end{pmatrix}$$

**24A.1 Thermal diffusion**

*a*. From Table 24.2-3 it can be seen that, for positive $k_T$ and the conditions shown, hydrogen must be designated as species $A$. Then Eq. 24.2-13 gives:

$$x_{A2} - x_{A1} = -k_T \ln \frac{T_2}{T_1} = 0.0166 \ln \frac{600}{200} = 0.0183$$

Thus there is a higher concentration of $A$ (hydrogen) in the bulb at temperature $T_1$, the mole fraction difference being 0.0183.

*b*. The recommended mean temperature is obtained from Eq. 24.2-14:

$$T_m = \frac{T_1 T_2}{T_2 - T_1} \ln \frac{T_2}{T_1} = \frac{(200)(600)}{600 - 200} \ln \frac{600}{200} = 330\text{K}$$

**24A.2 Ultracentrifugation of proteins**

Recognizing that $x_B$ is very nearly equal to $x_{B0}$, and taking the $1/\overline{V}_B$ power of both sides of Eq. 24.2-22, we get

$$\frac{x_A}{x_{A0}} = \exp\left[\left(\frac{\overline{V}_A}{\overline{V}_B}M_B - M_A\right)\frac{g_\Omega z}{RT}\right]$$

The values of the quantities in this equation are for this problem

$$M_A = 45{,}000 \ \text{g/g-mole}$$
$$\overline{V}_A = \frac{M_A}{M_A/\overline{V}_A} = \frac{45{,}000}{1.34} = 33{,}582 \ \text{cm}^3/\text{g-mole}$$
$$\frac{M_B}{\overline{V}_B} = 1.00 \ \text{g/cm}^3$$
$$g_\Omega = (50{,}000)(980.665) = 4.903\times 10^7 \ \text{cm/s}$$
$$R = 8.314\times 10^7 \ \text{g}\cdot\text{cm}^2/\text{s}^2\text{g-mole}\cdot\text{K}$$
$$T = 75°\text{F} = 297\text{K}$$

Substitution into the above equation gives

$$\frac{x_A}{x_{A0}} = \exp\left[((33{,}582)(1.00) - 45{,}000)\frac{(4.903\times 10^7)z}{(8.314\times 10^7)(297)}\right] = e^{-22.7z}$$

in which $z$ has units of centimeters.

A somewhat more accurate results may be obtained by accounting for the variation of $x_B$. Then taking the $1/\overline{V}_B$ power of Eq. 24.2-22 (with no approximations) gives:

$$\frac{x_A}{x_{A0}} = \left(\frac{1-x_A}{1-x_{A0}}\right)^{\overline{V}_A/\overline{V}_B} \exp\left[\left(\frac{\overline{V}_A}{\overline{V}_B}M_B - M_A\right)\frac{g_\Omega z}{RT}\right]$$

This differs from the foregoing solution only by the first factor on the right side. For positive $z$ the two solutions agree within 0.93% for the conditions given.

24A.2 Ultracentrifugation of Proteins (Alternative Solution)

It is preferable here to begin with Eq. 24.4-18, and, in the absence of further information, to assume concentration independent activity coefficient. This is probably a reasonable assumption.

Now begin by writing the equation of motion in the form

$$\partial p / \partial r = \rho g_{eff}; g_{eff} = (1g / cm^3) \cdot 50,000 \cdot 980cm / s^2$$

$$\partial p / \partial r = 4.9 \cdot 10^7 g / cm^2 s^2$$

Next note that 75 F = 23.8C = 297 K.

Then, neglecting the effect of pressure on the density of water we find from App. F, p. 867, that

$$cRT =$$

$$(55.5 \cdot 10^{-3} g - moles / cm^3) \cdot 297K \cdot (8.31451 \cdot 10^7 \frac{g - cm^2}{s^2 - gmole - K})$$

or $cRT = 1.371 \cdot 10^9 g / s^2 - cm$.

Finally we note that

$$M_P / M_W = 2,500; \hat{V}_P / \hat{V}_W = 1 / 1.34 = 0.746$$

Putting all of these results into Eq. 24.4-18 gives

$$\partial \ln x_P / \partial r = 22.68 / cm$$

Integrating this expression with the given BC at a reference value, z, of r yields

$$x_P = (5.6 \cdot 10^{-6}) e^{22.68 z / cm}$$

which completes the solution.

Prob. 24A.3 Ionic diffusivities.

| Ion: | $Na^+$ | $K^+$ | $Cl^-$ | |
|---|---|---|---|---|
| Diffusivity: | 1.33 | 1.95 | 2.00 | $\times 10^{-5}$ $cm^2/s$ |

Prob. 24B.1 Dimensions of the Lorentz force

In the international system of units,

$c_i$ [=]g-moles/$m^3$, $z_i$ [=] g-equivalents/g-mole, F = 96,500 coulombs/g-equivalent

$v_i$ [=] m/s

It follows that

$c_i\, z_i\, F\, [v_i \times B]$ [=] Newtons/$m^3$.

This is in agreement with the definition of the $d_i$.

24B.2 Junction potentials.

Integrating across the stagnant region gives

$$\phi_2 - \phi_1 = \left(\frac{RT}{F}\right)\left[\frac{\mathcal{D}_{mw} - \mathcal{D}_{xw}}{\mathcal{D}_{mw} + \mathcal{D}_{xw}}\right]\ln\left(\frac{c_1}{c_2}\right)$$

For 25 C, (RT/F) = 25.692 mV, ln ($c_1/c_2$) = 2.303. It follows that the junction potential ($\phi_2$-$\phi_1$) is –11.8 mV for the sodium chloride and 0.75 for the potassium chloride. The sign of this potential arises from the tendency of the chloride ion to move faster then either cation. The very small junction potential for KCl leads to the use of saturated KCl solutions (salt bridges) between electrodes and the point in a physiological solution where potential is to be measured. The concentrated KCl tends to overwhelm the effects of solution electrolytes, which are typically on about 0.15 N.

Prob. 24B.3 Donnan Exclusion

The external salt mole fraction is

$$x_{NaCl} = x_{Na^+} = x_{Cl^-} = 0.00180$$

The internal molar concentrations are

$$c_{NaR} = 1.03 \ \ eqts/L$$

$$c_{NaCl} = 0.001$$

$$c_W = 13.2$$

Then total molar concentration is

$$c_{tot} = 14.23$$

$$x_{Na^+} = 1.03/14.23 = 0.072$$

$$x_{Cl^-} = 0.01/14.23 = 0.0007027$$

It follows that

$$K_D = (0.0018^2)/[(0.000703)(0.0724)]$$

$$\doteq 0.064$$

24B.4 Osmotic Pressure

Begin with Eq. 24.5-4, and limit consideration to equilibrium between two electrically uncharged solutions. Under these conditions the equation reduces to

$$\nabla \ln(a_W)_{T,P} = -(\bar{V}_W / RT)\nabla p$$

where the subscript "T,p" is a reminder that the activities are to evaluated at the same reference temperature and pressure. Next note that for all practical purposes water may be considered incompressible and the sea water quite dilute. Then

$$\bar{V} \doteq 1/c$$

and we may write

$$(a_{W1})_{T,p} = (a_{W2})_{T,p} = p^0_{W1} / p^2_{W2} = e^{(p_2 - p_1)/cRT}$$

where $p^0_{Wi}$ is the vapor pressure of water over phase "i", and $p_i$ is the hydrostatic pressure on phase "i". This equation then relates the pressures on pure water and sea water required to make them kin equilibrium with each other. This could be accomplished for example by pressing down on the sea water with a hydrophobic screen with holes large enough to permit free passage of vapor but small enough that surface tension forces prevent movement of liquid. This would be one type of ideal membrane.

The osmotic pressure, $\pi$ , is $(p_2-p_1)$ if "2" refers to the sea water and "1" to pure water. Then

$$\pi = cRT \ln(1/0.9816)$$

No temperature was specified, and 300 K will be assumed here for illustrative purposes. Then, from App. F, p.867, we may write

$$\pi = (55.5 \cdot 10^{-3})(300)(82.0578)atm\big(\ln(1/0.9816)\big)$$
$$= 1366 \cdot 0.0185 \doteq 25atm$$

This will of course vary with salinity as well as temperature.

Prob. 24B.5 Permeability of a perfectly selective filtration membrane.

$$x_W = c\bar{V}_W = 1$$

$$x_S = N_S = 0$$

Any ultrafiltrate will now be pure water, and the single operative diffusion equation reduces to

$$\frac{d}{dz}\ln a_W + \frac{p\bar{V}_W}{RT} = -N_W / \mathcal{D}'_{Wm}$$

with

$$a_W = 1$$

at

$$z = \delta$$

It follows that

$$K_H = (\mathcal{D}'_{wm})_{avg} / RT\delta$$

where δ is membrane thickness, and

$$(\mathcal{D}_{im}) = \left[\frac{1}{\delta}\int_0^{\delta}\frac{dz}{\mathcal{D}_{im}}\right]^{-1}$$

In practice $K_H$ is just an empirical quantity.

24B.6 Model insensitivity

The ratio of fluxes predicted by Eqs. 1 and 2 is

$$\frac{N_A(pore)}{N_A(Diff)} = \frac{2\pi}{3}\frac{(1-\varepsilon)}{(1+\pi/8)} \doteq 1.5(1-\varepsilon)$$

At a porosity of 0.4 this ratio is about 0.9. Thus, although the two expressions predict different void fraction dependence they give nearly the same predictions for this common void fraction as well as the same dependence on particle diameter, temperature and molecular weight. Moreover, both fail to take into account the distributions of diameter and uniformity. From a practical standpoint they are closely equivalent.

**24C.1 Expressions for the mass flux**

*a*. First rewrite the left side of Eq. 24.2-8 thus

$$\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{N} \frac{x_\alpha x_\beta}{Đ_{\alpha\beta}}\left(\mathbf{v}_\alpha - \mathbf{v}_\beta\right) = \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{N} \frac{x_\alpha x_\beta}{Đ_{\alpha\beta}}\left(\mathbf{v}_\gamma - \mathbf{v}_\beta\right) + x_\alpha\left(\mathbf{v}_\alpha - \mathbf{v}_\gamma\right)\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{N} \frac{x_\beta}{Đ_{\alpha\beta}}$$

$$= \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{N} \frac{x_\alpha x_\beta}{Đ_{\alpha\beta}}\left(\mathbf{v}_\gamma - \mathbf{v}_\beta\right) + x_\alpha\left(\mathbf{v}_\alpha - \mathbf{v}_\gamma\right)\sum_{\substack{\beta=1\\ (\text{all }\beta)}}^{N} \frac{x_\beta}{Đ_{\alpha\beta}} + x_\alpha\left(\mathbf{v}_\gamma - \mathbf{v}_\alpha\right)\frac{x_\alpha}{Đ_{\alpha\alpha}}$$

in which $\mathbf{v}_\gamma$ is the velocity of an arbitrarily chosen species. In the last step, we have introduced $Đ_{\alpha\alpha}$, which has not been defined. We now define it as follows:

$$\frac{x_\alpha}{Đ_{\alpha\alpha}} = -\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{N} \frac{x_\beta}{Đ_{\alpha\beta}} \qquad \text{or} \qquad \sum_{\substack{\beta=1\\ (\text{all }\beta)}}^{N} \frac{x_\beta}{Đ_{\alpha\beta}} = 0 \quad (*)$$

Then, in the last expression (for the left side of Eq. 124.2-8) the second term drops out, and the third term can be combined with the first, so that we obtain

$$\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{N} \frac{x_\alpha x_\beta}{Đ_{\alpha\beta}}\left(\mathbf{v}_\alpha - \mathbf{v}_\beta\right) = \sum_{\beta=1}^{N} \frac{x_\alpha x_\beta}{Đ_{\alpha\beta}}\left(\mathbf{v}_\gamma - \mathbf{v}_\beta\right)$$

Thus the first term of Eq. 24.2-9 is now obtained, along with the auxiliary relation (designated by (*)), which now has to be included because of the introduction of the $Đ_{\alpha\alpha}$.

*b*. In (*a*) we have not included the thermal diffusion terms. This is easy to do if, in the above development, we replace $\mathbf{v}_\beta$ by $\mathbf{v}_\beta + \left(D_\beta^T/\rho_\beta\right)\nabla \ln T$. Then the procedure in (*a*) can be followed exactly to get Eq. 24.2-9, including the thermal diffusion terms.

24C.2 Differential centrifugation.

Yes. From Eq. 24.4-22, migration velocities are proportional to

$$v_{migr} \propto R_P^2 \Delta\rho$$

Begin by noting that protein mass is equal to

$$m_P = (mass / mole) \cdot (moles / molecule)$$

$$=(40{,}000\text{g/mole})/\tilde{\text{N}}=[40{,}000/(6.022 \cdot 10^{23})]g$$

$$= 5.64 \cdot 10^{-20} g$$

Next note that, if we assume the particles to be spherical

$$m_P = (4/3)\pi\rho_P R_P^3$$

$$R_P^2 \propto (m_P / \rho_P)^{2/3}$$

Then taking the migration velocity of the cells as unity, we find the relative migration velocities to be 1, 0.0122 and 0.0000102 respectively, and diffusion is negligible for both cells and inclusion bodies. Shape will be a secondary consideration under these circumstances.

24C.3 Transport characteristics of sodium chloride

First put all data on the same concentration scale by writing in the molar equivalents of molality using the formula provided. Next put the diffusivities and conductances in comparable units of "10^5*cm$^2$/s by noting that

Faraday's constant (F or more commonly $\mathcal{F}$ ) equals 96,485 coulombs/g-eqt.

R = 8.3145 joules/g-mol,K

1 joule = 1 volt-coulomb = 1volt-ampere-s

1 volt/ampere = 1 ohm

It follows that $\mathcal{F}^2 / RT = 3.755 \cdot 10^6 \, ohm - eqt$

It also follows that conductance is the sum of the two ionic conductances while diffusivity is the average defined by Eq. 24.4-32. Therefore conductances should be divided by 71.1 relative to diffusivities which are given by the table of the problem statement in units of 10^5*cm$^2$/s.

In the accompanying spread sheet and graph three quantities are given in terms of molality: diffusivity/activity, diffusivity and conductance. It may be seen that all three are equal to each other at the limit of infinite dilution to the accuracy of the figure.

However, conductance decreases rapidly with increasing concentration, because of an increasing effect of diffusional ion-ion interaction.

There is no such interaction for diffusivity, and the observed diffusivity of salt can be predicted surprisingly well from ionic conductances at infinite dilution. Salt diffusivity is equal to $1.54 \pm 0.07$ over the whole range of salt concentrations from zero to 3 N. Clearly ionic diffusivities are not changing much with concentration.

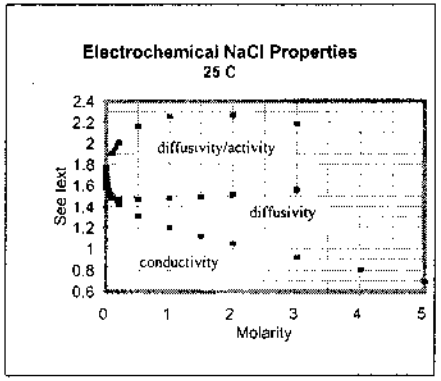
At the same time it can be seen that the activity correction is significant and masks some of the variation in transport properties.

Electrochemical Characteristics of Aqueous NaCl at 25 C

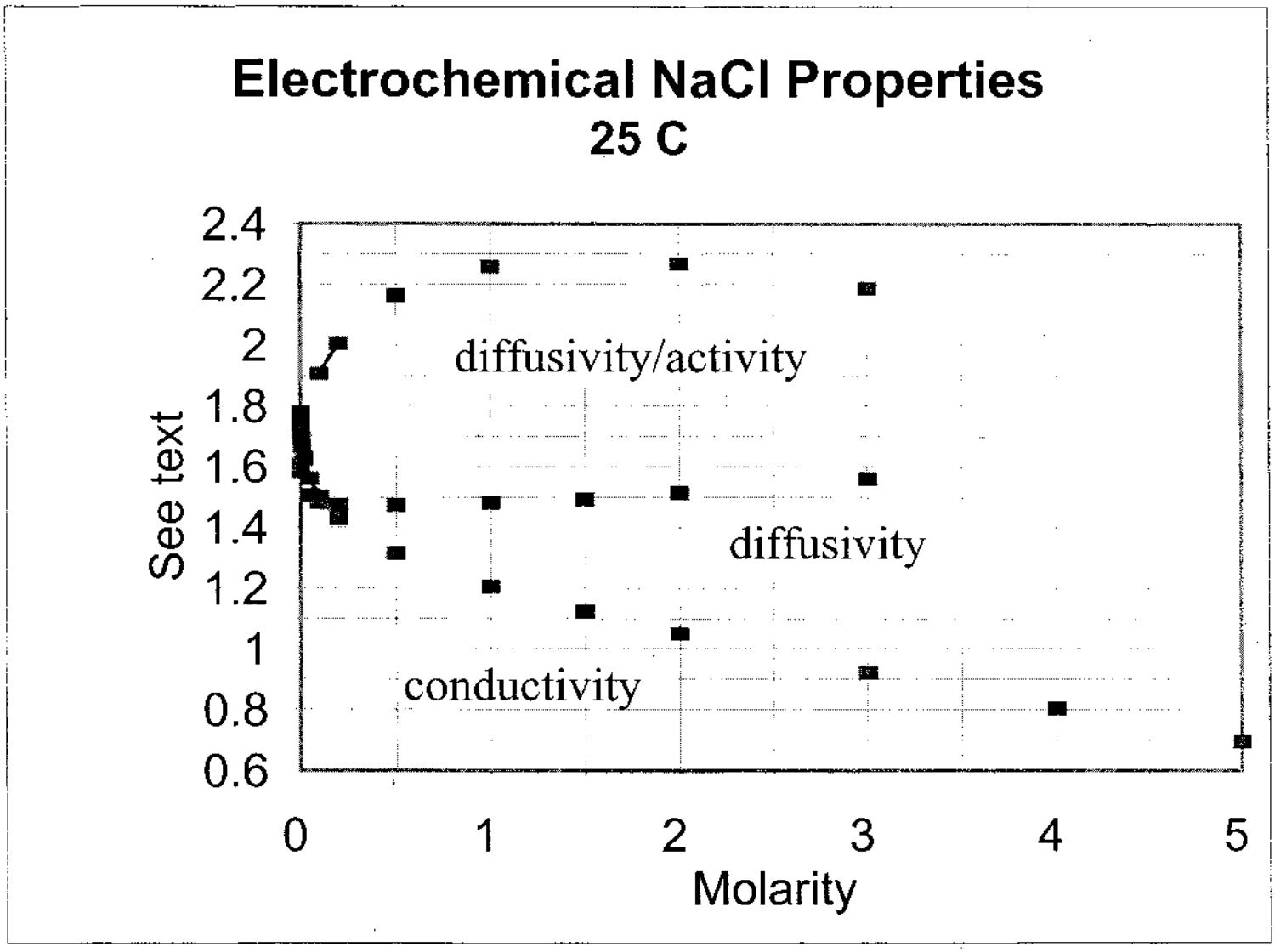
Diffusivity is in (cm^2/s)*10^5

Lamda = equivalent conductance in cm^2/ohm-equivalent

| c(molar) | Lambda | Diffusivity | c(molal) | Act. coeff. | dif/act | B/71.1 |
|---|---|---|---|---|---|---|
| 0 | 126.45 | 1.61 | 0 | 1 | 1.61 | 1.778481 |
| 0.00055 | 124.51 | | 0.00055 | | | 1.751195 |
| 0.001 | 123.74 | 1.585 | 0.001 | | | 1.740366 |
| 0.005 | 120.64 | | 0.005 | | | 1.696765 |
| 0.01 | 118.53 | | 0.01 | | | 1.667089 |
| 0.02 | 115.76 | | 0.02 | | | 1.628129 |
| 0.05 | 111.06 | 1.507 | 0.05 | | | 1.562025 |
| 0.1 | 106.74 | 1.483 | 0.1 | 0.778 | 1.90617 | 1.501266 |
| 0.2 | 101.71 | 1.475 | 0.2 | 0.735 | 2.006803 | 1.43052 |
| 0.298 | | | 0.3 | 0.71 | | |
| 0.39696 | | | 0.4 | 0.693 | | |
| 0.5 | 93.62 | 1.474 | 0.5 | 0.681 | 2.164464 | 1.316737 |
| 0.59316 | | | 0.6 | 0.673 | | |
| 0.69069 | | | 0.7 | 0.667 | | |
| 0.78784 | | | 0.8 | 0.659 | | |
| 0.88461 | | | 0.9 | 0.657 | | |
| 1 | 85.76 | 1.484 | | 0.657 | 2.258752 | 1.206188 |
| 1.17264 | | | 1.2 | 0.654 | | |
| 1.36276 | | | 1.4 | 0.655 | | |
| 1.5 | 79.86 | 1.495 | | | | 1.123207 |
| 1.55136 | | | 1.6 | 0.657 | | |
| 1.73844 | | | 1.8 | 0.662 | | |
| 2 | 74.71 | 1.516 | | 0.668 | 2.269461 | 1.050774 |
| 2.38125 | | | 2.5 | 0.688 | | |
| 3 | 65.57 | 1.563 | | 0.714 | 2.189076 | 0.922222 |
| 3.26725 | | | 3.5 | 0.746 | | |
| 4 | 57.23 | | 4 | 0.783 | | 0.804923 |
| 4.11525 | | | 4.5 | 0.826 | | |
| 5 | 49.46 | | 5 | 0.874 | | 0.69564 |




24-14


Electrochemical NaCl Properties
25 C
See text
2.4
2.2
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0
1
2
3
4
5
Molarity
diffusivity/activity
diffusivity
conductivity

Prob. 24C.4 Departures from electroneutrality

Begin by writing

$$d\phi / dz = \frac{d\phi}{dc_s}\frac{dc_s}{dz} = -\left(\frac{RT}{F}\right)\left[\frac{\mathcal{D}_{mw} - \mathcal{D}_{xw}}{\mathcal{D}_{mw} + \mathcal{D}_{xw}}\right]\left(\frac{1}{c_s}\right)\frac{dc_s}{dz}$$

$$d^2\phi / dz^2 = \left(\frac{RT}{F}\right)\left[\frac{\mathcal{D}_{mw} - \mathcal{D}_{xw}}{\mathcal{D}_{mw} + \mathcal{D}_{xw}}\right]\left(\frac{1}{c_s^2}\right)\left(\frac{dc_s}{dz}\right)^2$$

Now for NaCl,

$$\left(\frac{RT}{F}\right)\left[\frac{\mathcal{D}_{mw} - \mathcal{D}_{xw}}{\mathcal{D}_{mw} + \mathcal{D}_{xw}}\right] = 25.692mV \bullet 0.2875 = 7.3878mV$$

while

$$(dc_s / dz)^2 = 0.81(eqts / liter)^2 / L^2$$
$$c_s^2 = [1 - 0.9(z / L)]^2 (eqts / liter)^2$$

It follows that for our system

$$\frac{d^2\phi}{dz^2} = 7.3878mV\left(\frac{0.81}{L^2}\right)[1 - (0.9z / L)]^2$$

or, at the center of the stagnant region

$$\frac{d^2\phi}{dz^2} = 1.788mV / L^2$$

$$c_+ - c_- = (1.788 \bullet 10^{-3} V)(1/L^2)/(1.392 \bullet 10^{16} V - cm / eqt)$$
$$= 1.128 \bullet 10^{-18} \text{eqts/L}^2 \text{ cm}$$

This tends to be a very small number, but for thin membranes, it can become appreciable.

24C.5 Dielectrophoretic Driving Forces

Begin by writing

$$\nabla^2\phi = \frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = 0$$

with the boundary conditions that

At $r = R_1$, $\phi = \phi_1$ , and at $r = R_2$ , $\phi = \phi_2$,

It then follows directly that

$$\frac{\phi - \phi_1}{\phi_2 - \phi_1} = \frac{\ln(r/R_1)}{\ln(R_2/R_1)}$$

and that

$$E = \nabla\phi = \underline{\delta}_r \frac{d\phi}{dr} = \underline{\delta}_r \frac{(\phi_2 - \phi_1)}{\ln(R_2/R_1)}\frac{1}{r}$$

Next note from Eq. S, Table A.7-2

$$\nabla\underline{E} = \delta_r\delta_r \frac{d}{dr}\frac{d\phi}{dr} = \underline{\delta}_r\underline{\delta}_r\left(\frac{\phi_2 - \phi_1}{\ln(R_2/R_1)}\right)\left(\frac{-1}{r^2}\right)$$

and, finally

$$[\underline{E} \bullet \nabla\underline{E}] = -\delta_r\left(\frac{\phi_2 - \phi_1}{\ln(R_2/R_1)}\right)^2\left(\frac{1}{r^3}\right)$$

We see then that the dielectrophoretic force is in the negative r-direction, whatever the direction of **E**, and that it increases very strongly with decrease in r.

24C.6 Effects of small inclusions in a dielectric medium

The first requirement here is to complete the problem statement by adding: "Develop an expression for $\phi_s$ if $\phi_c \rightarrow \Phi z$ at large r with $\Phi$ a constant."

One must next recognize that for an uncharged medium

$$(\nabla \cdot E) = \nabla^2 \phi = 0$$

This equation and the given boundary conditions make an exact analog of the corresponding heat conduction problem, treated for example in Carslaw and Jaeger, Sect. 16.4 I, p. 426. We thus proceed with trial solutions of the form

$$\phi_c = \Phi r \cos\theta + \frac{B}{r^2} \cos\theta$$

$$\phi_s = A r \cos\theta$$

and the boundary condition

$$\varepsilon_s \left.\frac{d\phi_s}{dr}\right|_{r=R} = \varepsilon_c \left.\frac{d\phi_c}{dr}\right|_{r=R}$$

where R is the radius of the inclusion. This boundary condition plus that at large r permit determination of A and B, and the final result is

$$\phi_c = \Phi \left[ r \cos\theta + \frac{(\varepsilon_c - \varepsilon_s) R^3 \cos\theta}{r^2 (2\varepsilon_c + \varepsilon_s)} \right]$$

$$\phi_s = \frac{3\Phi \varepsilon_c r \cos\theta}{2\varepsilon_c + \varepsilon_s} = \frac{3\Phi \varepsilon_c z}{2\varepsilon_c + \varepsilon_s}$$

This result can be used to obtain the effect dielectric constants of composite media just as was done for thermal conductivity in Section 9.6.

24C.7 Frictionally induced selective filtration

The first step is to review the development of Section 24.5 and to obtain a directly useable set of equations for pressure and activity gradients. Begin by adding Eqs. 24.5-16 and 17 with the aid of Eq. 24.5-19 in the form

$$x_S \nabla_{T,p} \ln a_s + x_W \nabla_{T,p} \ln a_W = 0$$

The result is

$$\frac{N_S}{c\mathcal{D}'_{Sm}} + \frac{N_W}{c\mathcal{D}'_{WM}} = -\frac{(\phi_S + \phi_W)}{cRT} \nabla p = -\frac{\nabla p}{cRT}$$

Next this expression for pressure gradient should be put into Eq. 24.5-16 to obtain

$$\frac{x_W N_S - x_S N_W}{c\mathcal{D}_{SW}} + \frac{N_S}{c\mathcal{D}'_{Sm}} = -x_S \nabla_{T,p} \ln a_S$$
$$+\phi_S \left[ \frac{N_S}{c\mathcal{D}'_{Sm}} + \frac{N_w}{c\mathcal{D}'_{Wm}} \right]$$

This expression can be rearranged to

$$N_S \left[ \frac{x_W}{c\mathcal{D}_{SW}} + \frac{\phi_W}{c\mathcal{D}'_{Sm}} \right] - N_W \left[ \frac{x_S}{c\mathcal{D}_{SW}} + \frac{\phi_S}{c\mathcal{D}'_{Wm}} \right] = -\left( 1 + \frac{\partial \ln \gamma_S}{\partial \ln x_S} \right) \frac{dx_S}{dz}$$

The solution will always be dilute enough that the activity coefficient can be considered constant. Moreover, for this system there is no change in activity or concentration across the solution membrane interface, so our concentration boundary condition is just that

$$z = 0, x_S = 0.01$$

The imposed pressure drop across the membrane, which is the same as that between the upstream and downstream fluid phases, determines the flow rate. Limiting behavior is readily determined.

For very low flow rates, both convective terms are very small. Hence the diffusional term and concentration changes across the membrane, are also very small. Then, as for all but truly ideal membranes the filtrate has the same composition as the feed. This is a general characteristic of all non-ideal membranes.

For very high flow rates, the two convective terms are both very large buyt nealr equal, and Eq. 24.5-26 governs, except for a very thin boundary layer near the downstream end of the membrane. The two may then be equated, and for high flow rates the filtrate composition is expressible as

$$Y_S = \frac{y_S}{y_W} = \frac{N_S}{N_W} = \left(\frac{x_S}{x_W}\right)\left[\frac{1 + c\bar{V}_S \mathcal{D}_{SW} / \mathcal{D}'_{Wm}}{1 + c\bar{V}_W \mathcal{D}_{SW} / \mathcal{D}'_{Sm}}\right]$$

Here $Y_S$ and $y_S$ refer to mole ratio and mole fraction of solute respectively in the filtrate. Now in our specific case, "S" equals "G", and, because the solution is mostly water "c" equals $\bar{V}_W$. Then with

$$\mathcal{D}_{GW} / \mathcal{D}'_{Gm} = 25; \mathcal{D}'_{Wm} = 100\mathcal{D}'_{Gm}$$

$$Y_S / X_S = 2/26 = 1/13 = 0.00076$$

where $X_S$ is feed mole ratio.

The concentration profile and filtrate compositions at intermediate pressure gradients can be obtained by replacing $N_W$ in Eq. 24.5-16 using Eq. 24.5-20.

24C.8 Thermodynamically induced selective filtration.

Consider now internal behavior so that the $x_i'$ are defined as the mole fractions in the matrix-free solution just inside the upstream surface of the membrane. Then for the conditions described in the problem statement at high flow rates, where concentration gradients are confined to a thin boundary layer just inside the downstream surface

$$Y_S = \frac{N_S}{N_W} = X_S' = \frac{x_S'}{x_W'}$$

However, for the dilute solutions normal in these systems

$$X_S' = 0.1X_S$$

and

$$Y_S - 0.1X_A$$

Again for very slow flow there is no enrichment, and for intermediate flows one must carry out the integration suggested for the previous problem.

24C.9 Facilitated Transport

The defining equations are

$$N_C = -N_{CS}$$

$$N_C = \mathcal{D}_{Cm}(c_{C0} - c_{C\delta}) / \delta$$

$$N_{CS} = \mathcal{D}_{CSm}(c_{CS0} - c_{CS\delta}) / \delta$$

These fluxes are to be determined with the aid of the boundary conditions

$$c_{CS0} / c_{C0} = K_D c_{S0}$$

$$c_{CS\delta} / c_{C\delta} = K_D c_{S\delta}$$

and the conservation constraint

$$\frac{1}{\delta}\int_0^{\delta}(c_C + c_{CS})d\delta = \overline{c}$$

Here $\overline{c}$, the sum of free and complexed carrier, is position invariant. For our present purposes it and the two external concentrations will be assumed known.

Normally the carrier is much larger than the solute, and the diffusivities of the carrier and carrier-solute complex are assumed to be equal. This is done both for lack of data and because the model is so oversimplified that its chief use is to gain insight.

We will now make this assumption, in order to concentrate on the qualitative aspects of system behavior:

$$\mathcal{D}_{Cm} = \mathcal{D}_{CSm} = D$$

We may now replace the above summational constraint with

$$c_C + c_{CS} = \overline{c}$$

Flux of the solute-carrier complex can then be determined by noting

**24D.1 Entropy flux and entropy production**

*a*. From Eq. 24.1-2 we get for a moving fluid element

$$\frac{D\hat{U}}{Dt} = T\frac{D\hat{S}}{Dt} - p\frac{D\hat{V}}{Dt} + \sum_{\alpha=1}^{N}\frac{\overline{G}_\alpha}{M_\alpha}\frac{D\omega_\alpha}{Dt}$$

We now solve this for $D\hat{S}/Dt$ and multiply by the density to obtain

$$\rho\frac{D\hat{S}}{Dt} = \frac{\rho}{T}\frac{D\hat{U}}{Dt} + \frac{p\rho}{T}\frac{D\hat{V}}{Dt} - \frac{\rho}{T}\sum_{\alpha=1}^{N}\frac{\overline{G}_\alpha}{M_\alpha}\frac{D\omega_\alpha}{Dt}$$

Next we rewrite each of the terms on the right side by using the equations of change:

$$\frac{\rho}{T}\frac{D\hat{U}}{Dt} = -\frac{1}{T}(\nabla\cdot\mathbf{q}) - \frac{1}{T}(\boldsymbol{\tau}:\nabla\mathbf{v}) - \frac{p}{T}(\nabla\cdot\mathbf{v}) + \sum_{\alpha=1}^{N}(\mathbf{j}_\alpha\cdot\mathbf{g}_\alpha)$$

$$\frac{p\rho}{T}\frac{D\hat{V}}{Dt} = \frac{p\rho}{T}\frac{D}{Dt}\left(\frac{1}{\rho}\right) = -\frac{p}{\rho T}\frac{D\rho}{Dt} = \frac{p}{T}(\nabla\cdot\mathbf{v})$$

$$-\frac{\rho}{T}\sum_{\alpha=1}^{N}\frac{\overline{G}_\alpha}{M_\alpha}\frac{D\omega_\alpha}{Dt} = \frac{1}{T}\sum_{\alpha=1}^{N}\frac{\overline{G}_\alpha}{M_\alpha}\left[(\nabla\cdot\mathbf{j}_\alpha) - r_\alpha\right]$$

Then the entropy equation becomes

$$\rho\frac{D\hat{S}}{Dt} = -\frac{1}{T}(\nabla\cdot\mathbf{q}) - \frac{1}{T}(\boldsymbol{\tau}:\nabla\mathbf{v}) + \sum_{\alpha=1}^{N}(\mathbf{j}_\alpha\cdot\mathbf{g}_\alpha) + \frac{1}{T}\sum_{\alpha=1}^{N}\frac{\overline{G}_\alpha}{M_\alpha}\left[(\nabla\cdot\mathbf{j}_\alpha) - r_\alpha\right]$$

We next want to force this equation into the form of Eq. 24.1-1 by creating a divergence term on the right side and then adding compensating terms (the dashed-underlined terms):

$$\rho\frac{D\hat{S}}{Dt} = -\left(\nabla\cdot\frac{1}{T}\left\{\mathbf{q} - \sum_{\alpha=1}^{N}\frac{\overline{G}_\alpha}{M_\alpha}\mathbf{j}_\alpha\right\}\right) - \left(\frac{1}{T^2}\mathbf{q}\cdot\nabla T\right) - \underline{\sum_{\alpha=1}^{N}\left(\mathbf{j}_\alpha\cdot\nabla\left(\frac{1}{T}\frac{\overline{G}_\alpha}{M_\alpha}\right)\right)}$$

$$+\frac{1}{T}\sum_{\alpha=1}^{N}(\mathbf{j}_\alpha\cdot\mathbf{g}_\alpha)-\frac{1}{T}(\boldsymbol{\tau}:\nabla\mathbf{v})-\frac{1}{T}\sum_{\alpha=1}^{N}\frac{\overline{G}_\alpha}{M_\alpha}r_\alpha$$

This is now of the form of Eq. 24.1-1, so that the entropy flux and entropy generation terms may be identified as in Eqs. 24.1-3 and 4.

*b*. Going from Eq. 24.1-5 to Eq. 24.1-3 is straightforward:

$$\mathbf{s}=\frac{1}{T}\mathbf{q}^{(h)}+\sum_{\alpha=1}^{N}\frac{\overline{S}_\alpha}{M_\alpha}\mathbf{j}_\alpha=\frac{1}{T}\left(\mathbf{q}-\sum_{\alpha=1}^{N}\frac{\overline{H}_\alpha}{M_\alpha}\mathbf{j}_\alpha\right)+\frac{1}{T}\sum_{\alpha=1}^{N}\frac{T\overline{S}_\alpha}{M_\alpha}\mathbf{j}_\alpha$$

Then use of the relation $\overline{G}_\alpha=\overline{H}_\alpha-T\overline{S}_\alpha$ gives Eq. 24.1-3.

Substitution of Eq. 24.1-7 and the first line of Eq. 21.4-8 into Eq. 24.1-6 gives

$$g_S=\underbrace{-\left(\mathbf{q}\cdot\frac{1}{T^2}\nabla T\right)}_{(1)}+\underbrace{\frac{1}{T^2}\left(\sum_{\alpha=1}^{N}\frac{\overline{H}_\alpha}{M_\alpha}\mathbf{j}_\alpha\cdot\nabla T\right)}_{(2)}$$

$$-\frac{1}{T}\sum_{\alpha=1}^{N}\left(\mathbf{j}_\alpha\cdot\left\{\underbrace{T\nabla\frac{\overline{G}_\alpha}{M_\alpha T}}_{(3)}+\underbrace{\frac{\overline{H}_\alpha}{M_\alpha T}\nabla T}_{(4)}\underbrace{-\frac{1}{\rho}\nabla p}_{(5)}\underbrace{-g_\alpha}_{(6)}+\underbrace{\frac{1}{\rho}\sum_{\beta=1}^{N}\rho_\beta\mathbf{g}_\beta}_{(7)}\right\}\right)$$

$$\underbrace{-\frac{1}{T}(\boldsymbol{\tau}:\nabla\mathbf{v})}_{(8)}\underbrace{-\frac{1}{T}\sum_{\alpha=1}^{N}\frac{\overline{G}_\alpha}{M_\alpha}r_\alpha}_{(9)}$$

Term (1) gives the first term in Eq. (24.1-4), and terms (8) and (9) are the same as the third and fourth term. Term (5 ) is the same has the second part of the second term. Terms (6) and (7) are identically zero because the sum of the mass fluxes is zero. This leaves us with terms (2), (3), and (4), which we have to show gives the first term in Eq. 24.1-4. That is, we have to verify that

$$\frac{1}{T^2}\left(\sum_{\alpha=1}^{N}\frac{\overline{H}_\alpha}{M_\alpha}\mathbf{j}_\alpha\cdot\nabla T\right)-\frac{1}{T}\sum_{\alpha=1}^{N}\left(\mathbf{j}_\alpha\cdot T\nabla\frac{\overline{G}_\alpha}{M_\alpha T}\right)-\frac{1}{T}\sum_{\alpha=1}^{N}\left(\mathbf{j}_\alpha\cdot\frac{\overline{H}_\alpha}{M_\alpha T}\nabla T\right)$$

$$= -\sum_{\alpha=1}^{N}\left(\mathbf{j}_\alpha \cdot \nabla\left(\frac{1}{T}\frac{\overline{G}_\alpha}{M_\alpha}\right)\right)$$

This is indeed an identity, and therefore the equivalence of Eqs. 24.1-1 and 24.1-6 has been affirmed.

**Appendix A: p. 810, Ex. 1**

*a.* $(\mathbf{v}\cdot\mathbf{w})$
The order is $1+1-2=0$, the 1s for the vectors and the $-2$ for the single dot. The result is a scalar.

*b.* $(\mathbf{v}-\mathbf{u})\mathbf{w}$
The order is $1+1=2$. The result is a second-order tensor.

*c.* $(\mathbf{ab}:\mathbf{cd})$
The order is $1+1+1+1-4=0$, the 1s for the vectors and the $-4$ for the double dot. The result is a scalar.

*d.* $[\mathbf{v}\cdot\rho\mathbf{wu}]$
The order is $1+0+1+1-2=1$, so that the result is a vector.

*e.* $[[\mathbf{a}\times\mathbf{f}]\times[\mathbf{b}\times\mathbf{g}]]$
The order is $1+1+1+1-1-1-1=1$, so that the result is a vector. There are four 1s for the four vectors and three crosses for the three cross-products.

**Appendix A: p. 810, Ex. 2**

Verify, by example, the inequality

$$(\mathbf{u}\cdot\mathbf{v})\mathbf{w} \neq \mathbf{u}(\mathbf{v}\cdot\mathbf{w})$$

by means of a specific illustration.

Let us take $\mathbf{u}$ to be a unit vector in the $x$ direction, and call this vector $\boldsymbol{\delta}_x$. Then take $\mathbf{v}$ to be $\boldsymbol{\delta}_x$ also. Further take $\mathbf{w}$ to be a unit vector in the $y$ direction, calling it $\boldsymbol{\delta}_y$. Then the equality becomes

$$(\boldsymbol{\delta}_x\cdot\boldsymbol{\delta}_x)\boldsymbol{\delta}_y \neq \boldsymbol{\delta}_x(\boldsymbol{\delta}_x\cdot\boldsymbol{\delta}_y)$$

Since, by the definition of the dot product $(\boldsymbol{\delta}_x\cdot\boldsymbol{\delta}_x)=1$ and $(\boldsymbol{\delta}_x\cdot\boldsymbol{\delta}_y)=0$, the above inequality becomes

$$\boldsymbol{\delta}_y \neq 0$$

This is a true statement, inasmuch as $\boldsymbol{\delta}_y$ is a vector of unit length in the $y$ direction.

If all three vectors are equal

$$\mathbf{u}=\mathbf{v}=\mathbf{w}=\mathbf{A}$$

then the above "inequality" becomes

$$A^2\mathbf{A} = \mathbf{A}A^2$$

which is clearly an equality.

If the three vectors are mutually perpendicular, then one gets the equality 0 = 0.

**Appendix A: p. 810, Ex. 3**

The volume rate of flow of a fluid flowing through a surface is given by the product of the surface area $S$ and the component of the velocity that is perpendicular to the surface. The components of velocity parallel to the surface contribute nothing to the volume rate of flow.

Since the unit normal vector for the surface is $\mathbf{n}$, the component of the fluid velocity $\mathbf{v}$ in the direction perpendicular to the surface is $(\mathbf{v} \cdot \mathbf{n})$ and the volume rate of flow through the surface is $(\mathbf{v} \cdot \mathbf{n})S$.

When the volume rate of flow is multiplied by the fluid density, we get the mass rate of flow $\rho(\mathbf{v} \cdot \mathbf{n})S$.

**Appendix A: p. 810, Ex. 4**

Let us establish a coordinate system in the rotating body at some time $t = t_0$, such that the $z$ axis is along the axis of rotation and pointing in the direction of the **W** vector. This is a coordinate system fixed in space.

Select now a point $P$ in the rotating body with coordinates $x$, $y$, $z$ at time $t = t_0$. As the body rotates, the point $P$ will have the same value of $z$ at all times, but $x$ and $y$ will change. Let the instantaneous location of point $P$ be given by its $z$ coordinate and a vector **s** perpendicular to the axis of rotation. The vector **s** changes with time, although its magnitude is constant.

The instantaneous velocity vector **v** for the point $P$ will have a magnitude equal to the product of the magnitude of the **W** vector and the magnitude vector **s**, and its direction will be perpendicular to the vectors **W** and **s**. That is, the vector **v** will increase proportionately to an increase in **W**, and will also increase as point $P$ moves outward from the axis of rotation. The relation

$$\mathbf{v} = [\mathbf{W} \times \mathbf{s}]$$

captures the ideas in the paragraph above. It not only shows the linear dependence of **v** on **W** and **s**, but also gives the direction of the velocity correctly (which $[\mathbf{s} \times \mathbf{W}]$ would not).

The location of the point $P$ with respect to the origin of coordinates is $\mathbf{r} = \boldsymbol{\delta}_z z + \mathbf{s}$ so that the velocity of the point is given by

$$\mathbf{v} = [\mathbf{W} \times \mathbf{s}] = [\mathbf{W} \times (\mathbf{r} - \boldsymbol{\delta}_z z)]$$

Since **W** and $\boldsymbol{\delta}_z z$ are vectors with the same direction, their cross product will be zero. Therefore we are left with

$$\mathbf{v} = [\mathbf{W} \times \mathbf{r}]$$

This, then, gives the instantaneous velocity of a point in the rotating solids in terms of the instantaneous position vector of the point with respect to the origin of coordinates located on the axis of rotation.

**Appendix A: p. 810, Ex. 5**

The formal expression for the differential amount work done on an object as it is moved through a differential displacement $d\mathbf{r}$ by a force $\mathbf{F}$ is

$$dW = (\mathbf{F} \cdot d\mathbf{r})$$

Note that, if the force acts in a direction perpendicular to the displacement vector, no work is done.

The rate of doing work is then

$$\frac{dW}{dt} = \left(\mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\right) = (\mathbf{F} \cdot \mathbf{v})$$

since the velocity is given by the time rate of change of the position vector (see Exercise 1, p. 839). This expression for the rate of doing work is used in §9.8 in connection with developing the rate of doing work on a fluid element (see Eq. 9.8-4), an essential ingredient in the combined energy flux vector **e**.

**Appendix A: p. 814, Ex. 1**

$$a. \quad \sum_{k=1}^{3} k^2 = 1^2 + 2^2 + 3^2 = 14$$

$$b. \quad \sum_{k=1}^{3} a_k^2 = a_1^2 + a_2^2 + a_3^2$$

$$c. \quad \sum_{j=1}^{3}\sum_{k=1}^{3} a_{jk} b_{kj} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}$$
$$+ a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} + a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33}$$

$$d. \quad \left(\sum_{j=1}^{3} a_j\right)^2 = \sum_{j=1}^{3}\sum_{k=1}^{3} a_j a_k = a_1a_1 + a_1a_2 + a_1a_3$$
$$+ a_2a_1 + a_2a_2 + a_2a_3 + a_3a_1 + a_3a_2 + a_3a_3$$
$$= a_1^2 + a_2^2 + a_3^2 + 2a_1a_2 + 2a_2a_3 + 2a_3a_1$$

**Appendix A: p. 814, Ex. 2**

*a.* $(\mathbf{v}\cdot\mathbf{w}) = (1)(3) + (2)(-1) + (-5)(1) = -4$

*b.* $$\begin{aligned}[\mathbf{v}\times\mathbf{w}] &= \boldsymbol{\delta}_x[(2)(1)-(-5)(-1)] + \boldsymbol{\delta}_y[(-5)(3)-(1)(1)] \\ &\quad + \boldsymbol{\delta}_z[(1)(-1)-(2)(3)] \\ &= \boldsymbol{\delta}_x(-3) + \boldsymbol{\delta}_y(-16) + \boldsymbol{\delta}_z(-7)\end{aligned}$$

*c.* $v = \sqrt{(1)^2 + (2)^2 + (-5)^2} = \sqrt{30}$; $w = \sqrt{(3)^2 + (-1)^2 + (1)^2} = \sqrt{11}$

*d* $(\boldsymbol{\delta}_x \cdot \mathbf{v}) = v_x = 1$

*e.* $$\begin{aligned}[\boldsymbol{\delta}_x\times\mathbf{w}] &= \boldsymbol{\delta}_x[(0)(1)-(0)(-1)] + \boldsymbol{\delta}_y[(0)(3)-(1)(1)] \\ &\quad + \boldsymbol{\delta}_z[(1)(-1)-(0)(3)] \\ &= \boldsymbol{\delta}_y(-1) + \boldsymbol{\delta}_z(-1)\end{aligned}$$

*f.* From Eq. A.1-6,

$$\cos\phi_{\mathbf{vw}} = \frac{(\mathbf{v}\cdot\mathbf{w})}{vw} = \frac{-4}{\sqrt{(30)(11)}} = \frac{-4}{\sqrt{330}} = \frac{-4}{18.166} = -0.220$$

Hence $\phi_{\mathbf{vw}} = 77.29°$. The minus sign is not affixed to $\phi_{\mathbf{vw}}$ since the angle is defined only as the angle *between* **v** and **w** and therefore has no sign.

*g.* $$\begin{aligned}[\mathbf{r}\times\mathbf{v}] &= \boldsymbol{\delta}_x[(y)(-5)-(z)(2)] + \boldsymbol{\delta}_y[(z)(1)-(x)(-5)] \\ &\quad + \boldsymbol{\delta}_z[(x)(2)-(y)(1)]\end{aligned}$$

**Appendix A: p. 814, Ex. 3**

$$a.\ \left(\left[\boldsymbol{\delta}_x \times \boldsymbol{\delta}_y\right] \cdot \boldsymbol{\delta}_z\right) = \left(\boldsymbol{\delta}_z \cdot \boldsymbol{\delta}_z\right) = 1$$

$$b.\ \left[\left[\boldsymbol{\delta}_y \times \boldsymbol{\delta}_z\right] \times \left[\boldsymbol{\delta}_x \times \boldsymbol{\delta}_z\right]\right] = \left[\boldsymbol{\delta}_x \times \left(-\boldsymbol{\delta}_y\right)\right] = -\boldsymbol{\delta}_z$$

**Appendix A: p. 815, Ex. 4**

*a.*

$$\sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon_{1jk}\varepsilon_{2jk} = \varepsilon_{111}\varepsilon_{211} + \varepsilon_{112}\varepsilon_{212} + \varepsilon_{113}\varepsilon_{213}$$
$$+\varepsilon_{121}\varepsilon_{221} + \varepsilon_{122}\varepsilon_{222} + \varepsilon_{123}\varepsilon_{223} + \varepsilon_{131}\varepsilon_{231} + \varepsilon_{132}\varepsilon_{232} + \varepsilon_{133}\varepsilon_{233} = 0$$

since every term contains a permutation symbol with two indices the same.

*b.*

$$\sum_{k=1}^{3}\varepsilon_{11k}\varepsilon_{12k} = \delta_{11}\delta_{12} - \delta_{21}\delta_{11}$$

The left side is zero, because each term contains a permutation symbol with two indices the same. The right side is zero, because every term contains a Kronecker delta with two indices that are different

**Appendix A: p. 815, Ex. 5**

This result (where $\varepsilon_{ijk}$ is defined by Eqs. A.2-3 to 5) is used quite a lot in proofs involving cross products. In the sum on $j$ and $k$, there are nine terms:

$$\sum_{j=1}^{3}\sum_{k=}^{3}\varepsilon_{ijk}\alpha_{jk} = \varepsilon_{i11}\alpha_{11} + \varepsilon_{i12}\alpha_{12} + \varepsilon_{i13}\alpha_{13}$$
$$+\varepsilon_{i21}\alpha_{21} + \varepsilon_{i22}\alpha_{22} + \varepsilon_{i23}\alpha_{23}$$
$$+\varepsilon_{i31}\alpha_{31} + \varepsilon_{i32}\alpha_{32} + \varepsilon_{i33}\alpha_{33}$$

The terms containing epsilon with two indices equal are zero by definition. The remaining terms may be written as follows, using the fact that the epsilons change sign if any two indices are reversed:

$$\sum_{j=1}^{3}\sum_{k=}^{3}\varepsilon_{ijk}\alpha_{jk} = \varepsilon_{i12}\alpha_{12} + \varepsilon_{i13}\alpha_{13} + \varepsilon_{i23}\alpha_{23} - \varepsilon_{i12}\alpha_{21} - \varepsilon_{i13}\alpha_{13} - \varepsilon_{i23}\alpha_{32}$$
$$= \varepsilon_{i12}(\alpha_{12} - \alpha_{21}) + \varepsilon_{i13}(\alpha_{13} - \alpha_{13}) + \varepsilon_{i23}(\alpha_{23} - \alpha_{32})$$

But the quantities $\alpha_{jk}$ are symmetrical, and therefore the coefficients $(\alpha_{12} - \alpha_{21})$, etc., are zero. Therefore the entire sum is zero.

**Appendix A : p. 815, Ex. 6**

The cross product, being a vector quantity, can be written as the sum of products of unit vectors and scalar components thus:

$$[\mathbf{v}\times\mathbf{w}]=\sum_{i=1}^{3}\boldsymbol{\delta}_i[\mathbf{v}\times\mathbf{w}]_i$$

From Eq. A.2-21 we have

$$[\mathbf{v}\times\mathbf{w}]=\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon_{ijk}\boldsymbol{\delta}_i v_j w_k=\sum_{i=1}^{3}\boldsymbol{\delta}_i\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon_{ijk}v_j w_k\right)$$

the second form being obtained by rearranging the original expression.

Now we can equate the coefficients of the unit vectors to get

$$[\mathbf{v}\times\mathbf{w}]_i=\sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon_{ijk}v_j w_k$$

which just states that the $i$th component of $[\mathbf{v}\times\mathbf{w}]$ is given by the double sum on the right side. This result is used in the first line of Eq. A.2-25.

**Appendix A: p. 815, Ex. 7**

First all terms in the "identity of Lagrange" are expanded in component form, thus

$$\sum_i \sum_j \sum_k \sum_m \sum_n \varepsilon_{ijk} v_j w_k \varepsilon_{imn} v_m w_n + \left( \sum_i v_i w_i \right) \left( \sum_j v_j w_j \right) = \left( \sum_i v_i v_i \right) \left( \sum_j w_j w_j \right)$$

Then perform the sum on $i$ and use Eq. A.2-7 in the first term on the left side to get

$$\sum_j \sum_k \sum_m \sum_n \left( \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km} \right) v_j v_m w_k w_n + \sum_i \sum_j v_i v_j w_i w_j = \sum_i \sum_j v_i v_i w_j w_j$$

Next, perform the sums on $m$ and $n$ in the first term on the left side:

$$\sum_j \sum_k \left( v_j v_j w_k w_k - v_j v_k w_k w_j \right) + \sum_i \sum_j v_i v_j w_i w_j = \sum_i \sum_j v_i v_i w_j w_j$$

Make the change of indices $i \to j$ and $j \to k$ in the second term on the left side and in the term on the right side to get:

$$\sum_j \sum_k \left( v_j v_j w_k w_k - v_j v_k w_k w_j \right) + \sum_j \sum_k v_j v_k w_j w_k = \sum_j \sum_k v_j v_j w_k w_k$$

which completes the proof.

**Appendix A: p. 819, Ex. 1**

*a.* $[\boldsymbol{\tau}\cdot\mathbf{v}]_x = (3)(5)+(2)(3)+(-1)(-2) = 23$

$[\boldsymbol{\tau}\cdot\mathbf{v}]_y = (2)(5)+(2)(3)+(1)(-2) = 14$

$[\boldsymbol{\tau}\cdot\mathbf{v}]_z = (-1)(5)+(1)(3)+(4)(-2) = -10$

*b* $[\mathbf{v}\cdot\boldsymbol{\tau}]_x = (5)(3)+(3)(2)+(-2)(-1) = 23$

$[\mathbf{v}\cdot\boldsymbol{\tau}]_y = (5)(2)+(3)(2)+(-2)(1) = 14$

$[\mathbf{v}\cdot\boldsymbol{\tau}]_z = (5)(-1)+(3)(1)+(-2)(4) = -10$

Note that these results are the same as those in part (*a*). Normally $[\boldsymbol{\tau}\cdot\mathbf{v}] \neq [\mathbf{v}\cdot\boldsymbol{\tau}]$, but since $\boldsymbol{\tau}$ is symmetric, the two operations give identical results.

*c.* $(\boldsymbol{\tau}:\boldsymbol{\tau}) = \sum_i \sum_j \tau_{ij}\tau_{ji} = \sum_i \sum_j \tau_{ij}^2$

since $\boldsymbol{\tau}$ is symmetric. Then

$$(\boldsymbol{\tau}:\boldsymbol{\tau}) = (3)^2+(2)^2+(-1)^2+(2)^2+(2)^2+(1)^2 \\ +(-1)^2+(1)^2+(4)^2 = 41$$

*d* $(\mathbf{v}\cdot[\boldsymbol{\tau}\cdot\mathbf{v}]) = \sum_i v_i[\boldsymbol{\tau}\cdot\mathbf{v}]_i$

$= (5)(23)+(3)(14)+(-2)(-10) = 117$

*e* $$\mathbf{vv} = \boldsymbol{\delta}_x\boldsymbol{\delta}_x(25)+\boldsymbol{\delta}_x\boldsymbol{\delta}_y(15)+\boldsymbol{\delta}_x\boldsymbol{\delta}_z(-10) \\ +\boldsymbol{\delta}_y\boldsymbol{\delta}_x(15)+\boldsymbol{\delta}_y\boldsymbol{\delta}_y(9)+\boldsymbol{\delta}_y\boldsymbol{\delta}_z(-6) \\ +\boldsymbol{\delta}_z\boldsymbol{\delta}_x(-10)+\boldsymbol{\delta}_z\boldsymbol{\delta}_y(-6)+\boldsymbol{\delta}_z\boldsymbol{\delta}_z(4)$$

*f.* $[\boldsymbol{\tau}\cdot\boldsymbol{\delta}_x]_x = (3)(1)+(2)(0)+(-1)(0) = 3$

$[\boldsymbol{\tau}\cdot\boldsymbol{\delta}_x]_y = (2)(1)+(2)(0)+(1)(0) = 2$

$[\boldsymbol{\tau}\cdot\boldsymbol{\delta}_x]_z = (-1)(1)+(1)(0)+(4)(0) = -1$

**Appendix A: p. 819, Ex. 2**

*a.* $\left[\left[\boldsymbol{\delta}_1\boldsymbol{\delta}_2 \cdot \boldsymbol{\delta}_2\right] \times \boldsymbol{\delta}_1\right] = \left[\boldsymbol{\delta}_1 \times \boldsymbol{\delta}_1\right] = 0$

*b.* $\left(\boldsymbol{\delta}:\boldsymbol{\delta}_1\boldsymbol{\delta}_2\right) = \sum_m \sum_n \delta_{mn}\delta_{1n}\delta_{2m} = \sum_m \delta_{m1}\delta_{2m} = \delta_{12} = 0$

*c* $\left(\boldsymbol{\delta}:\boldsymbol{\delta}\right) = \sum_m \sum_n \delta_{mn}\delta_{nm} = \sum_m \delta_{mm} = 3$

*d.*
$$\begin{aligned}
\{\boldsymbol{\delta} \cdot \boldsymbol{\delta}\} &= \left\{\left(\sum_m \sum_n \boldsymbol{\delta}_m \boldsymbol{\delta}_n \delta_{mn}\right) \cdot \left(\sum_p \sum_q \boldsymbol{\delta}_p \boldsymbol{\delta}_q \delta_{pq}\right)\right\} \\
&= \sum_m \sum_n \sum_p \sum_q \left\{\boldsymbol{\delta}_m \boldsymbol{\delta}_n \cdot \boldsymbol{\delta}_p \boldsymbol{\delta}_q\right\} \delta_{mn}\delta_{pq} \\
&= \left(\sum_m \sum_n \sum_p \sum_q \boldsymbol{\delta}_m \boldsymbol{\delta}_q \delta_{mn}\delta_{np}\delta_{pq}\right) \\
&= \sum_m \sum_n \sum_p \sum_q \boldsymbol{\delta}_m \boldsymbol{\delta}_q \delta_{mn}\delta_{np}\delta_{pq} \\
&= \sum_m \sum_n \boldsymbol{\delta}_m \boldsymbol{\delta}_q \delta_{mq} = \boldsymbol{\delta}
\end{aligned}$$

This is just a special case of $\{\boldsymbol{\delta} \cdot \boldsymbol{\tau}\} = \{\boldsymbol{\tau} \cdot \boldsymbol{\delta}\} = \boldsymbol{\tau}$. Note also Eq. A.3-20.

**Appendix A: p. 819, Ex. 3**

According to Eq. A.3-14

$$(\boldsymbol{\alpha}:\boldsymbol{\beta}) = \sum_m \sum_n \alpha_{mn}\beta_{nm}$$

Now we make use of the fact that $\boldsymbol{\alpha}$ is symmetric:

$$(\boldsymbol{\alpha}:\boldsymbol{\beta}) = \tfrac{1}{2}\sum_m \sum_n (\alpha_{mn} + \alpha_{nm})\beta_{nm} = \tfrac{1}{2}\sum_m \sum_n \alpha_{mn}\beta_{nm} + \tfrac{1}{2}\sum_m \sum_n \alpha_{nm}\beta_{nm}$$

In the second summation we now use the fact that $\boldsymbol{\beta}$ is antisymmetric, so that

$$(\boldsymbol{\alpha}:\boldsymbol{\beta}) = \tfrac{1}{2}\sum_m \sum_n \alpha_{mn}\beta_{nm} - \tfrac{1}{2}\sum_m \sum_n \alpha_{nm}\beta_{mn}$$

Next, in the second summation we interchange the indices $m$ and $n$ to obtain

$$(\boldsymbol{\alpha}:\boldsymbol{\beta}) = \tfrac{1}{2}\sum_m \sum_n \alpha_{mn}\beta_{nm} - \tfrac{1}{2}\sum_m \sum_n \alpha_{mn}\beta_{nm} = 0$$

Alternatively, one can use the method of Example 5 on p. 815.

**Appendix A: p. 819, Ex. 4**

When the tensor $\{\boldsymbol{\sigma}\cdot\boldsymbol{\tau}\}$ on the left side of Eq. A.3-17 is expanded in terms of its components, we have

$$\{\boldsymbol{\sigma}\cdot\boldsymbol{\tau}\} = \sum_i \sum_l \boldsymbol{\delta}_i \boldsymbol{\delta}_l \{\boldsymbol{\sigma}\cdot\boldsymbol{\tau}\}_{il}$$

But the last expression in Eq. A.3-17 is

$$\{\boldsymbol{\sigma}\cdot\boldsymbol{\tau}\} = \sum_i \sum_l \boldsymbol{\delta}_i \boldsymbol{\delta}_l \left( \sum_j \sigma_{ij} \tau_{jl} \right)$$

Then, equating the coefficients of $\boldsymbol{\delta}_i \boldsymbol{\delta}_l$ in these two expressions, we get

$$\{\boldsymbol{\sigma}\cdot\boldsymbol{\tau}\}_{il} = \sum_j \sigma_{ij} \tau_{jl}$$

**Appendix A: p. 819, Ex. 5**

The total angular momentum of the rigid structure is

$$\mathbf{L} = \sum_{\nu} m_{\nu}\left[\mathbf{R}_{\nu} \times \mathbf{v}_{\nu}\right]$$

and the velocity of an individual particle is $\mathbf{v}_{\nu} = \left[\mathbf{W} \times \mathbf{R}_{\nu}\right]$. Hence the angular momentum is

$$\mathbf{L} = \sum_{\nu} m_{\nu}\left[\mathbf{R}_{\nu} \times \left[\mathbf{W} \times \mathbf{R}_{\nu}\right]\right] \equiv \left[\mathbf{\Phi} \cdot \mathbf{W}\right]$$

in which $\mathbf{\Phi}$ is the second-order "moment of inertia" tensor. This form is chosen in order to be parallel to the linear-momentum expression $\mathbf{p} = m\mathbf{v}$.

We start by considering the $i$th component of the angular momentum, making use of the comment just after Eq. A.2-21 to write out the cross products explicitly:

$$\begin{aligned} L_i &= \sum_{\nu}\sum_{j}\sum_{k}\sum_{m}\sum_{n} m_{\nu}\varepsilon_{ijk}R_{\nu j}\varepsilon_{kmn}W_m R_{\nu n} \\ &= \sum_{\nu}\sum_{j}\sum_{m}\sum_{n} m_{\nu}\left(\delta_{im}\delta_{jn} - \delta_{jm}\delta_{in}\right)R_{\nu j}W_m R_{\nu n} \end{aligned}$$

To get the second line we have summed on $k$ using Eq. A.2-7. Next we perform the summations on $m$ and $n$ to get

$$\begin{aligned} L_i = \sum_{\nu}\sum_{j} m_{\nu}\left(R_{\nu j}W_i R_{\nu j} - R_{\nu j}W_j R_{\nu i}\right) &= \sum_{\nu} m_{\nu}\sum_{j}\left(R_{\nu j}R_{\nu j}W_i - R_{\nu i}R_{\nu j}W_j\right) \\ &= \sum_{\nu} m_{\nu}\left(\mathbf{R}_{\nu}\cdot\mathbf{R}_{\nu}\right)W_i - \sum_{\nu} m_{\nu}\left[\mathbf{R}_{\nu}\mathbf{R}_{\nu}\cdot\mathbf{W}\right]_i \end{aligned}$$

The second form is a rearrangement of the first form, enabling us to use the rule that a dot indicates a sum on adjacent indices (see top of p. 823). Then in the third form we have introduced the dot-product notations of Eqs. A.2-20 and A.3-18.

Finally we write the last result as a vector equation

$$\mathbf{L} = \sum_{\nu} m_{\nu}\left(\mathbf{R}_{\nu}\cdot\mathbf{R}_{\nu}\right)\mathbf{W} - \sum_{\nu} m_{\nu}\left[\mathbf{R}_{\nu}\mathbf{R}_{\nu}\cdot\mathbf{W}\right]$$

This is very close to the form that we are seeking, but the first term on the right will have to be written as a tensor dotted into a vector, in order to match up with the second term. We do this by making use of Eq. A.3-20 to write

$$\begin{aligned}\mathbf{L} &= \sum_v m_v(\mathbf{R}_v \cdot \mathbf{R}_v)[\boldsymbol{\delta} \cdot \mathbf{W}] - \sum_v m_v[\mathbf{R}_v\mathbf{R}_v \cdot \mathbf{W}] \\ &= \sum_v m_v\left[\left\{(\mathbf{R}_v \cdot \mathbf{R}_v)\boldsymbol{\delta} - \mathbf{R}_v\mathbf{R}_v\right\} \cdot \mathbf{W}\right]\end{aligned}$$

This equation gives the relation between the angular momentum and the moment-of-inertia tensor. Since the moment-of-inertia tensor is not isotropic, the angular momentum and the angular velocity are not pointing in the same direction.

We finally get for the moment-of-inertia tensor:

$$\boldsymbol{\Phi} = \sum_v m_v\left\{(\mathbf{R}_v \cdot \mathbf{R}_v)\boldsymbol{\delta} - \mathbf{R}_v\mathbf{R}_v\right\}$$

This is a good illustration of how the operator tensor-dot-vector arises naturally in a mechanics problem.

**Appendix A: p. 819, Ex. 6**

The expression for the kinetic energy can be written as follows:

$$K = \sum_v \tfrac{1}{2} m_v \left( \dot{\mathbf{R}}_v \cdot \dot{\mathbf{R}}_v \right) = \sum_v \tfrac{1}{2} m_v \left( \left[ \mathbf{W} \times \mathbf{R}_v \right] \cdot \left[ \mathbf{W} \times \mathbf{R}_v \right] \right)$$
$$= \sum_v \tfrac{1}{2} m_v \sum_i \left[ \mathbf{W} \times \mathbf{R}_v \right]_i \left[ \mathbf{W} \times \mathbf{R}_v \right]_i$$

Next we expand the cross products using the statement just after Eq. A.2-21 to get

$$K = \sum_v \tfrac{1}{2} m_v \sum_i \left[ \sum_j \sum_k \varepsilon_{ijk} W_j R_{vk} \right] \left[ \sum_m \sum_n \varepsilon_{imn} W_m R_{vn} \right]$$
$$= \sum_v \tfrac{1}{2} m_v \sum_j \sum_k \sum_m \sum_n \left( \sum_i \varepsilon_{ijk} \varepsilon_{imn} \right) R_{vk} R_{vn} W_j W_m$$

Application of Eq. A.2-7 then gives

$$K = \sum_v \tfrac{1}{2} m_v \sum_j \sum_k \sum_m \sum_n \left( \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km} \right) R_{vk} R_{vn} W_j W_m$$
$$= \sum_v \tfrac{1}{2} m_v \sum_j \sum_k \left( R_{vk} R_{vk} W_j W_j - R_{vk} R_{vj} W_j W_k \right)$$
$$= \sum_v \tfrac{1}{2} m_v \left[ \left( \mathbf{R}_v \cdot \mathbf{R}_v \right) \left( \mathbf{W} \cdot \mathbf{W} \right) - \left( \mathbf{R}_v \mathbf{R}_v : \mathbf{W}\mathbf{W} \right) \right]$$

Here we have used the rule that a dot indicates the sum on adjacent indices. Next we note that $(\mathbf{W} \cdot \mathbf{W}) = (\boldsymbol{\delta} : \mathbf{W}\mathbf{W})$, so that

$$K = \sum_v \tfrac{1}{2} m_v \left( \left\{ \left( \mathbf{R}_v \cdot \mathbf{R}_v \right) \boldsymbol{\delta} - \left( \mathbf{R}_v \mathbf{R}_v \right) \right\} : \mathbf{W}\mathbf{W} \right) = \tfrac{1}{2} \left( \boldsymbol{\Phi} : \mathbf{W}\mathbf{W} \right)$$

Thus the result has been written in terms of the moment-of-inertia tensor. This is the rigid-body analog of $K = \frac{1}{2} m v^2$ for a single particle of mass $m$.

This exercise has shown how a double-dot product arises naturally in a mechanics problem.

**Appendix A: p. 823, Ex. 1**

First we expand the del operator and the vector **v** in terms of their components to get

$$(\nabla \cdot \mathbf{v}) = \left( \left\{ \boldsymbol{\delta}_x \frac{\partial}{\partial x} + \boldsymbol{\delta}_y \frac{\partial}{\partial y} + \boldsymbol{\delta}_z \frac{\partial}{\partial z} \right\} \cdot \left\{ \boldsymbol{\delta}_x v_x + \boldsymbol{\delta}_y v_y + \boldsymbol{\delta}_z v_z \right\} \right)$$

Then we multiply these two polynomial expressions (in the sense of taking the dot products term by term) to get

$$\begin{aligned}(\nabla \cdot \mathbf{v}) = {} & (\boldsymbol{\delta}_x \cdot \boldsymbol{\delta}_x)\frac{\partial v_x}{\partial x} + (\boldsymbol{\delta}_x \cdot \boldsymbol{\delta}_y)\frac{\partial v_y}{\partial x} + (\boldsymbol{\delta}_x \cdot \boldsymbol{\delta}_z)\frac{\partial v_z}{\partial x} \\ & + (\boldsymbol{\delta}_y \cdot \boldsymbol{\delta}_x)\frac{\partial v_x}{\partial y} + (\boldsymbol{\delta}_y \cdot \boldsymbol{\delta}_y)\frac{\partial v_y}{\partial y} + (\boldsymbol{\delta}_y \cdot \boldsymbol{\delta}_z)\frac{\partial v_z}{\partial y} \\ & + (\boldsymbol{\delta}_z \cdot \boldsymbol{\delta}_x)\frac{\partial v_x}{\partial z} + (\boldsymbol{\delta}_z \cdot \boldsymbol{\delta}_y)\frac{\partial v_y}{\partial z} + (\boldsymbol{\delta}_z \cdot \boldsymbol{\delta}_z)\frac{\partial v_z}{\partial z}\end{aligned}$$

Next we evaluate the dot products of the unit vectors, using Eqs. A.2-9 and 10

$$\begin{aligned}(\nabla \cdot \mathbf{v}) = {} & (1)\frac{\partial v_x}{\partial x} + (0)\frac{\partial v_y}{\partial x} + (0)\frac{\partial v_z}{\partial x} \\ & + (0)\frac{\partial v_x}{\partial y} + (1)\frac{\partial v_y}{\partial y} + (0)\frac{\partial v_z}{\partial y} \\ & + (0)\frac{\partial v_x}{\partial z} + (0)\frac{\partial v_y}{\partial z} + (1)\frac{\partial v_z}{\partial z}\end{aligned}$$

Therefore, we get

$$(\nabla \cdot \mathbf{v}) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

**Appendix A: p. 823, Ex. 2**

*a.* $[\nabla \times \mathbf{v}]_x = 0$

$[\nabla \times \mathbf{v}]_y = 0$

$[\nabla \times \mathbf{v}]_z = -b$

Rotational

*b* $[\nabla \times \mathbf{v}]_x = 0$

$[\nabla \times \mathbf{v}]_y = 0$

$[\nabla \times \mathbf{v}]_z = 0$

Irrotational

*c.* $[\nabla \times \mathbf{v}]_x = 0$

$[\nabla \times \mathbf{v}]_y = 0$

$[\nabla \times \mathbf{v}]_z = b - b = 0$

Irrotational

*c.* $[\nabla \times \mathbf{v}]_x = 0$

$[\nabla \times \mathbf{v}]_y = 0$

$[\nabla \times \mathbf{v}]_z = b + b = 2b$

Rotational

**Appendix A: p. 823, Ex. 3**

*a*. $(\nabla \cdot \mathbf{v}) = 0$

$(\nabla \mathbf{v})_{xy} = b$ and all other component are zero

$$[\nabla \cdot \mathbf{vv}]_x = \frac{\partial}{\partial x} v_x v_x + \frac{\partial}{\partial y} v_y v_x + \frac{\partial}{\partial z} v_z v_x = 0$$
$$[\nabla \cdot \mathbf{vv}]_y = \frac{\partial}{\partial x} v_x v_y + \frac{\partial}{\partial y} v_y v_y + \frac{\partial}{\partial z} v_z v_y = 0$$
$$[\nabla \cdot \mathbf{vv}]_z = \frac{\partial}{\partial x} v_x v_z + \frac{\partial}{\partial y} v_y v_z + \frac{\partial}{\partial z} v_z v_z = 0$$

*b*. $(\nabla \cdot \mathbf{v}) = b$

$(\nabla \mathbf{v})_{xx} = b$ and all other components are zero

$$[\nabla \cdot \mathbf{vv}]_x = \frac{\partial}{\partial x} v_x v_x + \frac{\partial}{\partial y} v_y v_x + \frac{\partial}{\partial z} v_z v_x = 2b^2 x$$
$$[\nabla \cdot \mathbf{vv}]_y = \frac{\partial}{\partial x} v_x v_y + \frac{\partial}{\partial y} v_y v_y + \frac{\partial}{\partial z} v_z v_y = 0$$
$$[\nabla \cdot \mathbf{vv}]_z = \frac{\partial}{\partial x} v_x v_z + \frac{\partial}{\partial y} v_y v_z + \frac{\partial}{\partial z} v_z v_z = 0$$

*c*. $(\nabla \cdot \mathbf{v}) = 0$

$(\nabla \mathbf{v})_{xy} = b,\ (\nabla \mathbf{v})_{yx} = b$ and all others are zero

$$[\nabla \cdot \mathbf{vv}]_x = \frac{\partial}{\partial x} v_x v_x + \frac{\partial}{\partial y} v_y v_x + \frac{\partial}{\partial z} v_z v_x = b^2 x$$
$$[\nabla \cdot \mathbf{vv}]_y = \frac{\partial}{\partial x} v_x v_y + \frac{\partial}{\partial y} v_y v_y + \frac{\partial}{\partial z} v_z v_y = b^2 y$$
$$[\nabla \cdot \mathbf{vv}]_z = \frac{\partial}{\partial x} v_x v_z + \frac{\partial}{\partial y} v_y v_z + \frac{\partial}{\partial z} v_z v_z = 0$$

*d.* $(\nabla \cdot \mathbf{v}) = 0$

$(\nabla \mathbf{v})_{xy} = -b, \ (\nabla \mathbf{v})_{yx} = b$ and all others are zero

$$[\nabla \cdot \mathbf{vv}]_x = \frac{\partial}{\partial x} v_x v_x + \frac{\partial}{\partial y} v_y v_x + \frac{\partial}{\partial z} v_z v_x = -b^2 x$$

$$[\nabla \cdot \mathbf{vv}]_y = \frac{\partial}{\partial x} v_x v_y + \frac{\partial}{\partial y} v_y v_y + \frac{\partial}{\partial z} v_z v_y = -b^2 y$$

$$[\nabla \cdot \mathbf{vv}]_z = \frac{\partial}{\partial x} v_x v_z + \frac{\partial}{\partial y} v_y v_z + \frac{\partial}{\partial z} v_z v_z = 0$$

**Appendix A: p. 523, Ex. 4**

$$a.\quad \left(\nabla\cdot[\boldsymbol{\alpha}\cdot\mathbf{r}]\right)=\sum_i\sum_j\frac{\partial}{\partial x_i}\left(\alpha_{ij}x_j\right)=\sum_i\sum_j\alpha_{ij}\delta_{ij}=\sum_i\alpha_{ii}=0$$

$$b.\quad \left[\nabla\times[\boldsymbol{\alpha}\cdot\mathbf{r}]\right]_n=\sum_p\sum_q\varepsilon_{npq}\frac{\partial}{\partial x_p}\left(\sum_m\alpha_{qm}x_m\right)$$
$$=\sum_p\sum_q\sum_m\varepsilon_{npq}\alpha_{qm}\delta_{pm}=\sum_p\sum_q\varepsilon_{npq}\alpha_{qp}=0$$

or $\quad \left[\nabla\times[\boldsymbol{\alpha}\cdot\mathbf{r}]\right]=0$

$$c\quad \left\{\nabla[\boldsymbol{\alpha}\cdot\mathbf{r}]\right\}_{pq}=\sum_n\frac{\partial}{\partial x_p}\left(\alpha_{qn}x_n\right)=\sum_n\alpha_{qn}\delta_{pn}=\alpha_{qp}=\alpha_{pq}$$

or $\quad \nabla[\boldsymbol{\alpha}\cdot\mathbf{r}]=\boldsymbol{\alpha}$

$$d.\quad \left\{\nabla[\boldsymbol{\alpha}\cdot\mathbf{r}]\right\}^{\dagger}_{pq}=\left\{\nabla[\boldsymbol{\alpha}\cdot\mathbf{r}]\right\}_{qp}=\alpha_{pq}=\alpha_{qp}$$

$$e\quad \left[\nabla\cdot[\boldsymbol{\alpha}\cdot\mathbf{r}][\boldsymbol{\alpha}\cdot\mathbf{r}]\right]_n=\sum_p\sum_q\sum_m\frac{\partial}{\partial x_p}\left(\alpha_{pq}x_q\alpha_{nm}x_m\right)$$
$$=\sum_p\sum_q\sum_m\alpha_{nm}\alpha_{pq}\left(\delta_{pq}x_m+\delta_{pm}x_q\right)$$
$$=\sum_q\sum_m\alpha_{nm}\alpha_{qq}x_m+\sum_q\sum_m\alpha_{nm}\alpha_{mq}x_q$$
$$=0+\left[\{\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}\}\cdot\mathbf{r}\right]_n$$

or $\quad \left[\nabla\cdot[\boldsymbol{\alpha}\cdot\mathbf{r}][\boldsymbol{\alpha}\cdot\mathbf{r}]\right]=\left[\{\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}\}\cdot\mathbf{r}\right]$

**Appendix A: p. 823, Ex. 5**

*a*. First we write the expression on the left in component form:

$$\nabla^2(\nabla\cdot\mathbf{v}) = \sum_i \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}\left(\sum_j \frac{\partial}{\partial x_j} v_j\right) = \sum_i\sum_j \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} v_j$$

Then we interchange the order of differentiation to get

$$\nabla^2(\nabla\cdot\mathbf{v}) = \sum_i\sum_j \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i} v_j\right) = \sum_j \frac{\partial}{\partial x_j}\left(\nabla^2\mathbf{v}\right)_j = \left(\nabla\cdot\nabla^2\mathbf{v}\right)$$

*b*. We write the left side in component form, making use of the definition of the transpose given in Eq. A.3-8:

$$\left[\nabla\cdot(\nabla\mathbf{v})^\dagger\right] = \sum_i\sum_j \boldsymbol{\delta}_j \frac{\partial}{\partial x_i}(\nabla\mathbf{v})^\dagger_{ij} = \sum_i\sum_j \boldsymbol{\delta}_j \frac{\partial}{\partial x_i}(\nabla\mathbf{v})_{ji} = \sum_i\sum_j \boldsymbol{\delta}_j \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} v_i$$

Then we change the order of differentiation to get

$$\left[\nabla\cdot(\nabla\mathbf{v})^\dagger\right] = \sum_i\sum_j \boldsymbol{\delta}_j \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_i} v_i\right) = \sum_j \boldsymbol{\delta}_j \frac{\partial}{\partial x_j}(\nabla\cdot\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v})$$

where use has been made of Eq. A.4-6.

**Appendix A: p. 823, Ex. 6**

*a*. Proof that div curl $\mathbf{v} = 0$

$$\begin{aligned}(\nabla \cdot [\nabla \times \mathbf{v}]) &= \sum_m \frac{\partial}{\partial x_m}[\nabla \times \mathbf{v}]_m \\ &= \sum_m \frac{\partial}{\partial x_m} \sum_p \sum_q \varepsilon_{mpq} \frac{\partial}{\partial x_p} v_q \\ &= \sum_m \sum_p \sum_q \varepsilon_{mpq} \frac{\partial^2 v_q}{\partial x_m \partial x_p} = 0\end{aligned}$$

Here we have the double sum on $m$ and $p$, and in the summand we have the product of the permutation symbol (which is antisymmetric in the indices $m$ and $p$) and the second derivative operator (which is symmetric in the indices $m$ and $p$). We know that this sum will be zero (see Exercise 5 on p. 815).

*b*. Proof that curl grad $s = 0$

$$[\nabla \times \nabla s]_m = \sum_n \sum_p \varepsilon_{mnp} \frac{\partial}{\partial x_n} \frac{\partial}{\partial x_p} s = 0$$

Here again we have a sum on two indices of a product of a quantity that is symmetric with respect to the same two indices and a quantity that is antisymmetric on these same two indices.

**Appendix A: p. 823, Ex. 7**

$$a.\ (\nabla\cdot\mathbf{r})=\left(\left\{\sum_i \boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right\}\cdot\left\{\sum_j \boldsymbol{\delta}_j x_j\right\}\right)=\sum_i\sum_j\left(\boldsymbol{\delta}_i\cdot\boldsymbol{\delta}_j\right)\frac{\partial}{\partial x_i}x_j$$
$$=\sum_i\sum_j\left(\boldsymbol{\delta}_i\cdot\boldsymbol{\delta}_j\right)\delta_{ij}=\sum_i\left(\boldsymbol{\delta}_i\cdot\boldsymbol{\delta}_i\right)==\sum_i 1=3$$

$$b.\ [\nabla\times\mathbf{r}]_k=\left[\left\{\sum_i \boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right\}\times\left\{\sum_j \boldsymbol{\delta}_j x_j\right\}\right]_k$$
$$=\sum_i\sum_j\left[\boldsymbol{\delta}_i\times\boldsymbol{\delta}_j\right]_k\frac{\partial}{\partial x_i}x_j=\sum_i\sum_j\varepsilon_{kij}\frac{\partial}{\partial x_i}x_j=\sum_i\sum_j\varepsilon_{kij}\delta_{ij}=0$$

since the double sum on two indices (here $i$ and $j$) over the product of a quantity that is symmetrical in the two indices ($\delta_{ij}$) and one that is antisymmetric in the same two indices ($\varepsilon_{kij}$) is zero.

$$c.\ \left[\mathbf{r}\times[\nabla\cdot\mathbf{vv}]\right]_i=\sum_j\sum_k\varepsilon_{ijk}x_j[\nabla\cdot\mathbf{vv}]_k=\sum_j\sum_k\varepsilon_{ijk}x_j\sum_n\frac{\partial}{\partial x_n}\left(v_n v_k\right)$$
$$=\sum_j\sum_k\varepsilon_{ijk}\left[\sum_n\frac{\partial}{\partial x_n}\left(x_j v_n v_k\right)-\sum_n v_n v_k\frac{\partial}{\partial x_n}x_j\right]$$
$$=\sum_j\sum_k\sum_n\varepsilon_{ijk}\frac{\partial}{\partial x_n}\left(x_j v_n v_k\right)-\sum_j\sum_k\varepsilon_{ijk}v_j v_k$$
$$=\sum_n\frac{\partial}{\partial x_n}\left(v_n\sum_j\sum_k\varepsilon_{ijk}x_j v_k\right)=\sum_n\frac{\partial}{\partial x_n}\left(v_n[\mathbf{r}\times\mathbf{v}]_i\right)$$
$$=\left(\nabla\cdot\mathbf{v}[\mathbf{r}\times\mathbf{v}]_i\right)=\left[\nabla\cdot\mathbf{v}[\mathbf{r}\times\mathbf{v}]\right]_i$$

Therefore, since the component $i$ was selected arbitrarily, the relation is true for all three components. As a consequence

$$\left[\mathbf{r}\times[\nabla\cdot\mathbf{vv}]\right]=\left[\nabla\cdot\mathbf{v}[\mathbf{r}\times\mathbf{v}]\right]$$

**Appendix A: p. 823, Ex. 8**

First, we let $s\mathbf{v} = \mathbf{w}$, and then we write the $i$th component in expanded form:

$$
\begin{aligned}
\left[\nabla \times [\nabla \cdot \mathbf{vw}]\right]_i &= \sum_j \sum_k \varepsilon_{ijk} \frac{\partial}{\partial x_j} [\nabla \cdot \mathbf{vw}]_k \\
&= \sum_j \sum_k \varepsilon_{ijk} \frac{\partial}{\partial x_j} \left( \sum_n \frac{\partial}{\partial x_n} v_n w_k \right) \\
&= \sum_n \frac{\partial}{\partial x_n} \sum_j \sum_k \varepsilon_{ijk} \frac{\partial}{\partial x_j} (w_k v_n) \\
&= \sum_n \frac{\partial}{\partial x_n} \{\nabla \times \mathbf{wv}\}_{in} \\
&= \sum_n \frac{\partial}{\partial x_n} \{\nabla \times \mathbf{wv}\}^{\dagger}_{ni} \\
&= \left[\nabla \cdot \{\nabla \times \mathbf{wv}\}^{\dagger}\right]_i
\end{aligned}
$$

The derivation was performed for the $i$th component only. Since the choice of $i$ was arbitrary, the same result holds for all components. Therefore

$$\left[\nabla \times [\nabla \cdot s\mathbf{vv}]\right] = \left[\nabla \cdot \{\nabla \times s\mathbf{vv}\}^{\dagger}\right]$$

In this development we have made use of the statement after Eq A.3-19 about the components of $\{\mathbf{v} \times \boldsymbol{\tau}\}$.

**Appendix A: p. 823, Ex. 9**

*a*. First we write the left side of the equation in terms of the components and then differentiate term by term

$$\nabla\frac{1}{r}=\sum_i \boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\frac{1}{\left(\Sigma_k x_k^2\right)^{1/2}}=\sum_i \boldsymbol{\delta}_i\left(-\frac{1}{2}\frac{2x_i}{\left(\Sigma_k x_k^2\right)^{3/2}}\right)$$

Then put the result back into the original notation to get

$$\nabla\frac{1}{r}=-\sum_i \boldsymbol{\delta}_i\left(\frac{x_i}{r^3}\right)=-\frac{1}{r^3}\sum_i \boldsymbol{\delta}_i x_i=-\frac{1}{r^3}\mathbf{r}$$

*b*. Write out the gradient operator, and then perform the differentiation

$$\nabla f(r)=\sum_i \boldsymbol{\delta}_i\frac{\partial f(r)}{\partial x_i}=\sum_i \boldsymbol{\delta}_i\frac{df}{dr}\left(\frac{\partial r}{\partial x_i}\right)$$

$$=\sum_i \boldsymbol{\delta}_i\frac{df}{dr}\left(\frac{\partial}{\partial x_i}\left(\Sigma_j x_j^2\right)^{1/2}\right)=\frac{df}{dr}\sum_i \boldsymbol{\delta}_i\left(\frac{1}{2}\frac{2x_i}{\left(\Sigma_j x_j^2\right)^{1/2}}\right)$$

Next we rewrite the result in the original notation

$$\nabla f(r)=\frac{df}{dr}\frac{\Sigma_i \boldsymbol{\delta}_i x_i}{\left(\Sigma_j x_j^2\right)^{1/2}}=\frac{df}{dr}\frac{\mathbf{r}}{r}$$

*c*. We write the expression in component form and then perform the differentiation

$$\nabla(\mathbf{a}\cdot\mathbf{r})=\sum_i \boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\left(\sum_j a_j x_j\right)=\sum_i\sum_j \boldsymbol{\delta}_i a_j\frac{\partial x_j}{\partial x_i}=\sum_i\sum_j \boldsymbol{\delta}_i a_j\delta_{ij}=\sum_i \boldsymbol{\delta}_i a_i=\mathbf{a}$$

**Appendix A: p. 824, Ex. 10**

*a.*
$$\frac{\partial}{\partial t}\rho v_x = -\left(\frac{\partial}{\partial x}\rho v_x v_x + \frac{\partial}{\partial y}\rho v_y v_x + \frac{\partial}{\partial z}\rho v_z v_x\right) - \frac{\partial p}{\partial x} - \left(\frac{\partial}{\partial x}\tau_{xx} + \frac{\partial}{\partial y}\tau_{yx} + \frac{\partial}{\partial z}\tau_{zx}\right) + \rho g_x$$

$$\frac{\partial}{\partial t}\rho v_y = -\left(\frac{\partial}{\partial x}\rho v_x v_y + \frac{\partial}{\partial y}\rho v_y v_y + \frac{\partial}{\partial z}\rho v_z v_y\right) - \frac{\partial p}{\partial y} - \left(\frac{\partial}{\partial x}\tau_{xy} + \frac{\partial}{\partial y}\tau_{yy} + \frac{\partial}{\partial z}\tau_{zy}\right) + \rho g_y$$

$$\frac{\partial}{\partial t}\rho v_z = -\left(\frac{\partial}{\partial x}\rho v_x v_z + \frac{\partial}{\partial y}\rho v_y v_z + \frac{\partial}{\partial z}\rho v_z v_z\right) - \frac{\partial p}{\partial z} - \left(\frac{\partial}{\partial x}\tau_{xz} + \frac{\partial}{\partial y}\tau_{yz} + \frac{\partial}{\partial z}\tau_{zz}\right) + \rho g_z$$

*b.* The components are written out in full in Eqs. B.1-1 through B.1-7 on p. 843.

**Appendix A: p. 825, Ex. 1**

*a* The divergence of **v** is given by

$$(\nabla \cdot \mathbf{v}) = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = \frac{\partial x_1}{\partial x_1} + \frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3} = 1$$

Then the integral of the divergence over the volume is

$$\int_0^4\int_0^2\int_0^1 (\nabla \cdot \mathbf{v}) dxdydz - 8$$

*b*. There will be six surface integrals, $I_k^{\pm}$ : the subscript $k$ (= 1,2,3) gives the coordinate direction perpendicular to the face, and the superscript indicates the direction (positive or negative) in which the face is "looking."

$$I_1^- = \int_0^4\int_0^2 (-\delta_x \cdot \mathbf{v})\Big|_{x=0} dydz = \int_0^4\int_0^2 (-\delta_x \cdot \delta_x x)\Big|_{x=0} dydz = 0$$

$$I_1^+ = \int_0^4\int_0^2 (\delta_x \cdot \mathbf{v})\Big|_{x=1} dydz = \int_0^4\int_0^2 v_x\Big|_{x=1} dydz = 8$$

$$I_2^- = \int_0^1\int_0^4 (-\delta_y \cdot \mathbf{v})\Big|_{y=0} dzdx = -\int_0^1\int_0^4 zdzdx = -8$$

$$I_2^+ = \int_0^1\int_0^4 (\delta_y \cdot \mathbf{v})\Big|_{y=2} dzdx = \int_0^1\int_0^4 zdzdx = 8$$

$$I_3^- = \int_0^2\int_0^1 (-\delta_z \cdot \mathbf{v})\Big|_{z=0} dxdy = -\int_0^2\int_0^1 ydxdy = -2$$

$$I_3^+ = \int_0^2\int_0^1 (\ \delta_z \cdot \mathbf{v})\Big|_{z=4} dxdy = \int_0^2\int_0^1 ydxdy = 2$$

When we add up the individual surface integralls we get +8.

**Appendix A: p. 825, Ex. 2**

The surface integral on the left side is evaluated as follows:

$$\int(\mathbf{n}\cdot[\nabla\times\mathbf{v}])dS = \int_0^2\int_0^4(\boldsymbol{\delta}_x\cdot[\nabla\times\mathbf{v}])dydz$$

$$= \int_0^2\int_0^4[\nabla\times\mathbf{v}]_x\,dydz = \int_0^2\int_0^4\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)dydz$$

$$= \int_0^2\int_0^4\left(\frac{\partial y}{\partial y}-\frac{\partial z}{\partial z}\right)dydz = \int_0^2\int_0^4(1-1)dydz = 0$$

The line integral has to be broken up into a sum of 4 integrals:

$$\oint(\mathbf{t}\cdot\mathbf{v})dC = \int_0^2(\boldsymbol{\delta}_y\cdot\mathbf{v})\Big|_{\substack{x=1\\z=0}}dy + \int_0^4(\boldsymbol{\delta}_z\cdot\mathbf{v})\Big|_{\substack{x=1\\y=2}}dz$$

$$+\int_0^2(-\boldsymbol{\delta}_y\cdot\mathbf{v})\Big|_{\substack{x=1\\z=4}}dy' + \int_0^4(-\boldsymbol{\delta}_z\cdot\mathbf{v})\Big|_{\substack{x=1\\y=0}}dz'$$

The primes on $y$ and $z$ in the last two integrals are added to indicate that these are variables in the $y$ and $z$ directions, but taken in the positive direction on the curve $C$. The dot products with the unit vectors select the desired component of the vector $\mathbf{v}$. Therefore we get:

$$\oint(\mathbf{t}\cdot\mathbf{v})dC = \int_0^2(z)\Big|_{\substack{x=1\\z=0}}dy + \int_0^4(y)\Big|_{\substack{x=1\\y=2}}dz + \int_0^2(-z)\Big|_{\substack{x=1\\z=4}}dy' + \int_0^4(-y)\Big|_{\substack{x=1\\y=0}}dz'$$

$$= 0 + 2\int_0^4 dz - 4\int_0^2 dy' + 0 = 0$$

Thus, Stokes' theorem is satisfied.

**Appendix A: p. 825, Ex. 3**

We begin by evaluating the integral on the left side of Eq. A.5-5:

$$\int_V s dV = \int_0^{4t}\int_0^{2t}\int_0^{t}(x+y+zt)dxdydz$$
$$= \tfrac{1}{2}x^2\Big|_0^t \cdot 8t^2 + \tfrac{1}{2}y^2\Big|_0^{2t} \cdot 4t^2 + t\cdot\tfrac{1}{2}z^2\Big|_0^{4t} \cdot 2t^2$$
$$= 4t^4 + 8t^4 + 16t^5 = 12t^4 + 16t^5$$

Then the left side of Eq. A.5-5 is:

$$\frac{d}{dt}\int_V s dV = 48t^3 + 80t^4$$

The volume integral on the right side of Eq. A.5-5 is:

$$\int_V \frac{\partial s}{\partial t} dV = \int_0^{4t}\int_0^{2t}\int_0^{t} z dxdydz = (t)(2t)\left(\tfrac{1}{2}z^2\Big|_0^{2t}\right) = 16t^4$$

The surface integral on the right side of Eq. A.5-5 has to be written as the sum of 6 integrals, over the six faces of the expanding solid. However, since three of the surfaces ($x = 0$, $y = 0$, and $z = 0$) are not moving, those faces do not need to be considered. The three surface integrals are then evaluated one by one:

$$\int_{S_x} s(\mathbf{v}_S \cdot \mathbf{n})dS = \int_0^{4t}\int_0^{2t}(x+y+zt)\Big|_{x=t} dydz$$
$$= (t)(2t)(4t) + \left(\tfrac{1}{2}(2t)^2\right)(4t) + (t)(2t)\left(\tfrac{1}{2}(4t)^2\right)$$
$$= 16t^3 + 16t^4$$

$$\int_{S_y} s(\mathbf{v}_S \cdot \mathbf{n})dS = 2\int_0^{4t}\int_0^{t}(x+y+zt)\Big|_{y=2t} dxdz$$
$$= 2\left[\left(\tfrac{1}{2}t^2\right)(4t) + (2t)(t)(4t) + (t)\left(\tfrac{1}{2}(4t)^2\right)(t)\right]$$
$$= 20t^3 + 16t^4$$

$$\int_{S_z} s(\mathbf{v}_S \cdot \mathbf{n}) dS = 4\int_0^{2t}\int_0^t (x+y+zt)\Big|_{z=4t} dxdy$$

$$= 4\left[\left(\tfrac{1}{2}t^2\right)(2t) + \left(\tfrac{1}{2}(2t)^2\right)(t) + \left(4t^2\right)(t)(2t)\right]$$

$$= 12t^3 + 32t^4$$

When these surface integrals are summed we get

$$\int_S s(\mathbf{v}_S \cdot \mathbf{n}) dS = 48t^3 + 64t^4$$

Then the complete right side of Eq. A.5-5 becomes

$$\int_V \frac{\partial s}{\partial t} dV + \int_S s(\mathbf{v}_S \cdot \mathbf{n}) dS = \left(16t^4\right) + \left(48t^3 + 64t^4\right) = 48t^3 + 80t^4$$

in agreement with the expression obtained from the left side.

**Appendix A: p. 825, Ex. 4**

The tensor analog of Eq. A.5-4 is

$$\int_S [\mathbf{n}\cdot\{\nabla\times\boldsymbol{\tau}\}]dS = \oint_C [\mathbf{t}\cdot\boldsymbol{\tau}]dC$$

We now simplify the integrand in the surface integral on the left side, for the particular choice of $\boldsymbol{\tau}$ indicated:

$$\begin{aligned}
[\mathbf{n}\cdot\{\nabla\times\boldsymbol{\tau}\}]_k &= \sum_l\sum_i\sum_j n_l\varepsilon_{ijl}\frac{\partial}{\partial x_i}\tau_{jk} \\
&= \sum_l\sum_i\sum_j n_l\varepsilon_{ijl}\frac{\partial}{\partial x_i}\left(\sum_m \varepsilon_{kmj}x_m\right) \\
&= \sum_l\sum_i\sum_j\sum_m n_l\varepsilon_{ijl}\varepsilon_{kmj}\delta_{im} \\
&= \sum_l\sum_i\sum_j n_l\varepsilon_{ijl}\varepsilon_{kij} \\
&= \sum_l n_l(2\delta_{kl}) = 2n_k
\end{aligned}$$

In going from the 4th line to the 5th line we have made use of Eq. A.2-6--one of the properties of the permutation symbols.

Now we simplify, for the specific choice of $\boldsymbol{\tau}$ indicated, the integrand on the right side:

$$[\mathbf{t}\cdot\boldsymbol{\tau}]_k = \sum_j t_j\tau_{jk} = \sum_j\sum_m t_j\varepsilon_{kmj}x_m = \sum_j\sum_m \varepsilon_{kmj}x_m t_j = \sum_j\sum_m \varepsilon_{kmj}x_m t_j = [\mathbf{r}\times\mathbf{t}]_k$$

Therefore

$$2\int_S \mathbf{n}dS = \oint_C [\mathbf{r}\times\mathbf{t}]dC$$

which is the desired result.

**Appendix A: p. 825, Ex. 5**

First we evaluate the volume integral ($I$) on the left side of Eq. A.5-2:

$$I = \int_{-2}^{+2}\int_0^1\int_0^{3-3y}\left(2x\boldsymbol{\delta}_x + 2y\boldsymbol{\delta}_y + 2z\boldsymbol{\delta}_z\right)dzdydx$$

The integral over the first term ($I_x$) will be zero, since the integrand is odd in $x$ and the interval over which the integation occurs is even. The integrals ($I_y$ and $I_z$) over the second and third terms are obtained as follows:

$$I_y = 2\boldsymbol{\delta}_y\int_{-2}^{+2}\int_0^1\int_0^{3-3y} ydzdydx = 8\boldsymbol{\delta}_y\int_0^1\int_0^{3-3y} ydzdy = 8\boldsymbol{\delta}_y\int_0^1 y(3-3y)dy$$
$$= 24\boldsymbol{\delta}_y\left(\tfrac{1}{2}y^2 - \tfrac{1}{3}y^3\right)\Big|_0^1 = 4\boldsymbol{\delta}_y$$

$$I_z = 2\boldsymbol{\delta}_z\int_{-2}^{+2}\int_0^1\int_0^{3-3y} zdzdydx = 4\boldsymbol{\delta}_z\int_0^1\left(z^2\right)\Big|_0^{3-3y}dy$$
$$= 36\boldsymbol{\delta}_z\int_0^1\left(1-2y+y^2\right)dy = 12\boldsymbol{\delta}_z$$

Hence the volume integral is

$$I = 4\boldsymbol{\delta}_y + 12\boldsymbol{\delta}_z.$$

Next we evaluate the surface integral ($J$) on the right side of Eq. A.5-2. First we evaluate the integral ($J_x$) over the triangular surfaces at the ends:

$$J_x = \boldsymbol{\delta}_x\int_0^1\int_0^{3-3y} s\big|_{x=+2}dzdy - \boldsymbol{\delta}_x\int_0^1\int_0^{3-3y} s\big|_{x=-2}dzdy = 0$$

since $s\big|_{x=+2} = s\big|_{x=-2}$. Next we evaluate the integrals $J_y$ and $J_z$, over the surfaces at $y=0$ and $z=0$:

$$J_y = -\boldsymbol{\delta}_y\int_{-2}^{+2}\int_0^3 s\big|_{y=0}dzdx = -\boldsymbol{\delta}_y\int_{-2}^{+2}\int_0^3\left(x^2+z^2\right)dzdx = -\boldsymbol{\delta}_y\int_{-2}^{+2}\left(x^2z + \tfrac{1}{3}z^3\right)\Big|_0^3 dx$$
$$= -\boldsymbol{\delta}_y\int_{-2}^{+2}\left(3x^2+9\right)dx = -\boldsymbol{\delta}_y(16+36) = -52\boldsymbol{\delta}_y$$

$$J_z = -\boldsymbol{\delta}_z \int_{-2}^{+2}\int_0^1 s\big|_{z=0}\, dydx = -\boldsymbol{\delta}_z \int_{-2}^{+2}\int_0^1 \left(x^2+y^2\right) dydx = -\boldsymbol{\delta}_z \int_{-2}^{+2}\left(x^2 y + \tfrac{1}{3}y^3\right)\Big|_0^1 dx$$
$$= -\boldsymbol{\delta}_z \int_{-2}^{+2}\left(x^2+\tfrac{1}{3}\right)dx = -\tfrac{20}{3}\boldsymbol{\delta}_z$$

Finally we get the contribution $J_s$ of the "slanting surface," which is described by the equation $F(x,y,z) = 3-3y-z = 0$ . The outwardly directed normal unit vector is then

$$\mathbf{n} = \pm\frac{\nabla F}{|\nabla F|} = \pm\frac{-3\boldsymbol{\delta}_y - \boldsymbol{\delta}_z}{\left|\sqrt{9+1}\right|} = \frac{3\boldsymbol{\delta}_y + \boldsymbol{\delta}_z}{\sqrt{10}}$$

and

$$\mathbf{n}s\big|_{\text{surface}} = \frac{3\boldsymbol{\delta}_y + \boldsymbol{\delta}_z}{\sqrt{10}}\left(x^2+y^2+z^2\right)\Big|_{z=3-3y}$$
$$= \frac{3\boldsymbol{\delta}_y + \boldsymbol{\delta}_z}{\sqrt{10}}\left(x^2+10y^2-18y+9\right)$$

The contribution of the slanting surface to the surface integral is then obtained by integrating over the slanting surface (or equivalently over the projection of the slanting surface onto the $xy$plane and then multiplying by the appropriate factor) to get

$$J_s = \int_S \mathbf{n}s dS = \sqrt{10}\int_{-2}^{+2}\int_0^1 \left(\frac{3\boldsymbol{\delta}_y + \boldsymbol{\delta}_z}{\sqrt{10}}\right)\left(x^2+10y^2-18y+9\right) dydx$$
$$= \left(3\boldsymbol{\delta}_y + \boldsymbol{\delta}_z\right)\int_{-2}^{+2}\left(x^2 y + \tfrac{10}{3}y^3 - 9y^2 + 9y\right)\Big|_0^1 dx$$
$$= \int_{-2}^{+2}\left(x^2+\tfrac{10}{3}\right)dx = \left(3\boldsymbol{\delta}_y + \boldsymbol{\delta}_z\right)\left(\tfrac{1}{3}x^3 + \tfrac{10}{3}x\right)\Big|_{-2}^{+2}$$
$$= 56\boldsymbol{\delta}_y + \tfrac{56}{3}\boldsymbol{\delta}_z$$

Thus the surface integral is

$$J = J_y + J_y + J_z + J_s = 0\boldsymbol{\delta}_x - 52\boldsymbol{\delta}_y - \tfrac{20}{3}\boldsymbol{\delta}_z + 56\boldsymbol{\delta}_y + \tfrac{56}{3}\boldsymbol{\delta}_z = 4\boldsymbol{\delta}_y + 12\boldsymbol{\delta}_z$$

in agreement with the expression for the volume integral.

**Appendix A: p. 828, Ex. 1**

*a*. We rewrite the integral by using Eq. A.6-10

$$\int_0^{2\pi}\int_0^{\pi}\boldsymbol{\delta}_r \sin\theta d\theta d\phi$$
$$= \int_0^{2\pi}\int_0^{\pi}\left(\boldsymbol{\delta}_x \sin\theta\cos\phi + \boldsymbol{\delta}_y \sin\theta\sin\phi + \boldsymbol{\delta}_z \cos\theta\right)\sin\theta d\theta d\phi$$
$$= \boldsymbol{\delta}_x \int_0^{\pi}\sin^2\theta d\theta \int_0^{2\pi}\cos\phi d\phi + \boldsymbol{\delta}_y \int_0^{\pi}\sin^2\theta d\theta \int_0^{2\pi}\sin\phi d\phi$$
$$+\boldsymbol{\delta}_z \int_0^{\pi}\cos\theta\sin\theta d\theta \int_0^{2\pi} d\phi$$

The $\phi$-integrals in the first two terms are zero, and the $\theta$-integral in the last term is zero. Note that the unit vectors in Cartesian coordinates can be taken out of the integrals, since they do not depend on the position in space.

The problem may also be solved by noting that for every value of $\theta$ and $\phi$, there is another pair of $\theta$ and $\phi$ values for which $\boldsymbol{\delta}_r$ points in the opposite direction. Therefore the integral over the top half of the sphere will just cancel the integral over the bottom half of the sphere.

*b*. This problem may be solved by the same method used in (*a*), that is, by writing out the expression for $\boldsymbol{\delta}_r$ and evaluating all the integrals that appear. This is very tedious.

We can do the problem by another method that is much easier. One has to realize in advance that the integral will be isotropic, inasmuch as there is nothing in the integral that depends on the orientation in space. Recognizing this we may state then that

$$\int_0^{2\pi}\int_0^{\pi}\boldsymbol{\delta}_r\boldsymbol{\delta}_r \sin\theta d\theta d\phi = A\boldsymbol{\delta}$$

That is, the integral will be some scalar constant $A$ multiplied by the unit tensor. Now we take the trace of the above expression to get

$$\int_0^{2\pi}\int_0^{\pi}(\boldsymbol{\delta}_r\cdot\boldsymbol{\delta}_r)\sin\theta d\theta d\phi = 3A \qquad \text{or} \qquad \int_0^{2\pi}\int_0^{\pi}\sin\theta d\theta d\phi = 3A$$

The left side is then $4\pi$, and $A = \frac{4}{3}\pi$.

**Appendix A: p. 828, Ex. 2**

The unit tensor in defined in Eq. A.3-10. If we perform one of the summations we get:

$$\boldsymbol{\delta} = \sum_i \boldsymbol{\delta}_i \boldsymbol{\delta}_i = \boldsymbol{\delta}_x \boldsymbol{\delta}_x + \boldsymbol{\delta}_y \boldsymbol{\delta}_y + \boldsymbol{\delta}_z \boldsymbol{\delta}_z$$

To convert this expression to spherical coordinates, we use Eqs. A.6-31 to 33. It is convenient to use the following abbreviations: $S = \sin\theta$, $C = \cos\theta$, $s = \sin\phi$, and $c = \cos\phi$.

$$\begin{aligned}\boldsymbol{\delta} = &\left(Sc\boldsymbol{\delta}_r + Cc\boldsymbol{\delta}_\theta - s\boldsymbol{\delta}_\phi\right)\left(Sc\boldsymbol{\delta}_r + Cc\boldsymbol{\delta}_\theta - s\boldsymbol{\delta}_\phi\right)\\ &+\left(Ss\boldsymbol{\delta}_r + Cs\boldsymbol{\delta}_\theta + c\boldsymbol{\delta}_\phi\right)\left(Ss\boldsymbol{\delta}_r + Cs\boldsymbol{\delta}_\theta + c\boldsymbol{\delta}_\phi\right)\\ &+\left(C\boldsymbol{\delta}_r - S\boldsymbol{\delta}_\theta\right)\left(C\boldsymbol{\delta}_r - S\boldsymbol{\delta}_\theta\right)\end{aligned}$$

Next we collect the terms with the same dyadic coefficients:

$$\begin{aligned}\boldsymbol{\delta} = &\boldsymbol{\delta}_r\boldsymbol{\delta}_r\left(S^2c^2 + S^2s^2 + C^2\right) + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta\left(C^2c^2 + C^2s^2 + S^2\right) + \boldsymbol{\delta}_\phi\boldsymbol{\delta}_\phi\left(s^2 + c^2\right)\\ &+\left(\boldsymbol{\delta}_r\boldsymbol{\delta}_\theta + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_r\right)\left(SCc^2 + SCs^2 - SC\right)\\ &+\left(\boldsymbol{\delta}_\theta\boldsymbol{\delta}_\phi + \boldsymbol{\delta}_\phi\boldsymbol{\delta}_\theta\right)\left(-Csc + Csc\right)\\ &+\left(\boldsymbol{\delta}_\phi\boldsymbol{\delta}_r + \boldsymbol{\delta}_r\boldsymbol{\delta}_\phi\right)\left(-Ssc + Ssc\right)\end{aligned}$$

Then standard trigonometric formulas can be used to get the final result:

$$\boldsymbol{\delta} = \boldsymbol{\delta}_r\boldsymbol{\delta}_r + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta + \boldsymbol{\delta}_\phi\boldsymbol{\delta}_\phi$$

**Appendix A: p. 839, Ex. 1**

*a*. We start with the expression for the position vector in cylindrical coordinates (see the caption of Fig. A.6-1):

$$\mathbf{r} = r\boldsymbol{\delta}_r + z\boldsymbol{\delta}_z$$

As the particle moves, $r$, $z$, and $\boldsymbol{\delta}_r$ change with time, but $\boldsymbol{\delta}_z$ does not. Hence, from Eqs. A.6-10 to 12 we can obtain the time derivative as follows:

$$\begin{aligned}\frac{d}{dt}\boldsymbol{\delta}_r &= \frac{d}{dt}\left(\boldsymbol{\delta}_x \cos\theta + \boldsymbol{\delta}_y \sin\theta\right)\\ &= \boldsymbol{\delta}_x\left(\frac{d}{d\theta}\cos\theta\right)\left(\frac{d\theta}{dt}\right) + \boldsymbol{\delta}_y\left(\frac{d}{d\theta}\sin\theta\right)\left(\frac{d\theta}{dt}\right)\\ &= \boldsymbol{\delta}_x(-\sin\theta)\dot{\theta} + \boldsymbol{\delta}_y(\cos\theta)\dot{\theta} = \boldsymbol{\delta}_\theta\dot{\theta}\end{aligned}$$

In the last step, Eq. A.6-11 has been used. Then the time derivative of the position vector--that is, the velocity of the particle--is

$$\mathbf{v} = \frac{d}{dt}\mathbf{r} = \dot{r}\boldsymbol{\delta}_r + r\dot{\boldsymbol{\delta}}_r + \dot{z}\boldsymbol{\delta}_z = \boldsymbol{\delta}_r\dot{r} + \boldsymbol{\delta}_\theta r\dot{\theta} + \boldsymbol{\delta}_z\dot{z}$$

*b*. To get the particle acceleration, we have to differentiate the velocity with respect to the time. But $\boldsymbol{\delta}_\theta$ has to be differentiated with respect to time, thus:

$$\frac{d}{dt}\boldsymbol{\delta}_\theta = \frac{d}{dt}\left(-\boldsymbol{\delta}_x \sin\theta + \boldsymbol{\delta}_y \cos\theta\right) = -\boldsymbol{\delta}_x(\cos\theta)\dot{\theta} - \boldsymbol{\delta}_y(\sin\theta)\dot{\theta} = -\boldsymbol{\delta}_r\dot{\theta}$$

In the last step, Eq. A.6-10 has been used.

The particle acceleration can now be obtained:

$$\begin{aligned}\mathbf{a} = \frac{d}{dt}\mathbf{v} &= \dot{\boldsymbol{\delta}}_r\dot{r} + \boldsymbol{\delta}_r\ddot{r} + \dot{\boldsymbol{\delta}}_\theta r\dot{\theta} + \boldsymbol{\delta}_\theta\dot{r}\dot{\theta} + \boldsymbol{\delta}_\theta r\ddot{\theta} + \boldsymbol{\delta}_z\ddot{z}\\ &= \left(\boldsymbol{\delta}_\theta\dot{\theta}\right)\dot{r} + \boldsymbol{\delta}_r\ddot{r} + \left(-\boldsymbol{\delta}_r\dot{\theta}\right)r\dot{\theta} + \boldsymbol{\delta}_\theta\dot{r}\dot{\theta} + \boldsymbol{\delta}_\theta r\ddot{\theta} + \boldsymbol{\delta}_z\ddot{z}\\ &= \boldsymbol{\delta}_r\left(\ddot{r} - r\dot{\theta}^2\right) + \boldsymbol{\delta}_\theta\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) + \boldsymbol{\delta}_z\ddot{z}\end{aligned}$$

**Appendix A: p. 839, Ex. 2**

*a.* $(\nabla \cdot \mathbf{v})$ *in spherical coordinates*

$$(\nabla \cdot \mathbf{v}) = \left( \left\{ \boldsymbol{\delta}_r \frac{\partial}{\partial r} + \boldsymbol{\delta}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{\delta}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \right\} \cdot \left\{ \boldsymbol{\delta}_r v_r + \boldsymbol{\delta}_\theta v_\theta + \boldsymbol{\delta}_\phi v_\phi \right\} \right)$$

$$= \left( \boldsymbol{\delta}_r \frac{\partial}{\partial r} \cdot \boldsymbol{\delta}_r v_r \right) + \left( \boldsymbol{\delta}_r \frac{\partial}{\partial r} \cdot \boldsymbol{\delta}_\theta v_\theta \right) + \left( \boldsymbol{\delta}_r \frac{\partial}{\partial r} \cdot \boldsymbol{\delta}_\phi v_\phi \right)$$
$$+ \left( \boldsymbol{\delta}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} \cdot \boldsymbol{\delta}_r v_r \right) + \left( \boldsymbol{\delta}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} \cdot \boldsymbol{\delta}_\theta v_\theta \right) + \left( \boldsymbol{\delta}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} \cdot \boldsymbol{\delta}_\phi v_\phi \right)$$
$$+ \left( \boldsymbol{\delta}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \cdot \boldsymbol{\delta}_r v_r \right) + \left( \boldsymbol{\delta}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \cdot \boldsymbol{\delta}_\theta v_\theta \right) + \left( \boldsymbol{\delta}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \cdot \boldsymbol{\delta}_\phi v_\phi \right)$$

$$= (\boldsymbol{\delta}_r \cdot \boldsymbol{\delta}_r)\frac{\partial v_r}{\partial r} + (\boldsymbol{\delta}_r \cdot \boldsymbol{\delta}_\theta)\frac{\partial v_\theta}{\partial r} + (\boldsymbol{\delta}_r \cdot \boldsymbol{\delta}_\phi)\frac{\partial v_\phi}{\partial r}$$
$$+ (\boldsymbol{\delta}_\theta \cdot \boldsymbol{\delta}_r)\frac{1}{r}\frac{\partial v_r}{\partial \theta} + (\boldsymbol{\delta}_\theta \cdot \boldsymbol{\delta}_\theta)\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + (\boldsymbol{\delta}_\theta \cdot \boldsymbol{\delta}_\phi)\frac{1}{r}\frac{\partial v_\phi}{\partial \theta}$$
$$+ (\boldsymbol{\delta}_\theta \cdot \boldsymbol{\delta}_\theta)\frac{v_r}{r} - (\boldsymbol{\delta}_\theta \cdot \boldsymbol{\delta}_r)\frac{v_\theta}{r}$$
$$+ (\boldsymbol{\delta}_\phi \cdot \boldsymbol{\delta}_r)\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} + (\boldsymbol{\delta}_\phi \cdot \boldsymbol{\delta}_\theta)\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + (\boldsymbol{\delta}_\phi \cdot \boldsymbol{\delta}_\phi)\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$
$$+ (\boldsymbol{\delta}_\phi \cdot \boldsymbol{\delta}_\phi)\frac{v_r}{r} + (\boldsymbol{\delta}_\phi \cdot \boldsymbol{\delta}_\phi)\frac{v_\theta \cos\theta}{r\sin\theta}$$
$$+ \left( \boldsymbol{\delta}_\phi \cdot (-\boldsymbol{\delta}_r \sin\theta + \boldsymbol{\delta}_\theta \cos\theta) \right)\frac{v_\phi}{r\sin\theta}$$

$$= \frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_r}{r} + \frac{v_\theta \cos\theta}{r\sin\theta}$$

This can easily be shown to be the same as Eq. (A) of Table A.7-3, which is more suitable for use in solving many problems.

*b.* $[\nabla \times \mathbf{v}]$ *in spherical coordinates*

$$[\nabla \times \mathbf{v}] = \left[ \left\{ \boldsymbol{\delta}_r \frac{\partial}{\partial r} + \boldsymbol{\delta}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{\delta}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \right\} \times \left\{ \boldsymbol{\delta}_r v_r + \boldsymbol{\delta}_\theta v_\theta + \boldsymbol{\delta}_\phi v_\phi \right\} \right]$$

$$
\begin{aligned}
&= \left[\boldsymbol{\delta}_r \frac{\partial}{\partial r} \times \boldsymbol{\delta}_r v_r\right] + \left[\boldsymbol{\delta}_r \frac{\partial}{\partial r} \times \boldsymbol{\delta}_\theta v_\theta\right] + \left[\boldsymbol{\delta}_r \frac{\partial}{\partial r} \times \boldsymbol{\delta}_\phi v_\phi\right] \\
&+ \left[\boldsymbol{\delta}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} \times \boldsymbol{\delta}_r v_r\right] + \left[\boldsymbol{\delta}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} \times \boldsymbol{\delta}_\theta v_\theta\right] + \left[\boldsymbol{\delta}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} \times \boldsymbol{\delta}_\phi v_\phi\right] \\
&+ \left[\boldsymbol{\delta}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \times \boldsymbol{\delta}_r v_r\right] + \left[\boldsymbol{\delta}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \times \boldsymbol{\delta}_\theta v_\theta\right] + \left[\boldsymbol{\delta}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \times \boldsymbol{\delta}_\phi v_\phi\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \boldsymbol{\delta}_\phi \frac{\partial v_\theta}{\partial r} - \boldsymbol{\delta}_\theta \frac{\partial v_\phi}{\partial r} - \boldsymbol{\delta}_\phi \frac{1}{r}\frac{\partial v_r}{\partial \theta} + \boldsymbol{\delta}_r \frac{1}{r}\frac{\partial v_\phi}{\partial \theta} + \boldsymbol{\delta}_\theta \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \boldsymbol{\delta}_r \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} \\
&\qquad + \boldsymbol{\delta}_\phi \frac{v_\theta}{r} - \boldsymbol{\delta}_\theta \frac{v_\theta}{r} + \boldsymbol{\delta}_r \frac{v_\theta \cos\theta}{r\sin\theta}
\end{aligned}
$$

$$
\begin{aligned}
&= \boldsymbol{\delta}_r \left(\frac{1}{r}\frac{\partial v_\phi}{\partial \theta} - \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_\phi \cos\theta}{r\sin\theta}\right) + \boldsymbol{\delta}_\theta \left(\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r}\right) \\
&\qquad + \boldsymbol{\delta}_\phi \left(\frac{\partial v_\theta}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\theta}{r}\right)
\end{aligned}
$$

The components of this expression can be rearranged to give Eqs. (G), (H), and (I) of Table A.7-3.

*c.* $\nabla\mathbf{v}$ *in spherical coordinates*

$$
\nabla\mathbf{v} = \left\{\boldsymbol{\delta}_r \frac{\partial}{\partial r} + \boldsymbol{\delta}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{\delta}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right\}\left\{\boldsymbol{\delta}_r v_r + \boldsymbol{\delta}_\theta v_\theta + \boldsymbol{\delta}_\phi v_\phi\right\}
$$

$$
\begin{aligned}
&= \boldsymbol{\delta}_r \frac{\partial}{\partial r}(\boldsymbol{\delta}_r v_r) + \boldsymbol{\delta}_r \frac{\partial}{\partial r}(\boldsymbol{\delta}_\theta v_\theta) + \boldsymbol{\delta}_r \frac{\partial}{\partial r}(\boldsymbol{\delta}_\phi v_\phi) \\
&+ \boldsymbol{\delta}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}(\boldsymbol{\delta}_r v_r) + \boldsymbol{\delta}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}(\boldsymbol{\delta}_\theta v_\theta) + \boldsymbol{\delta}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}(\boldsymbol{\delta}_\phi v_\phi) \\
&+ \boldsymbol{\delta}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(\boldsymbol{\delta}_r v_r) + \boldsymbol{\delta}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(\boldsymbol{\delta}_\theta v_\theta) + \boldsymbol{\delta}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(\boldsymbol{\delta}_\phi v_\phi)
\end{aligned}
$$

$$
\begin{aligned}
&= \boldsymbol{\delta}_r\boldsymbol{\delta}_r \frac{\partial v_r}{\partial r} + \boldsymbol{\delta}_r\boldsymbol{\delta}_\theta \frac{\partial v_\theta}{\partial r} + \boldsymbol{\delta}_r\boldsymbol{\delta}_\phi \frac{\partial v_\phi}{\partial r} \\
&+ \boldsymbol{\delta}_\theta\boldsymbol{\delta}_r \frac{1}{r}\frac{\partial v_r}{\partial \theta} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\phi \frac{1}{r}\frac{\partial v_\phi}{\partial \theta} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta \frac{v_r}{r} - \boldsymbol{\delta}_\theta\boldsymbol{\delta}_r \frac{v_\theta}{r}
\end{aligned}
$$

$$+\boldsymbol{\delta}_\phi\boldsymbol{\delta}_r\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi}+\boldsymbol{\delta}_\phi\boldsymbol{\delta}_\theta\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}+\boldsymbol{\delta}_\phi\boldsymbol{\delta}_\phi\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$
$$+\boldsymbol{\delta}_\phi\boldsymbol{\delta}_\phi\frac{v_r}{r}+\boldsymbol{\delta}_\phi\boldsymbol{\delta}_\phi\frac{v_\theta}{r}-\boldsymbol{\delta}_\phi\boldsymbol{\delta}_r\frac{v_\phi}{r}-\boldsymbol{\delta}_\phi\boldsymbol{\delta}_\theta\frac{v_\phi\cos\theta}{r\sin\theta}$$

$$=\boldsymbol{\delta}_r\boldsymbol{\delta}_r\frac{\partial v_r}{\partial r}+\boldsymbol{\delta}_r\boldsymbol{\delta}_\theta\frac{\partial v_\theta}{\partial r}+\boldsymbol{\delta}_r\boldsymbol{\delta}_\phi\frac{\partial v_\phi}{\partial r}$$
$$+\boldsymbol{\delta}_\theta\boldsymbol{\delta}_r\left(\frac{1}{r}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta}{r}\right)+\boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta\left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_\theta}{r}\right)+\boldsymbol{\delta}_\theta\boldsymbol{\delta}_\phi\frac{1}{r}\frac{\partial v_\phi}{\partial\theta}$$
$$+\boldsymbol{\delta}_\phi\boldsymbol{\delta}_r\left(\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi}-\frac{v_\phi}{r}\right)+\boldsymbol{\delta}_\phi\boldsymbol{\delta}_\theta\left(\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}-\frac{v_\phi\cos\theta}{r\sin\theta}\right)$$
$$+\boldsymbol{\delta}_\phi\boldsymbol{\delta}_\phi\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}+\frac{v_r}{r}+\frac{v_\theta\cos\theta}{r\sin\theta}\right)$$

The components of this expression agree with Eqs. (S) to (AA) in Table A.7-3.

*d.* $[\nabla\cdot\boldsymbol{\tau}]$ *in cylindrical coordinates*

$$[\nabla\cdot\boldsymbol{\tau}]=\left[\left\{\boldsymbol{\delta}_r\frac{\partial}{\partial r}+\boldsymbol{\delta}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}+\boldsymbol{\delta}_z\frac{\partial}{\partial z}\right\}\cdot\left\{\boldsymbol{\delta}_r\boldsymbol{\delta}_r\tau_{rr}+\boldsymbol{\delta}_r\boldsymbol{\delta}_\theta\tau_{r\theta}+\cdots\right\}\right]$$

$$=\boldsymbol{\delta}_r\left(\frac{\partial\tau_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\tau_{\theta r}}{\partial\theta}+\frac{\partial\tau_{zr}}{\partial z}\right)$$
$$+\boldsymbol{\delta}_\theta\left(\frac{\partial\tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\tau_{\theta\theta}}{\partial\theta}+\frac{\partial\tau_{z\theta}}{\partial z}\right)$$
$$+\boldsymbol{\delta}_z\left(\frac{\partial\tau_{rz}}{\partial r}+\frac{1}{r}\frac{\partial\tau_{\theta z}}{\partial\theta}+\frac{\partial\tau_{zz}}{\partial z}\right)$$
$$+\frac{\tau_{rr}}{r}\left[\boldsymbol{\delta}_\theta\frac{\partial}{\partial\theta}\cdot\boldsymbol{\delta}_r\boldsymbol{\delta}_r\right]+\frac{\tau_{r\theta}}{r}\left[\boldsymbol{\delta}_\theta\frac{\partial}{\partial\theta}\cdot\boldsymbol{\delta}_r\boldsymbol{\delta}_\theta\right]+\frac{\tau_{rz}}{r}\left[\boldsymbol{\delta}_\theta\frac{\partial}{\partial\theta}\cdot\boldsymbol{\delta}_r\boldsymbol{\delta}_z\right]$$
$$+\frac{\tau_{\theta r}}{r}\left[\boldsymbol{\delta}_\theta\frac{\partial}{\partial\theta}\cdot\boldsymbol{\delta}_\theta\boldsymbol{\delta}_r\right]+\frac{\tau_{\theta\theta}}{r}\left[\boldsymbol{\delta}_\theta\frac{\partial}{\partial\theta}\cdot\boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta\right]+\frac{\tau_{\theta z}}{r}\left[\boldsymbol{\delta}_\theta\frac{\partial}{\partial\theta}\cdot\boldsymbol{\delta}_\theta\boldsymbol{\delta}_z\right]$$
$$+\frac{\tau_{zr}}{r}\left[\boldsymbol{\delta}_\theta\frac{\partial}{\partial\theta}\cdot\boldsymbol{\delta}_z\boldsymbol{\delta}_r\right]+\frac{\tau_{z\theta}}{r}\left[\boldsymbol{\delta}_\theta\frac{\partial}{\partial\theta}\cdot\boldsymbol{\delta}_z\boldsymbol{\delta}_\theta\right]+\frac{\tau_{zz}}{r}\left[\boldsymbol{\delta}_\theta\frac{\partial}{\partial\theta}\cdot\boldsymbol{\delta}_z\boldsymbol{\delta}_z\right]$$

That is, we first differentiate the components of the tensor, and then we indicate the differentiation of the unit dyads. Next we actually perform the differentiation of the unit dyads:

$$
\begin{aligned}
[\nabla \cdot \boldsymbol{\tau}] = \boldsymbol{\delta}_r &\left( \frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta r}}{\partial \theta} + \frac{\partial \tau_{zr}}{\partial z} \right) \\
&+ \boldsymbol{\delta}_\theta \left( \frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{z\theta}}{\partial z} \right) \\
&+ \boldsymbol{\delta}_z \left( \frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} \right) \\
&\quad + \boldsymbol{\delta}_r \frac{\tau_{rr}}{r} + \boldsymbol{\delta}_\theta \frac{\tau_{r\theta}}{r} + \boldsymbol{\delta}_z \frac{\tau_{rz}}{r} + \boldsymbol{\delta}_\theta \frac{\tau_{\theta r}}{r}
\end{aligned}
$$

Next we collect the terms that belong to each unit vector to get:

$$
\begin{aligned}
[\nabla \cdot \boldsymbol{\tau}] = \boldsymbol{\delta}_r &\left( \frac{\partial \tau_{rr}}{\partial r} + \frac{\tau_{rr}}{r} + \frac{1}{r}\frac{\partial \tau_{\theta r}}{\partial \theta} + \frac{\partial \tau_{zr}}{\partial z} \right) \\
&+ \boldsymbol{\delta}_\theta \left( \frac{\partial \tau_{r\theta}}{\partial r} + \frac{\tau_{r\theta}}{r} + \frac{\tau_{\theta r}}{r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{z\theta}}{\partial z} \right) \\
&+ \boldsymbol{\delta}_z \left( \frac{\partial \tau_{rz}}{\partial r} + \frac{\tau_{rz}}{r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} \right)
\end{aligned}
$$

The components of this vector can now be put into the form shown in Eqs. (J), (K), and (L) of Table A.7-2.

**Appendix A: p. 839, Ex. 3**

*a*. From Eq. (A) with **v** replaced by $\rho\mathbf{v}$

$$(\nabla\cdot\rho\mathbf{v})=\frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r)+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho v_\theta)+\frac{\partial}{\partial z}(\rho v_z)$$

*b*. From Eq. (J) with $\boldsymbol{\tau}$ replaced by $\rho\mathbf{vv}$

$$[\nabla\cdot\rho\mathbf{vv}]_r=\frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r^2)+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho v_\theta v_r)+\frac{\partial}{\partial z}(\rho v_z v_r)-\frac{\rho v_\theta^2}{r}$$

*c*. From Eq. (J) with $\boldsymbol{\tau}$ replaced by $p\boldsymbol{\delta}$

$$[\nabla\cdot p\boldsymbol{\delta}]_r=\frac{1}{r}\frac{\partial}{\partial r}(rp)-\frac{1}{r}=\frac{\partial p}{\partial r}\qquad\text{(see also Eq. A.4-26)}$$

*d*. From Eq (A) with **v** replaced by $[\boldsymbol{\tau}\cdot\mathbf{v}]$

$$\begin{aligned}(\nabla\cdot[\boldsymbol{\tau}\cdot\mathbf{v}])=&\frac{1}{r}\frac{\partial}{\partial r}\left(r[\tau_{rr}v_r+\tau_{r\theta}v_\theta+\tau_{rz}v_z]\right)\\&+\frac{1}{r}\frac{\partial}{\partial\theta}[\tau_{\theta r}v_r+\tau_{\theta\theta}v_\theta+\tau_{\theta z}v_z]\\&+\frac{\partial}{\partial z}[\tau_{zr}v_r+\tau_{z\theta}v_\theta+\tau_{zz}v_z]\end{aligned}$$

*e*. From Eq. (Q) with **w** replaced by **v**

$$[\mathbf{v}\cdot\nabla\mathbf{v}]_\theta=v_r\left(\frac{\partial v_\theta}{\partial r}\right)+v_\theta\left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r}{r}\right)+v_z\left(\frac{\partial v_z}{\partial z}\right)$$

*f*. From Eqs. (S) to (AA)

$$\left[\nabla\mathbf{v}+(\nabla\mathbf{v})^\dagger\right]_{rr}=2\left(\frac{\partial v_r}{\partial r}\right)$$

$$\left[\nabla\mathbf{v}+(\nabla\mathbf{v})^\dagger\right]_{\theta\theta}=2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r}{r}\right)$$

$$\left[\nabla \mathbf{v} + (\nabla \mathbf{v})^{\dagger}\right]_{zz} = 2\left(\frac{\partial v_z}{\partial z}\right)$$

$$\left[\nabla \mathbf{v} + (\nabla \mathbf{v})^{\dagger}\right]_{r\theta} = \left[\nabla \mathbf{v} + (\nabla \mathbf{v})^{\dagger}\right]_{\theta r} = \left(\frac{\partial v_\theta}{\partial r} + \frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_r}{r}\right)$$

$$\left[\nabla \mathbf{v} + (\nabla \mathbf{v})^{\dagger}\right]_{\theta z} = \left[\nabla \mathbf{v} + (\nabla \mathbf{v})^{\dagger}\right]_{z\theta} = \left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right)$$

$$\left[\nabla \mathbf{v} + (\nabla \mathbf{v})^{\dagger}\right]_{zr} = \left[\nabla \mathbf{v} + (\nabla \mathbf{v})^{\dagger}\right]_{rz} = \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)$$

**Appendix A: p. 839, Ex. 4**

*a*. We develop the Laplace operator from the definition of the del-operator as follows:

$$
\begin{aligned}
(\nabla\cdot\nabla) &= \left(\boldsymbol{\delta}_r\frac{\partial}{\partial r}+\boldsymbol{\delta}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}+\boldsymbol{\delta}_z\frac{\partial}{\partial z}\right)\cdot\left(\boldsymbol{\delta}_r\frac{\partial}{\partial r}+\boldsymbol{\delta}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}+\boldsymbol{\delta}_z\frac{\partial}{\partial z}\right)\\
&= \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}
\end{aligned}
$$

The second term in the second line arises from differentiating $\boldsymbol{\delta}_r$ with respect to $\theta$. The result agrees with Eq. A.7-36.

Next we apply this operator to the vector $\mathbf{v}$ thus:

$$
\nabla^2\mathbf{v}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\mathbf{v}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\mathbf{v}+\frac{\partial^2}{\partial z^2}\mathbf{v}
$$

To do this, we operate on the three terms seriatim:

$$
\begin{aligned}
&\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}[\boldsymbol{\delta}_r v_r+\boldsymbol{\delta}_\theta v_\theta+\boldsymbol{\delta}_z v_z]\right)\\
&\qquad=\boldsymbol{\delta}_r\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right)+\boldsymbol{\delta}_\theta\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right)+\boldsymbol{\delta}_z\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}[\boldsymbol{\delta}_r v_r+\boldsymbol{\delta}_\theta v_\theta+\boldsymbol{\delta}_z v_z]\\
&\qquad=\frac{1}{r^2}\frac{\partial}{\partial\theta}\left[\boldsymbol{\delta}_\theta v_r+\boldsymbol{\delta}_r\frac{\partial v_r}{\partial\theta}-\boldsymbol{\delta}_r v_\theta+\boldsymbol{\delta}_\theta\frac{\partial v_\theta}{\partial\theta}+\boldsymbol{\delta}_z\frac{\partial v_z}{\partial\theta}\right]\\
&\qquad=\frac{1}{r^2}\left[-\boldsymbol{\delta}_r v_r+\boldsymbol{\delta}_\theta\frac{\partial v_r}{\partial\theta}+\boldsymbol{\delta}_\theta\frac{\partial v_r}{\partial\theta}+\boldsymbol{\delta}_r\frac{\partial^2 v_r}{\partial\theta^2}-\boldsymbol{\delta}_\theta v_\theta-\boldsymbol{\delta}_r\frac{\partial v_\theta}{\partial\theta}\right.\\
&\qquad\qquad\left.-\boldsymbol{\delta}_r\frac{\partial v_\theta}{\partial\theta}+\boldsymbol{\delta}_\theta\frac{\partial^2 v_\theta}{\partial\theta^2}+\boldsymbol{\delta}_z\frac{\partial^2 v_z}{\partial\theta^2}\right]\\
&\qquad\left[\boldsymbol{\delta}_r\left(-v_r+\frac{\partial^2 v_r}{\partial\theta^2}-2\frac{\partial v_\theta}{\partial\theta}\right)+\boldsymbol{\delta}_\theta\left(-v_\theta+2\frac{\partial v_r}{\partial\theta}+\frac{\partial^2 v_\theta}{\partial\theta^2}\right)\right.
\end{aligned}
$$

$$+\boldsymbol{\delta}_z \frac{\partial^2 v_z}{\partial \theta^2}\Bigg]$$

$$\frac{\partial^2}{\partial z^2}\left[\boldsymbol{\delta}_r v_r + \boldsymbol{\delta}_\theta v_\theta + \boldsymbol{\delta}_z v_z\right] = \boldsymbol{\delta}_r \frac{\partial^2 v_r}{\partial z^2} + \boldsymbol{\delta}_\theta \frac{\partial^2 v_\theta}{\partial z^2} + \boldsymbol{\delta}_z \frac{\partial^2 v_z}{\partial z^2}$$

Next we collect all the terms of the three components of $\nabla^2\mathbf{v}$:

$$\left[\nabla^2\mathbf{v}\right]_r = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) + \frac{1}{r^2}\left(-v_r + \frac{\partial^2 v_r}{\partial \theta^2} - 2\frac{\partial v_\theta}{\partial \theta}\right) + \frac{\partial^2 v_r}{\partial z^2}$$
$$= \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}$$

$$\left[\nabla^2\mathbf{v}\right]_\theta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\left(-v_\theta + 2\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial \theta^2}\right) + \frac{\partial^2 v_\theta}{\partial z^2}$$
$$= \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}$$

$$\left[\nabla^2\mathbf{v}\right]_z = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}$$

These results should be compared with Eqs. (M), (N), and (O) of Table A.7-2.

*b*. Use Eqs. (J), (K), and (L) together with Eqs. (S) to (AA) for the components of the velocity gradient tensor to get the same results as given above in (*a*).

*c*. Use gradient operations of Eqs. (D), (E), and (F), the divergence operation of Eq. (A), and the curl operations of Eqs. (G), (H), and (I) together with Eq. A.7-37 to get the results in (*a*).