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Q) Qn 1 $\quad P = 4$

$Q \rightarrow$ Backward

using undetermined coefficients, we have

$$f''''(x_i) = a_0 f(x_{i-4}) + a_1 f(x_{i-3}) + a_2 f(x_{i-2}) + a_3 f(x_{i-1}) + a_4 f(x_i)$$

∴ we will write each f_i by taylor series expansion, to later equate coefficients,

$h = \Delta x$

$$f(x_{i-1}) = f(x_i) - h f'(\xi) + \frac{h^2}{2!} f''(\xi) - \frac{h^3}{3!} f'''(\xi) + \frac{h^4}{4!} f''''(\xi)$$

$$f(x_{i-2}) = f(x_i) - 2h f'(\xi) + \frac{4h^2}{2!} f''(\xi) - \frac{8h^3}{3!} f'''(\xi) + \frac{16h^4}{4!} f''''(\xi)$$

$$f(x_{i-3}) = f(x_i) - 3h f'(\xi) + \frac{9h^2}{2} f''(\xi) - \frac{27}{3!} f'''(\xi) + \frac{81}{4!} f''''(\xi)$$

$$f(x_{i-4}) = f(x_i) - 4h f'(\xi) + \frac{16h^2}{2} f''(\xi) - \frac{64}{3!} f'''(\xi) + \frac{256}{4!} f''''(\xi)$$

Now, we'll use,

$$f''''(x) = a_0 f(x_i) + a_1 f(x_{i-1}) + a_2 f(x_{i-2}) + a_3 f(x_{i-3}) + a_4 f(x_{i-4})$$

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Substituting values of $f(x_i)$ computed so far;

$a_0 + a_1 + a_2 + a_3 + a_4 = 0$; — (1)

$a_1 + 2a_2 + 3a_3 + 4a_4 = 0$; — (2)

$a_1 + 4a_2 + 9a_3 + 16a_4 = 0$; — (3)

$a_1 + 8a_2 + 27a_3 + 64a_4 = 0$; — (4)

While → $a_1 + 16a_2 + 81a_3 + 256a_4 = \frac{4!}{h^4}$

Assuming $h = 1$ since we've to incorporate it into $\frac{1}{h^4}$ in each term [for sake of simplicity]

$a_0 = 1$; $a_1 = -4$; $a_2 = 6$; $a_3 = -4$ & $a_4 = 1$

$$\boxed{\therefore f''''(x_i) = 1 \times f(x_{i-4}) + (-4) f(x_{i-3}) + (6) f(x_{i-2}) + (-4) f(x_{i-1}) + 1 f(x_i)}$$

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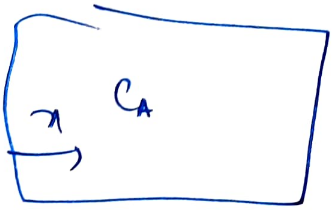
Ans Q (A) $\dfrac{d^2 C_A}{dx^2} - \dfrac{K}{D} C_A = 0$

clearly, $\dfrac{d^2 C_A}{dx^2} = \dfrac{K}{D} C_A$



where $K/D > 0$ therefore $\quad x=0,\ C_A = 1 \quad\quad x=1,\ \dfrac{dC_A}{dx} = 0$

$\therefore C_A(x) = C_1 \cosh\left(\sqrt{\dfrac{K}{D}}\, x\right) + C_2 \sinh\left(\sqrt{\dfrac{K}{D}}\, x\right)$

using boundary cond^n's to find C_1 & C_2;

$\therefore C_A(x=0) = 1$ & $\left.\dfrac{dC_A}{dx}\right|_{x=1} = 0$

$\dfrac{dC_A}{dx} = C_1\sqrt{\dfrac{K}{D}} \sinh\left(\sqrt{\dfrac{K}{D}}\, x\right) + C_2\sqrt{\dfrac{K}{D}} \cosh\left(\sqrt{\dfrac{K}{D}}\, x\right)$

$\therefore \dfrac{dC_A}{dx} = \sqrt{\dfrac{K}{D}}\left(C_1 \sinh\sqrt{\dfrac{K}{D}}\, x\right) + C_2 \cosh\left(\sqrt{\dfrac{K}{D}}\, x\right)$

Now, $C_A(x=0) = 1$, implies

$1 = C_1 \cdot 1 + C_2 \cdot 0 \rightarrow C_1 = 1$

$\therefore \left.\dfrac{dC_A}{dx}\right|_{x=0} = 0$ implies

$\sqrt{\dfrac{K}{D}}\,(0 + C_2 \cdot 1) = 0$

$\therefore C_2 = 0$

So, $C_1 = 1$ & $C_2 = 0$, $\therefore C_A(x) = \cosh\left(\sqrt{\dfrac{K}{D}}\, x\right)$

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By definition, let $\sqrt{\frac{k}{D}} := n$; so,

$C_A(x) = \cos^{-1}(nx)$ i.e. $x \in [0,1]$

The factor η

$\therefore$ $\eta := \dfrac{\int_0^1 k\, C_A\, dx}{k\, C_{A_S}}$, $C_{A_S} = 1$, k is constant

$\therefore$ $\eta = \int_0^1 C_A(x)\, dx$

so, $\eta = \int_0^1 \cos^{-1}(nx)\, dx$, $n \to$ constant positive

$= \left[\frac{1}{n} \sin^{-1}(nx) \right]_0^1 = \frac{1}{n} \left[\sin^{-1}(nx)\right]_0^1$

$= \frac{1}{n} [\sin^{-1}(n) - 0] = \dfrac{\sin^{-1}(n)}{n}$

so, $\boxed{\eta = \dfrac{\sin^{-1}(n)}{n}}$

effectiveness factor

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Assn 2

$$\frac{d^2c}{dt^2} = 0 \qquad c\,(\text{at } x = 0) = M$$

$$\frac{dc}{dx}\,(\text{at } x = 1) = P$$

Analytical Ans

$$c(x) = \frac{(Me^{-1} + P)\cdot e^{x} + (Me^{1} - P)\,e^{-x}}{e^{1} + e^{-1}}$$

$$\boxed{M = 1 \;;\; P = 1}$$

(A) Central Diff Scheme

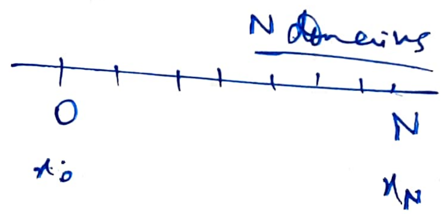
CDS

$$\boxed{\frac{-d^2u}{dx^2} = f} \qquad c(0) = 1, \quad \frac{dc}{dx}(x=1) = 1$$

$$\Rightarrow -\frac{(u_{i-1} - 2u_i + u_{i+1})}{\Delta x^2} \quad \text{[In general]}$$

$$\boxed{\frac{u_{n+1} - u_{n-1}}{\Delta x} = 1} \quad (\text{at } x = 1)$$

while



$$-\left(\frac{u_{N-1} - 2u_N + u_{N+1}}{\Delta x^2}\right) = f_N$$

$\therefore$ u_{N+1} gets eliminated.

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$\boxed{f = -c}$ Here,

$$\frac{d^2\mu}{dx^2} = \frac{d^2 c}{dx^2}$$

$\therefore$ $$-\frac{\left(\mu_{N-1} - 2\mu_N + \Delta x + \Delta\mu_{N-1}\right)}{\Delta x^2} = f_N$$

$$-\frac{\left(2\mu_{N-1} - 2\mu_N + \Delta x\right)}{\Delta x^2} = f_N$$

$\boxed{A\mu = F}$

$\therefore$ $$A = \frac{1}{\Delta x^2}\begin{bmatrix} -1 & 2 & & & & \\ & -1 & 2 & & & \\ & & -1 & 2 & & \\ & & & \ddots & & \\ & & & -1 & 2 & \\ & & & & -1 & 2 \end{bmatrix} \quad \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \end{bmatrix}$$

while $$F = \begin{bmatrix} -c_1 \\ -c_2 \\ \vdots \\ -c_N \end{bmatrix}$$

& we have,

μ_0 (at $x=0$)

$\Rightarrow$ c_1 (at $x=0$) $= 1$

Ans 3

RK_2 mid point

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$h = 1$ $\qquad$ tol $= 0.1$

we have

$$\frac{d^2c}{dx^2} = c \qquad c(\text{at } x=0) = 1 \longrightarrow \underline{BC\ 1}$$

$$\frac{dc}{dx}(\text{at } x=1) = 1 \longrightarrow \underline{BC\ 2}$$

BC 1 remains same

we'll assume some BC 2 i.e. $C_2 = 0$ (at $x=0$)

fix assumption

$C_1 = c$

$$\frac{dc_1}{dx} = c_2 \quad \& \quad \frac{dc_2}{dx} = \frac{d^2 c_1}{dx^2} = c_1$$

Pair of D.E. $\rightarrow$

$$\frac{d}{dx}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_2 \\ c_1 \end{bmatrix}$$

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4 Q No 4

[Old solution]

$$\frac{dy}{dt} = dy$$

$y(t=0) = P$

solve till $t = 5$

Given $d = 2$, $P = 2$

[A] Euler Explicit → $h = \left(\frac{M}{d}\right)$

(i) $M = 0.5$	(ii) $M = 1.25$	(iii) $M = 2.5$
$y(t=0) = 2$	$y(t=0) = 2$	$y(t=0) = 2$
till $t = 5$	till $t = 5$	till $t = 5$
$y' = 2y$	$y' = 2y$	$y' = 2y$
$h = \frac{0.5}{2}$	$h = \frac{1.25}{2}$	$h = \frac{2.5}{2}$

using iterative codes & hard wiring them into given problem ~~statements~~ for Euler Explicit Method.

$$y_{i+1} = y_i + h\frac{dy}{dt}(x_i, y_i)$$

we obtain →

$M = 0.5$ (i), $h = 0.25$	$y_{t=5} = \cancel{66\,50.51}$ $1.9e^{-6}$
$M = 1.25$ (ii), $h = 0.625$	$y_{t=5} = \cancel{1313.68}$ $3e^{-5}$
$M = 2.5$ (iii), $h = 1.25$	$y_{t=5} = \cancel{300.125}$ 10.125

Euler Explicit

OLD

$M = 0.5$ (i)	$y_{t=5} = \cancel{54701}$ 2.2 $7.3e^{-5}$
$M = 1.25$ (ii)	$y_{t=5} = \cancel{248669}$ $9.6e^{-5}$
$M = 2.5$ (iii)	$y_{t=5} = \cancel{13\,122}$ 0.00304832

Crank Nicholson Method

Values Output Printed in CODE

[B] Crank Nicholson method with ~~same~~ step sizes.

$$y_{i+1} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_{i+1}, y_{i+1})\right]$$

Similarly, by iterative code;

correction (done in main code)

$$y_{i+1} = y_i + \frac{h}{2}\left[-2y_i + -2y_{i+1}\right]$$

$$y_{i+1} = y_i + h(-y_i + -y_{i+1})$$

$$y_{i+1}(1+h) = (1-h)y_i$$

$$y_{i+1} = \left(\frac{1-h}{1+h}\right)y_i$$ → substitute into code

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Q2D h = 0.25

x	explicit	crank
0	2	2
0.25	3	3.33
0.5	4.5	5.55
0.75	6.75	9.25
1	10.125	15.43
1.25	15.1875	25.72
1.5	22.78	42.86
1.75	34.17	71.44
2	51.25	119.07
2.25	76.88	198.45
2.5	115.33	330.76
2.75	172.99	551.27
3	259.49	918.78
3.25	389.23	1531.31
3.5	583.85	2552.19
3.75	875.78	4253.64
4	1313.68	7089.41
4.25	1970.52	11815.7
4.5	2955.78	19692.8
4.75	4433.68	32821.33
5	6650.51	54702.2

h = 0.625

x	explicit	crank
0	2	2
0.625	4.5	8.66
1.25	10.12	37.55
1.875	22.78	162.74
2.5	51.25	705.21
3.125	115.33	3055.91
3.75	259.49	13242.3
4.375	583.85	57383.2
5	1313.68	248660

h = 1.25

x	explicit	crank
0	2	2
1.25	7	-18
2.5	24.5	162
3.75	85.75	-1458
5	300.125	13122

Ans 4 (c)

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Since, we've been given

$\lambda > 0$ in DECAY equation,

So it turns out to be exponential equation instead;

∴ we observe that Crank exceeds at a much faster than euler explicit

This happens due to ~~eul~~ crank incorporating y_{i+1} inside y', besides standard contribution of y_i into both the standard optimum functions of these both.

Also, it being exponential instead of decay;

∴ crank grows at much higher rate instead of matching more accurately with true value than euler explicit method

Due to this situation of the step size > 1 in case (iii);

∴ the scale factor $\frac{1+h}{1-h}$ shall be one due to $h > 1$;

∴ crank method dissolves in this case.

(iii) is incorrect, ∵ $h > 1$ ∴ that's taking wrong values for both explicit & crank

However in (i), (ii) ⟶ explicit grows slowly than crank, due to contribution of y_i & y_{i+1} both in y' of crank.

UPDATED code for Q.4 (as per changed de condition)

↳ uploaded into Qn as well.

Analysis → situation inverses completely and crank gives better results than euler, matching with true value more accurately.

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Ans 3 –

$f(x) = \ln(Px)$

fit a cubic polynomial

(a) Lagrangian Interpolation

step size $= 0.5$

$P = 5$

$\therefore f(x) = \ln(5x)$
$x_1 = 1$
$x_2 = 3$ $\Big] \rightarrow$ function

and $x = 2 \rightarrow$ required to be calculated $\rightarrow \boxed{f(2)}$

$\therefore$ table comes out to be ;

x	1	1.5	2	2.5	3
$f(x)$	1.6094	2.0149	2.3026	2.5257	2.7081

Lagrange Interpolating Polynomial

value of x at which $f(x) = \boxed{x=2}$

$$\therefore f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} \times f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} \times f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} \times f(x_2)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} \times f(x_3)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} \times f(x_4)$$

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$f(x)$ for $x = 2$

$$f(2) = \frac{(2-1.5)(2-2)(2-2.5)(2-3)}{(1-1.5)(1-2)(1-2.5)(1-3)} \times 1.6094 + \frac{(2-1)(2-2)(2-2.5)(2-3)}{(1.5-1)(1.5-2)(1.5-2.5)(1.5-3)} \times 2.0149$$

$$+ \frac{(2-1)(2-1.5)(2-2.5)(2-3)}{(2-1)(2-1.5)(2-2.5)(2-3)} \times 2.3026$$

$$+ \frac{(2-1)(2-1.5)(2-2)(2-3)}{(\cancel{2}-1)(\cancel{2}-1.5)(\cancel{2}-2)(\cancel{2}-2.5)} \times 2.5257$$

$$+ \frac{(2-1)(2-1.5)(2-2)(2-2.5)}{(3-1)(3-1.5)(3-2)(3-2.5)} \times 2.7081$$

$$\rightarrow f(2) = \frac{0.5 \times 0 \times -0.5 \times -1}{-0.5 \times (-1) \times -1.5 \times (-2)} \times 1.6094 + \frac{1 \times 0 \times (-0.5) \times (-1)}{(0.5)(-0.5)(-1)(5)} \times 2.0149$$

$$+ \frac{1 \times 0.5 \times (-0.5) \times (-1)}{1 \times 0.5 \times 0.5 \times (-1)} \times 2.3026 + \frac{1 \times (0.5) \times 0 \times (-1)}{(1.5)(1)(0.5)(-0.5)} \times 2.5257$$

$$+ \frac{1 \times 0.5 \times 0 \times (0.5)}{2 \times 1.5 \times (1) \times 0.5} \times 2.7081$$

$$f(2) = \frac{0}{1.5} \times 1.6094 + \frac{0}{-0.375} \times 2.0149 + \frac{0.25}{0.25} \times 2.3026$$

$$+ \frac{0}{-0.375} \times 2.527 + \frac{0}{1.5} \times 2.7081$$

$$\boxed{f(2) = 2.3026}$$

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Solⁿ 5

(A)

Lagrange therefore are

$$f_3(x) = y_1 P_1(x) + y_2 P_2(x) + y_3 P_3(x) + y_4 P_4(x)$$

$\therefore (x_1, y_1) = (1, 0)$

$(x_2, y_2) = (1.5, 2.014)$

$(x_3, y_3) = (2, 2.3025)$

$(x_4, y_4) = (2.5, 2.525)$

$\{ y = \ln(Px)$
$P = 5 \}$

Now, $x = 2$

$P_1(x) = 0$
$P_2(x) = 0$
$P_3(x) = 1$
$P_4 = 0$

$\rightarrow \therefore f_3(x) = -2.305(4)(x^3 - 5x^2 + 7.75x - 3.75)$

So at $x = 2$

$$\boxed{\begin{aligned} f_3(2) &= -2.305 \times 4 \times 1/4 \times (-1) \\ &= 2.305 = f_3(2) \end{aligned}}$$

Cubic Interpolating Polynomial [Newton's divided difference Interpolation]

$$f_n(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$

i	x_i	$f(x_i)$	$f[x_1, x_0]$	$f[x_2, x_1, x_0]$	$f[x_3, x_2, x_1, x_0]$
0	x_0	$f(x_0)$	$f[x_2, x_1]$	$f[x_3, x_2, x_1]$	
1	x_1	$f(x_1)$	$f[x_3, x_2]$		
2	x_2	$f(x_2)$			
3	x_3	$f(x_3)$			

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i	x_i	$f(x_i)$	First	Second	Third
0	1	0	4.028	-3.451	2.2126
1	1.5	2.013	0.577	-0.132	
2	2	2.3025	0.445		
3	2.5	2.525			

using $f[x_i, x_j] = \dfrac{f(x_i) - f(x_j)}{x_i - x_j}$ (notes)

while $f[x_i, x_j, x_k] = \dfrac{f[x_i, x_j] - f[x_j, x_k]}{x_i - x_k}$

$\therefore$ $b_0 = f(x_0) = 0$

$b_1 = f[x_1, x_0] = 4.028$

$b_2 = f[x_2, x_1, x_0] = -3.451$

$b_3 = f[x_3, x_2, x_1, x_0] = 2.2126$

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Ans 6 $I = a_1 f(x_1) + a_2 f(x_2) + a_3 f(x_3)$

True $\rightarrow \int_{-1}^{1} f(x)\,dx$ $\quad x_1, x_2, x_3 \in \{-1, 1\}$

Numerical $\rightarrow a_1 f(x_1) + a_2 f(x_2) + a_3 f(x_3)$

$f(x) = 1 \quad \int_{-1}^{1} 1\,dx = x\Big|_{-1}^{1} = 2 \quad a_1 \cdot 1 + a_2 \cdot 1 + a_3 \cdot 1$ - ①

$f(x) = x \quad \int_{-1}^{1} x\,dx = \frac{x^2}{2}\Big|_{-1}^{1} = 0 \quad a_1 u_1 + a_2 u_2 + a_3 u_3$ - ②

$f(x) = x^2 \quad \int_{-1}^{1} x^2\,dx = \frac{x^3}{3}\Big|_{-1}^{1} = 2/3 \quad a_1 u_1^2 + a_2 u_2^2 + a_3 u_3^2$ ③

$f(x) = x^3 \quad \int_{-1}^{1} x^3\,dx = \frac{x^4}{4}\Big|_{-1}^{1} = 0 \quad a_1 u_1^3 + a_2 u_2^3 + a_3 u_3^3$ - ④

$f(x) = x^4 \quad \int_{-1}^{1} x^4\,dx = x^5/5\Big|_{-1}^{1} = 2/5 \quad a_1 u_1^4 + a_2 u_2^4 + a_3 u_3^4$ - ⑤

$f(x) = x^5 \quad \int_{-1}^{1} x^5\,dx = x^6/6\Big|_{-1}^{1} = 0 \quad a_1 u_1^5 + a_2 u_2^5 + a_3 u_3^5$ - ⑥

we have ⑥ variables & ⑥ equation

$u_1^2 \times$ ④ - ⑥

$a_2 u_2^3(u_1^2 - u_2^2) + a_3 u_3^3(u_1^2 - u_3^2) = 0$

$$\boxed{a_2 u_2^3 (u_1^2 - u_2^2) = -a_3 u_3^3 (u_1^2 - u_3^2)} \quad \text{- ⑧}$$

Also, $u_1^2 \times$ ② - ④

$\Rightarrow a_2 u_2 (u_1^2 - u_2^2) + a_3 u_3 (u_1^2 - u_3^2) = 0$

$$\boxed{a_2 u_2 (u_1^2 - u_2^2) = -a_3 u_3 (u_1^2 - u_3^2)} \quad \text{- ⑦}$$

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Eq^n (7) $\times u_2^2$ - Eq^n (8)

$0 = -a_3 u_3 u_2^2 (u_1^2 - u_3^2) + a_3 u_3^3 (u_1^2 - u_3^2)$

$\Rightarrow a_3 u_3 (u_1^2 - u_3^2)(-u_2^2 + u_3^2) = 0$

Since, we are taking 3 symmetric points around 0;

$\therefore$ $u_2 = 0$

$\Rightarrow a_3 u_3^3 (u_1^2 - u_3^2) = 0$

$\Rightarrow \boxed{u_1 = \pm u_3}$

Eq^n's turn out to be $\rightarrow$

$a_1 + a_2 + a_3 = 2$ — (α) — (1)

$a_1 u_1 - a_3 u_1 = 0$ — (β) — (2)

$(a_1 - a_3) u_1 = 0$

$a_1 = a_3$ — (γ) — (3)

In (α)-(1), $2a_1 + a_2 = 2$

In (γ)-(3) $a_1 u_1^2 + a_3 u_3^2 = 2/3$

$2a_1 u_1^2 = 2/3$

$\Rightarrow \boxed{a_1 u_1^2 = 1/3}$ — (10)

In (5)

$\cancel{2} a_1 u_1^4 = \frac{\cancel{2}1}{5}$ — (11)

$\therefore$ Eq^n (10) $\times u_1^2$ - Eq^n (11)

$\therefore$ $\frac{1}{3} u_1^2 - \frac{1}{5} = 0$

$\therefore$ $u_1^2 = 3/5$ $\rightarrow \boxed{u_1 = \sqrt{\frac{3}{5}}} \Rightarrow \boxed{u_2 = -\sqrt{\frac{3}{5}}}$

$\boxed{\begin{array}{l} \therefore u_3 = \pm u_1 \\ u_3 = -\sqrt{3/5} \end{array}}$