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Q1 Method of undetermined coefficients

$P = 3^{rd}$

$$f'''(x_i) = a_1 f(x_i) + a_2 f(x_{i-1}) + a_3 f(x_{i-2}) + a_4 f(x_{i-3}) + a_5 f(x_{i-4})$$

On expanding each term using Taylor series.

$$f(x_{i-1}) = f(x_i) - \Delta x f'(x_i) + \frac{\Delta x^2}{2!} f''(x_i) - \frac{\Delta x^3}{3!} f'''(x_i) + \frac{\Delta x^4}{4!} f''''(x_i)$$

$$f(x_{i-2}) = f(x_i) - 2\Delta x f'(x_i) + \frac{4\Delta x^2}{2!} f''(x_i) - \frac{8\Delta x^3}{3!} f'''(x_i) + \frac{16\Delta x^4}{4!} f''''(x_i)$$

$$f(x_{i-3}) = f(x_i) - 3\Delta x f'(x_i) + \frac{9\Delta x^2}{2!} f''(x_i) - \frac{27\Delta x^3}{3!} f'''(x_i) + \frac{81}{4!} \Delta x^4 f''''(x_i)$$

$$f(x_{i-4}) = f(x_i) - 4\Delta x f'(x_i) + \frac{16\Delta x^2}{2!} f''(x_i) - \frac{64\Delta x^3}{3!} f'''(x_i) + \frac{256}{4!} \Delta x^4 f''''(x_i)$$

If these eq^n are substituted in main eq^n & compared.

then

$$a_1 + a_2 + a_3 + a_4 + a_5 = 0 \quad (1)$$

$$a_2 + 2a_3 + 3a_4 + 4a_5 = 0 \quad (2)$$

$$a_2 + 4a_3 + 9a_4 + 16a_5 = 0 \quad (3)$$

$$-\frac{\Delta x^3}{3!}(a_2 + 8a_3 + 27a_4 + 64a_5) = 1$$

$$\Rightarrow a_2 + 8a_3 + 27a_4 + 64a_5 = -\frac{6}{\Delta x^3} \quad (4)$$

$$a_2 + 16a_3 + 81a_4 + 256a_5 = 0 \quad (5)$$

Q2

$$\frac{d^2C}{dx^2} - C = 0$$

B.C $C(x=0) = 1$, $\frac{dC}{dx}(x=1) = 1$.

$x_0 \quad x_1 \quad x_2 \quad x_3$

$x_0 = 0$

$x_3 = 1$.

$\Delta x = \frac{1}{3}$

Using CDS

$$-\left(\frac{C_{i-1} - 2C_i + C_{i+1}}{\Delta x^2}\right) = -C_i$$

C_3 can be calculated using a fictitious point C_4

$$\frac{C_4 - C_2}{2\Delta x} = 1 \quad [\text{central diff. at } C_3]$$

So, $C_4 = C_2 + 2\Delta x$ (1)

Applying double derivative formula at C_3.

$$-\left(\frac{C_2 - 2C_3 + C_4}{\Delta x^2}\right) = -C_N$$

Putting (1) here.

$$\Rightarrow \boxed{\frac{-C_2 + C_3}{\Delta x^2} = -\frac{C_N}{2} + \frac{1}{\Delta x}}$$

x_1 can be calculated using double derivative formula

$$-\left(\frac{C_0 - 2C_1 + C_2}{\Delta x^2}\right) = -C_1$$

Putting $C_0 = 1$.

$$\Rightarrow -\left(\frac{1 - 2C_1 + C_2}{\Delta x^2}\right) = -C_1$$

$$\Rightarrow \boxed{\frac{2C_1 - C_2}{\Delta x^2} = -C_1 + \frac{1}{(\Delta x)^2}}$$

A system of linear equations can be established to solve them.

$$A\,C = f$$

$$A = \frac{1}{\Delta x^2}\begin{bmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$$

$$f = \begin{bmatrix} -C_1 + \frac{1}{(\Delta x)^2} \\ -C_2 \\ -\frac{C_3}{2} + \frac{1}{\Delta x} \end{bmatrix}$$

Q3 $$\frac{d^2C}{dx^2} = C$$

let $y_1 = C$

$$y_2 = \frac{dy_1}{dx}$$

$$\cancel{\frac{dy_2}{dx}} \Rightarrow \cdot \frac{d}{dx}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ y_1 \end{bmatrix}$$

$C(0) = 1, \quad y_2(1) = 1$

let $y_2(0) = 0$ [Initial guess]

Tried to solve on excel - "Shooting Method".

Q4

$$\frac{dy}{dt} + 3y = 0$$

$$y(t=0) = 3$$

$f(t,y) = -3y$

$t_0 = 0, \ y_0 = 3$

True solution

$\frac{dy}{dt} = -3y$

$\ln y = -3t + C$

At $t=0, \ y=3$

$\Rightarrow \ln 3 = C$

$\Rightarrow \ln y = -3t + \ln 3$

$\Rightarrow \ln \frac{y}{3} = -3t$

$\boxed{y = 3e^{-3t}}$

Step size, $h = \frac{M}{\lambda}$

A.

(i) $M = 0.5$

$h = \frac{0.5}{3} = 0.1667$

Formula for Euler's Explicit Method

$$y_{i+1} = y_i + h f(t_i, y_i)$$

i goes from 0 to $\frac{5}{h} = \frac{5}{\frac{0.5}{3}} = 30$

Formula for Crank Nicholson Method

$$y_{i+1} = y_i + \frac{h}{2}\left(f(t_i, y_i) + f(t_{i+1}, y_{i+1})\right)$$

$$\Rightarrow y_{i+1} = y_i + \frac{h}{2}\left(f(t_i, y_i) - 3y_{i+1}\right)$$

$$\Rightarrow y_{i+1} + \frac{3}{2}h y_{i+1} = y_i + \frac{hf}{2}(t_i, y_i)$$

$$\Rightarrow y_{i+1}\left(\frac{3}{2}h + 1\right) = y_i + \frac{hf(t_i, y_i)}{2}$$

$$\Rightarrow y_{i+1} = \frac{y_i + \frac{hf}{2}(t_i, y_i)}{1 + \frac{3}{2}h}$$

i goes from 0 to $\frac{5}{h}$

Data for each time step & for both methods is calculated using code " ODE - IVP"

(Data is printed on screen)

(c) Nicholson Method gives more accurate solutions.

* As h increases, error in each value also increases and that too by both methods.

*

Q5 → Please Refer to the code - "curveFitting.cpp"

$$f(x) = \ln(P^* x)$$
$$\Rightarrow f(x) = \ln(3x)$$

$$\Delta x = 0.3$$

Considering equally spaced data points as - (Generated from code)

(0.5, 0.405465)
(0.8, 0.875469)
(1.1, 1.19392)
(1.4, 1.43508)

(a) For Lagrange's Polynomial

$$f_{n-1}(x) = y_1 P_1(x) + y_2 P_2(x) + \dots + y_n P_n(x)$$

For $n = 4$ (No. of data points)

$$f_3(x) = y_1 P_1(x) + y_2 P_2(x) + y_3 P_3(x) + y_4 P_4(x)$$

$$P_1(2) = \frac{\cancel{(2-x_1)}(2-x_2)(2-x_3)(2-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)}$$

x_1, x_2, x_3, x_4 all are abscissa's of data points

Similarly other $P(x)$ are calculated

$$P_i(x) = \prod_{\substack{j=1 \\ i \neq j}}^{4} \frac{(x - x_j)}{(x_i - x_j)} \qquad \boxed{x = 2}$$

So, $f_3(2) = 1.98257$ (From code)

Newton's Divided Difference

$$f_3(x) = b_0 + b_1(x-x_1) + b_2(x-x_1)(x-x_2) + b_3(x-x_1)(x-x_2)(x-x_3)$$

where $b_0 = y_1$

& others are calculated by divided differences

$b_1 = f[x_2, x_1]$

$b_2 = f[x_3, x_2, x_1]$

$b_3 = f[x_4, x_3, x_2, x_1]$

Result obtained from code :

$b0 = 0.405465$

$b1 = 1.56668$

$b2 = -0.841944$

$b3 = 0.458384$

& $f_3(2) = 1.98257$