Q1. $\frac{dN}{dt} = 3t^2$ $\quad (P=7 \dots (19+9+6=34=7=3+4))$

→ 1^{st} Method : $h=0.1$ (Raltson).

$N|_{t=0.5} = ?$

$y_R(i+1) = y(i) + h\phi[y(i), x(i)]$ .... (Raltson Method) — (i)

$\because f(t,N) = \frac{dN}{dt} = 3t^2$

$\therefore h = 0.1$

Using (i) we modify it we get,

$y_R(i+1) = y(i) + h[\omega_1 k_1 + \omega_2 k_2]$; $\omega_1 = 1/3$; $\omega_2 = 2/3$

$k_1 = f(t,N)$

$= y(i) + \frac{1}{10}\left[\frac{k_1}{3} + \frac{2k_2}{3}\right]$ $\qquad k_2 = f\left(t + \frac{3}{4}hk_1, N + \frac{3}{4}h\right)$

$= y(i) + \frac{1}{30} \times [k_1 + 2k_2]$

$= y(i) + \frac{1}{30} \times \left[3t^2 + 2 \times 3 \times \left(t + \frac{3}{40} \times k_1\right)^2\right]$

$$\boxed{y_R(i+1) = y(i) + \frac{1}{\cancel{30}_{10}} \times \left[\cancel{3}t^2 + \cancel{6}^{2}\left(t + \frac{3}{40} \times 3t^2\right)^2\right] = y(i) + \frac{1}{10}\left[t^2 + 2\left(t + \frac{9t^2}{40}\right)^2\right]}$$

here $y(i) = N(i)$ & $N_R(i+1) = y_R(i+1)$

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$$\boxed{N_R(i+1) = N_R(i) + \frac{1}{10}\left[t_i^2 + 2\left(t_i + \frac{9t_i^2}{40}\right)^2\right]}$$ → Iteration formula

$t = 0$ to 0.5

Assuming $N_R|_{t=0} = 0$

$N_R(1) = 0 + \frac{1}{10} \times 0 = 0$

$N_R(2) = 0 + \frac{1}{10} \times \left[(0.1)^2 + 2\left(0.1 + \frac{9 \times 0.1^2}{40}\right)^2\right]$

$= 3.091 \times 10^{-3}$

$$N_R(3) = N_R(2) + \frac{1}{10} \times \left[(0.2)^2 + 2\left(0.2 + \frac{9}{40}(0.2)^2\right)^2\right] = 3.0910 \times 10^{-3} + 0.0127$$

$$= 0.015827$$

$$N_R(4) = 0.01587 + \frac{1}{10}\left[(0.3)^2 + 2\left(0.3 + \frac{9}{40}(0.3)^2\right)^2\right] = 0.05438$$

$$N_R(5) = 0.05438 + \frac{1}{10}\left[(0.4)^2 + 2\times\left(0.4 + \frac{9}{40}(0.4)^2\right)^2\right] =$$

$$N_R(6) = \quad + \frac{1}{10}\left[(0.5)^2 + 2\times\left(0.5 + \frac{9}{40}(0.5)^2\right)^2\right] =$$

2nd Method (Richardson's $h = 0.1$):-

$$\overline{y}_{p+1} = N_{p+1} = N_{p+1}\left(h/_2\right) + \frac{N_{p+1} - N_{p+1}(h/_2)}{2^{n-1} - 1} \rightarrow \text{Iteration formula}$$

3rd Method (Adaptive step size, $h = 0.1$) $\varepsilon_{tol} = 10^{-3}$

$$\frac{dN}{dt} = 3t^2 \; ; \; f(t) = -3t^2 \; ; \; \lambda =$$

$$f(N) = f(N_0) + \frac{\partial f}{\partial N_{N_0}}(N - N_0)$$

$$= 0 +$$

Q. $y(0)=1$

$$\frac{dy(x)}{dx} = -\lambda Y(x) + (1+\lambda)\cos(x) - (1-\lambda)\sin(x) = f(x,y)$$

$h = \pi/4$

∴ 1st Parameter ; $\lambda = 22/\pi$

$\longrightarrow$ $x = 0, \pi/4, \pi/2, 3\pi/4, \pi$; $y_0 = 1$ at $x=0$

$$\therefore y_{i+1} = y_i + hf(x_i, y_i) = y_i + \frac{\pi}{4}\left[-\frac{22}{\pi}\times y_i + \left(1+\frac{22}{\pi}\right)\cos(x_i) - \left(1-\frac{22}{\pi}\right)\sin(x_i)\right]$$

Euler Explicit Method

$$\therefore \boxed{y_{i+1} = y_i + \frac{1}{4}\left[-y_i\times 22 + (\pi+22)\cos(x_i) + (22-\pi)\sin(x_i)\right]}$$

$$y_1 = 1 + \frac{1}{4}\left[-22\times 1 + (\pi+22)\cos(0) + (22-\pi)\sin(0)\right]$$

$$\boxed{y_1 = 1 + \frac{1}{4}\left[-\cancel{22} + \pi + \cancel{22} + 0\right] = 1 + \pi/4 = 1.7853}$$

$$y_2 = (1+\pi/4) + \frac{1}{4}\times\left[-(1+\pi/4)22 + (\pi+22)\cos(\pi/4) + (22-\pi)\sin(\pi/4)\right]$$

$$= (1+\pi/4) + \frac{1}{4}\left[-22 - \frac{22\pi}{4} + \frac{\pi+22}{\sqrt{2}} + \frac{22-\pi}{\sqrt{2}}\right]$$

$$\boxed{y_2 = (1+\pi/4) + \frac{1}{4}\left[-22 - \frac{22\pi}{4} + \sqrt{2}\times 22\right] = -0.2562}$$

$$y_3 = -0.2562 + \frac{1}{4}\times\left[-22\times(-0.2562) + (\pi+22)\cos(\pi/2) + (22-\pi)\sin(\pi/2)\right]$$

$$= -0.2562 + \frac{1}{4}\times\left[5.6364 + 0 + 22-\pi\right]$$

$$\boxed{y_3 = 5.867}$$

$$y_4 = 5.86750 + \frac{1}{4}\left[-22\times(5.867) + (\pi+22)\cos(3\pi/4) + (22-\pi)\sin(3\pi/4)\right]$$

$$= \quad '' \quad + 1/4\left[-129.085 + (-17.77) + (13.33)\right]$$

$$\therefore \boxed{y_4 = -27.5142}$$

$$\therefore y_5 = y_4 + \frac{1}{4}\left[-22\times y_4 + (\pi+22)(-1) + (22-\pi)\times(0)\right]$$

$$\therefore \boxed{y_5 = y_4 + \frac{1}{4}\left[605.31 - 25.141\right] = 117.5279}$$

for Parameter 2 i.e $\lambda = 4/\pi$

$$y_{i+1} = y_i + \frac{\pi}{4}\left[-\frac{4}{\pi}y_i + (\pi+4)\cos(x_i) + (4-\pi)\sin(x_i)\right]$$

$y_1 = 0 + \ldots$

$y_2 =$

$y_3 =$

$y_4 =$

$y_5 =$

Similarly
We calculate
for this
all with
$\lambda = 4/\pi$

the conclusion comes that the distribution $Y(x)$ numerically from $x=0$, $x=\pi$; $h=\pi/4$
as the values of $Y(x)$ does match with the analytical results as they diverge overly high from True value (As seen in Code output)

Q4. $P=7$, Derivative $=3$
$n=5$, Central finite Difference
$f(x)=\sin(x)$

$$f'''(x_i)=a_1 f(x_{i+2})+a_2 f(x_{i+1})+a_3 f(x_i)+a_4 f(x_{i-1})+a_5 f(x_{i-2}) \quad —(1)$$

$$f(x_{i+2})=\left[f(x_i)+2\Delta x f'(x_i)+\frac{(2\Delta x)^2}{2!}f''(x_i)+\frac{(2\Delta x)^3}{3!}f'''(x_i)\right]+\frac{(2\Delta x)^4}{3!\times 4}f^{iv}(x_i)$$

$$f(x_{i+1})=f(x_i)+\Delta x f'(x_i)+\frac{\Delta x^2}{2!}\times f''(x_i)+\frac{(\Delta x)^3}{3!}f'''(x_i)+\frac{(\Delta x)^4}{4!}\times f^{iv}(x_i)$$

$$f(x_{i-1})=f(x_i)-\Delta x f'(x_i)+\frac{\Delta x^2}{2!}\times f''(x_i)-\frac{(\Delta x)^3}{3!}\times f'''(x_i)+\frac{(\Delta x)^4}{4!}\times f^{iv}(x_i)$$

$$f(x_{i-2})=f(x_i)-2\Delta x f'(x_i)+\frac{(-2\Delta x)^2}{2!}\times f''(x_i)+\frac{(-2\Delta x)^3}{3!}\times f'''(x_i)+\frac{(2\Delta x)^4}{4!}\times f^{iv}(x_i)$$

In (1) substituting these are Expansions:

$$f'''(x_i)=f(x_i)\left[a_1+a_2+a_3+a_4+a_5\right]+f'(x_i)\left[2a_1+a_2-a_4-2a_5\right]\Delta x+\frac{f''(x_i)\Delta x^2}{2!}\left[4a_1+a_2+a_4+4a_5\right]+\frac{f'''(x_i)}{3!}\left[8a_1+a_2-a_4-8a_5\right]\times\Delta x^3$$

$$+ f^{iv} \frac{\times \Delta x^4}{4!} \left[16a_1 + a_2 + a_4 + 16a_5\right] + \dots$$

$\therefore a_1 + a_2 + a_3 + a_4 + a_5 = 0$

$\therefore 2a_1 + a_2 - a_4 - 2a_5 = 0$

$\therefore 4a_1 + a_2 + a_4 + 4a_5 = 0$

$\therefore 8a_1 + a_2 - a_4 - 8a_5 = \frac{6}{\Delta x^3}$

$\therefore 16a_1 + a_2 + a_4 + 16a_5 = 0$

$\rightarrow$ $\boxed{a_1 = -a_5}$

$-3a_5 + a_5 = \frac{6}{\Delta x^3}$

$\boxed{+a_5 = \frac{-3}{\Delta x^3}}$

$a_1 = \frac{3}{\Delta x^3}$ $\quad a_2 = -2a_1$

$a_2 = -\frac{6}{\Delta x^3}$

$\therefore 2\left(\frac{3}{\Delta x^3}\right) + \left(-\frac{6}{\Delta x^3}\right) - a_4 - 2 \times \left(-\frac{3}{\Delta x^3}\right) = 0$

$\boxed{a_4 = \frac{12}{\Delta x^3} - \frac{6}{\Delta x^3} = \frac{6}{\Delta x^3}} \longrightarrow a_3 = -\cancel{a_1} - a_2 - a_4 - \cancel{a_5}$

$= \cancel{+\frac{6}{\Delta x^3}} - \cancel{\frac{6}{\Delta x^3}}$

$\boxed{a_3 = 0}$

$\therefore$

$$f'''(x_p) = \frac{3}{\Delta x^3} f(x_{p+2}) - \frac{6}{\Delta x^3} f(x_{p+1}) + 0 + \frac{6}{\Delta x^3} f(x_{p-1}) - \frac{3}{\Delta x^3} f(x_{p-2})$$

$$\therefore \boxed{f'''(x_p) = \frac{3f(x_{p+2}) - 6f(x_{p+1}) + 6f(x_{p-1}) - 3f(x_{p-2})}{\Delta x^3}} \rightarrow \text{put } f(x_p) = \sin(x_p$$

$$\text{error} = f^{iv} \frac{\times \Delta x^4}{4!} \times \left[\cancel{16a_1} + \cancel{a_2} + \cancel{a_4} + \cancel{16a_5}\right] + f^{v} \frac{\Delta x^5}{5!} \left[32a_1 + a_2 - a_4 - 32a_5\right]$$

$$= f^{v}(x_i)\frac{\times \Delta x^5}{5!}\times\left[64a_1 + 2a_2\right]$$

$$= f^{v}(x_i)\times\frac{\cancel{\Delta x^5}^{\Delta x^2}}{5!}\times\frac{1\times 2}{\cancel{\Delta x^3}}\times\left[32\times 3 + (-6)\right]$$

$$= f^{v}(x_i)\times\frac{2\Delta x^2}{5!}\times\left[96-6\right] = f^{v}(x_i)\frac{\times\cancel{2}\Delta x^2\times \cancel{90}^{\cancel{30}^{\cancel{15}^{3}}}}{\cancel{2}\times\cancel{3}\times\cancel{4}^{2}\times\cancel{5}}$$

$$= \frac{3}{2}\times f^{v}(x_i)\Delta x^2$$

$\therefore$ Truncation error $= \left|\frac{3}{2}\Delta x^2 f^{v}(x_i)\right|$

$\therefore$ Round off error $\simeq \left|\frac{\varepsilon_p}{\Delta x^3}\right|$

$\therefore$ Total error $\simeq \frac{3}{2}\Delta x^2 f^{v}(x_i) + \frac{\varepsilon_p}{\Delta x^3} = E(\Delta x)$

$$\frac{dE(\Delta x)}{d\Delta x} = \cancel{3}\Delta x f^{v}(x_i) - \frac{\cancel{3}\varepsilon_p}{\Delta x^4} = 0$$

$$\therefore \Delta x f^{v}(x_i) = \frac{\varepsilon_p}{\Delta x^4}$$

$\therefore$ ~~$\Delta x \propto \varepsilon_p^{1/5}$~~

$$\boxed{\Delta x = \left(\frac{2\times 10^{-16}}{\underbrace{\sin(x_i)^{v}}_{5^{th}\text{ derivative}}}\right)^{1/5} = \left(\frac{2\times 10^{-16}}{\cos(x_i)}\right)^{1/5}}$$

$\therefore$ Hence from here we can see the Δx which we found is numerically same with the Δx for $\varepsilon_p = 2\times 10^{-16}$

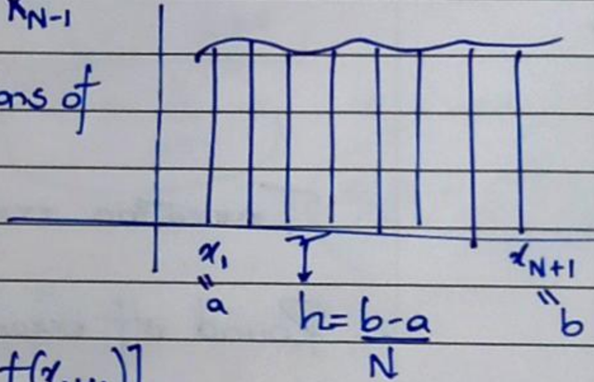
Q5   n = 1, 2, 3, 4 I_{true} ∴ $\boxed{\int_1^2 (x+1/x)^2\,dx = \frac{16\times 2}{6} - \frac{3}{6} = \frac{29}{6} = 4.833}$

$\int_1^2 \underbrace{(x+1/x)^2}_{y}\,dx$ .... 1/3rd simpson Rule

for repeated use of simpson 1/3rd Rule we have

$$I = \int_{x_1}^{x_{N+1}} y\,dx = \int_{x_1}^{x_3} y\,dx + \int_{x_3}^{x_5} y\,dx + \dots + \int_{x_{N-1}}^{x_{N+1}} y\,dx$$

We get I on deriving for N divisions of the area as



$$I = \frac{(b-a)}{3N}\left[f(x_1) + 4\sum_{i=2,4,6}^{N} f(x_i) + 2\sum_{i=1,3,5}^{N-1} f(x_i) + f(x_{N+1})\right]$$

$$\text{error} = -\frac{(b-a)^5}{180N^4} \times \bar{f}^{iv} \longrightarrow \text{for } N \quad \text{---} (*)$$

$f(x) = (x + 1/x)^2$

$f'(x) = 2(x + 1/x)\left[1 - 1/x^2\right] = 2\left(\frac{x^2+1}{x}\right) \times \left(\frac{x^2-1}{x^2}\right) = \frac{2}{x^3}(x^4 - 1)$

$= 2x - 2/x^3$

$f''(x) = 2 + 3\times 2/x^4$

$f'''(x) = -4!/x^5$

$f''''(x) = f^{iv} = 5!/x^6$

$$\bar{f}^{iv}(x) = \frac{\int_1^2 5!/x^6\,dx}{1} = \frac{5!}{-5} \times \left(\frac{1}{2^5} - \frac{1}{1^5}\right)$$

$= +4!\left(1 - 1/32\right)$

$= 24\left(\frac{31}{32}\right)$

∴ for $n=1$,

$$\alpha = \frac{x-x_1}{h} \; ; \; dx = h\,d\alpha \; ; \; h = (b-a)/2$$

$$x : x_1 \longrightarrow x_2 \qquad \int_{x_1}^{x_2} y\,dx \longrightarrow \int_1^2 yh\,d\alpha$$

$$\alpha : 1 \longrightarrow 2$$

$$E(\alpha) = \int_1^2 \frac{h^2 f'(\xi)}{2!}\,\alpha(1)\,d\alpha + \int_1^2 \frac{h^3}{3!} \times f''(\xi)\,\alpha(\alpha-1)\,d\alpha$$

$$= \frac{h^2}{2!}\int_1^2 f'(\xi) \times \alpha\,d\alpha + \frac{h^3}{3!} \times \int_1^2 f''(\xi)\left(\alpha^2-\alpha\right)d\alpha$$

$$= \frac{h^2}{2!} \times f'(\xi) \times \left[\frac{\alpha^2}{2}\right]_1^2 + \frac{h^3 \times f''(\xi)}{3!} \times \left[\frac{\alpha^3}{3} - \frac{\alpha^2}{2}\right]_1^2$$

$$= \frac{h^2}{2!} \times f'(\xi) \times \frac{1}{2} \times [4-1] + \frac{h^3 \times f''(\xi)}{3!} \times \left[\frac{8}{3} - \frac{4}{2} - \left(\frac{1}{3} - \frac{1}{2}\right)\right]$$

$$\boxed{E(\;) = \frac{3h^2}{4} \times f'(\xi) + \frac{h^3}{6} \times f''(\xi)\left[\frac{2}{3} - \frac{1}{3} + \frac{1}{2}\right] = \frac{3h^2}{4} \times f'(\xi) + \frac{5}{36}h^3 f''(\xi)}$$

for $n=2$

$$\boxed{E(\alpha) = \int_1^2 \frac{h^3}{3!} \times f''(\xi)\,\alpha(\alpha-1)\,d\alpha + \int_1^2 \frac{h^4}{4!} \times f'''(\xi)\,\alpha(\alpha-1)(\alpha-2)\,d\alpha}$$

$$E = \frac{h^3}{3!} \times f''(\xi) \times \left[\frac{5}{6}\right] + \frac{h^4}{4!} \times f'''(\xi)\,[\text{Const}]$$

for $n=3$

$E(\alpha) = \ldots$ similarly we will calculate for $n=4$

definitely with increasing N the error decreases with order N^4 (for any N) $= \frac{-(1)^5}{180 \times N^4} \times \frac{24 \times 31}{32}$

n=1 estimated error $= \frac{0.1291}{4.833}$ $= 0.0267 \times 100\%$ $= \left| -\left(\frac{0.1291}{N^4}\right) \right|$

estimate

n=2 estimated error $= \frac{0.1291}{4.833} \times \frac{1}{(2)^4} = 1.67 \times 10^{-3} \times 100\%$

n=3 " $= 0.0267 \times \frac{1}{(3)^4} = 3.2962 \times 10^{-4} \times 100\%$

n=4 " $= 0.0267 \times \frac{1}{(4)^4} = 1.042 \times 10^{-4} \times 100\%$

$\therefore$ We can calculate the Actual error likely in same way.