CLL110, Transport Phenomena

Lecture 1

(25th July, Thursday)

Vector and Tensor Analysis

Introduction

Transport Phenomena is the subject which deals with the movement of different physical quantities in any chemical or mechanical process and describes the basic principles and laws of transport. It also describes the relations and similarities among different types of transport that may occur in any system. Transport in a chemical or mechanical process can be classified into three types:

1. **Momentum transport** deals with the transport of momentum in fluids and is also known as fluid dynamics
2. **Energy transport** deals with the transport of different forms of energy in a system and is also known as heat transfer
3. **Mass transport** deals with the transport of various chemical species themselves

Three different types of physical quantities are used in transport phenomena: scalars (e.g. temperature, pressure and concentration), vectors (e.g. velocity, momentum and force) and second order tensors (e.g. stress or momentum flux and velocity gradient). It is essential to have a primary knowledge of the mathematical operations of scalar, vector and tensor quantities for solving the problems of transport phenomena.

Vectors : In contrast, consider the velocity of a particle or element of fluid; to describe it fully, we need to specify both its magnitude (in some suitable units) and its instantaneous spatial direction. Other examples are momentum, heat flux, and mass flux. These quantities are described by vectors. In books, vectors are printed in boldface. In ordinary writing, we may represent a vector in different ways.

Gibbs notation: $\bar{v},\ \vec{v},\ \underline{v},\ \underset{\sim}{v}$

Index notation: v_i

In this course, we will be using the following notations for scalar, vector and tensor quantities:

Notation:	a, b, c	scalar quantities
	$\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}$	vector quantities
	$\underset{\sim}{\tau}, \underset{\sim}{\sigma}$	2nd order tensor quantities

We represent the vectorial quantity with a symbol, we often know it only via its components in some basis set. Note that the vector as an entity has an **invariant** identity independent of the basis set in which we choose to represent it.

Basis Sets

The most common basis set in three-dimensional space is the orthogonal triad $\left(\underline{i}, \underline{j}, \underline{k}\right)$ corresponding to a rectangular Cartesian coordinate system. $\underline{i}$ stands for a unit vector in the *x*-direction and $\underline{j}, \underline{k}$ represent unit vectors in *y, z* directions respectively. Note that this is not a unique basis set. The directions of $\underline{i}, \underline{j}, \underline{k}$ depend on our choice of the coordinate directions.

There is no reason for the basis set to be composed of orthogonal vectors. The only requirement is that the three vectors chosen do not lie in a plane. Orthogonal sets are the most convenient, however.

In fact, the use of the indicial notation which we will be studying in the next few lectures will enable us to express the long formulae encountered in transport phenomena in a concise and compact fashion

Tensor quantities

Most of us might have already encountered scalars and vectors in the study of high-school physics. The essential difference between these two, it was pointed out, was that the vectors also have a direction associated with them along with a magnitude, whereas scalars only have a magnitude but no direction. Extending this definition, we can loosely define a 2nd order tensor as a physical quantity which has a magnitude and two different directions associated with it. To better understand why we might need two different directions for specifying a particular physical quantity, let us take the example of the stresses which may arise in a solid body, or a fluid. Clearly, the stresses are associated with a force, as well as with an area, whose direction is specified by the outward normal to the face on which it is acting. Hence, we will require 3^2, i.e. 9 components to specify a stress completely. In general, an nth order tensor will be specified by 3^n components (in a 3-dimensional system[1]). However, the number of components alone cannot determine whether a physical quantity is a vector or a tensor. The additional requirement is that there should be a linear transformation rule for obtaining the corresponding tensors if we rotate the coordinate system about the origin. Thus, tensor quantities can be defined by two essential conditions:

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[1] In an m-dimensional system, an nth order tensor will have m^n components. In transport phenomena, we deal with 3D systems in general, and hence m=3.

1. These quantities should have 3^n components. According to this definition, scalar quantities are zero order tensors and have $3^0 = 1$ component. Vector quantities are first order tensors and have $3^1 = 3$ components. Second order tensors have $3^2 = 9$ components and third order tensors have $3^3 = 27$ components. Third and higher order tensors are not used in transport phenomena, and are hence not dealt with here
2. The second necessary requirement of any tensor quantity is that it should follow some transformation rule for changing coordinate systems which will discussed later

Kronecker delta & Alternating Unit Tensor

There are two particular tensors which are quite useful in conveniently and concisely expressing several mathematical operations on tensors. These are the Kronecker delta and the Alternating Unit Tensor.

Kronecker delta ($\boldsymbol{\delta_{ij}}$)

Kronecker delta or kronecker's delta is a function of two variables, usually integer, which is 1 if they are equal and 0 otherwise.
It is expressed as a symbol δ_{ij}
$\delta_{ij} = 1$, if $i = j$
$\delta_{ij} = 0$ if $i \neq j$

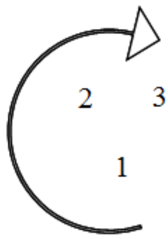
Thus, in three dimensions, we may also express the Kronecker delta as

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

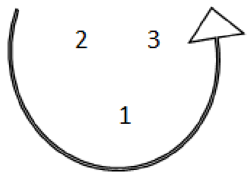
Alternating Unit Tensor ($\boldsymbol{\varepsilon_{ijk}}$) (also known as Levi Civita symbol or Permutation Symbol)

The alternating unit tensor ε_{ijk} is useful when expressing certain results in a compact form in index notation. Since it has three indices, it has 3 x 3 x 3 possible combinations. The value of these combinations are as follows

- $\varepsilon_{ijk} = 0$ if any two of indices *i, j, k* are equal. For example $\varepsilon_{113}, \varepsilon_{131}, \varepsilon_{111}, \varepsilon_{222} = 0$
- $\varepsilon_{ijk} = +1$ when the indices *i, j, k* are different and form an even permutation of (123), i.e. are in cyclic order. For example ε_{123}



- $\varepsilon_{ijk} = -1$ when the indices *i, j, k* are different and form odd permutation of (123), i.e. are in acyclic order. For example ε_{213}



Free indices and Dummy indices

Free indices

Free indices are the indices which occur only once in each tensor term. For example, *i* is the free index in following tensor term

$v_{ij} w_j$

In any tensorial expression, every term should have an equal number of free indices. For example, $v_{ij} w_j = c_j d_j$ is not a valid tensorial expression since the number of free indices is not equal in both terms.

Any free indices in a tensorial expression can be replaced by any other symbol as long as this symbol does not already occur in the expression. For example, $A_{ij} B_j = C_i D_j E_j$ is equivalent to $A_{kj} B_j = C_k D_j E_j$. The number of free indices in an indicial equation gives the actual number of mathematical equations that will arise from it.

Dummy indices

Dummy indices are the indices that occur twice in the tensor terms. For example, j is the dummy index in $A_{ij} B_j$.

Any dummy index implies the summation of all components of that tensor term associated with each coordinate axis. Thus, when we write $A_i \delta_i$, we actually imply $\sum_{i=1}^{3} A_i \delta_i$.

Any dummy index in a tensor term can be replaced by any other symbol as long as this symbol does not already occur in that term. For example, $A_{ijk} \delta_j \delta_k = A_{ipq} \delta_p \delta_q$

Note: The dummy indices can be renamed in each term separately, but free indices should be renamed for all terms in the tensor expression. For example, $A_{ij} B_j = C_i D_j E_j$ can be replaced by $A_{kp} B_p = C_k D_j E_j$

Here, i is the free index which should be replaced by k in both terms but j is a dummy index so it can be replaced in only one term by p.

Summation convention in vector and tensor analysis

According to the Summation convention rule, if k is a dummy index which repeats itself in the expression then there should be a summation sign with it. Therefore, we can eliminate the implied summation sign and can write the expression in a more compact way. For example, using the Summation convention

$\sum_k \sum_j \varepsilon_{ijk} \varepsilon_{ljk}$ can be simply written as $\varepsilon_{ijk} \varepsilon_{ljk}$

Lecture 2 (29th July, Monday)

Rules of the summation convention

I. Omit all summation signs.

II. If a suffix appears twice, a summation is implied, e.g. $a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$

Here i is a dummy or repeated index.

III. If a suffix appears only once it can take any value e.g. $a_i = b_i$ holds for i = 1; 2; 3.

Here i is a free index. Note that there may be more than one free index.

Always check that the free indices match on both sides of an equation.

IV. A given suffix must not appear more than twice in any term in an expression.

Always check that there aren't more than two identical indices e.g. $a_i b_i c_i$ is simply WRONG in our indicial notation.

(Note: For it appears it has to be explicitly written with a summation sign i.e. $\sum_{i=1}^{3} a_i b_i c_i$

This is not our indicial notation)

Multiplication of two vectors

Any two vectors can be multiplied in three different ways: (a) dyadic product, (b) dot product, and (c) cross product.

Dyadic Product of two vectors

The dyadic product is a mathematical operation on two vectors, which does not change the order of the resultant quantity. Since the order of the two vectors is one each, the order of tensor of the resultant is 2. Thus, the dyadic product of two vectors gives a second order tensor. To mathematically denote the dyadic product, we simply write the two vectors next to each other without any sign in between.

Example:

$$\underset{\sim}{\delta}_i \frac{\partial}{\partial x_i}\left(v_j \underset{\sim}{\delta}_j\right) \tag{2.1}$$

Here, $\underset{\sim}{v} = v_j \underset{\sim}{\delta}_j$ is velocity, i.e. a vector quantity and $\underset{\sim}{\delta}_i \frac{\partial}{\partial x_i}$ is the gradient operator, also a vector quantity. Hence, the resultant $\frac{\partial v_j}{\partial x_i} \underset{\sim}{\delta}_i \underset{\sim}{\delta}_j$ (which physically represents the velocity gradient) is a second order tensor quantity.

Scalar product or dot product of two vectors

The dot product is a mathematical operation on two vectors, which reduces the order of tensor of the resultant quantity by two. Hence, dot product of two vectors has zero order, i.e. it is a scalar quantity. Mathematically, the dot product is defined as

$$\underset{\sim}{v}.\underset{\sim}{w} = vw\, cos(\phi_{vw})$$

where v and w denote the respective magnitudes of the two vectors, and ϕ_{vw} denotes the angle formed between the two vectors.

Vector product or cross product of two vectors

The cross product is a mathematical operation on two vectors, which reduces the order of tensor of the resultant quantity by one. Hence, cross product of two vectors has order unity, i.e. it is a vector quantity. Mathematically, the cross product is defined as

$$\underset{\sim}{v}\times\underset{\sim}{w} = v_i\underset{\sim}{\delta}_i \times w_j\underset{\sim}{\delta}_j = v_i w_j \underset{\sim}{\delta}_i \times \underset{\sim}{\delta}_j = vw\, sin(\phi_{vw})\underset{\sim}{n}_{vw} \qquad (2.2)$$

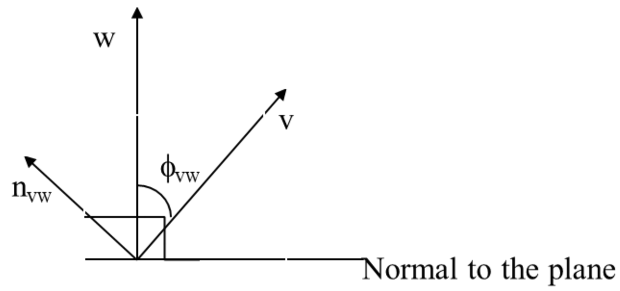


Figure 2.1: Cross product of vector v and w

Here, v and w denote the respective magnitudes of the two vectors, ϕ_{vw} denotes the angle formed between the two vectors, and $\underset{\sim}{n}_{vw}$ is a unit vector which is normal to $\underset{\sim}{v}$ and $\underset{\sim}{w}$.

Vector operation in components

Dot product of two unit vectors

If $\underset{\sim}{\delta}_1$, $\underset{\sim}{\delta}_2$ and $\underset{\sim}{\delta}_3$ are the three unit vectors along the axes in a rectangular coordinate system, then the dot product of these vectors has 9 possibilities

$$\underset{\sim}{\delta}_1.\underset{\sim}{\delta}_2,\ \underset{\sim}{\delta}_1.\underset{\sim}{\delta}_3,\ \underset{\sim}{\delta}_2.\underset{\sim}{\delta}_1,\ \underset{\sim}{\delta}_2.\underset{\sim}{\delta}_3,\ \underset{\sim}{\delta}_3.\underset{\sim}{\delta}_1,\ \underset{\sim}{\delta}_3.\underset{\sim}{\delta}_2 = 0$$

and $\underset{\sim}{\delta}_1.\underset{\sim}{\delta}_1,\ \underset{\sim}{\delta}_2.\underset{\sim}{\delta}_2,\ \underset{\sim}{\delta}_3.\underset{\sim}{\delta}_3 = 1$

Hence, all the nine terms can be written in cosine form by using the Kronecker delta. If i and j are the free indices, then

$\underset{\sim}{\delta}_i.\underset{\sim}{\delta}_j = \delta_{ij}$ because if $i = j$ then $\delta_{ij} = 1$;

$$i \neq j \text{ then } \delta_{ij} = 0$$

Notes:

I. Since there are two free indices *i* and *j*, $\underline{\delta_i} \cdot \underline{\delta_j} = \delta_{ij}$ is equivalent to 9 equations.

II. $\delta_{ij} = \delta_{ji}$. We say δ_{ij} is symmetric in its indices.

III. $\sum_{i=1}^{3} \delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$

IV. $\sum_{j=1}^{3} a_j\delta_{jk} = a_1\delta_{1k} + a_2\delta_{2k} + a_3\delta_{3k}$, Note that *k* is a free index.

If k = 1, then only the first term on RHS contributes and the RHS = a_1.

Similarly, if k =2 then the RHS =a_2 and if k = 3 the RHS = a_3. Hence

$$\sum_{j=1}^{3} a_j\delta_{jk} = a_k$$

Compaction rule

The repeated index is implicitly summed over, can be written as a compaction rule

$$[\ldots.]_j\delta_{jk} = [\ldots.]_k$$

Example

$$\delta_{ij}\delta_{jk} = \delta_{ik}$$

$$a_i\delta_{ij} = a_j$$

Cross product of two unit vectors

If $\underset{\sim}{\delta}_1$, $\underset{\sim}{\delta}_2$ and $\underset{\sim}{\delta}_3$ are the three unit vectors along the axes in a rectangular coordinate system, then the cross product of these vectors has 9 possibilities

$\underset{\sim}{\delta}_1 \times \underset{\sim}{\delta}_2 = \underset{\sim}{\delta}_3$ $\qquad$ $\underset{\sim}{\delta}_2 \times \underset{\sim}{\delta}_3 = \underset{\sim}{\delta}_1$ $\qquad$ $\underset{\sim}{\delta}_3 \times \underset{\sim}{\delta}_1 = \underset{\sim}{\delta}_2$ $\qquad$ (2.3)

If we change the order, the sign will change

$\underset{\sim}{\delta}_2 \times \underset{\sim}{\delta}_1 = -\underset{\sim}{\delta}_3$ $\qquad$ $\underset{\sim}{\delta}_3 \times \underset{\sim}{\delta}_1 = -\underset{\sim}{\delta}_1$ $\qquad$ $\underset{\sim}{\delta}_1 \times \underset{\sim}{\delta}_3 = -\underset{\sim}{\delta}_2$ $\qquad$ (2.4)

Since the angle between two same unit vectors is zero, $\underset{\sim}{\delta}_1 \times \underset{\sim}{\delta}_1 = \underset{\sim}{\delta}_2 \times \underset{\sim}{\delta}_2 = \underset{\sim}{\delta}_3 \times \underset{\sim}{\delta}_3 = 0$

Hence, all nine terms can be written in sine form by using alternating unit tensors.

$\underset{\sim}{\delta}_i \times \underset{\sim}{\delta}_j = \varepsilon_{ijk}\underset{\sim}{\delta}_k$ (9 equations)

Here, i and j are free indices and k is a dummy index.

Example:

$$\underset{\sim}{\delta}_1 \times \underset{\sim}{\delta}_2 = \varepsilon_{12k}\underset{\sim}{\delta}_k = \sum_{k=1}^{3} \varepsilon_{12k}\underset{\sim}{\delta}_k = \varepsilon_{121}\underset{\sim}{\delta}_1 + \varepsilon_{122}\underset{\sim}{\delta}_2 + \varepsilon_{123}\underset{\sim}{\delta}_3 = \delta_3 \text{ since, } e_{123} = 1, e_{121} = 0, e_{122} = 0 \quad (2.5)$$

$$\underset{\sim}{\delta}_1 \times \underset{\sim}{\delta}_1 = \varepsilon_{11k}\underset{\sim}{\delta}_k = \sum \varepsilon_{11k}\delta_k = 0 + 0 + 0 = 0 \tag{2.6}$$

It must be noted that the order of terms should not be changed as it might affect the final answer.

Example:

$$\underset{\sim}{v} \times \underset{\sim}{w} = v_i\underset{\sim}{\delta}_i \times w_j\underset{\sim}{\delta}_j$$

$$= v_i w_j \varepsilon_{ijk}\underset{\sim}{\delta}_k = \underset{\sim}{u} = u_k\underset{\sim}{\delta}_k \qquad \text{where, } u_k = v_i w_j \varepsilon_{ijk} \tag{2.7}$$

In the above equation, k is the free index, whereas i and j are the dummy indices.

If $k = 1$,

$$u_1 = v_i w_j \varepsilon_{ij1} = \sum_i^3 \sum_j^3 v_i w_j \varepsilon_{ij1} = v_2 w_3 \varepsilon_{231} + v_3 w_2 \varepsilon_{321} = v_2 w_3 - v_3 w_2 \text{ (all other terms are zero)}$$

Similarly, if $k = 2$,

$$u_2 = \sum_i \sum_j v_i w_j \varepsilon_{ij2} = -v_1 w_3 + w_1 v_3$$

For $k = 3$,

$$u_3 = \sum_i \sum_j v_i w_j \varepsilon_{ij3} = w_2 v_1 - v_2 w_1$$

Hence,

$$\underset{\sim}{v} \times \underset{\sim}{w} = (v_2 w_3 - v_3 w_2)\underset{\sim}{\delta}_1 + (w_1 v_3 - v_1 w_3)\underset{\sim}{\delta}_2 + (w_2 v_1 - v_2 w_1)\underset{\sim}{\delta}_3 \tag{2.8}$$

Note:

Symmetry of ε_{ijk} under cyclic permutations

$$\varepsilon_{ijk} = \varepsilon_{kji} = \varepsilon_{jki} = -\varepsilon_{jik} = -\varepsilon_{ikj} = -\varepsilon_{kji} \tag{2.9}$$

This holds for all values of i, j, k. Note that

If any two of the three indices i,j,k are the same, all term vanish

Equation (2.9) has three free indices so they represent 3^3 i.e 27 equations.

E.g. in $\varepsilon_{ijk} = \varepsilon_{kji}$, 3 equations says '1=1', 3 equations says '-1=-1' and 21 equations says 0=0

Mathematical operations on vectors

Addition of two vectors

$$\underset{\sim}{u} + \underset{\sim}{v} = u_i\underset{\sim}{\delta}_i + v_i\underset{\sim}{\delta}_i = \sum_{i=1}^3 (u_i + v_i)\underset{\sim}{\delta}_i \tag{2.10}$$

In the same way, subtraction of vectors is done as

$$\underset{\sim}{u} - \underset{\sim}{v} = u_i\underset{\sim}{\delta}_i - v_i\underset{\sim}{\delta}_i = \sum_{i=1}^3 (u_i - v_i)\underset{\sim}{\delta}_i \tag{2.11}$$

Lecture 2 (29th July, Monday)

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where v and w denote the respective magnitudes of the two vectors, and ϕ_{vw} denotes the angle formed between the two vectors.

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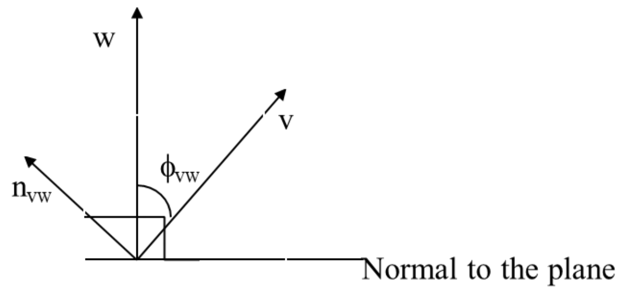

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$$\underset{\sim}{\delta_1}.\underset{\sim}{\delta_2},\ \underset{\sim}{\delta_1}.\underset{\sim}{\delta_3},\ \underset{\sim}{\delta_2}.\underset{\sim}{\delta_1},\ \underset{\sim}{\delta_2}.\underset{\sim}{\delta_3},\ \underset{\sim}{\delta_3}.\underset{\sim}{\delta_1},\ \underset{\sim}{\delta_3}.\underset{\sim}{\delta_2} = 0$$

and $\underset{\sim}{\delta_1}.\underset{\sim}{\delta_1},\ \underset{\sim}{\delta_2}.\underset{\sim}{\delta_2},\ \underset{\sim}{\delta_3}.\underset{\sim}{\delta_3} = 1$

Hence, all the nine terms can be written in cosine form by using the Kronecker delta. If i and j are the free indices, then

$\underset{\sim}{\delta_i}.\underset{\sim}{\delta_j} = \delta_{ij}$ because if $i = j$ then $\delta_{ij} = 1$;

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Notes:

I. Since there are two free indices *i* and *j*, $\underline{\delta_i} \cdot \underline{\delta_j} = \delta_{ij}$ is equivalent to 9 equations.

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IV. $\sum_{j=1}^{3} a_j \delta_{jk} = a_1\delta_{1k} + a_2\delta_{2k} + a_3\delta_{3k}$, Note that *k* is a free index.
If k = 1, then only the first term on RHS contributes and the RHS = a_1.
Similarly, if k =2 then the RHS =a_2 and if k = 3 the RHS = a_3. Hence

$$\sum_{j=1}^{3} a_j \delta_{jk} = a_k$$

Compaction rule

The repeated index is implicitly summed over, can be written as a compaction rule

$$[\ldots.]_j \delta_{jk} = [\ldots.]_k$$

Example

$$\delta_{ij}\delta_{jk} = \delta_{ik}$$

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Cross product of two unit vectors

If $\underset{\sim}{\delta}_1$, $\underset{\sim}{\delta}_2$ and $\underset{\sim}{\delta}_3$ are the three unit vectors along the axes in a rectangular coordinate system, then the cross product of these vectors has 9 possibilities

$$\underset{\sim}{\delta}_1 \times \underset{\sim}{\delta}_2 = \underset{\sim}{\delta}_3 \qquad \underset{\sim}{\delta}_2 \times \underset{\sim}{\delta}_3 = \underset{\sim}{\delta}_1 \qquad \underset{\sim}{\delta}_3 \times \underset{\sim}{\delta}_1 = \underset{\sim}{\delta}_2 \qquad (2.3)$$

If we change the order, the sign will change

$$\underset{\sim}{\delta}_2 \times \underset{\sim}{\delta}_1 = -\underset{\sim}{\delta}_3 \qquad \underset{\sim}{\delta}_3 \times \underset{\sim}{\delta}_1 = -\underset{\sim}{\delta}_1 \qquad \underset{\sim}{\delta}_1 \times \underset{\sim}{\delta}_3 = -\underset{\sim}{\delta}_2 \qquad (2.4)$$

Since the angle between two same unit vectors is zero, $\underset{\sim}{\delta}_1 \times \underset{\sim}{\delta}_1 = \underset{\sim}{\delta}_2 \times \underset{\sim}{\delta}_2 = \underset{\sim}{\delta}_3 \times \underset{\sim}{\delta}_3 = 0$

Hence, all nine terms can be written in sine form by using alternating unit tensors.

$\underset{\sim}{\delta}_i \times \underset{\sim}{\delta}_j = \varepsilon_{ijk} \underset{\sim}{\delta}_k$ (9 equations)

Here, i and j are free indices and k is a dummy index.

Example:

$$\underset{\sim}{\delta}_1 \times \underset{\sim}{\delta}_2 = \varepsilon_{12k}\underset{\sim}{\delta}_k = \sum_{k=1}^{3} \varepsilon_{12k}\underset{\sim}{\delta}_k = \varepsilon_{121}\underset{\sim}{\delta}_1 + \varepsilon_{122}\underset{\sim}{\delta}_2 + \varepsilon_{123}\underset{\sim}{\delta}_3 = \delta_3 \text{ since, } e_{123} = 1, e_{121} = 0, e_{122} = 0 \quad (2.5)$$

$$\underset{\sim}{\delta}_1 \times \underset{\sim}{\delta}_1 = \varepsilon_{11k}\underset{\sim}{\delta}_k = \sum \varepsilon_{11k}\delta_k = 0 + 0 + 0 = 0 \qquad (2.6)$$

It must be noted that the order of terms should not be changed as it might affect the final answer.

Example:

$$\underset{\sim}{v} \times \underset{\sim}{w} = v_i\underset{\sim}{\delta}_i \times w_j\underset{\sim}{\delta}_j$$

$$= v_i w_j \varepsilon_{ijk}\underset{\sim}{\delta}_k = \underset{\sim}{u} = u_k\underset{\sim}{\delta}_k \quad \text{where, } u_k = v_i w_j \varepsilon_{ijk} \qquad (2.7)$$

In the above equation, k is the free index, whereas i and j are the dummy indices.

If $k = 1$,

$$u_1 = v_i w_j \varepsilon_{ij1} = \sum_i^3 \sum_j^3 v_i w_j \varepsilon_{ij1} = v_2 w_3 \varepsilon_{231} + v_3 w_2 \varepsilon_{321} = v_2 w_3 - v_3 w_2 \text{ (all other terms are zero)}$$

Similarly, if $k = 2$,

$$u_2 = \sum_i \sum_j v_i w_j \varepsilon_{ij2} = -v_1 w_3 + w_1 v_3$$

For $k = 3$,

$$u_3 = \sum_i \sum_j v_i w_j \varepsilon_{ij3} = w_2 v_1 - v_2 w_1$$

Hence,

$$\underset{\sim}{v} \times \underset{\sim}{w} = (v_2 w_3 - v_3 w_2)\underset{\sim}{\delta}_1 + (w_1 v_3 - v_1 w_3)\underset{\sim}{\delta}_2 + (w_2 v_1 - v_2 w_1)\underset{\sim}{\delta}_3 \qquad (2.8)$$

Note:

Symmetry of ε_{ijk} under cyclic permutations

$$\varepsilon_{ijk} = \varepsilon_{kji} = \varepsilon_{jki} = -\varepsilon_{jik} = -\varepsilon_{ikj} = -\varepsilon_{kji} \qquad (2.9)$$

This holds for all values of i, j,k. Note that

If any two of the three indices i,j,k are the same, all term vanish

Equation (2.9) has three free indices so they represent 3^3 i.e 27 equations.

E.g. in $\varepsilon_{ijk} = \varepsilon_{kji}$, 3 equations says '1=1', 3 equations says '-1=-1' and 21 equations says 0=0

Mathematical operations on vectors

Addition of two vectors

$$\underset{\sim}{u} + \underset{\sim}{v} = u_i\underset{\sim}{\delta}_i + v_i\underset{\sim}{\delta}_i = \sum_{i=1}^3 (u_i + v_i)\underset{\sim}{\delta}_i \qquad (2.10)$$

In the same way, subtraction of vectors is done as

$$\underset{\sim}{u} - \underset{\sim}{v} = u_i\underset{\sim}{\delta}_i - v_i\underset{\sim}{\delta}_i = \sum_{i=1}^3 (u_i - v_i)\underset{\sim}{\delta}_i \qquad (2.11)$$

Lecture 3 (1st August)

Relation between alternating unit tensor and Kronecker delta

When two indices are common between the two alternating unit tensors

$$\varepsilon_{ijk}\varepsilon_{ljk} = \sum_k \sum_j \varepsilon_{ijk}\varepsilon_{ljk} = 2\delta_{il} \tag{3.1}$$

When one index is common between the two alternating unit tensors

$$\varepsilon_{ijk}\varepsilon_{mnk} = \sum_k \varepsilon_{ijk}\varepsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm} \tag{3.2}$$

A three by three determinant may be written in terms of the ε_{ijk}

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \sum_i^3 \sum_j^3 \sum_k^3 \varepsilon_{ijk} a_{1i} a_{2j} a_{3k} \tag{3.3}$$

Vector differential operation or Del operation

The vector differential operation (∇), also known as the 'del operation' is defined in the rectangular coordinate system as follows

$$\underset{\sim}{\nabla} = \underset{\sim}{\delta}_1 \frac{\partial}{\partial x_1} + \underset{\sim}{\delta}_2 \frac{\partial}{\partial x_2} + \underset{\sim}{\delta}_3 \frac{\partial}{\partial x_3} = \underset{\sim}{\delta}_i \frac{\partial}{\partial x_i} \tag{3.4}$$

The del operation is followed by a quantity which can be a scalar or vector or tensor.

Del operation on scalars

Del operation of a scalar field is called the gradient and is defined as follows:

$$\underset{\sim}{\nabla} s = \underset{\sim}{\delta}_i \frac{\partial}{\partial x_i} s = \frac{\partial s}{\partial x_1}\underset{\sim}{\delta}_1 + \frac{\partial s}{\partial x_2}\underset{\sim}{\delta}_2 + \frac{\partial s}{\partial x_3}\underset{\sim}{\delta}_3 \tag{3.5}$$

Here, s is a scalar quantity

Del operation on vectors

Three types of del operations are possible for vector quantities: (a) gradient, (b) divergence, and (c) curl of a vector field

Gradient of vector field

Dyadic product of the del operator and a vector quantity is called the gradient of vector field. It is a second order tensor quantity. If $\underset{\sim}{v}$ is a vector quantity then gradient of vector field is

$$\underset{\sim}{\nabla}\underset{\sim}{v} = \underset{\sim}{\delta}_i \frac{\partial}{\partial x_i}(v_j \underset{\sim}{\delta}_j) = \underset{\sim}{\delta}_i \underset{\sim}{\delta}_j \frac{\partial v_j}{\partial x_i} \tag{3.6}$$

This is called the gradient of the vector $\underset{\sim}{v}$.

Divergence of vector field

Dot product of the del operator and a vector quantity is called the divergence of vector field. It is a scalar quantity. If $\underset{\sim}{v}$ is a vector quantity then divergence of vector field is

$$\underset{\sim}{\nabla}.\underset{\sim}{v} = \underset{\sim}{\delta}_i \frac{\partial}{\partial x_i}.(v_j \underset{\sim}{\delta}_j)$$

$$= \frac{\partial v_j}{\partial x_i} \underset{\sim}{\delta}_i . \underset{\sim}{\delta}_j = \frac{\partial v_j}{\partial x_i} \delta_{ij} = \frac{\partial v_j}{\partial x_j}$$

Hence,

$$\underset{\sim}{\nabla}.\underset{\sim}{v} = \frac{\partial v_j}{\partial x_j} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \tag{3.7}$$

Curl of vector field

Cross product of the del operator and a vector quantity is called curl of vector field. It is a vector quantity. If $\underset{\sim}{v}$ is a vector quantity then curl of vector field is

$$\underset{\sim}{\nabla} \times \underset{\sim}{v} = \underset{\sim}{\delta}_i \frac{\partial}{\partial x_i} \times v_j \underset{\sim}{\delta}_j = \frac{\partial v_j}{\partial x_i} \underset{\sim}{\delta}_i \times \underset{\sim}{\delta}_j \tag{3.8}$$

$$= \frac{\partial v_j}{\partial x_i} \varepsilon_{ijk} \underset{\sim}{\delta}_k = \underset{\sim}{w} = w_k \underset{\sim}{\delta}_k$$

$$w_k = \frac{\partial v_j}{\partial x_i} \varepsilon_{ijk} = \sum_i \sum_j \frac{\partial v_j}{\partial x_i} \varepsilon_{ijk} \tag{3.9}$$

Laplacian of a scalar field

If we compute the divergence of the gradient of a scalar field then it is called the Laplacian of scalar field. Let $\underset{\sim}{\nabla}.\underset{\sim}{v}$ denote the divergence of a vector field $\underset{\sim}{v}$ which is the gradient of a scalar quantity s, i.e. $\underset{\sim}{v} = \underset{\sim}{\nabla} s$

Then,

$$\underset{\sim}{\nabla}.\underset{\sim}{v} = \underset{\sim}{\nabla}.(\underset{\sim}{\nabla} s) = \underset{\sim}{\delta}_i \frac{\partial}{\partial x_i}\left(\underset{\sim}{\delta}_j \frac{\partial}{\partial x_j} s \right) \tag{3.10}$$

$$= (\underset{\sim}{\delta}_i . \underset{\sim}{\delta}_j)\left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} s\right)$$

$$= \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} s \quad \delta_{ij}$$

$$= \frac{\P}{\P x_i}\frac{\P}{\P x_i} s$$

$$= \frac{\partial^2 s}{\partial x_i{}^2} \tag{3.11}$$

$$= \underset{\sim}{\nabla}.\underset{\sim}{\nabla} s \; = \frac{\partial^2 s}{\partial x_1{}^2} + \frac{\partial^2 s}{\partial x_2{}^2} + \frac{\partial^2 s}{\partial x_3{}^2} \tag{3.12}$$

Therefore, the Laplacian operator is defined as

$$\underset{\sim}{\nabla}.\underset{\sim}{\nabla} = \underset{\sim}{\nabla}^2 = \frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial x_2{}^2} + \frac{\partial^2}{\partial x_3{}^2} \tag{3.13}$$

Laplacian of a vector

Laplacian operator for a vector field is defined as follows

$$\underset{\sim}{\nabla}^2 v \; = \frac{\partial^2 v_j}{\partial x_i{}^2} \underset{\sim}{\delta}_j \tag{3.14}$$

Second necessary requirement of a vector quantity

Before proceeding further, we would like to revisit the definition of a vector. As we have discussed earlier there are two necessary requirements for a physical quantity to be a vector. The first requirement is that any vector quantity should have three components. The second necessary requirement is that it should follow certain transformation rule when the axis is rotated by some angle. We derive that transformation rule below.

$3^1 = 3$ components

$$v_p{}' = l_{ip} v_i \tag{3.15}$$

i is the dummy index

p is the free index

v_p is vector in new rotated coordinate system and v_i is vector in old system.

$l_{ip} = cos\angle\left(i^{th}\text{old axis } - \, p^{th}\text{new axis}\right)$

Proof of transformation rule in two dimension system

$p = 1$

$$v_1\text{'} = l_{i1}v_i = l_{11}v_1 + l_{21}v_2 + l_{31}v_3 \tag{3.16}$$

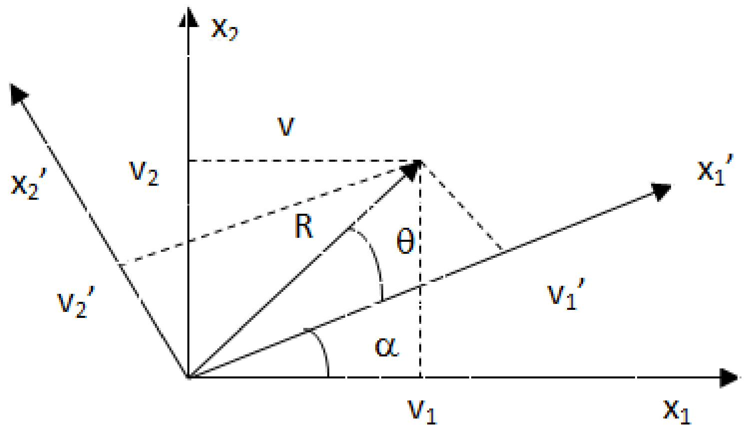

Fig 3.1 Rotation of axes

as shown in figure x_1 and x_2 are the old axes and v_1 and v_2 are coordinate of vector v of old axes. If axes are rotated by angle α. Now, new axis are x_1'and x_2'. And the coordinate of vector v in new axis are v_1' and v_2'. The angle of x_1' axis to line which join origin to v is θ.

$$tan\theta = \frac{v_2'}{v_1'} \tag{3.17}$$

$$v_1 = R\ cos\ (\theta + \alpha)$$

$$v_1\text{'} = R\ cos\theta \tag{3.18}$$

Therefore

$$\frac{v_1}{v_1'} = \frac{\cos(\theta+\alpha)}{\cos\theta} = cos\alpha - tan\theta sin\alpha$$

(3.17)

$$\frac{v_1}{v_1'} = \cos\alpha - \frac{v_2'}{v_1'}\sin\alpha$$

$$v_1 = v_1\text{'}cos\alpha - v_2\text{'}sin\alpha \tag{3.19}$$

Similarly, for component v_2

$$v_2 = R\ sin\ (\theta + \alpha) \tag{3.20}$$

$$v_2\text{'} = R\ sin\theta \tag{3.21}$$

$$\frac{v_2}{v_2'} = cos\beta + cot\theta sin\alpha$$

$$v_2 = v_1\text{'}sin\alpha + v_2\text{'}cos\alpha \tag{3.22}$$

Similar way component of vector v in new axis in terms of old axis

$$= v_1 cos\alpha + v_2 sin\alpha \tag{3.23}$$

$$= v_1' cos^2\alpha - v_2' sin\alpha cos\alpha + v_1' sin^2\alpha + v_2' cos\alpha sin\alpha = v_1' \tag{3.24}$$

$$v_1' = v_1 cos\alpha + v_2 sin\alpha = v_1 cos\alpha + v_2 cos\ (90\ - \alpha)$$

$$v_2' = -v_1 sin\alpha + v_2 cos\alpha = v_1 cos\ (90 + \alpha) + v_2 cos\alpha \tag{3.25}$$

Therefore generalized relation is

$$v_p' = l_{ip} v_i \tag{3.26}$$

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Lecture 3 (1st August)

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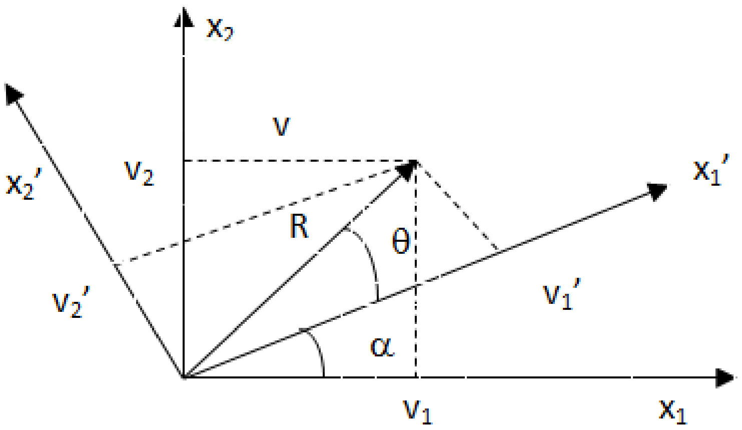

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Therefore generalized relation is

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Lecture 4 (5th August, Monday)

Second order tensor

As we have discussed earlier, second order tensors also follow similar rules. Firstly, they should contain $3^2 = 9$ components and secondly, they should also follow some transformation rule as follows.

$$\tau_{mn}' = l_{im} l_{jn} \tau_{ij} \tag{4.1}$$

$$l_{im} = cos\angle\left(i^{th}\text{old axis } - m^{th}\text{new axis}\right) \tag{4.2}$$

$$l_{jn} = cos\angle\left(j^{th}\text{old axis } - n^{th}\text{new axis}\right) \tag{4.3}$$

$$m = 1,\ n = 1$$

$$\tau'_{mn} = \tau_{11}'$$

$$\tau_{11}' = l_{i1} l_{j1} \tau_{ij}$$

$$\tau_{11}' = l_{i1} l_{j1} \tau_{ij} + \ldots\ldots..$$

$$= \sum_i \sum_j l_{i1} l_{j1} \tau_{ij} \text{ \{here, } i \text{ and } j \text{ are dummy indices\}} \tag{4.4}$$

τ_{11}, τ_{22}, τ_{33} are the normal component of second order tensor. The second order tensor is symmetric in nature.

$$\tau_{ij} = \tau_{ji} \tag{4.5}$$

Tensor as dyadic product of two vectors

As we have discussed earlier, the dyadic product of two vectors is a 2nd order tensor quantity. For example, if $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are two vectors, then

$$\underset{\sim}{v}\,\underset{\sim}{w} = (v_1\underset{\sim}{\delta}_1 + v_2\underset{\sim}{\delta}_2 + v_3\underset{\sim}{\delta}_3)\ (w_1\underset{\sim}{\delta}_1 + w_2\underset{\sim}{\delta}_2 + w_3\underset{\sim}{\delta}_3)$$

$$\begin{aligned} &= v_1 w_1 \underset{\sim}{\delta}_1\underset{\sim}{\delta}_1 + v_1 w_2 \underset{\sim}{\delta}_1\underset{\sim}{\delta}_2 + v_1 w_3 \underset{\sim}{\delta}_1\underset{\sim}{\delta}_3 \\ &+ v_2 w_1 \underset{\sim}{\delta}_2\underset{\sim}{\delta}_1 + v_2 w_2 \underset{\sim}{\delta}_2\underset{\sim}{\delta}_2 + v_2 w_3 \underset{\sim}{\delta}_2\underset{\sim}{\delta}_3 \\ &+ v_3 w_1 \underset{\sim}{\delta}_3\underset{\sim}{\delta}_1 + v_3 w_2 \underset{\sim}{\delta}_3\underset{\sim}{\delta}_2 + v_3 w_3 \underset{\sim}{\delta}_3\underset{\sim}{\delta}_3 \end{aligned} \tag{4.6}$$

We can write it in component form as

$$\begin{pmatrix} v_1w_1 & v_1w_2 & v_1w_3 \\ v_2w_1 & v_2w_2 & v_2w_3 \\ v_3w_1 & v_3w_2 & v_3w_3 \end{pmatrix}$$

$$\underset{\sim}{v}\,\underset{\sim}{w} = v_i\underset{\sim}{\delta}_i\, w_j\underset{\sim}{\delta}_j = v_i w_j \underset{\sim}{\delta}_i\underset{\sim}{\delta}_j \tag{4.7}$$

Transformation rule for vector

$$v_i \text{ to } v_m' = l_{im} v_i \tag{4.8}$$

w_j to $w_n' = l_{jn} w_j$ (4.9)

Dyadic product of both vectors (v_m' and w_n') gives the position of dyadic tensor according to new rotated axis.

$$v_m' w_n' = l_{im} v_i l_{jn} w_j = v_i w_j l_{im} l_{jn} \quad (4.10)$$

Above equation proves that product of 2 vectors in second order tensor. Thus, we can write any tensor $\underset{\sim}{\tau}$ as

$$\underset{\sim}{\tau} = \tau_{ij} \underset{\sim}{\delta}_i \underset{\sim}{\delta}_j \quad (4.11)$$

Mathematical operations for tensors

The following mathematical operations are possible for tensor quantities

Addition of tensors

Tensors of the same order can be added or subtracted as follows:

$$\underset{\sim}{\tau} + \underset{\sim}{\sigma} = \tau_{ij} \underset{\sim}{\delta}_i \underset{\sim}{\delta}_j + \sigma_{ij} \underset{\sim}{\delta}_i \underset{\sim}{\delta}_j$$

$$= (\tau_{ij} + \sigma_{ij}) \underset{\sim}{\delta}_i \underset{\sim}{\delta}_j \quad (4.12)$$

Similarly,

$$\underset{\sim}{\tau} - \underset{\sim}{\sigma} = \tau_{ij} \underset{\sim}{\delta}_i \underset{\sim}{\delta}_j - \sigma_{ij} \underset{\sim}{\delta}_i \underset{\sim}{\delta}_j$$

$$= (\tau_{ij} - \sigma_{ij}) \underset{\sim}{\delta}_i \underset{\sim}{\delta}_j \quad (4.13)$$

Here indices used in both tensors should be identical.

Multiplication of tensors

Multiplication operations are possible between two different order tensors. Some important operations, which are used in this book, are as follows.

Multiplication of a tensor by a scalar

A scalar and a 2nd order tensor quantity can be multiplied as follows

$$a \underset{\sim}{\tau} = (a \tau_{ij}) \underset{\sim}{\delta}_i \underset{\sim}{\delta}_j \quad (4.14)$$

Multiplication of a tensor by a vector

A vector and a 2nd order tensor can be multiplied in three ways:

Dyadic product of a vector and tensor

Dyadic product of vector and tensor is a third order tensor which is not discussed here.

$$\underset{\sim}{\tau} \underset{\sim}{v} = \tau_{ij} \underset{\sim}{\delta}_i \underset{\sim}{\delta}_j v_k \underset{\sim}{\delta}_k = \tau_{ij} v_k \underset{\sim}{\delta}_i \underset{\sim}{\delta}_j \underset{\sim}{\delta}_k \quad (4.15)$$

Dot product of a vector and tensor

Dot operation reduces the order of resulting quantity by two. Hence, the dot product of a vector and a tensor is a vector quantity. For example, if $\underset{\sim}{\tau}$ is tensor and $\underset{\sim}{v}$ is vector quantity, then

$$\underset{\sim}{\tau} . \underset{\sim}{v} = \tau_{ij} \underset{\sim}{\delta}_i \underset{\sim}{\delta}_j . v_k \underset{\sim}{\delta}_k = \tau_{ij} v_k \underset{\sim}{\delta}_i (\underset{\sim}{\delta}_j . \underset{\sim}{\delta}_k) \tag{4.16}$$

Here, the nearest unit vectors take part in the dot operation

$\underset{\sim}{\tau} . \underset{\sim}{v} = \tau_{ij} v_k \underset{\sim}{\delta}_i \delta_{jk} \rightarrow$ Replace *j* by *k* or *k* by *j*

$$= \tau_{ij} v_j \underset{\sim}{\delta}_i = \underset{\sim}{w} = w_i \underset{\sim}{\delta}_i \quad \text{Thus, } w_i = \tau_{ij} v_j \tag{4.17}$$

$$w_l = t_{lj} v_j = \sum_{j=1}^{3} t_{lj} v_j \tag{4.18}$$

For example, if *i=1*, then

$$w_1 = \tau_{11} v_1 + \tau_{12} v_2 + \tau_{13} v_3 \tag{4.19}$$

Cross product of a vector and tensor

Cross operation reduces the order of resulting quantity by one. Hence, the cross product of a vector and second order tensor is a second order tensor. For example, if $\underset{\sim}{\tau}$ is tensor and $\underset{\sim}{v}$ is vector quantity, then

$$\begin{aligned} \underset{\sim}{\tau} \times \underset{\sim}{v} &= \tau_{ij} \underset{\sim}{\delta}_i \underset{\sim}{\delta}_j \times v_k \underset{\sim}{\delta}_k \\ &= \tau_{ij} v_k \underset{\sim}{\delta}_i (\underset{\sim}{\delta}_j \times \underset{\sim}{\delta}_k) \\ &= \tau_{ij} v_k \underset{\sim}{\delta}_i (\varepsilon_{jkn} \underset{\sim}{\delta}_n) \\ &= \tau_{ij} v_k \varepsilon_{jkn} \underset{\sim}{\delta}_i \underset{\sim}{\delta}_n \end{aligned} \tag{4.20}$$

This is a second order tensor quantity.

Multiplication of two tensors

Four types of multiplication operations can be performed between two second order tensor quantities.

Dyadic Product of two tensors

Dyadic product of two second order tensors is a fourth order tensor quantity. It is not discussed here.

$$\underset{\sim}{\tau} \underset{\sim}{\sigma} \quad \rightarrow 4^{\text{th}} \text{ order tensor} \tag{5.1}$$

Cross product of two tensors

Cross product of two second order tensors is a third order tensor quantity and is not discussed here.

$$\underset{\sim}{\tau} \times \underset{\sim}{\sigma} \quad \rightarrow 3^{rd} \text{ order} \tag{5.2}$$

Dot product of tensors (Tensor product)

Dot reduces the order of resultant quantity by two. Thus, the dot product of two second order tensors is a second order tensor quantity. If $\underset{\sim}{\tau}$ and $\underset{\sim}{\sigma}$ are two second order tensors, then

$$\underset{\sim}{\tau}.\underset{\sim}{\sigma} = \tau_{ij}\underset{\sim}{\delta}_i\underset{\sim}{\delta}_j.\sigma_{kl}\underset{\sim}{\delta}_k\underset{\sim}{\delta}_l$$

$$= \tau_{ij}\sigma_{kl}\,\underset{\sim}{\delta}_i\,(\underset{\sim}{\delta}_j.\underset{\sim}{\delta}_k)\,\underset{\sim}{\delta}_l \tag{5.5}$$

Here the order of indices should not be changed and the dot product should take the two nearest indices.

$$\underset{\sim}{\tau}.\underset{\sim}{\sigma} = \tau_{ij}\sigma_{kl}\delta_{jk}\underset{\sim}{\delta}_i\underset{\sim}{\delta}_l \tag{5.6}$$

Replace *k* by *j* (or you may also replace *j* by *k*) (compaction operation)

$$\Omega = \underset{\sim}{\tau}.\underset{\sim}{\sigma} = \tau_{ij}\sigma_{jl}\underset{\sim}{\delta}_i\underset{\sim}{\delta}_l$$

$$= \Omega_{il}\underset{\sim}{\delta}_i\underset{\sim}{\delta}_l \qquad \text{where} \quad \Omega_{il} = \tau_{ij}\sigma_{jl}$$

For example,

$$\Omega_{11} = \tau_{1j}\sigma_{j1} = \tau_{11}\sigma_{11} + \tau_{12}\sigma_{21} + \tau_{13}\sigma_{31} \tag{5.7}$$

Double dot product or Scalar product of second order tensors

Double dot operation reduces the order of resultant quantity by four. Thus, the double dot product of two second order tensors is a scalar quantity. If $\underset{\sim}{\tau}$ and $\underset{\sim}{\sigma}$ are two second order tensors, then

$$\underset{\sim}{\tau} : \underset{\sim}{\sigma} = \tau_{ij}\underset{\sim}{\delta}_i\underset{\sim}{\delta}_j : \sigma_{kl}\underset{\sim}{\delta}_k\underset{\sim}{\delta}_l$$

$$= \tau_{ij}\sigma_{kl}\,(\underset{\sim}{\delta}_j.\underset{\sim}{\delta}_k)(\underset{\sim}{\delta}_i.\underset{\sim}{\delta}_l) \tag{5.8}$$

First dot operation should take place between the two nearer vectors and the next dot operation should take place in between two remaining vectors. Hence,

$$\underset{\sim}{\tau} : \underset{\sim}{\sigma} = \tau_{ij}\sigma_{kl}\delta_{jk}\,(\underset{\sim}{\delta}_i.\underset{\sim}{\delta}_l) \qquad \text{Replace } k \text{ by } j$$

$$= \tau_{ij}\sigma_{jl}\delta_{il} \qquad \text{Replace } l \text{ by } i$$

$$= \tau_{ij}\sigma_{ji} \tag{5.9}$$

This is a scalar quantity

Del operations involving tensors & dyads

Divergence of a second order tensor field

If $\underset{\sim}{\tau}$ is a second order tensor then divergence of the tensor field is

$$\underset{\sim}{\nabla}.\underset{\sim}{\tau} = \left(\underset{\sim}{\delta}_i \frac{\partial}{\partial x_i}\right).\left(\tau_{jk}\underset{\sim}{\delta}_j\underset{\sim}{\delta}_k\right) = \left(\underset{\sim}{\delta}_i.\underset{\sim}{\delta}_j\right)\underset{\sim}{\delta}_k \frac{\partial \tau_{jk}}{\partial x_i} \text{ (Apply compaction operation)} \tag{5.10}$$

$$= \underset{\sim}{\delta}_k \frac{\partial \tau_{ik}}{\partial x_i}$$

$$\underset{\sim}{\nabla}.\underset{\sim}{\tau} = \underset{\sim}{\delta}_k \frac{\partial \tau_{ik}}{\partial x_i} = \underset{\sim}{w} = w_k\underset{\sim}{\delta}_k \tag{5.11}$$

$$w_k = \frac{\partial \tau_{ik}}{\partial x_i} \tag{5.12}$$

Now we will try to prove some examples of vector and tensor identities.

Examples:

(1) Prove: $\underset{\sim}{\nabla}.\left(s\underset{\sim}{v}\right) = \underset{\sim}{\nabla}s.\underset{\sim}{v} + s\underset{\sim}{\nabla}.\underset{\sim}{v}$

Order $= 1+1-2 = 0$

LHS

$$= \underset{\sim}{\nabla}.(s\underset{\sim}{v}) \tag{5.13}$$

$$= \underset{\sim}{\delta}_i \frac{\partial}{\partial x_i}.(sv_j\underset{\sim}{\delta}_j)$$

$$= \frac{\partial}{\partial x_i}(sv_j)\underset{\sim}{\delta}_i.\underset{\sim}{\delta}_j$$

$$= \frac{\partial}{\partial x_i}(sv_j)\delta_{ij}$$

$$= \frac{\partial}{\partial x_i}(sv_i)$$

$$= s\frac{\partial v_i}{\partial x_i} + v_i\frac{\partial s}{\partial x_i} \tag{5.14}$$

RHS

$$= \underset{\sim}{\nabla}s.\underset{\sim}{v} + s\underset{\sim}{\nabla}.\underset{\sim}{v} \tag{5.15}$$

$$= \underset{\sim}{\delta}_i \frac{\partial s}{\partial x_i}.v_j\underset{\sim}{\delta}_j + s\underset{\sim}{\delta}_i \frac{\partial}{\partial x_i}.v_j\underset{\sim}{\delta}_j$$

$$= v_j\frac{\partial s}{\partial x_i}\delta_{ij} + s\frac{\partial v_j}{\partial x_i}\delta_{ij}$$

$$= v_i \frac{\partial s}{\partial x_i} + s \frac{\partial v_i}{\partial x_i} \tag{5.16}$$

From Eq. (5.14) and Eq. (5.16)

$\therefore$ LHS = RHS

(2) $\quad \underset{\sim}{\nabla}.\left(\underset{\sim}{v}\ \underset{\sim}{w}\right) = \underset{\sim}{v}.\underset{\sim}{\nabla}\underset{\sim}{w} + \underset{\sim}{w}\underset{\sim}{\nabla}.\underset{\sim}{v}$

LHS

$$= \underset{\sim}{\nabla}.\left(\underset{\sim}{v}\ \underset{\sim}{w}\right) \tag{5.17}$$

$$= \underset{\sim}{\delta}_i \frac{\partial}{\partial x_i}.(v_j \underset{\sim}{\delta}_j w_k \underset{\sim}{\delta}_k)$$

$$= \frac{\partial}{\partial x_i}\left(v_j w_k\right)(\underset{\sim}{\delta}_i.\underset{\sim}{\delta}_j)\underset{\sim}{\delta}_k$$

$$= \frac{\partial}{\partial x_i}\left(v_j w_k\right)\delta_{ij}\underset{\sim}{\delta}_k$$

$$= \frac{\partial}{\partial x_i}\left(v_i w_k\right)\underset{\sim}{\delta}_k$$

$$= \left(v_i \frac{\partial}{\partial x_i} w_k + w_k \frac{\partial v_i}{\partial x_i}\right)\underset{\sim}{\delta}_k \tag{5.18}$$

RHS

$$= \underset{\sim}{v}.\underset{\sim}{\nabla}\underset{\sim}{w} + \underset{\sim}{w}\underset{\sim}{\nabla}.\underset{\sim}{v} \tag{5.19}$$

$$= v_i \underset{\sim}{\delta}_i.\underset{\sim}{\delta}_j \frac{\partial}{\partial x_j} w_k \underset{\sim}{\delta}_k + w_k \underset{\sim}{\delta}_k \underset{\sim}{\delta}_j \frac{\partial}{\partial x_j}.v_i \underset{\sim}{\delta}_i$$

$$= v_i \delta_{ij} \frac{\partial}{\partial x_j} w_k \underset{\sim}{\delta}_k + w_k \underset{\sim}{\delta}_k \delta_{ji} \frac{\partial}{\partial x_j} v_i$$

$$= \left(v_i \frac{\partial w_k}{\partial x_i} + w_k \frac{\partial v_i}{\partial x_i}\right)\underset{\sim}{\delta}_k \tag{5.20}$$

From Eq. (5.18) and Eq. (5.20)

$\therefore$ LHS = RHS

(3) $\underset{\sim}{\nabla}.(\underset{\sim}{\tau}.\underset{\sim}{v}) = \underset{\sim}{v}.(\underset{\sim}{\nabla}.\underset{\sim}{\tau}) + \underset{\sim}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{v}$ if $\underset{\sim}{\tau}$ is a symmetric second order tensor

LHS

$$= \underset{\sim}{\delta}_k \frac{\partial}{\partial x_k}.(\tau_{ij}\underset{\sim}{\delta}_i\underset{\sim}{\delta}_j.v_l\underset{\sim}{\delta}_l) \tag{5.21}$$

$$= \underset{\sim}{\delta}_k \frac{\partial}{\partial x_k}.(\tau_{ij}v_l\underset{\sim}{\delta}_i\delta_{jl})$$

$$= \underset{\sim}{\delta}_k \frac{\partial}{\partial x_k}.(\tau_{ij}v_j\underset{\sim}{\delta}_i)$$

$$= \frac{\partial}{\partial x_k}.(\tau_{ij}v_j)\delta_{ki}$$

$$= \frac{\partial}{\partial x_i}(\tau_{ij}v_j)$$

$$= \tau_{ij}\frac{\partial v_j}{\partial x_i} + v_j\frac{\partial \tau_{ij}}{\partial x_i} \tag{5.22}$$

RHS

$$= v_l\underset{\sim}{\delta}_l.\left(\underset{\sim}{\delta}_k \frac{\partial}{\partial x_k}.\tau_{ij}\underset{\sim}{\delta}_i\underset{\sim}{\delta}_j\right) + \tau_{ij}\underset{\sim}{\delta}_i\underset{\sim}{\delta}_j : \left(\underset{\sim}{\delta}_k \frac{\partial}{\partial x_k}\right)(v_l\underset{\sim}{\delta}_l) \tag{5.23}$$

$$= v_l\underset{\sim}{\delta}_l.\left(\frac{\partial \tau_{ij}}{\partial x_k}\delta_{ki}\underset{\sim}{\delta}_j\right) + \tau_{ij}\delta_{il}\delta_{jk}\frac{\partial v_l}{\partial x_k}$$

$$= v_l\frac{\partial \tau_{ij}}{\partial x_i}\delta_{lj} + \tau_{ij}\delta_{il}\frac{\partial v_l}{\partial x_j}$$

$$= v_j\frac{\partial \tau_{ij}}{\partial x_i} + \tau_{ij}\frac{\partial v_i}{\partial x_j}$$

$$= v_j\frac{\partial \tau_{ij}}{\partial x_i} + \tau_{ji}\frac{\partial v_j}{\partial x_i}$$

$$= v_j\frac{\partial \tau_{ij}}{\partial x_i} + \tau_{ij}\frac{\partial v_j}{\partial x_i} \qquad (\underset{\sim}{\tau} \text{ is a symmetric tensor}) \tag{5.24}$$

From Eq. (5.22) and Eq. (5.24)

$\therefore$ LHS = RHS

Lecture 5 (08th August, Thursday)

Time derivatives in transport phenomena

Many times, we are interested to know how fast any physical quantity or property is changing with time. However, that the property is also a function of space makes it complicated to measure the changes in that property.

In this section, different types of time derivatives are discussed. Three types of time derivatives are mainly used in transport phenomena:

1. Partial derivative, denoted as $\frac{\partial C}{\partial t}$
2. Total derivative, denoted as $\frac{dC}{dt}$
3. Substantial derivative, denoted as $\frac{DC}{Dt}$

To understand the differences between these time derivatives, let us consider a hypothetical experiment. A chimney produces SO_2 gas and we want to study the change in SO_2 concentration with time in the atmosphere.

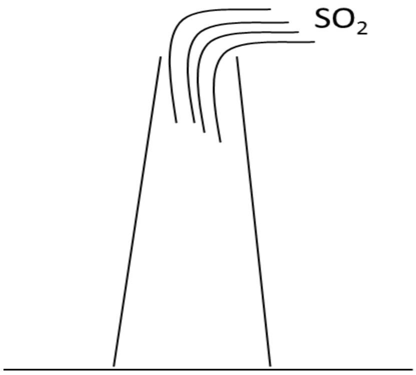


Fig 6.1 Partial derivative with constant observer at point C

Partial derivative

If the observer remains fixed at a particular position and determines the change in concentration of SO_2 after some time interval Δt, it is the partial derivative which is measured. At time t, let the concentration of SO_2 be C_1 and at time $t+\Delta t$, let it be $C_1+\Delta C_1$. Thus, the time derivative is

$$\lim_{\Delta t \to 0} \frac{C_1 + \Delta C_1 - C_1}{\Delta t} = \frac{\partial C}{\partial t} \qquad (6.1)$$

This is a partial derivative since the concentration of SO_2 also changes with spatial coordinate but we consider only the change with time.

Total derivative

If, however, the observer also changes the observation position with time, it is the total derivative which is measured. Suppose at any time $t = t$, the observer is situated at the point A and the measured concentration of SO_2 is C_1. At time $t= t+\Delta t$, the observer reaches point B and measures the concentration of SO_2 as C_2. In this case, the time derivative is

$$\lim_{\Delta t \to 0} \frac{C_2 - C_1}{\Delta t} = \frac{dC}{dt} \tag{6.2}$$

This is a total derivative since we are considering changes in the concentration with respect to both space and time. Therefore, it should also include the effects of the velocity of observer and the velocity of air, and can be expressed mathematically as:

$$\frac{dC}{dt} = \frac{\partial C}{\partial t} + v_x \frac{\partial C}{\partial x} + v_y \frac{\partial C}{\partial y} + v_z \frac{\partial C}{\partial z} \tag{6.3}$$

Here, v_x, v_y and v_z are the components of the velocity of the observer, v in the x, y and z directions respectively.

Substantial derivative

It is a special case of the total derivative when the observer moves in a balloon with the speed of the fluid flow itself. Thus, the velocity of fluid will be the same as the velocity of observer.

$$\lim_{\Delta t \to 0} \frac{(C_1 + \Delta C_1) - C_1}{\Delta t} = \frac{\partial C}{\partial t} + v_x \frac{\partial C}{\partial x} + v_y \frac{\partial C}{\partial y} + v_z \frac{\partial C}{\partial z} = \frac{DC}{Dt} \tag{6.4}$$

Here, v_x, v_y and v_z are the components of the velocity of the fluid, v in the x, y and z directions respectively.

To understand the difference between partial and total / substantial derivatives, let us take a simple one-dimensional problem. Let the point A be at the position 'x', and point B be at the position '$x+\Delta x$'. The concentration of SO_2 is a function of both time t as well as spatial coordinate x. As shown in Figure 6.2, the concentration profile (plot of C *vs.* x) changes with time. Let the concentration of SO_2 at the point A be recorded at time t as C_1 and at time $t+\Delta t$ as C_2. In the same way, the concentration of SO_2 is recorded at point B at time t as C_3 and at time $t=t+\Delta t$ as C_4 as shown in Figure 6.2

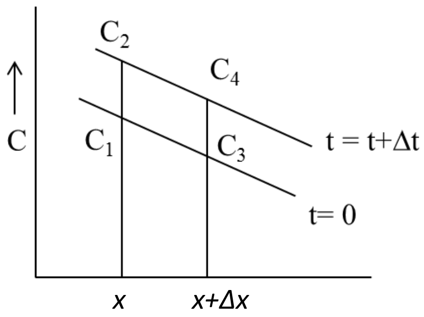


Fig 6.2 Position of observer A (C_1 & C_2) and B (C_3 and C_4)

The observer, starting from point A, reaches the point B in time Δt. If the velocity of the observer is *u*, the distance traversed in time Δt will be $\Delta x = u_x \Delta t$. Here consider observer is moving with the fluid or velocity equal to fluid.

The partial derivatives can thus be computed as

$$\frac{\partial C}{\partial t} = \frac{C_2 - C_1}{\Delta t} \qquad \text{at point A} \qquad (6.5)$$

$$\frac{\partial C}{\partial t} = \frac{C_4 - C_3}{\Delta t} \qquad \text{at point B} \qquad (6.6)$$

The substantial derivative, is however computed as

$$\frac{DC}{Dt} = \frac{C_4 - C_1}{\Delta t} \qquad (6.7)$$

Equation (6.7) shows the difference between the partial and substantial derivative. In order to relate the two mathematically, we can proceed as follows.

From the Figure 6.2, C_3 can be written as (using Taylor expension)

$$C_3 = C_1 + \frac{\partial C}{\partial x}\Delta x \qquad (6.7)$$

Further, C_4 can be written in terms of C_3

$$C_4 = C_3 + \frac{\partial C}{\partial t}\Delta t$$

$$= C_1 + \frac{\partial C}{\partial x}\Delta x + \frac{\partial C}{\partial t}\Delta t \qquad (6.8)$$

Therefore,

$$C_4 - C_1 = \frac{\partial C}{\partial x}\Delta x + \frac{\partial C}{\partial t}\Delta t = \frac{\partial C}{\partial x}u_x\Delta t + \frac{\partial C}{\partial t}\Delta t \qquad (6.9)$$

Dividing the equation by Δt and taking the limit as $\Delta t \rightarrow 0$, we have

$$\frac{DC}{Dt} = \lim_{\Delta t \to 0} \frac{C_4 - C_1}{\Delta t} = u_x \frac{\partial C}{\partial x} + \frac{\partial C}{\partial t} \tag{6.10}$$

Generalizing the equation for three dimensional space, and making it independent of the coordinate system, we can write it in vector and tensor form as

$$\frac{DC}{Dt} = \frac{\partial C}{\partial t} + \underset{\sim}{v}.\underset{\sim}{\nabla} C \tag{6.11}$$

For a quantity like velocity, which is a vector, the relation remains the same.

$$\frac{D\underset{\sim}{v}}{Dt} = \frac{\partial \underset{\sim}{v}}{\partial t} + \underset{\sim}{v}.\underset{\sim}{\nabla}\underset{\sim}{v} \tag{6.12}$$

Momentum Transport

Momentum transport deals with the transport of momentum which is responsible for flow in fluids. Momentum transport describes the science of fluid flow also called *fluid dynamics*. A few basic assumptions are involved in fluid flow, and these are discussed below.

No slip boundary condition

This is the first basic assumption used in momentum transport. It deals with fluid flowing over a solid surface, and states that whenever fluid comes in contact with any solid boundary, the adjacent layer of fluid which is in contact with the solid surface has the same velocity as the solid surface. Hence, we can say that there is no slip or relative velocity is zero at the fluid–solid interface.

For example, consider a fluid flowing inside a stationary tube of radius *R* as shown in Fig 7.1. Since the wall of the tube at $r=R$ is stationary, according to the no-slip condition, the fluid velocity at $r=R$ is also zero.

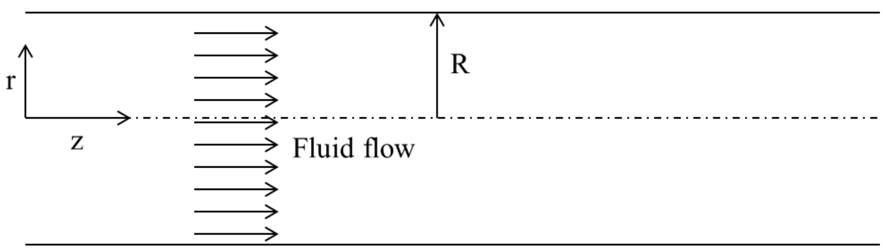


Fig 7.1 Fluid flow in a circular tube of radius *R*

In the second example of Figure 7.2, there are two plates which are separated by a distance *h*, and some fluid is present in between them. Now, if the lower plate is made to move with a velocity *V* and the upper plate is kept stationary, then no-slip will be applicable at both boundaries $y=0$ and $y=H$.

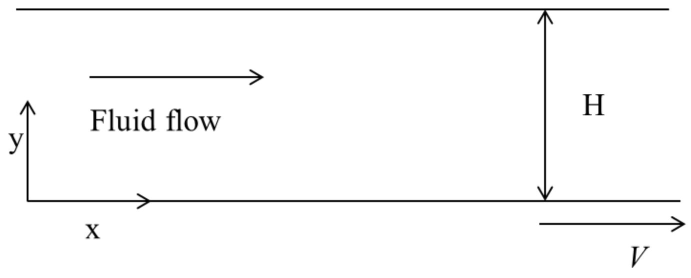


Fig 7.1 Two parallel plates at stationary condition

The fluid velocity at *y=0* will be *V* and fluid velocity at *y=H* will be *0*, since the top plate is stationary and bottom plate is moving at a velocity *V*.

If we observe closely, the no slip boundary conditions in the second example have led to the development of a *velocity gradient* in the flow.

$y = 0,\ v_x\,|_{y=0} = v$

$y = h,\ v_x\,|_{y=h} = 0$

Thus, every layer of fluid is moving at a different velocity. This would lead to shear force or friction force which is described in the next section.

Eulerian (Continuum) and Lagrangian description of a fluid

In reality we see fluid as continuum in nature, that is fluid if liquid takes a shape of container or gaseous occupies the container volume. Imagin the fluid at microscopic scale, where we see fluid composed of atom/molecules or a fluid particle as seen in Fig 7.2. Liquids are closely spaced molecules with large intermolecular forces. Retain volume and take shape of container. Gaseous are widely spaced molecules with small intermolecular forces. Take volume and shape of container. When we see the container fluid (liquid/gas) appears are continuum however if we look at smallest scale (at rage of molecular size) it appears as discrete fluid particles. There are two ways to describe the fluid motion. One is called Lagrangian, where one follows all fluid particles and describes the variations around each fluid particle along its trajectory. The other is Eulerian, where the variations are described at all fixed stations as a function of time. In the second, different particles pass the same station at different times.

Density of a fluid

$$\rho(\underline{x}) = \lim_{\delta V \to \delta V^*} \frac{\delta m}{\delta V}$$

$\delta V^* \approx 10^{-12} - 10^{-9}\ mm^3$ is size of an element volume

We see that below δV^* density of fluid fluctuates because of thermal fluctuations between molecules, above δV^* density become constant due to averaging effect of molecules in the element volume. Therefore, properties of continuum filed such as density, pressure velocity etc. are taken as well defined at infinitesimally small point.

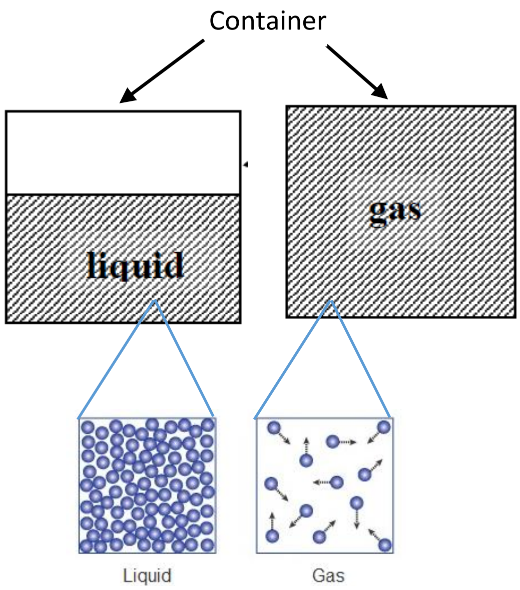


Fig 7.2 Fluid (liquid or gas) behavior

Lecture 6 (12th August, Monday)

Momentum Transport

Momentum transport deals with the transport of momentum which is responsible for flow in fluids. Momentum transport describes the science of fluid flow also called *fluid dynamics*. A few basic assumptions are involved in fluid flow, and these are discussed below.

No slip boundary condition

This is the first basic assumption used in momentum transport. It deals with fluid flowing over a solid surface, and states that whenever fluid comes in contact with any solid boundary, the adjacent layer of fluid which is in contact with the solid surface has the same velocity as the solid surface. Hence, we can say that there is no slip or relative velocity is zero at the fluid–solid interface.

For example, consider a fluid flowing inside a stationary tube of radius R as shown in Fig 7.1. Since the wall of the tube at $r=R$ is stationary, according to the no-slip condition, the fluid velocity at $r=R$ is also zero.

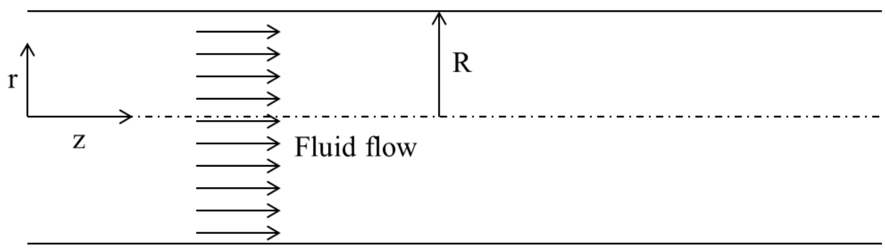


Fig 7.1 Fluid flow in a circular tube of radius *R*

In the second example of Fig 7.2, there are two plates which are separated by a distance h, and some fluid is present in between them. Now, if the lower plate is made to move with a velocity V and the upper plate is kept stationary, then no-slip will be applicable at both boundaries $y=0$ and $y=H$.

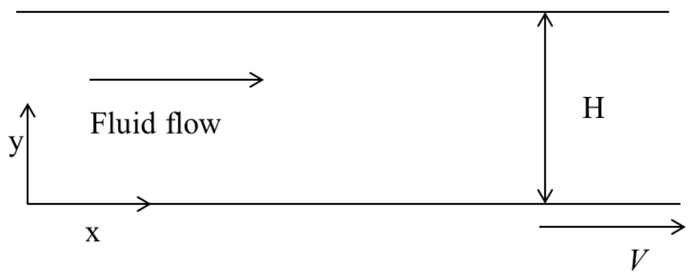


Fig 7.2 Two parallel plates at stationary condition

The fluid velocity at $y=0$ will be V and fluid velocity at $y=H$ will be 0, since the top plate is stationary and bottom plate is moving at a velocity V.

If we observe closely, the no slip boundary conditions in the second example have led to the development of a *velocity gradient* in the flow.

$y = 0, \; v_x \,|_{y=0} = v$

$y = h, \; v_x \,|_{y=h} = 0$

Thus, every layer of fluid is moving at a different velocity. This would lead to shear force or friction force which is described in the next section.

Newton's Law of Viscosity

Newton's law of viscosity is the basic consideration for solving problem of momentum transport in real fluids. Most of the fluids which we deal with in chemical engineering are Newtonian in nature. To understand the concept of Newtonian fluid, let us consider a following hypothetical experiment, in which there are two infinitely large plates situated parallel each other, separated by a distance H.

Note: If you are familiar with material sciences, you may notice that F/A (Force/Area) is essentially the term for stress. There are in fact two types of stresses; tensile and shear. A simple way to distinguish the two is that tensile stress acts perpendicularly on the surface of an object/molecule, whereas shear stress acts tangentially to a surface. As we are dealing with molecules "dragging" and "sliding" past each other to drive flow, we are more interested in shear forces acting on the molecule, generally denoted by $\boldsymbol{\tau}$.

Parallel Plate Example

Let's consider hypothetically a pair of large parallel plates. They each have a surface area of A and are separated by a distance of H. In between, there is some fluid (which again, could be a liquid *or* a gas) that reside in *layers* between the plates.

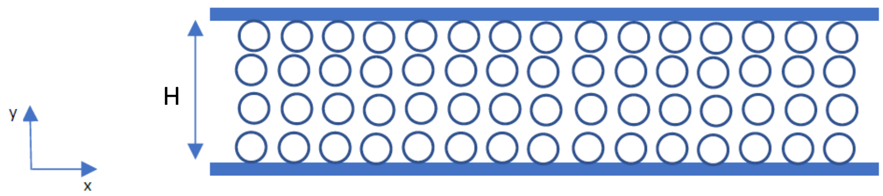


Fig. 7.3 Fluid between the plate is seen as fluid layers of well-organized molecules/fluid particles.

Now let's do a thought experiment; Consider a vertical imaginary line in a fluid when fluid is stationary at time t (see Fig. 7.4). A constant force F_x is now applied on the upper plate, and after some time it starts to move with a constant velocity V_x. We assumed the "no-slip" condition, where the layer of molecules in direct contact with the moving plate and move together with the same velocity in the *x*-direction i.e. V_x. The first layer of molecules "drag" the subsequent layers, each which has a slightly lower velocity – this creates a velocity gradient**.** Because of this velocity gradient molecules on an imaginary line are also moving at different velocity at different height. That creates deformation with a small angle $\Delta\beta$, Same force F_x is in place as the time progress to $t + 2\Delta t$, $t + 3\Delta t$, $t + 4\Delta t$ so on. Fluid will continue to deform, let say with an angle $2\Delta\beta$, $3\Delta\beta, 4\Delta\beta$, respectively. At long enough time velocity gradient exist with velocity of moving plate to stationary plate.

Now consider increasing force F_x at observed time $t + \Delta t$ to $2F_x$, $3F_x$ and so on in different experiments, deformation to fluid with angle $2\Delta\beta$, $2\Delta\beta$ will take place respectively, (Higher the shear force higher the fluid deformation).

If we see the relation between $\frac{F_x}{A}$,$\Delta\beta$ and Δt, it logical to say

$$\frac{F_x}{A} \propto \frac{\Delta\beta}{\Delta t} \qquad (7.1)$$

Force per unit area is proportional to rate of deformation. The fluid will not come to its initial state as force is removed rather it will keep deforming or in right sense it will flow. Shear stress (force per unit area) acting on a fluid will make fluid to flow. Note that velocity of fluid obtained is at steady state (independent of time) and laminar flow (low flow condition) that will be introduce in later lecture.

And for very small angle $\Delta\beta$, $\Delta\beta = \frac{\Delta L}{H} = \frac{V_x \Delta t}{H}$ $(tan\Delta\beta = \Delta\beta)$

So

$$\frac{F_x}{A} \propto \frac{\Delta\beta}{\Delta t} \propto \frac{V_x \Delta t}{H \Delta t} \qquad (7.2)$$

Equation 7.2 is Force/area is propositional to rate of deformation. This can be verified in the thought experiment that increasing F_x or Δt give the same expression($\frac{2F_x}{A} \propto \frac{2\Delta\beta}{\Delta t}$ or $\frac{F_x}{A} \propto \frac{2\Delta\beta}{2\Delta t}$)

For an infinitesimally small Δy, we have ΔV_x Therefore

$$\frac{F_x}{A} \propto \lim_{\Delta y \to 0} \frac{\Delta V_x}{\Delta y}$$

$$\frac{F_x}{A} \propto \frac{dV_x}{dy} \qquad (7.3)$$

Removing proportionality, we get

$$\frac{F_x}{A} = \mu \frac{dV_x}{dy} \tag{7.4}$$

$\frac{F_x}{A}$ is the shear stress i.e τ_{yx} (x-direction tangential force on an area whose normal is in y direction)

$$\tau_{yx} = \mu \frac{dV_x}{dy} \tag{7.5}$$

μ is the viscosity of a fluid. Subscript *x* represents force and *y* represent direction of area unit normal. This expression is known as newtons law of viscosity i.e Shear stress is proportional to velocity gradients.

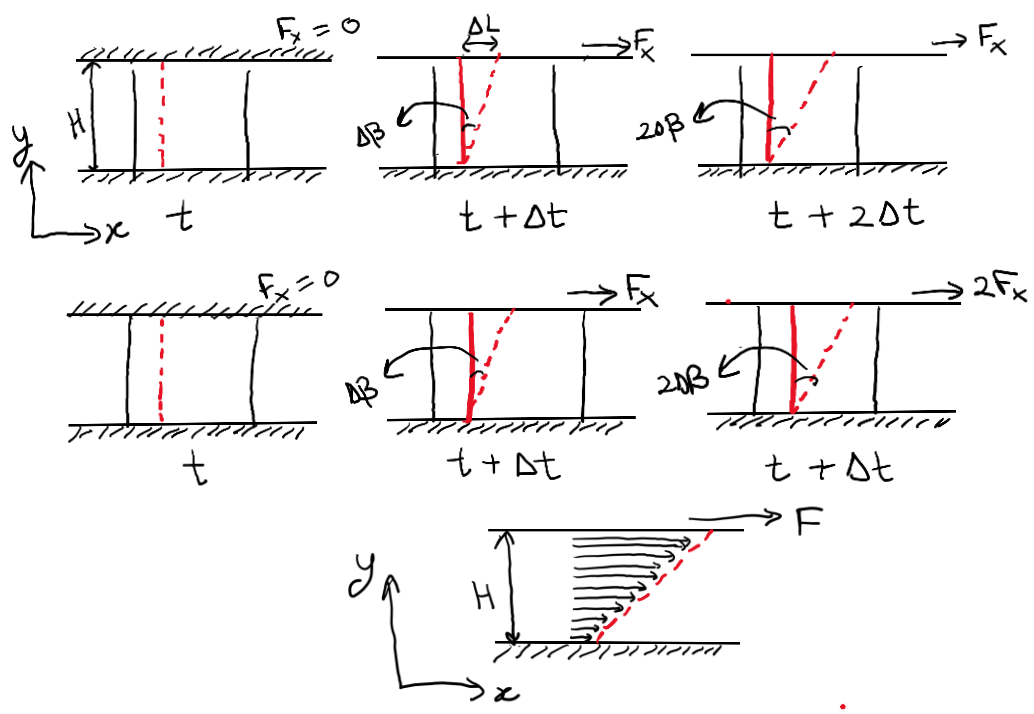


Fig. 7.4: Hypothetically fluid between parallel plate with upper plate moving and bottom plate stationary

Simple shear experiment

A real fluid is filled in between both plates. A constant force F_1 is now applied on the lower plate, and after some time it starts to move with a constant velocity V_1. The force is then changed, and the new velocity of the plate associated with that force is measured. The experiment is then repeated to take sufficient readings as shown in the following table.

Table No 7.1 Applied force vs. velocity

F	V	F/A	v/h
F_1	v_1	F_1/A	v_1/h
F_2	v_2	F_2/A	v_2/h
F_3	v_3	F_3/A	v_3/h
F_4	v_4	F_4/A	v_4/h
F_5	v_5	F_5/A	v_5/h
.	.	.	.
F_n	v_n	F_n/A	v_n/h

If the readings *F/A* are plotted against *V/h*, it is observed that they lie on a straight line passing through the origin.

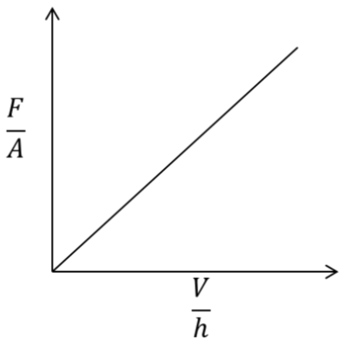


Fig 7.5 Shear stress vs. shear stain

Thus, it is concluded that *F/A* is proportional to *v/h* for a Newtonian fluid (consider linear relation to velocity of fluid u(y)).

$$\frac{F}{A} \propto \frac{v}{h} \tag{7.6}$$

Since in the last section, we had stated that it is the velocity gradient which leads to the development of shear force, the above equation can be re-written as

$$\Rightarrow \frac{F_1}{A} \propto \frac{v_1 - 0}{h - 0} = \frac{\Delta v_x}{\Delta y} \tag{7.7}$$

In the limiting form, as the gap between the plates is reduced, i.e. $h \to 0$, we have

$$\frac{F}{A} \propto \frac{dv_x}{dy}$$

$$\frac{F}{A} = \pm \mu \frac{dv_x}{dy} \tag{7.8}$$

Where, μ is a constant of proportionality, and is known as the *viscosity* of the fluid.

The quantity F/A is called the *shear stress* and is represented as τ_{yx}. The subscript x indicates the direction of force and subscript y indicates the direction of outward normal of the surface on which the force is acting. The quantity $\frac{dv_x}{dy}$ or *velocity gradient* is also called the *shear rate* of the fluid or *rate of deformation* or *strain rate*, and μ is a property of the fluid material which is called *viscosity*, and represents the resistance offered by the fluid to flow. Viscosity is constant for all Newtonian fluids (fluids which follow Newton's law of viscosity) and changes only with temperature.

Now; **What is viscosity?** Viscosity is a physical property of fluids (liquids and gasses) that pretty much measures their resistance to flow. On a more fundamental level, the term "flow" refers to molecules moving along due to some net force. The more a fluid is able to withstand that force, the greater its viscosity. Viscosity is very much tied in with velocity of a fluid. If you were to pour honey and water out of a cup, obviously the honey would flow slower and would be deemed more viscous than water.

Newton's law of viscosity, in its most basic form is given as

$$\tau_{yx} = \pm\mu\frac{dv_x}{dy} \tag{7.9}$$

Here, both '+' or '–' sign are valid. The positive sign is used in many fluid mechanics books whereas the negative sign in used in transport phenomena. If the positive sign is used then τ_{yx} is called *shear force* which is acting outwards to a solid surface but if the negative sign is used then τ_{yx} is called *momentum flux* which is flowing from a higher value to a lower value.

$$\tau_{yx} = +\mu\frac{dv_x}{dy} \quad \rightarrow \text{Shear force} \tag{7.10}$$

$$\tau_{yx} = -\mu\frac{dv_x}{dy} \quad \rightarrow \text{Momentum flux} \tag{7.11}$$

Equation may be interpreted in another fashion. In the neighborhood of the moving solid surface at y = 0 the fluid acquires a certain amount of x-momentum. This fluid, in turn, imparts momentum to the adjacent layer of liquid, causing it to remain in motion in the x direction. Hence x-momentum is being transmitted through the fluid in the positive y direction. Therefore this may also be interpreted as the flux of x-momentum in the positive y direction, where the term "flux" means "flow per unit area." This interpretation is consistent with the molecular picture of momentum transport and the kinetic theories of gases and liquids.

The reason for having a negative sign for shear stress in transport phenomena is to relate momentum transport to heat transport and mass transport. According to the second law of thermodynamics, heat transfer occurs from a higher temperature to a lower temperature. Similarly, mass transport occurs from a region of higher chemical potential to a region of lower chemical potential. Thereby deriving an analogy between momentum transport and heat/mass transport, it can be said that fluid momentum flows from higher value to lower value. Thus, we can say τ_{yx} is x- directional momentum flowing in y direction.

Lecture 6 (13th August, Thursday's timetable)

Generalization of Newton's Law of Viscosity

In the parallel plate example, we only needed to consider v_x (velocity of fluid moving in the x-direction), however there exists other situations where v_y and v_z come into play for momentum transport. In order to apply Newton's Law of Viscosity to all scenarios, we need to take into account the transport of momentum from *all* forces in *all* directions.

Assume you have an infinitesimally small cubic element in a fluid with axes in the x, y, and z direction. We can use this as a model to describe pressure and viscous forces.

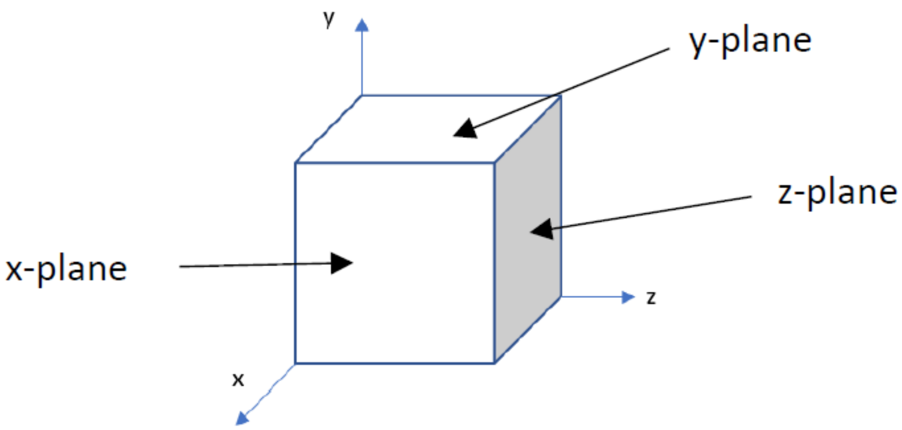


Fig. 8.1 Small cubic volume

Shear force (τ) also know term as **viscous force** can act on a plane at any angle (i.e any direction). Fortunately, as with all vectors in Cartesian coordinates, you can decompose shear stresses at any angle into terms of x-, y- and z-components. Be careful here – although we're saying that shear force acts in tangential direction we have a component that directly act perpendicularly to a plane (but we will group these with pressure forces later on).

Pressure forces act *perpendicularly* to a plane or surface. In relation to fluids, this is also known as hydrostatic pressure, which acts on a fluid whether it is stationary or moving. It can also be used to drive fluid flow, as pressure exerts a stress in the direction of high-pressure to low-pressure (think about how a straw works). Pressure forces are denoted by P.

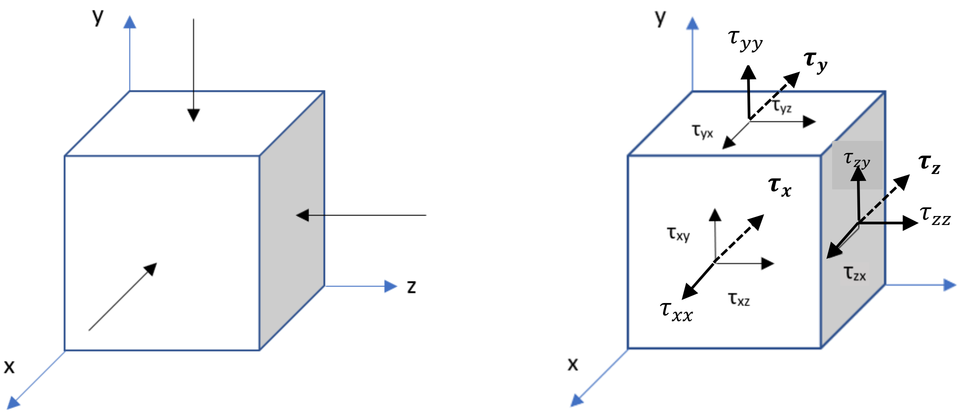


Fig 8.2 Here's a picture distinguishing between pressure forces (left) and the components of viscous forces (right) relative to a typical xyz axis.

Going one step further, you can imagine how both pressure and shear stresses can move fluids. We can combine the two forces, and write out the components that will act on each plane:

X-Plane	Y-Plane	Z-Plane
$\tau_{xx} + P$	τ_{yx}	τ_{zx}
τ_{xy}	$\tau_{yy} + P$	τ_{zy}
τ_{xz}	τ_{yz}	$\tau_{zz} + P$

Remember that τ_{ij} means "motion in the *j-direction* with flux in the *i-direction*". OR force acting in *j-direction* on and area whose normal is in *i-direction*. Refer to parallel plate experiment.

Stresses that include both normal and shear stresses can we written as $\sigma_{ij} = P\delta_{ij} + \tau_{ij}$ here δ_{ij} is kroneckar delta and *i,j* may be x, y and z.

Now viscous stress is a linear combination of all the velocity gradients, and viscous stress is a symmetric tensor it can be written as (for detail refer Transport phenomena, BSL,, Chapter 1)

$$\tau_{ij} = A\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + B\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right)\delta_{ij} \tag{8.1}$$

Constant A and B are μ and $\left(\frac{2}{3}\mu - \kappa\right)$ respectively. μ is viscosity and κ is dilatational viscosity

Therefore

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \left(\frac{2}{3}\mu - \kappa\right)\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right)\delta_{ij} \tag{8.2}$$

Equation is generalized expression for newtons law of viscosity, which has nine components (six independent, due to τ_{ij} symmetric tensor).

Writing expression in coordinate free

$$\boldsymbol{\tau} = \mu(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T) + \left(\frac{2}{3}\mu - \kappa\right)(\nabla \cdot \boldsymbol{v})\boldsymbol{\delta} \tag{8.3}$$

in which $\boldsymbol{\delta}$ is the unit tensor with component δ_{ij} and $\nabla \boldsymbol{v}$ is the velocity gradient tensor with components $\frac{\partial u_i}{\partial x_j}$, *and* $(\nabla \boldsymbol{v})^T$ is the "transposer' of the velocity gradient tensor with components $\frac{\partial u_j}{\partial x_i}$ and $(\nabla \cdot \boldsymbol{v})$ is the divergence of the velocity vector.

The important conclusion is that we have a generalization of $\tau_{yx} = \pm\mu\frac{dv_x}{dy}$, and this generalization involves not one but two coefficients characterizing the fluid: the viscosity μ and the dilatational viscosity κ. Usually, in solving fluid dynamics problems, it is not necessary to know κ. If the fluid is a gas, we often assume it to act as an ideal monoatomic gas, for which κ is identically zero. If the fluid is a liquid, we often assume that it is incompressible, and in later course we show that for incompressible liquids $(\nabla \cdot \boldsymbol{v}) = 0$, and therefore the term containing κ is discarded anyway. The dilational viscosity is important in describing sound absorption in polyatomic gases and in describing the fluid dynamics of liquids containing gas bubbles.

The SI unit of viscosity is *kg/m.s* or *Pa.s*. The CGS unit is *g/cm.s* and is commonly known as *poise (P)*, and is related to the SI unit as *1 P = 0.1 kg/m.s.*The unit poise is also used with the prefix centi-, which refers to one-hundredth of a poise, i.e. 1 cP = 0.01 P. The dimensions of viscosity are as follows:

$$\mu = \frac{Force/Area}{\left[\frac{dv_x}{dy}\right]} = \frac{MLT^{-1}L^{-2}}{LT^{-1}L^{-1}} = ML^{-1}T^{-1}$$

The viscosity of air at *25°C* is *0.018 cP* and for polymer melts it ranges from *1000* to *100,000 cP*, thus showing the long range of viscosity. The viscosity of water at *25°C* is *1 cP*.

Newtons law of viscosity is applicable only for certain class of fluid called Newtonian fluid. Most of the fluids in chemical engineering applications are classified under Newtonian fluid like water, air, gasoline, motor oil etc.

Fluids for which the newtons law is not applicable lies in the category of non-Newtonian fluids. For non-Newtonian fluid shear stresses are not linearly proportional to velocity gradients or shear stresses.

The viscosity of a non-Newtonian time independent fluid is dependent not only on temperature but also on shear rate. We will discuss a bit on model/relation of non-Newtonian fluid in our later lecture. Here we characterised fluids into different class.

Depending on how viscosity changes with shear rate the flow behaviour is characterised as:

- Shear thinning - the viscosity decreases with increased shear rate
- Shear thickening - the viscosity increases with increased shear rate
- Bingham plastic - exhibits a so-called yield value, *i.e.* a certain shear stress must be applied before flow occurs

Shear thinning fluids are also called *pseudoplastic* and shear thickening fluids are also called *dilatant*.

Examples of shear thinning fluids: paint, shampoo, blood, slurries, fruit juice concentrates, ketchup etc.

Examples of shear thickening fluids: wet sand, concentrated starch suspensions etc

Examples of Bingham plastic fluids: quark, tomato paste, toothpaste, hand cream, grease etc

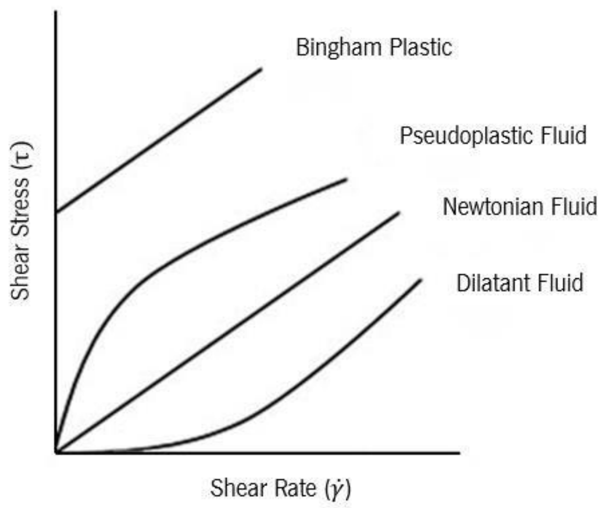


Fig. 8.3. Shear stress vs shear rate, characterizing the fluid

Coordinate systems

Till now, we have mostly seen and dealt with the Cartesian (or rectangular) coordinate system. However, depending on the geometry of the problem we intend to solve, we might find the Cartesian coordinate system cumbersome to use. For example, consider the flow of a fluid in a cylindrical pipe. In such a case, it is difficult to use the Cartesian coordinate because the boundary conditions become extremely complicated and involve all three variables, x, y, and z. Hence, a coordinate system describing the space in terms of distance from a fixed axis might prove to be much more useful. This is the cylindrical coordinate system. Therefore, depending upon the geometry in consideration, there are three types of coordinate systems which are used in transport phenomena:

1. Cartesian coordinate system
2. Cylindrical coordinate system
3. Spherical coordinate system

Cartesian coordinate system

Cartesian coordinate system is a three dimensional coordinate system. In this coordinate system, the space is defined by three lines, called *axes* which are mutually perpendicular to each other. These axes intersect each other at a point called the *origin*. By convention, the axes are termed as the x-axis, y-axis, and z-axis. Any point in space can then be defined by the distance of the point from the planes made by any two axes. For example, if any point P is located in space such that the distance of P to the yz plane is x, to the xy plane is z, and to the zx plane is y then the coordinates of point P are (x, y, z).

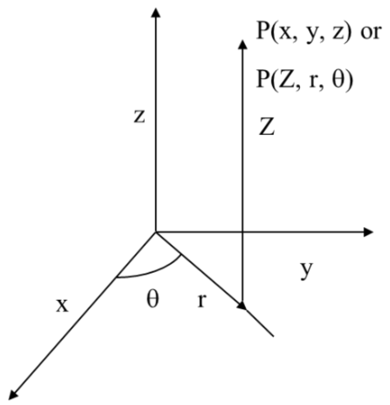


Fig 8.4 Cartesian and cylindrical coordinate system

Cylindrical coordinate system

Cylindrical coordinate system is a three dimensional coordinate system. In this coordinate system, the space can be defined by an axis z, and a direction r and an angle θ, both of which lie in a plane A perpendicular to the *z*-axis. The *origin* is the point lying at the intersection of the *z*-axis and the plane A. Any point P can be defined as P(r,θ,z) where *r* is the perpendicular (radial) distance from the *z*-axis to the point P; θ is the angle between the reference direction on the selected plane A and the line from the origin to the projection of point P on the plane A; *z* is the perpendicular distance from the point P to the plane A.

Spherical coordinate system

In this coordinate system, three dimensional space can be defined by an axis z, a direction *r* and a plane A (the xy plane of the Cartesian system). The axis z and the plane A are perpendicular to each other and intersect at a point called the *origin*.

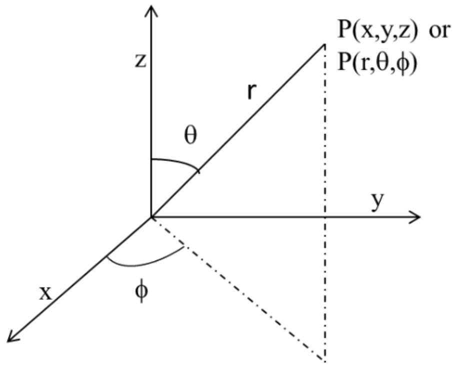


Fig 8.5 Spherical coordinate system

Any point P can be defined as P(r,θ,ϕ) where *r* is the distance of the point from the origin; θ is the angle made between z-axis and the line passing through origin and point P; ϕ is the angle made between x direction and the line which joins the origin to projection of point P on the xy plane.

Laminar and turbulent flow

Fluid flow can broadly be categorized into two kinds: laminar and turbulent. In laminar flow, the fluid layers do not inter-mix, and flow separately. This is the flow encountered when a tap is just opened and water is allowed to flow very slowly. As the flow increases, it becomes much more irregular and the different layers start mixing with each other, and this is called turbulent flow.

Osborne Reynolds distinguished between the two kinds of flow using an ingenious experiment which is known as the Reynolds's experiment, and is described below.

Reynolds's experiment

In this experiment, a circular transparent tube is taken with fluid flowing through it.

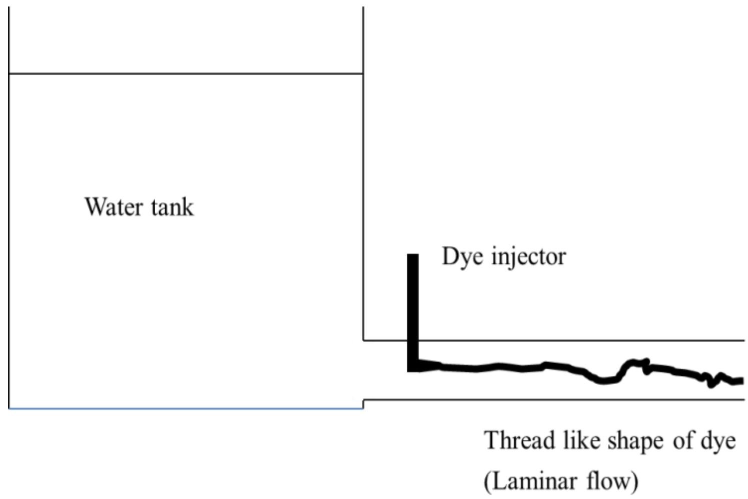


Fig 8.6 Reynolds's experiments

The velocity of fluid (flow rate) and the pipe diameter can be controlled. There is an arrangement to inject a coloured dye at the center of the pipe. The profile of the dye is observed along the pipe with different velocities for different fluids. It is found that for certain physical conditions, the dye shows a regular thread type profile. This regular thread type flow at low velocity is called as laminar flow. For some other conditions, it is found that the dye is mixed with the fluid and simply disappears. This is known as turbulent flow.

From the variables – average velocity of fluid $v_{z\,avg}$, pipe diameter D, fluid density ρ, and fluid viscosity μ, Reynolds found a dimensionless group which could be used to characterize the type of fluid flow in the tube. This dimensionless quantity is known as the *Reynolds number*. It was observed that if *Re >2100*, the dye simply disappeared.

$$\mathrm{Re} = \frac{\rho v_{z,avg} D}{\mu} \tag{8.4}$$

$\mathrm{Re} < 2100$ - Laminar flow, i.e. no mixing in the radial direction leading to a thread like flow

$\mathrm{Re} > 2100$ - Turbulent flow, i.e. mixing in the radial direction between layers of fluid

The critical value of Reynolds number at which the flow changes its trend from laminar to turbulent is different for different geometries. It is *2100* for flow through a pipe and 5×10^5 for flow over a flat surface. However, it should be noted that this critical value is independent of the fluid, i.e. it would be the same whether it is air or water flowing through the system.

In laminar flow, the fluid flows as a stream line flow with no disruption between layers. This means that there is no radial mixing of the fluid in the tube. In turbulent flow, the fluid is mixed rapidly due to several fluctuations and disturbances in the flow. The disturbance might be due to pumps, friction of the solid surface or any type of noise present in the surroundings. To understand the kind of velocity profile that we may obtain in the two kinds of flow, let us consider fluid flowing through a horizontal cylindrical tube. If the fluid is flowing in the *z* direction at steady state condition, then

For laminar flow:

$v_z = v_z(r)$

$v_r = 0$

$v_\theta = 0$

For turbulent flow:

$v_z = v_z(r, z, \theta, t)$

$v_r = v_r(r, z, \theta, t)$

$v_\theta = v_\theta(r, z, \theta, t)$

Thus, we see that laminar flow problems can be attempted to be solved analytically whereas turbulent flow problems are very complex. Turbulent flow problems cannot be solved completely; however, some reasonable answer or prediction can be calculated using complex computational methods.

For turbulent flow, if the fluid is flowing in the *z* direction then why are the velocity components in *r* and θ direction non-zero? The mathematical answer for this question can be deciphered from the equation of motion. The equation of motion is a non-linear partial differential equation. This non-linear nature of the equation causes instability in the system which produces flow in other directions. The instability in the system may occur due to disturbances or noise present in the environment. If a disturbance appears in a turbulent flow at $t=0$, due to high fluid velocity the deviations due to disturbances are high, and hence the effect due to disturbance never goes to zero. Thus, $v_r \neq 0$ and $v_\theta \neq 0$, even in z-directed flow.

On the other hand, if the velocity of fluid is very low then the deviation due to disturbances decays with time, and becomes negligible after some time and thus the flow is laminar. Consider a practical example in which some cars are moving on the highway in the same direction but in different lanes at different speeds. If suddenly, some obstacle comes on the road, then if the car’s speed is sufficiently low then it can move on to the other lane smoothly and comes back to its original lane after the obstacle is crossed. This is the regular laminar case. On the other hand, if the car is moving at a high speed and suddenly encounters an obstacle, then the driver may lose control, and the car might move haphazardly and hit the other cars. This is the turbulent case

Internal and external flows

Depending on how the fluid and the solid boundaries contact each other, the flow can be classified as internal flow, and external flow. In internal flows, the fluid is moving between solid boundaries. For example, the flow through a pipe or a duct is an internal flow. In external flows, however, the fluid is flowing over an external solid surface. For example, the flow of fluid over a sphere as shown in Figure 8.7 is an external flow.

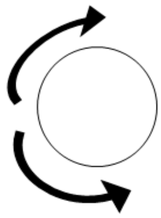


Fig 8.7 External flow around a sphere

Boundary layers and fully developed regions

Let us now consider the example of fluid flowing over a horizontal flat plate as shown in Figure 8.8. The velocity of the fluid is V_∞ before it comes in contact with the plate. As the fluid touches the plate, the velocity of the fluid layer just adjacent to the plate surface will become zero due to the no slip boundary condition. Now, this layer will drag the next fluid layer above and will reduce its speed. As the fluid proceeds along the plate (in x-direction), every layer starts to drag its adjacent layer but the effect of drag will reduce as we go away from the plate (in y-direction). Therefore, at some distance from the plate this drag effect will disappear or become insignificant. This distance denotes the extent of the boundary layer. In other words, the region close to the wall where velocity gradients are significant is called the *boundary layer* and the region beyond where velocity gradients are insignificant is called the inviscid flow or *potential flow* region.

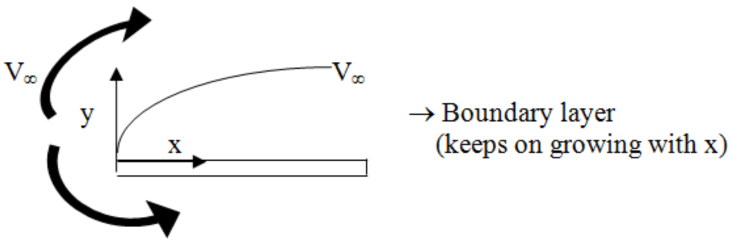


Fig 8.8 External flow over a flat plate

As depicted in Figure 8.8, the boundary layer keeps growing along the x-direction, and this is referred to as the developing flow region.

In internal flows (e.g. flow through a pipe), the boundary layers will merge into each other after flow over a distance as shown in Fig. 8.9 below.

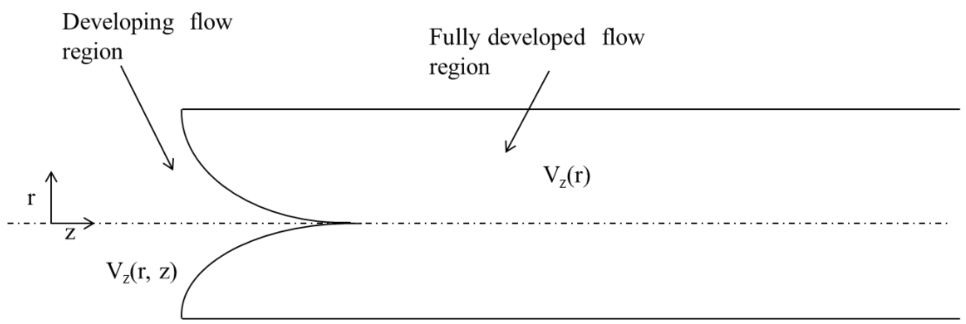


Fig 8.9 Developing flow and fully developed flow region

The region after the point at which the layers merge is called the *fully developed flow* region and before it is called the *developing flow* region. In fact, fully developed flow is another important assumption which is usually taken while solving problems of momentum transport. In the fully developed flow region (as shown in Figure 8.9), velocity v_z is a function of r direction only where shear forces work due to solid boundaries. But in the developing flow region, velocity v_z is also changing in the z direction along the direction of flow.

Lecture 8 (22^{nd} August, Thursday)

Axioms

The five fundamental **axioms** of transport phenomena are as follows:

1. *Mass is conserved*, leads to the equation of continuity
2. *Momentum is conserved*, leads to the equation of motion
3. *Moment of momentum is conserved*, leads to the fact that the 2^{nd} order stress tensor $\underset{\sim}{\tau}$ is symmetric
4. *Energy is conserved*, leads to the equation of energy, which can be combined with the equation of mechanical energy to give the equation of thermal energy
5. *In a multi-component system, mass of component i is conserved*, leads to the convective diffusion equation

Axiom-1

Mass is conserved

This axiom states that the rate of change of mass in any control volume is equal to the net inflow of mass plus the accumulation of mass in that control volume. The mathematical equation derived from this axiom is called the equation of continuity.

Consider a fluid of density ρ flowing with velocity $\underset{\sim}{v}$. Here, ρ and $\underset{\sim}{v}$ are functions of space (x,y,z) and time t. There are three types of *control volumes (CV)* which can be taken.

1. *Rectangular volume element*

   In this case, the control volume is rectangular in shape and is fixed in space. This method is the easiest way to solve the problem but requires more number of steps.

2. *Regular shape control volume element*

   In this case, the shape of the control volume is not fixed. It may be in any shape, but it is fixed in space. This method is somewhat more difficult than the previous method but it will solve the problem within lesser number of steps.

3. *Irregular shape control volume element*

   In this case, the control volume can be in any shape and it is also flowing with the speed of the fluid. This method is the most difficult of the three and includes some integration terms, but it requires the least number of steps to solve the problem.

In all the methods, the difference between the rate of mass entering and leaving from the control volume (net rate of inflow) has to be evaluated and this should be equal to the rate of accumulation of mass in the control volume (CV). The equation should then be divided by the volume of the CV and then converted into a differential equation by taking the limit as all dimensions go to zero. The limit effectively means that the CV collapses to a point, thereby making the equation valid at every point in the system.

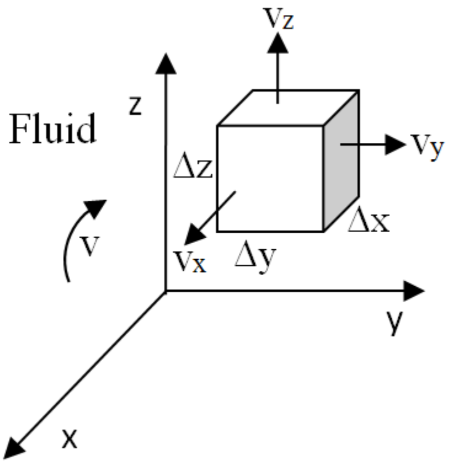


Fig 8.4 Fixed rectangular volume element through which fluid is flowing

We will consider the first method (i.e. a fixed, rectangular CV as depicted in Figure 8.4) throughout the course as it can be understood easily.

The conservation equation can be written in words as

$$\begin{pmatrix}\text{rate of accumulation of}\\ \text{mass in control volume}\end{pmatrix} = \begin{pmatrix}\text{rate of inflow of mass}\\ \text{into control volume}\end{pmatrix} - \begin{pmatrix}\text{rate of outflow of mass}\\ \text{out of control volume}\end{pmatrix}$$

Let m and $m+\Delta m$ be the mass of the control volume at time t and $t+\Delta t$ respectively. Then, the rate of accumulation, i.e. the left hand side of the equation is

$$\frac{\partial m}{\partial t} = \frac{\partial}{\partial t}(\rho \Delta x \Delta y \Delta z)$$

In order to evaluate the rate of inflow of mass into the control volume, we need to inspect how mass enters the control volume. Since fluid velocity has three components v_x, v_y and v_z in general, we will need to identify the components which will cause inflow at each of the six faces of the rectangular CV. For example, it is the component v_x which forces the liquid to flow in the x direction, and thus it will make the fluid enter/exit through the faces having area $\Delta y \Delta z$ at $x = x$ and $x = x+\Delta x$ respectively. The component v_y forces the liquid in y direction, and thus it will make the fluid enter/exit through the faces having area $\Delta x \Delta z$ at $y = y$ and $y = y+\Delta y$

respectively. Similarly, the component v_z forces the liquid in z direction, and thus it will make the fluid enter/exit through the faces having area $\Delta x\Delta y$ at $z = z$ and $z = z+\Delta z$ respectively.

The mass entering from x direction through the surface $\Delta y\Delta z$ is ($\rho v_x\ \Delta y\Delta z|_x$), mass entering from y direction through the surface $\Delta x\Delta z$ is ($\rho\ v_y\ \Delta x\Delta z|_y$) and mass entering from z direction through the surface $\Delta x\Delta y$ is ($\rho\ v_z\ \Delta x\Delta y|_z$). In a similar manner, expressions for mass leaving from the control volume can be written. The mass leaving from x direction through the surface $\Delta y\Delta z$ is ($\rho v_x\ \Delta y\Delta z|_{x+\Delta x}$), mass leaving from y direction through the surface $\Delta x\Delta z$ is ($\rho\ v_y\ \Delta x\Delta z|_{y+\Delta y}$) and mass leaving from z direction through the surface $\Delta x\Delta y$ is ($\rho\ v_z\ \Delta x\Delta y|_{z+\Delta z}$).

All the mathematical expressions can now be substituted in to the original equation in words

$$\frac{\partial}{\partial t}(\rho\Delta x\Delta y\Delta z)=\left[\rho v_x\Delta y\Delta z\big|_x - \rho v_x\Delta y\Delta z\big|_{x+\Delta x}+\rho v_y\Delta x\Delta z\big|_y - \rho v_y\Delta x\Delta z\big|_{y+\Delta y}+\rho v_z\Delta x\Delta y\big|_z - \rho v_z\Delta x\Delta y\big|_{z+\Delta z}\right] \quad (8.1)$$

Dividing the equation (8.1) by the volume $\Delta x\Delta y\Delta z$

$$\frac{\partial\rho}{\partial t}=\left[\frac{(\rho v_x)|_x-(\rho v_x)|_{x+\Delta x}}{\Delta x}\right]+\left[\frac{(\rho v_y)|_y-(\rho v_y)|_{y+\Delta y}}{\Delta y}\right]+\left[\frac{(\rho v_z)|_z-(\rho v_z)|_{z+\Delta z}}{\Delta z}\right] \quad (8.2)$$

Equation (8.2) shows the conservation of mass per unit volume of the system. Taking the independent limits $\Delta x\to 0$, $\Delta y\to 0$ and $\Delta z\to 0$, we get

$$\frac{\partial\rho}{\partial t}=\lim_{\Delta x\to 0}\left[\frac{(\rho v_x)|_x-(\rho v_x)|_{x+\Delta x}}{\Delta x}\right]+\lim_{\Delta y\to 0}\left[\frac{(\rho v_y)|_y-(\rho v_y)|_{y+\Delta y}}{\Delta y}\right]+\lim_{\Delta z\to 0}\left[\frac{(\rho v_z)|_z-(\rho v_z)|_{z+\Delta z}}{\Delta z}\right] \quad (8.3)$$

Using the definition of derivative, we can write

$$\frac{\partial\rho}{\partial t}=-\left[\frac{\partial(\rho v_x)}{\partial x}+\frac{\partial(\rho v_y)}{\partial y}+\frac{\partial(\rho v_z)}{\partial z}\right] \quad (8.4)$$

Equation (8.4) shows that mass is conserved at each point in the system. Rearranging the terms, we get the equation of continuity for Cartesian coordinates as given below.

Cartesian coordinates (x, y, z)

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho v_x)+\frac{\partial}{\partial y}(\rho v_y)+\frac{\partial}{\partial z}(\rho v_z)=0 \quad (8.5)$$

To use Equation (8.5) in other coordinate systems (that is, spherical or cylindrical), it should be written in vector and tensor form as

$$\frac{\partial\rho}{\partial t}+\underset{\sim}{\nabla}.(\rho\underset{\sim}{v})=0 \quad (8.6)$$

This is the generalized form of the equation of continuity in vector and tensor form which can be used for all types of coordinate systems. To write the above equation in cylindrical/spherical form, the divergence operation should be written for the cylindrical/spherical coordinate system. Vector and tensor analysis of cylindrical and spherical coordinate systems is not done here, and can be looked up elsewhere. Thus, the final expressions in cylindrical and spherical coordinates are given as below.

Cylindrical coordinates (r, θ, z)

$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r)+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho v_\theta)+\frac{\partial}{\partial z}(\rho v_z)=0 \tag{8.7}$$

Spherical coordinates (r, θ, φ)

$$\frac{\partial\rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(\rho r v_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho v_\theta\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}(\rho v_\varphi)=0 \tag{8.8}$$

Equation of continuity in substantial derivative form

The partial derivative present in the equation (8.6) can be converted into substantial derivative using vector and tensor identities.

$$\frac{d\rho}{dt}+\underset{\sim}{\nabla}.(\rho\underset{\sim}{v})=0 \tag{8.6}$$

$$\frac{d\rho}{dt}+\underset{\sim}{v}.\underset{\sim}{\nabla}\rho+\rho\underset{\sim}{\nabla}.\underset{\sim}{v}=0 \tag{8.10}$$

Since the substantial derivative of density ρ is defined as $\frac{D\rho}{Dt}=\frac{d\rho}{dt}+\underset{\sim}{v}.\underset{\sim}{\nabla}\rho$, we have

$$\frac{D\rho}{Dt}+\rho\underset{\sim}{\nabla}.\underset{\sim}{v}=0 \tag{8.11}$$

In many cases, the fluid is incompressible, i.e. density ρ is constant with time as well as space. For example, water is an incompressible fluid – its density can not be changed by applying pressure. In fact, all real liquids are incompressible fluids. For this special case, the equation of continuity is further simplified as below

$$\underset{\sim}{\nabla}.\underset{\sim}{v}=0 \quad (\rho \text{ is constant}) \tag{8.12}$$

The equation of continuity for an incompressible fluid does not mean that the system is in steady state condition. The velocity of the fluid might still be a function of time. What it implies is that if the velocity of the fluid changes in a particular direction *(x, y* or *z)* then it should also change in the other directions so that mass will be conserved without changing density.

Lecture 9 (28th August, Thursday)

During our last lecture we have seen conservation of mass, by applying on a small control volume CV (cubic, ΔxΔyΔz), Please refer to previous lecture notes, equation 8.6.
We saw equation of continuity.

$$\frac{d\rho}{dt} + \underset{\sim}{\nabla}.(\rho\underset{\sim}{v}) = 0 \qquad (8.6)$$

Equation of continuity in substantial derivative form
The partial derivative present in the equation (8.6) can be converted into substantial derivative using vector and tensor identities.

$$\frac{d\rho}{dt} + \underset{\sim}{\nabla}.(\rho\underset{\sim}{v}) = 0$$

$$\frac{d\rho}{dt} + \underset{\sim}{v}.\underset{\sim}{\nabla}\rho + \rho\underset{\sim}{\nabla}.\underset{\sim}{v} = 0$$

Since the substantial derivative of density ρ is defined as $\frac{D\rho}{Dt} = \frac{d\rho}{dt} + \underset{\sim}{v}.\underset{\sim}{\nabla}\rho$, we have

$$\frac{D\rho}{Dt} + \rho\underset{\sim}{\nabla}.\underset{\sim}{v} = 0$$

Material derivative of a fluid mass per unit volume (density) as experienced by a fluid particle. Let say `p' denote a fluid particle. A fluid particle is always travelling with the local fluid velocity $\underline{v}_p(t) = \underline{v}(\underline{\mathrm{x}}_p, t)$. The material derivative of a fluid density $\rho(\underline{\mathrm{x}}, t)$ as experienced by this fluid particle is given by:

$$\underbrace{\frac{D\rho}{Dt}}_{\text{Lagrangian rate of change}} = \underbrace{\frac{d\rho}{dt}}_{\text{Eulerian rate of change}} + \underbrace{\underset{\sim}{v}.\underset{\sim}{\nabla}\rho}_{\text{Convective rate of change}}$$

In many cases, the fluid is incompressible, i.e. density ρ is constant with time as well as space. For example, water is an incompressible fluid – its density can not be changed by applying pressure. In fact, all real liquids are incompressible fluids. For this special case, the equation of continuity is further simplified as below

$$\underset{\sim}{\nabla}.\underset{\sim}{v} = 0 \quad (\rho \text{ is constant})$$

The equation of continuity for an incompressible fluid does not mean that the system is in steady state condition. The velocity of the fluid might still be a function of time. What it implies is that if the velocity of the fluid changes in a particular direction *(x, y* or *z)* then it should also change in the other directions so that mass will be conserved without changing density.

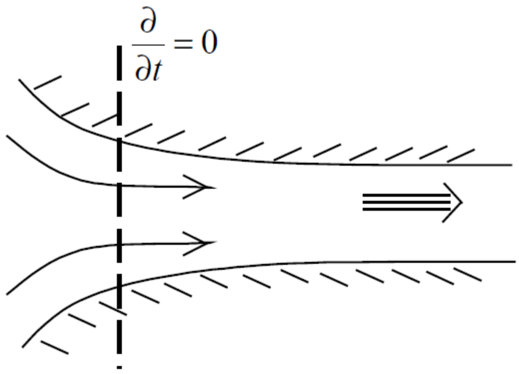


Fig: fixed in position and observing the flow with time

A steady flow is a strictly Eulerian concept $\left(\frac{d\rho}{dt} = 0\right)$. Assume a steady flow where the flow is observed from a fixed position $\frac{d\rho}{dt} = 0$. Note that $\frac{D}{Dt}$ is something following a particle in fluid flow and $\frac{D\rho}{Dt} \equiv 0$ does not mean steady since the flow could speed up at some points and slow down at others.

An incompressible flow is a Lagrangian concept $\left(\frac{D\rho}{Dt} \equiv 0\right)$. Assume a flow where the density of *each* fluid particle is constant in time. Be careful not to confuse this $\frac{d\rho}{dt} = 0$ which means that the density at a particular point in the flow is constant and would allow particles to change density as they flow from point to point.

Axiom-2

Momentum is conserved

This axiom states that momentum should be conserved in any system, i.e. the rate of change of momentum is equal to the sum of all applied forces plus the net inflow of momentum due to convection (when a fluid enters the control volume, it brings momentum into it, and when it leaves the control volume, it takes momentum away from it).

The conservation equation can be written in words as

$$\begin{pmatrix}\text{rate of accummulation}\\ \text{of momentum in CV}\end{pmatrix} = \begin{pmatrix}\text{rate of momentum}\\ \text{entering CV}\\ \text{by convection}\end{pmatrix} - \begin{pmatrix}\text{rate of momentum}\\ \text{leaving CV}\\ \text{by convection}\end{pmatrix} + \left(\sum \text{applied forces}\right)$$

The forces which are encountered in fluid systems are mainly of two types:

1. *Body force*: it depends on the mass of system and acts on the whole volume of the system. The most common example of body force is the gravity force.
2. *Surface force*: it does not depend on the mass of the system and act on surfaces. Surface forces can be classified into two types:
   a. Pressure forces
   b. Shear force (velocity gradients)

Pressure forces are the surface forces which are produced due to pressure in the fluid. It is given by the product of pressure and the area of surface on which it acts. The direction of pressure force is always inwards to the control volume. Therefore, it tries to compress the control volume.

Shear forces are the surface forces which are produced by shear stresses, acting on different surfaces of the control volume. In general, a shear force $\underset{\sim}{T}_n$ acting on a surface, whose unit normal vector is $\underset{\sim}{\delta}_n$, and caused by the shear stress $\underset{\sim}{\tau}$ is given by

$$\underset{\sim}{T}_n = \underset{\sim}{\delta}_n . \underset{\sim}{\tau} \tag{9.2}$$

In Cartesian coordinates (and for rectangular CVs), it will be convenient to take surfaces whose unit normal vectors are in the x, y, and z directions themselves. Let us now consider the surface which has its outward normal in the *x* direction. Since the fluid is present outside the control volume, it will apply the force $\underset{\sim}{T}_x$ on the surface, which will be given as

$$\underset{\sim}{T}_x = \underset{\sim}{T}_1 = \underset{\sim}{\delta}_1 . \underset{\sim}{\tau} = \underset{\sim}{\delta}_1 . \tau_{ij} \underset{\sim}{\delta}_i \underset{\sim}{\delta}_j \; = \tau_{1j} \underset{\sim}{\delta}_j$$

$$= \tau_{11} \underset{\sim}{\delta}_1 + \tau_{12} \underset{\sim}{\delta}_2 + \tau_{13} \underset{\sim}{\delta}_3 \tag{9.3}$$

In x, y and z coordinates $\underset{\sim}{T}_x = \tau_{xx} \underset{\sim}{\delta}_x + \tau_{xy} \underset{\sim}{\delta}_y + \tau_{xz} \underset{\sim}{\delta}_z$ (9.4)

Therefore, the first index (x) shows the face on which the force is acting, and the second index (x, y, z) shows the actual direction of the force.

Similarly, we can write, $\underset{\sim}{T}_y = \tau_{yx} \underset{\sim}{\delta}_x + \tau_{yy} \underset{\sim}{\delta}_y + \tau_{yz} \underset{\sim}{\delta}_z$ and $\underset{\sim}{T}_z = \tau_{zx} \underset{\sim}{\delta}_x + \tau_{zy} \underset{\sim}{\delta}_y + \tau_{zz} \underset{\sim}{\delta}_z$. Thereby, we can conclude that 9 components of shear stress are important in general:

τ_{xx} τ_{xy} τ_{xz} act on *x* face
τ_{yx} τ_{yy} τ_{yz} act on *y* face
τ_{zx} τ_{zy} τ_{zz} act on z face

Recall in newton law of viscosity, parallel plate experiment we had seen only one component to be present in the Newton's law of viscosity. How do we reconcile this difference? To understand it, let us again consider the experiment in which a fluid is filled between two parallel plates, which are separated by a distance h. We had applied the force F in the x direction (along the plate). There might be 9 components of shear forces in general, but the fluid in consideration is moving in x direction only. Therefore, only those components will appear which have x directional forces. Thus the stress tensor reduces to

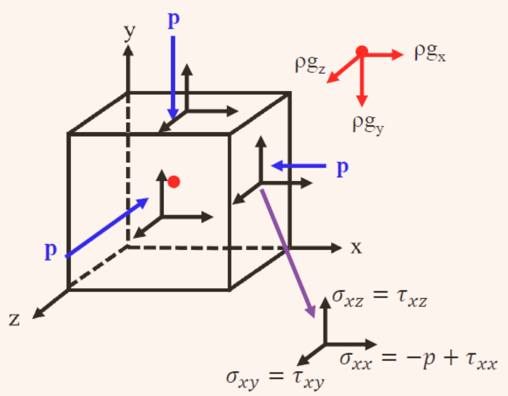
$$\begin{pmatrix} \tau_{xx} & 0 & 0 \\ \tau_{yx} & 0 & 0 \\ \tau_{zx} & 0 & 0 \end{pmatrix}$$

Now, we are left with only three components of shear forces. If we move in the x direction, there is no solid boundary present which can provide a velocity gradient. Hence, there is no shear force on the plane having normal in the x direction (consequently $\tau_{xx} = 0$). In the same way, there is no boundary present in the z direction, so there is no shear force on the plane having normal in the z direction (consequently $\tau_{zx} = 0$). Therefore τ_{yx} is the only non-zero component of the shear stress.

$$\tau_{yx} = -\mu \frac{dv_x}{dy} \qquad (9.5)$$

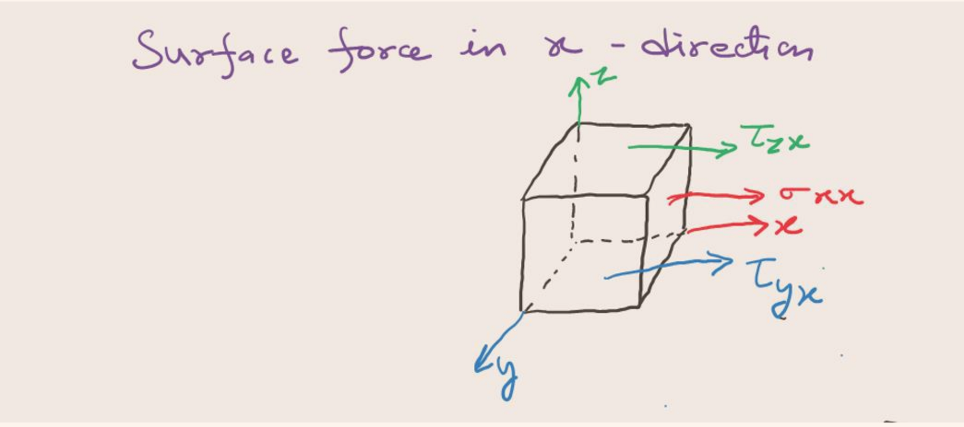
Below figures for your reference to understand stresses and forces due to stresses

Looking at stresses (both normal and tangential)



Flowing fluid

$$\sigma_{xx} = -p + \tau_{xx}$$
$$\sigma_{yy} = -p + \tau_{yy}$$
$$\sigma_{zz} = -p + \tau_{zz}$$



Now let's look at convective term: Momentum can, in addition, be transported by the bulk flow of the fluid, and this process is called convective transport. To discuss this, we use Fig. 2 and focus our attention on a cube-shaped region in space through which the fluid is flowing. At the center of the cube (located at x, y, z) the fluid velocity vector is $\underline{v}$we consider three mutually perpendicular planes (the shaded planes) through the point **x,** y, z, and we ask how much momentum is flowing through each of them. Each of the planes is taken to have unit area

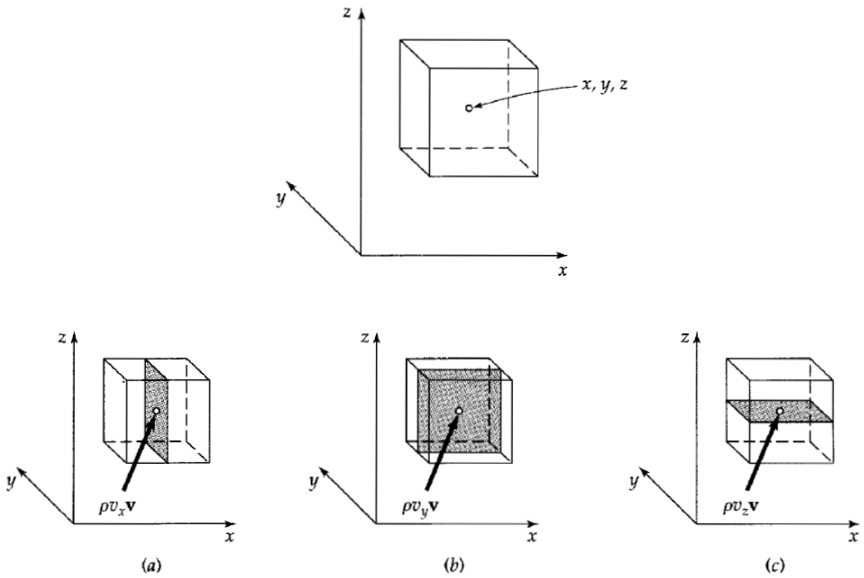


Fig. 2: The convective momentum fluxes through planes of unit area perpendicular to the coordinate directions.

The volume rate of flow across the shaded unit area in Fig 2 (a) is v_x . This fluid carries with it momentum $\rho\underline{v}$ per unit volume. Hence the momentum flux across the shaded area is $v_x\rho\underline{v}$. Similarly, the momentum flux across the shaded area in Fig 2 (b) is $v_y\rho\underline{v}$, and the momentum flux across the shaded area in Fig 2 (c) $v_z\rho\underline{v}$.

These three vectors $v_x\rho\underline{v}$ $v_y\rho\underline{v}$ and $v_z\rho\underline{v}$ describe the momentum flux across the three areas perpendicular to the respective axes. Each of these vectors has an x, y, and z component. These components can be arranged as shown in Table below. The quantity $\rho v_x v_y$ is the convective flux of y-momentum across a surface perpendicular to the x direction. This should be compared with the quantity τ_{xy} which is the molecular flux of y-momentum across a surface perpendicular to the x direction. The sign convention for both modes of transport is the same. In table this can be related as $\rho\underline{v}\ \underline{v}$ a second order tensor, it is called the convective momentum-flux tensor

Direction normal to the shaded surface	Flux of momentum through the shaded surface	Convective momentum flux components		
		x-component	y-component	z-component
x	$\rho v_x \mathbf{v}$	$\rho v_x v_x$	$\rho v_x v_y$	$\rho v_x v_z$
y	$\rho v_y \mathbf{v}$	$\rho v_y v_x$	$\rho v_y v_y$	$\rho v_y v_z$
z	$\rho v_z \mathbf{v}$	$\rho v_z v_x$	$\rho v_z v_y$	$\rho v_z v_z$

We will focus and understand to solve a few problems in momentum transport involving fluid flow in simple geometries using the shell momentum balance approach. However, without going into problem solving or derivation if we look and make analogy from mass conservation where we looked at total derivative for incompressible flow, here for momentum conservation for per unit volume.

$$\rho \frac{D\underset{\sim}{v}}{Dt} = \rho \frac{d\underset{\sim}{v}}{dt} + \rho \underset{\sim}{v}.\underset{\sim}{\nabla}\underset{\sim}{v} + \sum F_{per\ unit\ vol} \qquad (9.6)$$

$$\underbrace{\frac{D\underset{\sim}{v}}{Dt}}_{\text{Lagrangian acceleration}} = \underbrace{\frac{d\underset{\sim}{v}}{dt}}_{\text{Eulerian acceleration}} + \underbrace{\underset{\sim}{v}.\underset{\sim}{\nabla}\underset{\sim}{v}}_{\text{Convective acceleration}} \qquad (9.7)$$

For steady flow, eulerian acceleration will be zero and fluid still may have convective acceleration

Lecture 10 (31th August, Sat/Mon)

Shell Balances

When fluid flow occurs in a single direction everywhere in a system, shell balances are useful devices for applying the principle of conservation of momentum. An example is incompressible laminar flow of fluid in a straight circular pipe. Other examples include flow between two wide parallel plates or flow of a liquid film down an inclined plane.

In the above situations, fluid velocity varies across the cross-section only in one coordinate direction and is uniform in the other direction normal to the flow direction. For flow through a straight circular tube, there is variation with the radial coordinate, but not with the polar angle. Similarly, for flow between wide parallel plates, the velocity varies with the distance coordinate between the two plates. If the plates are sufficiently wide, we can ignore variations in the other direction normal to the flow which runs parallel to the surfaces of the plates. If we neglect entrance and exit effects, the velocity does not vary with distance in the flow direction in both cases; this is the definition of fully developed flow.

A momentum balance can be written for a control volume called a shell, which is constructed by translating a differential cross-sectional area (normal to the flow) in the direction of the flow over a finite distance.

The key idea is that we use a differential distance in the direction in which velocity varies. Later, we consider the limit as this distance approaches zero and obtain a differential equation. Typically, this is an equation for the shear stress. By inserting a suitable rheological model connecting the shear stress to the velocity gradient (like newton law of viscosity for Newtonian fluid), we can obtain a differential equation for the velocity distribution. This is then integrated with the boundary conditions relevant to the problem to obtain the velocity profile. Once the profile is known, we can calculate the volumetric flow rate and the average velocity as well as the maximum velocity. If desired, the shear stress distribution across the cross-section can be written as well. In the case of pipe flow, we shall see how this yields the well-known Hagen-Poiseuille equation connecting the pressure drop and the volumetric flow rate.

Flow through horizontal pipe

Step 1: Assumptions for formulating the problem

1. Steady state flow (velocities are not dependent on time)
2. Laminar flow (no flow in θ and r directions)
3. Incompressible (constant density), Newtonian (constant viscosity) fluid

4. Isothermal condition
5. Fully developed flow

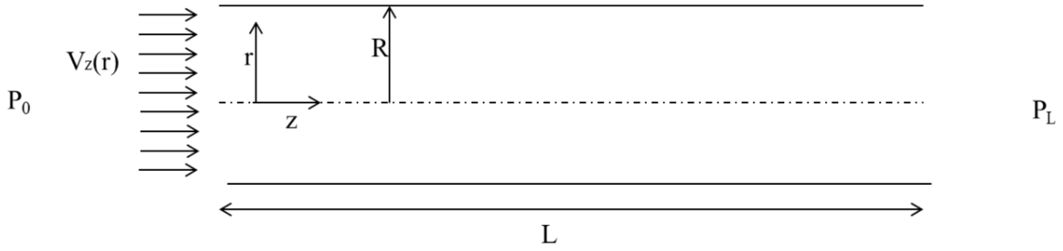


Fig 10.1 Laminar flow in a horizontal pipe

Step 2: We have to intuitively guess the velocity profile.

Since the flow is steady and laminar, we can say intuitively that the velocities in *r* direction and θ direction are zero. Due to steady state conditions, fluid velocity in *z* direction v_z is not dependent on *t*. Due to the axisymmetric geometry fluid velocity v_z is independent of θ.

$$v_r = 0, \; v_\theta = 0, \; v_z = v_z(r, \; z, \cancel{\theta}, \cancel{t})$$

$$v_z = v_z\left(r, \; z\right) \tag{10.1}$$

By applying the equation of continuity in cylindrical coordinates

$$\frac{\partial v_z}{\partial z} = 0$$

Hence, $v_z = v\left(r\right)$ (10.2)

This is also a situation of fully developed flow, where velocity varies only radial and remains same axially.

By using Axiom 2, there are three components of momentum balance.

1. Since $v_r = 0$, *r* directional momentum balance will not appear
2. Since $v_\theta = 0$, θ directional momentum balance will not appear
3. Since $v_z \neq 0$, *z* directional momentum balance will be used

Step 3: To decide the control volume.

The control volume should be decided very carefully. The geometry and size of the control volume should be taken according to the geometry of the system and based on the conditions given in the problem. In this problem, the geometry of the pipe is cylindrical, hence we use the cylindrical control volume. The fluid is flowing in the *z* direction but velocity is changing only in *r* direction. The control volume is taken in such a way that the variable thickness of control volume (differential element) is in the r direction so that we can integrate the property of flow in the r direction. As the flow is not depending on z and θ direction, we are free to choose any dimension in z or θ directions. This means that z can be any length in the system. It may be $L/4$, $L/2$ or L. In a similar manner, any value of θ can be taken. It may be 2π or π or $\pi/2$ or $\pi/4$. But in the r direction, we need to take the differential element dr so that we can integrate it for differential flow properties. (Note: try to take control volume in symmetric form if flow properties are not constant.) We consider a cylindrical strip as shown in Fig 10.1 as the control volume. The length of strip is L which is equal to length of pipe. The thickness of the strip is dr.

Step 4: Momentum balance for the control volume

As discussed earlier, the momentum balance can be written in two ways:

1. Taking velocity gradients due to viscosity as shear forces.
2. Taking velocity gradients due to viscosity as momentum flux.

We take both methods one by one and prove that both methods give the same result

Momentum balance using velocity gradients as shear force

Momentum coming into the control volume due to convection $=(\rho v_z^2 2\pi r dr)\big|_{z=0}$ (10.3)

Momentum leaving from control volume due to convection $=(\rho v_z^2 2\pi r dr)\big|_{z=L}$ (10.4)

Body forces = 0: Since the pipe is horizontal, gravity is not affecting the flow. No other body forces are acting on the control volume.

Surface forces

1. Pressure force – Fluid is flowing in z direction only. So pressure forces are working on the surface normal to z direction. The area of these surfaces is $2\pi r dr$ at $z=0$ and $2\pi r dr$ at $z=L$. Pressure is compressive force acting on the surface. Fluid surrounding the control volume is compressing it.

   Pressure force at $z=0$ is $P_0 2\pi r dr\big|_{z=0}$ (10.5)

   Pressure force at $z=L$ is $P_L 2\pi r dr\big|_{z=L}$ (10.6)

2. Shear forces:

   9 components are possible for the shear stress tensor which is given below.

$$\begin{pmatrix} \tau_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta z} \\ \tau_{zr} & \tau_{z\theta} & \tau_{zz} \end{pmatrix}$$

Among all *9* components the first column of stresses leads to *r* directional flow, the second column of stresses leads to θ directional flow, and the third column is leads to *z* directional flow. Since the fluid is flowing in the *z* direction, only the third column is non-zero. Velocity gradient is present only in the *r*-direction, hence only τ_{rz} is non-zero, the remaining two terms are zero. We now need to decide the direction in which the shear forces are acting.

$$\underset{\sim}{T}_n = \underset{\sim}{\delta}_n . \underset{\sim}{\tau} \tag{10.7}$$

If the unit vector is positive then T_n is also positive and if the unit vector is negative then T_n is negative. So that τ_{rz} at $r+dr$ is positive and τ_{rz} at *r* is negative.

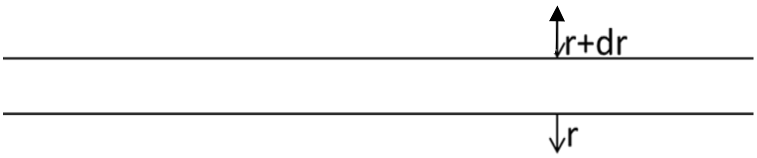


Fig 10.2 Control volume for flow through pipe

Accumulation term:

The system is in steady state, and hence the rate of accumulation of momentum equals 0

Substituting the mathematical expressions into the word equation for momentum balance, we have

$$0 \ = \ (rv_z{}^2 2\pi r dr)|_{z=0} - (rv_z{}^2 2\pi r dr)|_{z=L} + 0 \ + \ P_0 2\pi r dr \ - \ P_L 2\pi r dr \ + \ (\tau_{rz} 2\pi r L)|_{r+dr} - (\tau_{rz} 2\pi r L)|_r \tag{10.8}$$

The velocity is not changing along the axial direction of the pipe which indicates that the convective momenta at both ends of the pipe are equal and cancel out.

$$0 \ = \ 2\pi L[(\tau_{rz} r)|_{r+dr} - (\tau_{rz} r)|_r] \ + \ P_0 2\pi r dr \ - \ P_L 2\pi r dr \tag{10.9}$$

$$0 = \frac{\mathrm{P_0 - P_L}}{L} + \frac{(\tau_{\mathrm{rz}}\mathrm{r})|_{\mathrm{r+dr}} - (\tau_{\mathrm{rz}}\mathrm{r})|_{\mathrm{r}}}{rdr} \tag{10.10}$$

Taking the limit as $dr \rightarrow 0$, we note that τ_{rz} is a function of r only which means we get the total derivative instead of the partial derivative.

$$\frac{d(\tau_{\mathrm{rz}}\mathrm{r})}{dr} = \frac{-\mathrm{r}\,(\mathrm{P_0} - \mathrm{P_L})}{L} \tag{10.11}$$

Integrating once with respect to the variable *r*

$$\tau_{\mathrm{rz}}\mathrm{r} = -\frac{r^2}{2}\frac{(\mathrm{P_0} - \mathrm{P_L})}{L} + c_1$$

$$\tau_{\mathrm{rz}} = -\frac{r}{2}\frac{(\mathrm{P_0} - \mathrm{P_L})}{L} + \frac{c_1}{r} \tag{10.12}$$

Here, c_1 is a constant of integration. Equation (10.12) shows that if $r=0$, the value of τ_{rz} will be infinite, which is not physically possible. Therefore, c_1 must be zero. Hence,

$$\tau_{\mathrm{rz}} = -\frac{r}{2}\frac{(\mathrm{P_0} - \mathrm{P_L})}{L} \tag{10.13}$$

Now applying Newton's law of viscosity,

$$\tau_{\mathrm{rz}} = +\mu\frac{dv_{\mathrm{z}}}{dr} = -\frac{r}{2}\frac{(\mathrm{P_0} - \mathrm{P_L})}{L} \tag{10.14}$$

We have taken positive sign in the Newton's law of viscosity as we have taken the viscous forces to be the shear stresses.

Momentum balance using velocity gradients as momentum flux

Now we will employ the second method where velocity gradients due to viscosity are defined as viscous momentum flux. This momentum flux is entering the control volume through the surface $2\pi rL$ at $r = r$ and leaving through the surface $2\pi rL$ at $r = r + dr$, as shown in Figure 10.3.

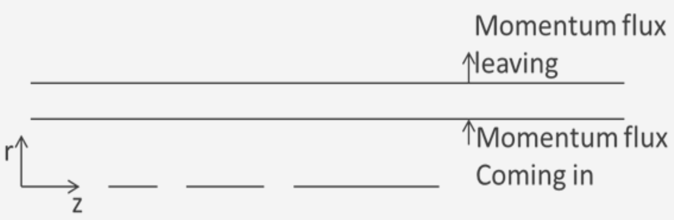


Fig 10.3 Momentum flux applied on control volume

Momentum flux at $r = r$ is $(\tau_{rz} 2\pi rL)\big|_r$ (10.15)

Momentum flux at $r = r + dr$ is $(\tau_{rz} 2\pi rL)\big|_{r+dr}$ (10.16)

In this case, the momentum balance equation in words is

$$\begin{pmatrix}\text{rate of accummulation}\\ \text{of momentum in CV}\end{pmatrix} = \begin{pmatrix}\text{rate of momentum}\\ \text{entering CV}\\ \text{by convection}\end{pmatrix} - \begin{pmatrix}\text{rate of momentum}\\ \text{leaving CV}\\ \text{by convection}\end{pmatrix} + \left(\sum \text{applied forces}\right)$$

$$+ \begin{pmatrix}\text{rate of viscous}\\ \text{momentum}\\ \text{entering CV}\end{pmatrix} - \begin{pmatrix}\text{rate of viscous}\\ \text{momentum}\\ \text{leaving CV}\end{pmatrix}$$

Substituting the mathematical expressions into this equation, we get

$$0 = 0 + 0 + (\tau_{rz} 2\pi rL)\big|_r - (\tau_{rz} 2\pi rL)\big|_{r+dr} + \left(P_0 - P_L\right) 2\pi r dr \tag{10.17}$$

$$0 = (\tau_{rz} r)\big|_r - (\tau_{rz} r)\big|_{r+dr} + \frac{\left(P_0 - P_L\right)}{L}$$

$$0 = -\frac{d(\tau_{rz} \mathrm{r})}{rdr} + \frac{(\mathrm{P_0} - \mathrm{P_L})}{L}$$

$$\frac{d(\tau_{rz} \mathrm{r})}{dr} = \frac{(\mathrm{P_0} - \mathrm{P_L})}{L} r$$

$$\tau_{rz} = \frac{(\mathrm{P_0} - \mathrm{P_L})}{2L} r + \frac{c_1}{r} \tag{10.18}$$

As we discussed earlier, c_1 should be zero.

$$\tau_{rz} = \frac{(\mathrm{P_0} - \mathrm{P_L})}{2L} r \tag{10.19}$$

Now applying Newton's law of viscosity for viscous momentum flux

$$\tau_{rz} = -\mu \frac{dv_z}{dr} = \frac{(\mathrm{P_0} - \mathrm{P_L})}{2L} r$$

$$\mu \frac{dv_z}{dr} = -\frac{(\mathrm{P_0} - \mathrm{P_L})}{2L} r \tag{10.20}$$

Equation (10.14) and (10.20) are identical and hence show that both methods give the same result.

Integrating once more with respect to the variable r, we get

$$v_z = -\frac{(\mathrm{P_0} - \mathrm{P_L})}{2\mu L} \frac{r^2}{2} + c_2 \tag{10.21}$$

Step 5: Boundary conditions

By applying the no-slip boundary condition, $v_z = 0$ at $r = R$

$$0 = -\frac{(\mathrm{P_0} - \mathrm{P_L})}{4\mu L} R^2 + c_2$$

$$c_2 = \frac{(P_0 - P_L)}{4\mu L} R^2 \tag{10.22}$$

Substituting the value of C_2 in Equation (10.21), we get

$$v_z = \frac{(P_0 - P_L)}{4\mu L} (R^2 - r^2) \tag{10.23}$$

Note: C_1 can also be calculated by using the boundary condition in terms of velocity v_z: either v_z is finite at $r=0$ or $\left.\frac{dv_z}{dr}\right|_{r=0} = 0$ (since the velocity profile is symmetric about $r=0$). All typesof boundary conditions give the same answer.

$$v_z = \frac{(P_0 - P_L)}{4\mu L} R^2 \left(1 - \frac{r^2}{R^2}\right) \tag{10.24}$$

Since, the shear stress is zero at the centre of pipe, the maximum velocity of the fluid will be exhibited at the centre of the pipe.

$$v_{z\max} = v_z \big|_{r=0} = \frac{(P_0 - P_L)}{4\mu L} R^2 \tag{10.25}$$

Thus, the velocity profile can also be expressed in terms of the maximum velocity as

$$v_z = v_{z,\max} \left(1 - \frac{r^2}{R^2}\right) \tag{10.26}$$

The average velocity of the fluid in the pipe is the average of local velocities. Thus, the average velocity can be calculated by estimating the volumetric flow rate through the pipe and then dividing it by the cross sectional area of the pipe. The total volumetric flow in the system is -

$$Q = \int dQ = \int v_z 2\pi r dr = \int v_{z\max} \left[1 - \frac{r^2}{R^2}\right] 2\pi r dr$$

$$= \int_0^R 2\pi v_{z\max} \left[r - \frac{r^3}{R^2}\right] dr = 2\pi v_{z\max} \left[\frac{r^2}{2} - \frac{r^4}{4R^2}\right]_0^R$$

$$= 2\pi v_{z\max} \left[\frac{R^2}{2} - \frac{R^2}{4}\right] = \pi v_{z\max} \frac{R^2}{2}$$

$$v_{z,avg} = \frac{Q}{A_c} = \pi v_{z,\max} \frac{R^2}{2} / \pi R^2 = \frac{v_{z,\max}}{2} \tag{10.26}$$

The velocity profile for laminar flow in a circular pipe is thus parabolic in shape (Figure 10.5).

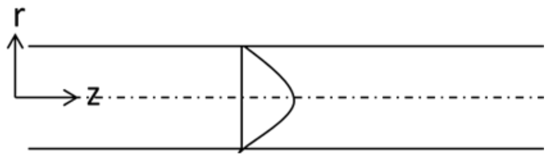


Fig 10.5 Velocity profile in horizontal pipe

We can also find the radial distance at which the local velocity of fluid flow equals the average velocity. Substituting $v_z = v_{z,avg} = \frac{v_{z,\max}}{2}$ into Equation (10.26), we get

$$1 - \frac{r^2}{R^2} = \frac{1}{2}$$

$$\frac{r^2}{R^2} = \frac{1}{2}$$

or, $r = \frac{R}{\sqrt{2}}$ (10.27)

Equation (10.27) shows the position in the pipe where the local fluid velocity equals the average velocity of fluid flow.

The volumetric flow rate in terms of pressure drop is as follows

$$Q = v_{zavg} \pi R^2 = \frac{\pi(\mathrm{P_0 - P_L})}{8\mu L} R^4 \tag{10.28}$$

$$Q = \frac{\pi(\mathrm{P_0 - P_L})}{128\mu L} D^4 \tag{10.29}$$

Equation (10.29) is known as the **Hagen–Poiseuille equation**, and is useful for relating different flow parameters and fluid parameters. For example, if the pressure drop is given, we can calculate the volumetric flow rate in the pipe and vice-versa. This equation can also be used for the calculation of viscosity in capillary viscometer problems. However, the equation is valid only for fully developed and laminar flow. Therefore, when this equation is used for calculation of viscosity, there is some error due to developing flow at both ends of the pipe. Hence, this equation has to be modified for real situations. In practice, an effective length L' is used instead of the actual capillary length L in the calculations. In case of capillary viscometer, care must be taken so that the diameter of the capillary is small enough so that fully developed flow can be achieved within the length of the capillary.

Lecture 11 (2nd September, Mon) and

Lecture 12 (5th September, Thursday)

Friction factor

The friction factor is a dimensionless number in fluid mechanics and momentum transport which provides an idea about the magnitude of shear stress produced by a solid boundary on fluid flow. It is defined as the ratio of shear stress at the wall due to fluid flow and the kinetic energy head of fluid flow $\frac{1}{2}\rho v^2{}_{zavg}$. Here, ρ is the density of the fluid and $v_{z\,avg}$ is the average velocity of fluid flow. The friction factor is thereby defined as

$$f = \frac{\tau_w}{\frac{1}{2}\rho v^2{}_{zavg}} \tag{11.1}$$

Where τ_w is the shear stress on the solid wall due to fluid flow. Since, the wall also applies an equal and opposite shear stress on the fluid,

$$\tau_w = -\tau_{rz}\big|_{r=R} \tag{11.2}$$

For laminar, fully developed and steady flow through a horizontal, cylindrical pipe, the velocity profile is parabolic and is given by

$$v_z = v_{z\max}\left[1-\frac{r^2}{R^2}\right] \tag{11.3}$$

So that the velocity gradient at the wall (r=R) is

$$\left.\frac{dv_z}{dr}\right|_{r=R} = -\frac{2v_{z\max}}{R} \tag{11.4}$$

Consequently, the shear stress (given by Newton’s law of viscosity) is

$$\tau_{rz}\big|_{r=R} = +\mu\left.\frac{dv_z}{dr}\right|_{r=R} = -\frac{2\mu v_{z\max}}{R}$$

$$\tau_{\mathrm{w}} = \ -\tau_{\mathrm{rz}}\big|_{r=R} = \frac{2\mu v_{z\max}}{R} = \frac{2\mu(\mathrm{P}_0-\ \mathrm{P}_{\mathrm{L}})}{4\mu L}\frac{R^2}{R} = \frac{(\mathrm{P}_0-\ \mathrm{P}_{\mathrm{L}})}{2L}R \tag{11.5}$$

Hence, the friction factor for laminar, fully developed and steady flow through a cylindrical pipe is given as

$$f = \frac{\tau_w}{\frac{1}{2}\rho v^2{}_{zavg}} = \frac{\frac{(\mathrm{P}_0-\ \mathrm{P}_{\mathrm{L}})}{2L}R}{\frac{1}{2}\frac{\rho(\mathrm{P}_0-\ \mathrm{P}_{\mathrm{L}})}{8\mu L}R^2} = \frac{8\mu}{\rho R v_{zavg}} \tag{11.6}$$

$$f = \frac{16\mu}{\rho D v_{zavg}} = \frac{16}{\mathrm{Re}} \tag{11.7}$$

Equation (11.7) shows that the friction factor in laminar flow region depends only on the Reynolds number. Clearly, friction factor is also a dimensionless number (as mentioned earlier).

Friction factor in turbulent flow

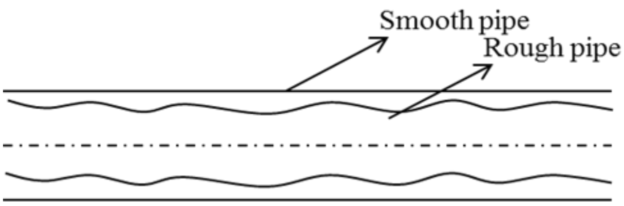


Fig 11.1 Smooth and rough surface of pipe

In turbulent flow, the friction factor also depends on the surface of the pipe. A rough pipe leads to higher turbulence than a smooth pipe, so that the friction factor for smooth pipes is less than that for rough pipes. Therefore, the ratio of surface roughness height ($\in$) to pipe diameter (D) is used to quantify the "roughness" of the pipe surface. However, there is no analytical equation for calculating the friction factor in turbulent flow. In practice, the shear stress on the wall is calculated by measuring the pressure drop across the pipe and flow rate through the pipe. If we plot friction factor vs Reynolds number on a log-log plot for constant surface roughness, we get a smooth curve. The curves are different for different surface roughness as given in Figure 11.2. The collection of these *f-Re* plots is called Moody Chart, and can be used for estimating the friction factor given the flow parameters.

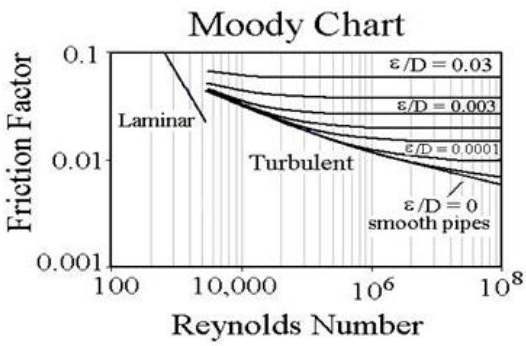


Fig 11.2 Moody chart for flow through cylindrical pipe

Ref: http://www.brighthub.com/engineering/civil/articles

Solution of momentum transport problems by shell momentum balance in laminar flow

In this section, we will solve a few momentum transport problems for laminar flow in simple geometries using the shell momentum balance approach. The detailed procedure is outlined below.

1. *Make a diagram of the flow geometry, clearly indicating all boundaries.*
2. *Select the appropriate coordinate system and mark the origin and axes on the diagram:*
   The coordinate system should be selected according to the geometry of the system. For example, if the fluid is flowing in between two parallel plates, the rectangular coordinate system should be employed; for pipe flow problems, the cylindrical coordinate system should be used; for fluid flow around a sphere, the spherical coordinate system should be used. The origin and axes should be placed according to the system boundaries so that they can be appropriately labeled. For example, in a pipe flow, the origin should be on central axes of the pipe so that the pipe wall can be defined by a radius R from the axis.
3. *Specify all assumptions according to the problem statement and formulate the problem mathematically for obtaining an analytical solution.*
   Some common assumptions for momentum transport problems are as follows:
   - Constant density, viscosity and steady state
   - Laminar flow (simple shear flow)
   - Newton's law is applicable
   - No slip boundary condition

   These assumptions simplify the problem, and enable us to solve it analytically.
4. *Guess the non-zero velocity components:*
   This is an important step to solve these problems. Since laminar flow is regular, and the fluid layers flow parallel to each other without mixing, it is easy to guess the non-zero velocities by intuition and most of the times these intuitions are correct. In general, the fluid can flow only in the direction where solid boundaries do not obstruct its path. Additionally, the velocities are a function of the direction in which it encounters solid boundaries which produce resistance to the flow due to the no slip boundary condition.

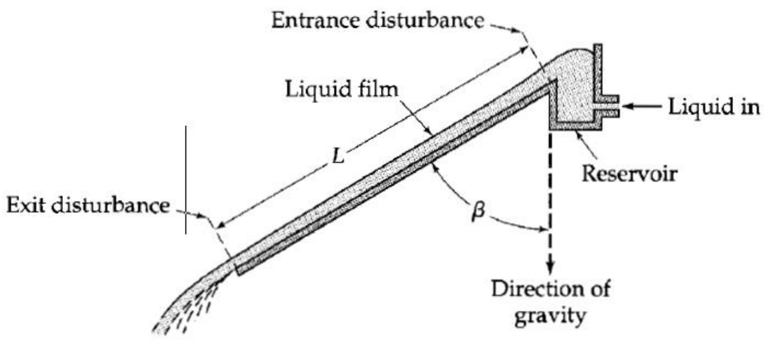
5. *Apply the equation of continuity:* Application of the equation of continuity for the non-zero velocity components further simplifies the problem by eliminating the functional dependence of the non-zero velocity components on certain spatial coordinate(s).
6. *Determine the non-zero shear stress component(s)*. The shear stress components have to be decided based on the non-zero component(s) of velocities.
7. *Determine all surface and body forces acting on the fluid*. According to problem statement, write down all surface and body force acting on the fluid carefully.
8. *Determine control volume for problem:*

   Draw control volume in system diagram according to system shape, size and problem statement. The selection of proper control volume is very important to solve problem correctly. The control volume should be select in such way that it can be easily integrated for whole system easily. The differential length of control volume should be taken in direction of changing velocity. For example, for solving pipe flow problem, control volume should be taken for full axial length of pipe and differential thickness of control volume should be in radial direction. Therefore, control volume should be in the shape of hollow circular tube of full length of pipe "L" and differential thickness "dr". If we take half length of pipe "L/2" and half cross section of tube then we can solve the problem but problem have to integrate again and again for determine solution for full pipe.
9. Write momentum balance for control volume. Write down momentum balance equation for given problem. The shear stress can be taken as shear force or momentum flux, both gives the same results.
10. Divide full equation by volume of control volume. Take limit of differential thickness tends to zero and develop differential form of equation.
11. Substitute Newton's low of viscosity for shear stress component.
12. Determine boundary condition: the boundary condition should be defined for solving the problems.
13. Solve the equation and determine the velocity profile.

Falling Film problem:

a) Falling film on inclined flat surface

A inclined surface of length L, width W are situated to angle β from direction of gravity as shown in Fig 11.3. Newtonian fluid is freely falling on the surface as a film of thicknessδ. The liquid is flowing in laminar condition. Determine velocity profile, maximum flow rate and shear force on the plate by fluid. Write appropriate assumption for solving the problem analytically.



Solution:

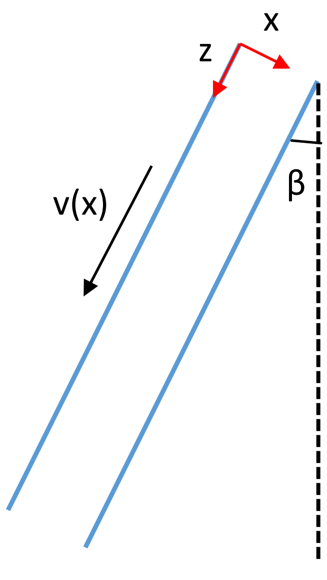


Fig 11.3 Laminar flow on inclined surface

Assumption:

- Steady and incompressible flow
- Laminar flow
- Newton's law is applicable
- No slip boundary condition at the surface

Non- zero velocity component:

The fluid is flowing in z direction only so only z component of velocity is non-zero.

Profile:

$v_x = 0$

$v_y = 0$

$$v_z = v_z(x, y, z, t) \quad (11.8)$$

Since there is no solid boundary at y direction so we can assume initially that v_z can't depends upon y direction.

Now

$$v_z = v_z(x, z) \quad (11.9)$$

Use equation of continuity for Cartesian coordinate system.

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

$$= \frac{\partial v_z}{\partial z} = 0 \quad (11.10)$$

Since, density is constant.

Equation (11.10) indicate that v_z is not depends on the z.

So, $v_z = v_z(x)$ (11.12)

There are nine components of shear stress

$\tau_{xx}\ \tau_{xy}\ \tau_{xz} \rightarrow$ acting on x face

$\tau_{yx}\ \tau_{yy}\ \tau_{yz} \rightarrow$ acting on y face

$\tau_{zx}\ \tau_{zy}\ \tau_{zz} \rightarrow$ acting on z face

Since v_z is only non-zero velocity which depends on x direction only. τ_{xz} is only non-zero component of shear stress and others are zero.

The liquid is falling freely due to gravity. The pressure is same at both ends of inclined plane so there is no pressure force on fluid. We can solve this problem by assuming shear stress as force or shear stress as momentum flux. We will use both methods one by one.

1. Assume τ_{xz} as momentum flux

Draw control volume of length L, width W and differential thickness δ.

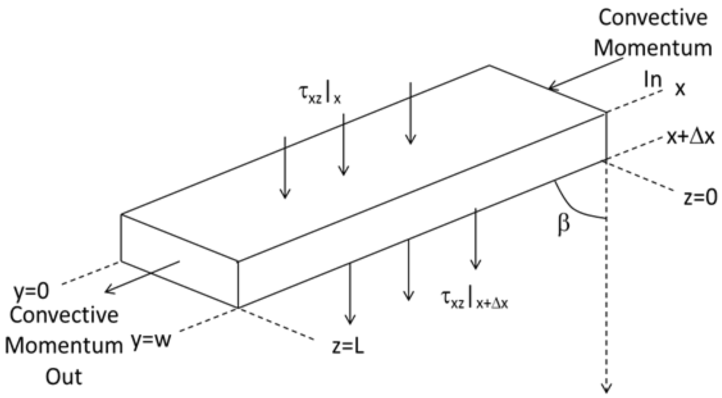


Fig 11.3 Control volume for falling film problem

The direction of momentum flux should be taken in positive direction of axes.

Rate of momentum in across a surface at $x \; = \; Lw\tau_{xz}|_x$ (11.13)

Rate of momentum out across a surface at $x + \Delta x = Lw\tau_{xz}|_{x+\Delta x}$ (11.14)

The gravity force acting on fluid in z direction $(Lw\Delta x)(\rho g \; cos\beta)$ (11.15)

Here ρ is the density of fluid.

Rate of momentum enter due to convection transport $= (\rho v_z w\Delta x \; v_z)|_{z=0}$ (11.16)

Rate of momentum out due to convection transport $= (\rho v_z w\Delta x \; v_z)|_{z=L}$ (11.17)

Now write the momentum transport equation

When these terms are substitute into the z- momentum balance

$$(\rho v_z w dx v_z)|_{z=0} - (\rho v_z w dx \; v_z)|_{z=L} + Lw(\tau_{xz}|_x - \tau_{xz}|_{x+\Delta x}) + (Lw\Delta x)\,(\rho g \; cos\beta) \; = \; 0 \tag{11.18}$$

Since velocity v_z is not depend on z. Thus, equation (11.16) and (11.17) are equal. So they will be canceling out in equation (11.18).

$$LW(\tau_{xz}|_x - \tau_{xz}|_{x+\Delta x}) \; + \; (LW\Delta x)(\rho g \; cos\beta) = 0 \tag{11.19}$$

Divide equation (11.19) by volume of control volume (LWΔx) and taking limit

$$\lim_{\Delta x \to 0} \frac{(\tau_{xz}|_x - \tau_{xz}|_{x+Dx})}{\Delta x} + \; (\rho g \cos\beta) = 0 \tag{11.20}$$

$$-\frac{\partial \tau_{xz}}{\partial x} = -\rho g \; cos \; \beta \tag{11.21}$$

τ_{xz} is function of x only.

$$-\frac{d\tau_{xz}}{dx} = -\rho g \; cos \; \beta \tag{11.22}$$

$\tau_{xz} = -\mu \dfrac{dv_z}{dx}$ as a momentum flux

$$\frac{d}{dx}\left(-\mu\frac{dv_z}{dx}\right)=\rho g\cos\beta \tag{11.23}$$

$$-\mu\frac{d^2v_z}{dx^2}=\rho g\cos\beta \tag{11.24}$$

$$\frac{d^2v_z}{dx^2}=-\left(\frac{\rho g\cos\beta}{\mu}\right) \tag{11.25}$$

$$\frac{dv_z}{dx}=-\left(\frac{\rho g\cos\beta}{\mu}\right)x+c_1 \tag{11.26}$$

$$v_z=-\left(\frac{\rho g\cos\beta}{\mu}\right)\frac{x^2}{2}+c_1x+c_2 \tag{11.27}$$

Now we have to define boundary condition.

First boundary condition at $x=0$ liquid surface is in contact with air so that shear stress at both surface should be equal.

So $\tau_{xz}\text{air}\big|_{x=0}=\tau_{xz}\text{liquid}\big|_{x=0}$ (11.28)

Both are Newtonian fluids.

$$\mu_g\frac{dv_{z,air}}{dx}\bigg|_{x=0}=\mu\frac{dv_{z,liquid}}{dx}\bigg|_{x=0} \tag{11.29}$$

Here ρ_g is the density and μ_g is the viscosity of air.

$$\frac{dv_{z,liquid}}{dx}\bigg|_{x=0}=\frac{\mu_g}{\mu}\frac{dvzair}{dx}\bigg|_{x=0} \tag{11.30}$$

In equation (11.30) μ_g is very smaller than μ, and velocity gradient for air is also much smaller so RHS in 11.30 is very small i.e near to zero.

$$\frac{dv_{zliquid}}{dx}\bigg|_{x=0}=0 \tag{11.31}$$

By substituting value of equ.(11.31) in equ.(11.26), we get

$$C_1=0 \tag{11.32}$$

Boundary condition -2

At solid surface, due to no slip boundary condition, velocity of fluid is equal to velocity of solids. Since in this problem, the solid surface at $x=\delta$ is stationary. So velocity of fluid at $x=\delta$ is zero.

At $x=\delta$; $v_z=0$ (11.33)

From equation (11.27), we get

$$0 = -\left(\frac{\rho g \cos\beta}{\mu}\right)\frac{\delta^2}{2}x + c_2 \qquad (11.34)$$

$$c_2 = \left(\frac{\rho g \cos\beta}{\mu}\right)\frac{\delta^2}{2} \qquad (11.35)$$

By substituting the value of equation (11.32) and equation (11.35) in equation (11.27)

$$v_z = -\left(\frac{\rho g \cos\beta}{\mu}\right)\frac{x^2}{2} + \left(\frac{\rho g \cos\beta}{\mu}\right)\frac{\delta^2}{2} \qquad (11.36)$$

$$v_z = \frac{\rho g \delta^2 \cos\beta}{2\mu}\left(1 - \left(\frac{x}{\delta}\right)^2\right) \qquad (11.37)$$

Equation (11.37) shows the velocity of falling film.

Now we will solve same problem with assumption that shear stress is force. Draw same control volume

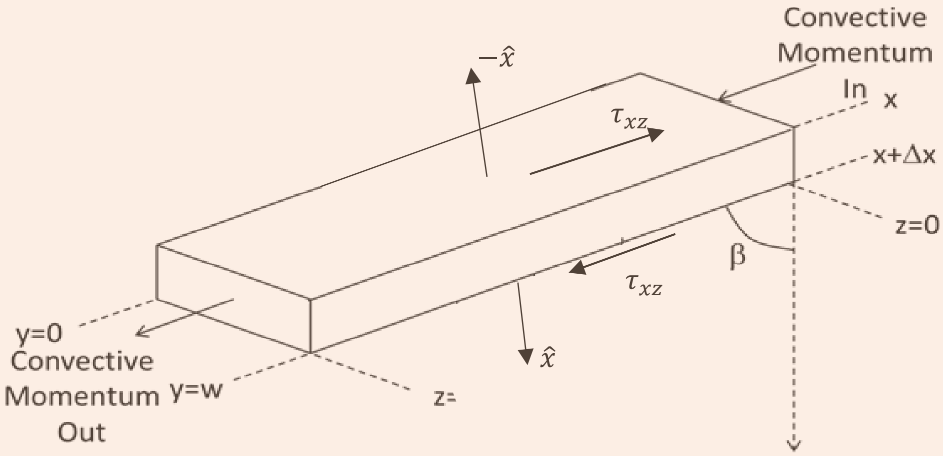


Fig 12.1 Control volume for falling film with shear stresses

Solid surface is situated at $x = \delta$ which forced shear on fluid. Therefore every layer of fluid is forced shear in decreasing direction of x co-ordinate.

So shear force at $x = x$ is $-L\tau_{xz}|_x$ (12.1)

Shear force at $x = x + \Delta x$ is $Lw\tau_{xz}|_{x+\Delta x}$ (12.2)

Body forces and convective momentum balance equation are also same as previous case.

Now momentum balance for this case.

Momentum in due to convective transport	-	Momentum out due to convective transport	+	Body forces	+	Shear forces	= 0

$$LW(\tau_{xz}|_{x+\Delta x} - \tau_{xz}|_{x}) + (LW\Delta x)(\rho g \cos\beta) = 0 \quad (12.3)$$

Divide equation (12.3) by volume of control volume WLΔx

$$\frac{(\tau_{xz}|_{x+\Delta x} - \tau_{xz}|_{x})}{\Delta x} + (\rho g \cos\beta) = 0 \quad (12.4)$$

Take limit $\Delta x \to 0$

$$\frac{d\tau_{xz}}{dx} = -\rho g \cos\beta \quad (12.5)$$

Now substitute Newton’s low of viscosity as shear force.

$$\tau_{xz} = \mu \frac{dv_z}{dx} \text{ as a shear force} \quad (12.6)$$

$$\mu \frac{d^2 v_z}{dx^2} = -\rho g \cos\beta \quad (12.7)$$

Equation (11.24) and equation (12.7) are same equation which proves that both approach gives same results.

Determine the maximum velocity:

The shear stress is minimum at x=0 so velocity should maximum at same point.

So Putting x=0 in equation (11.37)

$$v_{z,\max} = \frac{\rho g \delta^2 \cos\beta}{2\mu} \quad (12.8)$$

Average velocity and volumetric flow rate of falling film:

v_z is linear velocity in z direction. So volumetric flow rate can be determine by integrate it for full cross section of flow ($W\delta$)

So volumetric flow rate is

$$Q = \int_0^w \int_0^\delta v_z \, dx dy \quad (12.9)$$

From equation (11.37)

$$Q = \int_0^w \int_0^\delta \frac{\rho g \delta^2 \cos\beta}{2\mu} \left(1 - \left(\frac{x}{\delta}\right)^2\right) dx dy \quad (12.10)$$

$$Q = \frac{w \rho g \delta^2 \cos\beta}{2\mu} \left(x - \left(\frac{x^3}{3\delta^2}\right)\right)\Bigg|_0^\delta$$

$$Q = \frac{w \rho g \delta^2 \cos\beta}{2\mu} \left(\delta - \frac{\delta}{3}\right)$$

$$Q = \frac{w\rho g\delta^2 \cos\beta}{2\mu}\left(2\frac{\delta}{3}\right)$$

$$Q = \frac{w\rho g\delta^3 \cos\beta}{3\mu} \tag{12.11}$$

For average velocity, divide volumetric flow rate to cross section of following section.

$$\langle v_z \rangle_{avg} = \frac{Q}{\int_0^w \int_0^\delta dxdy} \tag{12.12}$$

$$\langle v_z \rangle_{avg} = \frac{\dfrac{w\rho g\delta^3 \cos\beta}{3\mu}}{w\delta}$$

$$\langle v_z \rangle_{avg} = \frac{\rho g\delta^2 \cos\beta}{3\mu}$$

$$\langle v_z \rangle_{avg} = \frac{2}{3} v_{z,\max} \tag{12.13}$$

For acting on solid surface via fluid

$$F = \int_0^L \int_0^w \left(\tau_{xz}\big|_{x=\delta}\right) dydz \tag{12.14}$$

$$F = \int_0^L \int_0^w \left(\mu \frac{dv_z}{dx}\bigg|_{x=\delta}\right) dydz$$

$$F = \rho g\delta L w \cos\beta \tag{12.15}$$

Laminar flow in a narrow slit:

A Newtonian fluid is flowing from narrow slit, formed by the two parallel plates as given in diagram, due to combine effect of gravity and pressure. Determine velocity profile, average velocity and mass flow rate for laminar flow.

$B<<W<<L$

Assumption:

- Constant density, viscosity and steady state
- Laminar flow (simple shear flow)
- Newton’s low is applicable
- No slip boundary condition

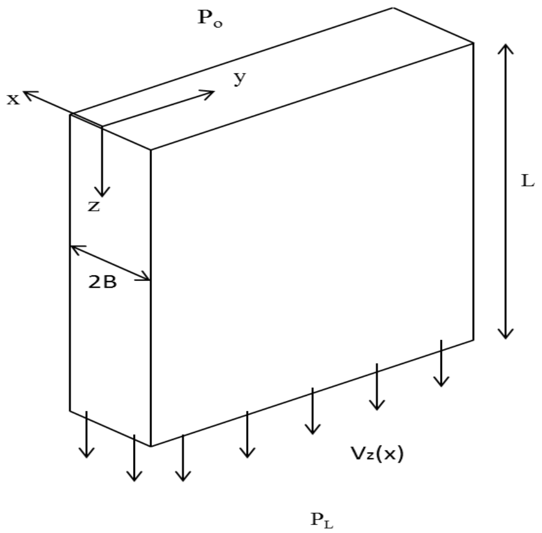


Fig 13.1 Laminar flow in narrow slit

Fluid is flowing in *z* direction only due to gravity and pressure difference. Therefore v_z is only non-zero velocity component. Since, slit is very narrow ($B<<W<<L$). We can assume that end effects are negligible in *y* direction and v_z is not the function of *y*.

$$v_z = v_z(x, z)$$

$$v_x = 0$$

$$v_y = 0 \tag{12.16}$$

Now use equation of continuity.

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0 \tag{12.17}$$

$$\rho \frac{\partial}{\partial z}(v_z) = 0 \tag{12.18}$$

Equation (12.18) shows that v_z is not the function of z. Now v_z is changing with *x* only.

$$v_z = v_z(x) \tag{12.19}$$

Therefore τ_{xz} is only non- zero shear stress component.

Draw control volume:

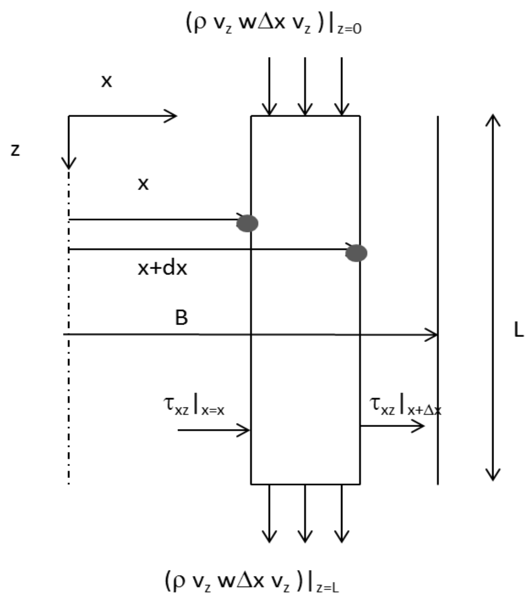


Fig 13.2 Control volume for laminar flow in narrow slit.

Momentum balance equation for control volume:

Convective momentum flux is flowing in control volume at $z=0$ is $(\rho v_z w\Delta x v_z)\big|_{z=0}$ (12.20)

Convective momentum flux leaving from CV at $z=L$ is $(\rho v_z w\Delta x v_z)\big|_{z=0}$ (12.21)

Momentum flux by molecular transport entering CV at $x=x$ is $(Lw\tau_{xz})|_{x=x}$ (12.22)

Momentum flux by molecular transport leaving from CV at $x=x+\Delta x$ is $(Lw\tau_{xz})|_{x=x+\Delta x}$ (12.23)

Pressure force at $z=0$ is $P_0\Delta xw$ (12.24)

Pressure force at $z=L$ is $-P_L\Delta xw$ (12.25)

Gravity force on CV is $\rho g\Delta xLw$ (12.26)

Now equation for momentum balance

$$(\rho v_z w\Delta x v_z)|_{z=0} - (\rho v_z w\Delta x v_z)|_{z=L} + (Lw\tau_{xz})|_{r=r} - (Lw\tau_{xz})|_{r=r+\Delta_r} + P_0 w\Delta x - P_L w\Delta x + \rho g\Delta xLw = 0 \quad (12.27)$$

v_z is not the function of z so equation (12.20) and (12.21) are equal and convective momentum balance terms are cancelling out from above equation

$$(Lw\tau_{xz})|_{x=x} - (Lw\tau_{xz})|_{x=x+\Delta_x} + P_0 w\Delta x - P_L w\Delta x + \rho g\Delta xLw = 0 \quad (12.28)$$

Divide equation (12.28) by volume of control volume ΔxLw

$$\frac{(\tau_{xz}|_x - \tau_{xz}|_{x+\Delta x})}{\Delta x} = \frac{P_L - P_0}{L} - \rho g \quad (12.29)$$

(to be continued in next lecture…..)

Lecture 13th (9th September, Monday)

Laminar flow in a narrow slit:

A Newtonian fluid is flowing from narrow slit, formed by the two parallel plates as given in diagram, due to combine effect of gravity and pressure. Determine velocity profile, average velocity and mass flow rate for laminar flow.

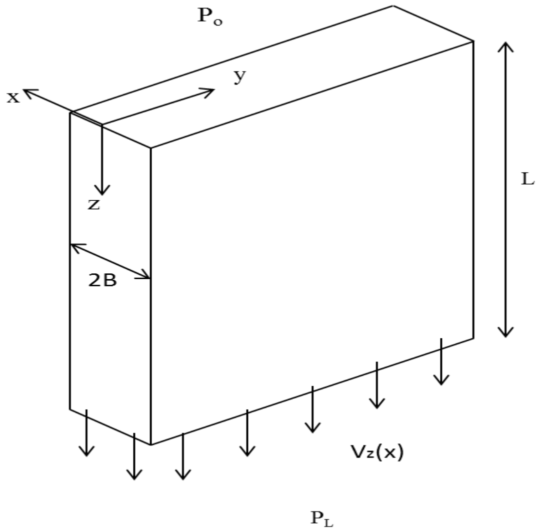


Fig 13.1 Laminar flow in narrow slit

B<<W<<L

Assumption:

- Constant density, viscosity and steady state
- Laminar flow (simple shear flow)
- Newton's low is applicable
- No slip boundary condition
- Fluid is flowing in *z* direction only due to gravity and pressure difference. Therefore v_z is only non-zero velocity component. Since, slit is very narrow (*B<<W<<L*). We can assume that end effects are negligible in *y* direction and v_z is not the function of *y*.
- $v_z = v_z\left(x, z\right)$
- $v_x = 0$
- $v_y = 0$ (13.1)
- Now use equation of continuity.
- $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0$ (13.2)
- $\rho \frac{\partial}{\partial z}(v_z) = 0$ (13.3)
- Equation (12.18) shows that v_z is not the function of z. Now v_z is changing with *x* only.

- $v_z = v_z(x)$ (13.4)
- Therefore τ_{xz} is only non- zero shear stress component.
- Draw control volume:

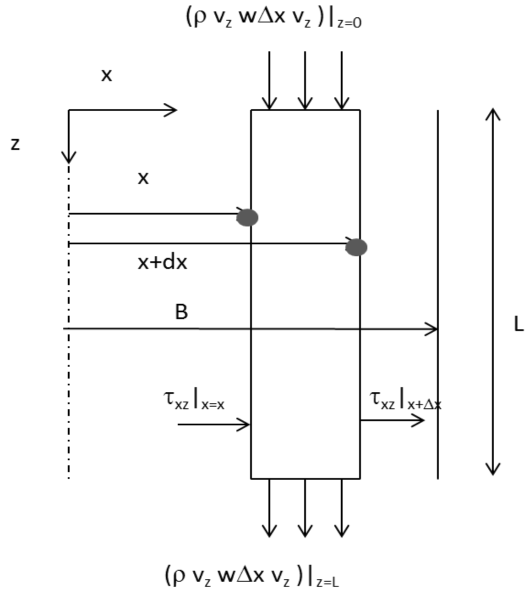


Fig 13.2 Control volume for laminar flow in narrow slit.

Momentum balance equation for control volume:

Convective momentum flux is flowing in control volume at $z = 0$ is $(\rho v_z w \Delta x v_z)\big|_{z=0}$ (13.5)

Convective momentum flux leaving from CV at $z = L$ is $(\rho v_z w \Delta x v_z)\big|_{z=0}$ (13.6)

Momentum flux by molecular transport entering CV at $x = x$ is $(Lw\tau_{xz})|_{x=x}$ (13.7)

Momentum flux by molecular transport leaving from CV at $x = x + \Delta x$ is $(Lw\tau_{xz})|_{x=x+\Delta x}$ (13.8)

Pressure force at $z = 0$ is $P_0 \Delta x w$ (13.9)

Pressure force at $z = L$ is $-P_L \Delta x w$ (13.10)

Gravity force on CV is $\rho g \Delta x L w$ (13.11)

Now equation for momentum balance

$$(\rho v_z w \Delta x v_z)|_{z=0} - (\rho v_z w \Delta x v_z)|_{z=L} + (Lw\tau_{xz})|_{r=r} - (Lw\tau_{xz})|_{r=r+\Delta_r} + P_0 w \Delta x - P_L w \Delta x + \rho g \Delta x L w = 0 \quad (13.12)$$

v_z is not the function of z so equation (13.5) and (13.6) are equal and convective momentum balance terms are cancelling out from above equation

$$(Lw\tau_{xz})|_{x=x} - (Lw\tau_{xz})|_{x=x+\Delta_x} + P_0 w \Delta x - P_L w \Delta x + \rho g \Delta x L w = 0 \quad (13.13)$$

Divide equation (13.13) by volume of control volume $\Delta x L w$

$$\frac{(\tau_{xz}\big|_x - \tau_{xz}\big|_{x+\Delta x})}{\Delta x} = \frac{P_L - P_0}{L} - \rho g \qquad (13.14)$$

Force due to gravity can be added to pressure force, like

$P_{effective} = P + \rho g z$, here z and g are in opposite direction or z is in direction of opposing gravity. However in our reference frame both g and z are in same direction therefore here

$$P_{effective} = P - \rho g z \qquad 13.15$$

So now we can add pressure force with gravity and take limit *Δx→0*

$$-\frac{\partial}{\partial x}(\tau_{xz}) = \frac{\big(P_L - \rho g z(L)\big) - \big(P_0 - \rho g z(0)\big)}{L} \qquad 13.16$$

We can keep $\big(P_L - \rho g z(L)\big) = P_{cL}$ and $\big(P_0 - \rho g z(0)\big) = P_{c0}$

$$\frac{\partial}{\partial x}(\tau_{xz}) = \left(\frac{P_{c0} - P_{cL}}{L}\right) \qquad (13.17)$$

$$\frac{\partial}{\partial x}(\tau_{xz}) = \left(\frac{P_{c0} - P_{cL}}{L}\right)x + c_1 \qquad (13.18)$$

Substituting Newton's low of viscosity

$$\tau_{xz} = -\mu \frac{dv_z}{dx} \qquad (13.19)$$

$$-\mu \frac{dv_z}{dx} = \left(\frac{P_{cL} - P_{c0}}{L}\right)x + c_1 \qquad (13.20)$$

$$v_z = -\left(\frac{P_{c0} - P_{cL}}{\mu L}\right)\frac{x^2}{2} - \frac{c_1}{\mu}x + c_2 \qquad (13.21)$$

B.C-1

At $x = 0$, velocity profile must be symmetric therefore $\frac{dv_z}{dx} = 0$ (13.22)

$C_1 = 0$

B.C-2

At $x = B,\ v_z = 0$ (No slip)

$$c_2 = \left(\frac{P_{c0} - P_{cL}}{\mu L}\right)\frac{B^2}{2} \qquad (13.23)$$

Substitute the value of c_1 and c_2 in equation (13.21)

$$v_z = \left(\frac{P_{c0} - P_{cL}}{\mu L}\right)\frac{B^2}{2}\left[1 - \left(\frac{x}{B}\right)^2\right] \qquad (13.24)$$

Equation (13.24) describes the velocity profile.

Mass flow rate and average velocity:

Mass flow rate = Volumetric flow rate * Density

$$= \rho \int_0^w \int_{-B}^{B} v_z dx dy$$

$$= \int_0^w \int_{-B}^{B} \left[\left(\frac{P_0 - P_L}{\mu L} \right) \frac{B^2}{2} \left[1 - \left(\frac{x}{B} \right)^2 \right] dx dy \right] \rho$$

$$= \frac{2\rho w (P_0 - P_L) B^2}{2\mu L} \left[x - \frac{x^3}{3B^2} \right]_0^B \tag{13.25}$$

Mass flow rate

$$W = \frac{2}{3} \frac{\rho w (P_0 - P_L) B^3}{\mu L} \tag{13.26}$$

Average velocity = Volumetric flow rate / area of cross section

$$= \frac{\frac{2}{3} \frac{w (P_0 - P_L) B^3}{\mu L}}{2BW}$$

$$\langle v_z \rangle = \frac{1}{3} \frac{w (P_0 - P_L) B^3}{\mu L} \tag{13.27}$$

Flow of two adjacent immiscible fluids: (BSL 2.5 page 56-58)

Two immiscible liquid is flowing in between two adjacent plates. Solve the problem for velocity profile and mass flow rate

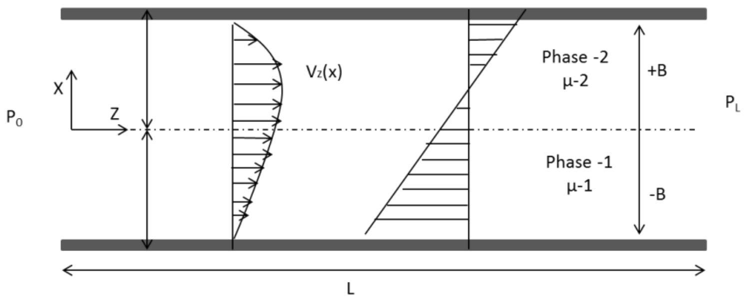


Fig 13.2 Flow of two immiscible fluids between a pair of horizontal plates

Assumption:

- Constant density, viscosity and steady state
- Laminar flow (simple shear flow)
- Newton’s low is applicable
- No slip boundary condition

Since fluid is flowing in z direction only. Therefore v_z is only non-zero velocity component. We can assume that end effects are negligible in y direction and v_z is not the function of y.

$$v_z = v_z(x, z)$$

$$v_x = 0$$

$$v_y = 0 \qquad (13.28)$$

Now use equation of continuity for cylindrical coordinate system.

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

$$\rho \frac{\partial}{\partial z}(v_z) = 0 \qquad \text{(steady state and incompressible flow)} \qquad (13.29)$$

Since in Equation (13.92) gradient of v_z is zero that shows v_z is not the function of z. Now v_z is changing with x only.

$$v_z = v_z(x) \qquad (13.30)$$

Therefore τ_{xz} is only non- zero shear stress component.

Draw control volume: (since v_z is changing with x only, so differential element will be in x direction)

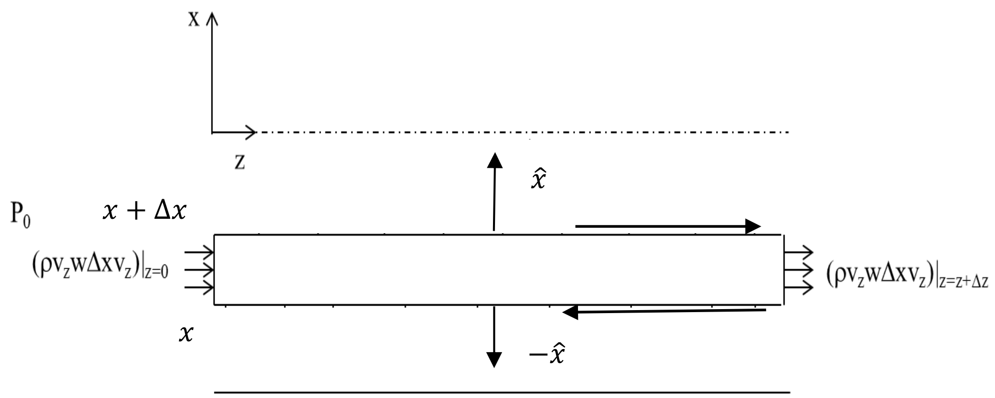


Fig 13.3 Control volume for Flow of two immiscible fluid between a pair of horizontal plates.

Here shear as a force

Momentum balance equation for control volume:

Convective momentum flux is flowing in control volume at z =0 is $(\rho v_z w \Delta x v_z)|_{z=0}$ (13.31)

Convective momentum flux leaving from CV at $z=L$ is $(\rho v_z w \Delta x v_z)|_{z=L}$ (13.32)

Here we have taken shear stress as force.

Shear force at $x=x$ is $-(LW\tau_{xz})|_{x=x}$ (13.33)

Shear force at $x=x+\Delta x$ is $(LW\tau_{xz})|_{x=x+\Delta x}$ (13.34)

Pressure force at $z=0$ is $P_0 \Delta x w$ (13.35)

Pressure force at $z=L$ is $-P_L \Delta x w$ (13.36)

Now equation for momentum balance

$$(\rho v_z w\Delta x v_z)|_{z=0} - (\rho v_z w\Delta x v_z)|_{z=L} - (Lw\tau_{rz})|_{x=x} + (Lw\tau_{rz})|_{x=x+\Delta x} + P_0 w\Delta x - P_L w\Delta x = 0 \tag{13.37}$$

v_z is not the function of z so equation (14.5) and (14.6) are equal and convective momentum balance terms are cancel out from equation (14.11).

$$(\mathrm{Lw}\tau_{\mathrm{rz}})|_{x=x+D_x} - (\mathrm{Lw}\tau_{\mathrm{rz}})|_{x=x} + P_0 w\Delta x - P_L w\Delta x = 0 \tag{13.38}$$

Divide equation (14.12) by volume of control volume ΔxLW

$$\frac{(\tau_{\mathrm{xz}}|_{x+\Delta x} - \tau_{\mathrm{xz}}|_{x})}{\Delta x} = \frac{P_0 - P_L}{L} \tag{13.39}$$

Now we can add pressure force with gravity and take limit $\Delta x \rightarrow 0$

$$\frac{\partial}{\partial x}(\tau_{\mathrm{xz}}) = \left(\frac{P_0 - P_L}{L}\right) \tag{13.40}$$

Since τ_{xz} is function of x only so partial differential converts to ordinary differential

Substituting Newton's low of viscosity

$$v_z = \left(\frac{P_0 - P_L}{\mu L}\right)\frac{x^2}{2} + \frac{c_1}{\mu}x + c_2 \tag{13.41}$$

This equation is valid for both regions. Therefore,

$$v_z^I = \left(\frac{P_0 - P_L}{\mu L}\right)\frac{x^2}{2} + \frac{c_1{}^I}{\mu_I}x + c_2{}^I \tag{13.42}$$

$$v_z^{II} = \left(\frac{P_0 - P_L}{\mu L}\right)\frac{x^2}{2} + \frac{c_1{}^{II}}{\mu_{II}}x + c_2{}^{II} \tag{13.43}$$

Boundary conditions:

There are four boundary condition used to solve the problem.

1. $x=0,\ v_z^I = v_z^{II}$
2. $x=-b,\ v_z^I = 0$
3. $x=+b,\ v_z^{II} = 0$
4. $x=0,\ \mu_I \frac{dv_z^I}{dx}|_{x=0} = \mu_{II}\frac{dv_z^{II}}{dx}|_{x=0}$ (13.44)

For solution to this problem refer to Book "Transport Phenomena" BSL Edition 2nd, Page number 56-58

Lecture 14 (19th September, Thursday)

Derivation of equation of motion:

In this section, we will derive a generalized equation which can be used to solve all type of momentum transport problem. This equation is based on axiom 2.

Axiom 2: momentum is conserved.

Take a control volume of $\Delta x \Delta y \Delta z$, fixed in space.

(1) Rate of accumulation of momentum in control volume = (2) Net rate of inflow of momentum by convection + (3) Net rate of inflow of momentum by viscous transport + (4) Pressure forces + (5) Force due to gravity

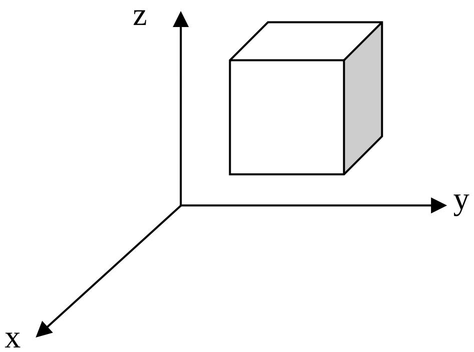


Fig 15.1 Cubical control volume fixed in space

Rate of change of momentum in control volume = $\frac{d}{dt}(\rho \Delta x \Delta y \Delta z v_x)$ (15.1)

Net rate of x-directed momentum inflow in control volume from x-plane = $[(\rho v_x \Delta z \Delta y) v_x]_x - [(\rho v_x \Delta z \Delta y) v_x]_{x+\Delta x}$ (15.2)

Net rate of x-directed momentum inflow in control volume from y-plane = $[(\rho v_y \Delta x \Delta z) v_x]_y - [(\rho v_y \Delta x \Delta z) v_x]_{y+\Delta y}$ (15.3)

Net rate of x-directed momentum inflow in control volume from z-plane = $[(\rho v_z \Delta x \Delta y) v_x]_z - [(\rho v_z \Delta x \Delta y) v_x]_{z+\Delta y}$ (15.4)

Net rate of change of momentum in control volume due to viscous transport = Shear stress has 9 components.

$$\tau = \begin{Bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{Bmatrix} \qquad (15.5)$$

Here, a second index shows the direction of momentum and first index shows the surface from which momentum is flowing. For example, τ_{yx} shows *x* directed force working on *y* surface. Therefore, x directed shear stresses are τ_{xx} τ_{yx} and τ_{zx}.

Net rate of change of x directed momentum flowing in x plane

$$(\tau_{xx}\Delta z\Delta y)|_{x} - (\tau_{xx}\Delta z\Delta y)|_{x+\Delta x} \tag{15.6}$$

Net rate of change of x directed momentum flowing in y plane

$$(\tau_{yx}\Delta x\Delta z)|_{y} - (\tau_{yx}\Delta x\Delta z)|_{y+\Delta y} \tag{15.7}$$

Net rate of change of x directed momentum flowing in z plane

$$(\tau_{zx}\Delta x\Delta y)|_{z} - (\tau_{zx}\Delta x\Delta y)|_{z+\Delta z} \tag{15.8}$$

Change of x directed momentum due to pressure force = $(p\Delta y\Delta z)|_{x} - (p\Delta y\Delta z)|_{x+\Delta x}$

(15.9)

Net rate of change x directed momentum due to gravity = $(\rho\Delta x\Delta y\Delta z)\ g_x$ (15.10)

Now add all above terms and divide all the terms by volume of control volume $\Delta x\ \Delta y\ \Delta z$

$$\frac{\partial(\rho v_x)}{\partial t} \tag{15.11}$$

$$\frac{(\rho v_x v_x)|_{x} - (\rho v_x v_x)|_{x+\Delta x}}{\Delta x} + \frac{(\rho v_y v_x)|_{y} - (\rho v_y v_x)|_{y+\Delta y}}{\Delta y} + \frac{(\rho v_z v_x)|_{z} - (\rho v_z v_x)|_{z+\Delta z}}{\Delta x}$$

$$= -\frac{\partial(v_x v_x)}{\partial x} - \frac{\partial(v_y v_x)}{\partial y} - \frac{\partial(v_z v_x)}{\partial x} \tag{15.12}$$

$$-\frac{\partial\tau_{xx}}{\partial x} - \frac{\partial\tau_{yx}}{\partial y} - \frac{\partial\tau_{zx}}{\partial z} \tag{15.13}$$

$$\frac{p_x - p_{x+\Delta x}}{\Delta x} = \frac{-\partial p}{\partial x} \tag{15.14}$$

$$\rho g_x \tag{15.15}$$

Take the individual limits

$\Delta x \rightarrow 0,\ \Delta y \rightarrow 0$ and $\Delta z \rightarrow 0$

x-direction:

$$\frac{\partial(\rho v_x)}{\partial t} = -\left[\frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_y v_x)}{\partial y} + \frac{\partial(\rho v_z v_x)}{\partial z}\right] - \left[\frac{\partial\tau_{xx}}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z}\right] - \frac{\partial p}{\partial x} + \rho g_x \tag{15.16}$$

Above equation represents equation of motion for x directed momentum for Cartesian coordinate system.

Similarly for *y*-direction:

$$\frac{\partial(\rho v_y)}{\partial t}=-\left[\frac{\partial(\rho v_x v_y)}{\partial x}+\frac{\partial(\rho v_y v_y)}{\partial y}+\frac{\partial(\rho v_z v_y)}{\partial z}\right]-\left[\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}\right]-\frac{\partial p}{\partial y}+\rho g_y \quad (15.17)$$

Similarly for z-direction:

$$\frac{\partial(\rho v_z)}{\partial t}=-\left[\frac{\partial(\rho v_x v_z)}{\partial x}+\frac{\partial(\rho v_y v_z)}{\partial y}+\frac{\partial(\rho v_z v_z)}{\partial z}\right]-\left[\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z}\right]-\frac{\partial p}{\partial z}+\rho g_z \quad (15.18)$$

We can write equation of motion in tensor form

$$\frac{\partial(\rho v)}{\partial t}=-\nabla.(\rho vv)-\nabla.\tau-\nabla p+\rho g \quad (15.19)$$

Above equation can be used for any coordinate system.

Equation in terms of substantial derivative

$$\frac{\partial(\rho v)}{\partial t}+\nabla.(\rho vv)=-\nabla.\tau-\nabla p+\rho g=\frac{\rho Dv}{Dt} \quad (15.20)$$

Use vector identities

$$\nabla.(vw)=v.\nabla w+w(\nabla.v)$$

$$\nabla.(xz)=x.\nabla z+z(\nabla.x) \quad (15.21)$$

Replace x by ρv & z by v

$$\nabla.(xz)=\nabla.(\rho vv)=\rho v.\nabla v+v(\nabla.\rho v) \quad (15.22)$$

$$\frac{\partial \rho v}{\partial t}=\rho\frac{\partial v}{\partial t}+v\frac{\partial \rho}{\partial t} \quad (15.23)$$

Substituting equation (15.22) and (15.23) in equation (15.19)

$$v\frac{\partial \ell}{\partial t}+\rho\frac{\partial v}{\partial t}=\rho v.\nabla v+v(\nabla.\rho v)-\nabla.\tau-\nabla p+\rho g \quad (15.24)$$

Rearrange the above expression

$$=\rho\left[\frac{\partial v}{\partial t}+v.\nabla v\right]+v\left[\frac{\partial \rho}{\partial t}+\nabla.\rho v\right]=-\nabla.\tau-\nabla p+\rho g \quad (15.25)$$

From equation of continuity

$$\frac{\partial \rho}{\partial t}+\nabla.\rho v=0$$

$$\rho\left[\frac{\partial v}{\partial t}+v.\nabla v\right]=-\nabla.\tau-\nabla p+\rho g \quad (15.26)$$

Above equation can be written in terms of substantial derivative

$$=\rho\frac{Dv}{Dt}=-\nabla.\tau-\nabla p+\rho g \quad (15.27)$$

Equation (15.26) and (15.27) are the generalized equation without any assumption.

Special case of equation of motion for Newtonian fluids is discussed below.

Newtonian Fluid:

Here, we have to substitute the value of shear stress according to Newton's law. Initially we derived the Newton's law for a one dimensional system as follows

$$v_y = 0, v_z = 0 \tag{16.1}$$

$$\tau_{yx} = -\frac{\mu \partial v_x}{\partial y} \tag{16.2}$$

Where, τ_{yx} is x directed shear stress working on y direction plane; v_x is velocity of fluid in x direction and μ is viscosity.

But for a three dimensional flow, the shear stress tensor has 9 components

$$\tau = \begin{Bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{Bmatrix} \tag{16.3}$$

τ_{xx} τ_{yy} and τ_{zz} are normal stresses and rest are shear stresses. Following axiom helps in simplifying the value of shear stresses.

Axiom 3: Moment of momentum is conserved.

This axiom leads to conclusion that the stress tensor τ is symmetric tensor;

$$\begin{aligned} \tau_{\mathrm{ij}} &= \tau_{\mathrm{ji}} \\ \tau_{\mathrm{xy}} &= \tau_{\mathrm{yx}} \\ \tau_{\mathrm{xz}} &= \tau_{\mathrm{zx}} \\ \tau_{\mathrm{yz}} &= \tau_{\mathrm{zy}} \end{aligned} \tag{16.4}$$

With this, the Newton's law viscosity for a three dimensional flow may be defined as follow.

$$\tau = -\mu(\Delta) + \frac{2}{3}\mu\nabla.\mathrm{vd} \qquad [\delta = \delta_{\mathrm{ij}}\delta_{\mathrm{i}}\delta_{\mathrm{j}}] \tag{16.5}$$

$$\Delta = \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\delta_i\delta_j \tag{16.6}$$

$$\tau_{\mathrm{xx}} = -2\mu\frac{\partial v_x}{\partial x} + \frac{2}{3}\mu\nabla.v \tag{16.7}$$

$$\tau_{\mathrm{yy}} = -2\mu\frac{\partial v_y}{\partial y} + \frac{2}{3}\mu\nabla.v \tag{16.8}$$

$$\tau_{\mathrm{zz}} = -2\mu\frac{\partial v_y}{\partial z} + \frac{2}{3}\mu\nabla.v \tag{16.9}$$

$$\tau_{\mathrm{xy}} = \tau_{\mathrm{yx}} = -\mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) \qquad (16.10)$$

$$\tau_{\mathrm{yz}} = \tau_{\mathrm{zy}} = -\mu\left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right) \qquad (16.11)$$

$$\tau_{\mathrm{xz}} = \tau_{\mathrm{zx}} = -\mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) \qquad (16.12)$$

The above equation for the Newton's law of viscosity experiment simplifies to equation (16.2) as shown below

$v_x = v_x(y)$, $v_y = 0$, $v_z = 0$ and ρ is constant.

$\tau_{\mathrm{yz}} = \tau_{\mathrm{zy}} = 0$

$\tau_{\mathrm{xz}} = \tau_{\mathrm{zx}} = 0$

or

$$\tau_{yx} = -\frac{\mu \partial v_x}{\partial y} \qquad (16.13)$$

Now we are in a position to determine the value of -∇τ in the general equation of motion for the case where we have Newtonian fluids.

Consider the x component of $-\nabla.\tau = -\left[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right]$ (16.14)

Assume ρ and μ are constant which is true for most of the Newtonian fluids.

The x component of $-\nabla.\tau = -\left[\frac{\partial}{\partial x}\left(-2\mu\frac{\partial v_x}{\partial x}\right) + \frac{\partial}{\partial y}\left(-\mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right) + \frac{\partial}{\partial z}\left(-\mu\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right)\right)\right]$

(16.15)

$$= \mu\left[\frac{\partial}{\partial x}\frac{\partial v_x}{\partial x} + \frac{\partial}{\partial y}\frac{\partial v_x}{\partial y} + \frac{\partial}{\partial y}\frac{\partial v_y}{\partial x} + \frac{\partial}{\partial z}\frac{\partial v_x}{\partial z} + \frac{\partial}{\partial z}\frac{\partial v_z}{\partial x} + \frac{\partial}{\partial x}\frac{\partial v_x}{\partial x}\right] \qquad (16.16)$$

$$= \mu\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} + \frac{\partial}{\partial x}\left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right]\right] \qquad (16.17)$$

From equation of continuity for incompressible fluid

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \qquad (16.18)$$

$$= \mu\left[\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right] \tag{16.19}$$

$$= \mu\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]v_x \tag{16.20}$$

Thus, $-\left(\nabla.\tau\right)_x = \mu\nabla^2 \mathrm{v_x}$ (16.21)

Similarly,

$$-\left(\nabla.\tau\right)_y = \mu\nabla^2 \mathrm{v_y} \tag{16.22}$$

And $-\left(\nabla.\tau\right)_z = \mu\nabla^2 \mathrm{v_z}$ (16.23)

In generalized vector and tensor form

$$-\left(\nabla.\tau\right) = \mu\nabla^2 \mathrm{v} \tag{16.24}$$

Now, equation of motion is

$$\frac{\rho Dv}{Dt} = -\nabla p + \mu\nabla^2 v + \rho g \tag{16.25}$$

Equation (16.25) is also called as Navier Stokes equation and used for solving problems of Newtonian fluid with constant density and viscosity. For other types of non-Newtonian and compressible fluids, generalized form of equation of motion should be used.

Detail form of equation of motion and Navier Stokes equation in Cartesian, cylindrical and spherical coordinates are given in appendix.

Lecture 15 (23rd September)

Solution of Momentum Transport problem by using Navier stokes equation

In this section, transport problem for Newtonian fluid is solved by using equation of motion or Navier – Stokes equation. First we will solve falling film problem and flow through circular tube for compare shell momentum balance method with equation of motion method. Then we will solve some typical momentum transport problems.

Falling film on a inclined surface

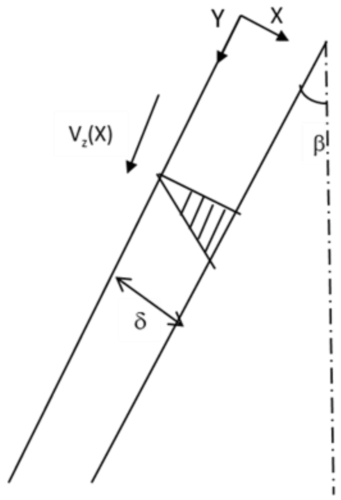


Fig 17.1 Falling film on inclined surface

This problem have been solved earlier via shell and momentum balance. Here we are solving this problem by using Navier stokes equation.

First assume all necessary assumption for solving problem. Determine non- zero velocity profile and non zero shear stress profile. These steps are same as shell momentum balance problem which shows that $v_z = v_z(r)$ and τ_{rz} are only non-zero component of velocity and shear stress vice versa.

Now use Navier stokes equation for Cartesian coordinate after cancelling all zero terms.

x- component of Navier stokes equation $\rho g_x = 0$ (17.1)

y- component of Navier stokes equation $\rho g_y = 0$ (17.2)

z- component of Navier stokes equation

$$\mu \frac{d^2 v_z}{dx^2} + \rho g_z = 0 \tag{17.3}$$

$$g_z = g \cos\beta \tag{17.4}$$

$$\frac{dv_z}{dx} = -g\cos\beta x + c_1 \tag{17.5}$$

$$v_z = -g\cos\beta x + c_1 x + c_2 \tag{17.6}$$

Boundary conditions are also same as previous method.

1. $x = 0,\ \left.\frac{dv_z}{dx}\right|_{x=0} = 0$
2. $x = \delta,\ v_z = 0$ (17.7)

This leads to the solution for velocity profile.

$$v_z = -\frac{g\cos\beta\delta^2}{2\mu}\left[1-\left(\frac{x}{\delta}\right)^2\right] \tag{17.8}$$

Fluid flow in vertical tube

A Newtonian fluid is flowing from circular cross section vertical tube due to pressure difference and gravity. Formulate the problem for momentum transport by Navier stokes equation.

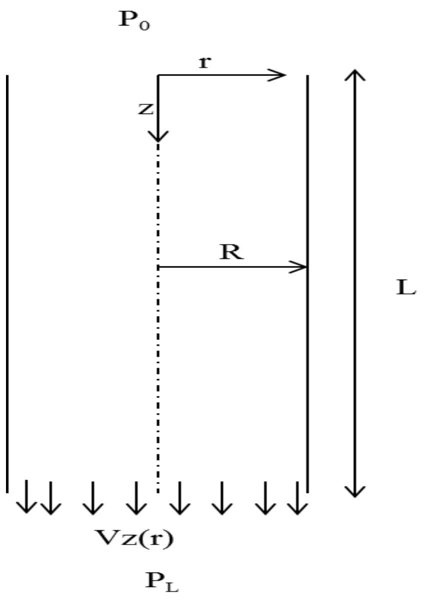


Fig 14.2 Flow through a vertical circular tube

Same types of problem have been solved by shell momentum balance for horizontal tube flow. So the initial steps are same which includes: make appropriate assumption and determine non-zero velocity profile. As we show earlier it leads to conclusion that $v_z = v_z(r)$.

Now use Navier stokes equation for cylindrical co-ordinate after cancel out all zero terms.

r- component of Navier stokes equation

$$\frac{\partial P}{\partial r} = 0 \quad (17.9)$$

θ- component of Navier stokes equation

$$\frac{\partial P}{\partial \theta} = 0 \quad (17.10)$$

z- component of Navier stokes equation

$$-\frac{\partial P}{\partial z} + \mu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r} \right) + \rho g_z = 0 \quad (17.11)$$

We can combine gravity to pressure force and rewrite equation (17.11)

$$-\frac{\partial P_c}{\partial z} + \mu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r} \right) = 0 \quad (17.12)$$

In equation (17.12) first term is only function of z and second term is only function of r.

$$\mathrm{F}(\mathrm{z}) \ + \ \mathrm{F}(\mathrm{r}) \ = \ 0 \quad (17.13)$$

This leads to result that

$$\mathrm{F}(\mathrm{z}) \ = \ \mathrm{c}_1; \ \mathrm{F}(\mathrm{r}) \ = \ \mathrm{c}_2 \quad (17.14)$$

$$\frac{\partial P_c}{\partial z} = c_1 \quad (17.15)$$

$$P_c = c_1 z + c_2 \quad (17.16)$$

Boundary conditions are

$z = 0 \quad P_c = P_{c\,0}$

$z = 0 \quad P_c = P_{c\,L}$ (17.17)

This leads to solution that

$$\frac{\mathrm{P}_{c0} - \mathrm{P}_{cL}}{L} = c_1 \quad (17.18)$$

By substituting in equation (17.12)

$$-\left(\frac{\mathrm{P}_{c0} - \mathrm{P}_{cL}}{L} \right) + \mu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r} \right) = 0 \quad (17.19)$$

$$v_z = -\left(\frac{\mathrm{P}_{c0} - \mathrm{P}_{cL}}{4\mu L} \right) r^2 + c_3 \ln r + c_4 \quad (17.20)$$

Boundary conditions are

At $r = 0$; v_z is finite

At $r = R$; $v_z = 0$ (17.21)

This leads to

$$v_z = \left(\frac{\mathrm{P}_{c0} - \mathrm{P}_{cL}}{4\mu L}\right) R^2 \left[1 - \left(\frac{r}{R}\right)^2\right]$$

Rotational viscometer problems

Operation of Couette viscometer

The formulation of couette viscometer for Newtonian fluid: Couette viscometer contains the two concentric cups. The fluid is placed in the cap and cap is then made to rotate with constant angular velocity Ω_0.

This put the force on inner cylinder

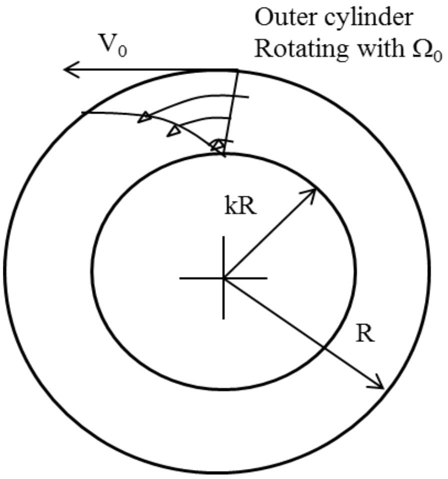


Fig 18.1 Top view of couette viscometer

Assumption:

- Isothermal condition, constant density, viscosity and steady state
- Laminar flow (simple shear flow)
- Newton’s low is applicable
- No slip boundary condition
- End effects are negligible.

Non-zero velocity profile:

The fluid is flowing in θ direction due to angular velocity of outer cylinder. So v_θ is only non-zero velocity in this case. As we assume end effects are negligible. It may not depend on z direction. So $\mathrm{v}_\theta = \mathrm{v}_\theta(\theta, \mathrm{r})$ (18.1)

By using equation of continuity in cylindrical co-ordinate system:

$$\frac{\partial\rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0 \qquad (18.2)$$

$$\frac{\rho}{r}\frac{\partial}{\partial\theta}(v_\theta)=0 \quad (18.3)$$

Equation (18.3) shows that v_z is not depending on z.

$$\mathrm{v}_\theta = \mathrm{v}_\theta(\mathrm{r})\,\text{only} \quad (18.4)$$

Use Navier stokes equation after cancel out all zero terms.

r- component of Navier stokes equation

$$\frac{\rho}{r}{v_\theta}^2 = \frac{\partial P}{\partial r} \quad (18.5)$$

θ- component of Navier stokes equation

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}(rv_\theta)\right)=0 \quad (18.6)$$

z- component of Navier stokes equation

$$\frac{\partial P}{\partial z}-\rho g_z=0 \quad (18.7)$$

Equation (18.5) and (18.7) shows that the effect of gravity and centrifuge on pressure. Equation (18.6) shows the velocity profile.

Solve the equation (18.6)

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)=c_1 \quad (18.8)$$

$$\frac{\partial}{\partial r}(rv_\theta)=c_1 r \quad (18.9)$$

$$rv_\theta=\frac{1}{2}c_1 r^2+c_2 \quad (18.10)$$

$$v_\theta=\frac{1}{2}c_1 r+\frac{c_2}{r} \quad (18.11)$$

Boundary conditions:

First boundary condition is no slip boundary condition at inner cylinder.

$r = kR$, $v_\theta = 0$ (18.12)

We can take second boundary condition in two ways.

1. $r = R$, $v_\theta = 0$

   or

   $r = R$ Torque $T|_{r=R} = T$

where $$\mathrm{T} = -2\pi \mathrm{RL}\mu\left.\frac{\partial v_\theta}{\partial r}\right|_{r=R} \quad (18.13)$$

This boundary condition gives the solution

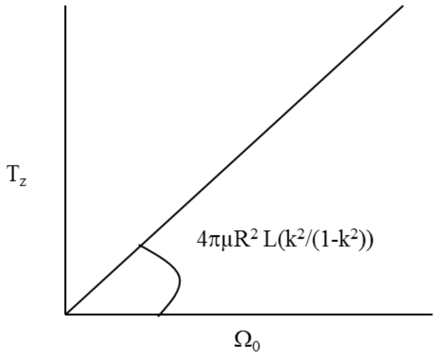
$$v_\theta = \Omega_0 R \frac{\left(\frac{r}{kR} - \frac{kR}{r}\right)}{\left(\frac{1}{k} - k\right)} \tag{18.14}$$

The torque is acing on the inner cylinder, is given by the product of the inward momentum flux *($-\tau_{r\theta}$)* and surface area of the cylinder

$$T_z = (-\tau_{r\theta})|_{r=kR}\ .2\pi kRL.kR = 4\pi\mu\Omega_0 R^2 L\left(\frac{k^2}{1-k^2}\right) \tag{18.15}$$

Equation (18.15) is used to determine viscosity by using different velocity and different torque. The curve between T_z Ω_0 gives the value of μ.

Ω_0	T_z
Ω_{01}	T_{z1}
Ω_{02}	T_{z2}
Ω_{03}	T_{z3}
Ω_{04}	T_{z4}
Ω_{05}	T_{z5}



Lecture 16 (26^{th} September)

Cone and Plate viscometer

A cone and plate viscometer consists of a stationary flat plate and an inverted cone, whose apex just contact at center of the plate. The liquid whose viscosity is to be measured is placed in the gap between the cone and plate. The cone is rotated at known angular velocity Ω, and the torque T_z is measured, required turning the cone. Find an expression for the viscosity of fluid in terms of Ω, T_z, and ψ_0 between the cone and plate. For commercial instrument, ψ_0 is about 1 degree.

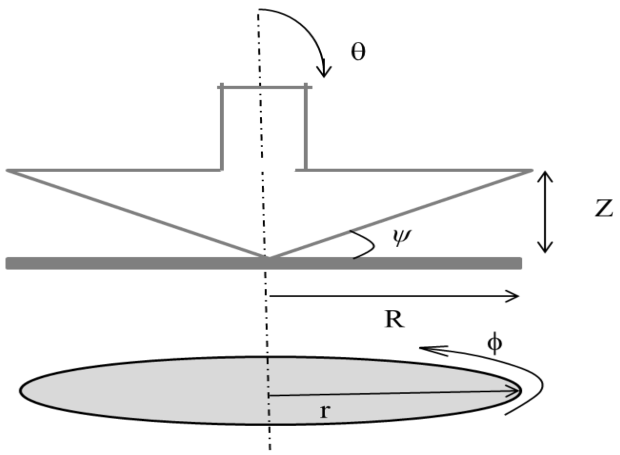


Fig 18.2 Cone and plate viscometer

Formulate the problem for determining viscosity

Assumptions:

- Constant density, viscosity and steady state
- Laminar flow (simple shear flow)
- Newton's low is applicable
- No slip boundary condition

Non- zero velocity profile:

Here we see spherical coordinates suits the problem (*r, θ and ϕ*)

Liquid is rotating in ϕ direction which will depend upon r and θ

So non- zero velocity is $v_\phi = v_\phi(\theta, r)$ (18.16)

Now applying equation of continuity for spherical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\rho v_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(\rho v_\phi) = 0 \quad (18.17)$$

$$\frac{\rho}{r\sin\theta}\frac{\partial}{\partial \phi}(v_\phi) = 0 \quad (18.18)$$

Equ.(1) shows that v_ϕ is not depend on ϕ.

$$v_\phi = v_\phi(\theta, r) \tag{18.19}$$

Here velocity is depending on two direction θ and r. It is not easy to deal with solution of equations available to us. So we can use other approaches to solve this problem. This is the limiting case where cone angle ψ is very small (which is true for real viscometers) so we can assume that ψ is negligible therefore problem converted to circular flow between two parallel plates. But in this case also non-zero velocity v_θ depends on θ and r. Therefore we use a further simplification. We assume that the effect of curvature is negligible (angular velocity Ω *is very small*). Since the linear velocity of the cone's top varies with r ($V = r\Omega$), we can treat each clice of the circle with circumference $2\pi r$ and thickness Δr as a linear element with length L $=2\pi r$ and thickness Δr moving at speed V. Now solve this problem for flow between two parallel plates in rectangular coordinates and then convert it in spherical coordinates.

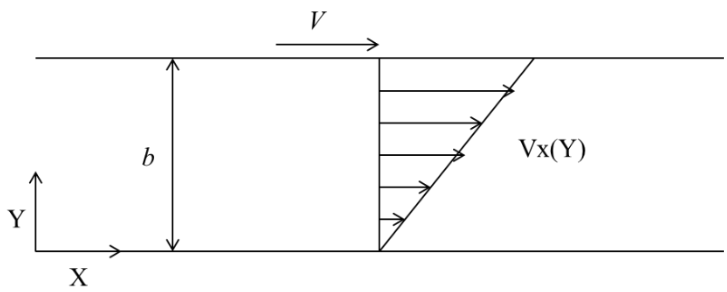


Fig 18.3 Cone and plate viscometer in cartesian coordinate

(neglecting curvature effect)

Here fluid is present in between two parallel plates and above plate is moving a constant velocity *v* in *x* direction so non zero velocity profile is

$$v_x = v_x(y) \tag{18.20}$$

Use Navier stokes equ.

$$\mu \frac{d^2 v_x}{dy^2} = 0 \tag{18.21}$$

$$v_x = c_1 y + c_2 \tag{18.22}$$

Boundary condition:

$y = 0, v_x = 0$

$$c_2 = 0 \tag{18.23}$$

$y = b, v_x = v$

$$c_1 = v/b \tag{18.24}$$

By substituting boundary conditions in equation (18.22)

$$\frac{v_x}{v} = \frac{y}{b} \Rightarrow v_x(y) = \frac{v}{b} y \tag{18.25}$$

Nowe we can change back to rotational flow in spherical coordinates, that is, the fluid actually flows in the ϕ direction and its velocity varies with r $v_\phi = v_\phi(r)$. Convert equation (18.25) in spherical coordinate

$$v_x = v_\theta \tag{18.26}$$

$$v = \Omega r \tag{18.27}$$

$$b = r \sin \psi = r \psi \tag{18.28}$$

$$y = r \sin \psi = r \psi = \left(\frac{\pi}{2} - \theta\right) \tag{18.29}$$

Substituting equation (18.26) – (18.29) in equation (18.25) which leads to the solution

$$\frac{v_\phi}{r} = \Omega \left(\frac{\left(\frac{\pi}{2}\right) - \theta}{\psi} \right) \tag{18.30}$$

Calculation for shear stress: The shear stress on the surface element f the cone by the fluid is given by $\tau_{\theta\phi}$ (the θ direction is the perpendicular to the surface element and the shear in the ϕ direction.

$$\tau_{\theta\phi} = \frac{-\mu \sin\theta}{r} \frac{\partial}{\partial\theta} \left(\frac{v_\phi}{\sin\theta} \right) \tag{18.31}$$

Since θ is closed to $\pi/2$ so $sin\theta$ is closed to unity

$$\tau_{\theta\phi} = \frac{-\mu}{r} \frac{\partial v_\phi}{\partial\theta} = -\mu \frac{1}{r} \frac{\partial}{\partial\theta} \left(\frac{\Omega r \left(\frac{\pi}{2} - \theta\right)}{\psi} \right) \tag{18.32}$$

$$\tau_{\theta\phi} = \mu \frac{\Omega}{\psi} \tag{18.33}$$

The torque required to rotate the cone can be measured by force times entire area of plate.

$$T_z = \int_0^{2\pi} \int_0^R (-\tau_{\theta\phi}) r \,|_{\theta=\pi/2} r dr d\phi = 2\pi \left(\frac{\mu\Omega}{\psi} \right) \int_0^R r^2 dr$$

$$T_z = 2\pi \left(\frac{\mu\Omega}{\psi} \right) \left(\frac{R^3}{3} \right) \tag{18.34}$$

By plotting the angular velocity Ω vs. torque T_z. Viscosity can be determined

Parallel – disc viscometer

A fluid which viscosity is to be measured is placed in the gap of thickness *B* between the two parallel discs of radius *R*. Lower plate is stationary and upper plate is rotating with an angular velocityΩ. Formulate problem for determining viscosity at low shear rate.

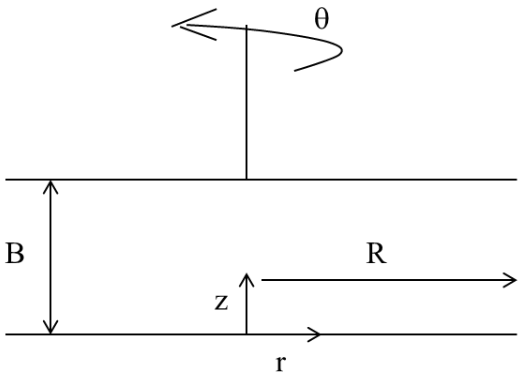


Fig 19.2 Front view of two plate viscometer

Assumptions:

- Constant density, viscosity and steady state
- Laminar flow (simple shear flow)
- Newton's low is applicable
- No slip boundary condition

Non- zero velocity profile:

Fluid is rotating in θ direction so v_θ is non-zero component which can be depends on *r*, *z* and θ

Apply equation of continuity for cylindrical coordinates

$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r)+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho v_\theta)+\frac{\partial}{\partial z}(\rho v_z)=0 \tag{19.23}$$

$$\frac{1}{r}\frac{\partial}{\partial\theta}(\rho r v_\theta)=0 \tag{19.24}$$

So $v_\theta$is not depends on θ.

$$\mathrm{v}_\theta = \mathrm{v}_\theta(\mathrm{r},\ \mathrm{z}) \tag{19.25}$$

For simplifying problem further, we can assume that

Now $v_\theta = r f(z)$ for low shear rate.

Use Navier – stokes equation for θcomponent in cylindrical co-ordinate systems.

$$0=\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(rv_\theta\right)\right)+\frac{\partial^2 v_\theta}{\partial z^2} \tag{19.26}$$

$$\mathrm{v}_\theta = \mathrm{r\,f}\left(\mathrm{z}\right) \tag{19.27}$$

$$0=\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r^2 f(\mathrm{z})\right)\right)+\frac{r\partial^2 f(\mathrm{z})}{\partial \mathrm{z}^2} \tag{19.28}$$

$$f(z)\frac{\partial}{\partial r}\left(\frac{2r}{r}\right)+\frac{r\partial^2 f(z)}{\partial z^2}=0 \tag{19.29}$$

$$\frac{\partial^2 f(z)}{\partial z^2}=0 \tag{19.30}$$

$$\mathrm{f}\left(\mathrm{z}\right) = \mathrm{c_1 z + c_2} \tag{19.31}$$

Boundary Conditions are:

(19.32)

At z = 0, v_θ *= 0, rf(z) = 0*

f(z) = 0

This leads to solution that $c_2 = 0$

$$At\ z = B,\ v_\theta = v,\ f\left(z\right) = v/r = \Omega \tag{19.33}$$

This leads to solution

$$\mathrm{C_1 = \Omega / B} \tag{19.34}$$

$$\mathrm{f}\left(\mathrm{z}\right) = \Omega \mathrm{z/B} \tag{19.35}$$

$$\mathrm{v}_\theta = \Omega \mathrm{zr / B} \tag{19.36}$$

Now the *z*-component of the torque exerted on the fluid by the upper rotating disc.

$$T_z = \int_0^{2\pi}\int_0^R \tau_{z\theta} r \,|_{z=B}\, r dr d\theta$$

$$= 2\pi\int_0^R\left[\mu\left[\frac{\partial \mathrm{v}_\theta}{\partial \mathrm{z}}\right]r\right]rd\theta$$

$$= 2\pi\mu\frac{\Omega}{B}\int_0^R r^3 dr$$

$$T_z = \pi\mu\frac{\Omega}{2B}R^2$$

(19.37)

From equation (19.37), the slope of Ω Vs. T_z gives the viscosity.

Lecture 17 (30th September)

Non-Newtonian Fluids

Non-Newtonian fluids are the fluid which does not obeys Newton's law of viscosity. For describing Non-Newtonian fluids, let's assume a hypothetical experiment. There are two initially long parallel plate situated at distance h to each other. One plate is stationary and other is moving as shown in diagram.

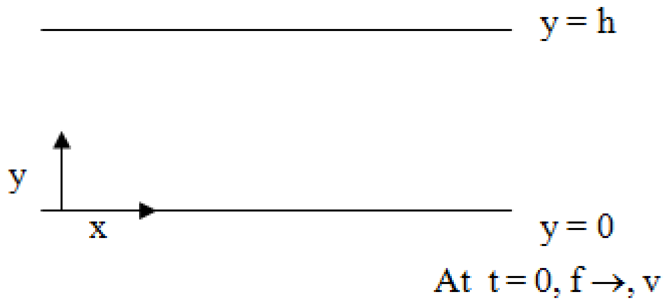


Fig 20.1 Non-Newtonian flow between two parallel plates

If force is required to move plate is *F* and plate velocity is v then $\tau_{xy} = F/A$

$$\frac{dv_x}{dy} = \frac{v}{H} \tag{20.1}$$

Now take the reading at different forces and measure associated velocity. Calculate τ_{xy} and $\frac{dv_x}{dy}$, and plot curve as given below

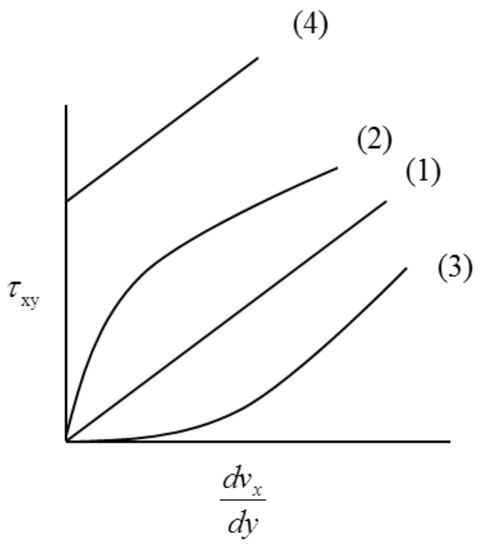


Fig 20.2 Shear stress vs. shear strain diagram for Newtonian and non-Newtonian fluids

If fluids shows curve like (1) then it is Newtonian fluids. But all other fluids are non-Newtonian fluids. Curve (2) is pseudo plastic fluid Curve (3) is dilatant fluid and curve (4) is Bingham plastic fluid.

These are called non-Newtonian fluids. There are several Theoretical and empirical models available to describe Rheological behavior of non-Newtonian fluids. Here we have discussed some of them which comes under the group of generalized Newtonian model. Basic equation for generalized non-Newtonian fluids are given below

$$\tau_{yx} = -\eta \frac{dv_x}{dy} \qquad (20.2)$$

Here η is apparent viscosity. This is changing with shear rate Therefore, $\eta = f(\frac{dv_x}{dy})$

If η is increasing with shear rate $\frac{dv_x}{dy}$ then fluid is dilatant fluid. If η is decreasing with increasing shear rate $\frac{dv_x}{dy}$ then fluid is pseudo plastic.

If η is the function of $\frac{dv_x}{dy}$ plus some constant then fluid is Bingham plastic

(I) Power Law Or Ostwald De Warle Model

Power low or Ostwald de warle model is most generalized model for non-Newtonian fluids. This is given as below

$$\tau_{yx} = -m \left| \left(\frac{dv_x}{dy} \right) \right|^{n-1} \frac{dv_x}{dy} \qquad (20.3)$$

Apparent viscosity η is

$$\eta = m \left| \left(\frac{dv_x}{dy} \right) \right|^{n-1} \qquad (20.4)$$

This is two parameter model where m and n are two parameters. n is called power law index.

If $n = 1$ Then

$$\eta = m \text{ (constant)} \qquad (20.5)$$

Therefore m is the viscosity of fluid and fluid is Newtonian fluid.

If $n > 1$ then

η will increases with shear rate and fluid is dilatant fluid.

If $n < 1$ then

η will decrease which shear rate and fluid is pseudo plastic fluid.

Modulus Sign

In power law model, modulus sign can be removed according to value of shear rate.

If

If $\frac{dv_x}{dy}$ is positive.

$$\eta = m\left(\frac{dv_x}{dy}\right)^{n-1} \tag{20.6}$$

If $\frac{dv_x}{dy}$ is negative.

$$\eta = m\left(-\frac{dv_x}{dy}\right)^{n-1} \tag{20.7}$$

Several fluids do not show single type of Rheological behavior. They show Newtonian behavior for some value of shear stress and non Newtonian behavior for some other shear stress. Several different models have been suggested for these types of fluids.

(II) Eyring Model

Eyring model is also two parameter model. The equation of Eyring model is as follow

$$\sinh\left(\frac{\tau_{yx}}{A}\right) = -\frac{1}{B}\frac{dv_x}{dy} \tag{20.8}$$

Where A, B are the two parameters.

In Eyring model if $\tau_{yx} \to 0$ which means if shear force are very low

$$\sinh\left\{\frac{\tau_{yx}}{A}\right\} \to \frac{\tau_{yx}}{A} \tag{20.9}$$

As $\tau_{yx} \to 0$, fluid shows Newtonian behavior

Therefore for low shear stress

Eyring model convert to

$$\tau_{yx} = -\frac{A}{B}\frac{dv_x}{dy} \tag{20.10}$$

This is same as Newtonian law where (A/B) is viscosity of fluids.

If τ_{yx} is very large, fluid shows Non Newtonian behavior.

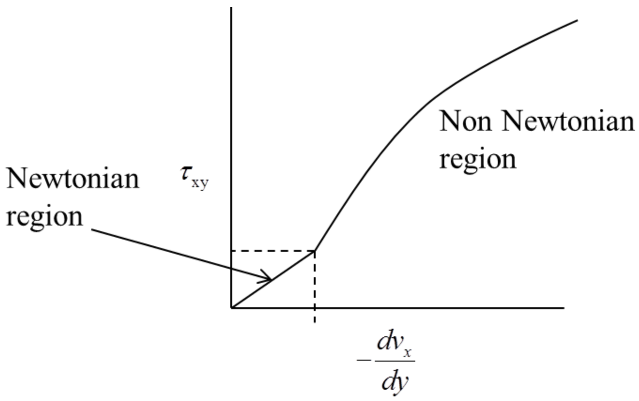


Fig 20.3 Shear stress vs. shear strain diagram for Eyring model

Therefore Eyring model is true for fluid which shows Newtonian behavior at low shear rate or low flow and non- Newtonian behavior at high shear rate or high flow.

(III) Ellis Model

Ellis model is a three parameter model. The equation of Ellis model is as follow

$$-\frac{dv_x}{dy} = \left\{\varphi_0 + \varphi_1 \left|\tau_{yx}\right|^{\alpha-1}\right\}\tau_{yx} \tag{20.11}$$

Here ϕ_0, ϕ_1, α are the three parameters

In Ellis model

If $\phi_1 = 0$,

$$\frac{dv_x}{dy} = -\varphi_0 \tau_{yx} \tag{20.12}$$

$$\tau_{yx} = -\frac{1}{\varphi_0}\frac{dv_x}{dy} \tag{20.13}$$

This is same as Newton's law of viscosity where $(1/\phi_0)$ is viscosity of fluid.

If $\phi_0 = 0$,

$$-\frac{dv_x}{dy} = -\varphi_1 \left|\tau_{yx}\right|^{a-1} \tau_{yx} \tag{20.14}$$

this equation is similar to Power law model

If $\alpha > 1$ and τ_{yx} is small then second term of Ellis model will be zero.

Now equation will be

$$\tau_{yx} = -\frac{1}{\varphi_0}\frac{dv_x}{dy} \tag{20.15}$$

This is again similar to Newton's law of viscosity

If $\alpha < <1$ and $\tau_{yx} \to \infty$ (very large)

Then again second term will be negligible. Therefore it again shows same equation

$$\tau_{yx} = -\frac{1}{\varphi_0}\frac{dv_x}{dy}$$

Therefore Ellis model are true for the fluids which follow Newton's model at very low shear stress and very high shear stress but non Newtonian behavior at intermediate shear stress.

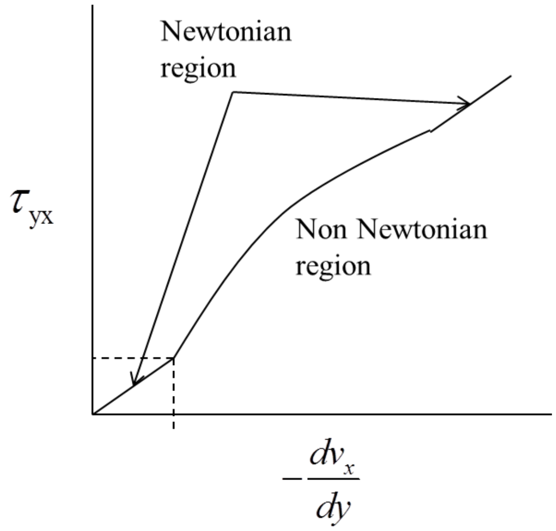


Fig 20.4 Shear stress vs. shear strain diagram for Ellis model

This type of behavior is shown by most of the melt polymers

(IV) Reiner Philippoff Model

This is also three parameter model

$$-\frac{dv_x}{dy} = \left[\frac{1}{\mu_\infty + \frac{\mu_0 - \mu_\infty}{1+\left(\frac{\tau_{yx}}{\tau_s}\right)^2}}\right]\tau_{yx} \qquad (20.16)$$

μ_0, μ_∞ and τ_s are the three parameters.

In Reiner Philippoff model,

If τ_{yx} is very large

The equation turns to

$$-\frac{dv_x}{dy} = \frac{1}{\mu_\infty}\tau_{yx} \qquad (20.17)$$

$$\tau_{yx} = -\mu_\infty \frac{dv_x}{dy}$$

Same as Newton's law

If τ_{yx} is very small then equation will be

$$-\frac{dv_x}{dy} = \frac{1}{\mu_0}\tau_{yx} \tag{20.18}$$

$$\tau_{yx} = -\mu_0 \frac{dv_x}{dy}$$

This is also same as Newton's low of viscosity

Therefore Reiner Philippoff model is true for fluid which shows Newtonian Behavior at very low and very high shear stress. But it is non Newtonian behavior for shear stress where μ_0 and μ_∞ are the viscosity of fluid at very low and very high shear stress respectively.

(V) Bingham Fluid Model

Bingham fluid is special type of fluids which require a constant shear stress to start flow.

In equation of Bingham fluid model is as follows

$\eta = \infty$ *if* $|\tau_{yx}| < \tau_0$

$$\eta = \left[-\mu \pm \frac{\tau_0}{\left|\frac{dv_x}{dy}\right|}\right]\frac{dv_x}{dy} \tag{20.19}$$

if $|\tau_{yx}| > \tau_0$ where

$$\tau_{yx} = -\eta \frac{dv_x}{dy} \tag{20.20}$$

Therefore, if shear stress is less than a fixed value τ_0 there will be no shear rate and fluid is flowing as plug flow. And if $|\tau_{yx}| > \tau_0$, fluid shows Newtonian type behavior. This phenomenon can be explained by shear stress vs. shear rate diagram.

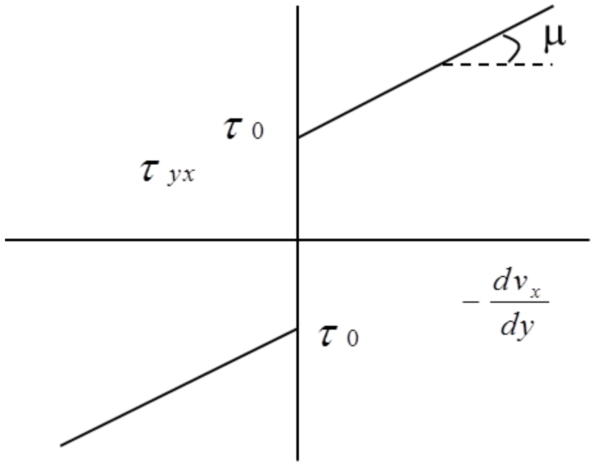


Fig 20.5 Shear stress vs. shear strain diagram for Bingham model

Lecture 18 (3rd October)

(extra solved problem on non-Newtonian fluid)

Momentum transport problem for power law and Bingham fluid:

In this section, we will solve momentum transport problem for power law fluid and Bingham plastic fluids.

Falling film on inclined plane

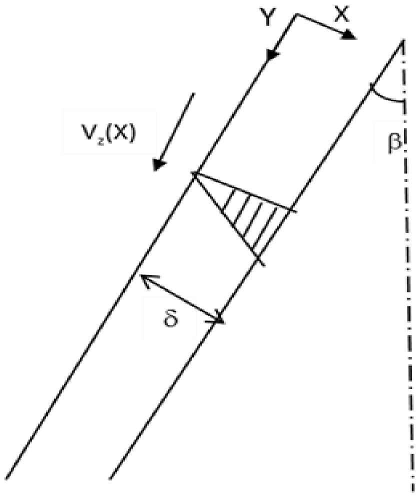


Fig 21.1 Falling film problem for non-Newtonian fluid

Initial steps are same as for Newtonian fluid solution which include make appropriate assumptions. Find non-zero velocity components. Apply equation of continuity and determine non-zero shear stress component. As we solve earlier $v_z = v_z$ (x) and τ_{xz} are the only non-zero velocity and shear stress respectively.

Now using generalized equation of motion for Cartesian coordinate

z-component

$$-\frac{d\tau_{xz}}{dx}+\rho g_z=0 \tag{21.1}$$

$$g_z = g\ \cos\beta \tag{21.2}$$

$$\tau_{xz}=\rho g\cos\beta x+c_1 \tag{21.3}$$

Solution for power low fluid

$$\tau_{xz} = -\eta \frac{dv_z}{dx} \tag{21.4}$$

$$\eta = m\left|\frac{dv_z}{dx}\right|^{n-1} \tag{21.5}$$

Since v_z is decreasing with increasing x so negative sign should be use

$$\tau_{xz} = -m\left(-\frac{dv_z}{dx}\right)^{n-1}\left(\frac{dv_z}{dx}\right) \tag{21.6}$$

$$\tau_{xz} = m\left(-\frac{dv_z}{dx}\right)^{n-1}\left(-\frac{dv_z}{dx}\right)$$

$$\tau_{xz} = m\left(-\frac{dv_z}{dx}\right)^{n} \tag{21.7}$$

Substitute equ (21.7) in equ. (21.1)

$$m\left(-\frac{dv_z}{dx}\right)^{n} = \rho g \cos\beta x + c_1$$

$$\left(-\frac{dv_z}{dx}\right)^{n} = \frac{\rho g \cos\beta}{m} x + c_1 \tag{21.8}$$

Boundary condition (1)

At $x = 0, \tau_{xz}\big|_{air} = \tau_{xz}\big|_{fluid}$

$$\left.\frac{dv_z}{dx}\right|_{x=0} = 0 \tag{21.9}$$

This Implies that

$$c_1 = 0$$

Now equ.(21.7) is

$$\left(-\frac{dv_z}{dx}\right)^{n} = \frac{\rho g \cos\beta}{m} x$$

$$-\frac{dv_z}{dx} = \sqrt[n]{\frac{\rho g \cos\beta}{m}}\sqrt[n]{x}$$

$$v_z = -\sqrt[n]{\frac{\rho g \cos\beta}{m}}\,\frac{x^{\frac{1}{n}+1}}{\frac{1}{n}+1} + c_2$$

$$v_z = \frac{n}{n+1}\sqrt[n]{\frac{\rho g \cos\beta}{m}}\,\delta^{\frac{n+1}{n}}\left[1-\left(\frac{x}{\delta}\right)^{\frac{n+1}{n}}\right] \qquad (21.10)$$

Tube Flow Problem For Power Law

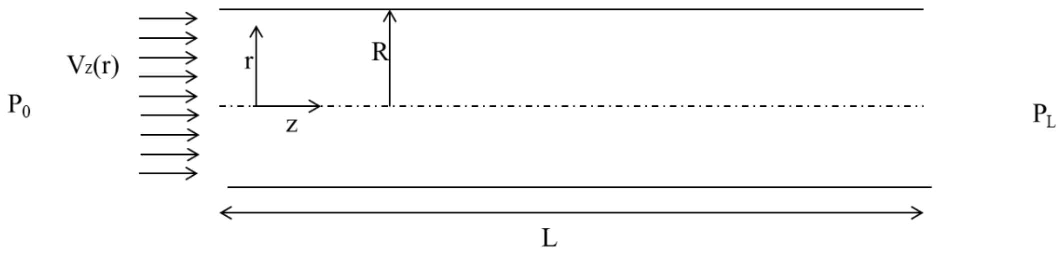


Fig 21.1 Flow through pipe for non-Newtonian fluid

As we solved previously non zero velocity is v_z which depends on r only. The non zero component of shear stress is τ_{rz}.

Apply general equation of motion in cylindrical co-ordinate.

$$\frac{\partial P}{\partial z} - \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) = 0 \qquad (21.11)$$

Equ. (21.11) will give the solution further

$$\frac{P_0 - P_L}{L} - \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) = 0$$

$$\tau_{rz} = \frac{(P_0 - P_L)}{2L}r + \frac{c_1}{r} \qquad (21.12)$$

Boundary condition 1:

At $r = 0$, velocity profile is symmetric so

$$\tau_{rz}\big|_{r=0} = 0 \qquad (21.13)$$

Substituting B.C 1 in equ (21.12)

$$C_1 = 0$$

$$\tau_{rz} = m\left(-\frac{dv_z}{dr}\right)^n \qquad (21.14)$$

Substituting equ (21.14) to eq (21.12)

$$m\left(-\frac{dv_z}{dr}\right)^n = \frac{(P_0 - P_L)}{2L}r$$

$$\left(-\frac{dv_z}{dr}\right)^n = \frac{(P_0 - P_L)}{2Lm} r$$

$$\frac{dv_z}{dr} = -\sqrt[n]{\frac{(P_0 - P_L)}{2Lm}} r^{\frac{1}{r}}$$

$$v_z = -\sqrt[n]{\frac{(P_0 - P_L)}{2Lm}} \frac{r^{\frac{n+1}{n}}}{\frac{n+1}{n}} + c_2 \tag{21.15}$$

Next boundary condition is

$$r = R,\ v_z = 0 \tag{21.16}$$

This boundary condition leads to solution

$$c_2 = -\sqrt[n]{\frac{(P_0 - P_L)}{2Lm}} \frac{R^{\frac{n+1}{n}}}{\frac{n+1}{n}} \tag{21.17}$$

So

$$v_z = \sqrt[n]{\frac{(P_0 - P_L)}{2Lm}} \frac{R^{\frac{n+1}{n}}}{\frac{n+1}{n}} \left[1 - \left(\frac{r}{R}\right)^{\frac{n+1}{n}}\right] \tag{21.18}$$

Tube Flow Problem For Bingham Fluid

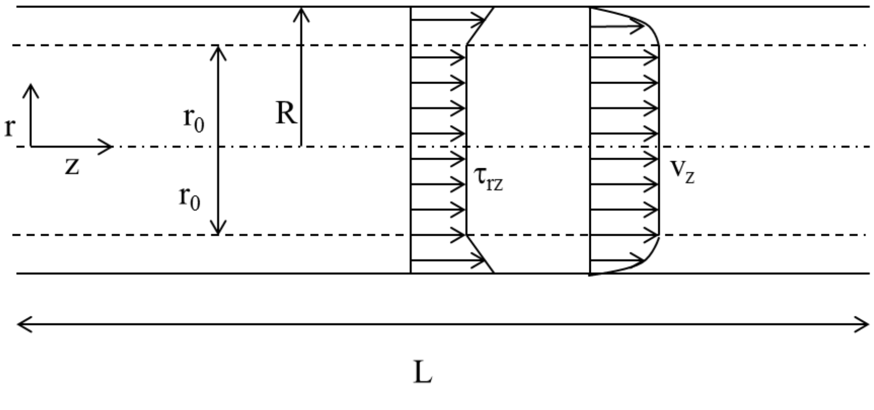


Fig 22.1 Flow through pipe for Bingham fluid

For Bingham fluid

$$\tau_{rz} = \eta\left(-\frac{dv_z}{dr}\right) \tag{22.1}$$

$$\eta = \infty \text{ when } \tau \le \tau_0 \text{ , } r \le r_0 \tag{22.2}$$

This leads to solution

$$\frac{dv_z}{dr} = 0 \text{, Therefore } v_z = v = \text{ constant} \tag{22.3}$$

$$\eta = \mu_0 \pm \frac{\tau_0}{\left|\frac{dv_z}{dr}\right|} \text{ if } \tau \ge \tau_0,\ r \le r_0 \tag{22.4}$$

Here (-) sign will be taken when $\frac{dv_z}{dr}$ is negative

And (+) sign will be taken when $\frac{dv_z}{dr}$ is positive

In this case, $\frac{dv_z}{dr}$ is negative.

$$\eta = \mu_0 - \frac{\tau_0}{\left|\frac{dv_z}{dr}\right|} \tag{22.5}$$

Substitute the value of equ (22.5) in equ (22.1)

$$\tau_{rz} = -\left\{\mu - \frac{\tau_0}{\frac{dv_z}{dr}}\right\}\frac{dv_z}{dr}$$

$$\tau_{rz} = -\mu\frac{dv_z}{dr} + \tau_0 \tag{22.6}$$

Condition for fluid movement

As we increase initial pressure P_0 fluid will not move initially. But after significant value of P_0, it starts to move. This value can be calculated when τ_{rz} is equal to τ_0.

From equ of motion

$$\frac{\partial P}{\partial z} - \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) = 0 \tag{22.7}$$

Equ. (7) will lead to the solution

$$\tau_{rz} = \frac{(P_0 - P_L)}{2L} r \tag{22.8}$$

So for initial movement of fluid $\tau_{rz} = \tau_0$ at $r = R$

$$\tau_{rz} = \frac{(P_0 - P_L)}{2L} R \tag{22.9}$$

Fluid will flow in the pipe if

$$\tau_0 \le \frac{(P_0 - P_L)}{2L} R \tag{22.10}$$

Now suppose ΔP exceed the minimum pressure required for flow. We can calculate the radius (r_0) of region where fluid have constant velocity (v) which can be calculated by equ. (22.8)

$$\tau_{rz} = \frac{(P_0 - P_L)}{2L} r \tag{22.11}$$

At $r = r_0, \tau_m = \tau_0$ (22.12)

$$\tau_0 = \frac{(P_0 - P_L)}{2L} r_0$$

$$r_0 = \frac{2\tau_0 L}{(P_0 - P_L)} \tag{22.13}$$

Eu.(22.13) shows that the effect of Δp to r_0 region. Therefore as we increase the (P_0-P_L) the plug flow region or r_0 will decrease.

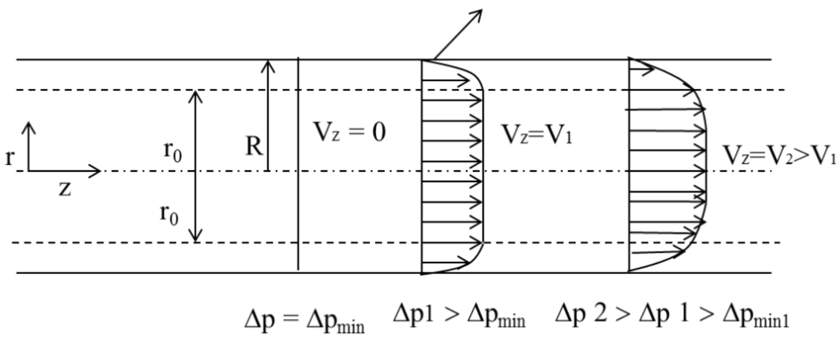


Fig 22.2 Effect of differential pressure flow through pipe for Bingham fluid

So plug flow region will continuously decreasing with increasing pressure drop. Now, solving the problem for Newtonian flow region

Where

(22.14)

$r > r_0$ $\tau_{rz} > \tau_0$

$$-\mu \frac{dv_z}{dr} + \tau_0 = \tau_{rz} = \frac{(P_0 - P_L)}{2L} r$$

$$\mu \frac{dv_z}{dr} = \tau_0 - \frac{(P_0 - P_L)}{2L} r$$

$$\mu \frac{dv_z}{dr} = \frac{\tau_0}{\mu} - \frac{(P_0 - P_L)}{2\mu L} r$$

$$v_z = \frac{\tau_0}{\mu} r - \left(\frac{(P_0 - P_L)}{4\mu L} r^2 \right) + c_1 \tag{22.15}$$

Boundary condition

$$r = R, \; v_z = 0 \tag{22.16}$$

$$0 = \frac{\tau_0}{\mu} R - \left(\frac{(P_0 - P_L)}{4\mu L} R^2 \right) + c_1$$

$$c_1 = \left(\frac{(P_0 - P_L)}{4\mu L} R^2 \right) - \frac{\tau_0}{\mu} R$$

$$v_z = \frac{\tau_0}{\mu} (r - R) + \left(\frac{(P_0 - P_L)}{4\mu L} R^2 \right) \left(1 - \left(\frac{r}{R} \right)^2 \right) \tag{22.17}$$

Falling Film Problem for Bingham Fluid

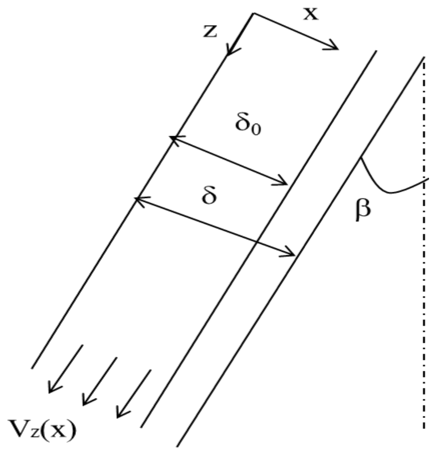


Fig 22.3 Flow on inclined surface for Bingham fluid

As we solved this problem for power law fluid

Shear stress problem will be

$$\tau_{xz} = \rho g \cos \beta x \tag{22.18}$$

Solution for Bingham fluid

$$\tau_{xz} = -\eta \frac{dv_z}{dx}$$

$$\eta = \infty \text{ when } \tau_{xz} < \tau_0 , \; x \le \delta_0 \tag{22.19}$$

$$\eta = \left\{ \mu \pm \frac{\tau_0}{\left| \frac{dv_z}{dx} \right|} \right\} \tau_{xz} > \tau_0 \text{ , } x \geq \delta_0 \tag{22.20}$$

δ_0 is unknown here but it depends on τ_0 which is fluid property

From equ. no (22.18)

$\tau_{xz} = \tau_0$, $x = \delta_0$

$\tau_0 = \rho g \cos \beta \delta_0$

$$\delta_0 = \frac{\tau_0}{\rho g \cos \beta}$$

(22.21)

For region (1) where

$\tau_{xz} < \tau_0$, $x < \delta_0$

So $\eta = \infty$

This shows that

$$\frac{dv_z}{dx} = 0$$

$$v_z = \text{constant} = v \tag{22.22}$$

For region (2) where

$\tau_{xz} > \tau_0$, $x > \delta_0$

$$\eta = \left\{ \mu \pm \frac{\tau_0}{\left| \frac{dv_z}{dx} \right|} \right\}$$

Here $\frac{dv_z}{dx}$ is negative therefore we have to take positive sign.

$$\tau_{xz} = -\left\{ \mu \frac{dv_z}{dx} - \tau_0 \right\} = \rho g \cos \beta x \tag{22.23}$$

$$-\mu \frac{dv_z}{dx} = \rho g \cos \beta x - \tau_0$$

$$\frac{dv_z}{dx} = \frac{-\rho g \cos \beta x}{\mu} + \tau_0$$

$$v_z = \frac{-\rho g \cos \beta}{\mu} \frac{x^2}{2} + \tau_0 x + c_2 \tag{22.24}$$

Boundary condition

At $x = \delta$, $v_z = 0$ (22.25)

$$c_2 = \frac{-\rho g \cos\beta}{\mu} \frac{\delta^2}{2} - \tau_0 \delta \tag{22.26}$$

$$v_z = \frac{\rho g \cos\beta \delta^2}{\mu} \left[1 - \frac{\delta^2}{x^2}\right] - \tau_0 \delta \left[1 - \frac{x}{\delta}\right] \tag{22.27}$$

Lecture 19 (14th October, Monday)

Heat Transfer

Heat transfer is the study of the flow of heat. In chemical engineering, we have to know how to predict rates of heat transfer in a variety of process situations that importantly helps in designing an equipment and processes. Now, what is heat transfer? Heat Transfer concern generation, use, conversion and exchange of thermal energy that is heat between physical systems. For example, HT occurs during cooling of fluids in industries. Seen in many different processes and industries. Understanding HT is very important to design efficient and cost-effective processes. Whenever there exists a temperature difference in a medium or between media, heat transfer must occur. Consequently, heat transport is defined as energy transfer due to a temperature difference.

In this section, we will study the transfer of energy or heat through different system. For this purpose we have to define some terms.

System a surrounding: System is precisely define region of universe which is being studied. The surrounding is the remainder of the universe outside the boundaries of the system

There are three types of system based on how system is interact with surrounding

(1) **Isolated system:** when system can't exchange energy and mass with surroundings.

(2) **Closed system:** in this system, the system can't exchange mass with surrounding but energy can be exchanged with surroundings.

(3) **Open system:** when system can exchange mass as well as energy with surroundings.

In isolated system, total energy can't be changed within two different state of system.

Therefore the first law of thermodynamics for isolated system is $\Delta E = E_1 - E_2 = 0$

Where, *ΔE* is change in energy at two different states 1 and 2.

For closed system, the first law of thermodynamic is $\Delta E = \Delta Q + \Delta W$ where, *ΔE* is change in energy of system. *ΔQ* is change in heat of system. *ΔW* is work done on the system or by the system. *ΔE* is the sum of all potential energy, kinetic energy and internal energy of system. Change in potential and kinetic energy of system is neglected then total energy *E* is converted to internal energy *U*. $\Delta U = \Delta Q + \Delta W$

We can say that conservation of energy is extended form of first law of thermodynamics.

First law of thermodynamics can be written for open system as follow.

$\Delta E = Q \pm W$ + inflow of energy in control system – out of energy from control system. It can be further expand in following manner.

Heat can be transferred by three ways

1. **Conduction:** It is the heat transfer due to temperature gradient without displaced the matter to its place. The conduction is described by Fourier equation of heat to transfer.

2. **Convection:** It is the heat transfer due to convective momentum of molecules between two points of system. The convection of heat is depends on convection of matter.

3. **Radiation:** Radiation is heat transfer as electromagnetic ways. Radiation is not required any medium to transport. A body is emitted radiation at very high temperature, and gain temperature due to radiation only if it comes in contacts with very high amount of radiations. Therefore, in chemical engineering, this type of effect is neglected.

Assume a solid block of thickness *H* and surface area *A*. The temperature of block at $x = 0$ is T_1, and $x = H$ is T_2 heat is flowing from higher temperature T_1 to lower temp T_2.

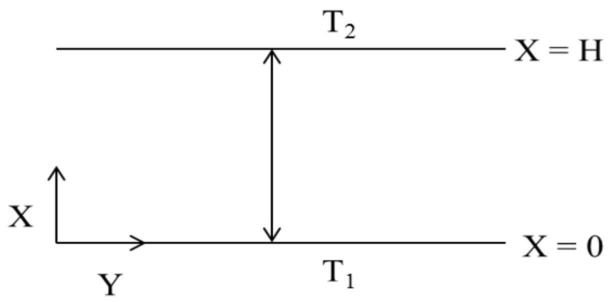


Fig 24.1 Flow of heat between two parallel plates

Measure heat flow at different temperature difference $\Delta T = T_2 - T_1$ and plot a graph Q/A Vs. $\Delta T/H$.

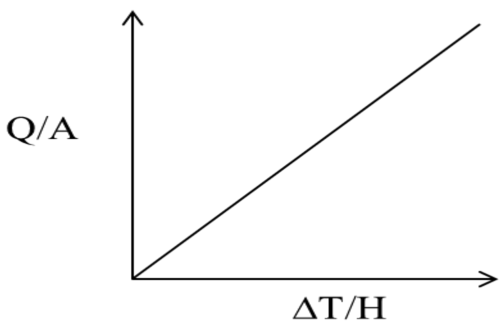


Fig 24.2 Heat flus vs. temperature gradient

We get a straight line passing through origin. Therefore

$$\frac{Q}{A} \propto \frac{\Delta T}{x} \tag{24.1}$$

In differential form, heat flux = heat flow per unit area = $\mathrm{Q/A} = q_x \propto \frac{\Delta T}{dx}$

$$q_x = -k\frac{dT}{dx} \tag{24.2}$$

K is Thermal conductivity. Here, negative sign indicates that heat is transferred in the direction of decreasing temperature.

This equation is called Fourier law of heat conduction. The same way of heat flow in *y* direction and *z* direction

$$\left.\begin{aligned} q_x &= -k\frac{dT}{dx} \\ q_y &= \frac{-kdT}{dy} \\ q_z &= \frac{-kdT}{dz} \end{aligned}\right\} \qquad \text{Generalization}$$

In vector and tensor form, Fourier's law of heat conduction is

$$\mathrm{q} = -\mathrm{k}\nabla\mathrm{T} \tag{24.3}$$

The unit of thermal conductivity is

$$k = [\text{Thermal Conductivity}] = \frac{[qx]}{\left[\frac{dT}{dx}\right]} = \frac{{}^{W}/_{m^2}}{\frac{K}{m}} = \frac{W}{mK} = \frac{J}{m\,s\,K} \tag{24.4}$$

One more term thermal diffusivity comes here which is equal to the ratio of thermal conductivity to density times heat capacity.

$$\alpha = [\text{Thermal Diffusivity}] = \frac{\text{Thermal Conductivity}}{\text{density}\times\text{heat capacity}} = \frac{k}{\rho Cp} = \frac{\frac{J}{m-s\,K}}{\frac{kg}{m^3}\frac{J}{kg\,K}} = \frac{m^2}{s} \tag{24.5}$$

The unit of thermal diffusivity is $\frac{m^2}{s}$, which is similar to unit of mass diffusivity *D* and kinematic viscosity (momentum viscosity) *k*. This similarity shows the analogy among all three types of transport.

Diffusivity in mass transport = $\frac{m^2}{s}$

$$\text{Momentum diffusivity} = \frac{\mu}{\rho} = = \frac{m^2}{s} \tag{24.6}$$

Thermal diffusivity in equation 24.5 is seen as ratio of material to conduct heat that of ability to store heat.

Axiom 4: Energy is conserved.

This axiom states first law of thermodynamics that is energy can't be destroyed or generated. It transfers from one form to another form. The detail equation of this axiom for open system is as follows:

Rate of change of energy = Net rate of inflow of energy – Net rate of out flow of energy $\pm$ generation of energy.

i.e

Rate of change of energy = Rate of inflow of energy by convection – Rate of outflow of energy by convection + heat added by conduction – heat removed by conduction ± work done by/on system + heat added by some heat source [by – & on +] (24.7)

In this section, we will solve some conduction heat transfer problem for simple geometry by using shell energy balance.

Heat conduction through composite wall

In a simple heat transfer problem, heat is flowing through a two rectangular composite wall as shown in figure. Formulate problem for heat loss per unit area of wall.

The temperature of inner wall at $x = 0$ is T_0 and temperature of outer surface of wall at $x = \delta$ is T_2. The thermal conductivity of material of inner wall is k_0 and outer wall is k_1

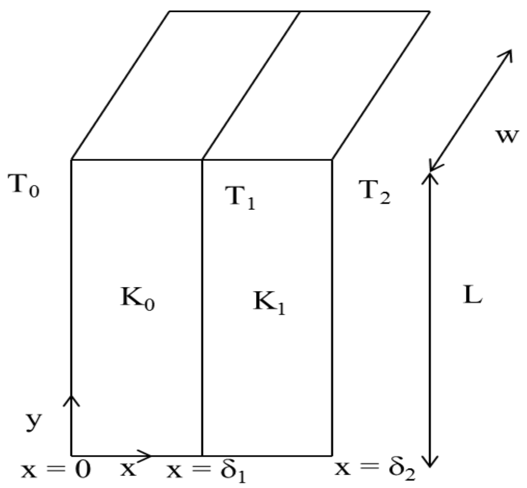


Fig 25.1 Heat conduction through a composite wall

First take necessary assumption for solving the problem.

1. System is in steady state with constant density for both phases.
2. Heat transfer coefficient of both phase k_0 and k_1 are constant.
3. System follow Fourier's low of thermal heat conduction
4. Heat loss from side wall are negligible

Draw the control volume

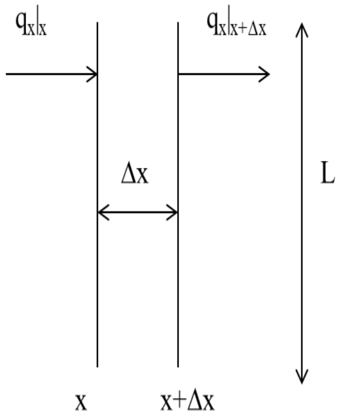


Fig 25.2 Control volume heat conduction through a composite wall

Since temperature are changing in x direction. So q_x is the non-zero heat flux. Therefore, control volume should be taken with differential thickness in x direction.

Heat balance in control volume:

Rate of heat flow entering to control volume $x = x = q_x LW|_x$ (25.1)

Rate of heat flow leaving from control volume $x = x+\Delta x = q_x LW|_{x+\Delta x}$ (25.2)

There is no source or sink of heat in the wall. There is no work done on the system or by the system. There is no other way of heat transfer in the wall. So from equ (25.1) and (25.2)

$$0 = \left(q_x LW\right)\Big|_{x=x} - \left(q_x LW\right)\Big|_{x+\Delta x} + 0\ (\text{No work done}) + 0(\text{Additional heat}) \quad (25.3)$$

$$0 = LW\left(q_x|_x - q_x|_{x+\Delta x}\right) \quad (25.4)$$

Divide by $LW\Delta x$ take limit Δx tends to 0

$$0 = \frac{dq_x}{dx} \Rightarrow q_x = C_1$$

$$q_x = \frac{-kdT}{dx} = C_1$$

$$\frac{dT}{dx} = \frac{-C_1}{k} \quad (25.5)$$

Now problem will be solved in both regions separately

Region (1) $0< x < \delta_1$

$k = k_0$

From eq (25.5)

$$\frac{dT}{dx} = \frac{-C_1}{k_0} \qquad (25.6)$$

Assume the temperature of inner wall at $x = \delta_1$ is T_{11}

Boundary conditions are

$$x = 0,\ T = T_0 \qquad (25.7)$$

$$x = \delta_1,\ T = T_{11} \qquad (25.8)$$

Substituting boundary conditions in equ(25.6) and integrate

$$\int_{T_0}^{T_{11}} dT = -\int_0^{\delta_1} \frac{C_1}{k_0} dx$$

$$\mathrm{T}_{11} - \mathrm{T}_0 = -\frac{\mathrm{C}_1\delta_1}{\mathrm{k}_0} \qquad (25.9)$$

In region 2,

Assume the temperature for outer wall

$$x = \delta_1 \text{ is } T_{12} \qquad (25.10)$$

$$x = \delta_2 \text{ is } T_2 \qquad (25.11)$$

$$\mathrm{T}_2 - \mathrm{T}_{12} = \frac{-\mathrm{C}_2(\delta_2 - \delta_1)}{\mathrm{k}_1} \qquad (25.12)$$

In equ. (25.9) and (25.12)

There is no heat loss in between wall. Therefore

$$\mathrm{T}_{11} = \mathrm{T}_{12} \qquad (25.13)$$

And rate of heat flow is constant (same), since area is same so heat flow is also same so

$$q_{x|1} = q_{x|2} \qquad (25.14)$$

Which leads to $c_1 = c_2 = q_0$ {constant heat flux}

Now using above boundary condition and substituting in equs (25.9) and (25.12))

We got the solution

$$\mathrm{T}_2 - \mathrm{T}_0 = -\mathrm{q}_0 \left\{ \frac{\delta_1}{k_0} + \frac{\delta_2 - \delta_1}{k_1} \right\} \qquad (25.15)$$

$$q_0 = \frac{T_0 - T_2}{\left\{\frac{\delta_1}{k_0} + \frac{\delta_2 - \delta_1}{k_1}\right\}} \tag{25.16}$$

Now total heat loss $Q_0 = q_0 A$

$$Q_0 = \frac{T_0 - T_2}{\left\{\frac{\delta_1}{Ak_0} + \frac{\delta_2 - \delta_1}{Ak_1}\right\}}$$

In the above equation of total heat loss, we see numerator as temperature difference and term in denominator as net (combined) resistance offered by material to heat transfer.

$$R_c = \left\{\frac{\delta_1}{Ak_0} + \frac{\delta_2 - \delta_1}{Ak_1}\right\}$$

So, resistance offered by any material for heat flow is $\frac{\text{Length}}{\text{Thermal conductivity X Area for heat flow}}$

Generally, we see

$$Q = \frac{\Delta T}{R_c}$$

What are the measurable quantities? If we see the measurable quantity that is of interest to us (or is in out control) is temperature difference (here T_0 and T_2). There when we do mathematical models, we are interested in finding a temperature profile. So, to find temperature profile (temperature at any point) we need combined resistance of the system.

Other examples on conductions

Heat transfer in cylindrical shell

Develop a formula for the overall heat transfer rate for the cylindrical shell as shown in given diagram

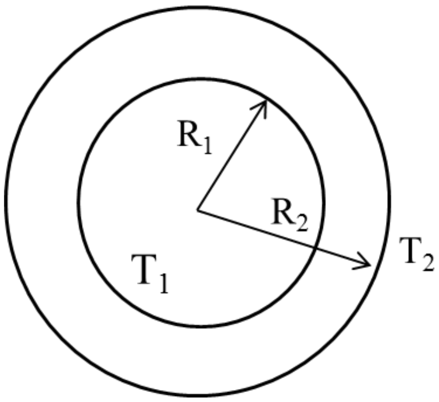


Fig 26.1 Heat transfer in cylindrical shell

Make assumption:

- System is in steady state with constant density.
- Thermal conductivity *k* is constant.
- System follow Fourier's law of thermal heat conduction
- Heat loss from side wall are negligible

Since temperature are changing in *r* direction

Therefore, q_r is non zero heat flux.

Draw control volume of differential thickness Δr.

k is constant

$T = T(r)$

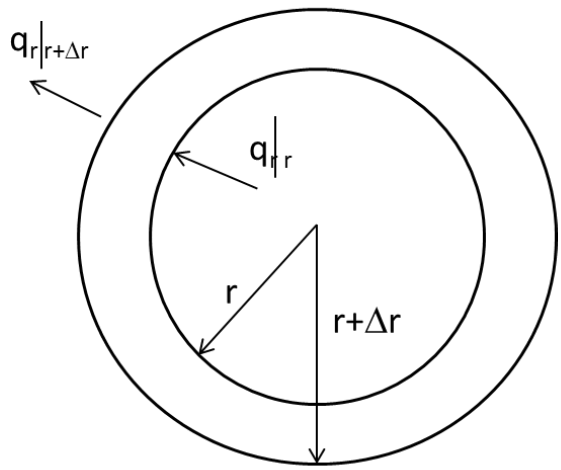


Fig 26.2 Control volume for heat transfer in cylindrical shell

Shell heat balance across the control volume:

Rate of heat flow entering into control volume from $r = r = \; 2\pi rLq_r|_r$ (26.1)

Rate of heat flow leaving from control volume from $r = r+\Delta r = \; 2\pi rLq_r|_{r+\Delta r}$ (26.2)

There is no source or sink of het in the wall. There is no work done on the system or by the system. There is no other way of heat transfer in the wall. So from equ (26.1) and (26.2)

$$0 = 0 + q_r 2\pi rL|_r - q_r 2\pi rL|_{r+\Delta r} + 0 + 0 \tag{26.3}$$

Divide by volume $2\pi rL\Delta r$ and take limit Δr tends to zero

$$0 = \frac{d(rq_r)}{dr} \tag{26.4}$$

$$q_r = \frac{c_1}{r} = \frac{-kdT}{dr} \tag{26.5}$$

By using Fourier's law of thermal conductivity

$$q_r = k\frac{dT}{dr} \tag{26.6}$$

$$kr\frac{dT}{dr} = c_1 \tag{26.7}$$

Boundary condition

at $r = R_1,\ T = T_1$ (26.8)

at $r = R_2,\ T = T_2$ (26.9)

This leads to solution

$$T_2 - T_1 = -c_1\frac{\ln(R_2/R_1)}{k}$$

$$c_1 = \frac{k\left(T_1 - T_2\right)}{\ln(R_2/R_1)} \tag{26.10}$$

$$q_r = \frac{c_1}{r} = \frac{k\left(T_1 - T_2\right)}{r\ln(R_2/R_1)}$$

$$rq_r = \frac{k\left(T_1 - T_2\right)}{r\ln(R_2/R_1)} \tag{26.11}$$

Total heat loss from cylindrical wall

$$Q_0 = 2\pi Lrq_r = \frac{2\pi Lk(T1-T2)}{ln(R2/R1)} \tag{26.12}$$

$$Q_0 = \frac{(T1-T2)}{\frac{ln(R_2/R_1)}{2\pi Lk}}$$

Develop a formula for heat loss from spherical shell as shown in Fig 26.3.

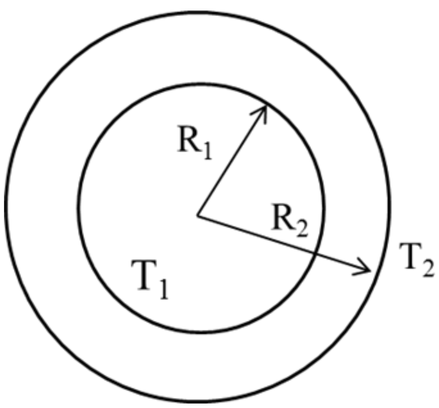


Fig 26.3 Heat transfer in spherical shell

Write assumption

- System is in steady state with constant density.
- Heat transfer coefficient k is constant.
- System follow Fourier's law of thermal heat conduction
- Heat loss from side wall are negligible

Since temperature changing in r direction

Therefore, q_r is non zero heat flux.

Draw control volume of differential thickness Δr.

Shell heat balance across the control volume

Rate f Heat entering into control volume from $r = r = \ 4\pi r^2 q_r|_r$ (26.13)

Rate of Heat leaving from control volume from $r = r+\Delta r = \ 4\pi r L q_r|_{r+\Delta r}$ (26.14)

k is constant $T = T(r)$

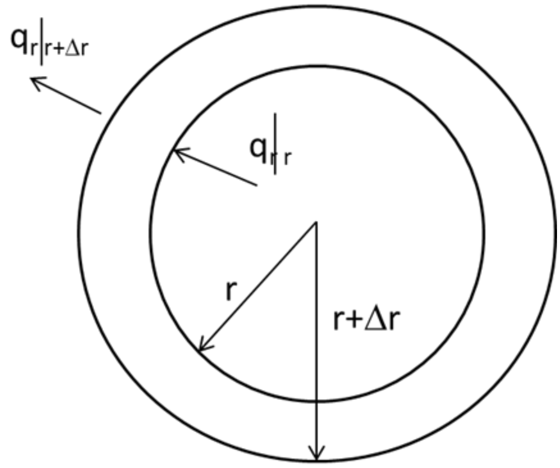


Fig 26.4 Control volume for heat transfer in spherical shell

There is no source or sink of heat in the wall. There is no work done on the system or by the system. There is no other way of heat transfer in the wall. So from equ (26.13) and (26.14)

$$0 = 0 + \left(q_r 4\pi r^2\right)_r - \left(q_r 4\pi r^2\right)_{r+dr} + 0 + 0 \tag{26.15}$$

Divide equ. (26.15) by volume $4\pi r^2 \Delta r$ and take limit Δr tends to zero

$$\frac{d}{dr}(r^2 q_r) = 0 \tag{26.16}$$

$$q_r = \frac{C_1}{r^2} \tag{26.17}$$

By substituting Fourier's law in equation (26.17)

$$q_r = k\frac{dT}{dr}$$

$$\Rightarrow \frac{dT}{dr} = \frac{-C_1}{kr^2}$$

$$T = \frac{C_1}{kr} + C_2 \qquad (26.18)$$

Boundary conditions are:

at $r = R_1$, $T = T_1$ (26.19)

at $r = R_2$, $T = T_2$ (26.20)

we get,

$$T_1 = \frac{C_1}{kR_1} + C_2 \qquad (26.21)$$

$$T_2 = \frac{C_1}{kR_2} + C_2 \qquad (26.22)$$

This leads to

$$c_1 = \frac{k\left(T_1 - T_2\right)}{\frac{1}{R_1} - \frac{1}{R_2}} \qquad (26.23)$$

From equation (26.23) and (26.17)

$$q_r = \frac{1}{r^2}\frac{k\left(T_1 - T_2\right)}{\frac{1}{R_1} - \frac{1}{R_2}} \qquad (26.24)$$

$$Q_0 = 4\pi r^2 q_r = \frac{4\pi k(T_1 - T_2)}{\frac{1}{R_1} - \frac{1}{R_2}} = \frac{(T_1 - T_2)}{\frac{1}{4\pi k}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)}$$

Lecture 20 (17th October, Thursday)

Lecture 21 (21th October, Monday)

Lecture 22 (24th October, Thursday)

Heat Transfer is transfer of thermal energy between two bodies/points that are at different temperatures. That is related with first law of thermodynamics and is extension of first law of thermodynamics.

Work done and heat are interrelated concepts, here work done is the force used to transfer energy between system and surrounding and is needed to create heat and transfer of thermal energy. Work and heat allow system to exchange energy. Also, when we look at the internal energy of a system i.e all energy withing a given system that is due to kinetic energy of molecules, energy stored in bonds of molecules etc. With the interaction of heat, work and internal energy there are energy transfer and conversions every time a change is made upon the system. No net energy is created or lost during these transfers.

Derivation of equation of energy

Axiom: energy is conserved:

In this section, we derive the equation of energy by using Axiom-4, which states that energy is conserved. The equation of total energy may be further divided into two parts. First is the equation of mechanical energy and second is the equation of thermal energy. The equation of mechanical energy is derived from equation of motion. The equation of thermal energy is derived by subtracting the equation of mechanical energy from the equation of total energy. Later, the equation of thermal energy is modified in temperature explicit form, which may be used for obtaining temperature profile

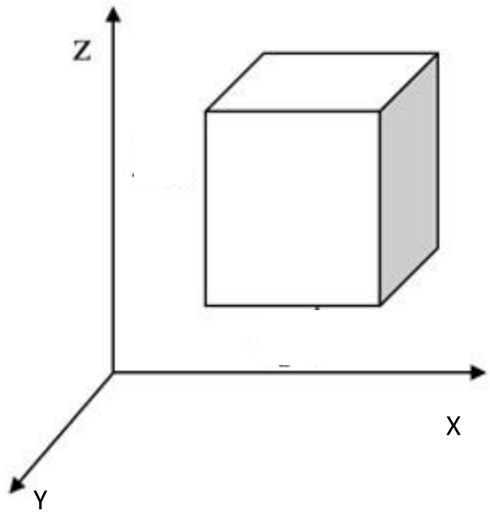


Fig 28.1 Cubical control volume

Now taking a control volume Δx Δy Δz fixed in space. The fluid flows with velocity **v** which has components v_x v_y v_z.

Total energy is addition of potential energy P, Internal energy $\widehat{U}$ and kinetic energy $\frac{1}{2}v^2$

Since control volume is fixed in space. Therefore change in potential energy in control volume is zero. Write energy balance for control volume.

$Rate$ of accumulation of internal & kinetic energy in control volume[1] =

Rate of net (difference between in & out flow) change of internal & kinetic energy by convection[2]

+ Net rate of heat addition by conduction[3]

± Work done by/on the system against various forces (Pressure, Gravity, Shear)[4]

±Rate of heat addition/removal by some heat sources[5] (28.1)

We will write all the terms one by one.

[1]. Rate of accumulation of internal and kinetic energy.

$$\frac{d}{dt}\left[(\rho\Delta x\Delta y\Delta z)\left(\frac{1}{2}v^2+\widehat{U}\right)\right] \qquad (28.2)$$

[2]. Rate of net change of kinetic and internal energy by convection.

$$\left[(\rho v_x \Delta y \Delta z)\left(\frac{1}{2}v^2+\hat{U}\right)\right]\Big|_x-\left[(\rho v_x \Delta y \Delta z)\left(\frac{1}{2}v^2+\hat{U}\right)\right]\Big|_{x+\Delta x}$$
$$+\left[(\rho v_y \Delta x \Delta z)\left(\frac{1}{2}v^2+\hat{U}\right)\right]\Big|_y-\left[(\rho v_y \Delta x \Delta z)\left(\frac{1}{2}v^2+\hat{U}\right)\right]\Big|_{y+\Delta y}$$
$$+\left[(\rho v_z \Delta y \Delta x)\left(\frac{1}{2}v^2+\hat{U}\right)\right]\Big|_z-\left[(\rho v_z \Delta x \Delta y)\left(\frac{1}{2}v^2+\hat{U}\right)\right]\Big|_{z+\Delta z} \quad (28.3)$$

[3]. Rate of heat addition by conduction:

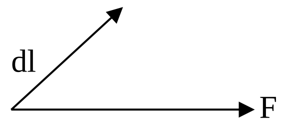
We have three components of heat q_x, q_y and q_z.

q_x is flowing from x face plane, q_y is flowing from y face plane and q_z is flowing from z face plane. Therefore net addition of heat by conduction is given below

$$(q_x\Delta y\Delta z)|_x-(q_x\Delta y\Delta z)|_{x+\Delta x}+(q_y\Delta x\Delta z)|_y-(q_y\Delta x\Delta z)|_{y+\Delta y}+(q_z\Delta x\Delta y)|_z-(q_z\Delta x\Delta y)|_{z+\Delta z}$$
$$(28.4)$$

Work done by the system

In mechanics, work is defined as a dot product of force and displacement. If F is force and dl is displacement then work w is



$$\delta w = F.dl$$

Fig 28.2 Work done by the system

In fluid mechanics, work done per unit time can be measured

$$\frac{\delta w}{\delta t}=F.\frac{\delta l}{\delta t}=\boldsymbol{F.v} \quad (28.5)$$

Work done by the system = $-\boldsymbol{F.v}$ (28.6)

$$\frac{dw}{dt}=-(F_xv_x+F_yv_y+F_zv_z) \quad (28.7)$$

There are three types of force working in fluids mechanics.

1. Pressure force
2. Gravity force
3. Shear force

Therefore, these three forces will produce work in different form. Here, we will take work done by these force one by one.

Work done against Gravity forces

The gravity force can be written as

Work done by gravity force can be written as

$= \rho \Delta x \Delta y \Delta z g_x v_x + \rho \Delta x \Delta y \Delta z g_y v_y + \rho \Delta x \Delta y \Delta z g_z v_z$

$$= \rho \Delta x \Delta y \Delta z\, \boldsymbol{g} \cdot \boldsymbol{v} \quad (28.8)$$

Pressure Forces

The pressure force is always working to the normal (inwards to surface) to surface. It is a compressible force. Therefore work done on the control volume by pressure force may be calculated as follows:

Rate of work done by x-direction pressure force:

$$= \left[(v_x P \Delta y \Delta z)|_x - [(v_x P \Delta y \Delta z)|_{x+\Delta x} \right] \quad (28.9)$$

$$= \left[(v_y P \Delta x \Delta z)|_y - [(v_y P \Delta x \Delta z)|_{y+\Delta y} \right] \quad (28.10)$$

$$= \left[(v_z P \Delta x \Delta y)|_z - [(v_z P \Delta x \Delta y)|_{z+\Delta z} \right] \quad (28.11)$$

Rate of Work done by shear forces

There are nine components of shear stress tensor. Three of these act on x directed face, similarly the net three acts on the y directed face and remaining three act on z directed faces (shown in Fig. 29.1) as discussed earlier.

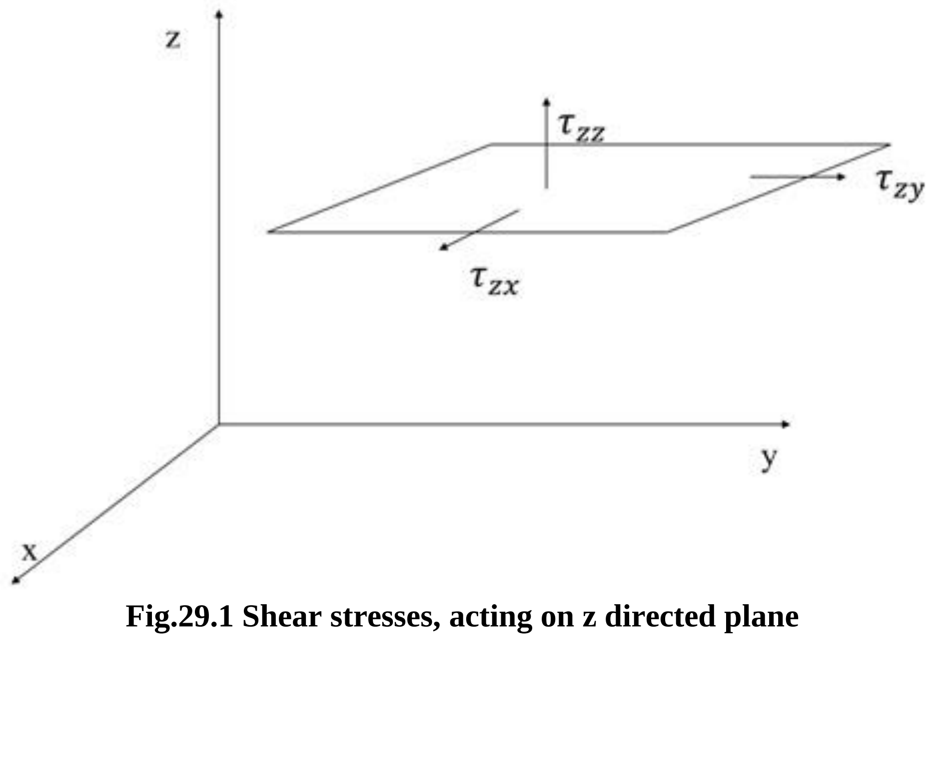


Fig.29.1 Shear stresses, acting on z directed plane

The shear force acting on x face plane is τ_{xx}, τ_{xy}, τ_{xz}.
Therefore, the work done by shear forces on the control volume, acting on *x* directed plane may be calculated as

$$-\left[\tau_{xx}v_x + \tau_{xy}v_y + \tau_{xz}v_z\right]\Delta y\Delta z|_{x=x} - \left[+(-\tau_{xx})v_x + (-\tau_{xy})v_y + (-\tau_{xz})\,v_z\right]\Delta y\Delta z|_{x+\Delta_x} \tag{29.1}$$

Similarly, the work done by shear forces on the control volume, acting on *y* directed plane may be calculated as

$$\left[\tau_{yx}v_x + \tau_{yy}v_y + \tau_{yz}v_z\right]|_{y+\Delta y}\Delta x\Delta z - \left[\tau_{yx}v_x + \tau_{yy}v_y + \tau_{yz}v_z\right]|_{y}\Delta x\Delta z \tag{29.2}$$

and the work done by shear force on the control volume, acting on *z* directed plane may be calculated as

$$\left[\tau_{zx}v_x + \tau_{zy}v_y + \tau_{zz}v_z\right]|_{z+\Delta z}\Delta x\Delta y - \left[\tau_{zx}v_x + \tau_{zy}v_y + \tau_{zz}v_z\right]|_{z}\Delta x\Delta y \tag{29.3}$$

If any heat source/sink is present in the control volume which generates the heat as *Sc* per unit volume, then heat generated in the control volume may be written as.

$$Sc\Delta x\Delta y\Delta z \tag{29.4}$$

We now substitute all terms given above in Equation (28.1) and then divide by ΔxΔyΔz. After taking the limits Δx, Δy and Δz going to zero, we obtain the following equation

$$\frac{\partial}{\partial t}\left[\rho\left(\widehat{U}+\frac{v^2}{2}\right)\right] = -\left[\frac{\partial}{\partial x}\left\{\rho v_x\left(\widehat{U}+\frac{v^2}{2}\right)\right\} + \frac{\partial}{\partial y}\left\{\rho v_y\left(\widehat{U}+\frac{v^2}{2}\right)\right\} + \frac{\partial}{\partial z}\left\{\rho v_z\left(\widehat{U}+\frac{v^2}{2}\right)\right\}\right] - \left[\frac{\partial}{\partial x}q_x + \frac{\partial}{\partial y}q_y + \frac{\partial}{\partial z}q_z\right] + \rho(v_x g_x + v_y g_y + v_z g_z) - \left[\frac{\partial}{\partial x}(Pv_x) + \frac{\partial}{\partial y}(Pv_y) + \frac{\partial}{\partial z}(Pv_z)\right] - \left\{\frac{\partial}{\partial x}(\tau_{xx}v_x + \tau_{xy}v_y + \tau_{xz}v_z) + \frac{\partial}{\partial y}(\tau_{yx}v_x + \tau_{yy}v_y + \tau_{yz}v_z) + \frac{\partial}{\partial z}(\tau_{zx}v_x + \tau_{zy}v_y + \tau_{zz}v_z)\right\} + S_c \tag{29.5}$$

In Equation (29.5), the stress tensor $\underline{\tau}$ was taken as shear forces. To change it into momentum flux, we replace all components of $\underline{\tau}$ with a minus sign. In addition, if we rewrite the Equation (29.5) in vector and tensor form , we obtain the following result for equation of energy.

$$\frac{\partial}{\partial t}\left[\rho\left(\widehat{U}+\frac{v^2}{2}\right)\right] = -\underline{\nabla}.\left[\rho\left(\hat{v}+\frac{v^2}{2}\right)\right]\underline{v} - \underline{\nabla}.\underline{q} + \rho\underline{v}.\underline{g} - \underline{\nabla}.(P\underline{v}) - \underline{\nabla}.(\underline{\tau}.\underline{v}) + Sc \tag{29.6}$$

We can further simplify this equation (29.6) by combining the internal energy and kinetic energy terms as shown below

Assume $s = \widehat{U} + \frac{v^2}{2}$ (29.7)

Then, Equation (29.6) may be written as

$$\frac{\partial}{\partial t}[\rho s] + \underline{\nabla}.[\rho s]\underline{v} = -\underline{\nabla}.\underline{q} + \rho\underline{v}.\underline{g} - \underline{\nabla}.(P\underline{v}) - \underline{\nabla}.(\underline{\tau}.\underline{v}) + Sc \quad (29.8)$$

The left hand side of Equation (29.8) may be modified as shown below

$$\frac{\partial(\rho s)}{\partial t} + \underline{\nabla}.(\rho s\underline{v}) = \frac{\rho\partial s}{\partial t} + \frac{s\partial\rho}{\partial t} + s\nabla.\left(\rho\underline{v}\right) + \rho\underline{v}\cdot\nabla s \quad (29.9)$$

OR

$$\frac{\partial(\rho s)}{\partial t} + \underline{\nabla}.(\rho s\underline{v}) = \rho\left[\frac{\partial s}{\partial t} + \underline{v}.\underline{\nabla}s\right] + s\left[\frac{\partial\rho}{\partial t} + \underline{v}.\underline{\nabla}\rho\underline{v}\right] \quad (29.10)$$

From equation of continuity we know

$\frac{\partial\rho}{\partial t} + \underline{\nabla}.\rho\underline{v} = 0$

Therefor, Equation (29.10) simplifies to

$$\frac{\partial(\rho s)}{\partial t} + \underline{\nabla}.\left(\rho s\underline{v}\right) = \rho\left[\frac{\partial s}{\partial t} + \underline{v}.\underline{\nabla}s\right] = \rho\frac{Ds}{Dt}$$

Therefor, Equation (29.08) simplifies to

$$\rho\frac{Ds}{Dt} = -\underline{\nabla}.\underline{q} + \rho\underline{v}.\underline{g} - \underline{\nabla}.(P\underline{v}) - \underline{\nabla}.(\underline{\tau}.\underline{v}) + Sc$$

Or

$$\rho\frac{D\left(\widehat{U} + \frac{v^2}{2}\right)}{Dt} = -\underline{\nabla}.\underline{q} + \rho\underline{v}.\underline{g} - \underline{\nabla}.(P\underline{v}) - \underline{\nabla}.(\underline{\tau}.\underline{v}) + Sc$$

Above equation represent the equation of total energy in terms of substantial derivative. Since observer is moving with fluid in the case of substantial derivative. Therefore, convective term will be vanishing from the above equation.

Earlier, we had derived the equation of energy which may be further divided into two parts

1. Equation of mechanical energy
2. Equation of thermal energy

Equation of mechanical energy

For understanding the nature of mechanical energy, consider a simple case of a single particle moving in one direction as shown in Fig. 30.1. Assume the particle has mass *m* and is located at height *h* from a reference plane and moving upward with velocity $\underline{v}$. Gravity is the only force working on the particle.

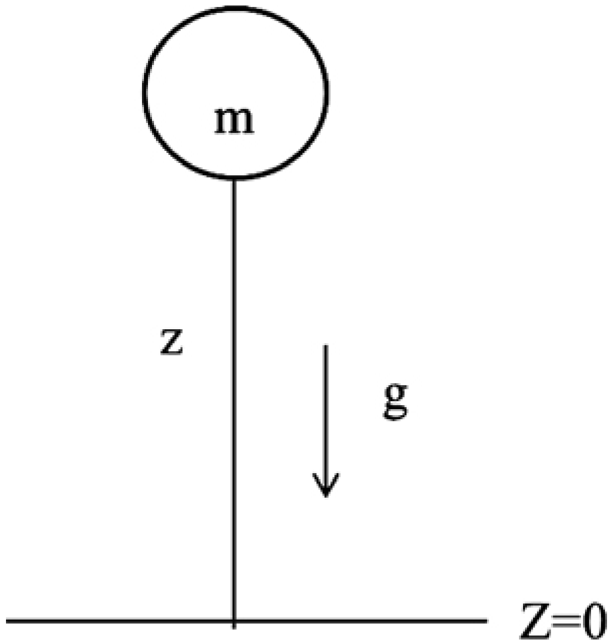


Fig 30.1 A particle of mass m situated at height z

Starting with Newton's second law of motion, we have

Force = mass x acceleration
Where

$$\underline{F} = m\underline{a} \quad (30.1)$$

OR

$$\underline{F} = m\frac{d\underline{v}}{dt} \quad (30.2)$$

By taking dot product of equation (30.2) with velocity , we find that

$$\underline{v} \cdot \left[\underline{F} = m\frac{d\underline{v}}{dt}\right]$$

Or

$$\underline{v} \cdot \underline{F} = \underline{v} \cdot \left[m \frac{d\underline{v}}{dt} \right] \tag{30.4}$$

Using vector identity, we have

$$\frac{d(\underline{v} \cdot \underline{v})}{dt} = \underline{v} \cdot \frac{d\underline{v}}{dt} + \frac{d\underline{v}}{dt} \cdot \underline{v} = 2\underline{v} \cdot \frac{d\underline{v}}{dt} \tag{30.5}$$

Or

$$\underline{v} \cdot \frac{d\underline{v}}{dt} = \frac{1}{2} \frac{d(v^2)}{dt} \tag{30.6}$$

where, v is the magnitude of the velocity vector $\underline{v}$.

By substituting Equation (30.6) in Equation (30.4), we obtain

$$\underline{v} \cdot \underline{F} = m \frac{d\left(\frac{v^2}{2}\right)}{dt} \tag{30.6}$$

or,

$$v_1 F_1 + v_2 F_2 + v_3 F_3 = m \frac{d\left(\frac{v^2}{2}\right)}{dt} \tag{30.7}$$

For the example given above, we have ,

$F_1 = 0, F_2 = 0, F_3 = -mg$ (30.8)

and

$v_1 = 0, v_2 = 0, v_3 = v$

Thus, Equation (30.7), reduces to

$$-vmg = m \frac{d\left(\frac{v^2}{2}\right)}{dt} \tag{30.9}$$

Substitute $v = (dz/dt)$. Thus we obtain,

$$-mg \frac{dz}{dt} = m \frac{d\left(\frac{v^2}{2}\right)}{dt} \tag{30.10}$$

Since, m and g are constants. We may rewrite above equation as,

$$-\frac{d(mgz)}{dt} = m\frac{d\left(\frac{v^2}{2}\right)}{dt}$$

or

$$\frac{d}{dt}\left(mgz + m\frac{v^2}{2}\right) = 0$$

$$mgz + m\frac{v^2}{2} = constant \tag{30.11}$$

First term in above equation is the potential energy and second term represents the kinetic energy. Therefore, the above equation states that the sum of kinetic and potential energy remains constant. This is the equation of mechanical energy for a particle and similar equation may be derived for fluids as shown below.

First term in above equation is the potential energy and second term represents the kinetic energy. Therefore, the above equation states that the sum of kinetic and potential energy remains constant. This is the equation of mechanical energy for a particle and similar equation may be derived for fluids as shown below.

Equation of mechanical energy for fluids

The equation of motion for a fluid is equivalent to Newton's second law of motion for solid bodies. Therefore, to derive the equation of mechanical energy for fluids we take the dot product of velocity with equation of motion for fluids. i.e.,

$$\underset{\sim}{v}.\left[\rho\frac{D\underset{\sim}{v}}{Dt} = \rho\underset{\sim}{g} - \underset{\sim}{\nabla}P - \underset{\sim}{\nabla}.\underset{\sim}{\tau}\right]$$

As before,

$$\underset{\sim}{v}.\frac{D\underset{\sim}{v}}{Dt} = \frac{D\left(\frac{v^2}{2}\right)}{Dt}$$

(Note: substantial derivatives behave like normal derivatives.). Thus,

$$\rho\frac{D}{Dt}\left(\frac{v^2}{2}\right) = \rho\underset{\sim}{g}.\underset{\sim}{v} - \underset{\sim}{v}.\left(\underset{\sim}{\nabla}P\right) - \underset{\sim}{v}.\left(\underset{\sim}{\nabla}.\underset{\sim}{\tau}\right)$$

The following vector and tensor identities may be used for simplifying Equation (30.14)

$$\underset{\sim}{\nabla}.\left(P\underset{\sim}{v}\right) = P\left(\underset{\sim}{\nabla}.\underset{\sim}{v}\right) + \underset{\sim}{v}.\left(\underset{\sim}{\nabla}P\right)$$

and if $\underset{\sim}{\tau}$ is a second order symmetric tensor then we also have

$$\underset{\sim}{\nabla}.(\underset{\sim}{\tau}.\underset{\sim}{v}) = \underset{\sim}{v}.\underset{\sim}{\nabla}.\underset{\sim}{\tau} + \underset{\sim}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{v}$$

Thus, we obtain

$$\rho \frac{D}{Dt}\left(\frac{v^2}{2}\right) = \underset{\sim}{v}.\left(\rho \underset{\sim}{g}\right) - \left[\underset{\sim}{\nabla}.\left(P\underset{\sim}{v}\right) + \left(-P\left(\underset{\sim}{\nabla}.\underset{\sim}{v}\right)\right)\right] - \left[\underset{\sim}{\nabla}.\left(\underset{\sim}{\tau}.\underset{\sim}{v}\right) + \left(-\underset{\sim}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{v}\right)\right]$$

or

$$\rho \frac{D}{Dt}\left(\frac{v^2}{2}\right) = \underset{\sim}{v}.\left(\rho \underset{\sim}{g}\right) - \underset{\sim}{\nabla}.\left(P\underset{\sim}{v}\right) + P\left(\underset{\sim}{\nabla}.\underset{\sim}{v}\right) - \underset{\sim}{\nabla}.\left(\underset{\sim}{\tau}.\underset{\sim}{v}\right) + \underset{\sim}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{v}$$

Equation (30.18) is called the equation of mechanical energy for fluids. Significance of each term is given below.

$$\begin{aligned}
&\rho \frac{D}{Dt}\left(\frac{v^2}{2}\right)\begin{Bmatrix}\text{Rate of change of}\\ \text{kinetic energy}\\ \text{per unit volume}\end{Bmatrix} = \underset{\sim}{v}.\left(\rho \underset{\sim}{g}\right)\begin{Bmatrix}\text{work done by gravity}\\ \text{force on the system}\end{Bmatrix}\\
&-\underset{\sim}{\nabla}.\left(P\underset{\sim}{v}\right)\begin{Bmatrix}\text{work done by pressure}\\ \text{force on the system}\end{Bmatrix} + P\left(\underset{\sim}{\nabla}.\underset{\sim}{v}\right)\begin{Bmatrix}\text{reversible conversion of}\\ \text{kinetic energy into}\\ \text{the internal energy}\end{Bmatrix}\\
&-\underset{\sim}{\nabla}.\left(\underset{\sim}{\tau}.\underset{\sim}{v}\right)\begin{Bmatrix}\text{work done by viscous}\\ \text{forces on system}\end{Bmatrix} + \underset{\sim}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{v}\begin{Bmatrix}\text{irreversible conversion of}\\ \text{kinetic energy in to the heat}\end{Bmatrix}
\end{aligned}$$

...**(30.19)**

As discussed earlier, the equation of thermal energy can be derived by subtracting the equation of mechanical energy from the equation of total energy, i.e.,

$$\begin{pmatrix}\text{Equation of thermal}\\ \text{energy}\end{pmatrix} = \begin{pmatrix}\text{Equation of}\\ \text{energy}\end{pmatrix} - \begin{pmatrix}\text{Equation of mechanical}\\ \text{energy}\end{pmatrix}$$

Thus

$$\rho \frac{D\left(\hat{U}\right)}{Dt} = -\underset{\sim}{\nabla}.\underset{\sim}{q} - \underset{\sim}{\nabla}.\left(P\underset{\sim}{v}\right) + S_c + \underset{\sim}{\nabla}.\left(P\underset{\sim}{v}\right) - P\left(\underset{\sim}{\nabla}.\underset{\sim}{v}\right) - \left(\underset{\sim}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{v}\right)$$

or

$$\rho\frac{D(\hat{U})}{Dt} = -\underset{\sim}{\nabla}.\underset{\sim}{q} + S_c - P(\underset{\sim}{\nabla}.\underset{\sim}{v}) - (\underset{\sim}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{v})$$

The significance of each term in equation of above thermal energy, Equation is given below

$$\rho\frac{D(\hat{U})}{Dt}\left\{\begin{array}{l}\text{Rate of change of}\\ \text{internal energy}\\ \text{per unit volume}\end{array}\right\} = -\underset{\sim}{\nabla}.\underset{\sim}{q}\left\{\begin{array}{l}\text{Heat transferred}\\ \text{by conduction}\end{array}\right\} + S_c\left\{\begin{array}{l}\text{Heat generated / removed}\\ \text{by source or sink}\end{array}\right\}$$
$$-P(\underset{\sim}{\nabla}.\underset{\sim}{v})\left\{\begin{array}{l}\text{reversible conversion of}\\ \text{kinetic energy into}\\ \text{the internal energy}\end{array}\right\} - (\underset{\sim}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{v})\left\{\begin{array}{l}\text{irreversible conversion of}\\ \text{kinetic energy in to the heat}\end{array}\right\}$$

Here $(\underset{\sim}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{v})$, is known as the viscous heat dissipation and the significance of this will be discussed later.

Equation of mechanical energy of fluids and its interpretation

If we consider a special case of non-viscous fluid, where the shear stress is zero, Equation simplifies as shown below

$$\rho\frac{D\left(\frac{v^2}{2}\right)}{Dt} = -\underset{\sim}{v}.\underset{\sim}{\nabla}P + \rho(\underset{\sim}{v}.\underset{\sim}{g})$$

Here, the gravity may be represented by gradient of a scalar quantity *Φ*, or

$$\underset{\sim}{g} = -\underset{\sim}{\nabla}\hat{\phi}$$

Then, Equation may be rewritten as

$$\rho\frac{D\left(\frac{v^2}{2}\right)}{Dt} = -\underset{\sim}{v}.\underset{\sim}{\nabla}P + \rho\left(-\underset{\sim}{v}.\underset{\sim}{\nabla}\hat{\phi}\right)$$

..

Further, if we assume that pressure and gravity do not depend on time. Thus, we have

$$\frac{\partial P(t)}{\partial t} = 0$$

and

$$\rho \frac{\partial \phi}{\partial t} = 0$$

After substituting these values we obtain

$$\rho \frac{D\left(\frac{v^2}{2}\right)}{Dt} = -\underset{\sim}{v}.\underset{\sim}{\nabla} P - \frac{\partial P}{\partial t} + \rho\left(-\underset{\sim}{v}.\underset{\sim}{\nabla}\hat{\phi}\right) - \rho \frac{\partial \hat{\phi}}{\partial t}$$

which may be further simplified as

$$\frac{D\left(\frac{v^2}{2}\right)}{Dt} = \frac{D\left(\frac{P}{\rho}\right)}{Dt} - \frac{D\hat{\phi}}{Dt}$$

or

$$\rho \frac{D}{Dt}\left(\frac{v^2}{2} + \frac{p}{\rho} + \hat{\phi}\right) = 0$$

which leads to

$$\left(\frac{v^2}{2} + \frac{p}{\rho} + \hat{\phi}\right) = constant$$

The above equation is called Bernoulli’s equation. This states that the sum of the kinetic energy, pressure and potential energy heads is constant for a non-viscous fluid.

Comparisons of mechanical and thermal energy

The term q is not present in equation of mechanical energy. Therefore if we heat something it comes as thermal energy and increase temperature. In the same way Sc (heat source) does not change mechanical energy and only increase temperature of body.

Now comes again equation of thermal energy:

Internal energy $\hat{U}$ is the state function which can be define in terms of pure component. It may be fixed by fixing two intensive variables. Intensive variable are temperature, pressure and volume. If we fix any two of them everything is fixed so all the properties will be fixed.

$\hat{U}(T, \hat{v})$

$\hat{U}$= internal energy per unit mass (Per unit mass is taken to make volume intensive)

$$d\hat{U} = \left(\frac{d\hat{U}}{dT}\right)_{\hat{U}} dT + \left(\frac{d\hat{U}}{d\hat{U}}\right)_T d\hat{U} \qquad (31.1)$$

For a real gas

$$d\hat{U} = C_v dT + \left[-p + T\left(\frac{dp}{dT}\right)_{\hat{v}}\right] d\hat{v} \qquad (31.2)$$

$$\left[-p + T\left(\frac{dp}{dT}\right)_{\hat{v}}\right] = -p + \frac{TR}{\hat{v}} = 0 \qquad (31.3)$$

For ideal gas $d\hat{v} = C_v DT$ (31.4)

$$\frac{D\hat{U}}{Dt} = C_v \frac{DT}{Dt} + \left[-p + T\left(\frac{dp}{dT}\right)_{\hat{v}}\right] \frac{D\hat{v}}{Dt} \qquad (31.5)$$

$$= C_v \frac{DT}{Dt} + \left[-pT\left(\frac{dp}{dT}\right)_v\right] \frac{D\hat{v}}{Dt} \qquad (31.6)$$

$$\hat{v} = \left(\frac{1}{\rho}\right) \qquad (31.7)$$

$$\frac{D\hat{v}}{Dt} = \frac{D\left(\frac{1}{\rho}\right)}{Dt} = \frac{-1}{\rho^2}\frac{D\rho}{Dt} \qquad (31.8)$$

$$\frac{D\hat{U}}{Dt} = C_v \frac{DT}{Dt} + \left[-p + T\left(\frac{dp}{dT}\right)_{\hat{v}}\right]\left(\frac{-1}{\rho^2}\frac{D\rho}{Dt}\right)$$

$$\rho\frac{D\hat{U}}{Dt} = \rho C_v \frac{DT}{Dt} + \left[-p + T\left(\frac{dp}{dT}\right)_{\hat{v}}\right]\left(\frac{-1}{\rho^2}\frac{D\rho}{Dt}\right)$$

$$= -\nabla.q - p(\nabla.v) + (-\rho : \nabla v) + S_c \qquad (31.9)$$

Substituting in thermal energy equation:-

$$\rho C_v \frac{DT}{Dt} = -\nabla.q - p(\nabla.v) + \frac{1}{\rho}\left[-p + T\left(\frac{dp}{dT}\right)_v\right]\frac{D\rho}{Dt} + (-\tau : \nabla v) + S_c \qquad (31.10)$$

From equation of continuity

$$\rho C_v \frac{DT}{Dt} = -\nabla.q - p(\nabla.v) + \frac{1}{\rho}\left[-p + T\left(\frac{dp}{dT}\right)_v\right]\left[\rho.\nabla v\right] + (-\tau : \nabla v) + S_c \qquad (31.11)$$

Further simplify the equation leads to:

$$= -\nabla.q - T\left(\frac{dp}{dT}\right)_v (\nabla.v) + (-\tau : \nabla v) + S_c$$

$$\rho C_v\left[\frac{dT}{dt} + v.\nabla T\right] = -\nabla.q - T\left(\frac{dp}{dT}\right)_v (\nabla.v) + (-\tau : \nabla v) + S_c \qquad (31.12)$$

We can simplify above equations according to some limiting cases

Case 1: heat conduction in solids

Since all the velocities are zero, therefore above equation will be

$$\rho C_v \frac{dT}{dt} = -\nabla.q + S_c \qquad (31.13)$$

$$q = -k\nabla.T$$

………Fourier low of heat conduction

$$\rho C_v \frac{dT}{dt} = +k\nabla^2.T + S_c \qquad (31.14)$$

This is also known as Fourier second law of heat conduction.

Case 2 Heat transfer in fluids with constant ρ and *k*.

From equation of continuity

$$\nabla.v = 0 \qquad (31.15)$$

$$C_v = C_p \qquad (31.16)$$

$$\rho C_v\left[\frac{dT}{dt} + v.\nabla T\right] = -\nabla.q + (-\tau : \nabla v) + S_c \qquad (31.17)$$

If k is also constant

$$-\nabla.q = -k\nabla^2 T$$

$$\rho C_v\left[\frac{dT}{dt} + v.\nabla T\right] = k\nabla^2 T + (-\tau : \nabla v) + S_c \qquad (31.18)$$

This equation is exactly same as Navior stoke equation.

$$\rho\frac{Dv}{Dt} = \mu\nabla^2 v - \nabla p + \rho g$$

$$\frac{\partial v}{\partial t} + \rho.\nabla v = \frac{\mu}{\rho}\nabla^2 v - \frac{\nabla p}{\rho} + g \qquad (31.19)$$

Thermal energy

$$\frac{\partial T}{\partial t} + v.\nabla T = \frac{k}{\rho C_p}\nabla^2 T + \frac{(-\tau : \nabla v)}{\rho C_p} + \frac{S_c}{\rho C_p} \qquad (31.20)$$

The nature of above both (thermal energy and Navior stoke) equations are same

Viscous dissipation / heating

The term $(-\tau : \nabla v)$ is represent conversion of mechanical energy in to thermal energy due to viscous forces. This term is always positive. For Newtonian fluids it can be calculated by using Newton's law.

For Newtonian fluids

$$\tau = -\mu \nabla v \tag{31.21}$$

$$-\tau : \nabla v = -\mu \phi_v \tag{31.22}$$

Where ,

$$\phi_v = \left[\left(\frac{\partial v_x}{\partial x} \right)^2 + \left(\frac{\partial v_y}{\partial y} \right)^2 + \left(\frac{\partial v_z}{\partial z} \right)^2 \right] + \left[\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right]^2 + \left[\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial z} \right]^2 + \left[\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right]^2 - \frac{2}{3} \left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right] \tag{31.23}$$

Note: Below is reference notes for understanding (covered theoretically in lecture, explained below with an example

Significance of Viscous dissipation / heating

Viscous dissipation term is present in equation of thermal energy for all fluids. For analyze the significance of viscous dissipation, we assume a hypothetical experiment. A fluid is flowing in between two parallel plates. As shown in diagram. Solve this problem for viscous heat dissipation.

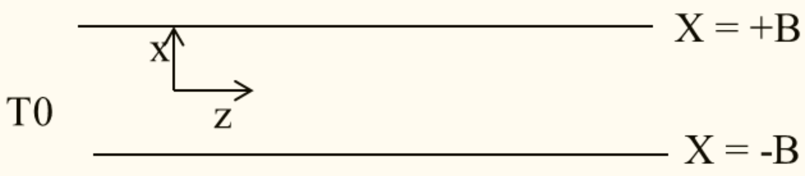


Fig 32.1 Viscous flow between two parallel plates

Assumptions:

- Constant density ρ, viscosity μ & thermal conductivity k
- Newtonian Fluid
- Steady state
- Fully developed flow

Fluid is flowing in z direction only. Therefore,

$$\mathrm{v_x = 0,\ v_y = 0,\ v_z = v_z(x)}$$

Velocity profile can be given by following formula

$$\mathrm{v_z = v_{z,max}}\left[1-\frac{x^2}{B^2}\right] \tag{32.1}$$

Here $V_{z\,max}$ is the maximum velocity of fluid.

Initially, both plates are at same temperature.

At steady state, heat produce by viscous dissipation will remove from both plates and temperature is not changing along the plate. Therefore, In fully developed region, temperature is the function of only *x* direction.

Fully developed temperature profile is

$$\mathrm{T = T(x)} \tag{32.2}$$

From equation of thermal energy

$$\rho C_p \frac{DT}{Dt} = k\nabla^2 T + \mu\varphi_v + S_c \tag{32.3}$$

Substitute the value of viscous dissipation in equation of thermal energy.

$$0 = k\frac{d^2T}{dx^2} + \mu\left(\frac{dv_z}{dx}\right)^2 \tag{32.4}$$

Now From Eq. (32.1)

$$\frac{dv_z}{dx} = v_{z\,\max}\left(\frac{-2x}{b^2}\right) \tag{32.5}$$

$$0 = k\frac{d^2T}{dx^2} + \mu\frac{4x^2}{B^4}v_{z,\max}^2 \tag{32.6}$$

$$\frac{d^2T}{dx^2} = \frac{-\left(\frac{4\mu}{k}\frac{v_{z,\max}^2}{B^4}\right)x^2}{A} = Ax^2$$

$$\frac{dT}{dx} = \frac{Ax^3}{3} + C_1$$

$$T = \frac{Ax^4}{12} + C_1 x + C_2 \tag{32.7}$$

Boundary conditions are

at $x = -B$, $T = T_0$ (32.8)

at $x = B$, $T = T_0$ (32.9)

$$T_0 = \frac{AB^4}{12} + C_1 B + C_2$$

$$\Rightarrow \mathrm{C}_1 = 0 \tag{32.10}$$

$$T_0 = \frac{AB^4}{12} - C_1 B + C_2$$

$$\Rightarrow \mathrm{C}_2 = \mathrm{T}_0 - \frac{AB^4}{12} \tag{32.11}$$

$$\mathrm{T} = \mathrm{T}_0 + \frac{AB^4}{12}\left(\frac{x^4}{B^4} - 1\right)$$

$$\mathrm{T} - \mathrm{T}_0 = \frac{-AB^4}{12}\left[1 - \left(\frac{x}{B}\right)^4\right]$$

$$\mathrm{T} - \mathrm{T}_0 = \frac{\mu v_{z,\max}^2}{3k}\left[1 - \left(\frac{x}{B}\right)^4\right] \tag{32.12}$$

At $x = 0$

$$\left(\mathrm{T} - \mathrm{T}_0\right)_{\max} = \frac{\mu v_{z,\max}^2}{3k} \tag{32.13}$$

Now by using above equation we can predict the amount of viscous dissipation.

Water is flowing at average flow rate 10 ft/sec in general application. We are substituting velocity 100 ft/sec and put value viscosity of water (1 cP) and conductivity of water in equation.

$$\left(\mathrm{T} - \mathrm{T}_0\right)_{\max} = \frac{\mu v_{z,\max}^2}{3k} = \frac{100^2}{3}\frac{\mu_{25^0 C}}{k_{23^0 C}} = 1^0 F \tag{32.14}$$

Even for such large velocities, temperature rise is only 1^0F for water. Therefore, viscous dissipation can be neglected here.

For highly viscous polymer melt (10,000 *Cp*) at 25^0

$$\left(\mathrm{T} - \mathrm{T}_0\right)_{\max} = 10{,}000^0\mathrm{F} \tag{32.15}$$

Therefore, viscous dissipation is considerable for highly viscous fluid.

Else for air, water and low viscosity this term can be neglected.

Lecture 23 (28th October, Monday)

MASS TRANSPORT

In this section, we will discuss the transport of different species in a multicomponent system. Later, we will take the axiom that mass of every species is conserved to obtain the convective diffusion equation.

Definitions

For multicomponent mass transport, we define the following terms for a diffusing mixture which has *1,2,3,4........N* components.

1. The mass concentration

Mass concentration of a component *"i"* in the mixture is defined as the mass of component *"i"* per unit volume of the mixture.

$$\rho_i = \frac{m_i}{V_T}$$

where, m_i is the mass of component i , V_T is the total volume of the mixture, and ρ_i is the mass concentration. It may be noted that mass concentration of the component i is not the density of the same component but has the same unit as density, i.e., mass per unit volume.

The mass concentration of the mixture ρ, may also be calculated in similar manner and given by

$$\rho = \rho_1 + \rho_2 + \rho_3 + \ldots.. + \rho_N$$

or

$$\rho = \sum_{i=1}^{N} \rho_i$$

which is also happened to be the density of the mixture. The mass fraction of component *"i"* , w_i, may be defined as

$$w_i = \frac{\rho_i}{\rho}$$

2. The molar concentration

Molar concentration of the component *"i"* is defined as the moles of component i per unit volume of the mixture, i.e.,

$$c_i = \frac{M_i}{V_T}$$

where M_i is the moles of component *"i"* , V_T is the total volume of the mixture, and c_i is the molar concentration. Thus, the molar concentration of the mixture, *c*, is given by

$$c = \sum_{i=1}^{N} \frac{M_i}{V_T} = \frac{M_1 + M_2 + M_3 + \dots + M_N}{V_T}$$

or

$$c = c_1 + c_2 + \dots + c_N$$

or

$$c = \sum_{i=1}^{N} c_i$$

The mole fraction of component *i*, x_i, may be defined as

$$x_i = \frac{c_i}{c}$$

We may also relate the molar concentration to mass concentration, as follows

$$c_i = \frac{\rho_i}{M_{wi}}$$

where M_{wi} is the molecular weight of component *"i"*.

3. Average velocities

In a diffusing mixture, the components are moving at different velocities due to the mass transfer of components from higher concentrations to lower concentrations. We may define the average velocities of the mixture in two different ways.

1. The mass average velocity

Mass average velocity is the actual velocity of a fluid which may be measured by experimental methods. The mass average velocity of a mixture is defined as

$$\underset{\sim}{v} = \frac{\rho_1 \underset{\sim}{v}_1 + \rho_2 \underset{\sim}{v}_2 + \ldots + \rho_N \underset{\sim}{v}_N}{\rho_1 + \rho_2 + \ldots + \rho_N}$$

or

$$\underset{\sim}{v} = \sum \frac{\rho_i \underset{\sim}{v}_i}{\rho}$$

where $\underset{\sim}{v}$ is the mass average velocity of the mixture and $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3, \text{------}, \underset{\sim}{v}_N$ are the velocity of components 1,2,3,------N respectively.

2. The molar average velocity

the molar average velocity is based on the molar concentration of species. It is a hypothetical velocity which can not be determined from experiments. It is defined as,

$$\underset{\sim}{v}^* = \frac{c_1 \underset{\sim}{v}_1 + c_2 \underset{\sim}{v}_2 + \ldots + c_N \underset{\sim}{v}_N}{c_1 + c_2 + \ldots + c_N}$$

or

$$\underset{\sim}{v}^* = \frac{\sum c_i \underset{\sim}{v}_i}{c}$$

where $\underset{\sim}{v}^*$ represents the molar average velocity of the mixture

4. The diffusion velocities

The diffusion velocities are the relative velocities of components with respect to mass or molar average velocity of the mixture. These may be calculated as $(\underset{\sim}{v}_i - \underset{\sim}{v})$ or $(\underset{\sim}{v}_i - \underset{\sim}{v}^*)$

5. Fluxes

Fluxes are defined as the mass or molar flow of any species per unit area, per unit time. The fluxes are differentiated as mass and molar fluxes, and based on the relative movement of the observer. Details of various types of fluxes are given in the following table.

Table. 34.1 Type of fluxes

Fluxes	Mass flux	Molar flux
With respective to stationary observer (Absolute/total flux)	$\underset{\sim}{n}_i = \rho_i \underset{\sim}{v}_i$	$\underset{\sim}{N}_i = c_i \underset{\sim}{v}_i$
With respective to observer moving with mass average velocity $\underset{\sim}{v}$. (Relative/ diffusive flux)	$\underset{\sim}{j}_i = \rho_i(\underset{\sim}{v}_i - \underset{\sim}{v})$	$\underset{\sim}{J}_i = c_i(\underset{\sim}{v}_i - \underset{\sim}{v})$
With respective to observer moving with molar average velocity $\underset{\sim}{v}^*$.	$\underset{\sim}{j}_i^* = \rho_i(\underset{\sim}{v}_i - \underset{\sim}{v}^*)$	$\underset{\sim}{J}_i^* = c_i(\underset{\sim}{v}_i - \underset{\sim}{v}^*)$

Fluxes in the above table are related to each other. Such relationships are very important for the solution of the mass transfer problems and are derived below

(1) The absolute mass flux in terms of the convective and diffusive mass fluxes

If $\underset{\sim}{v}_i$ is the velocity of component *i* and $\underset{\sim}{v}$ is the mass average velocity of the mixture. We may write,

$$\underset{\sim}{v}_i = \underset{\sim}{v}_i + \underset{\sim}{v} - \underset{\sim}{v}$$

Multiplying Equation (34.9) with ρ_i, we have

$$\underset{\sim}{n}_i = \rho_i \underset{\sim}{v}_i = \rho_i\left(\underset{\sim}{v}_i + \underset{\sim}{v} - \underset{\sim}{v}\right)$$

or

$$\underset{\sim}{n}_i = \rho_i \left[\left(\underset{\sim}{v}_i - \underset{\sim}{v} \right) + \underset{\sim}{v} \right]$$

or

$$\underset{\sim}{n}_i = \rho_i \underset{\sim}{v} + \rho_i \left(\underset{\sim}{v}_i - \underset{\sim}{v} \right)$$

In Equation (34.10) term $\rho_i \underset{\sim}{v}$ represents the convective mass flux of component *i* due to bulk transport and the term $\rho_i \left(\underset{\sim}{v}_i - \underset{\sim}{v} \right)$ represents the diffusive mass flux of component *i*. *The above equation may also be rewritten as shown below*

$$\underset{\sim}{n}_i = \rho_i \frac{\sum \rho_j \underset{\sim}{v}_j}{\rho} + \underset{\sim}{j}_i$$

Since $\dfrac{\rho_i}{\rho}$ is the mass fraction of component *i*, w_i, we have

$$\underset{\sim}{n}_i = w_i \sum \rho_j \underset{\sim}{v}_j + \underset{\sim}{j}_i$$

or

$$\underset{\sim}{n}_i = w_i \sum \underset{\sim}{n}_j + \underset{\sim}{j}_i$$

(2) The absolute molar flux in terms of the convective and diffusive molar fluxes

If $\underset{\sim}{v}_i^*$ is the molar velocity of component *i* and $\underset{\sim}{v}^*$ is the molar average velocity of the mixture, we may write,

$$\underset{\sim}{v}_i = \underset{\sim}{v}_i + \underset{\sim}{v}^* - \underset{\sim}{v}^*$$

Multiplying Equation (34.13) by c_i, we have

$$\underset{\sim}{N}_i = c_i \underset{\sim}{v}_i = c_i \left(\underset{\sim}{v}_i - \underset{\sim}{v}^* \right) + c_i \underset{\sim}{v}^*$$

In Equation (34.14) The term $c_i \underset{\sim}{v}^*$ represents the convective molar flux of component *i* and the term $c_i\left(\underset{\sim i}{v} - \underset{\sim}{v}^*\right)$ represents the diffusive molar flux of component *i*. Thus,

$$\underset{\sim i}{N} = \underset{\sim i}{J}^* + c_i \frac{\sum c_j \underset{\sim j}{v}}{c}$$

or

$$\underset{\sim i}{N} = \underset{\sim i}{J}^* + x_i \sum_{j=1}^{n} \underset{\sim j}{N}$$

where x_i is the mole fraction of component *i* in the mixture. Equation (34.12) and Equation (34.16) are commonly used for solving many simple mass transfer problems.

Fick's law of diffusion

The Fick's law of diffusion is widely used for describing the diffusive fluxes in terms of gradient of concentration. To understand the Fick's law of diffusion, consider a thin silica porous plate of area *A* and thickness *H* as shown in Fig. 35.1. Initially at $t<0$, the both surfaces of the plate are kept in contact with the air. After at time $t=0$, the lower plate is suddenly brought in the environment of Helium of mass fraction $w_{A,0}$. If the concentration of Helium out-side the upper plate is $w_{A,H}$, such as

$$\Delta w_{A,y} = w_{A,0} - w_{A,H}$$

then Helium will start to diffuse across the plate with mass flux $\Delta w_{He,y}$.

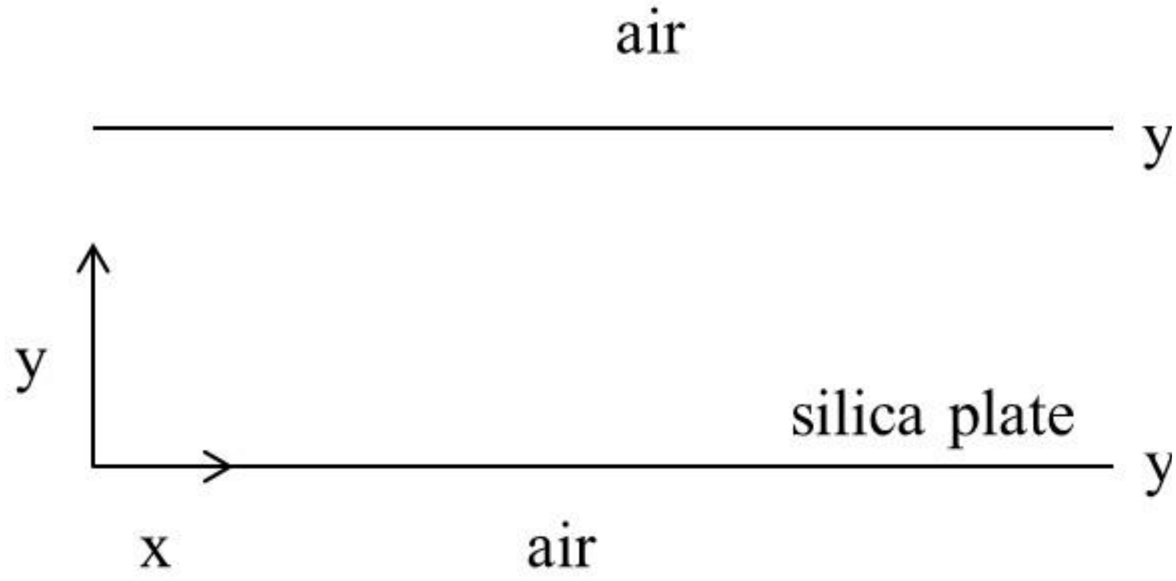


Fig 35.1 Mass transfer in a thin porous silica plate at *t =0*

After the system has reached a steady state, for a particular value of $\Delta w_{He,y}$, mass flux of Helium $j_{He,y}$ is measured. The experiment may be repeated for different values of $\Delta w_{He,y}$ and each time $j_{He,y}$ is measured. If $j_{He,y}$ is plotted against $\Delta w_{He,y}$, we find that we have a straight line passing through a origin, i.e.,

$$j_{He,y} \propto \frac{\Delta w_{He,y}}{\Delta y}$$

or

$$j_{Ay} = -\rho D_{AB} \frac{dw_A}{dy}$$

where ρ_i is the density of mixture and D_{AB} is the diffusion of *A* (Helium) in *B* (silica). The Equation (35.3) is known as the Fick's law of diffusion. It may be written in vector and tensor form that may be applied to any coordinate system, i.e.,

$$\Delta w = w_{A0} - w_{AH}$$

$$\underset{\sim}{j}_A = -\rho D_{AB} \underset{\sim}{\nabla} w_A$$

The detail form of Fick's law in all coordinate systems are given in Appendix - 7

Since diffusive fluxes are the relative fluxes, it implies that

$$\sum \underset{\sim}{j}_i = 0$$

or for a binary system

$$\underset{\sim}{j}_A + \underset{\sim}{j}_B = 0$$

where relative flux for species *B* may be defined as

$$\underset{\sim}{j}_B = -\rho D_{BA} \underset{\sim}{\nabla} w_B$$

Thus, from Equations (35.4), (35.5) and (34.6), we obtain

$$D_{AB} = D_{BA}$$

It may be shown that if Equations (35.4) and (35.6) is the Fick's law of diffusion in terms of mass fluxes then The following expressions are mathematically equivalent in terms of molar fluxes, i.e.,

$$\underset{\sim}{J}_A^* = -cD_{AB} \underset{\sim}{\nabla} x_A$$

and

$$\underset{\sim}{J}_B^* = -cD_{BA} \underset{\sim}{\nabla} x_B$$

where *c* is the total concentration of the mixture and x_A and x_B are the mole fractions of components A and B respectively. If *c* is constant, then we can further modify Equation (35.8) as

$$\underset{\sim}{J}_A^* = -D_{AB} \underset{\sim}{\nabla}(cx_A) = -D_{AB} \underset{\sim}{\nabla} c_A$$

Limitations of Fick’s law

Fick’s law of diffusion as defined above is applicable for a binary system and It may not be valid for a multicomponent system. To understand this, consider a system where a component A has to diffuse in the mixture of components A ,B and C. Thus, Fick's law of diffusion, if applicable, may be written for species A as follows

$$\underset{\sim}{J_A}^* = -cD_{A(BC)}\underset{\sim}{\nabla} x_A$$

or

$$\underset{\sim}{J_A}^* = -cD_{AM}\underset{\sim}{\nabla} x_A$$

where D_{AM} represents the diffusivity of species A in the mixture of B and C. Since, the concentrations of B and C may have different values, the diffusion coefficient D_{AM} may not be a constant. In this case, the diffusivity may be a function of concentrations of A, B and C. However, if the concentrations in the mixture are nearly constant, then D_{AM} may be a constant. In this case, the Fick's law may be applied. For example, if methane is diffusing through air, which is a mixture of Oxygen and Nitrogen, then we may use Fick's law of diffusion, as the composition of air is constant.

Second limitation of Fick's law is that it considers only the concentration gradients as the driving force for mass transport. Since, the real driving force in mass transport is the chemical potential (which may also include pressure differences, temperature and electrical gradients), the form of Fick's law as given in Equation (35.8) may not be valid. In such cases, the Fick's law may be in the extended in terms of gradients of chemical potential but again only for binary systems. To get better solution for a multi-component system, the Stefan Maxwell equations may be preferred. (Please refer to any mass transfer book for Stefan Maxwell equation and extended form of Fick's law of diffusion)

Lecture 24 (4th November Monday)

MASS TRANSPORT

In our last lecture we have seen, for multi component mixture we can define mass concentration (ρ_i) mass fraction (w_i) mole concentration (c_i), molar fraction(x_i) for the species "i". Velocities of species relative to mass average($\boldsymbol{v}_i - \boldsymbol{v}$) and molar average velocities($\boldsymbol{v}_i - \boldsymbol{v}^*$). From those we saw fluxes are defined as the mass or molar flow of any species per unit area, per unit time. The fluxes are differentiated as mass and molar fluxes as below.

Fluxes	Mass flux	Molar flux
With respective to stationary observer (Absolute/total flux)	$\underset{\sim}{n}_i = \rho_i \underset{\sim}{v}_i$	$\underset{\sim}{N}_i = c_i \underset{\sim}{v}_i$
With respective to observer moving with mass average velocity $\underset{\sim}{v}$. (Relative/ diffusive flux)	$\underset{\sim}{j}_i = \rho_i(\underset{\sim}{v}_i - \underset{\sim}{v})$	$\underset{\sim}{J}_i = c_i(\underset{\sim}{v}_{i-} - \underset{\sim}{v})$
With respective to observer moving with molar average velocity $\underset{\sim}{v}^*$.	$\underset{\sim}{j}_i^* = \rho_i(\underset{\sim}{v}_i - \underset{\sim}{v}^*)$	$\underset{\sim}{J}_i^* = c_i(\underset{\sim}{v}_i - \underset{\sim}{v}^*)$

Then we saw Ficks law of diffusion

$$\underset{\sim}{j}_A = -\rho D_{AB} \underset{\sim}{\nabla} w_A$$

$$\underset{\sim}{J}_A^* = -c D_{AB} \underset{\sim}{\nabla} x_A$$

Diffusive fluxes are the relative fluxes, it implies that

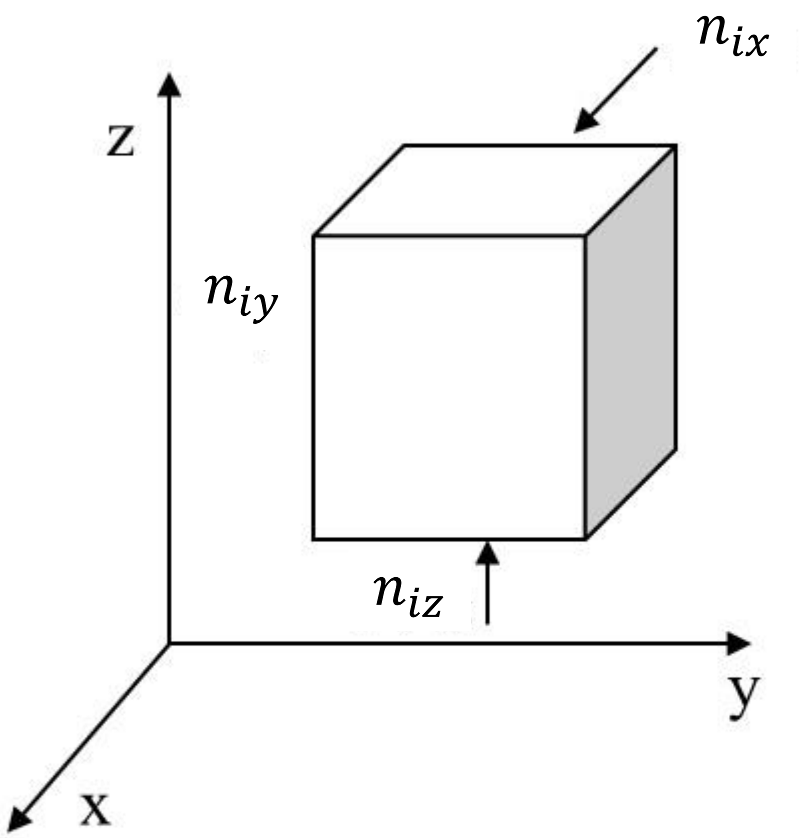
$$\underset{\sim}{j_A} + \underset{\sim}{j_B} = 0$$

$$D_{AB} = D_{BA}$$

Now we will see mass balance for each component

Equation of continuity for a multicomponent mixture

Now, we apply the Axiom 5, "The mass of each species in a mixture is conserved" to obtain the equation of continuity for a multicomponent mixture. The system considered is a volume element ΔxΔyΔz, fixed in space through which the fluid mixture is flowing as shown in Fig. below. The mixture contains the species i where i=1,2,3,4.......N



Following the axiom, we may write mass balance for a species "I" in the control volume as given below

Within this mixture, reactions among the various chemical species may be occurring, and we use the symbol r_i, to indicate the rate at which species a is being produced, with dimensions of mass/volume time

$$\begin{bmatrix} Rate\ of\ accumulation\ of \\ species\ \text{i}in\ the\ C.V \end{bmatrix} = \begin{bmatrix} Net\ rate\ of\ inflow\ of\ mass\ of \\ species \text{ by convection and diffusion} \\ i \end{bmatrix} + \begin{bmatrix} Net\ rate\ of\ deneration\ of\ specices\ \text{"i"} \\ by\ the\ chemical\ reaction\ in\ CV. \end{bmatrix}$$

Within this mixture, reactions among the various chemical species may be occurring, and we use the symbol r_i, to indicate the rate at which species a is being produced, with dimensions of mass/volume time

$$\begin{aligned} \frac{\partial}{\partial t}(\rho_i \Delta x \Delta y \Delta z) = {} & n_{ix}|_x \Delta y \Delta z - n_{ix}|_{x+\Delta x} \Delta y \Delta z \\ & + n_{iy}\big|_y \Delta x \Delta z - n_{iy}\big|_{y+\Delta y} \Delta x \Delta z \\ & + n_{iz}|_z \Delta x \Delta y - n_{iz}|_{z+\Delta z} \Delta x \Delta y + r_i \Delta x \Delta y \Delta z \end{aligned} \qquad (24.1)$$

The combined mass flux $n_{ix}, n_{iy}\ and\ n_{iz}$ includes both the molecular flux (diffusion) and the convective flux. When the entire mass balance is written down and divided by $\Delta x \Delta y \Delta z$ one obtains, after letting the size of the volume element decrease to zero

$$\frac{\partial}{\partial t}(\rho_i) = -\frac{\partial n_{ix}}{\partial x} - \frac{\partial n_{iy}}{\partial y} - \frac{\partial n_{iz}}{\partial z} + r_i, \quad here\ i = 1, 2, 3 \ldots \ldots N\ speicies \qquad (24.2)$$

Eq 24.2 is the equation of continuity for species *"i"* in a multicomponent reacting mixture. It describes the change in mass concentration of species "i" with time at a fixed point in space by the diffusion and convection of species "I", as well as by chemical reactions that produce or consume "i".

The quantities $n_{ix}, n_{iy}\ and\ n_{iz}$ are the Cartesian components of the mass flux vector $\boldsymbol{n}_i = \rho_i \boldsymbol{v}_i$

(Note: In bold notation indicates vector quantity)

Equation 24.2 may be rewritten in vector notation as

$$\frac{\partial}{\partial t}(\rho_i) = -\boldsymbol{\nabla}.\boldsymbol{n}_i + r_i \qquad (24.3)$$

The combined mass flux $\boldsymbol{n_i}$ includes both the molecular flux (diffusion) and the convective flux and can be written as

$$\frac{\partial}{\partial t}(\rho_i) = -\boldsymbol{\nabla}.\rho_i\boldsymbol{v} - \boldsymbol{\nabla}.\boldsymbol{j_i} + r_i \quad for \quad i = 1, 2, 3 \ldots\ldots N\ speicies$$

(24.4)

Addition for all *N* equations

$$\frac{\partial}{\partial t}(\rho_1 + \rho_2 + \rho_3 + \cdots.\rho_N) = -\boldsymbol{\nabla}.(\rho_1\boldsymbol{v} + \rho_2\boldsymbol{v} + \rho_3\boldsymbol{v} + \cdots\rho_N\boldsymbol{v}) - \boldsymbol{\nabla}.(\boldsymbol{j}_1 + \boldsymbol{j}_2 + \boldsymbol{j}_3 + \cdots\boldsymbol{j}_N) + r_1 + r_1 + r_1 + \cdots r_N$$

(24.5)

The fact that the law of conservation of total mass gives $\sum r_i = 0$

Also, we have seen in previous lecture that $\boldsymbol{j}_1 + \boldsymbol{j}_2 + \boldsymbol{j}_3 + \cdots\boldsymbol{j}_N = 0$

So, equation 23.4 equation of continuity reduced to equation 24.6, which is the equation of continuity for the mixture. This equation 24.6 is identical to the equation of continuity for a pure fluid that we show during Mass and Momentum balance (N-S equation).

$$\frac{\partial}{\partial t}(\rho) + \boldsymbol{\nabla}.(\rho\boldsymbol{v}) = 0 \qquad (24.6)$$

for a fluid mixture of constant mass density ρ equation becomes $\boldsymbol{\nabla}.\boldsymbol{v} = 0$.

So-far above we used mass units. However, a corresponding derivation is also possible in molar units. The equation of continuity for species "i" in molar quantities is

$$\frac{\partial}{\partial t}(c_i) = -\boldsymbol{\nabla}.\boldsymbol{N}_i + R_i \qquad for \quad i = 1, 2, 3 \ldots\ldots N\ speicies$$

(24.7)

where *R,* is the molar rate of production of species "i" per unit volume.

$$\frac{\partial}{\partial t}(c_i) = -\nabla . c_i \boldsymbol{v}^* - \nabla . \boldsymbol{J}_i^* + R_i \quad for \quad i = 1, 2, 3 \dots \dots N\ speicies$$

(24.8)

LHS of equation 24.8 is: Rate of increase in moles of species "i" per unit volume.

RHS of equation 24.8: First term is net rate f addition of moles of species "i" per unit volume by convection, second term rate of addition of moles of species "i" per unit volume by diffusion and third term is rate of production of moles of species "i" per unit volume by reaction.

Summing For all N components equation reduce to

$$\frac{\partial}{\partial t}(c) = -\nabla . c\boldsymbol{v}^* + \sum R_i \qquad (24.9)$$

Note that the chemical reaction R_i term does not drop out because the number of moles is not necessarily conserved in a chemical reaction.

For a fluid mixture of constant *molar density* c.

$$\nabla . \boldsymbol{v}^* = \frac{1}{c}\sum R_i \qquad (24.10)$$

We have thus seen that the equation of continuity for species *"i"* may be written in two forms i.e in mass and other in molar. Using the continuity relations and mass conc. Mass fraction, mole conc, molar fraction we may verify that the equation of continuity for species i can be put into two additional, equivalent forms:

$$\rho\frac{\partial}{\partial t}(w_i) + \mathbf{v}\cdot\nabla w_i = -\nabla . \boldsymbol{j}_i + r_i \quad for \quad i = 1, 2, 3 \dots \dots N\ speicies$$

(24.11)

$$c\frac{\partial}{\partial t}(x_i) + \mathbf{v}^*\cdot\nabla x_i = -\nabla . \boldsymbol{J}_i^* + R_i - x_i\sum R_j\ for\ i = 1, 2, 3 \dots \dots N\ speicies$$

(24.12)

These equations 24.11 and 24.12 express exactly the same physical content, but they are written in two different sets of notations-the first in mass quantities and the second in molar quantities. To use these equations, we have to insert the appropriate expressions for the fluxes and the chemical reaction terms.

We will follow these equations for binary system A and B with constant ρD_{AB}

Lecture 25 (7th November, Thursday)

Boundary Conditions in mass transport

To solve simple mass transport problems, we need to solve the equation of continuity for each component of the mixture. The resulting differential equations may be further simplified by using the classical assumptions. When these differential equations are integrated, the constants of integration appear which will have to be determine by using suitable boundary conditions. Depending on the problem, the following boundary conditions may be applicable.

1. The concentration or flux may be specified at a boundary, for example

$$x_A\big|_{z=0} = x_{A0} \quad \text{or} \quad N_{Az}\big|_{z=0} = N_{A0}$$

2. The fluxes are equal at interfaces.

For example, $N_{Az}^{Phase\,I}\Big|_{z=L} = N_{Az}^{Phase\,II}\Big|_{z=L}$.

In addition to this, a second boundary condition is required. It may be noted that at an interface, the concentrations may not be equal as was the case for velocities in the momentum transport and temperatures in the heat transports. The region for the this is that the driving forces in mass transport are the chemical potential . Thus,

$$\mu_{Az}^{Phase\,I}\Big|_{z=L} = \mu_{Az}^{Phase\,II}\Big|_{z=L}$$

It may be used as second boundary condition at the interface.

3. The mass transfer coefficient at a system boundary may be specified. For example,

$$N_{Az}\big|_{z=0} = k_C\left(c_{A0} - c_{Ab}\right)$$

where N_{A0} is the molar flux of species A, C_{A0} is the concentration of species A at the system boundary and C_{Ab} is the known concentration in the surrounding fluid. Here, K_c, is the mass transfer coefficient and must be known a priori. This boundary condition is similar to the convective boundary condition used earlier in heat transfer.

Now, we solve some simple mass transport problems for a binary system.

Diffusion of A through a stagnant gas B

Consider a system shown in Fig. 37.1 where liquid A is evaporating into a non-diffusing gas B. At steady state, the partial pressure of species A at gas-liquid interface is given as P_{A1}

while the partial pressure of species A at the top of vessel is given as P_{A2}. The liquid level in the vessel is maintained at $z=0$. Here, the mole fraction of species A at liquid gas interface may be evaluated by using Raoult's law or

$$y_{A1} = \frac{p_A^{sat}}{P}$$

where p_A^{sat}, is the saturation pressure and P is the total pressure in the system.

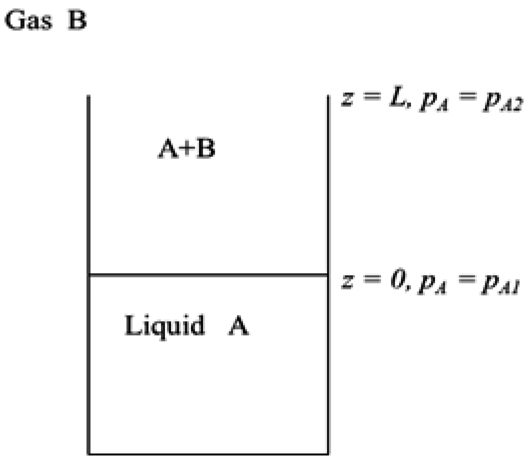


Fig 37.1 Diffusion through a stagnant gas

Assumption

1. Steady state
2. Mass transport in x and y directions are negligible. Thus, $N_{Az,}$ is the only non-zero component of the flux $(N_{Ax} = N_{Ay} = 0)$.
3. Density ρ and diffusivity D_{AB} are constants.
4. No chemical reaction-taking place in the system $(R_A = 0)$.

We start with the equation of continuity for component A , i.e.,

$$\frac{\partial c_A}{\partial t} + \underset{\sim}{\nabla} . \underset{\sim}{N}_A = R_A$$

By applying assumptions (1) to (4), Equation (37.2) may be simplified as,

$$\frac{dN_{Az}}{dz} = 0$$

After integration, we get

$$N_{AZ} = C_1$$

where c_1 is an integral constant.

The N_{Az}, may be calculated as

$$N_{Az} = y_A(N_{Az} + N_{Bz}) + J^*_{Az}$$

where y_A is the mole fraction of A in the gas column in the vessel. Because B is stagnant, the flux of species B , N_{Bz} , is zero. Thus, Equation (37.5) may be simplified as

$$N_{Az} = y_A(N_{Az}) + J^*_{Az}$$

or

$$N_{Az} = \frac{J^*_{Az}}{(1 - y_A)}$$

By substituting the Fick's law of diffusion in Equation (37.6), we get

$$N_{AZ} = \frac{-cD_{AB}\dfrac{dy_A}{dz}}{(1 - y_A)} = C_1$$

where I is assumed the total concentration of the mixture, c, is constant.

Thus, by integrating Equation (37.7), we obtain

$$ln(1 - y_A) = \frac{C_1 z}{cD_{AB}} + C_2$$

where c_2 is another integration constant.
Equation (37.8) may be written in the terms of partial pressures as

$$ln(1 - y_A) = ln\frac{P(1 - y_A)}{P} = ln\left(\frac{P - p_A}{P}\right) = \frac{C_1 z}{cD_{AB}} + C_2$$

The Boundary conditions are

at $z=0$,
$p_A = p_{A1}$

and at $z=L$,
$p_A = p_{A2}$

By using above boundary condition from Equation (37.9), we obtain

$$ln\left(\frac{P-p_{A1}}{P}\right)=C_2$$

and

$$ln\left(\frac{P-p_{A2}}{P}\right)=\frac{C_1 z}{cD_{AB}}+ln\left(\frac{P-p_{A1}}{P}\right)$$

or

$$C_1=\frac{cD_{AB}}{L}ln\left(\frac{P-p_{A2}}{P-p_{A1}}\right)$$

Finally, we get the concentration profile for component *A* as

$$ln(1-y_A)=\frac{z}{L}ln\left(\frac{P-p_{A2}}{P-p_{A1}}\right)+ln\left(\frac{P-p_{A1}}{P}\right)$$

Tapered tube problem

Now, we consider a slightly different problem where the vessel wall in above problem are tapered as shown in Fig. 37.2. Therefore, N_{Ax} and N_{Ay} may not be equal to zero as was assumed earlier. Here, we use the concept of averaged concentration across the cross section of the vessel and assume that it is a function of *z* only. Note, the cross sectional area of the vessel is changing. Similar types of arguments were used earlier to simply the problem of heat transfer in a rectangular fin. Once again, we start with the shell mass balance instead of the equation of continuity to solve the problem.

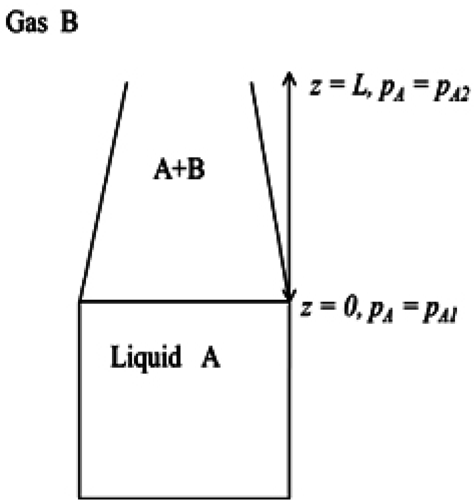


Fig 37.2 Diffusion through a taper tube

We write shell balance on component *A* over a control volume of thickness Δz , i.e.,

$$(SN_{AZ})\big|_z - (SN_{AZ})\big|_{z+\Delta z} = 0$$

where *s* is the cross sectional area of the vessel.

By taking the limit Δz going to zero, we have

$$\frac{d(SN_{AZ})}{dz} = 0$$

and after integration, we obtain

$$SN_{AZ} = C_1$$

or

$$N_{AZ} = \frac{C_1}{S}$$

Here, N_{Az} is the average mass flux of component *A* over the cross section of the tube. The cross section area *s* may be calculated as a function of *z*. After substituting this function in Equation (37.15), remaining part may be solved in a similar way as before.

Leaching of a component from a spherical particle

A spherical particle contains a compound *A* that is to be leached out by a solvent *B*. The concentration of *A* in the bulk stream is $c_{A\delta}$ at surface of particle is c_{AS} which is the saturation concentration of A in B. Assume the solvent B is flowing over the spherical particle under turbulent flow conditions. As before, we may assume a stagnant film of thickness δ around the spherical particle through which A is leached out into the bulk stream. Furthermore, if we neglect the curvature effects this problem reduces to a problem as shown in Fig 41.1. Determine the rate of leaching.

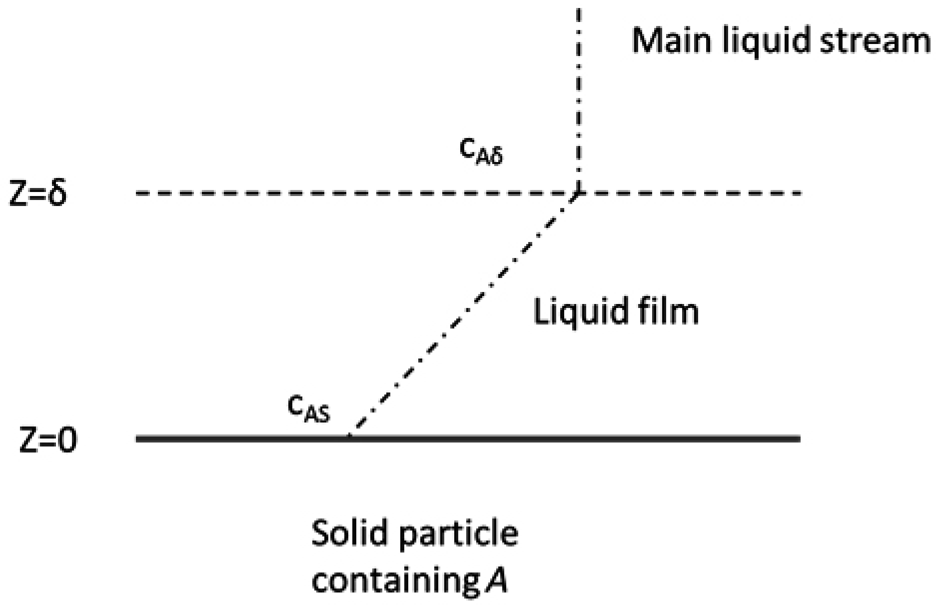


Fig 41.1 Mass transfer through a stagnant liquid film

Assumption

- steady state.
- Mass transfer is only by diffusion.
- B is stagnant.
- Diffusivity of *A* in *B*, D_{AB} is a constant.
- Mass transport in *x* and *y* directions are negligible
- No reaction taking place.

Using assumption (1) to (6), the equation of continuity for this system may be simplified as

$$\frac{d}{dz}\left(N_{Az}\right)=0$$

But here, total flux of *A* may be written as

$$N_{Az}=\left(N_{Az}+N_{Bz}\right)x_A+J_{Az}^{*}$$

Since N_{Bz} is zero and the transport of component *A* takes place only by diffusion, we may neglect the convective term in Equation (41.2), or

$$N_{Az} = J^*_{Az}$$

Substituting the Fick's law of diffusion in Equation (41.3), we get

$$N_{Az} = -D_{AB}\frac{dc_A}{dz}$$

From Equations (41.1) and (41.4), we obtain

$$\frac{d^2 c_A}{dz^2} = 0$$

and after integration, we have

$$c_A = C_1 z + C_2$$

The integration constants C_1 and C_2 may be determined by using the following boundary conditions.

at

$$z = 0, c_A = c_{AS}$$

and

at

$$z = \delta, c_A = c_{A\delta}$$

Using above boundary conditions, we obtain

$$C_1 = \frac{(c_{A\delta} - c_{AS})}{\delta}$$

and

$$C_2 = c_{AS}$$

Finally, the concentration profile along the z direction is given by

$$c_A = \frac{(c_{A\delta} - c_{AS})}{\delta} z + c_{AS}$$

The leaching rate may be calculated as

$$N_A = N_A\big|_{z=0} = \frac{(c_{A\delta} - c_{AS})}{\delta}$$

A slightly better solution of above problem may be found by not neglecting the convective term in Equation (41.2), i.e.,

$$N_{Az} = x_A N_{Az} + J^*_{Az}$$

or

$$N_{Az} = \frac{J^*_{Az}}{(1 - x_A)}$$

After applying the Fick’s law of diffusion, we get the expression

$$-\frac{cD_{AB}}{(1 - x_A)} \frac{dx_A}{dz} = C'_1$$

and after integration of Equation (41.14), we find

$$ln(1 - x_A) = \frac{C'_1 z}{cD_{AB}} + C'_2$$

or

$$ln\left(\frac{c - c_A}{c}\right) = \frac{C'_1 z}{cD_{AB}} + C'_2$$

Equation (41.15) is subjected to the boundary conditions given in Equations (41.7) and (41.8) which finally gives the leaching rate as

$$N_{Az} = C_1 = \frac{cD_{AB}}{\delta} ln\left(\frac{c - c_{A\delta}}{c - c_{AS}}\right)$$

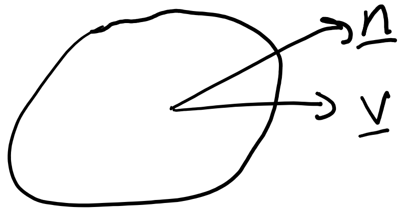
and the concentration profile as

$$ln\left(\frac{c-c_A}{c}\right)=\frac{z}{\delta}ln\left(\frac{c-c_{AS}}{c-c_{AS'}}\right)+ln\left(\frac{c-c_{AS'}}{c}\right)$$

→ Phenomena of Diffusion and Convection
→ / Molecular origin of two transport processes

Material
Energy
Momentum } Transport due to Bulk fluid velocity (i.e) Convection

Flux through an area ds



* material (c) transport $= (\underline{v} \cdot \underline{n})\, c$
* Energy (e) transport $= (\underline{v} \cdot \underline{n})\, e$
* Momentum transport $= (\underline{v} \cdot \underline{n})\, (\rho \underline{v})$

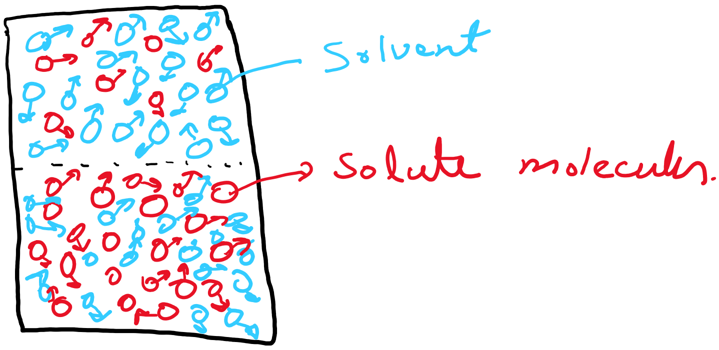
Material
Energy
momentum } Transport due to fluctuation of molecules i.e Molecular fluctuating velocities
So called Diffusion

Diffusion

Consider a container having concentration gradients.

Random motion of molecules

No net mean velocity $\Sigma = 0$



Solute molecules are at higher concentration in the lower region so there will be movement of solute from lower to upper region till it equilibrate

What is driving force for the motion?

Concentration difference?

In principle it is due to thermodynamic behind the system.

Entropy prefer all molecules distribute equally in all the region of space

(i.e conc. difference leads to net movement of molecules]

if 'c' is the fluctuating mean velocity of a molecules then $\frac{1}{2}mc^2 = \frac{3}{2}kT$

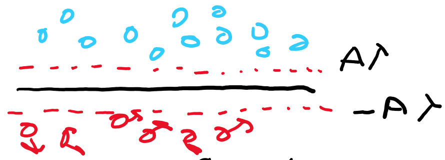
Molecules arrangment are different in liquid and gaseous so they behave differently.

If we think about Diffusion coefficient.

how molecules are crossing.

If we [illegible]

Consider how molecules are crossing.



These molecules actually are coming from distance away from surface

this distance will be order of their mean free path 'λ'

No net change of molecules

Molecules going from below to above

Same number of molecules coming from above to below

(No mean velocity)

$$\frac{\text{Number of Red molecules coming downward}}{\text{Area} \times \text{time}} = B c \, V_{rms}\Big|_{z=A\lambda}$$

$$\frac{\text{Number of Red molecules going upward}}{\text{Area} \times \text{time}} = B c \, V_{rms}\Big|_{z=-A\lambda}$$

constant

Net flux J $\quad B c \, V_{rms}\Big|_{z=-A\lambda} - B c \, V_{rms}\Big|_{z=A\lambda}$

from continuum approximation

(Taylor Series)

from continuum approximation

(Taylor Series)

$$C\Big|_{z=-A\lambda} = C\Big|_{z=0} - A\lambda \frac{dc}{dz}\Big|_{z=0} + \frac{(A\lambda)^2}{2!}\frac{d^2c}{dz^2}\Big|_{z=0} + \cdots$$

$$C\Big|_{z=A\lambda} = C\Big|_{z=0} + A\lambda \frac{dc}{dz}\Big|_{z=0} + \frac{(A\lambda)^2}{2!}\frac{d^2c}{dz^2} + \cdots$$

Net flux

$$= B V_{rms}\left[C\Big|_{z=z-A\lambda} - C\Big|_{z=z+A\lambda}\right]$$

$$J = -2AB\lambda V_{rms}\frac{dc}{dz}$$

Comparing with Fick's Law of Diffusion

$$D = 2AB\lambda V_{rms}$$ (Diffusion Co-efficient)

typically $A = 2/3$, $B = 1/4$ [for monoatomic gas]

$$D = \frac{1}{3}\lambda V_{rms}$$

and we know

$$\frac{1}{2} m V_{rms}^2 = \frac{3}{2} kT$$

$$V_{rms} = \sqrt{\frac{3kT}{m}}$$

$\lambda \rightarrow$ Mean free path for a molecules
Distance between two intermolecular collisions.

Lecture 26 (11^{th} November, Monday)

Some applications problems to understand

The effectiveness factor for a spherical catalyst particle

Fixed bed reactors are very common in chemical industries. These reactors are packed with catalyst particles of irregular shape and the reactants enter the reactor from one end while the products are withdrawn from other end. Here, we consider a spherical porous catalytic particle of radius R located at any point in this reactor. This particle is submerged in a gas stream containing the reactant A and the product B. A first order reaction $A \rightarrow B$ takes place inside the porous catalyst. There are two major mass transfer steps occurring around this particle as shown in Fig. 39.1. In the first step, the mass transfer of A occurs from bulk phase to the surface of the catalyst particle and in the second step, mass transfer occurs inside the catalyst particle where the chemical reaction takes place. In reality, this problem may be difficult to solve as both steps are coupled with each other. Here, we show how to handle both steps separately.

Diffusion and chemical reaction inside a porous spherical catalyst particle

Here, we solve the second part of the problem, where both components (A and B) are diffusing inside the pours of the particle and the first order chemical reaction ($R_A = - k_{1cA}$) occurs at inside surface of the pours particle. We assume that the concentration of the reactant A is known at the catalyst surface. This assumption actually decouples the problem from the first step, where the components are transferred from bulk phase to the catalyst surface.

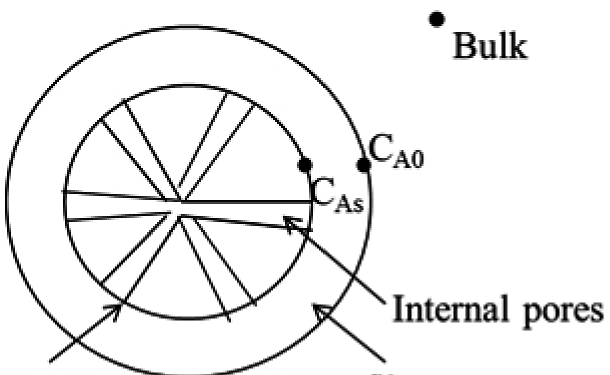


Fig 39.1 Diffusion and chemical reaction inside a porous catalyst

Assumption

- Pores are uniformly distributed and both components diffuse only in the radial direction. Thus fluxes in θ and Φ directions(N_θ and N_Φ)are zero.
- System is in steady state.
- Total concentration of both components, c and diffusivity of A in B, D_{AB}, are constants.

The first order chemical reaction takes place inside the catalyst with reaction rate, R_A, given by

$$R_A = -k_1 c_A$$

where k_1 is the reaction rate constant which depends on surface area, as shown below

$$k_1 = k_1'' a$$

or

$$R_A = -k_1'' a c_A$$

Here, *"a"* is the inside surface area per unit volume of the catalyst. It may be noted that as reactant *A* diffuses inside the catalyst, the concentration of reactant *A* decreases in the radial direction. Thus,

$$c_A = c_A(r)$$

The equation of continuity for component *A* may be written as follows

$$\frac{dc_A}{dt} + \nabla . N_A = R_A$$

By using the assumptions (1) to (3), we obtain the following simplified form of the equation of continuity in the spherical coordinate system.

$$\frac{1}{r^2}\frac{d}{dr}(r^2 N_{Ar}) = -k_1'' a c_A$$

where, N_{Ar} may be obtained as

$$N_{Ar} = (N_{Ar} + N_{Br})\, x_A + J_{Ar}^*$$

At steady state, we may further assume the equimolar counter diffusion. Thus,

$$N_{Ar} = -N_{Br}$$

and with this the Equation may be simplified as,

$$N_{Ar} = J^*_{Ar}$$

Substituting J^*_{Ar} from Fick's law, we obtain

$$N_{Ar} = -D_{AB}\frac{dc_A}{dr}$$

Equation may now be written as

$$\frac{1}{r^2}\frac{d}{dr}\left(-r^2 D_{AB}\frac{dc_A}{dr}\right) = -k_1''ac_A$$

or

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dc_A}{dr}\right) = \frac{k_1''a}{D_{AB}}c_A$$

or

$$\frac{d}{dr}\left(r^2\frac{dc_A}{dr}\right) = r^2\alpha^2 c_A$$

where

$$\alpha = \sqrt{\frac{k_1''a}{D_{AB}}}$$

Expending the Equation, we obtain

$$\frac{r^2 d^2 c_A}{dr^2} + 2r\frac{dc_A}{dr} - \alpha^2 r^2 c_A = 0$$

The Equation (39.12) is a second order differential equation. To solve it, we use the following transformation

$$c_A(r) = \frac{f(r)}{r}$$

Thus,

$$\frac{dc_A}{dr} = \frac{f'}{r} - \frac{f}{r^2}$$

and

$$\frac{d^2c_A}{dr^2} = \frac{f''}{r} - \frac{f'}{r^2} + \frac{f}{2r^3} - \frac{f'}{r^2} = \frac{f''}{r} - \frac{2f'}{r^2} + \frac{2f'}{r^3}$$

Substituting these in the Equation, we finally obtain

$$rf'' - 2f' + \frac{2f}{r} + 2f' - \frac{2f}{r} - \alpha^2 rf = 0$$

or

$$f'' - \alpha^2 f = 0$$

which has the following solution

$$f = C_1 e^{-\alpha r} + C_2 e^{\alpha r}$$

$$c_A = \frac{C_1 e^{-\alpha r} + C_2 e^{\alpha r}}{r}$$

The first boundary condition is at $r = 0, c_A$ is finite. This leads to the solution that

$$C_1 = -C_2$$

or

$$c_A = \frac{2C_2\left(e^{\alpha r} - e^{-\alpha r}\right)}{2r}$$

or

$$c_A = \frac{C'_2 \sinh(\alpha r)}{r}$$

where, $C'_2 = 2C_2$

The second boundary condition is given here that

at $r = R, c_A = c_{As}$

Thus,

$$C'_2 = \frac{R c_{AS}}{\sinh(\alpha R)}$$

we finally obtain the concentration profile of *"A*" inside the catalyst particle as

$$c_A = \frac{R c_{AS}}{r} \frac{\sinh(\alpha r)}{\sinh(\alpha R)}$$

The rate of conversion of *A* into *B* may now be calculated as,

w_{As} =Mass flux x Surface area of catalyst

or

$$w_{AS} = N_{AS} \times 4\pi R^2$$

or

$$w_{AS} = -D_{AB} \left.\frac{dc_A}{dr}\right|_{r=R} \times 4\pi R^2$$

When Equation is used in above expression, we finally obtain

$$w_{AS} = 4\pi R D_{AB} c_{AS} \left(1 - \alpha R \coth(\alpha R)\right)$$

To understand the effect of internal mass transfer resistance, we consider the ideal conditions where this resistance is absent and all available area inside the catalyst particle is exposed to the surface concentration c_{AS}. Thus, the molar rate of conversion may now be calculated as

$$w_{A0} = \left(\frac{4}{3} \pi R^3 \right) k_1'' a c_{AS}$$

We may define the effectiveness factor η for the catalyst particle as given below

$$\eta = \frac{w_{AS}}{w_{A0}} = \frac{3}{k^2} \left[k \coth k - 1 \right]$$

where,

$$k = \alpha R = R \sqrt{\frac{k_1'' a}{D_{AB}}}$$

For irregular shaped catalyst particles, the above results may be applied by reinterpreting the value of average radius R'. For a non-spherical particle, the radius R may be redefined as

$$R' = 3 \frac{V_P}{S_P}$$

where, V_P and S_P are the average volume and average external surface area of the catalyst particles respectively. Thus, the effectiveness factor may now be calculated as:

$$\eta = \frac{w_{AS}}{w_{A0}} = \frac{1}{3\Lambda^2} \left[3\Lambda \coth 3\Lambda - 1 \right]$$

where,

$$\Lambda = \sqrt{\frac{a k_1''}{D_{AB}}} \frac{V_P}{S_P}$$

It is clear that if effectiveness factor η is plotted as the function of Λ, as shown in Fig. 39.2.

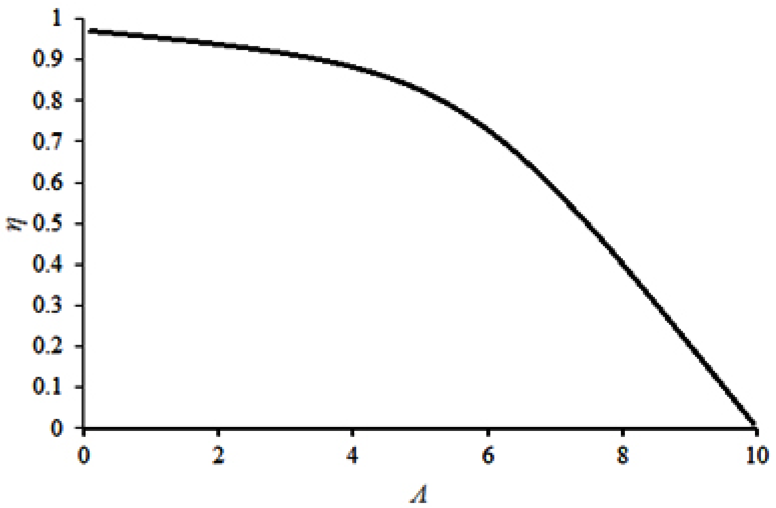


Fig. 39.2 Effectiveness factor η vs plot.

However, if Λ is too small which is possible for smaller size catalyst particles, the pressure drop across the reactor usually increases exponentially, which may not be desirable. Thus, the optimum size catalyst particles should be used which have the sufficiently high effectiveness factors but also have smaller pressure drops across the reactor.

Part -2 Solution of external mass transfer using film theory model

Here, we take the external mass transfer problem, disused earlier in the previous lecture. The problem is the coupling of the internal and external mass transfer problems due to the common boundary conditions at the surface of the catalyst particle. Here, we take a simple external mass transfer problem where it is decoupled from internal mass transfer by assuming instantaneous reaction at the surface of the non porous catalytic particle. Assume an instantaneous reaction $2A \rightarrow A_2$ is taking place on the catalyst surface. Further, we assume that a stagnant fluid film surrounds each catalyst particle. Component A must diffuse through this film to reach the catalyst surface, where the above reaction occurs. It may be noted that the assumption of a stagnant film is a reasonably accurate assumption for turbulent flows, where the concentrations are uniform outside the film due to good mixing provided by the turbulence. The film thickness (δ) decreases as the flow becomes more and more turbulent. For $\delta/R<<1$, we may further neglect the curvature effects and assume a flat surface, where the condition are as shown in Fig. (40.1).

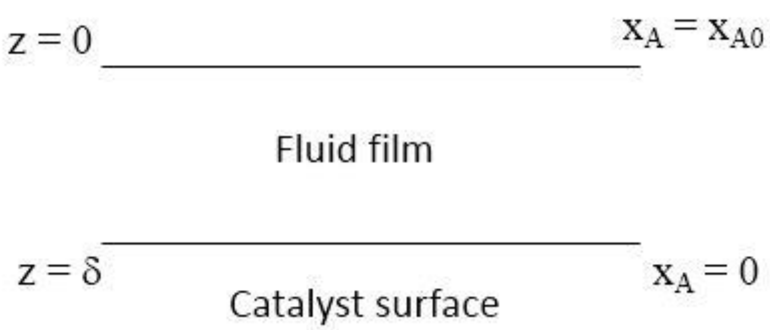


Fig 40.1 External mass transfer through fluid film up to catalyst surface

Assumption

- Both components diffuse only in the z direction. Thus, component of fluxes N_x and N_y are zero.
- System is at steady state.
- Diffusivity of A in A_2, D_{AA2}, is constant.

The equation of continuity for component A may be written as follows

$$\frac{dc_A}{dt} + \nabla . N_A = R_A$$

Applying assumptions (1) to (3), we obtain the following simplified form of equation of continuity

$$\frac{dN_{Az}}{dz} = 0$$

which may be further simplified as

$$N_{Az} = C_1$$

where C_1 is an integration constant. Further, N_{Az} may be obtained as

$$N_{Az} = \left(N_{Az} + N_{A_2z} \right) x_A + J^*_{Az}$$

and after substituting the Fick's law of diffusion, we get

$$N_{Az} = \left(N_{Az} + N_{A_2z} \right) x_A - c\, D_{AA_2} \frac{dx_A}{dz}$$

At steady state, we have

$$N_{A_2z} = -N_{Az} / 2$$

From above equations we can obtain

$$x_A \frac{N_{Az}}{2} - c\,D_{AA_2} \frac{dx_A}{dz} = C_1$$

or

$$N_{Az} = \frac{-cD_{AA_2}}{1 - \frac{x_A}{2}} \left(\frac{dx_A}{dz} \right)$$

Equations may be combined to give

$$-cD_{AA_2} \frac{dx_A}{dz} = \left(1 - \frac{x_A}{2} \right) C_1$$

Integrating above equation, we finally obtain

$$2\,ln \left(1 - \frac{x_A}{2} \right) = \frac{C_1 Z}{cD_{AA_2}} + C_2$$

where C_2 is another constant of integration. Since the reaction is instantaneous, we have the following two boundary conditions,

at

$$z = 0,\ x_A = x_{A0}$$

and at

$$z = \delta, x_A = 0$$

By applying the above boundary conditions, we may determine the constants of integration C_1 and C_2. Thus, the mass transfer rate is given by

$$N_{Az} = C_1 = \frac{2cD_{AA_2}}{\delta} ln \left(\frac{1}{1 - \frac{x_{A0}}{2}} \right)$$

Slow chemical reaction:

The previous problem may also be solved when the reaction $2A \rightarrow A_2$ is a slow reaction at the outer surface of the catalyst particle. Let the rate of disappearance of component A be proportional to the surface concentration of component A. In this case, the second boundary condition given above may be modified as shown below.

at

$$z = \delta, N_{Az} = k_1 c_A$$

Which leads to the solution

$$N_{AZ} = \frac{2cD_{AA_2}}{\delta} ln\left[\frac{\left(1 - \frac{N_{Az}}{2k_1 c}\right)}{\left(1 - \frac{x_{A0}}{2}\right)}\right]$$

The expression given in Equation is a transcendental equation for N_{Az} and may require numerical techniques for determining N_{Az} for the given values of x_{A0}, c, D_{AA2} and δ.

---------------End--------------------