Notes for Paul J. Atzberger "The Monte-Carlo Method"

Malte Brinch
University of Copenhagen
Septemper 2019

1. Article

It is possible to express the solution to many problems in math as the integration of a function.

$$I = \int_{\Omega} f(x)dx$$

in which Ω is the domain of integration, the integral I can be related to an expectation of a random variable with respect to some probability measure. it is possible to rewrite the integral in terms of the expectation value for probability measures of a random variable X that have a density ρ(x).

$$I = E(g(X))$$

Where $g(x) = \frac{f(x)}{\rho(x)}$. We can then use the law of large number, which states that for a collection of independent identically distributed random variables, to write

$$E(g(X)) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} g(X_i)$$

Which means we can approximate I with $I \approx \frac{1}{N}\sum_{i=1}^{N} g(X_i)$, this is the Monte-Carlo method.

the accuracy of the method is given by

$$error = |\frac{\sigma_g}{\sqrt{N}}\eta(0,1)|$$

Where

$$\sigma_g^2 = \int_{\Omega} (g(x) - I)^2 \rho(x)dx$$

and η(0,1) denotes a standard normal random variable (Gaussian random variable) with mean zero and variance 1. the convergence rate in The Monte-Carlo method is strongly influenced by the prefactor σ_g which depends on the function f(x) and the sampling distribution with density ρ(x) that is used. The prefactor is the main wy the accuracy can be improved.

For the Monte-Carlo method to work we need a way to generate random number or at least pseudo-random numbers there are a couple of ways to generate this:

- Linear Congruential Generator, which attempts to create a sequence of numbers in a range by recurrence.
- Lagged Fibonocci Generators, which also uses recurrence but a different form
- Transformation method, where samples for non-uniform random variables can be obtained from the samples generated for the uniform random variable, since transforming a random variables changes the underlying probability distribution.
- Exponentially Distributed Random Variables, with a probability density $\rho(x) = \lambda e^{-\lambda x}$
- Normally Distributed Random Variables: Box-Muller, with a probability density $\rho(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$
- Sampling by Rejection, where the desired random variates can be obtained by generating candidate samples which are either accepted or rejected to obtain the desired distribution.

Variance reduction is an important parts of the Monte-Carlo method and a way to reduce it is to use a probability density ρ(x) for which generation of variates X_i is not too difficult while making σ_g^2 small. one method is Importance sampling, which is concerned with the choosing ρ(x) for the random variates X_i so that regions which contribute significantly to the expectation of g(X) are sampled with greater frequency. for a given value f(x)>0 it is possible to find a $\rho(x)$ which makes $\sigma_g = 0$. This is possible is we chose $\rho(x) = \frac{f(x)}{I}$ since then $g(x) = I$ but then we would already know what I is. We can instead try to choose a $\rho(x)$ which comes as close to this as possible. A way to this is to use sampling with a Gaussian mixture where $\rho(x)$ is a multi-modal probability distributions. The method require generating the extra uniform random variate but it also decreases σ_g more than a standard Gaussian distribution.