Fast Convergence of Regularized Learning in Games

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The paper

Fast Convergence of Regularized Learning in Games

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Online learning

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General model

You try to map inputs $x \in \mathcal{X}$ to outputs $y \in \mathcal{Y}$.

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Your goal is minimal accumulated loss.

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Prediction with Expert Advice

Consider d experts giving you advices.

Choose the best advice.

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Regret

How do you know how good you are?

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$r(T) :=$ (your cumulated loss) - (the leader's cumulated loss)

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$r(T)$:= (your cumulated loss) - (the leader's cumulated loss)

No-regret algorithm

A no-regret algorithm always achieves regret that is sublinear in T.

Follow the Leader

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Always trust the currently leading expert with his advice for the next round.

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Worst case regret is not sublinear

Example:
Binary classification: $y \in \{A, B\}$
Two experts: one always predicts A, the other one always B
Your loss is 0 if you predict right or 1 if you predict wrong.

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Example:
Binary classification: $y \in \{A, B\}$
Two experts: one always predicts A, the other one always B
Your loss is 0 if you predict right or 1 if you predict wrong.

In the worst case your prediction is always false.
Your regret is at least $T/2$.

Deterministic or not?

An adversaries perspective

Finite amount of experts and deterministic behaviour allow easy construction of worst case scenario.
Always make the algorithm's prediction false.

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Idea: randomness

Instead of picking one expert just give the probabilities of choosing the experts.
The adversary is not allowed to know the draw.
We then try to minimize accumulated expected loss.

Stability is also important

Follow the Leader – regret bound by cheating

Let $f_1, \ldots, f_T$ be the sequence of loss functions and $w_1, \ldots, w_T$ be the probabilities determined by *Follow the Leader*, and w^* the leading probabilities.

$$r(T) = \sum_{t=1}^{T} (f_t(w_t) - f_t(w^*)) \leq \sum_{t=1}^{T} \left(\underbrace{f_t(w_t) - f_t(w_{t+1})} \right)$$

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Follow the Regularized Leader

$$w_T = \operatorname*{argmin}_{w \in \Delta} \left(\sum_{t=1}^{T-1} f_t(w) \right) + \frac{1}{\eta} \mathcal{R}(w)$$

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Follow the Regularized Leader with entropic regularizer

$$w_T = \underset{w \in \Delta}{\operatorname{argmin}} \left(\sum_{t=1}^{T-1} f_t(w) \right) + \frac{1}{\eta} \sum_{i=1}^{d} w_i log(w_i)$$

Online learning summary

- Expert Advice Framework

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Follow the Regularized Leader

Follow the Regularized Leader is a no-regret algorithm with $r(T) \in O(\sqrt{T})$.

Games

Matching pennies

$A \diagdown B$	Heads	Tails
Heads	$1 \diagdown -1$	$-1 \diagdown 1$
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Each player wants to maximize his accumulated utility.

Game

For a game G of n players:

Each player i has

- a finite strategy space S_i and a
- utility function $u_i : S_1 \times \ldots \times S_n \to [0, 1]$.

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Then the player receives $\mathbf{u}_i^t$, the expected utility for each of his strategies x: $\mathbf{u}_i^t = (u_{i,x}^t)_{x \in S_i}$ with $u_{i,x}^t = \mathbb{E}_{s_{-i} \sim w_{-i}^t} [u_i(x, s_{-i})]$

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The expected utility for a player i in iteration t is therefore $\langle w_i^t, u_i^t \rangle$.

Playing under nice conditions

Nice opponents

Assume all players to use no-regret algorithms.

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Can this be generalized?

RVU property

RVU – Regret bounded by Variation in Utilities

A vanishing regret algorithm has the RVU property with parameters $\alpha > 0$ and $0 < \beta \leq \gamma$ if for any sequence of utilities $\mathbf{u}^1, \mathbf{u}^2, \ldots, \mathbf{u}^T$ the regret is bounded as

$$r(T) \leq \alpha + \beta \sum_{t=1}^{T} \max_{x \in S_i} \left| u_{i,x}^t - u_{i,x}^{t-1} \right|_1^2 - \gamma \sum_{t=1}^{T} \left\| \mathbf{w}^t - \mathbf{w}^{t-1} \right\|_1^2$$

All players' average regret

Fast convergence of all players' average regret

Suppose that the algorithm of each player i satisfies the RVU property with parameters α, β and γ such that $\beta \leq \gamma/(n-1)^2$. Then $\sum_{i \in N} r_i(T) \leq \alpha n$ and therefore all players' average regret converges at a rate of $O(1/T)$.

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$$= \alpha n + \sum_{t=1}^{T} \left(\beta \sum_{i \in N} \max_{x \in S_i} \left| u_{i,x}^t - u_{i,x}^{t-1} \right|^2 - \gamma \sum_{i \in N} \left\| \mathbf{w}^t - \mathbf{w}^{t-1} \right\|_1^2 \right)$$

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Proof - intermediate step

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$$= \max_{x \in S_i} \left| \sum_{\tilde{s} \in \mathbf{s}_{-i}} u_i(x, \tilde{s}) \left(\mathrm{Prob}^{t}(\tilde{s}) - \mathrm{Prob}^{t-1}(\tilde{s}) \right) \right|$$

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$$\leq \sum_{\tilde{s} \in \mathbf{s}_{-i}} \left| \prod_{j \neq i} w_{j,\tilde{s}_j}^t - \prod_{j \neq i} w_{j,\tilde{s}_j}^{t-1} \right|$$

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\begin{aligned}
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&=\max_{x\in S_i}\left|\sum_{\tilde{s}\in\mathbf{s}_{-i}}u_i(x,\tilde{s})\left(\mathrm{Prob}^{t}(\tilde{s})-\mathrm{Prob}^{t-1}(\tilde{s})\right)\right| \\
&\leq\max_{x\in S_i}\sum_{\tilde{s}\in\mathbf{s}_{-i}}u_i(x,\tilde{s})\left|(\mathrm{Prob}^{t}(\tilde{s})-\mathrm{Prob}^{t-1}(\tilde{s}))\right| \\
&\leq\sum_{\tilde{s}\in\mathbf{s}_{-i}}\left|\prod_{j\neq i}w_{j,\tilde{s}_j}^{t}-\prod_{j\neq i}w_{j,\tilde{s}_j}^{t-1}\right|\leq\sum_{j\neq i}\|\mathbf{w}_j^{t}-\mathbf{w}_j^{t-1}\| \\
&\Rightarrow\sum_{i\in N}\max_{x\in S_i}\left|u_{i,x}^{t}-u_{i,x}^{t-1}\right|^2\leq(n-1)^2\sum_{i\in N}\left\|\mathbf{w}_i^{t}-\mathbf{w}_i^{t-1}\right\|_1^2
\end{aligned}
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Proof - continued

$$\sum_{i \in N} r_i(T) \leq \alpha n + \sum_{t=1}^{T} \left(\beta \sum_{i \in N} \max_{x \in S_i} \left| u_{i,x}^{t} - u_{i,x}^{t-1} \right|^2 - \gamma \sum_{i \in N} \left\| \mathbf{w}^{t} - \mathbf{w}^{t-1} \right\|_1^2 \right)$$

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={}&\alpha n + \sum_{t=1}^{T}\left(\underbrace{\left(\beta(n-1)^2 - \gamma\right)}_{\leq 0}\sum_{i\in N}\left\|\mathbf{w}^{t} - \mathbf{w}^{t-1}\right\|_1^2\right)\\
\leq{}&\alpha n
\end{aligned}
$$

Optimistic Follow the Regularized Leader

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Let $\mathcal{R}$ be a suitable regularizer and $\mathbf{M}_i^T$ be an adaptive prediction sequence:

$$\mathbf{w}_i^T = \operatorname*{argmax}_{\mathbf{w} \in \Delta(S_i)} \left\langle \mathbf{w}, \left(\sum_{t=1}^{T-1} \mathbf{u}_i^t \right) + \mathbf{M}_i^T \right\rangle - \frac{\mathcal{R}(\mathbf{w})}{\eta}.$$

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Recency bias

Optimistic Follow the Regularized Leader has the `RVU` property with

- one-step recency bias $\mathbf{M}_i^t = \mathbf{u}_i^{t-1}$
- H-step recency bias $\mathbf{M}_i^t = \sum_{\tau=t-H}^{t-1} \mathbf{u}_i^\tau / H$
- geometrically discounted recency bias $\mathbf{M}_i^t = \frac{1}{\sum_{\tau=0}^{t-1} \delta^{-\tau}} \sum_{\tau=0}^{t-1} \delta^{-\tau} \mathbf{u}_i^\tau$

One-step recency bias

With $\mathbf{M}_i^t = \mathbf{u}_i^{t-1}$ and using stepsize η, *Optimistic Follow the Regularized Leader* satisfies the `RVU` property with constants $\alpha = R/\eta$, $\beta = \eta$ and $\gamma = 1/(4\eta)$ where $R = \max_i \left(\sup_{\mathbf{f} \in \Delta(S_i)} \mathcal{R}(\mathbf{f}) - \inf_{\mathbf{f} \in \Delta(S_i)} \mathcal{R}(\mathbf{f}) \right)$.

Other contributions

Meta-algorithm

They show a meta-algorithm that uses any tunable algorithm that satisfies the RVU property so that the RVU property is preserved but also that the worst case rate against adversarial environments is $O(1/\sqrt{T})$.

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Fast convergence of each player's average regret

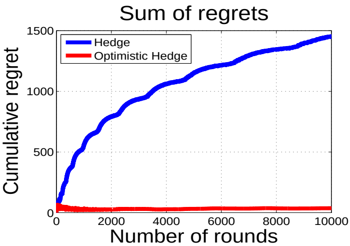
If either all players use

- *Optimistic Follow the Regularized Leader* with $\mathbf{M}_i^t = \mathbf{u}_i^{t-1}$ or
- all use the meta-algorithm with the same input algorithm that satisfies a certain stability condition

then each player's average regret converges at the rate of $O(T^{-3/4})$.

Experimental validation

Results I



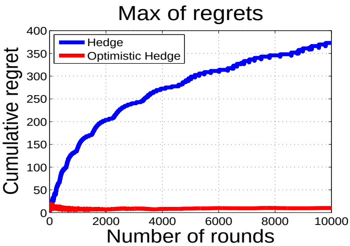


Figure: Maximum and sum of individual regrets over time under the Hedge (blue) and Optimistic Hedge (red) dynamics.

Results II

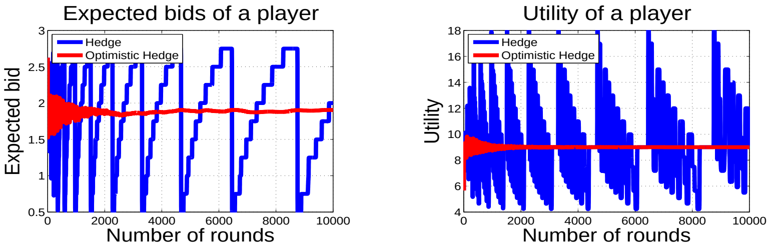


Figure: Expected bid and per-iteration utility of a player on one of the four items over time, under Hedge (blue) and Optimistic Hedge (red) dynamics.

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- Every tunable no-regret algorithm with the RVU property can be used by a **meta-algorithm** that than also satisfies the RVU property and guarantees a worst case rate of $O(1/\sqrt{T})$.
- When all players use the same algorithm chosen from OFRL with $\mathbf{M}_i^t = \mathbf{u}_i^{t-1}$ or the meta-algorithm with the same input algorithm that satisfies the stability condition: Each player's **individual regret** converges at rate $O(T^{-3/4})$ instead of $O(1/\sqrt{T})$.

Discussion

- Is `RVU` necessary? (probably not)
- Is observing only the other's players moves instead of the expected utilities also enough to get faster rates?
- A precise quantification of the desired behaviour, which is necessary for stable trajectories for $\mathbf{w}_i$, is of great interest.