▸ k-**Turing Machine** $M = (Q, \Sigma, \Gamma, \Sigma', \delta, \Box, q_0, q_+, q_-)$

- Q states, $q_0, q_+, q_- \in Q$
- Σ input symbols
- Γ working symbols, $\Box \in \Gamma$
- Σ' output symbols
- δ : $(Q \setminus \{q_+, q_-\}) \times (\Sigma \cup \{\vdash, \dashv\}) \times \Gamma^k \to \mathcal{B}$, no moving out of the input bounds
- $\mathcal{B}$ $= \mathcal{P}_{\neq\emptyset}(Q \times \Gamma^k \times \{-1, 0, 1\}^{k+1} \times (\Sigma' \cup \{\epsilon\}))$

Configuration $(q, B_1, \ldots, B_k, i_0, \ldots, i_k) \in Q \times (\Gamma^+)^k \times \mathbb{N}^{k+1}$

▸ **Number of configurations** There exists at most $a \cdot |w| \cdot b^n$ $(a, b \in \mathbb{N})$ configurations that have at most length $n \in \mathbb{N}$. For f-space-bound TM: $a \cdot |w| \cdot b^{f(|w|+2)}$.

▸ f **is time-constructible** if $f(n) \geq n \; \forall n \in \mathbb{N}$ and $\exists$ det. TM that stops on a^n after exactly $f(n+2)$ steps.

▸ f **is space-constructible** if $f(n) \geq \log n \; \forall n \in \mathbb{N}$ and $\exists$ f-space-bounded det. TM that computes $a^n \mapsto a^{f(n+2)}$.

▸ **Linear speed-up** Let $L \in$ DTIME(f) with $f(n) \geq n \; \forall n \in \mathbb{N}$ and $\epsilon > 0$. Then $L \in$ DTIME(g) with $g(n) = \lceil \epsilon f(n) \rceil \; \forall n \in \mathbb{N}$.

▸ **Linear tape compression** DSPACE$(f) \subseteq$ DSPACE(g) $\forall \; \epsilon > 0$ and $g(n) = \lceil \epsilon f(n) \rceil \; \forall n \in \mathbb{N}$

▸ **Space hierarchy** f space-constructible. $f(n) \geq \log n$.
Then DSPACE$(o(f)) \neq$ DSPACE$(\mathcal{O}(f))$.

▸ **Time hierarchy** DTIME$(o(f)) \subsetneq$ DTIME$(\mathcal{O}(f; g))$ with $g(n) = n \log n \; \forall n \in \mathbb{N}$

▸ **Savitch** NSPACE$(f) \subseteq$ DSPACE$(f; g)$ with $g(n) = n^2 \; \forall n \in \mathbb{N}$

▸ **Padding** $L \subseteq \Sigma^*$, $f(n) \geq n \; \forall n \in \mathbb{N}$, $\# \notin \Sigma$. $Pad_f^{\#}(L) = \{w\#^{f(|w|+2)-|w|} \mid w \in L\}$

▸ **Time - Translation** $f(n) \geq n \leq g(n) \; \forall n \in \mathbb{N}$, g time-constructible, output $\#^{f(n+2)}$ computable in time $g(f(n+2)+2)$ on input $\#^n$, $L \subseteq \Sigma^*$, $\# \notin \Sigma$.

- Pad$_f^{\#}(L) \in$ **D/N**TIME$(\mathcal{O}(g)) \Leftrightarrow L \in$ **D/N**TIME$(\mathcal{O}(f; g))$

▸ **Space - Translation** $f(n) \geq n$, $g(n) \geq \log n \; \forall n \in \mathbb{N}$, g space-constructible, binary repr. of $f(n+2)$ computable in space $g(f(n+2)+2)$ on input $\#^n$, $L \subseteq \Sigma^*$, $\# \notin \Sigma$.

- Pad$_f^{\#}(L) \in$ **D/N**SPACE$(g) \Leftrightarrow L \in$ **D/N**SPACE$(f; g)$

▸ **Reduction** $L \subseteq \Sigma^*$, $L' \subseteq (\Sigma')^*$. $L \preceq L'$ iff $w \in L \Leftrightarrow f(w) \in L'$ for some *Reduction* f. (log-space: $\preceq_{\mathsf{L}}$, polynomial-time: $\preceq_{\mathsf{P}}$) If $L' \in$ **P** and $L \preceq_{\mathsf{P}} L'$ then $L \in$ **P**.

▸ **Complete Problems** $\mathcal{C}$ a class, L a problem.

- $\mathcal{C}$-hard if $L' \preceq_{\mathsf{L}} L \; \forall L' \in \mathcal{C}$
- $\mathcal{C}$-complete if L $\mathcal{C}$-hard and $L \in \mathcal{C}$

▸ **Schöning** $L_1, L_2 \subseteq \Sigma^*$ decidable, $\mathcal{C}_1, \mathcal{C}_2$ (effective) recur. enum. classes of decidable languages over Σ^* and closed under finite variation, $L_1 \notin \mathcal{C}_1$, $L_2 \notin \mathcal{C}_2$. $\Rightarrow$ decidable $L \subseteq \Sigma^*$ with $L \notin \mathcal{C}_1 \cup \mathcal{C}_2$. If $L_1 \in$ **P** and $\emptyset \neq L_2 \neq \Sigma^*$, then also $L \preceq_{\mathsf{P}} L_2$.

▸ **Ladner** If **P** $\neq$ **NP** then there eff. exists $L \in$ **NP** $\setminus$ **P** that is not **NP**-complete under $\preceq_{\mathsf{P}}$.

▸ **Sparse** $L \subseteq \Sigma^*$ is *sparse* if there $\exists$ polynomial p s.t. $|L \cap \Sigma^n| \leq p(n) \; \forall n \in \mathbb{N}$.

▸ **Mahaney** If **P** $\neq$ **NP**. Then there exists no sparse language that is **NP**-hard under $\preceq_{\mathsf{P}}$ (or $\preceq_{\mathsf{L}}$).

▸ **Boolean circuit** Finite directed labeled graph $C = (V, E, l)$ for languages $\{0, 1\}^*$

- $V = \{1, \ldots, o\}$, $o \in \mathbb{N}$, $v < v' \; \forall (v, v') \in E$
- $l : V \to \{\wedge, \vee, \neg, 0, 1\} \cup \{x_1, \ldots, x_n\}$ labeling gates
- *binary* if input rank is ≤ 2, *monotone* if no negation, *constant* if no inputs
- $I(w)$ is evaluation of C for w
- *size* of C is $|C| = |V|$, *depth* $dp(C)$ of C is length of maximal path in (V, E)
- $\mathcal{C} = (C_n)_{n \in \mathbb{N}}$ is *family* of Boolean circuits
- There ex. circuit family $\mathcal{C} = (C_n)_{n \in \mathbb{N}}$ that defines L such that $|C_n| \leq 2^n + 2n + 3$ and $dp(C_n) \leq 3 \; \forall L \subseteq \{0, 1\}^*$
- There ex. circuit family $\mathcal{C} = (C_n)_{n \in \mathbb{N}}$ of binary circuits C_n that defines L and $|C_n| \leq n2^n + 2n + 1$ and $dp(C_n) \leq 1 + \rceil \log n^+ \rceil + n$

▸ c-**polynomial circuit family** $\mathcal{C} = (C_n)_{n \in \mathbb{N}}$, $f : \mathbb{N} \to \mathbb{N}$. $\mathcal{C}$ is f-*size-bounded* if $|C_n| \leq f(n) \; \forall n \in \mathbb{N}$. c-*polynomial* if there is polynomial p s.t. $\mathcal{C}$ is p-size-bounded.

▸ **Shannon** Fraction of n-ary Boolean functions with binary circuits of size smaller than $\frac{2^n}{3n}$ tends to 0 for $n \to \infty$

▸ **Uniform family** $\mathcal{C} = (C_n)_{n \in \mathbb{N}}$ is uniform if there exists log-space computable function f with $f(1^n) = C_n \; \forall n \in \mathbb{N}$

▸ **Nick's class** $k \in \mathbb{N}$. **NC**^{k} class of languages defined by c-polynomial uniform binary circuit families $\mathcal{C} = (C_n)_{n \in \mathbb{N}}$ with $dp(C_n) \in \mathcal{O}(\log^k n) \; \forall n \in \mathbb{N}$. **AC**^{k} for non-binary circuits.

▸ **Oracle TM** TM with additional write-only, auto-advance output tape and special states q_y, q_n, $q_?$. For $O \subseteq \Sigma^*$, spontaneous transition from $q_?$ to $q_{\overset{?}{w \in O}}$.

- **P**$^O = \{L(M^O) \mid$ det. polyn.-time oracle TM $M^O\}$
- **NP**$^O = \{L(M^O) \mid$ polyn.-time oracle TM $M^O\}$
- $O \in$ **P**^{O}
- There exists $O \in$ **PSPACE** s.t. **P**^{O} = **NP**^{O}.
- There exists decidable $O \subseteq \{0, 1\}^*$ s.t. **P**$^O \neq$ **NP**^{O}.

▸ **Polynomial Turing reduction** A, B problems. $A \preceq_{\mathsf{P}}^T B$ if there ex. det. polyn.-time oracle TM M with $A = L(M^B)$.

▸ **Probabilistic TM** TM time-bounded by polynomial p is *probabilistic* if all computations for input w have exactly length $p(|w|)$ and exactly 2 nondeterministic alternatives in each step.

- Monte-Carlo accepts $L \subseteq \Sigma^*$ if $\forall w \in \Sigma^*$, $w \notin L$ iff all computations reject and $w \in L$ iff at least $\frac{2}{3}$ of all computations accept. **RP** set of Monte-carlo accepted languages.
- PIT $\in$ co-**RP**
- Monte-Carlo algorithms have uncertain correctness, Las Vegas algorithms have uncertain runtime.
- „Zero-error Probabilistic Polynomial time“ = **ZPP** = **RP** $\cap$ co-**RP**
- majority accepts $L_{\mathsf{MAJ}}(M) = \{w \in \Sigma^* \mid$ more than half of all computations accept $w\}$
- **PP** set of majority accepted languages
- **P** $\subseteq$ **ZPP** = co-**ZPP** = **RP** $\cap$ co-**RP** $\subseteq$ **RP** $\subseteq$ **NP** $\subseteq$ **PP** = co-**PP** $\subseteq$ **PSPACE**
- **RP** $\subseteq$ **BPP** = co-**BPP** $\subseteq$ **PP**
- co-**RP** $\subseteq$ **BPP**
- **ZPP** $\subseteq$ co-**RP** $\subseteq$ co-**NP** $\subseteq$ **PP**

▸ **Randomized TM** det. TM with input tape Z of infinite random bits. f-time-bounded if classic def. holds for every content of Z. Accepts w with prop. $\frac{\# Z \in \{0,1\}^{f(|w|)} \text{ w accepted}}{2^{f(|w|)}}$

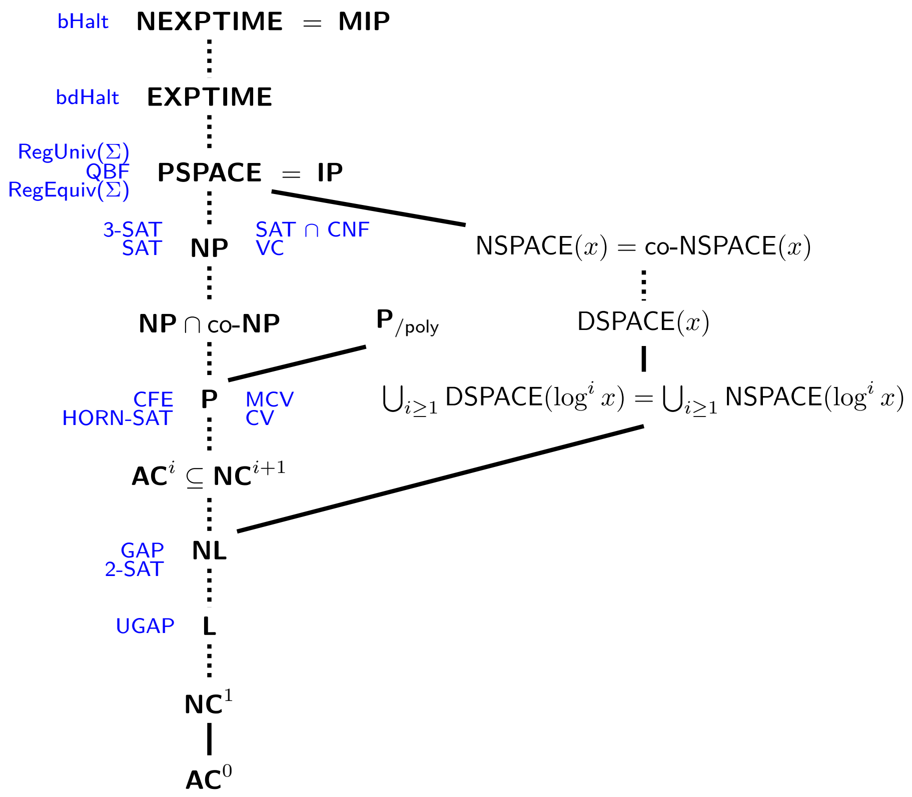
▸ **Interactive Proof System** Pair (A,B), $C=\{0,1\}$, $A:\bigcup_{i\in\mathbb{N}}(C^*)^{1+2i}\to C^*$, p,q polyn., p-time-bounded rand. TM B. $q(|w|)$ rounds. A and B share tape W. Round i:

$\rightarrow\ a_i = A(w,a_1,b_1,\ldots,a_{i-1},b_{i-1})$
$\leftarrow\ b_i = B(w,a_1,b_1,\ldots,a_{i-1},b_{i-1},Z_1,\ldots,Z_i)$

In round $q(|w|)$ Bob decides whether to accept w. (A,B) accepts L if for all w:

- $w\in L$, then Bob accepts input w with prop. $\geq 1-2^{-|w|}$
- $w\notin L$, then no (A',B) exists s.t. Bob accepts w with probability $\geq 2^{-|w|}$

- **IP** set of languages defined by IPs
- Non-isomorphism of finite graphs is in **IP**
- **IP** is closed under $\preceq_{\mathbf{P}}$



DTIME$(f)\subseteq$ NTIME$(f)\subseteq$ DSPACE$(f)\subseteq$ NSPACE$(f)\subseteq$ DTIME$(2^{\mathcal{O}(f)})$
Hennie & Stearns: We can emulate k-band in 2-band with log overhead
L $\subseteq$ **NL** $\subseteq$ DTIME$(2^{\mathcal{O}(\log x)})=$ **P**
NL $\subsetneq$ **CSL** $=$ **LBA** $=$ NSPACE$(x)\subseteq$ DTIME$(2^{\mathcal{O}(x)})$
DSPACE$(x^2)\subseteq$ DTIME$(2^{\mathcal{O}(x^2)})$
GAP $\in$ DSPACE$(\log^2 x)$
PSPACE $=\bigcup_{i\in\mathbb{N}_+}$ DSPACE$(x^i)=\bigcup_{i\in\mathbb{N}_+}$ NSPACE(x^i) since NSPACE$(x^i)\subseteq$ DSPACE(x^{2i}) by Savitch

L $\subsetneq$ DSPACE$(\log^2 x)\subsetneq$ DSPACE$(x)\subseteq$ NSPACE$(x)\subsetneq$ **PSPACE**
DTIME$(\mathcal{O}(x))\subsetneq$ DTIME$(\mathcal{O}(x^2))\subsetneq$ **P**
P $\subsetneq$ DTIME$(\mathcal{O}(2^x))\subsetneq$ DTIME$(\mathcal{O}(2+\epsilon)^x)$
DCSL $=$ DSPACE$(x)\neq$ NSPACE$(x)=$ **CSL** $\Rightarrow$ **L** $\neq$ **NL**
P $\neq$ DSPACE$(x)=$ **DCSL**
$L\in$ **L** $\Leftrightarrow\exists\phi\in$ FO(CTC): $L=\{w\in\Sigma^*\mid w\models\phi\}$
$L\in$ **NL** $\Leftrightarrow\exists\phi\in$ FO(TC): $L=\{w\in\Sigma^*\mid w\models\phi\}$
PRIM $\in$ **NP** $\cap$ co-**NP**
There are **NP**-complete problems $\Rightarrow$ there are **NP**-complete problems in NTIME(x)
If **P** $\neq$ **NP**, then there eff. exists $L\in$ **NP** $\setminus$ **P** that is not **NP**-complete under $\preceq_{\mathbf{L}}$

TODO: Razborov