- k-**Turing Machine** $M = (Q, \Sigma, \Gamma, \Sigma', \delta, \Box, q_0, q_+, q_-)$
  - Q states, $q_0, q_+, q_- \in Q$
  - Σ input symbols
  - Γ working symbols, $\Box \in \Gamma$
  - Σ' output symbols
  - δ : $(Q \setminus \{q_+, q_-\}) \times (\Sigma \cup \{\vdash, \dashv\}) \times \Gamma^k \to \mathcal{B}$, no moving out of the input bounds
  - $\mathcal{B}$ $= \mathcal{P}_{\neq\emptyset}(Q \times \Gamma^k \times \{-1, 0, 1\}^{k+1} \times (\Sigma' \cup \{\epsilon\}))$

Configuration $(q, B_1, \dots, B_k, i_0, \dots, i_k) \in Q \times (\Gamma^+)^k \times \mathbb{N}^{k+1}$

- **Number of configurations** There exists at most $a \cdot |w| \cdot b^n$ $(a, b \in \mathbb{N})$ configurations that have at most length $n \in \mathbb{N}$. For f-space-bound TM: $a \cdot |w| \cdot b^{f(|w|+2)}$.
- f **is time-constructible** if $f(n) \geq n$ $\forall n \in \mathbb{N}$ and $\exists$ det. TM that stops on a^n after exactly $f(n+2)$ steps.
- f **is space-constructible** if $f(n) \geq \log n$ $\forall n \in \mathbb{N}$ and $\exists$ f-space-bounded det. TM that computes $a^n \mapsto a^{f(n+2)}$.
- **Linear speed-up** Let $L \in$ DTIME(f) with $f(n) \geq n$ $\forall n \in \mathbb{N}$ and $\epsilon > 0$. Then $L \in$ DTIME(g) with $g(n) = \lceil \epsilon f(n) \rceil$ $\forall n \in \mathbb{N}$.
- **Linear tape compression** DSPACE$(f) \subseteq$ DSPACE(g) $\forall$ $\epsilon > 0$ and $g(n) = \lceil \epsilon f(n) \rceil$ $\forall n \in \mathbb{N}$
- **Space hierarchy** f space-constructible. $f(n) \geq \log n$. Then DSPACE$(o(f)) \neq$ DSPACE$(\mathcal{O}(f))$.
- **Time hierarchy** DTIME$(o(f)) \subsetneq$ DTIME$(\mathcal{O}(f; g))$ with $g(n) = n \log n$ $\forall n \in \mathbb{N}$
- **Klein-O** $g \in o(f)$ iff $\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0$. L'Hospital applicable if both converge to 0 or ∞.
- **Savitch** NSPACE$(f) \subseteq$ DSPACE$(f; g)$ with $g(n) = n^2$ $\forall n \in \mathbb{N}$
- **Immerman & Szelepcsényi** NSPACE(f) = co-NSPACE(f)
- **Padding** $L \subseteq \Sigma^*$, $f(n) \geq n$ $\forall n \in \mathbb{N}$, $\# \notin \Sigma$. $Pad_f^{\#}(L) = \{w\#^{f(|w|+2)-|w|} \mid w \in L\}$
- **Time - Translation** $f(n) \geq n \leq g(n)$ $\forall n \in \mathbb{N}$, g time-constructible, output $\#^{f(n+2)}$ computable in time $g(f(n+2)+2)$ on input $\#^n$, $L \subseteq \Sigma^*$, $\# \notin \Sigma$.
  - Pad$_f^{\#}(L) \in$ N/DTIME$(\mathcal{O}(g)) \Leftrightarrow L \in$ N/DTIME$(\mathcal{O}(f; g))$
- **Space - Translation** $f(n) \geq n$, $g(n) \geq \log n$ $\forall n \in \mathbb{N}$, g space-constructible, binary repr. of $f(n+2)$ computable in space $g(f(n+2)+2)$ on input $\#^n$, $L \subseteq \Sigma^*$, $\# \notin \Sigma$.
  - Pad$_f^{\#}(L) \in$ N/DSPACE$(g) \Leftrightarrow L \in$ N/DSPACE$(f; g)$
- **Reduction** $L \subseteq \Sigma^*$, $L' \subseteq (\Sigma')^*$. $L \preceq L'$ iff $w \in L \Leftrightarrow f(w) \in L'$ for some *Reduction* f. (log-space: $\preceq_{\mathbf{L}}$, polynomial-time: $\preceq_{\mathbf{P}}$) If $L' \in$ **P** and $L \preceq_{\mathbf{P}} L'$ then $L \in$ **P**.
- **Complete Problems** $\mathcal{C}$ a class, L a problem.
  - $\mathcal{C}$-hard if $L' \preceq_{\mathbf{L}} L$ $\forall L' \in \mathcal{C}$
  - $\mathcal{C}$-complete if L $\mathcal{C}$-hard and $L \in \mathcal{C}$
- **Schöning** $L_1, L_2 \subseteq \Sigma^*$ decidable, $\mathcal{C}_1, \mathcal{C}_2$ (effective) recur. enum. classes of decidable languages over Σ^* and closed under finite variation, $L_1 \notin \mathcal{C}_1$, $L_2 \notin \mathcal{C}_2$. $\Rightarrow$ decidable $L \subseteq \Sigma^*$ with $L \notin \mathcal{C}_1 \cup \mathcal{C}_2$. If $L_1 \in$ **P** and $\emptyset \neq L_2 \neq \Sigma^*$, then also $L \preceq_{\mathbf{P}} L_2$.
- **Ladner** If **P** $\neq$ **NP** then there eff. exists $L \in$ **NP** $\setminus$ **P** that is not **NP**-complete under $\preceq_{\mathbf{P}}$.
- **Sparse** $L \subseteq \Sigma^*$ is *sparse* if there $\exists$ polynomial p s.t. $|L \cap \Sigma^n| \leq p(n)$ $\forall n \in \mathbb{N}$.
- **Mahaney** If **P** $\neq$ **NP**. Then there exists no sparse language that is **NP**-hard under $\preceq_{\mathbf{P}}$ (or $\preceq_{\mathbf{L}}$).
- **Boolean circuit** Finite directed labeled graph $C = (V, E, l)$ for languages $\{0, 1\}^*$
  - $V = \{1, \dots, o\}$, $o \in \mathbb{N}$, $v < v'$ $\forall (v, v') \in E$
  - $l : V \to \{\wedge, \vee, \neg, 0, 1\} \cup \{x_1, \dots, x_n\}$ labeling gates
  - *binary* if input rank is ≤ 2, *monotone* if no negation, *constant* if no inputs
  - $I(w)$ is evaluation of C for w
  - *size* of C is $|C| = |V|$, *depth* $dp(C)$ of C is length of maximal path in (V, E)
  - $\mathcal{C} = (C_n)_{n\in\mathbb{N}}$ is *family* of Boolean circuits
  - There ex. circuit family $\mathcal{C} = (C_n)_{n\in\mathbb{N}}$ that defines L such that $|C_n| \leq 2^n + 2n + 3$ and $dp(C_n) \leq 3$ $\forall L \subseteq \{0, 1\}^*$
  - There ex. circuit family $\mathcal{C} = (C_n)_{n\in\mathbb{N}}$ of binary circuits C_n that defines L and $|C_n| \leq n2^n + 2n + 1$ and $dp(C_n) \leq 1 + \lceil \log n^+ \rceil + n$
- c-**polynomial circuit family** $\mathcal{C} = (C_n)_{n\in\mathbb{N}}$, $f : \mathbb{N} \to \mathbb{N}$. $\mathcal{C}$ is f*-size-bounded* if $|C_n| \leq f(n)$ $\forall n \in \mathbb{N}$. *c-polynomial* if there is polynomial p s.t. $\mathcal{C}$ is p-size-bounded.
- **Shannon** Fraction of n-ary Boolean functions with binary circuits of size smaller than $\frac{2^n}{3n}$ tends to 0 for $n \to \infty$
- **Uniform family** $\mathcal{C} = (C_n)_{n\in\mathbb{N}}$ is uniform if there exists log-space computable function f with $f(1^n) = C_n$ $\forall n \in \mathbb{N}$
- **Nick's class** $k \in \mathbb{N}$. **NC**^{k} class of languages defined by c-polynomial uniform binary circuit families $\mathcal{C} = (C_n)_{n\in\mathbb{N}}$ with $dp(C_n) \in \mathcal{O}(\log^k n)$ $\forall n \in \mathbb{N}$. **AC**^{k} for non-binary circuits.
- **Razbarov's Theorem** Attempt to prove **P** $\neq$ **NP**. He showed that there is a monotone language in **NP** that has no c-polynomial monotone bool. circuit. If one could show that all monotone languages in **P** have c-pol. mon. bool. circuit, then **P** $\neq$ **NP**.
- **Oracle TM** TM with additional write-only, auto-advance output tape and special states q_y, q_n, $q_?$. For $O \subseteq \Sigma^*$, spontaneous transition from $q_?$ to q_+ or q_-.
  - **P**$^O = \{L(M^O) \mid$ det. polyn.-time oracle TM $M^O\}$
  - **NP**$^O = \{L(M^O) \mid$ polyn.-time oracle TM $M^O\}$
  - $O \in$ **P**^{O}
  - There exists $O \in$ **PSPACE** s.t. **P**^{O} = **NP**^{O}.
  - There exists decidable $O \subseteq \{0, 1\}^*$ s.t. **P**$^O \neq$ **NP**^{O}.
- **Polynomial Turing reduction** A, B problems. $A \preceq_{\mathbf{P}}^T B$ if there ex. det. polyn.-time oracle TM M with $A = L(M^B)$. M uses problem B as Oracle.
- **Probabilistic TM** TM time-bounded by polynomial p is *probabilistic* if all computations for input w have exactly length $p(|w|)$ and exactly 2 nondeterministic alternatives in each step.
  - Monte-Carlo accepts $L \subseteq \Sigma^*$ if $\forall w \in \Sigma^*$, $w \notin L$ iff all computations reject and $w \in L$ iff at least $\frac{2}{3}$ of all computations accept. **RP** set of Monte-carlo accepted languages.
  - PIT $\in$ co-**RP**. Test if two polynomials are identical.
  - Monte-Carlo algorithms have uncertain correctness, Las Vegas algorithms have uncertain runtime.
  - „Zero-error Probabilistic Polynomial time" = **ZPP** = **RP** $\cap$ co-**RP**. Result is always correct, and expected runtime is in **P**. Run Monte-Carlo TMs for L and L^C in parallel until one accepts and the other rejects.
  - majority accepts $L_{\mathsf{MAJ}}(M) = \{w \in \Sigma^* \mid$ more than half of all computations accept $w\}$
  - **PP** set of majority accepted languages
  - **BPP** qualified majority e.g. $w \in L$ iff $> \frac{2}{3}$ runs accept w, $w \notin L$ iff $> \frac{2}{3}$ runs reject.
  - **P** $\subseteq$ **ZPP** = co-**ZPP** = **RP** $\cap$ co-**RP** $\subseteq$ **RP** $\subseteq$ **NP** $\subseteq$ **PP** = co-**PP** $\subseteq$ **PSPACE**
  - **RP** $\subseteq$ **BPP** = co-**BPP** $\subseteq$ **PP**

- co-**RP** $\subseteq$ **BPP**
- **ZPP** $\subseteq$ co-**RP** $\subseteq$ co-**NP** $\subseteq$ **PP**

▸ **Randomized TM** det. TM with input tape Z of infinite random bits. f-time-bounded if classic def. holds for every content of Z. Accepts w with prop. $\frac{\# Z \in \{0,1\}^{f(|w|)} \text{ w accepted}}{2^{f(|w|)}}$

▸ **Interactive Proof System** Pair (A, B), $C = \{0,1\}$, $A : \bigcup_{i \in \mathbb{N}} (C^*)^{1+2i} \to C^*$, p, q polyn., p-time-bounded rand. TM B. $q(|w|)$ rounds. A and B share tape W. Round i:

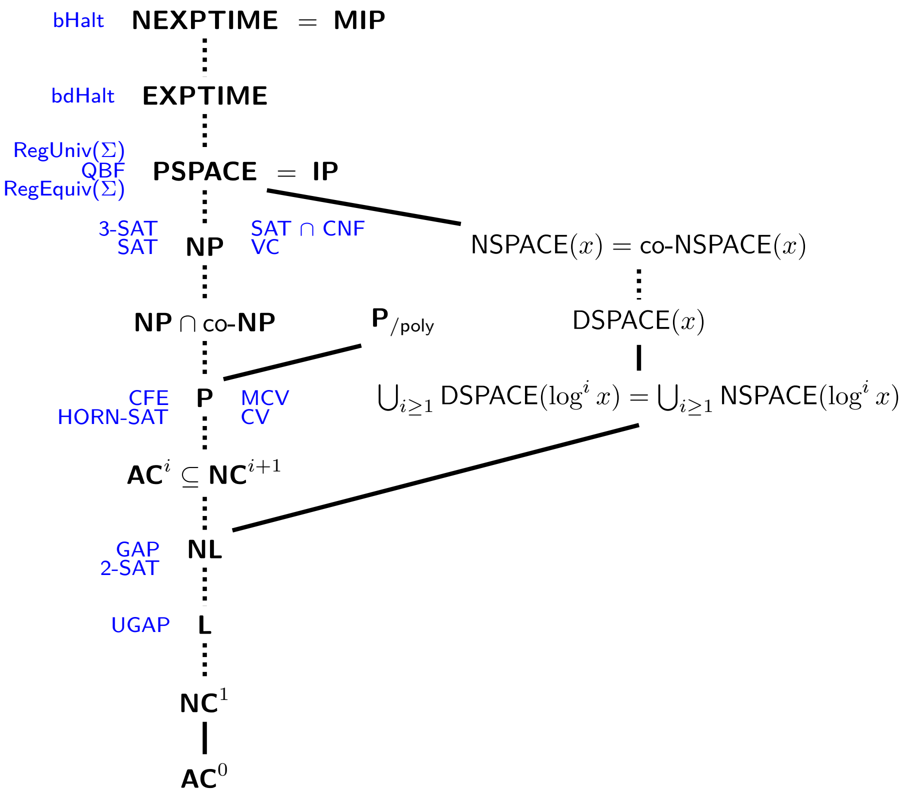
$\to$ $a_i = A(w, a_1, b_1, \ldots, a_{i-1}, b_{i-1})$
$\leftarrow$ $b_i = B(w, a_1, b_1, \ldots, a_{i-1}, b_{i-1}, Z_1, \ldots, Z_i)$

In round $q(|w|)$ Bob decides whether to accept w. (A, B) accepts L if for all w:

- $w \in L$, then Bob accepts input w with prop. $\geq 1 - \frac{1}{2^{|w|}}$
- $w \notin L$, then no (A', B) exists s.t. Bob accepts w with probability $\geq \frac{1}{2^{|w|}}$

- **IP** set of languages defined by IPs
- Non-isomorphism of finite graphs is in **IP**
- **IP** is closed under $\preceq_{\mathbf{P}}$

Note: Alice tries to convince Bob that $w \in L$. Bob can't trust Alice and has to verify. Usually this is done by sending Alice a decision problem, that she can only correctly decide with probability $\frac{1}{2^n}$. Switching equal colored socks n times, she has to decide if there was a switch or not.



DTIME$(f) \subseteq$ NTIME$(f) \subseteq$ DSPACE$(f) \subseteq$ NSPACE$(f) \subseteq$ DTIME$(2^{\mathcal{O}(f)})$

Hennie & Stearns: We can emulate k-band in 2-band with log overhead
L $\subseteq$ **NL** $\subseteq$ DTIME$(2^{\mathcal{O}(\log x)}) =$ **P**
NL $\subsetneq$ **CSL** $=$ **LBA** $=$ NSPACE$(x) \subseteq$ DTIME$(2^{\mathcal{O}(x)})$
DSPACE$(x^2) \subseteq$ DTIME$(2^{\mathcal{O}(x^2)})$
PSPACE $= \bigcup_{i \in \mathbb{N}_+}$ DSPACE$(x^i) = \bigcup_{i \in \mathbb{N}_+}$ NSPACE(x^i) since NSPACE$(x^i) \subseteq$ DSPACE(x^{2i}) by Savitch
L $\subsetneq$ DSPACE$(\log^2 x) \subsetneq$ DSPACE$(x) \subseteq$ NSPACE$(x) \subsetneq$ **PSPACE**
DTIME$(\mathcal{O}(x)) \subsetneq$ DTIME$(\mathcal{O}(x^2)) \subsetneq$ **P**
P $\subsetneq$ DTIME$(\mathcal{O}(2^x)) \subsetneq$ DTIME$(\mathcal{O}(2+\epsilon)^x)$
DCSL $=$ DSPACE$(x) \neq$ NSPACE$(x) =$ **CSL** $\Rightarrow$ **L** $\neq$ **NL**
P $\neq$ DSPACE$(x) =$ **DCSL**
$L \in$ **L** $\Leftrightarrow \exists \phi \in$ FO(CTC)$: L = \{w \in \Sigma^* \mid w \models \phi\}$
$L \in$ **NL** $\Leftrightarrow \exists \phi \in$ FO(TC)$: L = \{w \in \Sigma^* \mid w \models \phi\}$
PRIM $\in$ **NP** $\cap$ co-**NP**
There are **NP**-complete problems $\Rightarrow$ there are **NP**-complete problems in NTIME(x)
If **P** $\neq$ **NP**, then there eff. exists $L \in$ **NP** $\setminus$ **P** that is not **NP**-complete under $\preceq_{\mathbf{L}}$

▸ **NL-complete** GAP $\in$ DSPACE$(\log^2 x) \subseteq$ NSPACE$(\log x) \subseteq$ **NL**. Any $\log$-space-bounded TM can be transformed into a dir. graph with polynomial size. Each configuration (limited by $2^{\mathcal{O}(\log x)}$) resembles a vertex. Decide if accepting conf. is reachable from starting conf. with GAP.
We can reduce 2-SAT to GAP and vice-versa. Each disjunction $\neg x_1 \implies x_2$ of 2-SAT is edge in graph. If we can reach x_1 from $\neg x_1$ and vice-versa, the formula cotains contradiction.

▸ **P-complete** CFE $\in$ **P-c** because we can construct grammar G_w for TM M with input w that produces ϵ from final configuration in $\log$ space. If grammar produces empty language $w \notin L_M$.
CV $\in$ **P-c** because we can construct bool. circuit from G_w that evaluates to 1 iff $w \in L$.
HORN-SAT $\in$ **P-c** because we can reduce bool. circuit to HORN-SAT formula.

▸ **NP-complete** SAT $\in$ **NP-c** because we can construct a huge formula from any TM that is satisfiable iff $w \in L(M)$. Create constraints for the configurations at each point in time. We can also create CNF from every prop. formula which is also a 3-SAT formula, hence CNF, 3-SAT $\in$ **NP-c**. All of these are obviously in **NP** because we can guess an interpretation and verify it.

▸ **(N)EXPTIME-complete** bdHalt is **EXPTIME**-complete because we can simulate any TM M that is time-bounded by $2^{p(|w|+2)}$ and see if it halts on input w. Input of bdHalt would be binary repr. of M and binary repr. of $2^{p(|w|+2)}$ which can be calculated in $\log$-space. The same applies to bHalt $\in$ **NEXPTIME-c**.