Numerical methods for Differential equations Report 2

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1 Inhomogeneous poisson equation

In the this section, the following Boundary Value Problem (BVP) will be solved using a Finite volume method.

$$\begin{aligned}
-\nabla \cdot (k\nabla u) &= f, \quad (x,y) \in \Omega = (0,8)\times(0,4)\\
u(x,y) &= 0, \quad (x,y) \in \partial\Omega\\
k(x,y) &= 1+4x+6y, \quad (x,y) \in \bar{\Omega}\\
f(x,y) &= e^{\alpha(x-1)^2+\alpha(y-1)^2} + e^{\alpha(x-3)^2+\alpha(y-1)^2}\\
&\quad + e^{\alpha(x-1)^2+\alpha(y-1)^2} + e^{\alpha(x-3)^2+\alpha(y-3)^2}\\
&\quad + e^{\alpha(x-5)^2+\alpha(y-3)^2} + e^{\alpha(x-7)^2+\alpha(y-3)^2}\\
&\quad \text{with } \alpha = -5, \quad (x,y) \in \bar{\Omega}
\end{aligned} \tag{1}$$

With $\bar{\Omega} = [0,8]\times[0,4]$. In order to discretize the PDE, the following stencil on a doubly uniform grid with step h is considered.

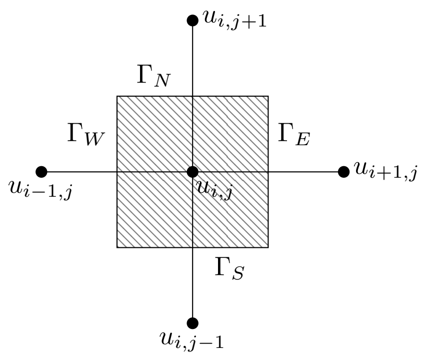


Figure 1: Stencil used for the discretization

In order to discretize the PDE with a finite volume method (FVM), it has to be integrated over a control volume $V_{i,j}$, which is represented by the hatched Area, whose boundary is $\Gamma_{i,j}$.

$$\iint_{V_{i,j}} [-\nabla\cdot(k\nabla u)]dV = \iint_{V_{i,j}} f dV \tag{2}$$

Using the Divergence Theorem

$$\oint_{\Gamma_{i,j}} (-k\nabla u\cdot\mathbf{n})d\Gamma = \iint_{V_{i,j}} f dV \tag{3}$$

Which can be rewritten as

$$\oint_{\Gamma_{ijj}} \left(-k\frac{\partial u}{\partial \mathbf{n}}\right) d\Gamma = \iint_{V_{i,j}} f dV \tag{4}$$

By splitting the boundary and using a midpoint approximation for the right hand side

$$\int_{\Gamma_W}\left(k\frac{\partial u}{\partial x}\right)dy + \int_{\Gamma_E}\left(-k\frac{\partial u}{\partial x}\right)dy + \int_{\Gamma_S}\left(k\frac{\partial u}{\partial y}\right)dx + \int_{\Gamma_N}\left(-k\frac{\partial u}{\partial y}\right)dx = f_{i,j}h^2 \tag{5}$$

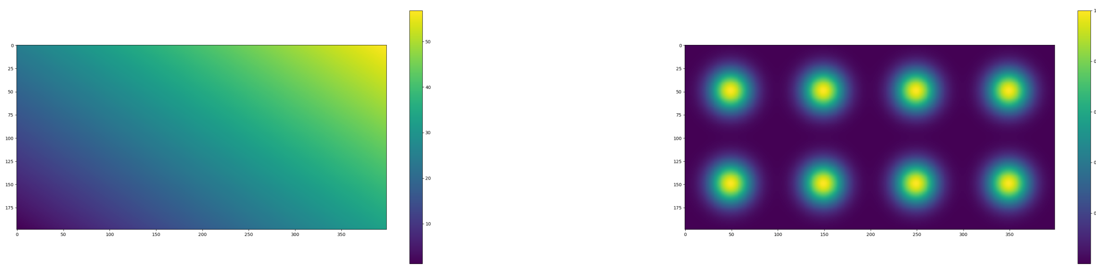
By approximating $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ with a central difference formula and using a midpoint approximation for the Integral, a discrete approximation can be found.

$$k_{i-1/2,j}\left(u_{i,j}-u_{i-1,j}\right)-k_{i+1/2,j}\left(u_{i+1,j}-u_{i,j}\right)+k_{i,j-1/2}\left(u_{i,j}-u_{i,j-1}\right)-k_{i,j+1/2}\left(u_{i,j+1}-u_{i,j}\right)=h^2 f_{i,j} \tag{6}$$

Which can be rearranged to

$$\begin{aligned} f_{ij} = &-\frac{k_{i-1/2,j}}{h^2}u_{i-1,j}-\frac{k_{i,j-1/2}}{h^2}u_{i,j-1} \\ &+\left(\frac{k_{i-1/2,j}}{h^2}+\frac{k_{i,j-1/2}}{h^2}+\frac{k_{i+1/2,j}}{h^2}+\frac{k_{i,j+1/2}}{h^2}\right)u_{i,j} \\ &-\frac{k_{i+1/2,j}}{h^2}u_{i+1,j}-\frac{k_{i,j+1/2}}{h^2}u_{i,j+1} \end{aligned} \tag{7}$$

When assembling a linear system on a lexicographic grid from this equation, a matrix $A \in \mathbb{R}^{(N_x-1)*(N_y-1)\times(N_x-1)*(N_y-1)}$, where $N_x = 8/h$ and $N_y = 4/h$. This matrix has 5 non-zero diagonals, located on the main diagonal, right next to the main diagonal and offset by $\pm(N_x-1)$. All these diagonals are full, except for the first off diagonals, which have a 0 on every N_x^{th} element. These diagonals can be made, by calculating k for a flattened shifted grid, which can be trimmed to include the proper coefficients in the matrix, which is implemented in the code.



(a) Coefficient K on the domain

(b) Source on the domain

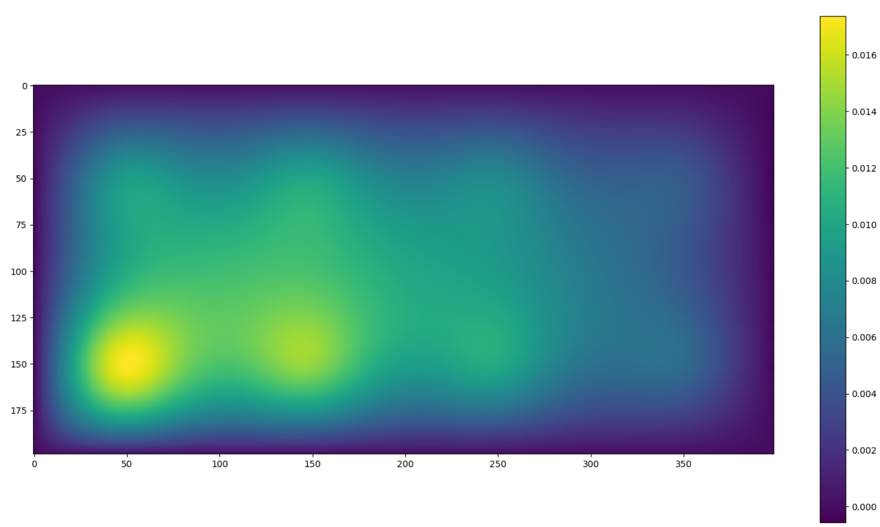


Figure 3: Solution to the BVP on the domain for $h = 0.02$

The solution seems reasonable, as the source function can be interpreted as a steady influx, that is diffused on the domain. As the rate of diffusion is influenced by k it is expected that the upper right corner will experience stronger diffusion than then the lower left corner, which is the case in the solution. Furthermore, zero Dirichlet boundary conditions are applied, which are also clearly visible in the solution.

2 Time evolution of the heat equation

In this section the unsteady heat equation will be solved. This problem can be formulated as the following BVP, with an initial condition.

$$\begin{gathered}\frac{\partial u}{\partial t} - \Delta u = 0, \quad (x,y) \in (0,4) \times (0.4), \quad t \in [0, 0.15] \\ u(x,y,t) = 0, \quad (x,y) \in \partial\Omega \\ u(x,y,0) = e^{\alpha(x-2)^2 + \alpha(y-2)^2} \\ \text{with } \alpha = -5, (x,y) \in \overline{\Omega}\end{gathered} \tag{8}$$

In which $\overline{\Omega} = [0,4] \times [0,4]$. This equation can be discretized on a doubly uniform grid with step h. The negative Laplacian in its discrete form can be expressed as A, such that the equation becomes

$$\frac{\partial \mathbf{u}}{\partial t} \approx -A\mathbf{u} \tag{9}$$

In order to solve this unsteady equation, a time-stepping scheme has to be employed. As a simple explicit scheme, Forward Euler (FE) is considered. In this scheme, $\mathbf{u}^{k+1}$ is calculated as

$$\mathbf{u}^{k+1} = \mathbf{u}^k + \Delta t * \frac{\partial \mathbf{u}^k}{\partial t} \tag{10}$$

By substituting Equation 9

$$\mathbf{u}^{k+1} = \mathbf{u}^k - \Delta t * A\mathbf{u}^k \tag{11}$$

$$\mathbf{u}^{k+1} = (I - \Delta t * A)\mathbf{u}^k \tag{12}$$

As an implicit method, Backward Euler (BE) is used. $\mathbf{u}^{k+1}$ in BE is calculated as follows.

$$\mathbf{u}^{k+1} = \mathbf{u}^k + \Delta t * \frac{\partial \mathbf{u}^{k+1}}{\partial t} \tag{13}$$

By substituting Equation 9

$$\mathbf{u}^{k+1} = \mathbf{u}^k - \Delta t * A\mathbf{u}^{k+1} \tag{14}$$

$$\mathbf{u}^{k+1} + \Delta t * A\mathbf{u}^{k+1} = \mathbf{u}^k \tag{15}$$

$$(I + \Delta t * A)\mathbf{u}^{k+1} = \mathbf{u}^k \tag{16}$$

In order to solve for $\mathbf{u}^{k+1}$ a linear system has to be solved, thus this is an implicit method.

In order to compare these 2 methods, the time evolution is simulate on a grid with $h = 0.08$ and $\delta t = 0.015$.

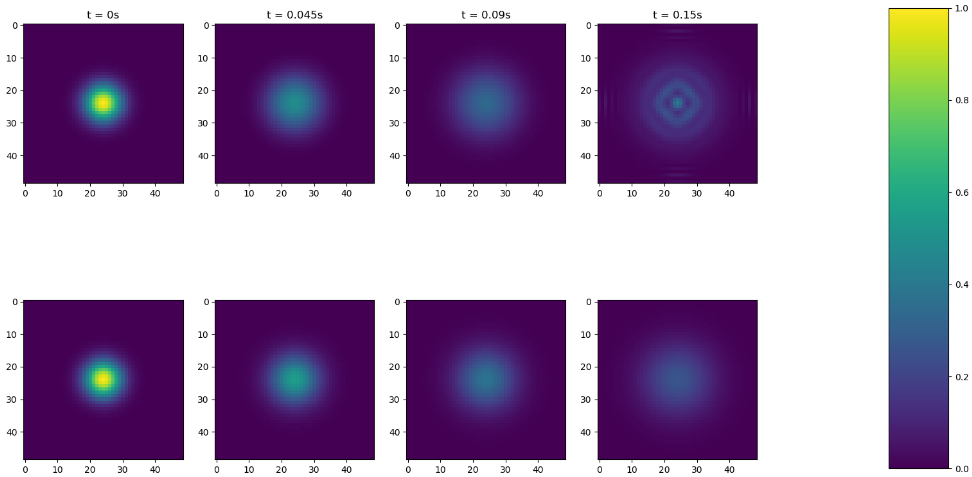
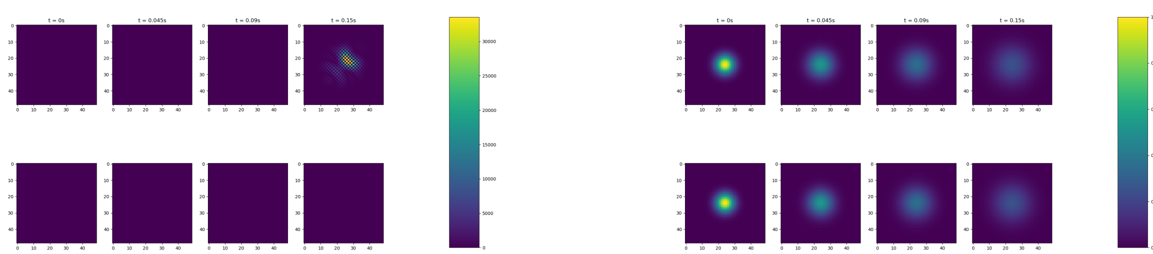


Figure 4: Time evolution of the Heat equation. First Row presents the Forward Euler, second row is computed with Backwards Euler. All plots are normalized to the same colorscale.

It can be seen that FE diffuses faster than BE. Furthermore, FE becomes unstable at $t = 0.15$ as the solution does not settle to reach 0 at all points in the domain, which should be expected from the maximum principle. In order to get stable results from the simulation is run with $\Delta t = 0.005, 0.003$ and 0.0015. For $\Delta t = 0.005 \text{and} 0.003$ the solution with FE blows up in the given time, however for $\Delta t = 0.0015$ the solution of FE matches the implicit solution very well.



(a) Unstable solution for $\Delta t = 0.005$, with all plots normalized to the same scale

(b) Stable solution for $\Delta t = 0.0015$, with all plots normalized to the same scale

Figure 5: Comparison of a stable and unstable solution for the BVP

In order to use appropriate timesteps for FE, a stability limit for Δt has to be derived. The stability of a time stepping scheme can be analyzed by looking at the eigenvalues of the stepping scheme. For this λ is considered to be an eigenvalue of the negative Laplacian and σ is considered to be an eigenvalue of the time stepping scheme.

$$C = I - \Delta t A \tag{17}$$

For some eigenvector $\mathbf{x}_m$ it holds that

$$\begin{aligned}
C\mathbf{x}_m &= (I - \Delta t A)\mathbf{x}_m && (18)\\
C\mathbf{x}_m &= I\mathbf{x}_m - \Delta t A\mathbf{x}_m && (19)\\
\sigma_m\mathbf{x}_m &= \mathbf{x}_m - \Delta t\lambda_m\mathbf{x}_m && (20)\\
\sigma_m\mathbf{x}_m &= (1 - \Delta t\lambda_m)\mathbf{x}_m && (21)\\
& && (22)
\end{aligned}$$

Thus the relationship for the eigenvalues of the negative Laplacian (λ) and the eigenvalues of the time stepping scheme (σ) are related by $\sigma_m = 1 - \Delta t\lambda_m$. For FE to be stable $|\sigma_m| \leq 1$ for all σ_m. Thus

$$\lambda_{k_x,k_y} = \frac{4}{h^2}\left[\sin^2\left(\frac{\pi k_x}{2N_x}\right) + \sin^2\left(\frac{\pi k_y}{2N_y}\right)\right] \tag{23}$$

Thus $lambda_m$ is bounded.

$$\begin{aligned}
0 \leq \lambda_m &\leq \frac{4}{h^2} && (24)\\
|1 - \Delta t\lambda_m| &\leq 1 && (25)\\
\Delta t\lambda_m &\leq 2 && (26)\\
\Delta t\frac{4}{h^2} &\leq 2 && (27)\\
\Delta t &\leq \frac{h^2}{2} && (28)
\end{aligned}$$

Calculating the stability limit for $h = 0.08$ it can be seen that $\Delta t \leq 0.0016$. As previously only one time step was stable, which explains the blow-up of the solution for $\Delta t = 0.005$ and 0.003.

By running the simulation for different values of h, with a stable time step for FE and a time step of 0.015 for BE, the speed of an implicit vs explicit method can be compared.

h	FE	BE
0.08	0.163 s	0.107 s
0.04	0.968 s	0.650 s
0.02	7.986 s	4.412 s

Table 1: Time to solve the BVP for different step-sizes with FE and BE

It can be seen that solving the system implicitly is always faster for this range of h than the explicit solution, as the required time step decreases rapidly such that solving a linear system is less computational effort than performing more steps.

3 Time evolution of the wave equation

In order to examine the time evolution of the wave equation, the following BVP is considered.

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= c^2\Delta u + f, \quad (x,y) \in (0,4)\times(0,12), \quad t \in [0,8]\\
u(x,y,t) &= 0, \quad (x,y) \in \partial\Omega, \quad t \in [0,8]\\
u(x,y,0) &= 0, \quad (x,y) \in \Omega\\
\left.\frac{\partial u}{\partial t}\right|_{t=0} &= 0, \quad (x,y) \in \Omega\\
f(x,y,t) &= \sin(2\pi\nu t)e^{\alpha(x-2)^2+\alpha(y-2)^2} \quad (x,y) \in \overline{\Omega}, \quad t \in [0,8]\\
&\text{with } \alpha = -50, \nu = 4
\end{aligned} \tag{29}$$

With $\overline{\Omega} = [0,4] \times [0,12]$. The discretization of the Laplacian Operator is performed using a central finite difference method on a doubly uniform grid with step h.

$$L = \frac{1}{h^2} \begin{bmatrix} -2 & 1 & 0 & \dots & 0 \\ 1 & -2 & 1 & & \vdots \\ 0 & 1 & -2 & 1 & \\ \vdots & & & \ddots & \\ 0 & \dots & 0 & 1 & -2 \end{bmatrix} \tag{30}$$

From this the space discrete form can be determined to be

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} \approx c^2 L \mathbf{u} + f \tag{31}$$

In order to use time stepping on the second derivative in time, a finite difference time discretization (FDTD) is used.

$$\frac{\partial^2 \mathbf{u}^k}{\partial t^2} = \frac{\mathbf{u}^{k-1} - 2\mathbf{u}^k + \mathbf{u}^{k+1}}{\Delta t^2} + \mathcal{O}(\Delta t^2) \tag{32}$$

$$c^2 L \mathbf{u}^k + f \approx \frac{\mathbf{u}^{k-1} - 2\mathbf{u}^k + \mathbf{u}^{k+1}}{\Delta t^2} \tag{33}$$

$$\mathbf{u}^{k+1} \approx c^2 \Delta t^2 L \mathbf{u}^k + \Delta t^2 f + 2\mathbf{u}^k - \mathbf{u}^{k-1} \tag{34}$$

$$\mathbf{u}^{k+1} \approx (c^2 \Delta t^2 L + 2I)\mathbf{u}^k - \mathbf{u}^{k-1} + \Delta t^2 f \tag{35}$$

In order to approximate the initial conditions with sufficient accuracy, $\mathbf{u}^1$ is expanded as a Taylor series around $\mathbf{u}^0$.

$$\mathbf{u}^1 = \mathbf{u}^0 + \Delta t \frac{\partial \mathbf{u}^0}{\partial t} + \frac{\Delta t^2}{2} \frac{\partial^2 \mathbf{u}^0}{\partial t^2} + \mathcal{O}(\Delta t^3) \tag{36}$$

As it is given that $\frac{\partial \mathbf{u}^0}{\partial t} = 0$ and that $\frac{\partial^2 \mathbf{u}^0}{\partial t^2} \approx c^2 L \mathbf{u}^0 + f$ The first step can be calculated as.

$$\mathbf{u}^1 = \mathbf{u}^0 + \frac{\Delta t^2 c^2}{2} L \mathbf{u}^0 + \frac{\Delta t^2}{2} f + \mathcal{O}(\Delta t^3) \tag{37}$$

In order to guarantee a stable time stepping, the simulation is run at 99% of the corresponding Courant–Friedrichs–Lewy (CFL) number. This number is a important dimensionless number, which relates the speed of the wave to the corresponding spatial and time steps used.

$$C = \frac{c \Delta t}{\sqrt{2} h} \tag{38}$$

Running the simulation for $c = 1$ on a grid with $h = 0.02$ yields the follwing solution.

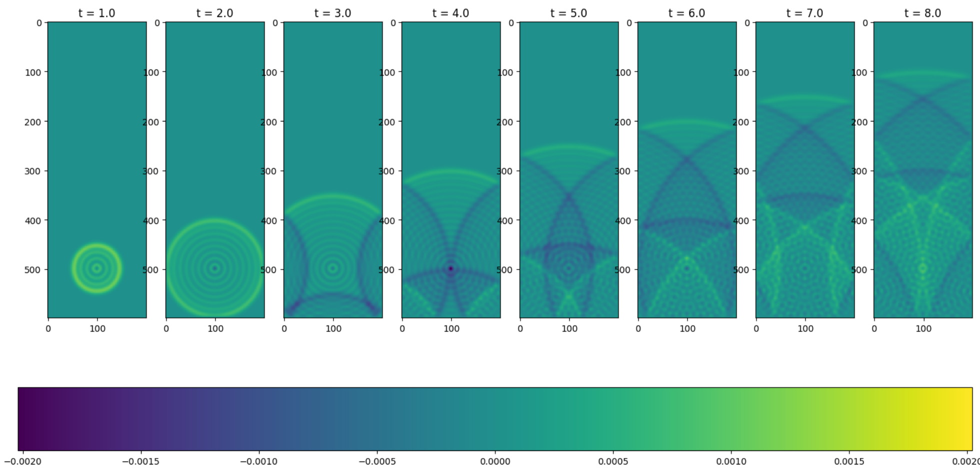


Figure 6: Solution of the wave equation on a uniform grid with step 0.02 and a wavespeed of 1

It can clearly be seen that the wavefront travels at a speed of 1 unit per secodn, which is expected as $c = 1$. Furthermore, reflection can be observed at the boundaries, which however flip the orientation of the wave, whcich would not be expected in a physical system. This behaviour comes from the use of homogeneous dirichlet conditions on the boundary of the domain. When performing a similar the simulation with $c = 2$ it can be seen that the wave travels twice as fast.

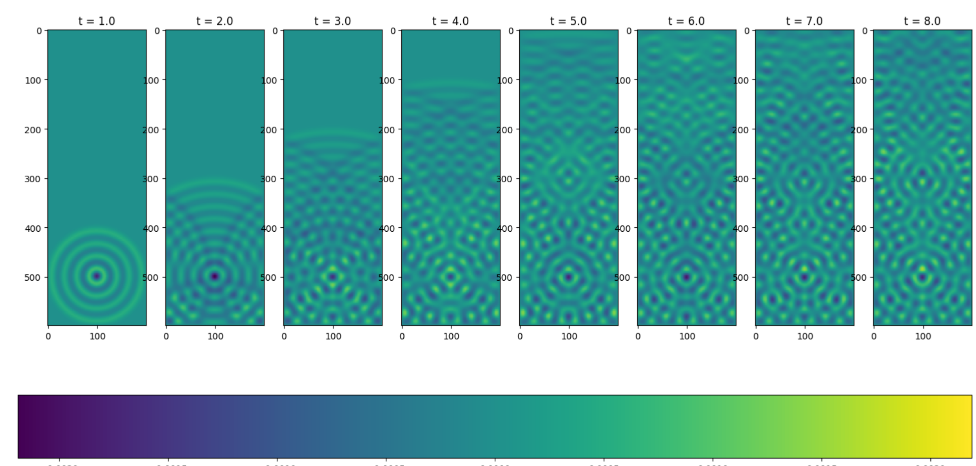
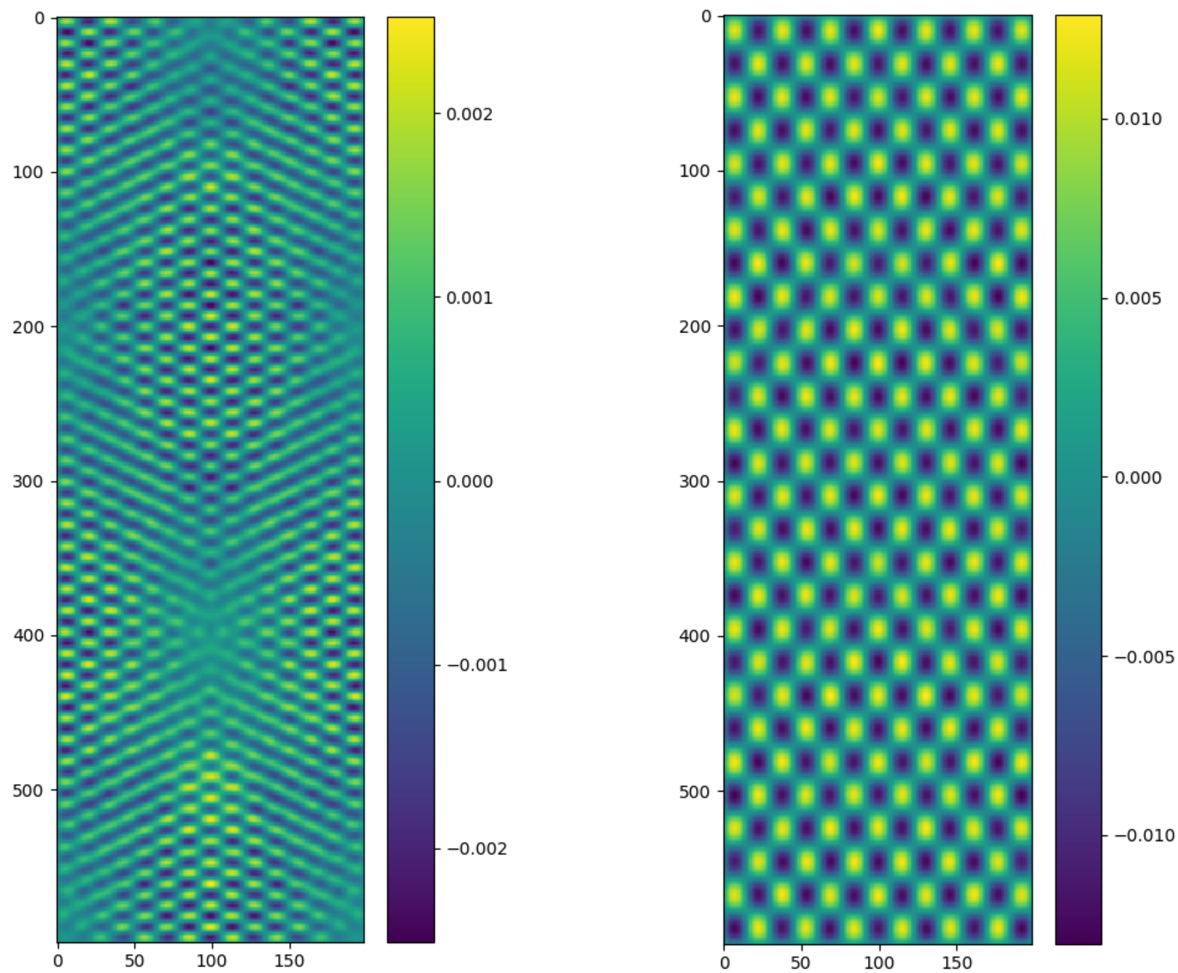


Figure 7: Solution of the wave equation on a uniform grid with step 0.02 and a wavespeed of 2

In contrast to the previous simulation, standing waves can be observed. Furthermore shifted replicas of the source are present in later solutions. Changing the speed of the wave slightly destroys these inferences. it can therefore be assumed that these patterns come from standing waves developing between the boundaries. This can be expected as the frequency of the source function permits an integer number of wavelengths to travel to each boundary. This however is also the case for $c = 1$ and no such pattern is visible at 8 seconds. Thus both simulations are run for 300 and 600 seconds to reach a quasi steady state and the results are compared. The simulation for $c = 1$ is run twice as long, to allow the waves to travel the same distance as for $c = 2$.



(a) Quasi-steady solution for c=1 at t=600s

(b) Quasi-steady solution for c=2 at t=300s

Figure 8: Comparison of the quasi-steady solutions

From these 2 figures it can be seen that the patterns emerging in page 7 and page 7, are indeed standing waves.