London School of Economics

FM442 – Quantitative Methods for Finance and Risk Analysis

Michaelmas 2019

Class solutions

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1 Risk measures

1. Write down the mathematical definition of VaR (value-at-risk) and derive ES (expected shortfall).

Solution

(a) VaR

$$\Pr[Q \leq -\mathrm{VaR}(p)] = p$$

(b) ES

$$\mathrm{ES} = \int_{-\infty}^{-\mathrm{VaR}(p)} x f_{\mathrm{VaR}}(x) dx$$

$$1 = \int_{-\infty}^{-\mathrm{VaR}(p)} f_{\mathrm{VaR}}(x) dx = \frac{1}{p} \int_{-\infty}^{-\mathrm{VaR}(p)} f_q(x) dx$$

2. What is a coherent risk measure?

Solution

Consider two real-valued random variables: X and Y. A function

$\varphi(.) : X, Y \to \mathbb{R}$ is called a coherent risk measure if it satisfies for X, Y and constant c

(a) Monotonicity

$$\text{if } X, Y \in V \text{ and } X \leq Y, \text{ then, } \varphi(X) \geq \varphi(Y)$$

(b) Subadditivity

$$\text{if } X, Y, (X+Y) \in V, \text{ then, } \varphi(X+Y) \leq \varphi(X) + \varphi(Y)$$

(c) Positive homogeneity

$$\text{if } X \in V \text{ and } c > 0, \text{ then, } \varphi(cX) = c\varphi(X)$$

(d) Translation invariance

$$\text{if } X \in V \text{ and } c \in \mathbb{R}, \text{ then, } \varphi(X+c) = \varphi(X) - c$$

3. Suppose you own two assets, A and B, with payoffs that are independent of each other, with each asset returning either 0 with probability 0.9 or -100. Is VaR(5%) sub–additive?

Solution

The VaR for A is:

$$\begin{aligned} \text{VaR}^{5\%}(A) =& 100 \\ \text{VaR}^{15\%}(A) =& 0 \end{aligned}$$

There are 2 possible asset allocations possible A or $0.5A + 0.5B$.

The possible outcomes for the portfolio $(0.5A + 0.5)$ are:

Asset	probability
$A = 0, B = 0$	$0.1^2 = 0.81$
$A = -100, B = -100$	$0.01^2 = 0.01$
$A = 0, B = -100 \vee A = -100, B = 0$	$1 - 0.81 - 0.01 = 0.18$

so

$$\text{VaR}^{5\%}(A+B) = 100 \leq \text{VaR}^{5\%}(A) + \text{VaR}^{5\%}(B) = 200$$

so subadditive.

4. Consider the assets in the last question. Is VaR(15%) sub–additive?

> **Solution**
> Given the the way the probabilities work out,
>
> $$\text{VaR}^{15\%}(A) = \text{VaR}^{15\%}(B) = 0$$
>
> and
>
> $$\text{VaR}^{15\%}(A + B) = 100$$
>
> so not subadditive.

5. Give one example of a traded asset that could lead to a sub–additivity violation of VaR, carefully explaining why the payoff structure of this asset would lead to that conclusion.

> **Solution**
> An actual asset that where most of the time the return is zero,
>
> but occasionally very negative). This could be the exchange the rate of a country with a peg, that occasionally devalues, short deep out of the money options, junk bonds, etc .

6. Suppose you own 1$ million worth of both stocks A and B. Stock A is a small cap stock, with a market capitalization of 3 million, while stock B is a large cap stock, with a market capitalization of 3 billion. As a consequence, one of the axioms of a coherent risk measure is likely to be violated for VaR on one of the stocks but not the other. Which stock is it and why would that be the case?

> **Solution**
> This is about homogeneity.
>
> $$c \times Risk(A) = Risk(c \times A)$$
>
> for an asset A and constant C. This would likely hold for the big cap, but not for the small cap.

7. State the three steps in VaR calculations.

Solution

(a) To specify probability p, commonly used are VaR levels of 1% – 5%, but it is mainly determined by how the risk managers wishes to interpret the VaR number

(b) To decide holding periods, commonly 1 day, but it can vary depending on different circumstances

(c) Identification of probability distribution of the profit and loss of the portfolio