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Assignment: Modifying the Exner Model

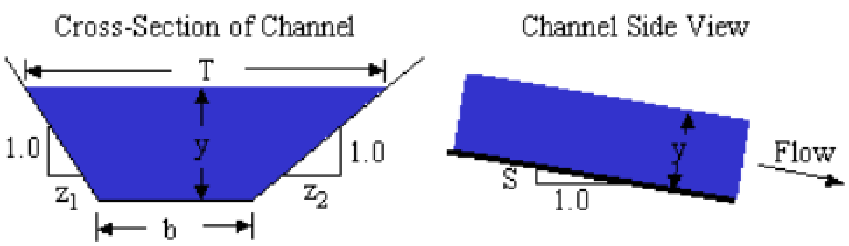
I. Introduction

The objective of this report is to give the reader insights into the functionality of the "Backwater_SS_Bedforms.xlsm" calculation sheet. Said sheet is able to model the sediment transport and evolution of a channel's bed level for a channel defined by the user. Apart from the channel's bed slope, the user is also able to define changes in the cross section (narrowing or widening), a bed rock level at which the channel bed will not lower any more in case its being eroded and variable levels of discharge, upstream and downstream boundary conditions. Finally, the user is also able to define a constant rate of extraction or addition of sediment to the channel to simulate "mining".

II. Theory

1. Hydraulics

From a given water depth, the geometric properties for a trapezoidal channel are calculated using the equations in Figure 1. These equations consider two different side slopes for the banks so even an irregular trapezoidal channel could be modelled. If the base is given a value of 0, a triangular cross section can also be modeled.



$$R_h = \frac{A}{P} \qquad A = \frac{y}{2}(b+T)$$

$$P = b + y\left(\sqrt{1+z_1^2} + \sqrt{1+z_2^2}\right) \qquad T = b + y(z_1 + z_2)$$

Figure 1. Geometric properties of the channel
Source: LMNOEngineering

To analyse the wave propagation and regime of the flow, calculations for the velocity, celerity and Froude number are used. The celerity describes the velocity of propagation of a perturbation or wave within the flow, and the Froude number is the ration between the velocity of the flow in the channel and the celerity.

$$v = \frac{Q}{A} \qquad C = \sqrt{\frac{gA}{T}} \qquad Fr = \frac{v}{C}$$

Total energy in the channel is the addition of the potential and kinematic energy. For its use in this model, the formulation considers each energy as a “head” and the final calculation has units of meters above sea level.

$$H = z + h + \frac{v^2}{2g}$$

To consider the resistance to the flow and roughness of the channel’s bed, the Manning formulation is used. The Manning formula describes the discharge that passes through a channel’s cross section, given its water depth, geometric properties and friction slope. The friction slope is the change in total energy from to subsequent cross sections along the channel.

$$Q = \frac{1}{n} A R_h^{\frac{2}{3}} \sqrt{S_f} \qquad S_f = \left(\frac{Qn}{A R_h^{\frac{2}{3}}} \right)^2$$

The resistance of the flow creates energy losses between subsequent cross sections. The idea for solving a backwater curve, is to find the correct water depths that correctly estimate the energy loss produced by the resistance to flow according to the Manning formulation. Starting from a certain cell (1) with known properties, the water depth in the subsequent cross section (2) can be calculated with the following approach.

$$H_1 = H_2 + \Delta H_{1-2} \qquad \Delta H_{1-2} = \frac{1}{2}(x_2 - x_1)(S_{f1} + S_{f2})$$

The backwater curve can be solved forwards and backwards starting from upstream and downstream boundary conditions respectively. Because of this, one of the solutions has to be picked as an overall correct solution. This selection is determined by the amount of momentum in a given cross section. The calculation for momentum is also generalized for a trapezoidal channel.

$$\bar{h} = \frac{h}{3} \frac{2b + T}{b + T} \qquad M = gA\bar{h} + \frac{Q^2}{A}$$

2. Sediment Transport

2.1 The Shields Diagram

Flow along the channel creates a shearing stress between the water and the materials on the channel’s bed and sides. If this stress is high enough, the water will start to erode material from the channel. Shear stress is calculated using the following equation:

$$\tau_0 = \rho_w g R_h S_f$$

Where ρ_w is the density of the water (1000 kg/m^3).

To assess if the shear stress is high enough to erode the bed of the channel and trigger a sediment transport process, the Shields diagram is used. The Shields diagram shows how the dimensionless critical shear stress (or Shields parameter) is a function of the shear Reynolds number.

$$\tau_* = \theta = \frac{\tau}{\rho_w g d_{50} R} \qquad Re_* = \frac{u_* d_{50}}{\nu} = \frac{\sqrt{\tau/\rho_w}}{\nu}$$

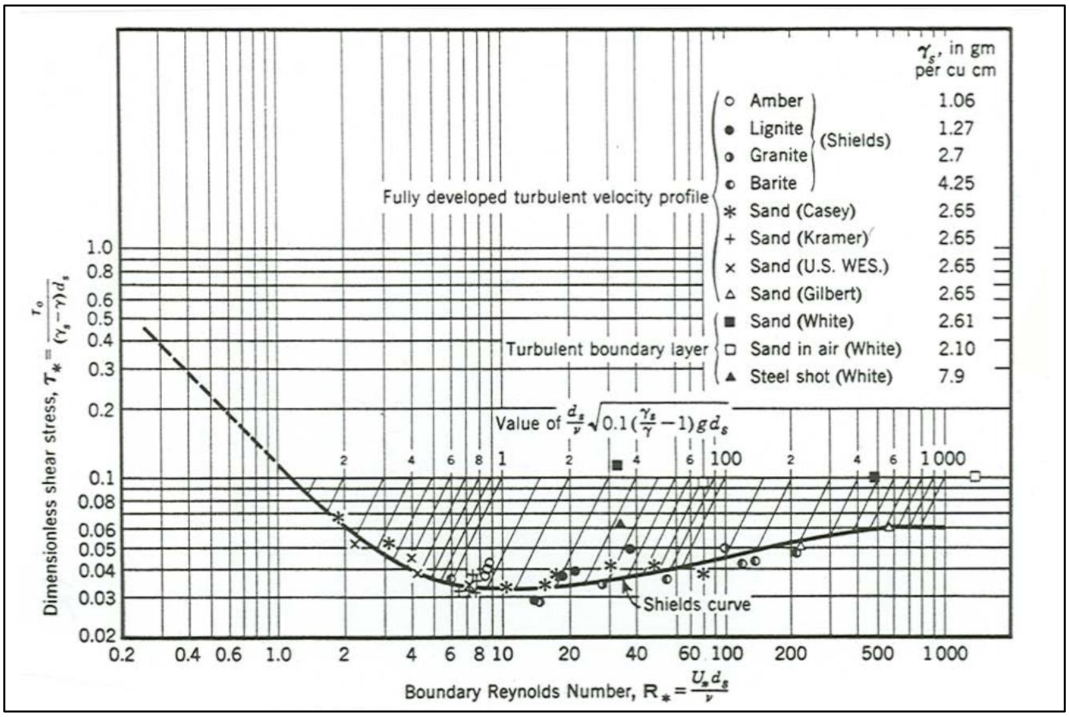


Figure 2.Shields diagram
Source: Vanoni (1975)

The R that appears in the Shields parameter calculation corresponds to the submerged specific weight of the sediment material. In general, this term can be used as the following:

$$R = \frac{(\rho_s - \rho_w)\cancel{g}}{\rho_w \cancel{g}} = \frac{2650 - 1000}{1000} = 1.65$$

The line that appears in Figure 2 shows a separation between regions also known as the “threshold of motion”. Any combination of Reynolds particle number and dimensionless shear stress plotted above this line signifies that the is a sediment transport process taking place.

To ease the use of the Shields diagram, Cao, Pender and Meng (2006) developed regression equations to calculate the critical dimensionless shear stress using the particle Reynolds number. This last parameter is solely determined by fluid and sediment characteristic, and is useful because knowing the critical Shields parameter of a particular sediment diameter makes the analysis of the threshold of motion easier.

$$Re_p = \frac{d_{50}\sqrt{Rgd_{50}}}{\nu}$$

$$\begin{array}{lll} \theta_c = 0.1414 Re_p^{-0.2306} & & Re_p < \approx 6.61 \\ \theta_c = \frac{\left[1+\left(0.0223 Re_p\right)^{2.8358}\right]^{0.3542}}{3.0946 Re_p^{0.6769}} & \text{for} & Re_p \in (6.61, 282.84) \\ \theta_c = 0.045 & & Re_p > \approx 282.84 \end{array}$$

2.2 Bedload Sediment Transport

To calculate a rate of sediment that is eroded from the bed and carried by the flow, the equations from Einstein and Brown (1950) can be used. In these equations, the dimensionless rate of solid discharge is calculated using only the Shields parameter. As a reminder to the reader, the Shields parameter (θ) can also be called the dimensionless shear stress (τ^*).

For different ranges of Shields parameter, different equations apply as follows:

$$
\begin{aligned}
q_s^* &= 2.15e^{-0.391/\tau^*} & & \tau^* < 0.18 \\
q_s^* &= 40\tau^{*3} & \text{for} \quad & 0.18 < \tau^* < 0.52 \\
q_s^* &= 15\tau^{*3/2} & & 0.52 < \tau^*
\end{aligned}
$$

The calculated solid discharge rate is dimensionless. In order to give it units of m^2/s, the following transformation is applied. If multiplied by a certain length, this las parameter will yield units of m^3/s which is a volumetric rate for sediment.

$$q_s = \sqrt{gd_{50}R}d_{50}q_s^*$$

To avoid confusion, this bedload sediment transport rate will be called q_b further in this report.

2.3 Bedforms

In order to consider the bedform roughness, the Einstein stress decomposition as proposed by Bateman (2020) can be used. This decomposition separates the total bed stress generated by the water in a channel into the stress generated by the skin friction and the surface of bed (or bedform).

Engelund and Hansen (1967) developed a relationship between stresses in which the stress to the skin friction can be calculated from the total stress.

$$\tau_s^* = 0.06 + 0.4(\tau^*)^2$$

The total velocity (U) in a channel can be calculated as a function of the depth of the water generated by the skin friction (H_s). In the following equation, $k_s = 3\ d_{90}$.

$$\frac{U}{\sqrt{gH_sS}} = 8.32\left(\frac{H_s}{k_s}\right)^{1/6}$$

Considering a rectangular channel. This last equation the be used to calculate the total water depth (H).

$$H = \frac{Q/b}{U}$$

Simplifying:

$$H = \frac{Q/b}{8.32\left(\frac{H_s}{k_s}\right)^{1/6}\sqrt{gH_sS}}$$

Usually, the term for the slope (S) is taken as the slope for the bottom of the channel (S_0) but, in a backwater calculation, the friction slope (S_f) can also be used.

2.4 Suspended Sediment Transport

To correctly consider the relation of entrainment of material in suspended sediment transport, equilibrium conditions have to be considered. The morphodynamic formulation given by Parker (2004) can be used.

$$q_{ss} = \frac{Eu_*H}{\kappa}I$$

In the previous equation E, is the entrainment relation of the sediment, u_* is the shear velocity of the flow, H is the total water depth in a cross section, κ is the von Karman constant (0.4) and I is an integration term that solves the Rouse-Vanoni profile for suspended sediment concentration.

$$u_* = \sqrt{\tau_0/\rho_w}$$

For the entrainment relation (E), the following two equations are given by Wright and Parker (2004):

$$E = \frac{AZ_u^5}{1 + \frac{A}{0.3} Z_u^5} \qquad Z_u = \frac{\sqrt{gH_s S_f}}{v_s(R_f)} Re_p^{0.6} S_f^{0.07}$$

$$A = 5.7 \times 10^{-7}$$

The fall velocity (v_s) is the rate at which a particle falls in still water and it depends mainly on the shape and size of the particle. In this case, it is calculated from the dimensionless fall velocity (R_f) for which Dietrich (1982) gives a regression equation using the particle Reynolds number.

$$v_s = R_f \sqrt{Rgd_{50}} \qquad R_f = exp\left\{-b_1 + b_2 \ln(Re_p) - b_3[\ln(Re_p)]^2 - b_4[\ln(Re_p)]^3 + b_5[\ln(Re_p)]^4\right\}$$

Where: b_1 = 2.891384; b_2 = 0.95296; b_3 = 0.056835; b_4 = 0.002892 and b_5 = 0.000245.

Finally, for the suspended sediment concentration profile the following integration term is presented in order to “sum up” different concentrations of sediment along the height of the water depth.

$$I\left(\frac{u_*}{v_s}, \frac{H}{k_c}, \varsigma_b\right) = \int_{\varsigma_b}^{1} \left[\frac{(1-\varsigma)/\varsigma}{(1-\varsigma_b)/\varsigma_b}\right]^{\frac{v_s}{\kappa u_*}} \ln\left(30\ \frac{H}{k_c}\varsigma\right) d\varsigma$$

Where ς is the height above the channel’s bed. Wright and Parker use the starting value for integration (ς_b) as 0.05 or 5% of the total water height.

Finally, the composite roughness height (k_c) can be computed from the friction resistance factor (Cz).

$$Cz = c_f^{-1/2} = \frac{v}{u_*} \qquad k_c = \frac{11H}{e^{\kappa\, Cz}}$$

2.5 Bed Evolution

To model the evolution of the bed due to sediment transport, the Exner model is used. The Exner model gives an equation for the conservation of mass of sediment in quasi-unsteady flow.

$$(1-\lambda)\frac{\partial z}{\partial t} + \frac{\partial q_t}{\partial x} = 0$$

λ is the porosity of the bed and can be used as 0.4 or 40%. q_t is the total rate of sediment discharge and is the addition of bedload sediment discharge and suspended sediment discharge.

In the quasi-unsteady flow approach, dt is calculated as the minimum ration between the distance between cross sections (dx) and the shear velocity in a cross section.

$$\partial t = \frac{\partial x}{u_*}$$

III. Implementation

The ***Backwater_SS_Bedforms.xlsm*** presents the user a main calculation sheet called ***Grid-BC***. In this sheet, the user is able to define their desired calculation grid as well as other input parameters for the application of the model, this is shown in Figure 3. All the cells in the sheet follow the colour codes:

Input
Linked Cell
Calculation

Aspects such as narrowing and widening of the channel and the bedrock level can be changed in the "user generated grid" before the calculation process starts. In addition, to solve a specific problem for this assignment, a constant rate of change for q_t has been added to simulate "mining". For example, the user can assign a constant rate of extraction of materials (sediment) to a particular cell in their grid.

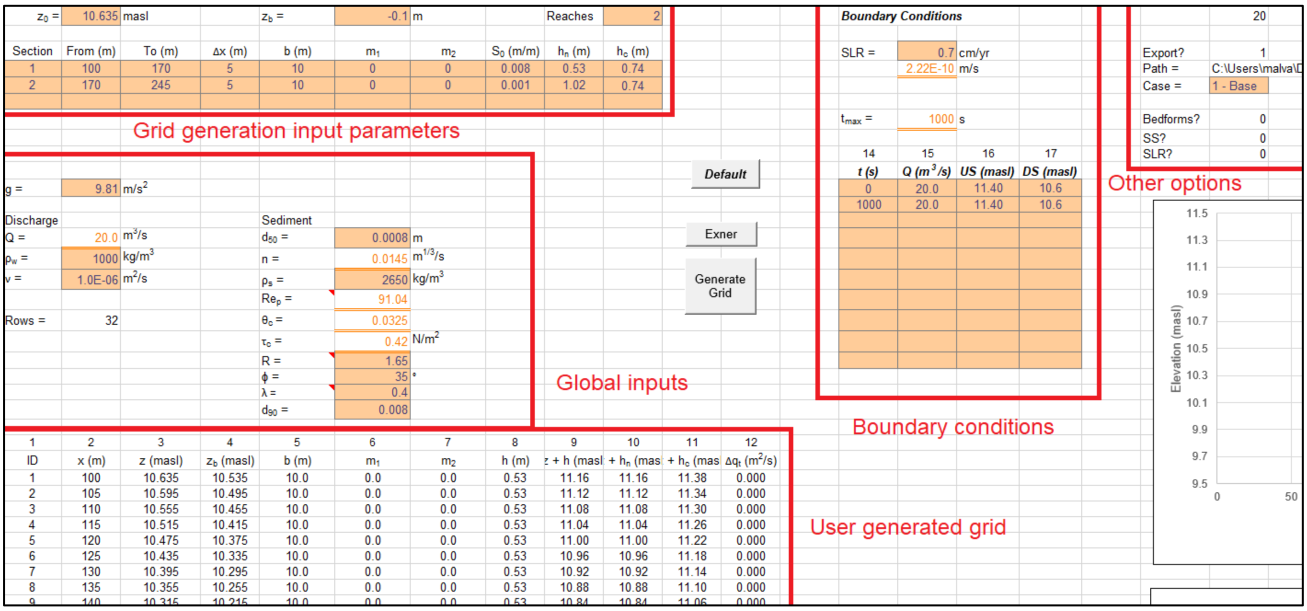


Figure 3. Main window of "Backwater_SS_Bedforms.xlsm"

As soon as the user clicks on the ***Generate Grid*** button, a calculation grid is generated and the graph on the right is updated to show the user a side of view of their channel as well as the computed normal and critical depths all along the channel for their initial discharge. This is done in order to help the user pick suitable boundary conditions for their simulation. An example is shown in Figure 4. Normal water depths all along the channel are also used as starting conditions for the calculation.

The normal and critical depths are calculated using a Newton-Raphson scheme. This scheme is implemented in Excel as a function and can be seen in ***Module 1*** of the VBA code. The routine to generate the grid from the user inputs can found in ***Module 2*** of the VBA code.

The ***Default*** button writes default input values adapted from the calculation sheet "BackWaterFRM8-V0.xlsm" developed by Bateman (2020).

The "global inputs" and "boundary conditions" areas are self-explanatory. In the latter, the user is allowed to input their desired hydrograph and boundary conditions. The downstream boundary condition can be used, for example, if the user wants to simulate waves as an oscillating water level downstream or even, sea level rise.

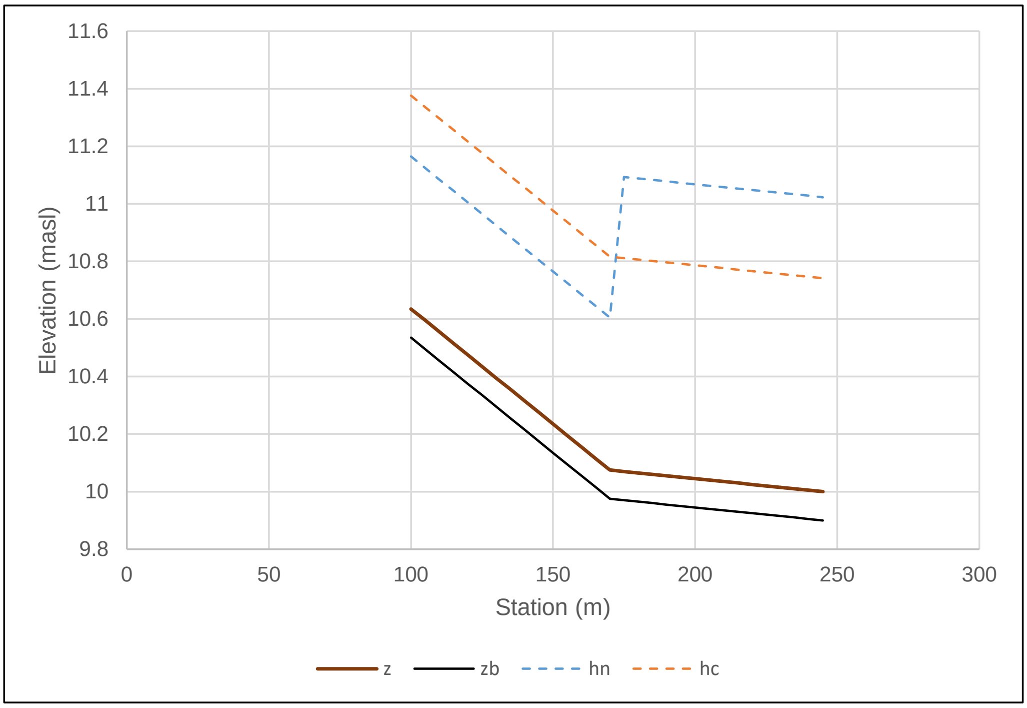


Figure 4. Example of a user defined rectangular channel of 145 m, with b = 10 m, two different slopes, a Manning roughness of 0.0145 s/m$^{1/3}$ and an initial discharge of 20 m^3/s

Continuing the explanation of Figure 3, in the "other options" area, the user is able to export a plot of the results for each simulation time step. The user must write down a *Path* pointing to a folder within their computer and inside that folder, another folder must exist where the plots will be exported. The name of this last folder must match the name in the *Case* field that appears in the "other options" area. Exporting this set of graphs is very helpful for the user to analyse their results and it is encouraged.

The *Bedforms*, *SS* and *SLR* fields activate different routines in the calculation engine of the worksheet. These are explained further along in this report. The user must activate these routines by writing a 1 next to the name of the routine. By default, all these routines are deactivated and therefore 0's appear next to their name.

The *SLR* routine adds a constant value to the downstream boundary condition based on the time of the current simulation as a way to describe sea level rise. The value is set to 0.7 cm/year which is the rate given for the IPCC scenario 8.5 for global mean sea level rise.

The ***US*** and ***DS*** sheets calculate the water level profile of the backwater curve. As mentioned before, the solution is based on the energy equation and is solved forwards and backwards depending on the boundary condition. For any two adjacent cross sections, all the values for cross section 1 are given by a set boundary condition or a previous solution, and the values for cross section 2 are calculated using the GRG Nonlinear method in Solver for Excel. The objective function for said method is stated as follows:

$$MIN\ f(h_2) = \left(H_1 - H_2(h_2) - \frac{1}{2}(x_2 - x_1)\left(S_{f1} + S_{f2}(h_2)\right)\right)^2$$

In ***Module 2*** of the VBA code, an iteration scheme has been developed for Solver to go through every cross section (or row) and find the correct water depth in order to fulfil either the objective function or to write the critical depth if a solution cannot be found. Additionally, restrictions for a positive value of water depth and the required Froude number depending on the boundary condition have been added. Starting from the upstream boundary condition, the solution is assumed to be supercritical so all cross sections should have a solution with Froude numbers equal or greater than 1. Starting from the downstream boundary condition, the solution is assumed to be subcritical and all cross sections should have a solution with Froude numbers equal or lesser than 1.

In the ***Solution*** sheet, a solution is picked between the values in the ***US*** and ***DS*** calculation sheets. The criteria to select said solution is the quantity of momentum calculated for the cross section, where the solution with highest momentum is always chosen.

Once a solution is found, all equations related to sediment transport are calculated within the ***Solution*** sheet. The threshold of movement is evaluated for each cross section at any given time step and if the conditions permit it; bedload sediment transport is always calculated using the Einstein and Brown equations described in Section 2.2.

If the *SS* subroutine is activated, the sheet will calculate suspended sediment transport for each cross section. The formulation follows the approach by Parker (2004) described in Section 2.4. In addition, a decomposition of the total water head is applied using, again, an approach detailed by Wright and Parker (2004). For this approach, the minimum τ^* for which the decomposition can be applied is calculated and is implemented in the sheet as the VBA code in ***Module 2***.

Knowing τ^*_{min}, the water depth related to skin friction can be calculated as follows:

$$\frac{H_s S_f}{R d_{50}} = \begin{cases} 0.05 + 0.7\left[\left(\frac{HS_f}{Rd_{50}}\right)\left(\frac{Q/B}{\sqrt{g}H^{3/2}}\right)^{0.7}\right]^{0.8}, & \frac{HS_f}{Rd_{50}} \geq \tau^*_{min} \\ \frac{HS_f}{Rd_{50}}, \quad \frac{HS_f}{Rd_{50}} < \tau^*_{min} & \end{cases}$$

With this decomposition, all equations in Section 2.4 can be applied for an estimation of suspended sediment transport. With this routine activated, the sum of total sediment transport is changed to take into account bedload and suspended sediment.

With the total sediment transport rate calculated for every cross section. The Exner equation is applied in order to calculate a new bed level (z_{new}) for the next time step. The bedrock level is imposed as a conditional statement where z_{new} cannot go lower than the bedrock (z_b). The procedure for the application of the Exner model was described in Section 2.5.

The *Bedforms* routine considers the Engelund-Hansen relationship between stresses described in Section 2.3, in order to update the Manning roughness of all the cross sections of the channel. The procedure was adapted from the sheet “Bed Forms.xlsx” developed by Bateman (2015). An initial value of H_s is estimated and then corrected using the characteristics of the flow in the channel in order to satisfy the Engelund-Hansen relationship. The correct H_s is used to calculate H, and then a new Manning value is calculated using the following equation:

$$n_{new} = \frac{1}{Q}(BH)H^{2/3}\sqrt{S_f}$$

The estimation and correction of H_s is implemented in the VBA code within ***Module 2***. The corrected Manning roughness values are used in the hydraulic and sediment transport calculations in the next iteration.

Module 3 in the VBA code contains mostly support routines that are used, for example: to save the results of an iteration, clear worksheets and copy values from the ***Solution*** sheet back to the ***US*** and ***DS*** calculation sheets.

Finally, coming back to the ***Grid-BC*** sheet, the ***Exner*** button launches the calculation. The Exner routine is contained within ***Module 2*** of the VBA code. This routine has different counters in order to manage time and iteration number. The routine can be summarized in the following steps:

1. Read the boundary conditions from ***Grid-BC*** for the current time step
2. Write the boundary conditions in the ***US*** and ***DS*** calculation sheets
3. Solve the backwater curve in ***US*** and ***DS***
4. The correct solution for each cross section is automatically chosen by the ***Solution*** sheet
5. Calculate bedforms, sediment transport and bed evolution within the ***Solution*** sheet
6. Calculate bed level and Manning roughness for the next iteration, update these values in the ***US*** and ***DS*** sheets
7. Calculate the minimum time step, advance time and proceed to the next iteration
8. Repeat from Step 1

IV. Results

1. Base Case

To simulate this case, the *Default* button in the ***Backwater_SS_Bedforms.xlsm*** file can be pressed and the simulation can be run using the *Exner* button. The simulation runs for 1000 s and all the input values are adapted from Bateman (2020).

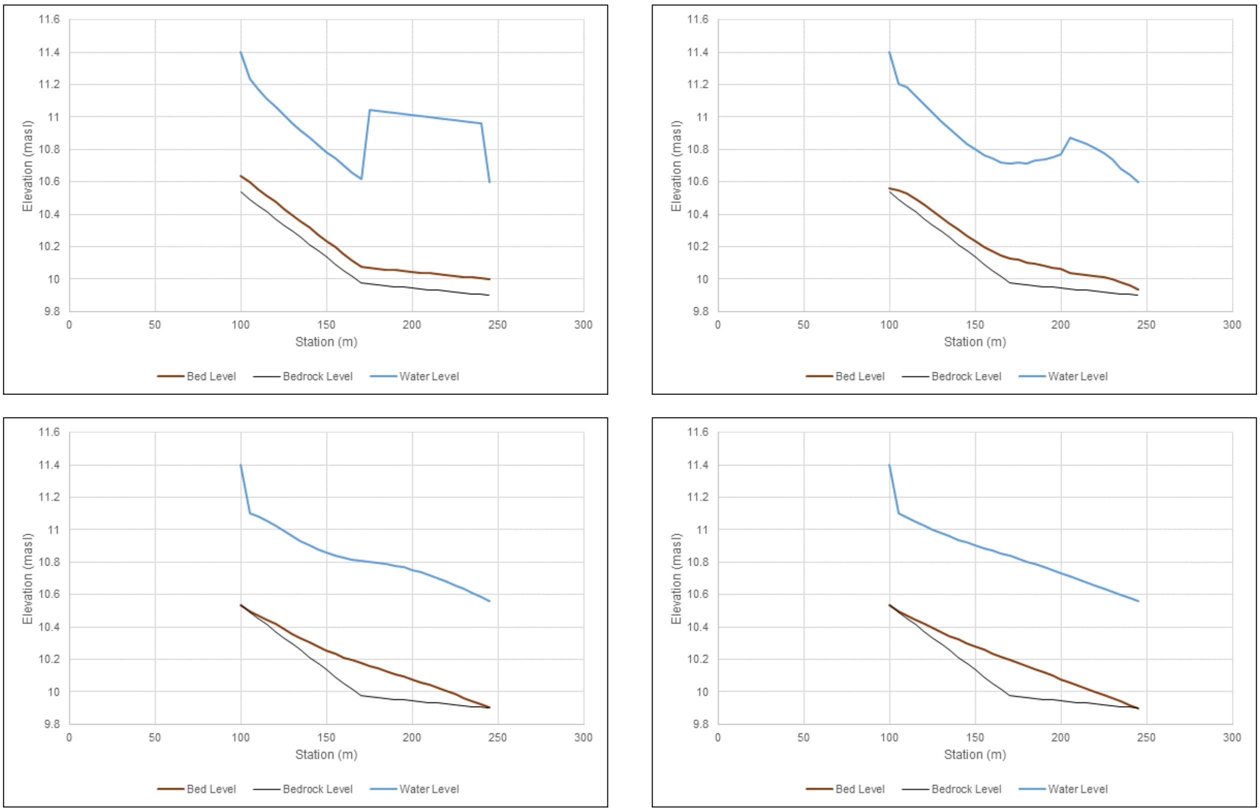


Figure 5. Results of the Base Case run for 0 s, 135 s, 500 s and 1000 s

In this case, due to the change of slope, there is a change in flow regime from supercritical to subcritical and a hydraulic jump is formed. Due to this change in flow regime and the hydraulic jump, sediment is deposited in the middle of the channel. Close to the boundary conditions, the bed is eroded until the bedrock is reached.

After some time, equilibrium is reached when a single slope for the channel's bed is formed.

2. Mining

This is the same as the base case but in the cell at 125 m, a constant extraction rate of sediment of 0.01 m^2/s has been added to simulate the extraction of materials from the channel. The results of the simulation are very similar to those in Figure 5 but, after some time and equilibrium is reached, a step has been formed in the channel's bed due to the effect of the "mining".

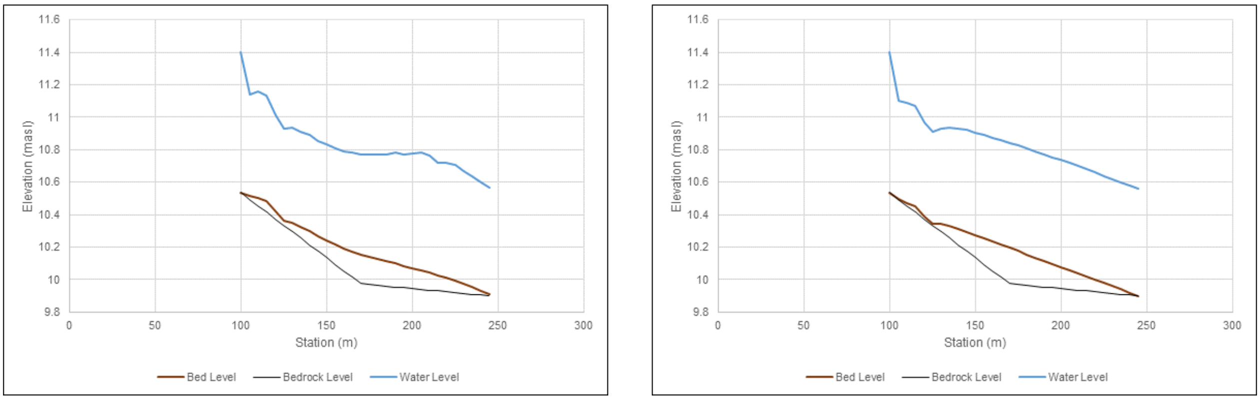


Figure 6. Results of the Mining Case run for 300 s and 1000 s

3. Narrowing and Widening

For this case, the Base Case was modified to include a narrowing of the channel from 10 m to 8 m from station 120 m to station 140 m, and a widening of the channel from 10 m to 12 m from station 200 m to station 220 m.

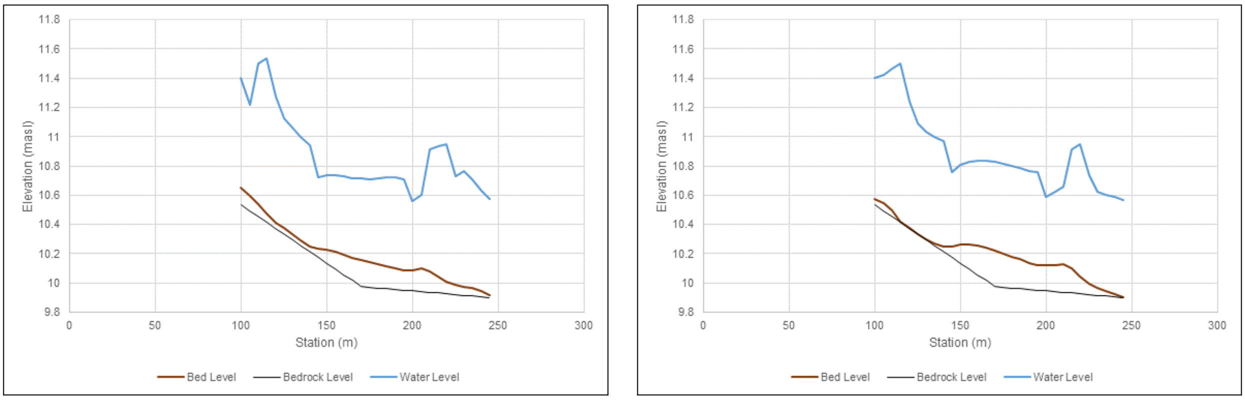


Figure 7. Results of the Narrowing and Widening Case run for 200 s and 1000 s

At the contraction, the bed is eroded until the bedrock is reached and immediately after the contraction sediment is deposited. In contrast, at the widening, due to the reduction in flow velocity and increased water levels, sediment is deposited within the 20 m where the widening takes place.

The water level profile is very variable due to the very irregular channel bed, but equilibrium is reached when the simulation is finished.

4. Hydrograph and Waves

In order to simulate a hydrograph, a gradual increase from 20 to 25 m^3/s in imposed in the inlet (upstream boundary condition) of the channel with the properties of the base case. To improve the behaviour of the model, a similar increase is also applied to the upstream boundary condition. To simulate the waves generated by the tides, an oscillation of 20 cm is added in the downstream boundary condition.

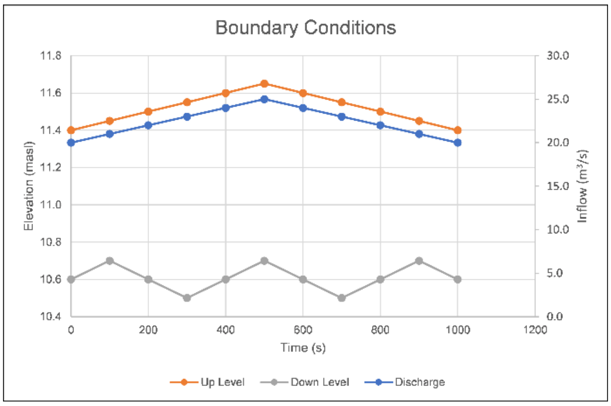


Figure 8. Boundary Conditions for Case 4

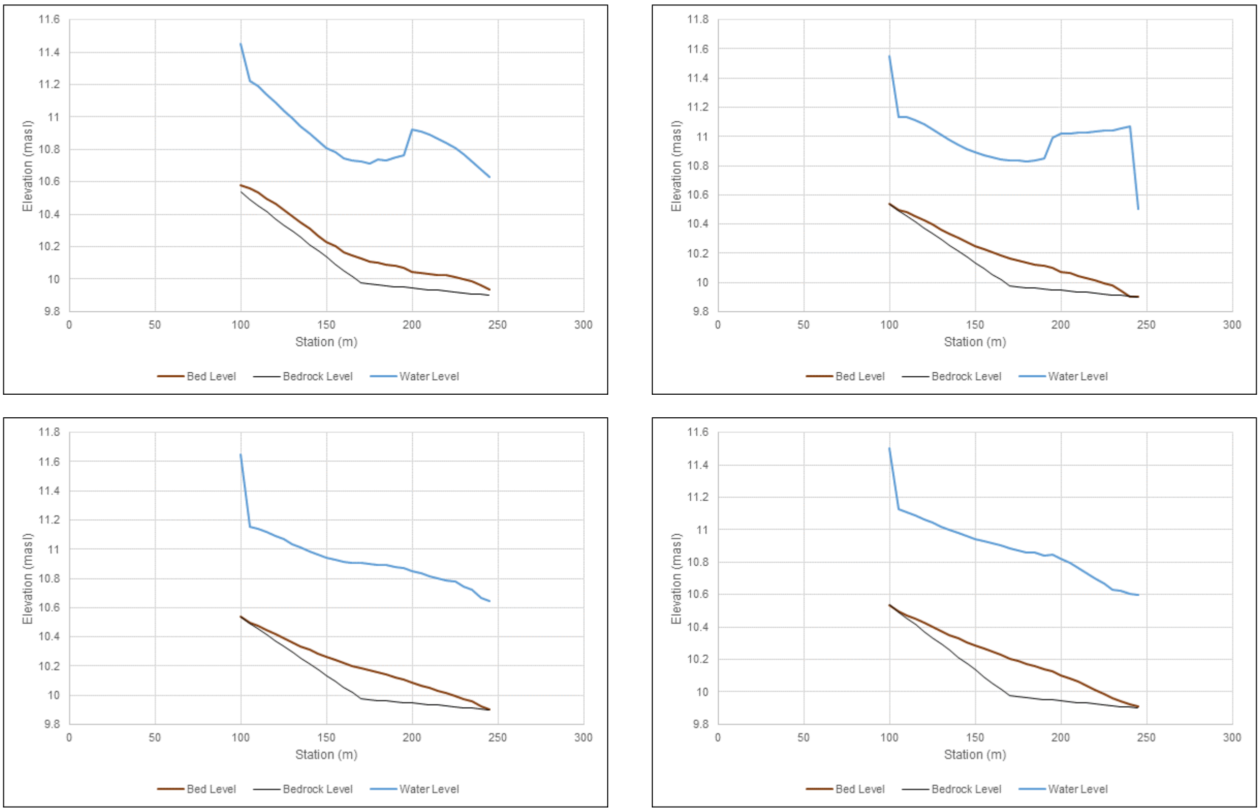


Figure 9. Results for the Hydrograph and Waves Case for 100 s, 350 s, 550 s and 850 s

The water levels calculated along the channel are very sensible to changes in the discharge and/or boundary conditions. As it can be seen in Figure 9, although an equilibrium is reached close to the end of the simulation, the results tend to change and deviate from the equilibrium every 100 s when the boundary conditions or discharge are changed.

5. Sea Level Rise in 100 years

In general, simulations of 1000 s using the ***Backwater_SS_Bedforms.xlsm*** sheet take between 15 to 18 min of real time. This means that a long-term simulation of 100 years is not feasible. A simplification is proposed where the final bed level from Case 2 is inputted into the model as the starting bed and the downstream boundary conditions is increased by 70 cm to comply with the estimation of mean sea level rise for the IPCC scenario 8.5.

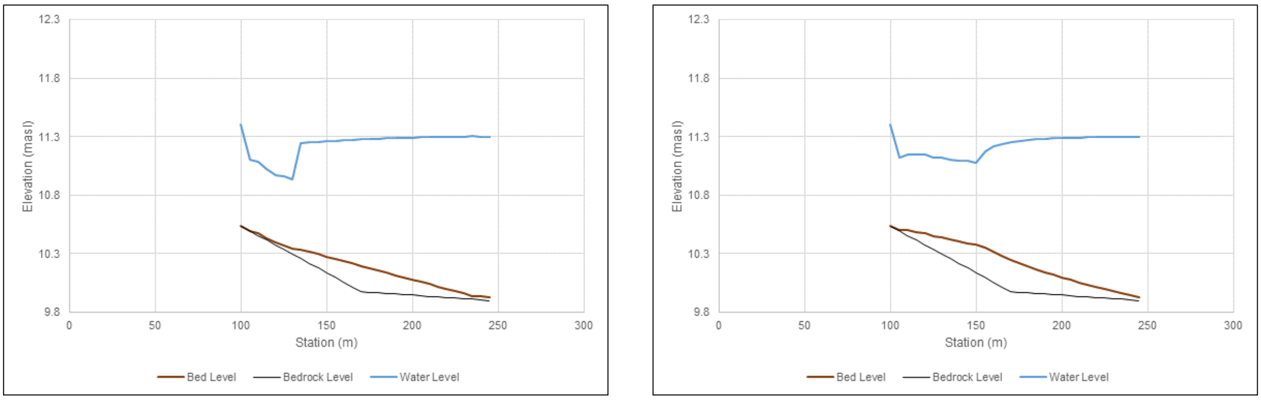


Figure 10. Results for the Sea Level Rise Case for 25 s and 1000 s

For a channel that has reached equilibrium, a large increase in the water level of the downstream boundary condition will make it so sediment starts to be deposited along the channel.

6. Bedforms

The implementation of Bedforms by themselves within the model are used to update the Manning roughness value at every time step in order to more accurately take into account the changing characteristics of the gradually varied flow profile. With the *Default* button, a Manning roughness of 0.0145 s/m$^{1/3}$ is used for all the cross sections of the channel, but, if the Bedforms routine is activated, before the first iteration and considering the slope and discharge in the channel, said value is corrected to 0.037 s/m$^{1/3}$ for the upstream reach of the channel and 0.035 s/m$^{1/3}$ for the downstream reach.

Higher Manning roughness values also tend to make the flow all along the channel subcritical. This is helpful for the suspended sediment routine because the equations shown in Section 2.4, where developed for applications in subcritical flow.

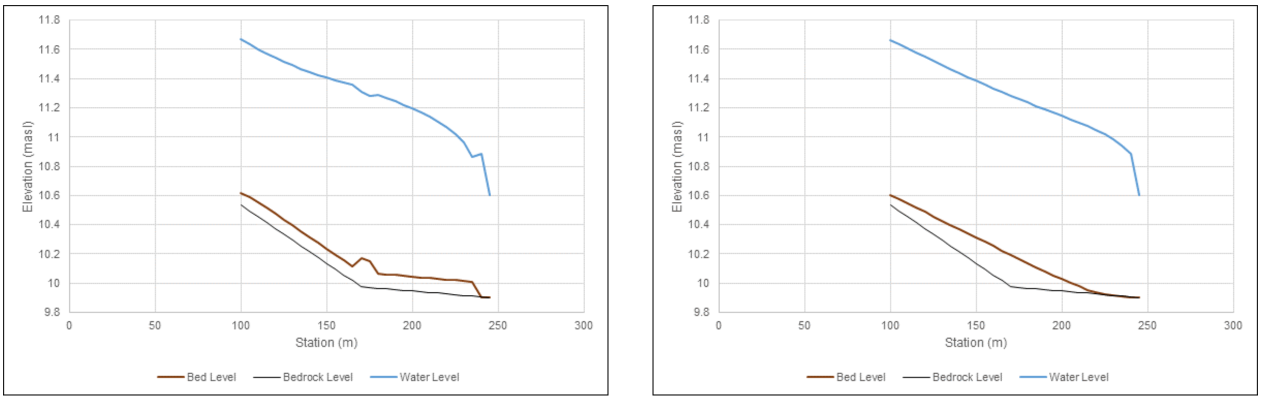


Figure 11. Results for the Bedforms Case for 15 s and 600 s

At 15 s, the program has corrected that Manning roughness value for all the cells in the calculation grid and thus, higher water levels are computed. The upstream boundary condition slowly makes it so the bed is eroded until the bedrock is reached, and a little bit of a steeper equilibrium slope is reached in comparison with the base case. Roughness values are updated all throughout the simulation.

7. Suspended Sediment

To incorporate bedforms into the base case, it is encouraged that the user defines a higher Manning roughness for the whole channel as a global parameter, or activates the Bedforms module. This is because the equations described in Section 2.4 were developed to be applicable in a subcritical flow regime.

If a section of the channel becomes supercritical, the model is still able to calculate but it will reach a "wrong" solution and start to oscillate. This can be seen in Figure 12.

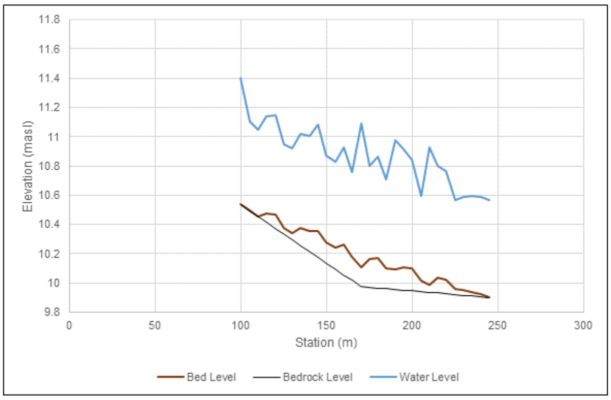


Figure 12. Incorrect results obtained for a supercritical flow regime by the Suspended Sediment module

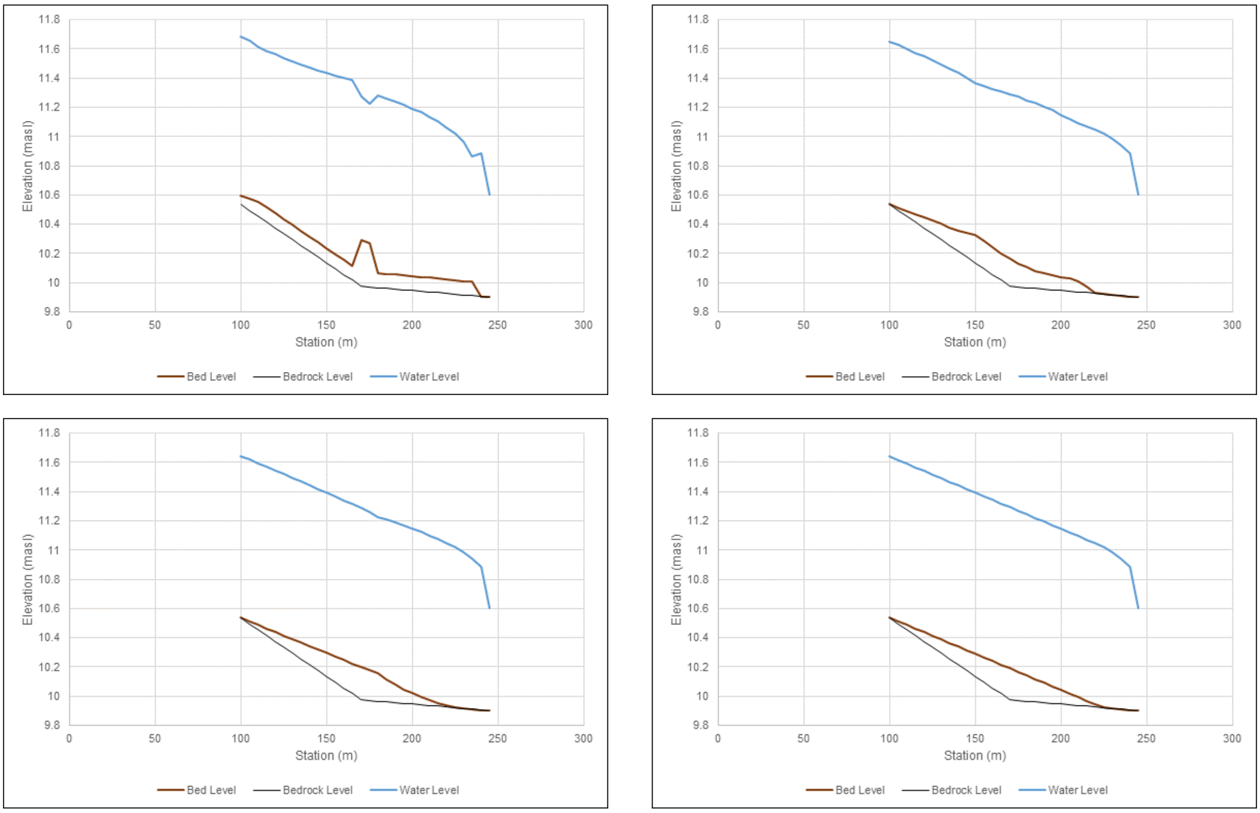


Figure 13. Results for the Suspended Sediment Case for 15 s, 175 s, 300 s and 1000 s

A correct solution with the Suspended Sediment module activated is shown in Figure 13. This solution does not differ very much from the solution with only the Bedforms routine solution activated. This is because the calculated rate of suspended sediment transport adds to the bedload sediment and the equilibrium of the bed is reached faster.

Between 175 and 300 s, the movement of a ripple downstream can be appreciated. This is the effect of the channel having a higher sediment transport rate due to the combined action of the bedload and suspended sediment.

8. Slot

Finally, to test the capabilities and the robustness of the model (and to have a little bit of fun) a highly irregular bed is set as a test case with the Bedforms and Suspended Sediment routines activated. The result is that sediment is deposited within the depression defined in the channel, and eventually equilibrium is reached once again.

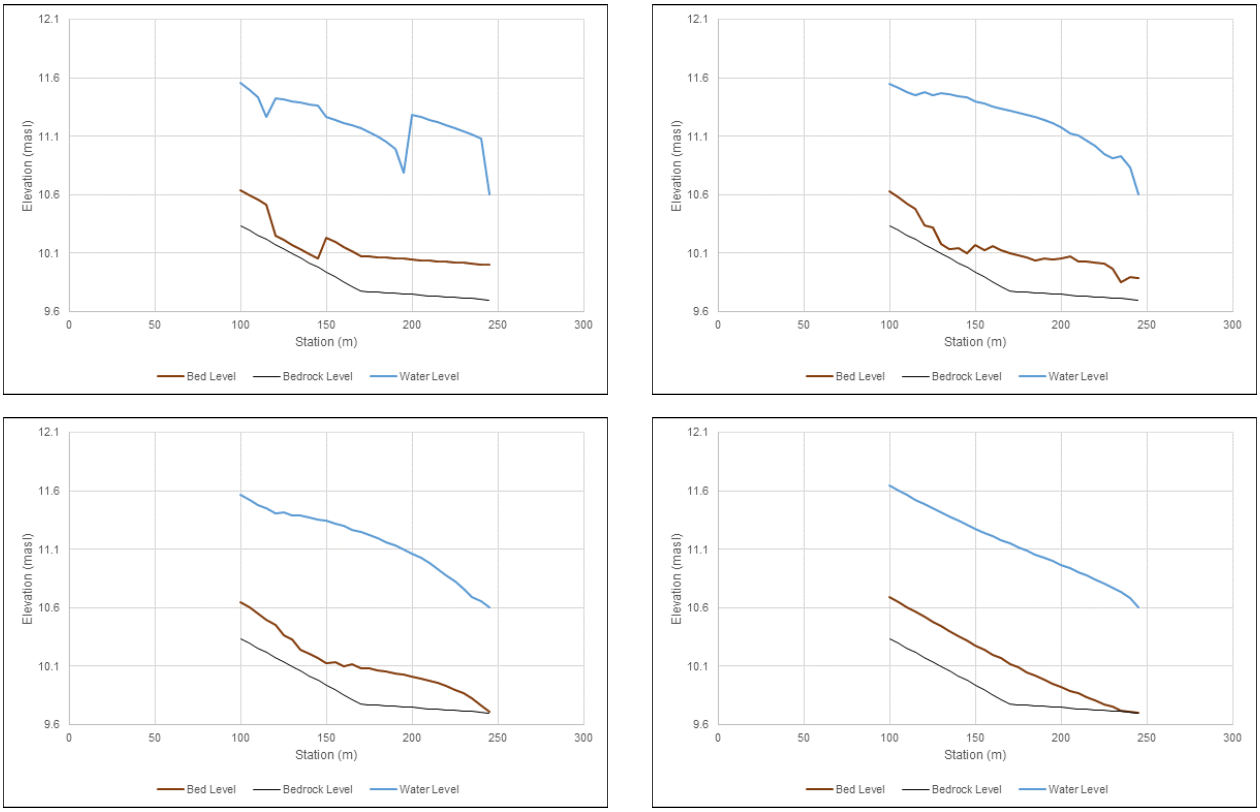


Figure 14. Results for the Slot Case for 0 s, 15 s, 250 s and 650 s

V. References

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