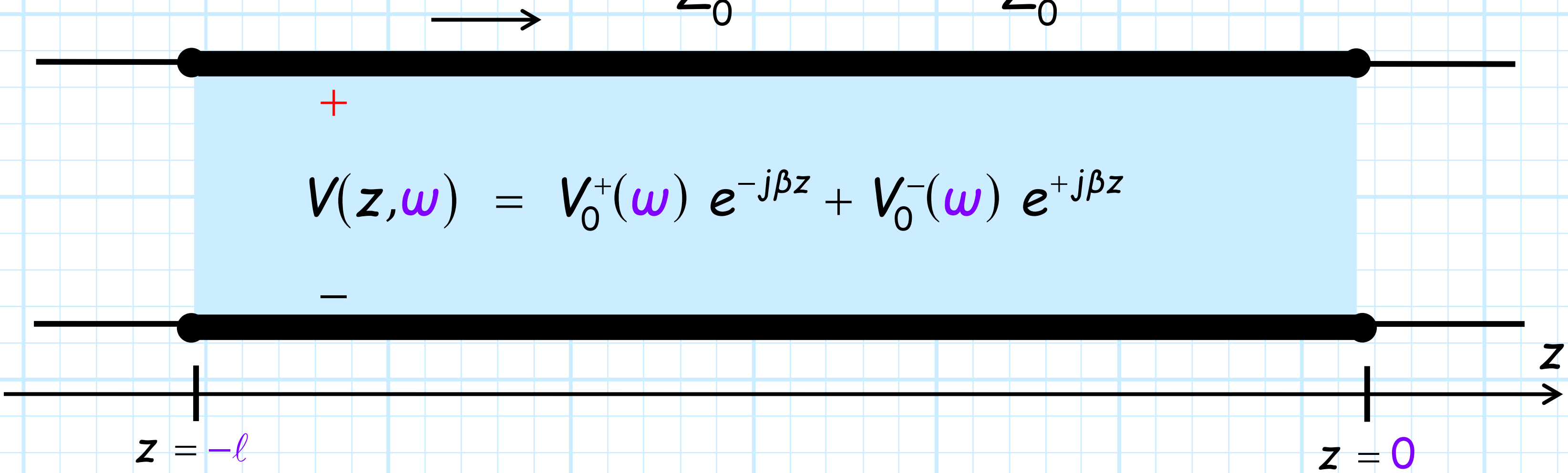
Energy Flow

Q: *At what* ***rate*** *does* ***energy*** *flow along a transmission line?*

$$I(z,\omega) = \frac{V_0^+(\omega)}{Z_0} e^{-j\beta z} - \frac{V_0^-(\omega)}{Z_0} e^{+j\beta z}$$

$$V(z,\omega) = V_0^+(\omega)\, e^{-j\beta z} + V_0^-(\omega)\, e^{+j\beta z}$$



A: **The** time averaged rate of energy flow (joules/sec)—at some point z along the transmission line—is:

$$P(z) = \frac{1}{2} Re\left\{ V(z)\, I^*(z) \right\}$$

Make this make sense to you

So, let's first examine **this** product:

$$V(z)I^*(z) = \left(V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}\right)\left(\frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{+j\beta z}\right)^*$$

$$= \left(V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}\right)\left(\frac{\left(V_0^+\right)^*}{Z_0} e^{+j\beta z} - \frac{\left(V_0^-\right)^*}{Z_0} e^{-j\beta z}\right)$$

$$= \frac{\left|V_0^+\right|^2 e^0}{Z_0} - \frac{V_0^+\left(V_0^-\right)^* e^{-j2\beta z}}{Z_0} + \frac{\left(V_0^+\right)^* V_0^- e^{+j2\beta z}}{Z_0} - \frac{\left|V_0^-\right|^2 e^0}{Z_0}$$

$$= \frac{\left|V_0^+\right|^2}{Z_0} - \frac{\left|V_0^-\right|^2}{Z_0} + \left[\frac{V_0^+\left(V_0^-\right)^*}{Z_0} e^{-j2\beta z}\right]^* - \left[\frac{V_0^+\left(V_0^-\right)^*}{Z_0} e^{-j2\beta z}\right]$$

Make this make sense as well

Now, using the fact that for a complex number c:

$$c^* - c = -j2\, Im\{c\}$$

Applying this to the **last two terms** of the previous result:

$$\left(\frac{V_0^+ \left(V_0^-\right)^*}{Z_0} e^{-j2\beta z}\right)^* - \left(\frac{V_0^+ \left(V_0^-\right)^*}{Z_0} e^{-j2\beta z}\right) = -j2\, Im\left\{\frac{V_0^+ \left(V_0^-\right)^*}{Z_0} e^{-j2\beta z}\right\}$$

And so we can finally conclude:

$$V(z)\, I^*(z) = \frac{\left|V_0^+\right|^2}{Z_0} - \frac{\left|V_0^-\right|^2}{Z_0} - j2\, Im\left\{\frac{V_0^+ \left(V_0^-\right)^*}{Z_0} e^{-j2\beta z}\right\}$$

Whew! This simplified nicely

Note of the **three terms** above, the **first two** are purely **real**, while the **third** term is purely **imaginary**.

Thus, power depends on the **first two terms only:**

$$P(z) = \frac{1}{2} Re\{V(z)\, I^*(z)\} = \frac{\left|V_0^+(\omega)\right|^2}{2Z_0} - \frac{\left|V_0^-(\omega)\right|^2}{2Z_0}$$

Q: *Wait a second! Does this even passes the **smell test**?*

*Total current $I(z)$ and total voltage $V(z)$ are **functions of position**, as is (apparently) $P(z)$.*

*But, the result above (the right side of the expression) is **not** dependent on z !?*

It passes the smell test

A: But it **does** pass the smell test!

The result:

$$P(z) = \frac{\left|V_0^+(\omega)\right|^2}{2\,Z_0} - \frac{\left|V_0^-(\omega)\right|^2}{2\,Z_0}$$

simply means that the rate of energy flow is a **constant** value—constant from **one end** of the transmission line to **the other**.

Of course, this **better** be the case, as a lossless transmission line is—um—**lossless**!

→ Therefore, this **lossless** device **cannot alter** the rate of energy flow.

Energy **leaves** the transmission line at the **same** rate at which it **enters**—and it remains at that rate at **every** single point in between!

A wave interpretation

Q: *Hey, the **first** term in this power expression depends on the squared amplitude of the **plus-wave** , and the **second** likewise depends on the **minus-wave** amplitude.*

$$P(z) = \frac{\left|V_0^+(\omega)\right|^2}{2\,Z_0} - \frac{\left|V_0^-(\omega)\right|^2}{2\,Z_0}$$

*Might there then be some **wave interpretation** of this power result?*

A: **Absolutely**!

Let's first determine the power associated with the **plus-wave** only→

$$\begin{aligned} P^+(z) &= \frac{1}{2} Re\left\{V^+(z)\; I^+(z)^*\right\} \\ &= \frac{1}{2} Re\left\{V_0^+(\omega)\, e^{-j\beta z}\; \frac{V_0^+(\omega)^*}{Z_0}\, e^{+j\beta z}\right\} \\ &= \frac{1}{2} Re\left\{\frac{\left|V_0^+(\omega)\right|^2}{Z_0}\, e^0\right\} \\ &= \frac{\left|V_0^+(\omega)\right|^2}{2\,Z_0} \end{aligned}$$

In the direction of decreasing z

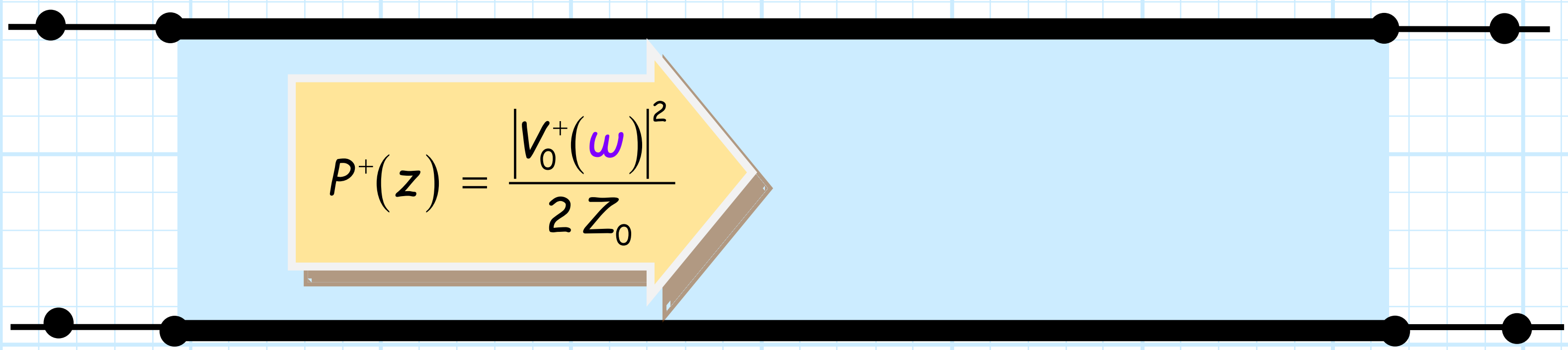
Likewise for the **minus-wave**:

$$
\begin{aligned}
P^-(z) &= \frac{1}{2} Re\left\{ V^-(z)\, I^-(z)^* \right\} \\
&= \frac{1}{2} Re\left\{ V_0^-\; e^{+j\beta z} \left(\frac{V_0^-(\omega)}{Z_0} \right)^* e^{-j\beta z} \right\} \\
&= \frac{1}{2} Re\left\{ \frac{\left|V_0^-(\omega)\right|^2}{Z_0}\; e^0 \right\} \\
&= \frac{\left|V_0^-(\omega)\right|^2}{2\,Z_0}
\end{aligned}
$$

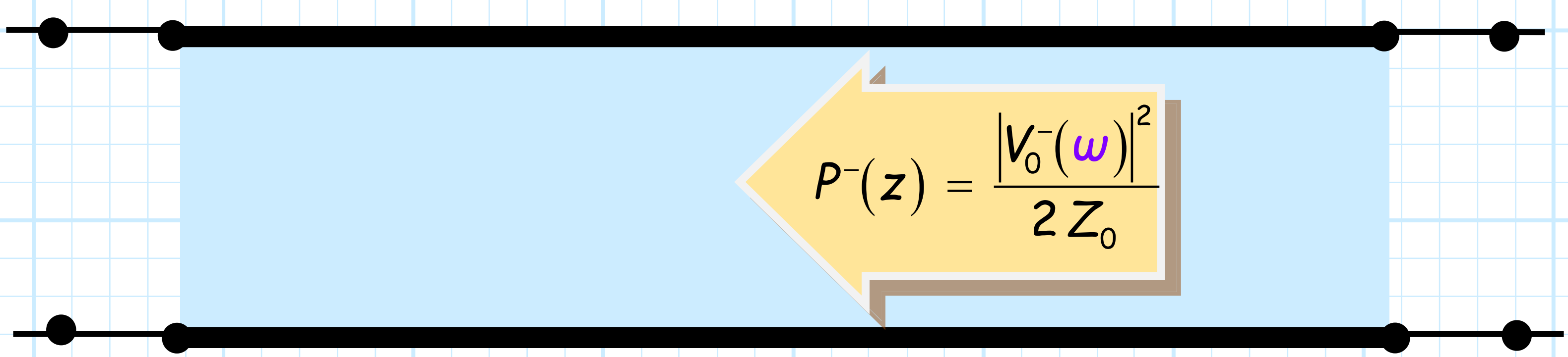
Note that this energy flows in the direction of current $I^-(z)$—that is, **flowing in the direction of decreasing z**!

Positive in opposite directions

The rate $P^{+}(z)$ is **always positive**, meaning electromagnetic energy flows in the direction of **increasing *z*** (the plus-wave direction of propagation!).



Conversely, the rate $P^{-}(z)$ is likewise **always positive**, meaning electromagnetic energy flows in the direction of **decreasing *z*** (the minus-wave direction of propagation!).

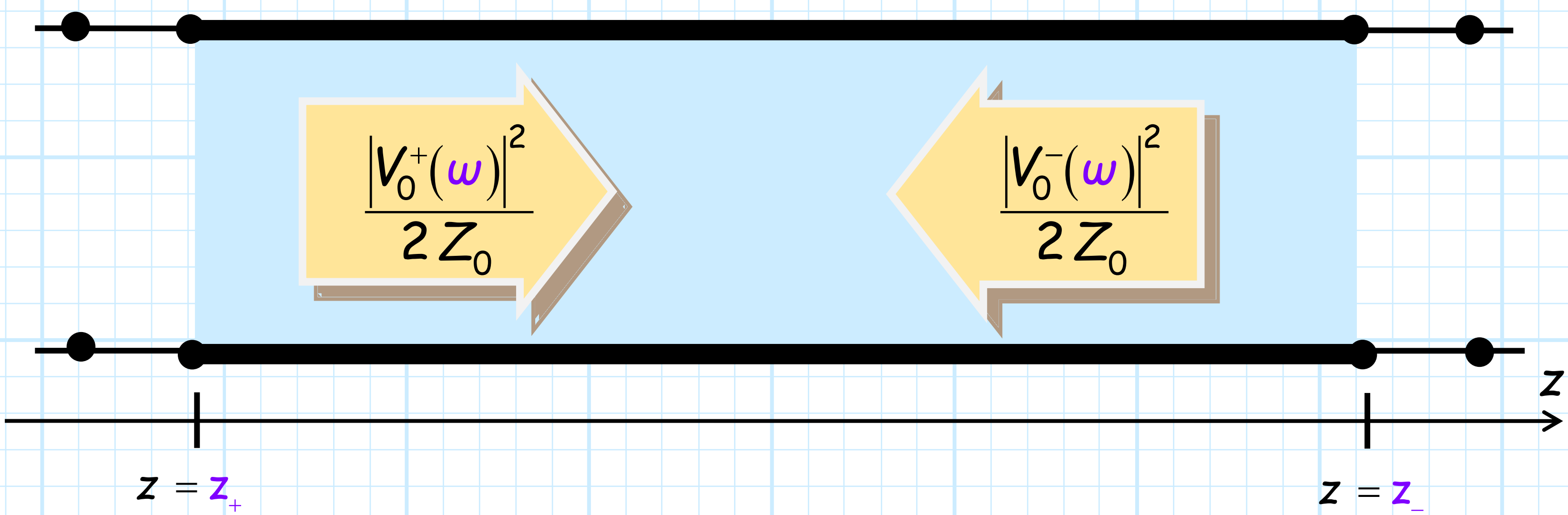


Net power is the difference

The **net** rate of energy flow along the transmission line is then just the **difference** of these two values→:

$$P_{net}(z) = \frac{\left|V_0^+(\omega)\right|^2}{2Z_0} - \frac{\left|V_0^-(\omega)\right|^2}{2Z_0}$$

$$= P^+(z) - P^-(z)$$



Q: *So* ***is*** *the "net" energy flow $P_{net}(z)$ a positive number,* ***or*** *a negative value?*

A: It depends!

Net power can be positive or negative

If the power associated with the plus-wave is **greater** than that of the minus-wave, then $P_{net}(z)$ will be **positive**:

$$P_{net}(z) = P^{+}(z) - P^{-}(z) > 0 \quad \text{if} \quad \frac{\left|V_0^{+}(\omega)\right|^2}{2\,Z_0} > \frac{\left|V_0^{-}(\omega)\right|^2}{2\,Z_0}$$

Conversely, **if** the power associated with the plus-wave is **less** than that of the minus-wave, then $P_{net}(z)$ will be **negative**:

$$P_{net}(z) = P^{+}(z) - P^{-}(z) < 0 \quad \text{if} \quad \frac{\left|V_0^{+}(\omega)\right|^2}{2\,Z_0} < \frac{\left|V_0^{-}(\omega)\right|^2}{2\,Z_0}$$

Q: ***Negative*** *power? What the heck does* ***that*** *mean?*

A: A **negative** value of $P_{net}(z)$ indicates that the "net" power is flowing in the direction of **decreasing** z.

In terms of Γ_0

Finally, recall that **wave amplitudes** V_0^+ and V_0^- are related by the reflection coefficient value $\Gamma_0(\omega) = \Gamma(z=0,\omega)$:

$$\Gamma_0(\omega) = \frac{V_0^-(\omega)}{V_0^+(\omega)} \qquad \Rightarrow \qquad V_0^-(\omega) = \Gamma_0(\omega)\, V_0^+(\omega)$$

Thus, their **magnitudes** are related as:

$$\left|\Gamma_0(\omega)\right| = \frac{\left|V_0^-(\omega)\right|}{\left|V_0^+(\omega)\right|} \qquad \Rightarrow \qquad \left|V_0^-(\omega)\right| = \left|\Gamma_0(\omega)\right| \left|V_0^+(\omega)\right|$$

And so:

$$P^-(z) = \frac{\left|V_0^-(\omega)\right|^2}{2\,Z_0} = \frac{\left|\Gamma_0(\omega)\right|^2 \left|V_0^+(\omega)\right|^2}{2\,Z_0} = \left|\Gamma_0(\omega)\right|^2\; P^+(z)$$

Power is simply related

Meaning the **plus-wave power** and **minus-wave power** are **simply** related as:

$$\boxed{\left|\Gamma_0(\omega)\right|^2 = \frac{P^-(z)}{P^+(z)}}$$

And therefore:

$$P_{net}(z) = P^+(z) - P^-(z) = P^+(z) - \left|\Gamma_0(\omega)\right|^2 P^+(z) = P^+(z)\left(1 - \left|\Gamma_0(\omega)\right|^2\right)$$

I **repeat**:

$$\boxed{P_{net}(z) = P^+(z)\left(1 - \left|\Gamma_0\right|^2\right)}$$