PHSX 531: Homework #9

November 5, 2024

Grant Saggars

Problem 1

(3 pts) Show that the electric field of a (perfect) dipole can be written in the coordinate-free form

$$\mathbf{E}_{\text{dip}}(\mathbf{p}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}\right]$$

Solution:

$$\begin{aligned}
\mathbf{E}_{\text{dip}}(\mathbf{p}) &= \frac{\mathbf{p}}{4\pi\epsilon_0 r^3} 2\cos\theta\hat{\boldsymbol{r}}\sin\theta\hat{\boldsymbol{\theta}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(p\cos\theta\hat{\boldsymbol{r}} - p\cos\theta\hat{\boldsymbol{r}}\ p\sin\theta\hat{\boldsymbol{\theta}}\right] \\
&= \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(p\cos\theta, p\sin\theta)\cdot(1,0)\hat{\boldsymbol{r}} - (p\cos\theta, p\sin\theta)\right] \\
\mathbf{E}_{\text{dip}}(\mathbf{p}) &= \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}\right]
\end{aligned}$$

Problem 2

A conducting sphere of radius a, at potential V_0, is surrounded by a thin concentric spherical shell of radius b, over which someone has glued a surface charge $\sigma(\theta) = k\cos\theta$, where k is a constant and θ is the usual spherical coordinate.

(a) (3 pts) Find the potential in each region: $r > b$ and $a < r < b$.

Solution:

We need to start with spherical harmonics. We have spherical solutions to Laplace's equation:

$$V(r,\theta) = \sum_{l}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

We have the boundary conditions that (1) finiteness of potential, (2) potential is continuous across boundaries, (3) that $E_{\text{above}} - E_{\text{below}} = \frac{\sigma}{\epsilon_0}$, and (4) that $V(a) = V_0$. Finiteness is pretty trival, and we drop the outside term which blows up at infinity.

We can begin by equating the potentials inside and outside as in boundary condition (2):

$$\sum_l \underbrace{\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)}_{\text{(inside)}} \cancel{P_l(\cos\theta)} = \sum_l \underbrace{\left(\frac{C_l}{r^{l+1}}\right)}_{\text{(outside)}} \cancel{P_l(\cos\theta)}$$

$$\boxed{C_l = (A_l r^{2l+1} + B_l)\big|_{r=b}} \tag{2}$$

Now the more complicated boundary condition (3) says that

$$E_{\text{above}} - E_{\text{below}} = \frac{\sigma(\theta)}{\epsilon_0} \tag{2.33}$$

$$\implies \nabla V|_{r=b} = -\frac{\sigma(\theta)}{\epsilon_0} \tag{2.35}$$

$$\sum \nabla \left(\frac{C_l}{r^{l+1}} P_l(\cos\theta)\right) = -\frac{\sigma(\theta)}{\epsilon_0}$$

$$\sum \nabla \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta) = -\frac{\sigma(\theta)}{\epsilon_0}$$

$$\sum \left(A_l r^{l-1} l - \frac{B_l(l+1)}{r^{l+2}}\right) P_l(\cos\theta) = -\frac{k}{\epsilon_0}\cos\theta \overset{\text{(this is } P \text{ one)}}{=} -\frac{k}{\epsilon_0} P_1(\cos\theta)$$

I will stop here to make a few notes before moving forward. Equation (3.85) showcases this technique to substitute in $P_1(\cos\theta)$, where it is shown that for $l \neq 0$ the right hand side of the equation with the charge density goes to zero, and the legendre polynomials drop out due to orthagonality. Also, we can start substituting $r \to b$.

$$\boxed{\begin{cases} A_l b^{l-1} l - \frac{B_l(l+1)}{b^{l+2}} = 0, & l \neq 1 \\ A_1 - \frac{2B_1}{b^3} = -\frac{k}{\epsilon_0}, & l = 1 \end{cases}} \tag{3}$$

Boundary condition (4) says

$$\sum \left(A_l a^l + \frac{B_l}{a^{l+1}}\right) P_l(\cos\theta) = V_0$$

$$\implies \begin{cases} \left(A_0 + \frac{B_0}{a}\right) = V_0, & l = 0 \\ \left(A_l a^l + \frac{B_l}{a^{l+1}}\right) P_l(\cos\theta) = 0, & l \neq 0 \end{cases}$$

$$\boxed{\begin{cases} B_0 = a(V_0 - A_0), & l = 0 \\ B_l = -A_l a^{2l+1}, & l \neq 0 \end{cases}} \tag{4}$$

Each of the boundary conditions has been worked, so we can start combining them to express coefficients.

$$0 \overset{(3)}{=} A_l b^{l-1} l - \frac{B_l(l+1)}{b^{l+2}}, \quad B_l \overset{(4)}{=} -A_l a^{2l+1}$$

$$\implies 0 = A_l b^{l-1} l - \frac{-A_l a^{2l+1}(l+1)}{b^{l+2}}$$

$$0 = A_l b^{2l+1} l + A_l a^{2l+1}(l+1)$$

$$0 = A_l \left[b^{2l+1} l + a^{2l+1}(l+1)\right]$$

Problem 2 continued on next page…

If we throw this back at the solution from (2) and (4), we have no solutions for $l > 1$:

$$\boxed{A_l, B_l, C_l = 0, \quad l \neq 0,\ l \neq 1}$$

$$C_0 - B_0 = 0(b-a)A_0 + aV_0 = aV_0 - aA_0$$

Which implies

$$\boxed{A_0 = 0,\ B_0 = C_0 = aV_0 \quad l = 0}$$

$$A_1 - \frac{2\overset{(3)}{B_1}}{b^3} = -\frac{k}{\epsilon_0}, \quad B_1 \overset{(4)}{=} -A_1 a^3$$

$$A_1 + 2A_1 = k \to A_1 = \frac{k}{3\epsilon_0}$$

$$B_1 = -\frac{ka^3}{3\epsilon_0}$$

$$C_1 = (b^3 - a^3)A_1 = \frac{(b^3 - a^3)k}{3\epsilon_0}$$

$$\boxed{A_1 = \frac{k}{3\epsilon_0},\ B_1 = -\frac{ka^3}{3\epsilon_0}\ C_1 = \frac{(b^3-a^3)k}{3\epsilon_0}, \quad l = 1}$$

This then gives solutions

$$\boxed{\begin{cases} V(r) = \frac{aV_0}{r} + \frac{(b^3-a^3)}{3r^2\epsilon_0}k\cos\theta, & r \geq b \\ V(r) = \frac{aV_0}{r} + \frac{(r^3-a^3)}{3r^2\epsilon_0}k\cos\theta, & r \leq b \end{cases}}$$

(b) (3 pts) Find the surface charge density $\sigma_i(\theta)$ on the conductor.

Solution:

$$\begin{aligned} \sigma_i(\theta) &= -\epsilon_0 \frac{\partial V}{\partial r}\hat{n} \\ &= -\epsilon_0 \frac{\partial}{\partial r}\left[\frac{aV_0}{r} + \frac{(r^3-a^3)}{3r^2\epsilon_0}k\cos\theta\right] \\ &= \frac{2aV_0\epsilon_0}{r^3} + \frac{2a^3}{r^4}k\cos\theta \end{aligned} \tag{2.49}$$

Problem 3

(3 pts) According to quantum mechanics, the electron cloud for a hydrogen atom in the ground state has a charge density

$$\rho(r) = \frac{q}{\pi a^3} e^{-2r/a}$$

where q is the charge of the electron and a is the Bohr radius. Find the atomic polarizability. *[Hint: First calculate the electric field of the electron cloud, $\mathbf{E}_e(r)$; then expand the exponential assuming $r \ll a$]*

Solution:

i. The charge enclosed is given by the integral

$$Q_{enc} = \frac{4\pi q}{\pi a^3} \int_0^r r^2 e^{-2r/a}\, dr = q\left[1 - e^{-2r/a}\left(1 + \frac{2r}{a} + \frac{2r^2}{a^2}\right)\right]$$

ii. Then the electric field is

$$\begin{aligned}
\oint E \cdot dA &= \frac{Q_{enc}}{\epsilon_0} \\
E \cdot 4\pi r^2 &= \frac{q}{\epsilon_0}\left[1 - e^{-2r/a}\left(1 + \frac{2r}{a} + \frac{2r^2}{a^2}\right)\right] \\
E &= \frac{q}{\epsilon_0 4\pi r^2}\left[1 - e^{-2r/a}\left(1 + \frac{2r}{a} + \frac{2r^2}{a^2}\right)\right]
\end{aligned}$$

iii. The linear term of the Taylor Expansion is polarizability, Taylor expand by expanding the product:

$$\begin{aligned}
A = e^{-2r/a} &= 1 + \left(-\frac{2r}{a}\right) + \frac{1}{2!}\left(\frac{2r}{a}\right)^2 + \dots \\
B &= 1 + \frac{2r}{a} + \frac{2r^2}{a^2} \\
1 - AB &= 1 - \frac{4}{3}\left(\frac{r}{a}\right)^3 + \dots
\end{aligned}$$

Then if we choose the linear term,

$$E \approx \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\left(\frac{4}{3}\left(\frac{r}{a}\right)^3\right)$$

Which reduces to

$$E = \frac{1}{3\pi\epsilon_0 a^3} qr = \frac{1}{3\pi\epsilon_0 a^3}\mathbf{p}$$

Problem 4

(4 pts) A (perfect) dipole $\mathbf{p}$ is situated a distance z above an infinite grounded conducting plane. The dipole makes an angle θ with the perpendicular to the plane. Find the torque on $\mathbf{p}$. If the dipole is free to rotate, in what orientation will it come to rest?

Problem 4 continued on next page...

Solution:

i. Electric field felt by the dipole is equal to the induced field by the dipole. We have

$$\boxed{\begin{aligned} E_{dip} &= \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta,\ \sin\theta,\ 0) \\ \mathbf{p} &= (p\cos\theta,\ p\sin\theta) \end{aligned}}$$

Then, by the image method,

$$\mathbf{E_i} = \frac{p}{4\pi\epsilon_0 (2z)^3}(2\cos\theta,\ \sin\theta,\ 0)$$

ii. Net torque is then

$$\begin{aligned} \mathbf{N} &= \mathbf{p} \times \mathbf{E_i} \\ &= (p\cos\theta,\ p\sin\theta,\ 0) \times \frac{p}{4\pi\epsilon_0 (2z)^3}(2\cos\theta,\ \sin\theta,\ 0) \\ &= \frac{p^2}{4\pi\epsilon_0 (2z)^3}\begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \cos\theta & \sin\theta & 0 \\ 2\cos\theta & \sin\theta & 0 \end{vmatrix} \\ &= \frac{p^2}{4\pi\epsilon_0 (2z)^3}\left(\sin\theta\cos\theta\hat{\boldsymbol{\phi}}\right) \\ &= \frac{p^2}{8\pi\epsilon_0 (2z)^3}\sin(2\theta) \end{aligned}$$

This has zeroes at integer multiples of $\pi/2$, so it will orient parallel to the surface.