590 Proofs & Definitions

Grant Saggars

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1 Definitions

Null Space / Kernel:

The null space of T is the set of vectors which are mapped to zero.

It can be found by reducing $Tx = 0$. Count the rows of all zeroes to determine nullity. For example, suppose matrix T reduces to the following:

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & -6 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

We can say that $x = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 6 \\ 2 \end{bmatrix}$. As a result, we can say that null $t = \text{span}\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 6 \\ 2 \end{bmatrix}\right\}$

Injectivity:

A function $T : V \to W$ is called injective if $Tu = Tv$ implies $u = v$ (every input maps to ONE output).

Surjectivity:

A function $T : V \to W$ is called surjective if its range equals W. This is to say that the range is unchanged after the map.

Projection:

$$proj_v(u) = \frac{u \cdot v}{|v|^2} v$$

Characteristic Polynomial:

$$f(\lambda) = \det(A - I\lambda)$$

Rotations:

$$\begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$$

Orthagonal Complement:

Let W be a subspace of $\mathbb{R}^n$. Its **orthagonal complement** is the subspace

$$W^\perp = \{v \in \mathbb{R}^n | v \cdot w = 0 \ \forall \ w \in W\}$$
$$W^\perp = \ker(A^T)$$

Theorem 3

Let $W = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right\}$. Therefore $W^\perp = \ker\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$. To find the kernel:

$$\ker\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \vec{0} \implies \ker(A) = \begin{pmatrix} 1 & -2 & 1 \end{pmatrix} A$$

Now we can show that $\ker\begin{pmatrix} 1 & -2 & 1 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}$, which we found earlier.

2 Injective/Surjective/Bijective

2.1 Injective (one-to-one)

Proof: Injectivity is Equivalent to Null Space Equals 0

We know that the zero vector is in any subspace, and because the nullity is a subspace it must contain the zero vector. Consequently, because $Tu = Tv$ implies $u = v$, $T(v) = 0 = T(0)$ (T can only map 0 to 0). Additionally, we have defined injective (one-to-one) maps as mapping one input to one output. Therefore, more than one vector cannot map to 0.

A map to a smaller dimensional space is not injective

2.2 Surjective (onto)

Proof: Check ranges:

By definition a surjective transformation does not change the range of the vector space. We can prove a transformation is surjective by simply verifying this.

By the rank-nullity theorem, Dim T + Null T equals number of columns in a matrix.

A map to a larger dimensional space is not surjective

3 Invertible Matrices are Bijective

Proof:

First, assume that T is invertible. To show that it is injective (one-to-one) suppose that $u,\ v \in V$ and $Tu = Tv$ (definition of injectivity). Then

$$u = T^{-1}(Tu) = T^{-1}(Tv) = v$$

Second, take $w \in W$. The following implies that w is in the range of T, and that implies range $T = W$. (surjectivity)

$$w = T^{-1}(Tw) = w$$

Third, assume that T is bijective. To prove it is invertible:

$$T((S \circ T)v) = (T \circ S)(Tv) = I(Tv) = Tv$$

Finally, to show that S is linear, suppose $w_1,\ w_2 \in W$

$$T(Sw_1 + Sw_2) = T(Sw_1) + T(Sw_2) = w_1 + w_2$$

This also implies that $S(w_1 + w_2) = Sw_1 + Sw_2$, so it therefore satisfies the additive property. To show homogeneity:

$$T(\lambda Sw) = \lambda T(Sw) = \lambda w$$

Thus, λSw is the unique element of V that T maps to λw. this implies that $S(\lambda w) = \lambda Sw$