PHSX 611: Homework #5

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Problem 1

Find the eigenfunctions and eigenvalues of position operator $\hat{x}$. (0.4 pts.)

Solution: The eigenvalues of $\hat{x}$ are pretty straightforward; set $f(x)$ the eigenfunction and u the eigenvalue:

$$xf(x) = uf(x)$$

The eigenfunction $f(x)$ which satisfies this is the piecewise function:

$$f(x) = \begin{cases} 0 & x \neq u \\ A & x = u \end{cases}$$

Or, more succinctly with a delta function:

$$f(x) = A\delta_{xu}$$

Eigenfunctions belonging to distinct eigenvalues ought to be complete and orthagonal. I will denote the other eigenvalue v (letting A equal 1):

$$\langle f_u' | f_u \rangle = \delta(u - u')$$

And finally completeness:

$$f(x) = \int_{-\infty}^{\infty} c(u) f_u(x)\, du = \int_{-\infty}^{\infty} c(u)\delta(x-u)\, du$$

$$c(y) = f(y)$$

Problem 2

Provide at least two examples of Hamiltonians from Chapter 2 that have both discrete and continuous parts of the spectra. Explain why they have both - e.g. explain how to find basis. (0.4 pts.)

Solution:

(I) **The Delta Function Potential:** When the delta function potential is configured such that it is a well ($-\alpha V(x)$), we get discrete spectra. However when configured to be a barrier ($\alpha V(x)$), we get something similar to a free particle where it can take on any energy, position, momenta, etc.

(II) **The Finite Square Well:** The same case emerges for the square well for the same reasons.

Problem 3

(Problem 3.4) Show that position and Hamiltonian operators (where potential V only depends on position and doesn't depend on time) are Hermitian. (0.4pts.)

Solution:

(I) **Their eigenvalues are real.**

This holds as we have seen previously.

(II) **To show that $\langle\psi|x\phi\rangle = \langle\phi|x\psi\rangle^*$:**

$$\begin{aligned}\int \psi^* x\phi \, dx &= \int (\phi^* x\psi)^* \, dx \\ &= \int \psi x\phi^* \, dx \\ &= \int \phi^* x\psi \, dx\end{aligned}$$

To show that $\langle\psi|H\phi\rangle = \langle\phi|h\psi\rangle^*$:

$$\begin{aligned}\int \psi^* \left(-i\hbar\frac{d\phi}{dx}\right) \, dx + \int \psi^* V(x)\phi \, dx &= \int \phi \left(i\hbar\frac{d\psi^*}{dx}\right) \, dx + \int \phi V(x)\psi^* \, dx \\ &= \psi^*\phi|_{-\infty}^{\infty} - i\hbar \int \psi^* \frac{d\phi}{dx} \, dx + \int \phi V(x)\psi^* \, dx \\ &= -i\hbar \int \psi^* \frac{d\phi}{dx} \, dx + \int \phi V(x)\psi^* \, dx\end{aligned}$$

Problem 4

(Problem 3.11) Find the momentum-space wave function $\Phi(p,t)$ for a particle in the ground state of the harmonic oscillator. (0.4pts.)

Solution:
By Fourier transform:

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(k,t) e^{+ikx}\, dk$$

Plancherel's theorem allows:

$$\Phi(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x,t) e^{-ikx}\, dx$$

(note that k represents the wave number, where $p = \hbar k$)

$$\begin{aligned}
\Phi(p,t) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \Psi_0(x,t)\, e^{-ipx/\hbar}\, dx \\
&= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar} \psi(t) e^{-ipx/\hbar}\, dx \\
&= \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \psi(t) \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\frac{m\omega}{\hbar}x^2 - \frac{ip}{\hbar}x\right)\, dx
\end{aligned}$$

Let a equal $\frac{m\omega}{\hbar}$ and J equal $-\frac{ip}{\hbar}$. The exponental then goes to:

$$\begin{aligned}
\left(-\frac{1}{2}ax^2 + Jx\right) = -\frac{1}{2}a\left(x^2 - \frac{2Jx}{a} + \frac{J^2}{a^2} - \frac{J^2}{a^2}\right) &= -\frac{1}{2}a\left(x - \frac{J}{a}\right)^2 + \frac{J^2}{2a} \\
\implies \exp\left(\frac{J^2}{2a}\right) \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}a\left(x - \frac{J}{a}\right)^2\right)\, dx& \\
= \exp\left(\frac{J^2}{2a}\right) \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}aw^2\right)\, dw& \\
= \left(\frac{2\pi}{a}\right)^{1/2} \exp\left(\frac{J^2}{2a}\right)&
\end{aligned}$$

Therefore the momentum-space wave function equals:

$$\frac{1}{\sqrt{2\pi\hbar}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-i\omega t/\hbar} \left(\frac{2\pi\hbar}{m\omega}\right)^{1/2} \exp\left(\frac{(-ip)^2}{2m\omega}\right)$$

Problem 5

(Problem 2.11) Compute the expectation value for coordinate and momentum operator for the ground state of Harmonic oscillator ψ_0. Check the uncertainty principle for those values. What do you expect to find for the first excited state of the Harmonic oscillator? (0.4 pts.)

Solution:

$$|\psi_0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar};$$

(I) $\langle \psi_0^* | \hat{x}\psi_0 \rangle$:

$$\begin{aligned}\langle \psi_0^* | \hat{x}\psi_0 \rangle &= \int_{-\infty}^{\infty} x\sqrt{\frac{m\omega}{\pi\hbar}} e^{-2m\omega x^2/2\hbar}\, dx \\ &= 0\end{aligned}$$

(II) $\langle \psi_0^* | \hat{p}\psi_0 \rangle$:

$$\begin{aligned}\langle \psi_0^* | \hat{p}\psi_0 \rangle &= -i\hbar \int_{-\infty}^{\infty} -\left(\frac{m\omega}{2\hbar}\right)^{3/2} x e^{-2m\omega x^2/2\hbar}\, dx \\ &= 0\end{aligned}$$

Evidently, both work out to zero due to odd parity.

(III) $\langle \psi_0^* | \hat{x}^2\psi_0 \rangle$:

$$\begin{aligned}\langle \psi_0^* | \hat{x}^2\psi_0 \rangle &= \sqrt{\frac{m\omega}{2\hbar}} \int_{-\infty}^{\infty} x^2 e^{-2m\omega x^2/2\hbar}\, dx \\ &= \frac{\hbar}{2m\omega}\end{aligned}$$

(IV) $\langle \psi_0^* | \hat{p}^2\psi_0 \rangle$:

$$\begin{aligned}\langle \psi_0^* | \hat{p}^2\psi_0 \rangle &= -\hbar^2\sqrt{\frac{m\omega}{2\hbar}} \int_{-\infty}^{\infty} \left(\frac{m\omega}{\hbar}x\right)^2 e^{-2m\omega x^2/2\hbar} - \left(\frac{m\omega}{\hbar}\right) e^{-2m\omega x^2/2\hbar}\, dx \\ &= -\hbar^2\sqrt{\frac{m^3\omega^3}{\pi\hbar^3}} \int_{-\infty}^{\infty} \left(\frac{m\omega}{\hbar}x^2 - 1\right) e^{-m\omega x^2/2\hbar}\, dx \\ &= \frac{\hbar m\omega}{2}\end{aligned}$$

(V) $\sigma_x\sigma_p \geq \hbar/2$

$$\begin{aligned}\sigma_x &= \sqrt{\frac{\hbar}{2m\omega}} \\ \sigma_p &= \sqrt{\frac{\hbar m\omega}{2}} \\ \sigma_x\sigma_p &= \frac{h}{2}\end{aligned}$$

Problem 6

(Problem 3.21) Test the energy-time uncertainty principle for the free particle wave packet in Problem 2.42 and the observable x by calculating σ_H, σ_x, and $d\langle x\rangle/dt$ exactly. (0.5 pts.)

Solution:

$$\Delta E \Delta t \geq \hbar/2 \rightarrow \sigma_H \frac{\sigma_x}{\left|\frac{d\langle x\rangle}{dt}\right|} \geq \hbar/2$$

The gaussian wave packet in question has a wave function:

$$\Psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4} \frac{1}{\sqrt{1+\frac{2i\hbar a t}{m}}} \exp\left[\frac{-a(x-\frac{\hbar l t}{m})^2}{1+\frac{2i\hbar a t}{m}}\right] \exp\left[il\left(x-\frac{\hbar l t}{2m}\right)\right]$$

(I) $\langle x\rangle$

$$\langle \psi^*|x\psi\rangle = \int_{-\infty}^{\infty} x\sqrt{\frac{2}{\pi}}\sqrt{\frac{a}{1+\frac{4\hbar^2a^2t^2}{m^2}}} \exp\left[-2\left(\frac{a}{1+\frac{4\hbar^2a^2t^2}{m^2}}\right)\left(x-\frac{\hbar l t}{m}\right)^2\right]\,dx$$

let $x-\frac{\hbar l t}{m} \rightarrow v$, and $dv = dx$

$$\sqrt{\frac{2m^2a}{\pi(m^2+4\hbar^2a^2t^2)]}}\int_{-\infty}^{\infty} v\exp\left(-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}v^2\right)\,dx + \frac{\hbar l t}{m}\int_{-\infty}^{\infty} v\exp\left(-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}v^2\right)\,dv$$

$$= 2\hbar l t\sqrt{\frac{2a}{\pi(m^2+4\hbar^2a^2t^2)}}\sqrt{\pi}\left(\frac{\sqrt{\frac{m^2+4\hbar^2a^2t^2}{2m^2a}}}{2}\right)$$

$$= \frac{\hbar l t}{m}$$

(II) $\langle x^2\rangle$

$$\langle \psi^*|x^2\psi\rangle = \sqrt{\frac{2m^2a}{\pi(m^2+4\hbar^2a^2t^2)]}}\int_{-\infty}^{\infty}(v+\frac{\hbar l t}{m})^2\exp\left(-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}v^2\right)\,dv$$

$$= \sqrt{\frac{2m^2a}{\pi(m^2+4\hbar^2a^2t^2)]}}\int_{-\infty}^{\infty}\left(v^2+\frac{2\hbar l t}{m}v+\frac{\hbar^2l^2t^2}{m62}\right)\exp\left(-\frac{2m^2a}{m^2+4\hbar^2a^2t^2}\right)\,dv$$

$$= \frac{m^2_{4\hbar^2a^2t^2}}{4m^2a}+\frac{\hbar^2l^2t^2}{m^2}$$

Therefore $\sigma_x = \sqrt{\frac{m^2+4\hbar^2a^2t^2}{4m^2a}}$

(III) $\langle p\rangle = m\frac{d\langle x\rangle}{dt} = m\frac{d}{dt}\left(\frac{\hbar l t}{m}\right) = \hbar l$

(IV) $\int_{-\infty}^{\infty}\Psi^*(x,t)\left(-i\hbar\frac{\partial}{\partial x}\right)^2\Psi(x,t)\,dx = \hbar^2(a+l^2)$

(V) $\sigma_p = \hbar\sqrt{a}$

(VI) Momentum space of the gaussian wave packet:

$$\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{-ipx/\hbar}\left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1+\frac{2i\hbar a t}{m}}}\exp\left[\frac{-a\left(x-\frac{\hbar l t}{m}\right)^2}{1+\frac{2i\hbar a t}{m}}\right]\exp\left[il\left(x-\frac{\hbar l t}{2m}\right)\right]dx$$

Problem 6 continued on next page...

This is a gaussian with a linear term in the exponent which can be written as:

$$\Phi(p,t) = \left(\frac{2a}{\pi}\right)^{1/4} \frac{1/\sqrt{2\pi\hbar}}{\sqrt{1+\frac{2i\hbar at}{m}}} \exp\left(-\frac{m+2i\hbar at}{4\hbar^2 am}p^2 + \frac{l}{2\hbar a}p - \frac{l^2}{4a}\right) \int_{-\infty}^{\infty} \exp\left(-\frac{a}{1+\frac{2i\hbar at}{m}}u^2\right) du$$

With $u = x + \frac{im(p-\hbar l)-2p\hbar at}{2\hbar am}$ and $du = dx$. This gives:

$$\Phi(p,t) = \frac{1}{\sqrt[4]{2\pi\hbar^2 a}} \exp\left(-\frac{m+2i\hbar at}{4\hbar^2 am}p^2 + \frac{l}{2\hbar a}p - \frac{l^2}{4a}\right).$$

(VII) $\langle H\rangle$: Expressed in momentum space:

$$\begin{aligned}
\langle H\rangle = \langle \Phi|\hat{H}\Phi\rangle &= \int_{-\infty}^{\infty} \Phi^*(p,t)\hat{H}\Phi(p,t)\,dp \\
&= \int_{-\infty}^{\infty} \Phi^*(p,t)\left(\frac{\hat{p}^2}{2m}\right)\Phi(p,t)\,dp \\
&= \frac{1}{2m}\langle \Phi|\hat{p}^2\Phi\rangle = \frac{1}{2m}\langle p^2\rangle
\end{aligned}$$

From before we have $\langle H\rangle = \frac{1}{2m}\langle p^2\rangle = \frac{\hbar^2}{2m}(a+l)^2$

(VIII) $\langle H^2\rangle$: Expressing in momentum space:

$$\begin{aligned}
\langle H^2\rangle = \langle \Phi|\hat{H}^2\Phi\rangle &= \int_{-\infty}^{\infty} \Phi^*(p,t)\hat{H}^2\Phi(p,t)\,dp \\
&= \int_{-\infty}^{\infty} \Phi^*(p,t)\left(\frac{\hat{p}^4}{4m^2}\right)\Phi(p,t)\,dp \\
&= \frac{1}{2m}\langle \Phi|p^4\Phi\rangle = \frac{1}{2m}\langle p^4\rangle
\end{aligned}$$

Using momentum space to calculate $\langle p^4\rangle$:

$$\begin{aligned}
\langle p^4\rangle &= \langle \Phi^*|p^4\Phi\rangle \\
&= \frac{1}{\hbar\sqrt{2\pi a}} \int_{-\infty}^{\infty} p^4 \exp\left(-\frac{1}{2\hbar^2 a}p^2 + \frac{l}{\hbar a}p - \frac{l^2}{2a}\right) dp
\end{aligned}$$

Yet another gaussian with a linear term:

$$\langle p^4\rangle = \hbar^4(3a^2 + 6al^2 + l^4)$$

(IX) Final result:

$$\begin{aligned}
\sqrt{\langle H^2\rangle - \langle H\rangle^2}\frac{\sqrt{\langle x^2\rangle - \langle x\rangle^2}}{\left|\frac{d\langle x\rangle}{dt}\right|} &= \sqrt{\left[\frac{\hbar^4}{4m^2}\left(3a^2+6al^2+l^4\right)\right] - \left[\frac{\hbar^2}{2m}\left(a+l^2\right)\right]^2}\,\frac{\sqrt{\left(\frac{m^2+4\hbar^2a^2t^2}{4m^2a} + \frac{\hbar^2l^2t^2}{m^2}\right) - \left(\frac{\hbar lt}{m}\right)^2}}{\left|\frac{\hbar l}{m}\right|} \\
&= \sqrt{\frac{\hbar^4 a\left(a+2l^2\right)}{2m^2}\frac{\sqrt{\frac{m^2+4\hbar^2a^2t^2}{4m^2a}}}{\frac{\hbar l}{m}}} \\
&= \frac{\hbar}{2}\sqrt{\left(1+\frac{a}{2l^2}\right)\left(1+\frac{4\hbar^2a^2t^2}{m^2}\right)}.
\end{aligned}$$

Problem 7

Compute the following commutators for operators in three dimensions: $[\widehat{x}_i, \widehat{x}_j]$, $[\widehat{p}_i, \widehat{p}_j]$, $[\widehat{x}_i, \widehat{p}_j]$, $[\widehat{H}, \hat{\boldsymbol{r}}]$, $[\widehat{H}, \widehat{\boldsymbol{p}}]$, where x_{i} and p_{i} are components of coordinate and momentum operator in three dimensions respectively, and $\boldsymbol{r}$ and $\boldsymbol{p}$ are vectors of position operator and momentum operator in three dimensions. ($0.5\mathrm{pts}$.)

Solution:

(I)

$$[\hat{x}_i, \hat{x}_j] = \hat{x}_i\hat{x}_j - \hat{x}_j\hat{x}_i = 0$$

(II)

$$\begin{aligned}
[\hat{p}_i, \hat{p}_j] &= \hat{p}_i\hat{p}_j - \hat{p}_j\hat{p}_i \\
&= -i\hbar\frac{\partial}{\partial x_i}\left(-i\hbar\frac{\partial}{\partial x_j}(f)\right) + -i\hbar\frac{\partial}{\partial x_j}\left(-i\hbar\frac{\partial}{\partial x_i}(f)\right) \\
&= -\hbar^2\nabla f + \hbar^2\nabla f \\
&= 0
\end{aligned}$$

(III)

$$\begin{aligned}
[\hat{x}_i, \hat{p}_j] &= \hat{x}_i\hat{p}_j - \hat{p}_j\hat{x}_i \\
&= x_i(-i\hbar)\frac{\partial}{\partial x_j}(f) - (-i\hbar)\frac{\partial}{\partial x_j}(x_i f) \\
&= x_i(-i\hbar)\frac{\partial f}{\partial x_j} + i\hbar\left(\frac{\partial x_i}{\partial x_j}f + \frac{\partial f}{\partial x_j}x_i\right) \\
&= 0
\end{aligned}$$

(IV)

$$\begin{aligned}
[\widehat{H}, \hat{r}] &= [\widehat{T} + \widehat{V}, \hat{r}] = [\widehat{T}, \hat{r}] + [\widehat{V}, \hat{r}] \\
&= -\frac{\hbar^2}{2m}\nabla(\nabla(rf)) + \frac{\hbar^2}{2m}r\nabla^2(f) + [\widehat{V}, \hat{r}] \\
&= -\frac{\hbar^2}{2m}\nabla(r\nabla f + f\cancelto{1}{\nabla r}) + \cancel{\frac{\hbar^2}{2m}\nabla^2}(f)r + [\widehat{V}, \hat{r}] \\
&= -\frac{\hbar^2}{2m}(r\nabla^2 f + 2\nabla f) + [\widehat{V}, \hat{r}] \\
&= -\frac{\hbar^2}{m}\nabla f + [\widehat{V}, \hat{r}] \\
&= -\frac{\hbar}{im}\left(\frac{\hbar}{i}\nabla f\right) + [\widehat{V}, \hat{r}] \\
&= -\frac{\hbar}{im}\hat{p} + Vr - rV \\
&= -\frac{\hbar}{im}\hat{p}
\end{aligned}$$

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(V)

$$\begin{aligned}
[\widehat{H}, \hat{p}] &= [\widehat{T} + \widehat{V}, \hat{p}] = \left[\frac{\hat{p}^2}{2m}, \hat{p}\right] + [\widehat{V}, \hat{p}] \\
&= \frac{1}{2m}(\hat{p}[\hat{p}, \hat{p}] - [\hat{p}, \hat{p}]\hat{p}) + [\widehat{V}, \hat{p}] \\
&= 0 + V(-i\hbar)\nabla(f) - (-i\hbar)\nabla(Vf) \\
&= -i\hbar(V\nabla(f) - V\nabla(f) - f\nabla(V)) \\
&= i\hbar\nabla V
\end{aligned}$$

Problem 8

(Problem 3.37, Virial theorem, bonus, see full text of the problem in Griffiths, Third Edition) (0.5 pt.)