PHSX 631: Homework #9

May 6, 2025

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Problem 1

Explicitly demonstrate (write it ALL out – it should take several lines of detailed math) that $p^\mu p_\mu$ is a Lorentz invariant by *explicitly* applying the Lorentz transform. p^μ is the 4-momentum. Use the x-direction for the relative frame motion.

Solution:
I work this in imaginary time notation, since I found it interesting and I think it makes this math easier.
We have the 4-velocity defined to be

$$\vec{U} = \frac{d\vec{r}}{d\tau} = \gamma(u)\left(\frac{d\mathbf{r}}{t}, ic\right) = \gamma(u)(\mathbf{u}, ic) \equiv (\mathbf{U}, U_4)$$

And the 4-momentum defined to be

$$\vec{p} = m\vec{U} = m(\mathbf{U}, U_4) = (\mathbf{p}, i\gamma/c)$$

This definition inherently obeys the dot product on minowski space, given in Griffiths without the need to separately write contravariant and covariant forms. It follows that

$$\vec{U}\cdot\vec{U} = \mathbf{U}\cdot\mathbf{U} + U_4^2 = \frac{\mathbf{u}\cdot\mathbf{u} - c^2}{1 - u^2/c^2} = -c^2$$

Finally we can reintroduce the mass term

$$\vec{p}\cdot\vec{p} = -m^2c^2$$

Problem 2

An energetic particle with the rest mass of a neutron $M_n = 937\,\text{MeV}/c^2$ and a kinetic energy of $5 \times 937\,\text{MeV}$ collides with a particle at rest that has a rest mass of $2M_n$. A composite particle is created in this collision. What is the rest mass and the kinetic energy of the new composite particle? What is its speed?

Solution:

1. Particle 1 will have momentum and energy defined by:

$$M_n c^2 = 937 \text{ MeV}$$
$$T = 5M_n c^2 \text{ MeV} = 4185 \text{ MeV}$$
$$E^2 = (T + M_n c^2)^2 = p^2c^2 + M_n^2 c^4$$

   This gives

$$E = 5622 \text{ MeV}$$
$$p = 5543 \text{ MeV/c}$$

2. Particle 2 will just have rest energy, no momentum:

$$E = 2M_n c^2 = 1874 \text{ MeV}$$

3. The composite particle will have conserved quanities,

$$p = 5543 \text{ MeV/c}$$
$$E = 5622 + 1874 = 7496 \text{ MeV}$$

4. Rest mass is given by

$$E^2 = p^2c^2 + m^2c^4 \implies m^2 = \frac{E^2 - p^2c^2}{c^4}$$

   Substituting values gives $m = 5046 \ \frac{\text{MeV}}{\text{c}^2}$

5. No potential, so we can just subtract off rest mass to obtain kinetic energy

$$T = E - 3mc^2 = 7096 \text{ MeV} - 5046 \text{ MeV} = 2450 \text{ MeV}$$

6. The speed of the composite particle is

$$\frac{v}{c} = \frac{p}{E} = \frac{5543}{7496} \approx 0.739$$

Problem 3

A charge is released from rest at the origin, in the presence of a uniform electric field $\mathbf{E} = E_0\hat{z}$ and a uniform magnetic field $\mathbf{B} = B_0\hat{x}$. Find the trajectory of the particle by transforming to a frame in which $\mathbf{E} = 0$, finding the path in that frame and then transforming back to the original frame. Assume that $E_0 < cB_0$.

Solution:
We seek a reference frame moving perpendicular to both fields with velocity given by

$$\mathbf{v} = \frac{\mathbf{E}\times\mathbf{B}}{B^2} = \frac{E_0}{B_0}\hat{\mathbf{y}}$$

The transformed fields will be

$$\mathbf{B} = \gamma\left(\mathbf{B} - \frac{\mathbf{v}\times\mathbf{E}}{c^2}\right) = \gamma\left(B_0 - \frac{vE_0}{c^2}\right)\hat{\mathbf{x}} = \gamma B_0\left(1 - \frac{E_0^2}{B_0^2c^2}\right)\hat{\mathbf{x}}$$

The particle is at rest in the lab frame, so it will have velocity $-\mathbf{v} = -\frac{E_0}{B_0}\hat{\mathbf{y}}$ in the transformed frame. This just represents a particle moving in a magnetic field, so we get cyclotron motion

$$y'(t') = r'\sin\left(\frac{qB_0'}{m}t'\right)$$
$$z'(t') = -r\cos\left(\frac{qB_0'}{m}t'\right)$$

Lorentz transform to get back to the lab frame

$$y = \gamma\left(y' + vt'\right)$$
$$t = \gamma\left(t' + \frac{vy'}{c^2}\right)$$
$$z = z'$$

Substituting,

$$y\left(t'\right) = \gamma\left[r'\sin\left(\omega_c't'\right) + vt'\right]$$
$$z\left(t'\right) = -r'\cos\left(\omega_c't'\right) + r'$$

Which are parameterized by t'.

Problem 4

Field problem.

(a) What are expressions for the scalar and vector potentials for a point charge q at rest at the origin. Consider this the frame K.

Option #1, Scalar:

We can write a scalar potential and no vector potential such that

$$\begin{cases} V = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \\ \mathbf{A} = 0 \end{cases}$$

Option #2, Vector:

We can write a vector potential and no scalar potential such that

$$\begin{cases} V = 0 \\ \mathbf{A} = \frac{1}{4\pi\epsilon_0}\frac{qt}{r} \end{cases}$$

(b) What are expressions for the electric and magnetic fields from part (a)?

Solution:

Both cases return field functions:

$$\begin{cases} \mathbf{E} = -\frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \\ \mathbf{B} = 0 \end{cases}$$

(c) An inertial frame K' is moving at speed v in the x-direction with respect to frame K. What are expressions (in K' coordinates) for the scalar and vector potentials in the K' frame in terms of x, y, z, t.

Solution:

Note, the 4-potential $\mathbf{A}$ in frame K is chosen to be

$$A^\mu = \left(\frac{V}{c}, 0, 0, 0\right)$$

$$\begin{cases} \mathbf{A}^\mu \equiv \left(\frac{V}{c}, \mathbf{A}\right) \\ V^\mu \equiv \gamma V \end{cases}$$

Calculating $\mathbf{A}$,

$$A^0 = \gamma\left(A^0 - \frac{v}{c}A^1\right) = \gamma\frac{V}{c}$$
$$A^1 = \gamma\left(A^1 - \frac{v}{c}A^0\right) = \gamma\left(\frac{V}{c}\cdot\frac{v}{c}\right)$$

$$V^{\mu} = cA^0 = \gamma V = \frac{\gamma q}{4\pi\varepsilon_0 R}$$
$$\mathbf{A}^{\mu} = \left(A^1, 0, 0\right) = \left(-\gamma\frac{v}{c^2}V, 0, 0\right)$$

Where,

$$R \equiv \left[\gamma^2(x' + vt')^2 + y'^2 + z'^2\right]$$

Since,

$$\begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix}$$

(d) What are expressions for the electric and magnetic fields from part (c) in terms of x, y, z, t.

Solution:

$$V^{\mu} = \gamma V = \frac{\gamma q}{4\pi\varepsilon_0\sqrt{\gamma^2(x' + vt)^2 + y'^2 + z'^2}}$$
$$\mathbf{A}' = \left(-\gamma\frac{v}{c^2}V, 0, 0\right)$$

(e) Using results from the earlier parts of this problem, demonstrate that the quantity $\left(B^2 - \frac{E^2}{c^2}\right)$ is a Lorentz invariant.

Solution:

From these equations, we may recover fields:

$$\mathbf{B}^{\mu}\nabla' \times \mathbf{A}' = \left(\frac{\partial A'_x}{\partial y'}\hat{z} - \frac{\partial A'_x}{\partial z'}\hat{y}\right)$$
$$\mathbf{E}^{\mu} \equiv -\nabla V^{\mu} - \frac{\partial \mathbf{A}^{\mu}}{\partial t}$$

$$\begin{aligned}
\frac{\partial A'_x}{\partial y'} &= -\frac{\gamma^2 qv}{4\pi\epsilon_0 c^2}\frac{\partial}{\partial y'}\frac{1}{R} \\
&= -\frac{\gamma^2 qv}{4\pi\epsilon_0 c^2}\frac{1}{R^2}\frac{\partial R}{\partial y'} \\
&= -\frac{\gamma^2 qv}{4\pi\epsilon_0 c^2}\frac{1}{R^2}\frac{1}{2R}\frac{\partial}{\partial y'}(R^2) \\
&= -\frac{\gamma^2 qv}{4\pi\epsilon_0 c^2}\frac{1}{R^2}\left(\frac{1}{R}y'\right) \\
&= -\frac{\gamma^2 qv}{4\pi\epsilon_0 c^2 R^3}y'
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\partial A'_x}{\partial z'} &= -\frac{\gamma^2 qv}{4\pi\epsilon_0 c^2}\frac{\partial}{\partial z'}\frac{1}{R} \\
&= -\frac{\gamma^2 qv}{4\pi\epsilon_0 c^2}\frac{1}{R^2}\frac{\partial R}{\partial z'} \\
&= -\frac{\gamma^2 qv}{4\pi\epsilon_0 c^2}\frac{1}{R^2}\frac{1}{2R}\frac{\partial}{\partial z'}(R^2) \\
&= -\frac{\gamma^2 qv}{4\pi\epsilon_0 c^2}\frac{1}{R^2}\left(\frac{1}{R}z'\right) \\
&= -\frac{\gamma^2 qv}{4\pi\epsilon_0 c^2 R^3}z'
\end{aligned}$$

So,

$$\mathbf{B}' = -\frac{\gamma^2 qv}{4\pi c^2\epsilon_0 R^3}\left(y'\hat{z} - z'\hat{y}\right)$$

for $\mathbf{E}'$, the time derivative portion is zero, so we will have a gradient

$$\begin{aligned}
-\nabla V &= -\frac{\gamma q}{4\pi\epsilon_0 c^2}\cdot\nabla\frac{1}{R} \\
&= -\frac{\gamma q}{4\pi\epsilon_0 c^2}\frac{1}{R^2}\left(\frac{\partial R}{\partial x'} + \frac{\partial R}{\partial y'} + \frac{\partial R}{\partial z'}\right) \\
&= -\frac{\gamma^2 q}{4\pi\epsilon_0 c^2 R^3}(x' + vt') + y' + z'
\end{aligned}$$

Finally I confirm invariance, where we're looking to see

$$B'^2 - \frac{E'^2}{c^2} = -\left(\frac{q}{4\pi\epsilon_0 R^3}\right)^2\left(\gamma^2(x'+vt')^2 + y'^2 + z'^2 + \frac{v^2}{c^2}\left(y'^2 - z'^2\right)\right)$$