PHSX 631: Homework #3

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Grant Saggars

Problem 1

Griffiths Problem 9.10 The intensity of sunlight hitting the earth is about $1300\,\mathrm{W/m^2}$. If sunlight strikes a perfect absorber, what pressure does it exert? How about a perfect reflector? What fraction of atmospheric pressure does this amount to?

Solution:
Quoting Griffiths on this: When light falls (at normal incidence) on a perfect absorber, it delivers its momentum to the surface. In a time Δt, the momentum transfer is (Fig. 9.12) $\Delta p = \langle \mathbf{g} \rangle Ac\Delta t$ so the radiation pressure (average force per unit area) is

$$P = \frac{1}{A}\frac{\Delta p}{\Delta t} = \frac{1}{2}\epsilon_0 E_0^2 = \frac{I}{c}$$

On a perfect reflector the pressure is twice as great, because the momentum switches direction, instead of simply being absorbed.

$$P_{\mathrm{abs}} = \frac{1300\,\mathrm{W/m^2}}{c} \approx 4.33 \times 10^{-6}\,\text{Pa} = 4.28 \times 10^{-11}\,\text{atm}$$
$$P_{\mathrm{ref}} = 2\frac{1300\,\mathrm{W/m^2}}{c} \approx 8.67 \times 10^{-6}\,\text{Pa} = 8.56 \times 10^{-11}\,\text{atm}$$

Problem 2

Consider EM plane waves in free space propagating in the z-direction with wave vector $\mathbf{k} = k\hat{\mathbf{z}}$ and frequency ω. The wave amplitude is E_0 and the wave polarization vector is given by: $\hat{\mathbf{n}} = \hat{\mathbf{x}} + i\hat{\mathbf{y}}$. This is complex. Find the electric and magnetic fields versus z and t (the real fields). Sketch $\mathbf{E}(z, 0)$ and $\mathbf{B}(z, 0)$ components as functions of z. What is the Poynting vector and the intensity for this wave?

Solution:
We derived fields for EM plane waves and found that:

$$\begin{aligned}\mathbf{E}(\mathbf{r}, t) &= E_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)\hat{\mathbf{n}}}\\ \mathbf{B}(\mathbf{r}, t) &= \frac{1}{c} E_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}(\hat{\mathbf{k}} \times \hat{\mathbf{n}}) = \frac{1}{c}\mathbf{k} \times \mathbf{E}\end{aligned}$$

I will just take the real component for the final field. In the case of the magnetic field, there will be a $\hat{\mathbf{k}} \times \hat{\mathbf{n}}$ term which is given as:

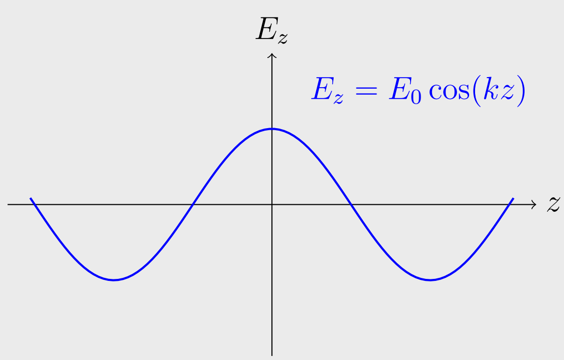
$$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & i & 0 \\ 0 & 0 & 1 \end{vmatrix} = i\hat{\mathbf{x}} + \hat{\mathbf{y}}$$

$$\begin{aligned}\mathbf{E}(\mathbf{r}, t) &= E_0[\cos{(kz - \omega t)} + i \sin{(kz - \omega t)}]\hat{\mathbf{z}}\\ &\underset{\text{Re}}{\to} E_0 \cos{(kz - \omega t)}\, \hat{\mathbf{z}}\\ \mathbf{B}(\mathbf{r}, t) &= \frac{E_0}{c}\left[\cos{(kz - \omega t)} + i \sin{(kz - \omega t)}\right](i\hat{\mathbf{x}} + \hat{\mathbf{y}})\\ &= \frac{E_0}{c}\left[(i\cos(kz - \omega t) - \sin(kz - \omega t))\hat{\mathbf{x}} + (\cos(kz - \omega t) + i\sin(kz - \omega t))\hat{\mathbf{y}}\right]\\ &\underset{\text{Re}}{\to} \frac{E_0}{c}\left[\cos{(kz - \omega t)}\,\hat{\mathbf{y}} - \sin{(kz - \omega t)}\,\hat{\mathbf{x}}\right]\end{aligned}$$

Since we have a lot of terms in the parenthesis flying around, let $\alpha = kz - \omega t$. The corresponding Poynting vector is

$$\begin{aligned}\mathbf{S} &= \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})\\ &= \frac{E_0^2}{\mu_0 c}(\cos^2\alpha\hat{\mathbf{x}} + \cos\alpha\sin\alpha\hat{\mathbf{y}})\end{aligned}$$

Interpreting these though, we get a very nice helical shape out of the magnetic fields that (appears to) propagate down the z-axis.



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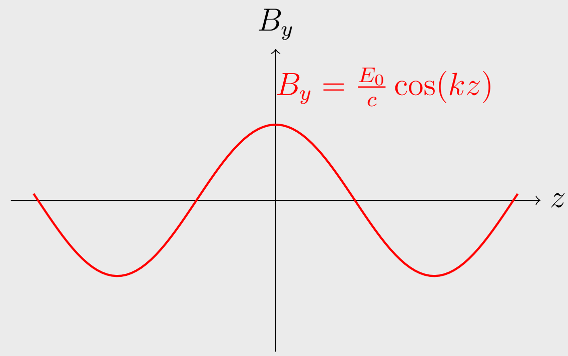

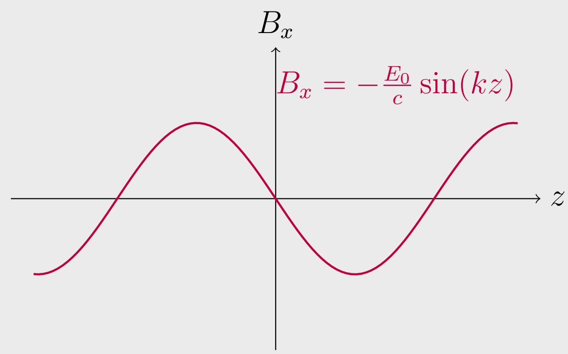

Problem 3

What are the units of momentum density? Units of Poynting vector $\mathbf{S}$? Units of Maxwell stress tensor $\mathbf{T}$? Demonstrate that the units of each term of the Poynting theorem and the momentum balance relation (with $\mathbf{T}$) are consistent.

Solution:

1. Momentum density: $\mathrm{kg}/(\mathrm{m}^2 \cdot \mathrm{s})$
2. Poynting vector: W/m^2 or $\mathrm{J}/(\mathrm{m}^2 \cdot \mathrm{s})$
3. Maxwell stress tensor: N/m^2 or J/m^3

Poynting theorem tells us that we will have units

$$[\mathrm{J}/(\mathrm{m}^2 \cdot \mathrm{s})]/\mathrm{m} + \mathrm{J}/(\mathrm{m}^3 \cdot \mathrm{s}) = (\mathrm{A}/\mathrm{m}^2) \cdot (\mathrm{V}/\mathrm{m}) = \mathrm{J}/(\mathrm{m}^3 \cdot \mathrm{s})$$

Momentum balance equation:

$$\frac{\partial \vec{p}}{\partial t} + \nabla \cdot \mathbf{T} = \vec{f}$$

will have units

$$[\mathrm{kg}/(\mathrm{m}^2 \cdot \mathrm{s})]/\mathrm{s} + [\mathrm{N}/\mathrm{m}^2]/\mathrm{m} = \mathrm{N}/\mathrm{m}^3 = \mathrm{kg}/(\mathrm{m}^2 \cdot \mathrm{s}^2)$$

Problem 4

A physical quantity, $y(x,t)$, is a function of distance x and time t, and also obeys the wave equation:

$$\frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2 y}{\partial x^2} = 0.$$

where c is a wave speed.

(a) What is a general solution to this wave equation?

Solution:

The wave equation is a classic example of a separable linear PDE. We suppose $u(x,t) = v(x)w(t)$. Then,

$$u_{tt} = v(x)w''(t), \qquad u_{xx} = v''(x)w(t) \tag{1}$$

so

$$u_{tt} = c^2 u_{xx}$$

I can rewrite this using the relations in equation 1, and separate my equations:

$$\frac{w''}{w} = c^2 \frac{v''}{v} = \lambda$$

This is equivalent of the two separate equations:

$$w'' = \lambda w$$
$$v'' = \frac{\lambda}{c^2} v$$

In the case $\lambda = 0$,

$$w(t) = A + Bt; \qquad v(x) = C + Dx$$

The linear combination of all solutions is in itself a solution. Now, there's also the case that $\lambda > 0$, and here $\lambda = \omega^2 > 0$, netting exponential solutions

$$w(t) = Ae^{\pm \omega t}; \qquad v(x) = Be^{\pm \frac{\omega}{c} x}$$

And finally there exist solutions where $\lambda = \omega^2 < 0$, corresponding to cos & sin equations:

$$w(t) = A\cos\omega t + B\sin\omega t; \qquad v(t) = C\cos\frac{\omega}{c}x + D\sin\frac{\omega}{c}x$$

So to summarize, depending on the λ, we either get linear, exponential, or sinusoidal functions. The product of w and v are the complete wave equation. Any combination of these solutions is valid, and just like the harmonic solutions this is a sum over all possible solutions thanks to superposition which we constrain to fit the physics. Usually we start by writing the equation as a sum of backwards and forwards travelling wave components. Formally writing the general solution:

$$u(x,t) = f(x+ct) + g(x-ct)$$

(b) The initial conditions on the y variable are given by the functions:

$$y(x, t=0) = \cos(2x) \quad \text{for} \quad -\pi/4 < x < +\pi/4;$$

and $y = 0$ for other x values.

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and

$$\frac{dy}{dt}(x, t = 0) = 0.$$

What is the wave solution, $y(x,t)$? Sketch $y(x,t)$ at a few times.

Solution:
I reference Strauss's PDE book as I work through this, as it is my first time working with such PDEs. I'd like to derive the general solution to the boundary value problem with the wave equations. Let the boundary conditions be

$$u(x,0) = \phi(x), \qquad u_t(x,0) = \psi(x)$$

We can start by analyzing the stationary problem, at time $t = 0$, where

$$\phi(x) = f(x) + g(x), \qquad \psi(x) = cf'(x) - cg(x)$$

equivalently,

$$\phi' = f' + g', \qquad \psi' = f' - g'$$

adding and subtracting these, we arrive at

$$f' = \frac{1}{2}\left(\phi' + \frac{\psi'}{c}\right), \qquad g' = \frac{1}{2}(\phi' - \frac{\psi'}{c})$$

integrating gives

$$f(s) = \frac{1}{2}\phi(s) + \frac{1}{2c}\int_0^s \phi + A$$
$$g(s) = \frac{1}{2}\phi(s) + \frac{1}{2c}\int_0^s \phi + B$$

substituting gives

$$u(x,t) = \frac{1}{2}\phi(x+ct) + \frac{1}{2c}\int_0^{x+ct}\psi + \frac{1}{2}\phi(x-ct) - \frac{1}{2c}\int_0^{x-ct}\psi$$

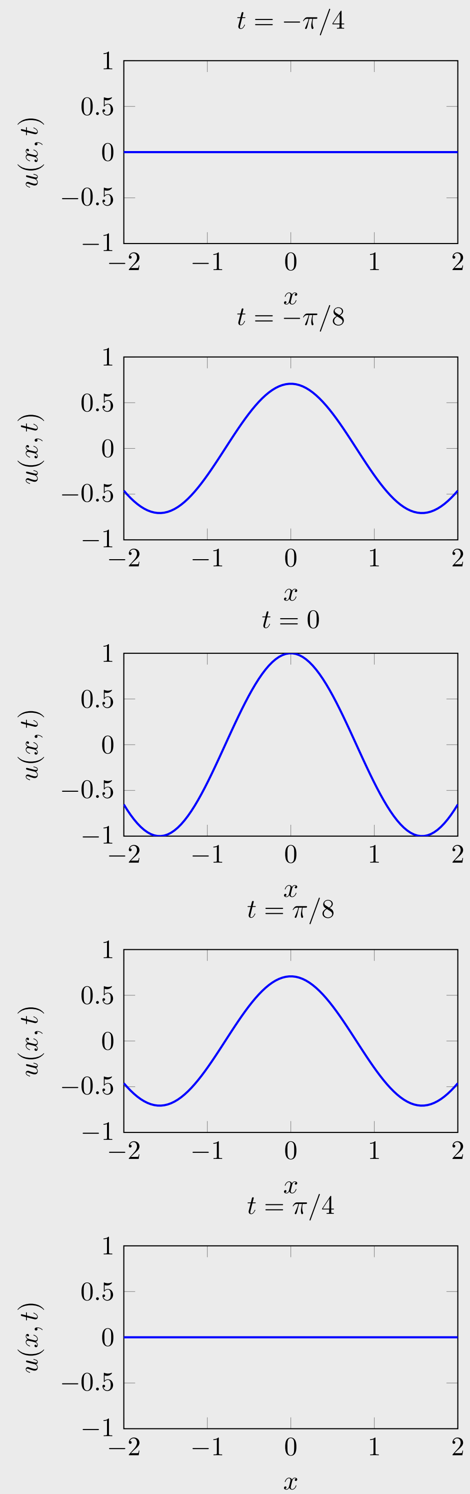
simplifies

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)\,ds$$

This is the solution formula for the initial-value problem, due to d'Alembert in 1746. In our case, $\phi = \cos(2x)$ and $\psi = 0$, and this solution is simply:

$$u(x,t) = \frac{1}{2}[\cos(2x + 2ct) + \cos(2x - 2ct)]$$

And this represents a stationary wave.


$t = -\pi/4$
$t = -\pi/8$
$t = 0$
$t = \pi/8$
$t = \pi/4$
$u(x,t)$
x
1
0.5
0
−0.5
−1
−2
−1
0
1
2