PHSX 521: Homework #1

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Problem 1

(1.2, Taylor) Two vectors are given as $\mathbf{b} = (1, 2, 3)$ and $\mathbf{c} = (3, 2, 1)$. Find $\mathbf{b} + \mathbf{c}$, $5\mathbf{b} - 2\mathbf{c}$, $\mathbf{b} \cdot \mathbf{c}$, and $\mathbf{b} \times \mathbf{c}$.

Solution:

i. $(1, 2, 3) + (3, 2, 1) = (4, 4, 4)$

ii. $(5, 10, 15) - (6, 4, 2) = (-1, 6, 13)$

iii. $(1, 2, 3) \cdot (3, 2, 1) = 3 + 4 + 3 = 10$

iv.

$$\begin{aligned}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix} &= (2-6)\hat{\mathbf{i}} - (1-9)\hat{\mathbf{j}} + (2-6)\hat{\mathbf{k}} \\ &= (-4, 8, -4)\end{aligned}$$

Problem 2

(1.5, Taylor) Find the angle between a body diagonal of a cube and any one of its face diagonals. [Hint: choose a cube with side 1 and with one corner at 0 and the opposite corner at the point (1,1,1). Write down the vector that represents a body diagonal and another that represents a face diagonal, then find the angle between them as in problem 1.4.]

Solution:
I will work with diagonal vector $\alpha = (1, 1, 1)$ and vertex at $\beta = (1, 1, 0)$. The angle between them is given by the following definition of the dot product

$$|\alpha||\beta| \cos\theta = \alpha \cdot \beta$$

$$\begin{aligned} |\alpha| &= \sqrt{3} \\ |\beta| &= \sqrt{2} \\ \alpha \cdot \beta &= 2 \end{aligned}$$

This gives

$$\theta = \cos\left(\frac{2}{\sqrt{6}}\right)$$

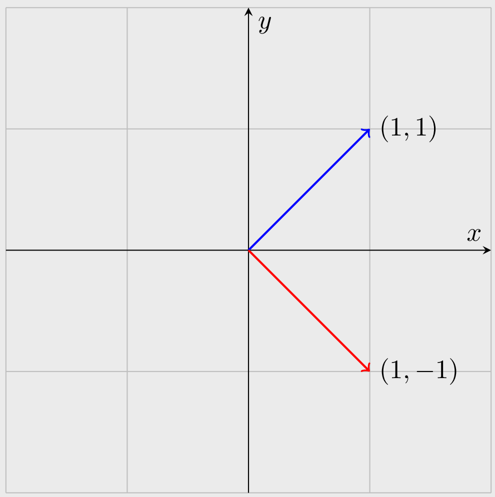
Problem 3

(1.6, Taylor) By evaluating their dot product, find the values of the scalar s for which the two vectors $\mathbf{b} = \hat{\mathbf{x}} + s\hat{\mathbf{y}}$ and $\mathbf{c} = \hat{\mathbf{x}} - s\hat{\mathbf{y}}$ are orthagonal. (remember that two vectors are orthagonal if and only if their dot product is zero.) Explain your answers with a sketch.

Solution:

$$\begin{aligned}(1, s) \cdot (1, -s) &= 0 \\ 1 - s^2 &= 0 \\ s^2 &= 1\end{aligned}$$

This implies s equals ± 1.



Problem 4

(1.19, Taylor) If **r**, **v**, **a** denote the position, velocity, and acceleration of a particle, prove that

$$\frac{d}{dt}[\mathbf{a} \cdot (\mathbf{v} \times \mathbf{r})] = \dot{\mathbf{a}} \cdot (\mathbf{v} \times \mathbf{r})$$

Solution:

$$\frac{d}{dt}[\mathbf{a} \cdot (\mathbf{v} \times \mathbf{r})] = \left[\frac{d}{dt}(\mathbf{a}) \cdot (\mathbf{v} \times \mathbf{r})\right] + \left[(\mathbf{a}) \cdot \frac{d}{dt}(\mathbf{v} \times \mathbf{r})\right] \tag{1}$$

The product rule applied to cross products makes the right hand side of the sum:

$$\begin{aligned} &= (\mathbf{a}) \cdot \left(\frac{d\mathbf{r}}{dt} \times \mathbf{v} + \mathbf{r} \times \frac{d\mathbf{v}}{dt}\right) \\ &= (\mathbf{a}) \cdot \left(\cancelto{0}{\mathbf{v} \times \mathbf{v}} + \mathbf{r} \times \mathbf{a}\right) \end{aligned}$$

Because $\mathbf{r} \times \mathbf{a}$ results in a vector orthagonal to **a**, the resulting dot product here will be zero. As a result the only part of equation (1) which remains is the left side of the sum. Therefore:

$$\frac{d}{dt}[\mathbf{a} \cdot (\mathbf{v} \times \mathbf{r})] = \dot{\mathbf{a}} \cdot (\mathbf{v} \times \mathbf{r})$$

Problem 5

(1.22, Taylor) The two vectors **a** and **b** lie in the xy plane and make angles α and β with the x axis. **(a)** By evaluating $\mathbf{a} \cdot \mathbf{b}$ in two ways [namely using (1.6) and (1.7)] prove the well-known trig identity

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

(b) By similarly evaluating $\mathbf{a} \times \mathbf{b}$ prove that

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$

Solution:

(a) The relevant trig identities here are:

$$\mathbf{a} \cdot \mathbf{b} = |a||b|\cos(\alpha - \beta) \tag{1}$$

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= a_x b_x + a_y b_y \\ &= (|a|\cos(\alpha))(|b|\cos(\beta)) + (|a|\sin(\alpha))(|b|\sin(\beta)) \\ &= |a||b|(\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)) \end{aligned} \tag{2}$$

Equating (1) and (2) allows us to get a relation in terms of angles alone:

$$\cancel{|a||b|}\cos(\alpha - \beta) = \cancel{|a||b|}(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$$

(b) Recall that components given in terms of (r, θ) in cartesian coordinates are $(|x|\cos\theta, |y|\sin\theta)$. The cross product between two such vectors is therefore:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ |a|\cos\alpha & |a|\sin\alpha & 0 \\ |b|\cos\beta & |b|\sin\beta & 0 \end{vmatrix} = \{(|a||b|\cos\alpha\sin\beta) - |a||b|(\sin\alpha\cos\beta)\}\hat{\mathbf{z}} \tag{3}$$

To complete the proof we need to utilize another definition of the cross product which I have never seen until now:

$$\mathbf{a} \times \mathbf{b} = |a||b|\sin(\alpha - \beta)(-\hat{\mathbf{z}}) \tag{4}$$

Equating (3) and (4) leaves us with:

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$