PHSX 531: Homework #12

December 10, 2024

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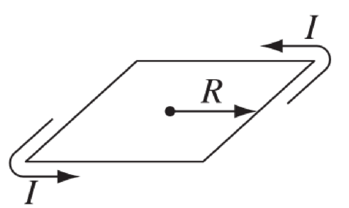
Problem 1

(3 pts) Find the magnetic field at the center of a square loop, which carries a current I. Let R be the (shortest) distance from the center to side.



Solution:
Each side has a length $\ell = 2R$. Taking $\hat{x}$ to be the horizontal, $\hat{y}$ to be the vertical, and $\hat{z}$ to be out of the page, the magnetic field points in the $\hat{z}$ direction (taking the line normal vector to be towards the middle of the loop).

$$\mathscr{r}^2 = x^2 + y^2, \quad \hat{\mathscr{r}} = \frac{x\hat{x} + y\hat{y}}{(x^2+y^2)^{1/2}}, \quad dl \times \hat{\mathscr{r}} = \frac{R}{(x^2+y^2)} \quad \text{(regardless of side)}$$

$$\frac{\mu_0}{4\pi} I \int_{-R}^{R} \frac{R\hat{z}}{(x^2+y^2)^{3/2}}\, dl$$

This integral is symmetric over each part, so it would be equivalent to taking 4 times the result of this integral. The integral over the bottom segment will be:

$$\begin{aligned}
\mathbf{B} &= \frac{\mu_0 I}{4\pi} \int_{-R}^{R} \frac{R\,dx}{(x^2+R^2)^{3/2}} \hat{z} \\
&= \frac{\mu_0 I}{4\pi} \int_{\pi/4}^{-\pi/4} \frac{R(R\sec^2\theta)}{(R)^{3/2}(1+\tan^2\theta)}\, d\theta && x = R\tan\theta \\
& && dx = R\sec^2\theta \\
&= \frac{\mu_0 I}{4\pi R} \int_{-\pi/4}^{\pi/4} \frac{sec^2\theta}{sec^3\theta}\, d\theta \\
&= \frac{\mu_0 I}{4\pi R} \int_{-\pi}^{\pi} \cos\theta\, d\theta \\
&= \frac{\mu_0 I}{4\pi R} [\sin(\pi/4) - \sin(-\pi/4)] \\
&= \frac{\sqrt{2}\mu_0 I}{\pi R}
\end{aligned}$$

Problem 2

(3 pts) Find the magnetic field at the center of a circular loop of radius R, which carries a counterclockwise current I when looking from ”above”.

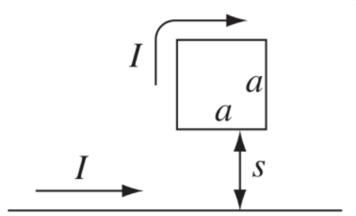
Solution:
Setting this up in the same way as before ($\hat{z}$ out of the page), magnetic field will be pointing into the page if $\hat{s}$ is towards the middle of the loop.

$$dl = Rd\phi\hat{\phi}, \qquad \mathscr{r} = s = R, \qquad \hat{\mathscr{r}} = \hat{s}, \qquad dl \times \hat{\mathscr{r}} = R\, d\phi\hat{z}$$

$$\mathbf{B} = -\frac{\mu_0 I}{4\pi}\int_0^{2\pi}\frac{R\, d\phi}{R^2}\hat{z} = \frac{\mu_0 I}{2R}\hat{z}$$

Problem 3

(4 pts) Find the force on a square loop placed near an infinite straight wire. Both the loop and the wire carry a steady current I.



Solution:
We have magnetic force $\mathbf{F} = I\int(d\mathbf{l}\times\mathbf{B})$. The force due to the magnetic magnetic fields on both sides cancel, since they have the same magnitude and opposite direction. We have already found the magnetic field due to a line of current a few times before,

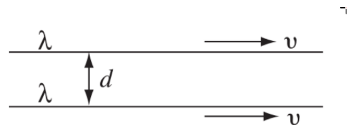
$$\mathbf{B}_{\text{bottom}} = \frac{\mu_0 I}{2\pi s}\hat{z}$$
$$\mathbf{B}_{\text{top}} = -\frac{\mu_0 I}{2\pi(s+a)}\hat{z}$$

This gives magnetic force:

$$\begin{aligned}
\mathbf{F} &= I\int_0^a \hat{x}\times\frac{\mu_0 I}{2\pi s}\hat{z}\, dx - I\int_0^a \hat{x}\times\frac{\mu_0 I}{2\pi(s+a)}\hat{z}\, dx \\
&= I\int_0^a \frac{\mu_0 I}{2\pi s}\hat{y}\, dx - I\int_0^a \frac{\mu_0 I}{2\pi(s+a)}\hat{y}\, dx \\
&= \frac{\mu_0 I^2 a}{2\pi s}\hat{y} - \frac{\mu_0 I^2 a}{2\pi(s+a)}\hat{y} \\
&= \frac{\mu_0 I^2 a}{2\pi}\left(\frac{1}{s} - \frac{1}{s+a}\right)
\end{aligned}$$

Problem 4

Extra Credit (4 pts) Suppose you have two infinite straight line charges λ, a distance d apart, moving along at a constant speed v. How great would v have to be in order for the magnetic attraction to balance the electric repulsion? Work out the actual number. Is this a reasonable sort of speed?



Solution:
In example 5.5, we found that in this configuration,

$$\mathbf{F} = I_2 \left(\frac{\mu_0 I_1}{2\pi d} \right) \int dl$$

and per unit length,

$$f = \frac{\mu_0}{2\pi} \frac{I_1 I_2}{d}$$

The electric field per length due to this configuration is (using $\hat{y}$ up)

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{2\lambda}{y} \hat{y}$$

And the charge distribution acting on this is λ from the other wire in the same direction. So they are repulsive with force:

$$f = \frac{1}{4\pi\epsilon_0} \frac{2\lambda^2}{d}$$

Now equating these, using $I = \lambda v$:

$$\begin{aligned} \frac{\mu_0}{2\pi} \frac{\lambda^2 v^2}{d} &= \frac{1}{4\pi\epsilon_0} \frac{2\lambda^2}{d} \\ v^2 &= \frac{1}{\mu_0 \epsilon_0} \end{aligned}$$

The actual number is

$$(8.85 \times 10^{-12} 4\pi \times 10^{-7})^{-1/2} = 2.998633 \cdots \times 10^8$$

Which is the speed of light. *I wonder if doing this calculation in a different unit system would actually put c there.*