

Homework 4

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1. (4 pts) Hydrogen emits light at certain frequencies. The so-called Balmer series of Hydrogen includes two frequencies of light of 457THz and 617THz, where $T = \text{Tera}$. (There are many other frequencies as well, but for now we will focus on these two.) These frequencies are measured in a lab when the Hydrogen atom is at rest. Say there's a galaxy at redshift $z = 3$. If on Earth we observe the Balmer series of Hydrogen in that the galaxy, what two frequencies will we observe? Assuming the galaxy is moving directly away from from us, what speed is it moving at?

$$1. \quad 1 + z = \frac{f_{\text{source}}}{f_{\text{observed}}} \implies 4 = \frac{457 \text{ THz}}{f_{\text{observed}}} \implies f_{\text{observed}} = \frac{457 \text{ THz}}{4} = 118.75 \text{ THz}$$

$$2. \quad 1 + z = \frac{f_{\text{source}}}{f_{\text{observed}}} \implies 4 = \frac{617 \text{ THz}}{f_{\text{observed}}} \implies f_{\text{observed}} = \frac{617 \text{ THz}}{4} = 154.25 \text{ THz}$$

$$3. \quad f_{\text{observed}} = f_{\text{source}} \sqrt{\frac{1 - u/c}{1 + u/c}} \implies 118.75 = 457 \sqrt{\frac{1 - u/c}{1 + u/c}} \implies 261.869 \times 10^6 \text{ m/s}$$

2. According to observer \mathcal{O} , a blue flash occurs at $x_b = 9.8 \text{ m}$ at time $t_b = 0.164\mu\text{s}$, and a red flash occurs at $x_r = 30.4 \text{ m}$ at time $t_r = 0.214\mu\text{s}$. According to an observer \mathcal{O}' , who is in motion relative to \mathcal{O} at velocity u , the two flashes appear to be simultaneous.

(a) (4 pts) Find the velocity u and distance between the flashes according to \mathcal{O}' .

$$1. \quad \gamma(u) = \sqrt{\frac{1}{1 - u^2/c^2}}$$

$$2. \quad \Delta t = \gamma(u) \left(t - \frac{u}{c^2} x \right) \implies \gamma(u) \neq 0 \quad \text{because } u \neq c$$

1. Solving for the velocity u :

$$\cancel{\gamma(u)} \left(0.164 \times 10^{-6} - \frac{u}{c^2} (9.8) \right) = \cancel{\gamma(u)} \left(0.214 \times 10^{-6} - \frac{u}{c^2} (30.4) \right) = 2.181441 \times 10^8 \text{ m/s}$$

$$2. \quad \text{Solving for the distance } L: L = \frac{L_0}{\gamma(u)} = \frac{30.4 - 9.8}{1.457836} = 14.13053 \text{ m}$$

(b) (3 pts) Use your answer from (a) to show that the invariant distance Δs is indeed the same according to the observers in \mathcal{O} and \mathcal{O}' .

$$\begin{aligned} 1. \quad & \text{The invariant distance is defined as: } s^2 = x^2 + y^2 + z^2 - c^2 t^2 \\ & \implies 20.6^2 - c^2 (0.05 \times 10^{-6})^2 = 14.13053^2 - c^2 (14.13053 / 2.181441 \times 10^8)^2 \\ & \implies -199.67 = -177.441 \text{ (I know this isn't correct but I can't figure out why)} \end{aligned}$$

3. The TARDIS is moving directly toward the Defiant at $0.76c$. Through some wibbly wobbly, timey wimey stuff, the Doctor on the TARDIS throws a ball 5 km forward in the same direction of motion as the TARDIS moving towards the Defiant and the ball stays in the air for $10 \text{ } \mu\text{s}$ according to the Doctor.

(a) (5 pts) How far does the ball move and for how long is it in the air according to Captain Sisko on the Defiant?

1. The time and distance given are in proper time and length:

$$L = \frac{L_0}{\gamma(0.76c)} = L_0 \sqrt{1 - \frac{u^2}{c^2}} = 3.25 \text{ km}$$

- 2.

$$t = \gamma(u)t_0 = 10(1.539) = 15.39 \text{ } \mu\text{s}$$

(b) (5 pts) What is the speed of the ball according to Commander Worf on the Defiant? Calculate in 2 different ways.

$$1. \ r = \frac{d}{t} \implies \frac{3.75 \times 10^3}{13.33 \times 10^{-6}} = 2.813203 \times 10^8 \text{ m/s}$$

$$2. \ v' = \frac{v + u}{1 + \frac{uv}{c^2}} \implies \frac{0.76c - 2.45 \times 10^8}{1 - \frac{(0.76c)(2.45 \times 10^8)}{c^2}}$$

(c) (3 pts) Show that the invariant distance Δs is the same according to Donna on the TARDIS and Lieutenant Commander Dax on the Defiant.

1. The invariant distance is defined as: $s^2 = x^2 + y^2 + z^2 - c^2 t^2$

$$(5 \times 10^3)^2 - (10 \times 10^{-6})^2 c^2 = (3.75 \times 10^3)^2 - (13.33 \times 10^{-6})^2 c^2 \\ \implies 1.6 \times 10^7 = -1.07 \times 10^6$$

Once again, I don't know why this isn't right.

4. Extra Credit: (4 pts) Observer \mathcal{O} fires a light beam at 20° above the x -axis. Find the three components of velocity according to an observer \mathcal{O}' moving at a speed of u in the x -direction. Show that \mathcal{O}' also measures the value c for the speed of light.

1. Break the velocity of the light beam into its components:

$$x : \ c \cos(20) = 2.817 \times 10^8 \qquad y : \ c \sin(20) = 1.025 \times 10^8$$

2. Add the velocities relativistically:

$$v'_x = \frac{v_x - u}{1 - \frac{v_x u}{c^2}} \implies \frac{2.817 \times 10^8 + u}{1 + \frac{(2.817 \times 10^8)(u)}{c^2}} \\ v'_y = \frac{v_y \sqrt{1 - \frac{u^2}{c^2}}}{1 - v_x \frac{u}{c^2}} \implies \frac{1.025 \times 10^8 \sqrt{1 - \frac{u^2}{c^2}}}{1 - \frac{2.817 \times 10^8 (u)}{c^2}} \\ v'_z = \frac{v_z \sqrt{1 - \frac{u^2}{c^2}}}{1 - v_x \frac{u}{c^2}} \implies 0$$