MATH 526: Homework #3

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Problem 3

Let W be a random variable giving the number of heads minus the number of tails in three tosses of a coin. List the elements of the sample space S for the three tosses of the coin and to each sample point assign a value w of W.

Solution:

$$\begin{aligned}
\{HHH\}&: \quad 3 \\
\{HTH\}&: \quad 1 \\
\{HHT\}&: \quad 1 \\
\{THH\}&: \quad 1 \\
\{HTT\}&: \quad -1 \\
\{THT\}&: \quad -1 \\
\{TTH\}&: \quad -1 \\
\{TTT\}&: \quad -3
\end{aligned}$$

Problem 7

The total number of hours, measured in units of 100 hours, that a family runs a vacuum cleaner over a period of one year is a continuous random variable X that has the density function

$$f(x) = \begin{cases} x, & 0 < x < 1, \\ 2 - x, & 1 \leq x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the probability that over a period of one year, a family runs their vacuum cleaner

(a) Less than 120 hours

Solution:

$$\int_0^{1.2} f(x)\,dx = \int_0^1 x\,dx + \int_1^{1.2} 2 - x\,dx = \frac{1}{2} + \frac{9}{50} = 68\%$$

(b) Between 50 and 100 hours.

Solution:

$$\int_{0.5}^{1} f(x)\,dx = \int_{0.5}^{1} x\,dx = 37.5\%$$

Problem 13

The probability distribution of X, the number of imperfections per 10 meters of a synthetic fabric in continuous rolls of uniform width, is given by

x	0	1	2	3	4
f(x)	0.41	0.37	0.16	0.05	0.01

Construct the cumulative distribution function of X.

Solution:

x	0	1	2	3	4
c(x)	0.41	0.78	0.94	0.99	1.0

Problem 14

The waiting time, in hours, between successive speeders spotted by a radar unit is a continuous random variable with cumulative distribution function $F(x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-8x}, & x \geq 0. \end{cases}$

Find the probability of waiting less than 12 minutes between successive speeders

Problem 14 continued on next page...

(a) Using the cumulative distribution function of X;

Solution:

$$\int 1 - e^{-8x}\, dx = 0.2 - \frac{1}{8}e^{-8x}\Big|_0^{0.2} \approx 10\%$$

(b) Using the probability density function of X.

Solution:

$$\int_0^{0.2} 1 - e^{-8x}\, dx = 0.2 - \frac{1}{8}e^{-1.6} + \frac{1}{8} \approx 10\%$$

Problem 17

A continuous random variable X that can assume values between $x = 1$ and $x = 3$ has a density function given by $f(x) = \frac{1}{2}$.

(a) Show that the area under the curve is equal to 1.

Solution:

$$\int_1^3 \frac{1}{2}\, dx = (1/2)(2) = 1$$

(b) Find $P(2 < X < 2.5)$.

Solution:

$$\int_2^{2.5} \frac{1}{2}\, dx = (1/2)(1/2) = 0.25$$

(c) Find $P(X \leq 1.6)$.

Solution:

$$\int_1^{1.6} (\frac{1}{2})\, dx = (1/2)(0.6) = 0.3$$

Problem 33

Suppose a certain type of small data processing firm is so specialized that some have difficulty making a profit in their first year of operation. The probability density function that characterizes the proportion Y that make a profit is given by

$$f(y) = \begin{cases} ky^4(1-y)^3, & 0 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) What is the value of k that renders the above a valid density function?

Solution:

$$\frac{1}{k} = \int_0^1 y^4(1-y)^3\ dx = \frac{1}{280} \implies k = 280$$

(b) Find the probability that at most 50% of the firms make a profit in the first year.

Solution:

$$\int_0^{0.5} 280y^4(1-y)^3\ dx \approx 36.4\%$$

(c) Find the probability that at least 80% of the firms make a profit in the first year.

Solution:

$$\int_0^{0.8} 280y^4(1-y)^3\ dx \approx 94\%$$

Problem 36

On a laboratory assignment, if the equipment is working, the density function of the observed outcome, X, is given by

$$f(x) = \begin{cases} 2(1-x), & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Calculate $P(X \leq \frac{1}{3})$.

Solution:

$$\int_0^{1/3} 2(1-x)\,dx = 5/9$$

(b) What is the probability that X will exceed 0.5?

Solution:

$$\int_{0.5}^{1} 2(1-x)\,dx = 1/4$$

(c) Given that $X \geq 0.5$, what is the probability that X will be less than 0.75?

Solution:

$$\int_{0.5}^{0.75} 2(1-x)\,dx = 3/16$$

Problem 38

If the joint probability distribution of X and Y is given by $f(x,y) = \frac{x+y}{30}$ for $x = 0, 1, 2, 3;\ y = 0, 1, 2$, find $P(0 < X < 1 | Y = 2)$.

Solution:

$$\int_0^1 \frac{x+2}{30}\,dx = 1/12$$

Problem 41

A candy company distributes boxes of chocolates with a mixture of creams, toffees, and cordials. Suppose that the weight of each box is 1 kilogram, but the individual weights of the creams, toffees, and cordials vary from box to box. For a randomly selected box, let X and Y represent the weights of the creams and the toffees, respectively, and suppose that the joint density function of these variables is

$$f(x, y) = \begin{cases} 24xy, & 0 \leq x \leq 1, 0 \leq y \leq 1, x + y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the probability that in a given box the cordials account for more than $1/2$ of the weight.

Solution:

We want to find the $P(1 - X - Y) > 0.5$ which simplifies to $P(X + Y) < 0.5$:

$$Z = \int_0^1 \int_0^{0.5-x} 24xy \; dy \; dx = \int_0^1 12x(-x + 0.5)^2 \; dx = 0.5$$

(b) Find the marginal density for the weight of the creams.

Solution:

$$g(x) = \int_0^1 24xy \; dy = 12x$$

(c) Find the probability that the weight of the toffees in a box is less than $1/8$ of a kilogram if it is known that creams constitute $3/4$ of the weight.

Solution:

$$\int_0^{1/8} 24(3/4)y \; dy = 9/64$$

Problem 43

Let X denote the reaction time, in seconds, to a certain stimulus and Y denote the temperature (°F) at which a certain reaction starts to take place. Suppose that two random variables X and Y have the joint density

$$f(x, y) = \begin{cases} 4xy, & 0 < x < 1, 0 < y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find

(a) $P(0 \le X \le \frac{1}{2}$ and $\frac{1}{4} \le Y \le \frac{1}{2})$;

Solution:

$$\int_0^{1/2} \int_{1/4}^{1/2} 4xy \; dy \; dx = 3/64$$

(b) $P(X < Y)$.

Solution: $P(X < Y)$ implies a region of $0 < y < 1$ and $0 < x < y$:

$$\int_0^1 \int_0^y 4xy \; dx \; dy = 1/2$$

Problem 45

Let X denote the diameter of an armored electric cable and Y denote the diameter of the ceramic mold that makes the cable. Both X and Y are scaled so that they range between 0 and 1. Suppose that X and Y have the joint density

$$f(x, y) = \begin{cases} \frac{1}{y}, & 0 < x < y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find $P(X + Y > \frac{1}{2})$.

Solution:

$$\int_0^1 \int_{1/2-y}^1 \frac{1}{y} \; dx \; dy = 2\ln(2) - 1 \approx 38\%$$

Problem 47

The amount of kerosene, in thousands of liters, in a tank at the beginning of any day is a random amount Y from which a random amount X is sold during that day. Suppose that the tank is not resupplied during the day so that $x \leq y$, and assume that the joint density function of these variables is

$$f(x, y) = \begin{cases} 2, & 0 < x \leq y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Determine if X and Y are independent.

Solution:

$$MDF(x) = \int_x^1 2\ dy = 2 - 2x$$
$$MDF(y) = \int_0^y 2\ dx = 2y$$

It is obvious that their JDF is not equal to the product of the MDF, therefore they are not independent.

Problem 56

The joint density function of the random variables X and Y is

$$f(x, y) = \begin{cases} 6x, & 0 < x < 1, 0 < y < 1 - x, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Show that X and Y are not independent.

Solution:

$$MDF(x) = \int_0^{1-x} 6x\ dy = 3(1-y)^2$$
$$MDF(y) = \int_0^1 6x\ dx = 3$$

Their product is clearly not the JDF, therefore they are not independent.

Problem 56 continued on next page. . .

(b) Find $P(X > 0.3|Y = 0.5)$.

Solution:

$$P(X > 0.3|Y = 0.5) = \frac{\int_{0.3}^{1} 6x\ dx}{\int_{0}^{1} 6x\ dx} = \frac{2.73}{3} = 0.91$$