Time-Harmonic Solutions of Linear Systems

We **electrical engineers** often assume that current and/or voltages sources are specifically **time harmonic** (i.e., sinusoidal), oscillating at some radial **frequency** ω (e.g., $\cos\omega t$):

$$\boxed{v(t) = V_0 \cos[\omega t + \varphi]}$$

Q: *But why **do** we seemingly **always** assume a **sinusoidal** function of time?*

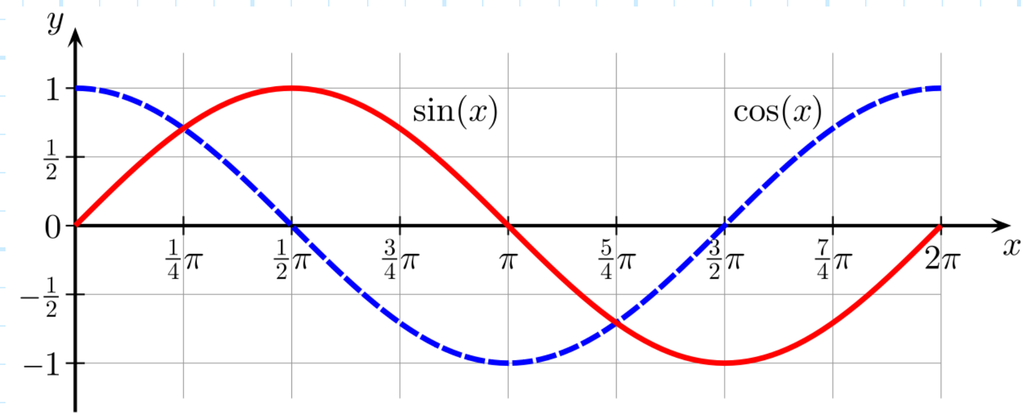
*Why not a **square wave**, or **triangle wave**, or a "sawtooth" function?*

A: We assume **sinusoids** because they have a **very** special property!

Eigen Functions and Linear Circuits

Sinusoidal time functions—and **only** sinusoidal time functions—are:

*the **eigen functions** of all **linear**, **time-invariant** systems!*



Q: ???

A: Say a **sinusoidal** voltage source with frequency ω is used to excite a **linear**, time-invariant circuit.

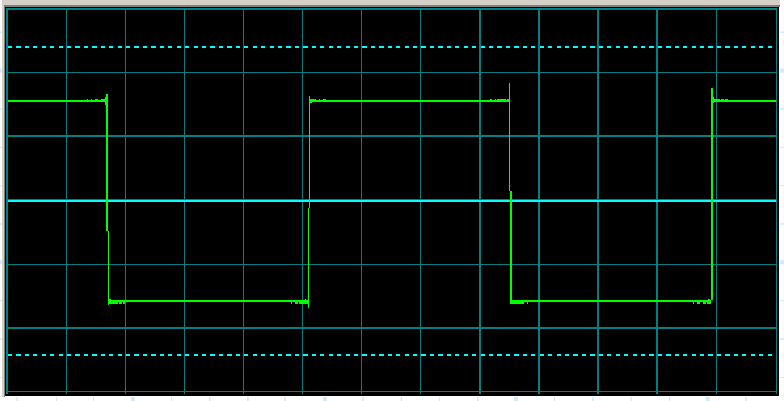
Then the current and voltage at each and **every** point with the circuit will **likewise** vary sinusoidally—at **precisely** the same frequency ω!

The heartbreak of linear distortion

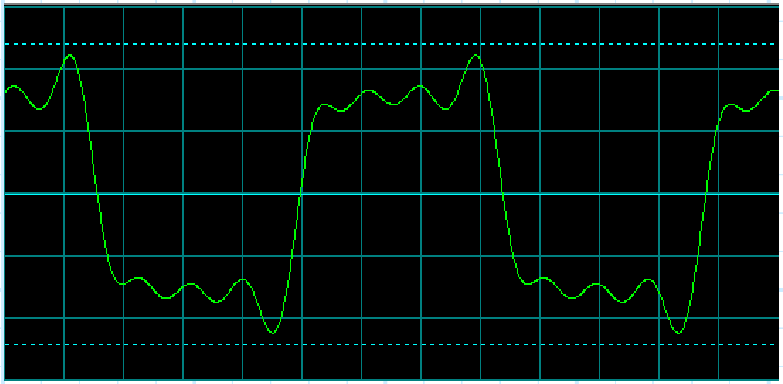
Q: *So what? Isn't that **obvious**?*

A: Not at all!

If you were to excite a linear circuit with a **square wave**, or triangle wave, or sawtooth, you would find that—generally speaking—**nowhere else** in the circuit is the voltage a perfect square wave, triangle wave, or sawtooth.



http://www.ultracad.com/square.htm

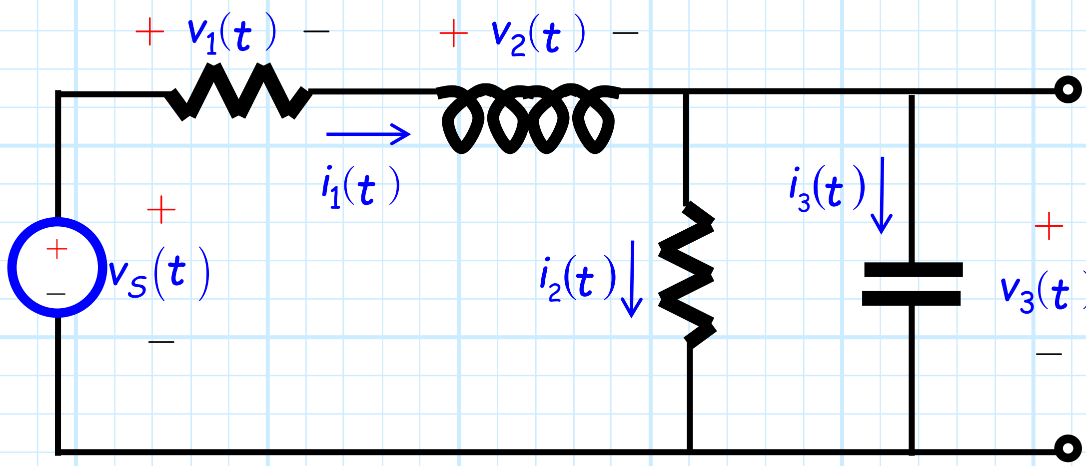


The linear circuit will effectively **distort** the input signal into **something else**!

Better brush up on your diffy-Q

Q: *Into **what function** will the input signal be distorted?*

A: It depends—both on the original form of the **input signal**, and the parameters of the **linear circuit**.



At **different** points within the circuit we will discover **different** functions of time—**unless**, of course, we use a **sinusoidal** input.

For a **sinusoidal** excitation, we find at **every** point within circuit an **undistorted** sinusoidal function!

Only the magnitudes and phases are different

Q: *So, the* ***sinusoidal*** *function at every point in the circuit is* ***exactly*** *the same as the* ***input*** *sinusoid?*

A: Not quite **exactly** the same.

Although at **every** point within the circuit the voltage will be **precisely** sinusoidal (with frequency ω), we find that:

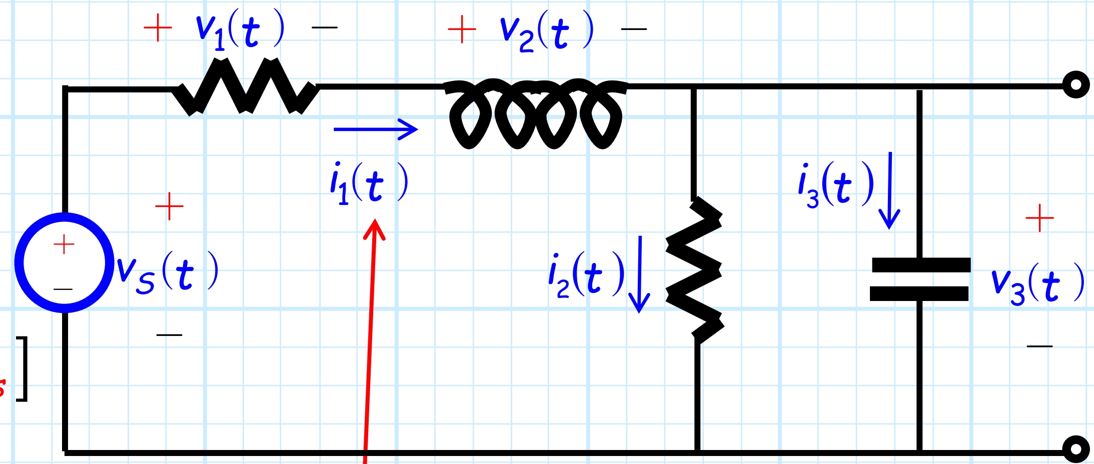
1. the **magnitude** of the sinusoid will generally be **different** at each and every point within the circuit, and

2. **relative phase** of the sinusoid will generally be **different** at each and every point within the circuit.

See what I mean?

Thus, *if* a linear circuit is excited—for example—by a **sinusoidal voltage source** of the form:

$$v_s(t) = v_{0s} \cos[\omega t + \varphi_s]$$



then the currents and voltages everywhere in the circuit will be precisely the same sinusoid, at precisely the same frequency, but with dissimilar **magnitudes** v_{0n} and **phases** φ_{vn} :

$$v_1(t) = v_{01} \cos[\omega t + \varphi_{v1}] \qquad i_1(t) = i_{01} \cos[\omega t + \varphi_{i1}]$$

$$v_2(t) = v_{02} \cos[\omega t + \varphi_{v2}] \qquad i_2(t) = i_{02} \cos[\omega t + \varphi_{i2}]$$

$$v_3(t) = v_{03} \cos[\omega t + \varphi_{v3}] \qquad i_3(t) = i_{03} \cos[\omega t + \varphi_{i3}]$$

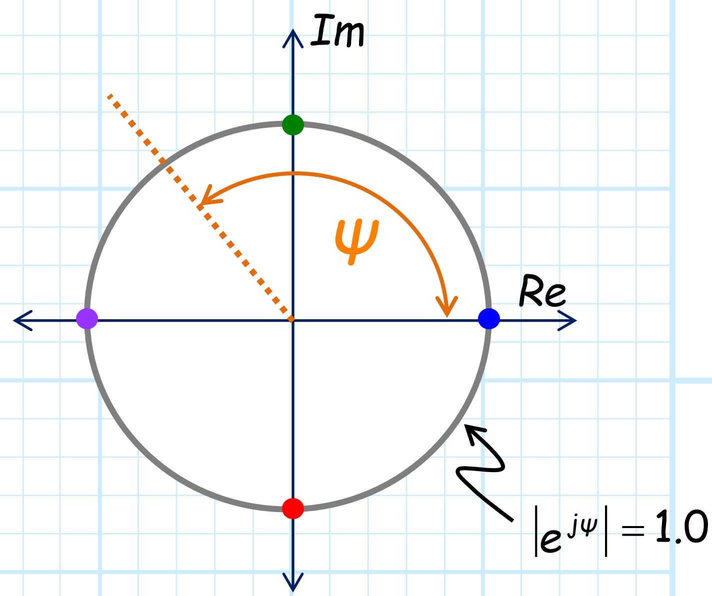
The Houston Eulers were named for him!

Now, consider **Euler's equation**, which states:

$$\boxed{e^{j\psi} = \cos\psi + j\sin\psi}$$

Thus, for example:

$$
\begin{aligned}
e^{j0} &= \cos 0 + j\sin 0 = 1 + j\,0 = 1 \\
e^{j\left(\pi/2\right)} &= \cos\left(\pi/2\right) + j\sin\left(\pi/2\right) = 0 + j\,1 = j \\
e^{j\pi} &= \cos\left(\pi\right) + j\sin\left(\pi\right) = -1 + j\,0 = -1 \\
e^{j\left(3\pi/2\right)} &= \cos\left(3\pi/2\right) + j\sin\left(3\pi/2\right) = 0 + j\left(-1\right) = -j \\
\left|e^{j\psi}\right|^2 &= \cos^2\psi + \sin^2\psi = 1.0
\end{aligned}
$$



Cosine: it's just the real part of an imaginary exponential

Thus, it is apparent that:

$$\cos\psi = Re\{\cos\psi + j\sin\psi\} = Re\{e^{j\psi}\}$$

and so **if**:

$$\psi = \omega t + \varphi$$

then we can conclude that a **time-harmonic** voltage **can be expressed as**:

$$
\begin{aligned}
v_n(t) &= V_{0n}\cos\left(\omega t + \varphi_n\right) \\
&= V_{0n}\, Re\left\{e^{j\left(\omega t + \varphi_n\right)}\right\} \\
&= Re\left\{\underbrace{V_{0n}\, e^{+j\varphi_n}}\; e^{j\omega t}\right\}
\end{aligned}
$$

Look at this !!

Hmmm… a complex number *also* has magnitude and phase—what a coincidence!

It is thus apparent that we can uniquely specify the time-harmonic voltage with the **complex value** V_n :

$$\boxed{V_n \doteq v_{0n} e^{+j\varphi_n}}$$

where the magnitude of the complex value is the magnitude of the sinusoid:

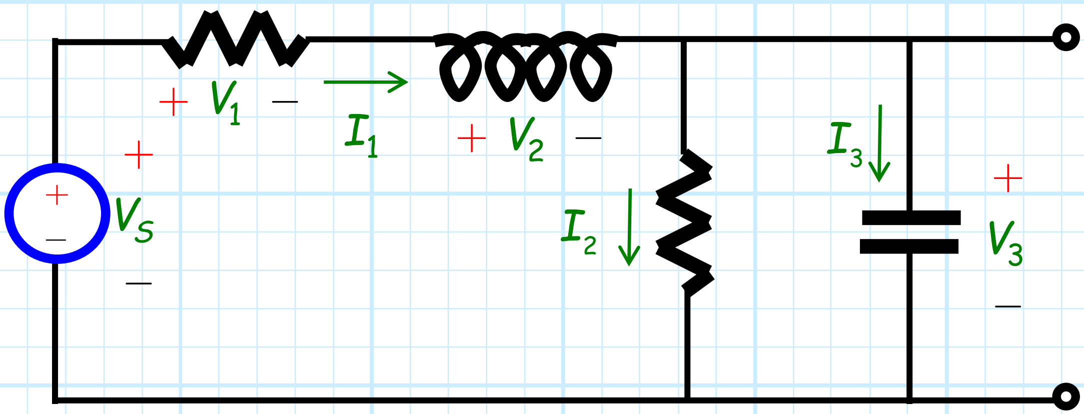
$$v_{0n} = \left|V_n\right|$$

and the phase of the complex value is the relative phase of the sinusoid:

$$\varphi_n = arg\left\{V_n\right\}$$

This is unfathomably important!!!!!!!!!!!!!!!!!!!!!

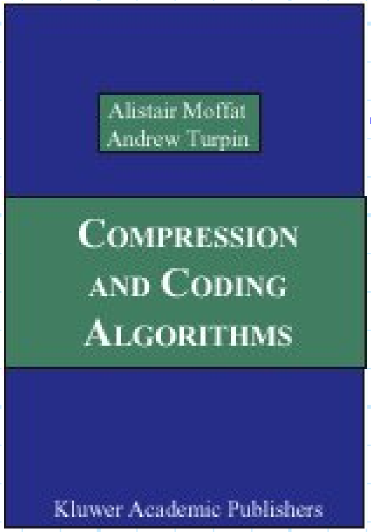
We **electrical engineers** almost **always** describe the activity of a linear circuit (**if** excited by time-harmonic sources) in terms of **complex values** V_n and I_n—and **only** in terms of these **complex** values.



It is unfathomably important that you understand what these complex values mean !!!!!!!!!!!!!!!!!!!!!!!!

You **must understand** what these **complex values** are telling you about the currents and voltages in a **linear** circuit.

Electrical engineers have compression algorithms too!



Perhaps it's helpful to think about these complex numbers as sort of a **compression algorithm**, with the important information **"embedded"** in the complex values.

To **recover** the information, we simply take the **magnitude** and **phase** of these complex values.

$$V_n$$

$$v_{n0} = |V_n|$$

$$\varphi_n = \arg\{V_n\}$$

$$v_n(t) = v_{0n} \cos(\omega t + \varphi_n)$$

Why we love our eigen functions!

Q: *Hey wait a minute!*

What happened to the time-harmonic function $e^{j\omega t}$??

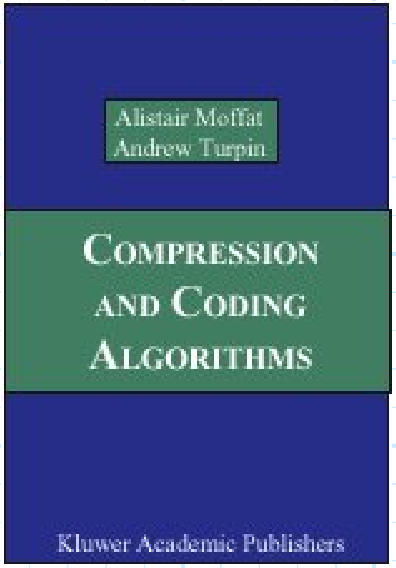
A: There is no reason to **explicitly** write the complex function $e^{j\omega t}$.

We know in fact (being the **eigen function** of linear systems and all) that if this is the time function at any **one** location (such as the excitation source) then this must be time function at **all locations** in the circuit!

→ The only **unknowns** are the **complex** values V_n and I_n

Real voltage: a function of time
Complex voltage: just a constant

Once we determine V_n, we **can** always (**if** we so desire) "recover" the **real** function $v_n(t)$ as:



$$
\begin{aligned}
v_n(t) &= Re\left\{ V_n \, e^{j\omega t} \right\} \\
&= v_n \cos\left(\omega t + \varphi_n\right)
\end{aligned}
$$

Thus, if we assume a **time-harmonic source**, finding the solution $v_n(t)$ reduces to solving for the **complex values** V_n !!!

A quiz!

See if **you** can determine what these complex values tell you about the **time-harmonic voltage** at different locations in a linear circuit:

$$V_0 = 3 \quad \Rightarrow \quad v_0(t) = \quad \cos(\omega t + \quad)$$

$$V_1 = j \quad \Rightarrow \quad v_1(t) = \quad \cos(\omega t + \quad)$$

$$V_2 = e^{j\pi/4} \quad \Rightarrow \quad v_2(t) = \quad \cos(\omega t + \quad)$$

$$V_3 = -2 \quad \Rightarrow \quad v_3(t) = \quad \cos(\omega t + \quad)$$

$$V_4 = \sqrt{2} + j\sqrt{2} \quad \Rightarrow \quad v_4(t) = \quad \cos(\omega t + \quad)$$

$$V_5 = 3e^{-j\pi/4} \quad \Rightarrow \quad v_5(t) = \quad \cos(\omega t + \quad)$$