PHSX 531: Homework #2

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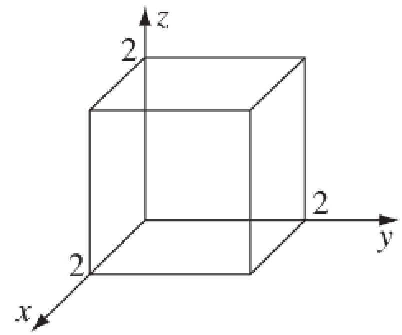
Problem 1

Consider the function $\mathbf{v} = (xy)\hat{x} + (2yz)\hat{y} + (3zx)\hat{z}$. NOTE: Test means to check both sides of the identity and show they are the same.

(a) (4 pts) Test Gauss's theorem for a cube with side of length 2:



Solution:

Gauss's theorem says that:

$$\int_V (\vec{\nabla} \cdot \vec{v})\, d\tau = \oint_S \vec{v} \cdot d\vec{a}$$

i. The left hand side of the equation is the integral

$$\int_0^2 \int_0^2 \int_0^2 (3x + y + 2z)\ dx\ dy\ dz = 48$$

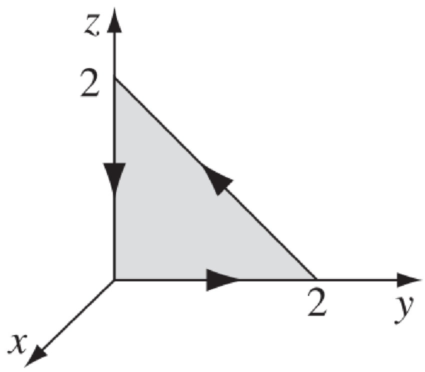
ii. The right hand side of the equation is the set of integrals:

$$\begin{aligned}
x = 2 &\implies 2\int_0^2 \int_0^2 y\ dy\ dz = 8 \\
x = 0 &\implies 0 \\
y = 2 &\implies 2\int_0^2 \int_0^2 2z\ dx\ dz = 16 \\
y = 0 &\implies 0 \\
z = 2 &\implies 2\int_0^2 \int_0^2 3x\ dx\ dy = 24 \\
z = 0 &\implies 0
\end{aligned}$$

(recall that for cases such as at the surface of $x = 2$, x goes to 2 in the vector field. Additionally, sign for vector field is flipped due to face orientation for the planes through $(x, y, z) = 0$. Use Griffiths page 27 for review on the subject.)

The sum of these is the total surface integral which equals 48.

(b) (4 pts) Test Stoke's theorem for the following shaded triangle:



Solution:

Stoke's theorem says that:

$$\int_S (\vec{\nabla} \times \vec{v})\, d\vec{a} = \oint_P \vec{v} \cdot d\vec{\ell}$$

i. The left hand side is the integral:

$$\int_0^2 \int_0^{2-y} (-2y, -3z, -x) \cdot (\ dz\ dy, 0, 0) = \int_0^2 \int_0^{2-y} -2y\ dz\ dy = -\frac{8}{3}$$

ii. The right hand side is the set of integrals:

$$(0,0,0) \to (0,2,0) : \int_0^2 2y\cancelto{0}{z}\,dy = 0$$

On the line $z = 2 - y$, $dy = dz$

$$\begin{aligned}(0,2,0) \to (0,0,2) : \int_0^2 (xy, 2yz, 3zx) \cdot (0, dy, dy)\ dy \\ = -\int_0^2 2y(2-y) + 0\ dy \\ = -\frac{8}{3}\end{aligned}$$

On the last line, $x = y = 0$, sending the integral to zero. The total line integral is indeed equal to the left hand side of the equation.

Problem 2

(2 pts) Using second derivatives, fundamental theorem of divergences, Gauss's theorem, and/or Stoke's theorem, show

$$\int_S f(\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \int_S [\mathbf{A} \times (\nabla f)] \cdot d\mathbf{a} + \oint_P f\mathbf{A} \cdot d\boldsymbol{\ell} \tag{1}$$

Solution:
The product rule for curl converts

$$f(\nabla \times A) \to A \times (\nabla f) + \nabla \times (fA)$$

This is aready half way to what we want:

$$\int_S f(\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \int_S [A \times (\nabla f)]\ d\mathbf{a} + \int_S [\nabla \times (fA)]\ d\mathbf{a}$$

Stokes theorem can now be used to get the last term into the desired loop integral:

$$\int_S f(\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \int_S [\mathbf{A} \times (\nabla f)] \cdot d\mathbf{a} + \oint_P f\mathbf{A} \cdot d\boldsymbol{\ell}$$