PHSX 611: Homework 2

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Grant Saggars

Problem 1

(Problem 2.1a) Prove the following theorem: for normalizable solutions of S.E., the separation constant E must be real. (0.5 pt.)

Solution:
A good place to start is to try proof by contradiction. Suppose a wavefunction exists with complex energy $E_0 + i\Gamma$.

$$\begin{aligned}\Psi(x,t) &= A\psi e^{-i(E_0+i\Gamma)t/\hbar} \\ &= A\psi e^{-iE_0t/\hbar}e^{\Gamma t/\hbar}\end{aligned}$$

Now to try and find the normalization constant A:

$$\begin{aligned}1 &= \int_{-\infty}^{\infty} \Psi\Psi^* \; dx \\ &= \int_{-\infty}^{\infty} (A\psi \cancel{e^{-iE_0t/\hbar}} e^{\Gamma t/\hbar})(A^*\psi^* \cancel{e^{iE_0t/\hbar}} e^{\Gamma t/\hbar}) \; dx \\ &= \int_{-\infty}^{\infty} |A|^2|\psi|^2 e^{2\Gamma t/\hbar} \; dx\end{aligned}$$

This integral sends $\psi(x)$ to one due to finiteness, and in order to make the equation true the exponential must equal one. Therefore the only possible value Γ can take is zero. Therefore there cannot be a complex component in a wavefunction's energy.

Problem 2

(Problem 2.1b) Prove the following theorem: the time-dependent wave function $\psi(x)$ can always be taken to be real (unlike $\Psi(x,t)$, which is complex in the general case). Note, that this doesn't mean that every solution of the time-independent Schrodinger equation is real; what this means is that if you have one that is not, it can always be expressed as a linear combination of solutions (with the same energy) that are. (0.5pt.)

Solution:
If a given $\psi(x)$ is a solution to the wave function, so too is its complex conjugate, since solutions exist for both:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + E\psi(x) = 0 \tag{1}$$
$$-\frac{\hbar^2}{2m}\frac{d^2\psi^*}{dx^2} + E\psi(x) = 0 \tag{2}$$

Now, because a linear combination of any two solutions of the Shrödinger equation are in itself a new solution, the linear combination of a solution and its complex conjugate will always eliminate the complex part.

$$a + bi + a - bi = 2a$$

Problem 3

(Problem 2.2) Show that E must exceed the minimum value of $V(x)$, for every normalizable solution to the time-independent Schrodinger equation. (0.5pt.)

Solution:
For the Shrödinger equation $\frac{\partial^2\psi}{\partial x^2} = \frac{2m}{\hbar^2}(V - E)\psi$, it is clear that if E is less than V, the right side of the equation will be of the same sign as the left. Griffiths points out the consequence of this is that the second derivative of ψ will always have the same sign as ψ. When this is the case, for positive ψ the function will either be concave up or concave down for negative ψ (for all x). This means that the function will either fly to infinity or negative infinity and never converge.

Problem 4

(Problem 1.8) Consider the following situation - you add a constant V_0 to potential energy (V_0 is independent of x and t). In classical mechanics, this won't change anything but in quantum mechanics, it may. Show that the wavefunction picks up a time-dependent phase factor: $e^{-iV_0t/\hbar}$. What effect does it have on the expectation value of a dynamic variable? (0.5 pt.)

Solution:

$$\frac{\partial\Psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{i}{h}V(x,t)\Psi(x,t)$$

I'll substitute a constant potential into the equation, separate, and see what happens:

Problem 4 continued on next page...

$$
\begin{aligned}
\frac{\partial \Psi}{\partial t} &= \frac{i\hbar}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar}[V(x,t) + V_0]\Psi(x,t) \\
i\hbar\psi\frac{d\phi}{dt} &= -\frac{\hbar}{2m}\frac{\partial^2 \psi}{\partial x^2}\phi + \frac{i}{\hbar}V\psi\phi + \frac{i}{\hbar}V_0\psi\phi \\
i\hbar\frac{1}{\phi}\frac{d\phi}{dt} - \frac{iV_0}{\hbar} &= -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\frac{1}{\psi} + V
\end{aligned}
$$

$$
\frac{i\hbar}{\phi}\frac{d\phi}{dt} - \frac{iV_0}{\hbar} = E \tag{1}
$$

$$
\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi \tag{2}
$$

We're focused on the time-dependent part here, (as hinted by Griffiths), so I will integrate to solve the differential equation:

$$
\begin{aligned}
\int \frac{1}{\phi}\, d\phi &= \frac{-iEt}{\hbar} + \frac{-iV_0 t}{\hbar} \\
\ln(\phi) &= \\
\phi = \exp\left(\frac{-iEt}{\hbar} + \frac{-iV_0 t}{\hbar}\right) &= \exp\left(\frac{-iEt}{\hbar}\right)\exp\left(\frac{-iV_0 t}{\hbar}\right)
\end{aligned}
$$

Now trying an operator Q with the new energy:

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \left[\Psi^{(}x,t)e^{-iV_0 t/\hbar}\right]^* Q\left[\Psi(x,t)e^{-iV_0 t/\hbar}\right]\, dx \\
&= \int_{-\infty}^{\infty} \left[\Psi^*(x,t)\right] Q\left[\Psi(x,t)\right] \cancel{e^{iV_0 t/\hbar}}\cancel{e^{-iV_0 t/\hbar}}\, dx
\end{aligned}
$$

In this case due to the conjugate, there is no difference made by the new potential.