# MATH 648: Homework #5

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- Section 10.3 #1, #2.
- Section 10.4 #2, #3.
- Section 10.6 #3, #4.

# Section 10.3

## Problem 1

Geodesics on a sphere of radius $R > 0$ correspond to the extremals of the functional

$$J(\phi) = \int_{\theta_0}^{\theta_1} \sqrt{1 + \sin^2 \theta \phi'^2} d\theta,$$

where $\phi$ is the polar angle, $\theta$ is the azimuth angle, and $\phi'$ denotes $d\phi/d\theta$. Show that $J$ satisfies the Legendre condition (10.7).

**Solution:**
Working with the assumption that the Legendre condition, as given, holds in polar coordinates, we find:

$$f_{\phi'} = \frac{\phi' \sin^2(\theta)}{\sqrt{1 + \phi'^2 \sin^2(\theta)}}$$

$$f_{\phi'\phi'} = \frac{\sin^2(\theta)}{\left(1 + \phi'^2 \sin^2(\theta)\right)^{3/2}} \geq 0$$

(This is never negative for real values, implying convexity)

# Problem 2

Let

$$J(y) = \int_{x_0}^{x_1} \frac{1 + y^2}{y'^2} dx.$$

Suppose that $J$ has a local extremum at $y$. Use the Legendre condition to determine the nature of the extremum.

**Solution:**

$$p(x) = f_{y'y'} = \frac{6(1 + y^2)}{y'^4}$$

(This is always greater than zero by the even powers, implying a minimal)

# Section 10.4

# Problem 1

Derive the Riccati equation (10.11) associated with the functional of Example 10.3.3. Solve the Riccati equation directly and show that there are no solutions $w$ defined for all $x \in [0, \ell]$ if $\ell > \pi$.

**Solution:**

## Part I

Our Riccati equation is:

$$w' + q(x) - \frac{w^2}{p(x)} = 0$$

To get to this, we start with the second variation, defined as:

$$\delta^2 J(\eta, y) = \int_{x_0}^{x_1} \left(p(x)\eta'^2 + q(x)\eta^2\right)\, dx$$

Well behaved functions (consider $w(x)$, for example) have $\eta$ at endpoints which vanish to zero. Jacobi makes the observation that for ANY smooth function $w$, one has

$$\int_{x_0}^{x_1} (w\eta^2)'\, dx$$

by the product rule, $2w\eta\eta' + w'\eta^2 = (w\eta^2)'$

$$\delta^2 J(\eta, y) = \int_{x_0}^{x_1} \left(p\eta'^2 + 2w\eta\eta' + (w' + q)\eta^2\right)\, dx$$

We know:

1. For $y$ to be a minimal, $p(x) \geq 0$ for $x \in [x_0, x_1]$
2. By the Legendre necessary condition $p(x)$ will be greater than zero for this domain.

Our integrand can therefore be rewritten as

$$\begin{aligned} & p\left(\eta'^2 + 2\frac{w}{p}\eta\eta' + \frac{w^2}{p^2}\eta^2\right) + \left(w' + q - \frac{w^2}{p}\right)\eta^2 \\ &= p\left(\eta' + \frac{w}{p}\eta\right)^2 + \left(w' + q - \frac{w^2}{p}\right)\eta^2 \end{aligned}$$

If we can find a function $w = w(x)$ so that

$$w' + q(x) - \frac{w^2}{p(x)} = 0, \qquad x \in [x_0, x_1]$$

Then the second variation $\delta^2 J(\eta, y) \geq 0$ for any $\eta$ since $p(x) > 0$

Problem 1 continued on next page... 

## Part II

To solve we convert the riccati equation into a 2nd order linear system by introducing $u = u(x)$:

$$w(x) = -\frac{p(x)u'(x)}{u(x)}, \quad \text{or} \quad u(x) = u_0 e^{-\int_{x_0}^{x} \frac{w(z)}{p(z)}\, dz}$$

This gives us our beloved Jacobi Accessory Equation:

$$(p(x)u')' - q(x)u = 0, \quad x \in [x_0, x_1]$$

## Part III

I am not sure how to prove that there are no solutions for all $x \in [0, \ell$ if $\ell > \pi$.

# Problem 2

Let

$$f\left(x, y, y'\right)=y'^{2}-y' y+y^{2}.$$

Show, using elementary arguments, that $\delta^2 J(\eta, y) \geq 0$ for all $\eta \in H$. Derive the Jacobi accessory equation and show by solving this equation that any nontrivial solution $u$ can have at most one zero.

**Solution:**
We've derived the Jacobi accessory equation in the prior problem. We can also apply the same arguments to show that the second variation is greater than zero for all $\eta$:

$$p(x)=f_{y'y'}=2 \geq 0$$

If we also want to show that there exists no conjugate points for any nontrivial solution, we ought to apply the jacobi accessory equation.

$$\begin{aligned}(p(x)u(x))'-q(x)u=0; \quad q(x)=f_{yy}-\frac{d}{dx}f_{yy'}=2 \\ (2u)'-2u=0\end{aligned}$$

This is evidently an exponential function which has only one root.

# Section 10.6

# Problem 3

Let

$$J(y) = \int_0^{\pi/4} \left(y^2 - y'^2 - 2y\cosh x\right) dx.$$

Find the extremals for $J$ and show that for the fixed endpoint problem these extremals produce weak local maxima.

**Solution:**

## E-L:

$$\begin{aligned} 2y - 2\cosh x - \frac{d}{dx}2y' &= 0 \\ y'' - y &= -\cosh x \\ y_c &= c_1 e^x + c_2 e^{-x} \\ y_p &= -\frac{1}{2}x\sinh(x) \\ y(x) &= c_1 e^x + c_2 e^{-x} - \frac{1}{2}x\sinh(x) \end{aligned}$$

## Categorizing Extrema:

$$\begin{aligned} p(x) &= f_{y'y'} = -2 \\ q(x) &= f_{yy} - \frac{d}{dx}f_{yy'} = 2 \\ (p(x)u')' + q(x)u &= -2u'' + 2u = 0 \\ u(x) &= c_1 e^x + c_2 e^{-x} \end{aligned}$$

Now, $u(0) = 0$ implies:

$$c_1 + c_2 = 0$$

We choose $c_1 = -c_2 = \frac{1}{2}$ which gives us:

$$u(x) = \sinh(x) > 0,\ x > 0$$

Thereby showing it is a local maxima.

# Problem 4

Let

$$J(y) = \int_{x_0}^{x_1} y' \left(1 + x^2 y'\right) dx,$$

where $0 < x_0 < x_1$. Find the extremals for $J$ and the general solution to the Jacobi accessory equation. Find any conjugate points to $x_0$ and determine the nature of the extremals for the fixed endpoint problem.

**Solution:**

## E-L:

$$1 + 2x^2 y' = c_1$$
$$y' = \frac{dy}{dx} = \frac{c_1 - 1}{2x^2}$$
$$y = -\frac{c_1 - 1}{2x} + c_2$$

## Categorizing Extremals:

$$p(x) = f_{y'y'} = x^2 > 0$$
$$q(x) = f_{yy} - \frac{d}{dx} f_{yy'} = 0$$
$$x^2 u''(x) = 0$$

We get $u(x) = c_1 x + c_2$. For $u(0) = 0$;

$$c_2 = 0$$

There are no other zeroes of this function except for when $c_1$ is 0, which has undefined behavior as $u(x) = 0$ for all $x$, so this must be a local minimal (meaning no conjugate points).