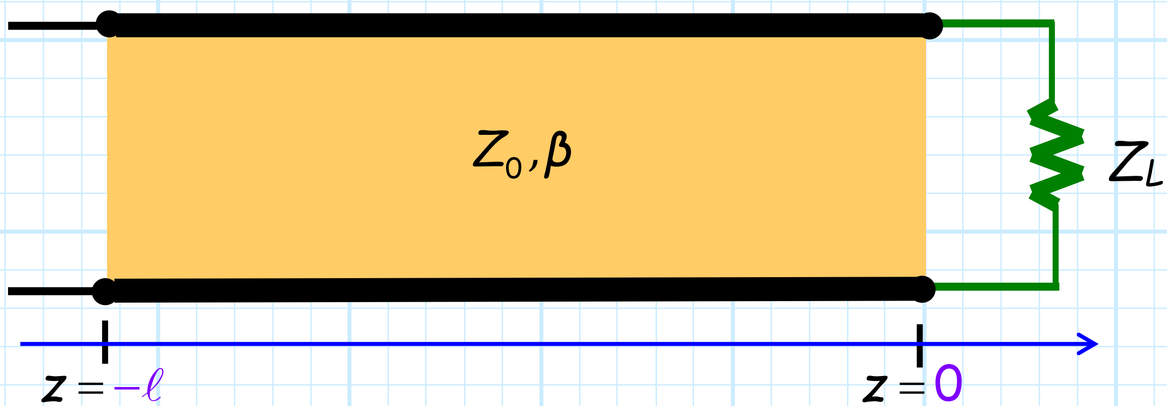
The Input Impedance of a Terminated Transmission Line

Consider a **lossless** line, length ℓ, terminated with **load** Z_L.

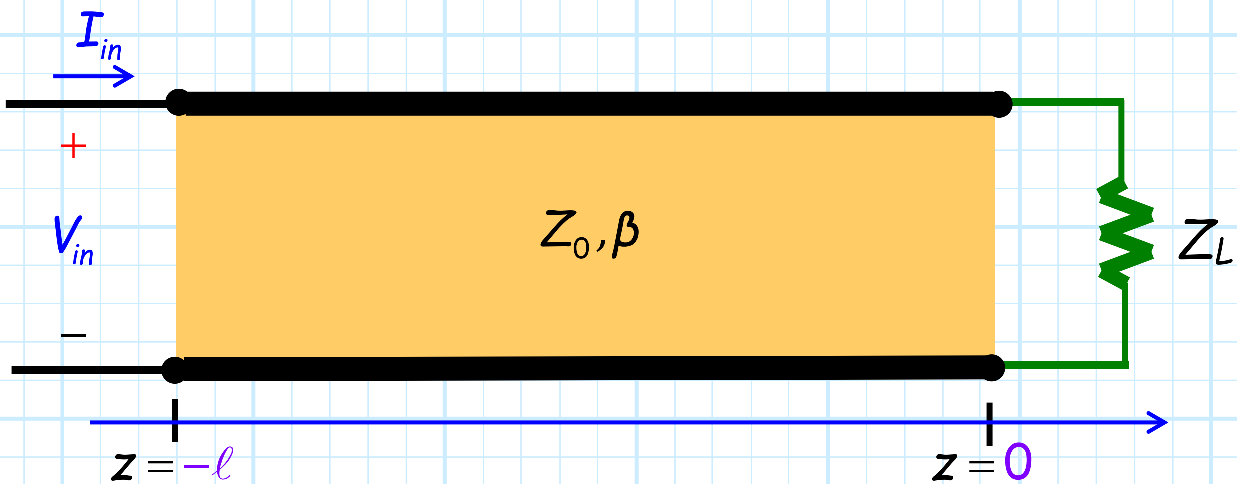


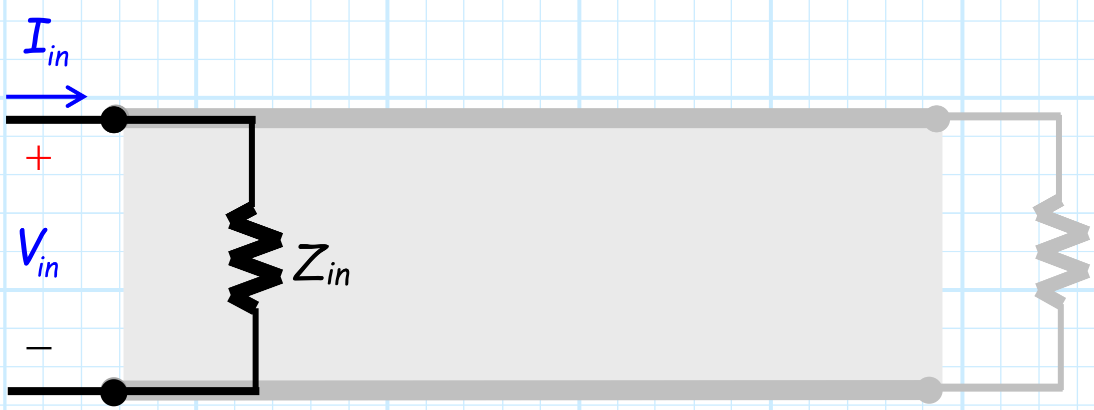
Note that this terminated transmission line is a **one-port device**!

It's not Z_{in} and it's not Z_0

The **two-port transmission line** acts as an **impedance transformer**—transforming the **load** impedance Z_L to an **input** impedance Z_{in}.

Let's determine the **input impedance** of the resulting **one-port** device!





The definition of input impedance

Q: *Just how do we determine this **input** impedance?*

A: The **input impedance** of a one-port device is simply:

$$Z_{in} = \frac{V_{in}}{I_{in}}$$

Q: *But what **is** V_{in} and I_{in} for this **terminated transmission line**???*

A: Sigh.

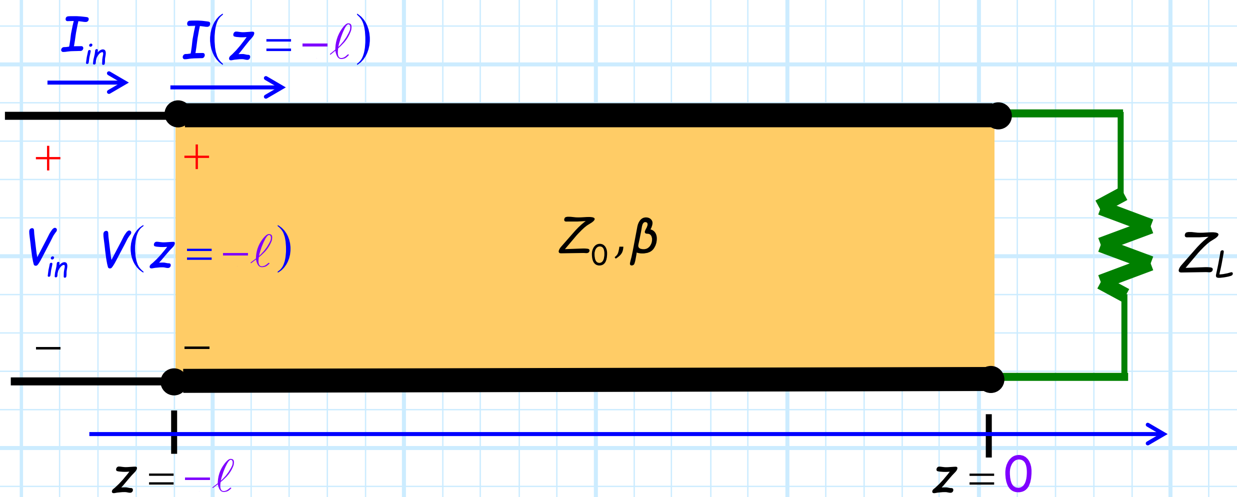
Boundary Conditions!

Using **BOUNDARY CONDITIONS**, we know that (from KVL):

$$V_{in} = V(z=-\ell)$$

And also (from KCL):

$$I_{in} = I(z=-\ell)$$



When evaluated at the beginning

Therefore, the **input impedance** of a terminated transmission line is simply the **ratio** of the total voltage and total current—when **evaluated** at the **beginning** of the transmission line (i.e., at index location $z=-\ell$):

$$Z_{in} = \frac{V_{in}}{I_{in}} = \frac{V(z=-\ell)}{I(z=-\ell)}$$

Q: *Hey wait, isn't the **ratio** of total voltage $V(z)$ and total current $I(z)$ simply **line impedance**?*

A: More precisely, their ratio is the line impedance **function**:

$$\frac{V(z)}{I(z)} = Z(z)$$

Say this if you wish to flunk

Q: *So the* ***input impedance*** Z_{in} *of a terminated transmission line is simply equal to its* ***line impedance*** $Z(z)$ *?*

A: NOOOOOO!!!!!!!!!!!!!OOOOOOOOO!!!!!!!OOOOOOOOOO!!!!!!!!!!!!!.

The input impedance of a terminated transmission line is equal to its line impedance **function**, when **evaluated** at the **beginning** of the transmission line (i.e., at index location $z = -\ell$):

$$Z_{in} = \frac{V(z=-\ell)}{I(z=-\ell)} = Z(z=-\ell)$$

Q: *You're always* ***so picky****.*

A: But this is a **critical distinction**!

Make this make sense

The **line impedance function** $Z(z)$ describes the line impedance value at **any** and **all locations** z along the transmission line.

In **contrast**, **input impedance** Z_{in} is the complex **value** of the line impedance function at **one specific location**—the beginning (i.e., the **input**) of the terminated transmission line.

* Line Impedance $Z(z)$ is a complex **function** of **position** z.

* Input Impedance Z_{in} is a **specific complex number**—it **cannot** be a **function** of index z!

Remember?

Now, **recalling** that the **line impedance function** is:

$$
\begin{aligned}
Z(z) &= Z_0\left(\frac{e^{-j\beta z} + \Gamma_L\, e^{+j\beta z}}{e^{-j\beta z} - \Gamma_L\, e^{+j\beta z}}\right) \\
&= Z_0\left(\frac{(Z_L + Z_0)\, e^{-j\beta z} + (Z_L - Z_0)\, e^{+j\beta z}}{(Z_L + Z_0)\, e^{-j\beta z} - (Z_L - Z_0)\, e^{+j\beta z}}\right) \\
&= Z_0\left(\frac{Z_L \cos\beta z - jZ_0 \sin\beta z}{Z_0 \cos\beta z - jZ_L \sin\beta z}\right)
\end{aligned}
$$

we can thus determine the **input impedance**:

$$Z_{in} = Z(z = -\ell)$$

Be careful of the signs!

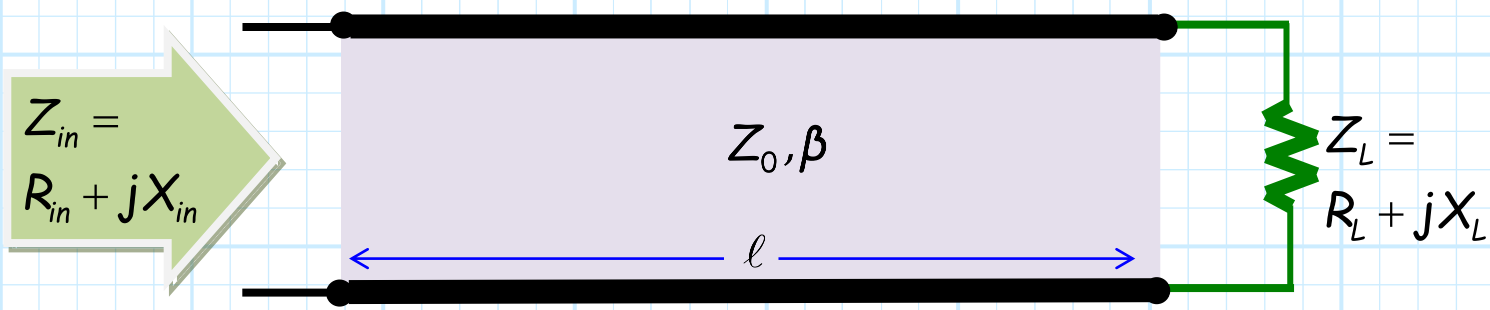
$$
\begin{aligned}
Z_{in} &= Z(z=-\ell) \\
&= Z_0 \left(\frac{e^{+j\beta\ell} + \Gamma_L \, e^{-j\beta\ell}}{e^{+j\beta\ell} - \Gamma_L \, e^{-j\beta\ell}} \right) \\
&= Z_0 \left(\frac{(Z_L + Z_0)\, e^{+j\beta\ell} + (Z_L - Z_0)\, e^{-j\beta\ell}}{(Z_L + Z_0)\, e^{+j\beta\ell} - (Z_L - Z_0)\, e^{-j\beta\ell}} \right) \\
&= Z_0 \left(\frac{Z_L \cos\beta\ell + j Z_0 \sin\beta\ell}{Z_0 \cos\beta\ell + j Z_L \sin\beta\ell} \right)
\end{aligned}
$$

Note Z_{in} is equal to **neither** the load impedance Z_L, **nor** the characteristic impedance Z_0!

$$Z_{in} \neq Z_L \qquad \text{and} \qquad Z_{in} \neq Z_0$$

It could be just about anything!

Depending on the values of $\beta\ell$ and Z_0, the **input** impedance Z_{in} can be **radically** different from the **load** impedance Z_L!



$$Z_{in} \neq Z_L \quad \text{(generally speaking)}$$

Not that we're surprised

Of course, this result is **not** at all surprising.

For when a transmission line is terminated in some arbitrary **load**, the line impedance function can **change dramatically** with position z.

$$Z(z) = Z_0 \left(\frac{(Z_L + Z_0)\, e^{-j\beta z} + (Z_L - Z_0)\, e^{+j\beta z}}{(Z_L + Z_0)\, e^{-j\beta z} - (Z_L - Z_0)\, e^{+j\beta z}} \right)$$

The **input** impedance is different from the **load** impedance because the **line** impedance changes significantly between the **beginning** of the line and its **end**!