PHSX 531: Homework #3

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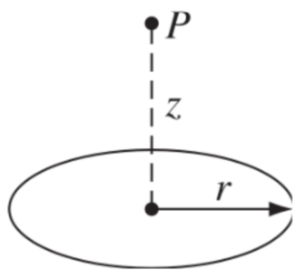
Problem 1

(4 pts) Find the electric field a distance z above the center of a circular loop:



Take the limit $z \gg r$. Does your answer make sense? Explain why or why not in detail.

Solution:
In this case it is probably easiest to define **r** as the difference in positions from the coordinate from the coordinate on the surface of the shape $(r - r')$. Electric field for some line charge is then given by

$$\frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}(\mathbf{r}-\mathbf{r}')\,dl'$$

Which in polar coordinates for this case becomes (notice the $rd\theta$):

$$\frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\frac{\lambda}{(r^2+z^2)^{3/2}}\left(\langle 0,0,z\rangle - \langle r'\cos\theta, r'\sin\theta, 0\rangle\right) r(d\theta')$$
$$\frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\frac{\lambda}{(r^2+z^2)^{3/2}}\langle -r'\cos\theta, r'\sin\theta, z\rangle\, r(d\theta')$$
$$= \frac{\lambda}{4\pi\epsilon_0}\frac{rz}{(r^2+z^2)^{3/2}} + \cancelto{0}{\int_0^{2\pi} r'\cos\theta\, d\theta'} + \cancelto{0}{\int_0^{2\pi} r'\sin\theta\, d\theta'} + \cancelto{2\pi}{\int_0^{2\pi} d\theta'}$$
$$= \frac{\lambda}{2\epsilon_0}\frac{rz}{(r^2+z^2)^{3/2}}\ \overset{(\hat{\mathbf{z}})}{\langle 0,0,1\rangle}$$

While we cannot just set $r \to 0$, we can take the ratio of r and z and do a series expansion:

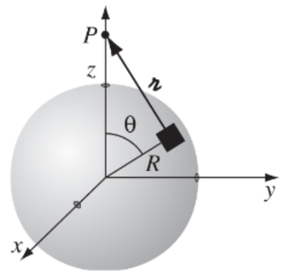
$$
\begin{aligned}
\mathbf{E} &= \frac{\lambda}{2\epsilon_0}\frac{rz}{[z^2(\frac{r^2}{z^2}+1)]^{3/2}} \\
&= \frac{\lambda}{2\epsilon_0}\frac{r}{z^2}\left(1+\frac{r^2}{z^2}\right)^{-3/2} \\
&= \frac{\lambda}{2\epsilon_0}\frac{r}{z^2}\left[\sum_{k=0}^{\infty}\frac{\Gamma(-3/2+1)}{\Gamma(k+1)\Gamma(-3/2-k+1)}\left(\frac{r^2}{z^2}\right)^k\right] \\
&= \frac{\lambda}{2\epsilon_0}\frac{r}{z^2}\left[\frac{\Gamma(-1/2)}{\Gamma(1)\Gamma(-1/2)}\left(\frac{r}{z}\right)^0+\frac{\Gamma(-1/2)}{\Gamma(2)\Gamma(-3/2)}\left(\frac{r}{z}\right)^2+\frac{\Gamma(-1/2)}{\Gamma(3)\Gamma(-5/2)}\left(\frac{r}{z}\right)^4+\dots\right] \\
&\approx \frac{\lambda}{2\epsilon_0}\frac{r}{z^2}\left[1-\frac{3r^2}{2z^2}+\frac{15r^4}{8r^4}\right] \\
&\approx \frac{Q}{4\pi\epsilon_0}\frac{1}{z^2}\left[1-\frac{3r^2}{2z^2}+\frac{15r^4}{8r^4}\right]
\end{aligned}
$$

Here we can see that the ratio of r/z gets inftesimally small when $z \gg r$, thereby giving an inverse square field akin to a point charge.

$$
E \approx \frac{Q}{4\pi\epsilon_0}\frac{1}{z^2}
$$

Problem 2

(4 pts) Find the electric field a distance z from the center of a spherical surface of radius R that carries a uniform surface charge density σ. Treat the case of $z < R$ (inside) as well as $z > R$ (outside). Express your answers in terms of the total charge q on the sphere. [Hint: Use the law of cosines to write r in terms of R and θ. Be sure to take the positive square root of $\sqrt{R^2 + z^2 - 2Rz\cos\theta}$, which depends on if you're inside or outside the sphere.]



Solution:
First we need to express some distance z along the $\hat{\mathbf{z}}$ axis for all points on the surface of the sphere. By the law of cosines, $\mathbf{r}^2 = z^2 + R^2 - 2zR\cos\theta$. Describing $\hat{\mathbf{r}}$ is a bit tricky as well. Thinking about it conceptually, it is the $z - component$ from the surface of the sphere to point z. This means that it is actually the z' component, if we consider $\mathbf{r} = r = r'$. Geometrically the angle from the top of the triangle (γ) can be used to find $\hat{r} = \cos\gamma = \frac{z'}{r} = \frac{z - R\cos\theta}{\sqrt{R^2+z^2-2Rz\cos\theta}}$. Now we get to integrate the monster that is:

$$\frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\mathbf{r})}{\mathbf{r}^2}\hat{\mathbf{r}}\, da' = \frac{\sigma}{4\pi\epsilon_0}\int_0^{\pi}\int_0^{2\pi} \frac{1}{R^2 + z^2 - 2Rz\cos\theta}\,\frac{z - R\cos\theta}{\sqrt{R^2 + z^2 - 2Rz\cos\theta}} R^2 \sin\theta \, d\phi \, d\theta$$

$$= \frac{\sigma}{4\pi\epsilon_0}\int_0^{\pi} \frac{R^2\sin\theta(z - r\cos\theta)}{(R^2 + z^2 - 2Rz\cos\theta)^{3/2}}\, d\theta \overbrace{\int_0^{2\pi} d\phi}^{2\pi}$$

Make the substitution $u = \cos\theta$ and $du = -\sin\theta$:

$$-\frac{R^2\sigma}{2\epsilon_0}\int_{u=1}^{u=-1} \frac{z - Ru}{(R^2 + z^2 - 2Rzu)^{3/2}}\, du$$

Problem 2 continued on next page...

Make a new substitution $v = R^2 + z^2 - 2Rzu$, $dv = -2Rz\ du$, $du = -\frac{dv}{2Rz}$, and $z - Ru = \frac{v - R^2 - z^2 + 2v^2}{2z}$:

$$
\begin{aligned}
-\frac{R^2\sigma}{2\epsilon_0}\frac{1}{2Rz}\int_{v=(R+z)^2}^{v=(R-z)^2} \frac{\frac{v-R^2-z^2+2v^2}{2z}}{(v)^{3/2}}\ dv &= -\frac{R^2\sigma}{2\epsilon_0}\frac{1}{4Rz^2}\int_{v=(R+z)^2}^{v=(R-z)^2} \frac{v - R^2 + z^2}{(v)^{3/2}}\ dv \\
&= -\frac{R^2\sigma}{2\epsilon_0}\frac{1}{4Rz^2}\int_{v=(R+z)^2}^{v=(R-z)^2} (v - R^2 + z^2)(v)^{-3/2}\ dv \\
&= -\frac{R^2\sigma}{2\epsilon_0}\frac{1}{4Rz^2}\int_{v=(R+z)^2}^{v=(R-z)^2} (R^2 + z^2)(v)^{-3/2} + (v)^{-1/2}\ dv \\
&= -\frac{R^2\sigma}{2\epsilon_0}\frac{1}{4Rz^2}\left[-2(R^2+z^2)(v)^{-1/2} + (2v)^{1/2}\right]\Big|_{(R-z)^2}^{(R+z)^2} \\
&= -\frac{R^2\sigma}{2\epsilon_0}\frac{1}{4Rz^2}\left[\frac{-2(R^2+z^2)}{R+z} + 2(R+z)\right] - \left[\frac{-2(R^2+z^2)}{R-z} + 2(R-z)\right]
\end{aligned}
$$

Simplification yields

$$
E = -\frac{\sigma R^2}{2\epsilon_0 z^2}\left(1 - \frac{R-z}{|R-z|}\right)
$$

Problem 3

(3 pts) Use your answer to Number 2 to find the field inside and outside a solid sphere of radius R that carries a uniform volume charge density ρ. Express your answers in terms of the total charge of the sphere, q. Draw a graph of $|E|$ as a function of the distance from the center.

Solution:

i. **For** $z < R$: E goes to zero, since the term inside the parenthesis will go to one, regardless of R and z.

ii. **For** $z > R$: Due to the absolute value, we will get $z - R$ under these conditions, giving

$$E = -\frac{\sigma R^2}{2\epsilon_0 z^2}\left(1 - \frac{R - z}{z - R}\right)$$