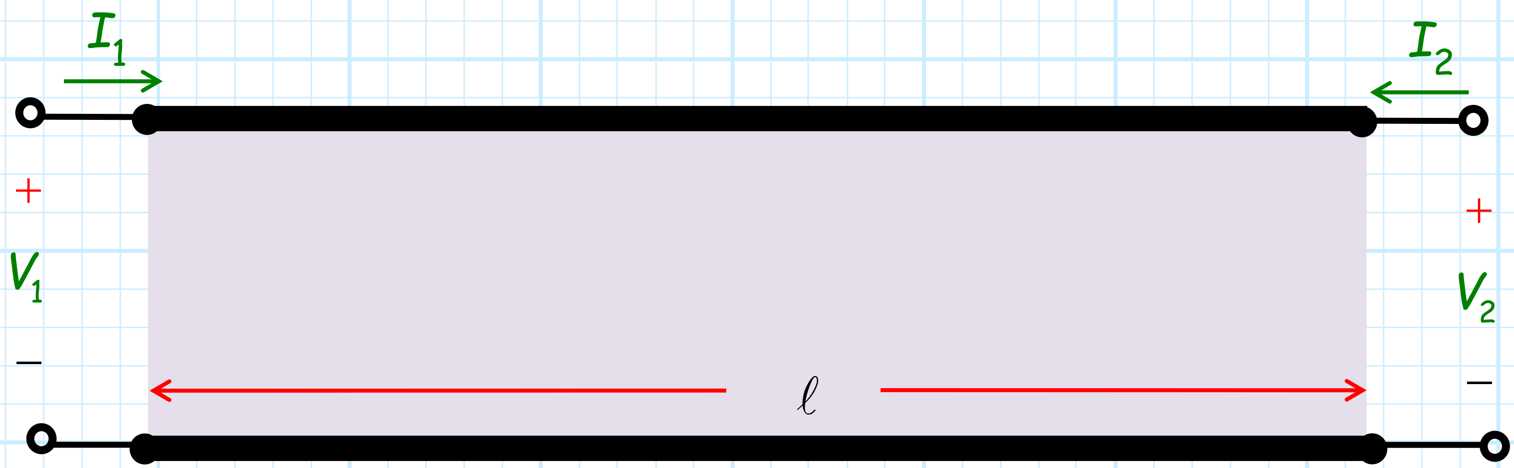
The Telegrapher Equations

Note that a transmission line is a **two-port** device!

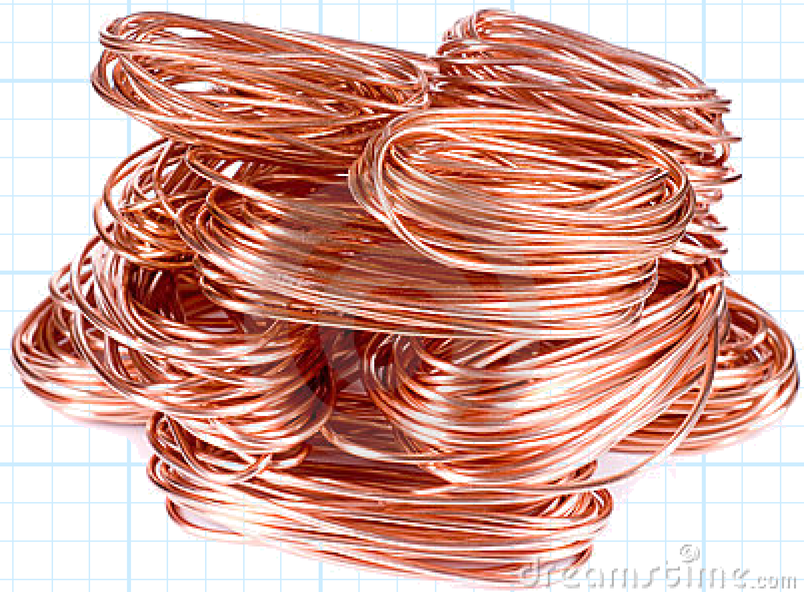


→This two-port device is characterized by **5 parameters**.

Normalized resistance

1. $R \doteq$ resistance/unit length (e.g., Ohms/meter)

Although the two **conductors** (wires) of a transmission line are typically made from materials exhibiting excellent **conductivity** (e.g., copper), they of course are **not** perfectly conducting.

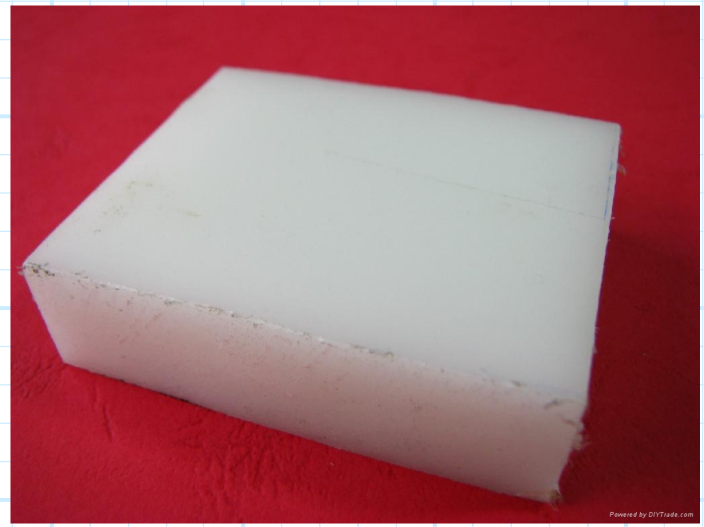


The value R thus specifies the **normalized resistance** (e.g., Ohms/meter) of the wires.

Normalized conductance

2. $G \doteq$ conductance/unit length (e.g., mhos/meter)

Generally speaking, the two transmission conductors are separated by a rigid **insulating** material.



The **conductivity** of this insulating material is **very low**, but of course is **not** zero.

→ The value G thus specifies the **normalized conductance** (e.g, mhos/meter) of this insulating material.

Normalized inductance and capacitance

3. $L \doteq$ inductance/unit length (e.g., henries/meter)

The value L specifies the **normalized (self) inductance** of each wire.

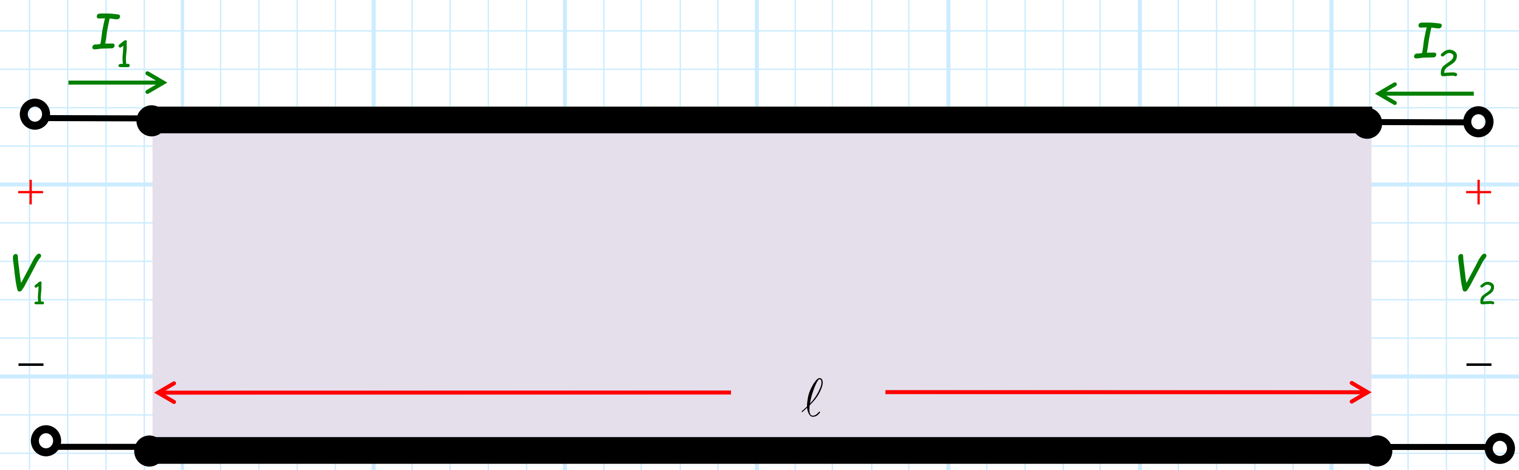
4. $C \doteq$ capacitance/unit length (e.g., farads/meter)

The value C specified the **normalized capacitance** of the two wires.

This value partially depends on the **dielectric** constant of the insulating material

Line Length

5. $\ell \doteq$ the physical length (e.g. meters) of the transmission line.

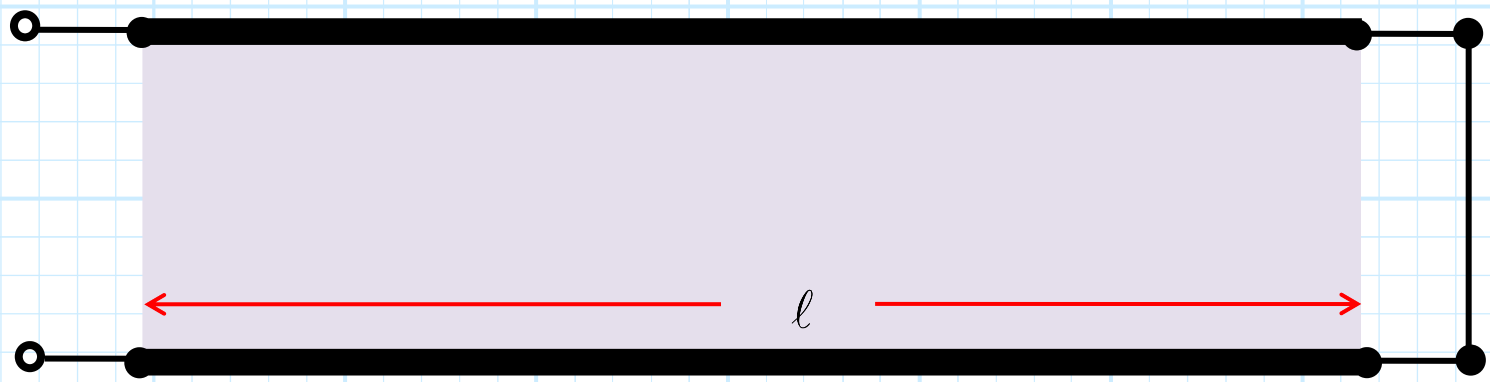


The resistance, inductance, conductance and capacitance of a transmission line is not localized (i.e.,**"lumped")** to any one location, but instead is distributed uniformly throughout the device.

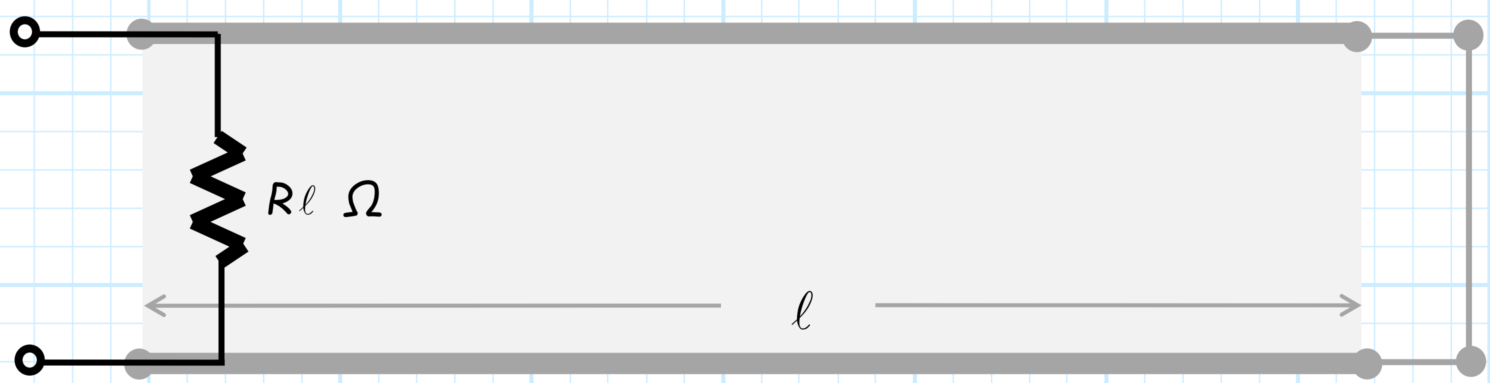
Thus, the **total** resistance, inductance, conductance, and capacitance of the transmission line is **directly proportion** to its length ℓ.

An example

For example, the **resistance** exhibited by **this** one-port device:



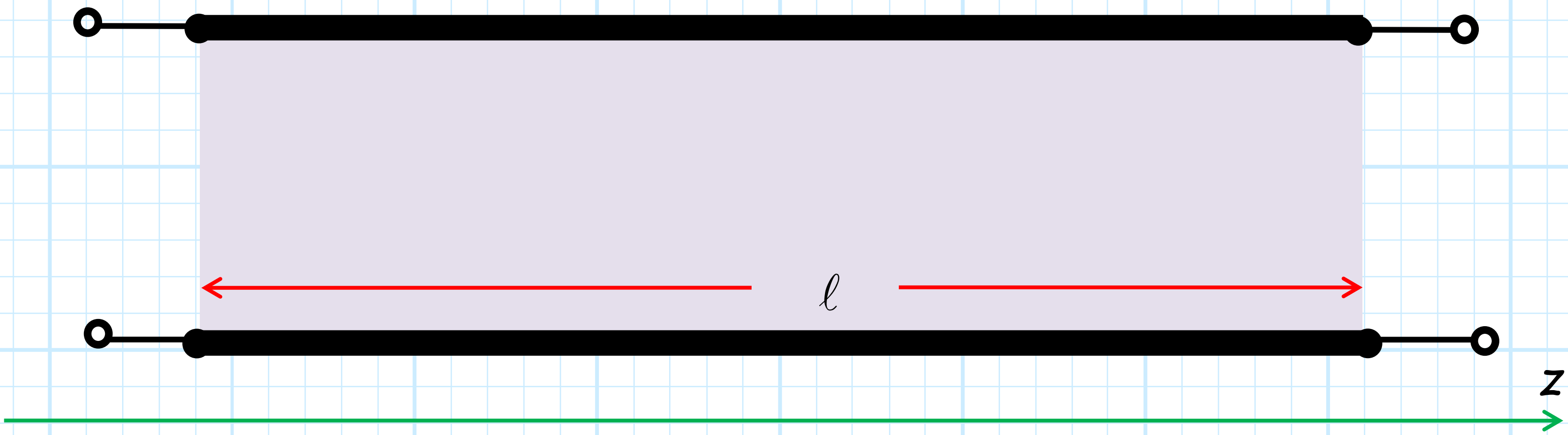
is the **product** of the **normalized** line resistance R (Ohms/meter), and line length ℓ (meters).



We must specify our location

In order to analyze a transmission line, we need to **uniquely** define **locations** at every **point** along the line.

→This is accomplished with a simple **real line**!

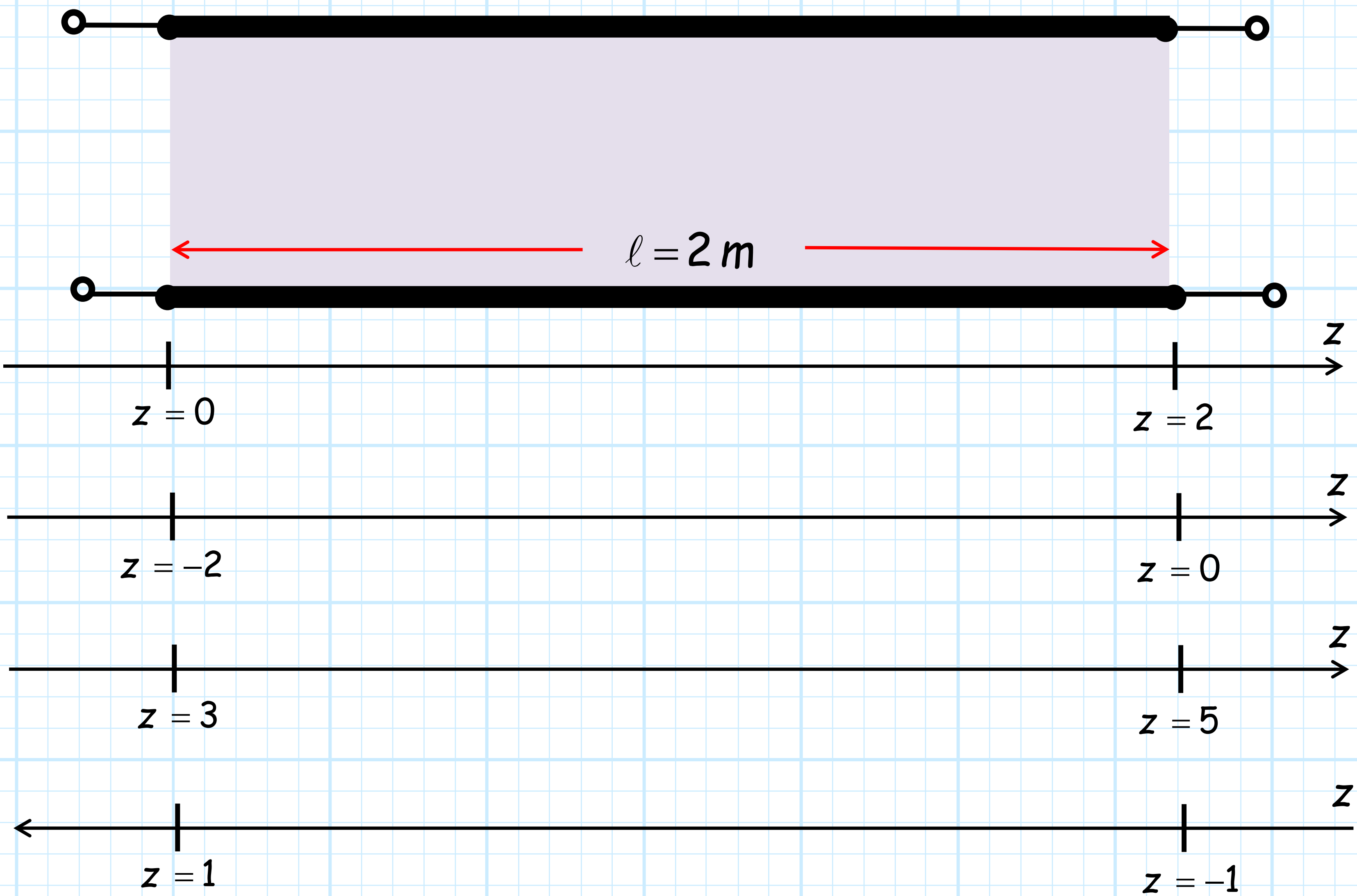


The **index** $\boldsymbol{z}$ is a continuous real value that indicates **distance** (e.g., in meters) along the transmission line.

Both the **direction** and the **center** (where z = 0) of this real line are completely **arbitrary**!

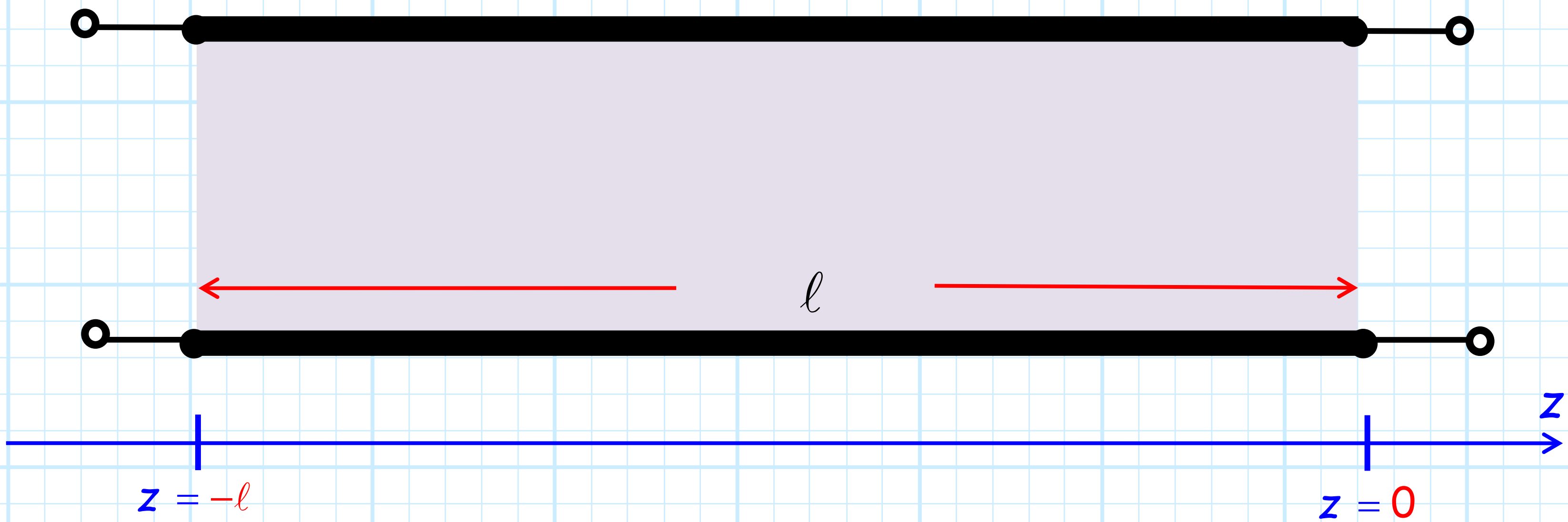
Like mile markers, they're arbitrary

For **example**, say the length of some transmission line is 2 meters. **Any** of the real lines shown below could index this transmission line:



This is fairly common

A **very common** alignment of the real line is shown below:



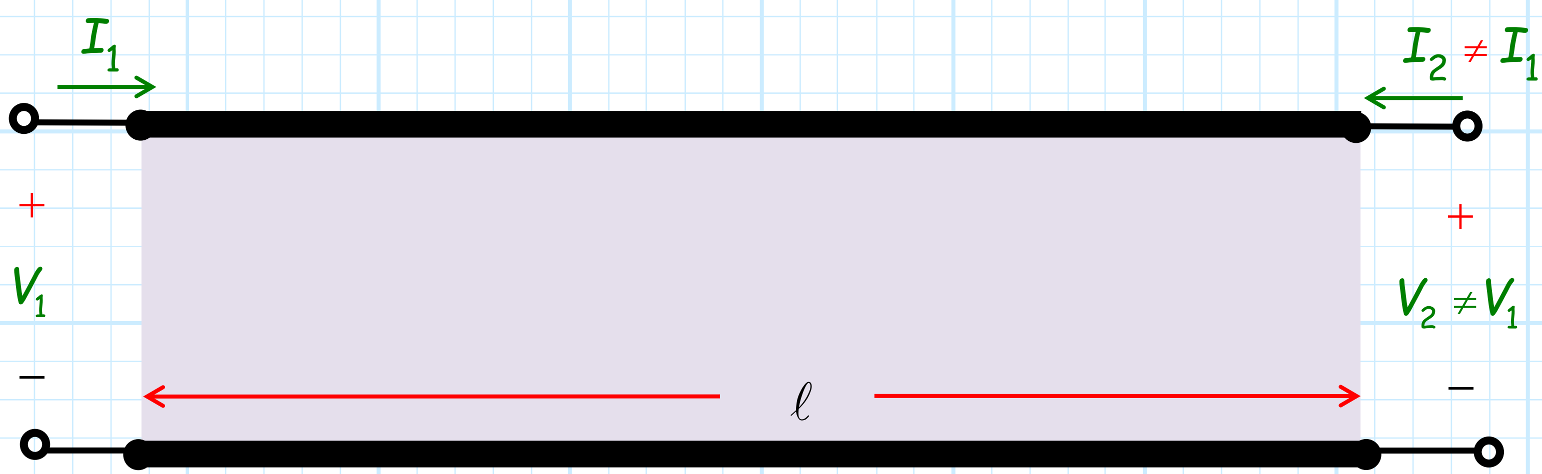
Note for **this** alignment, **all** the values of z along the transmission line are **negative**! I.E.,:

$$-\ell \leq z \leq 0$$

This is the problem

Q: *I don't understand. **Why** do we need this **index z** ?*

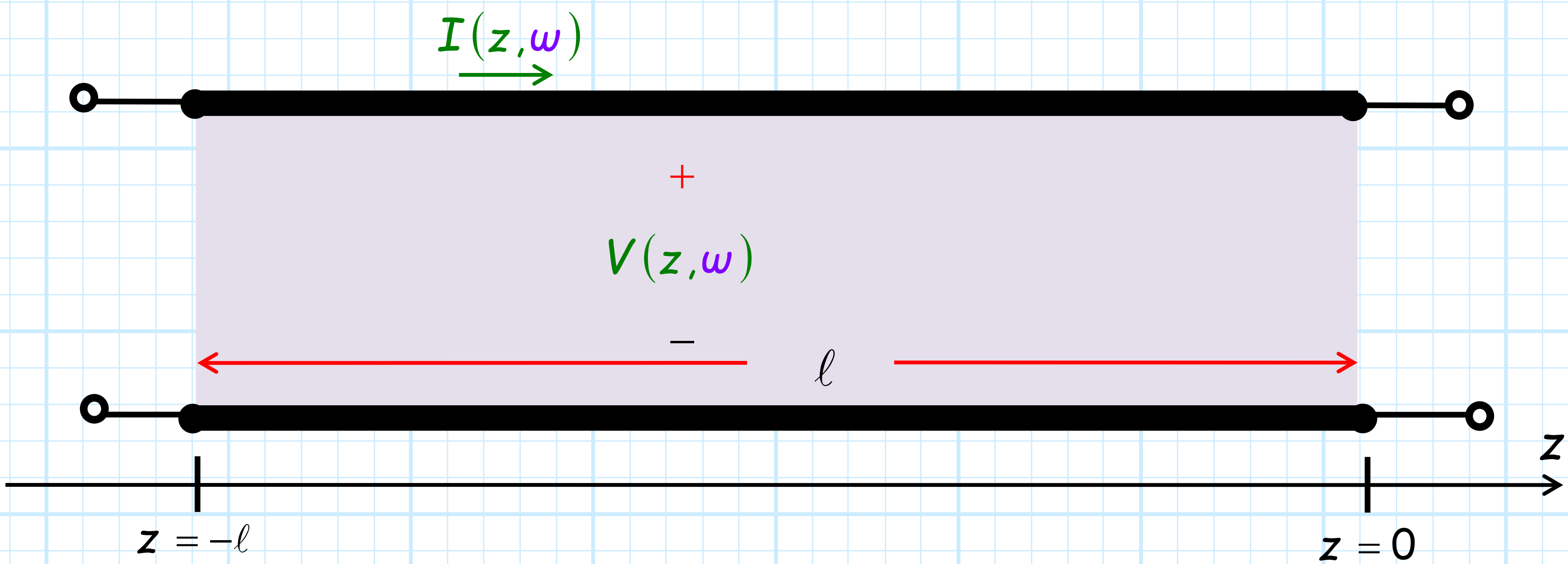
A: Remember, when signal **frequencies** ω are **very high**, the voltages and currents at either end of a transmission line are usually **quite different**.



And since the currents and voltages at either end of the line are **dissimilar**, it shouldn't be a surprise to learn that the currents and voltages are also different at **every point in between**.

The problem gets even worse

→The voltage and current of a transmission line are continuous **functions of position!!!**



Magnitude and phase depend on z

Q: *Voltages and currents are **complex functions of position**!*

*What the heck does **that mean**?*

A: Remember, a complex value has both a **magnitude** and **phase.**

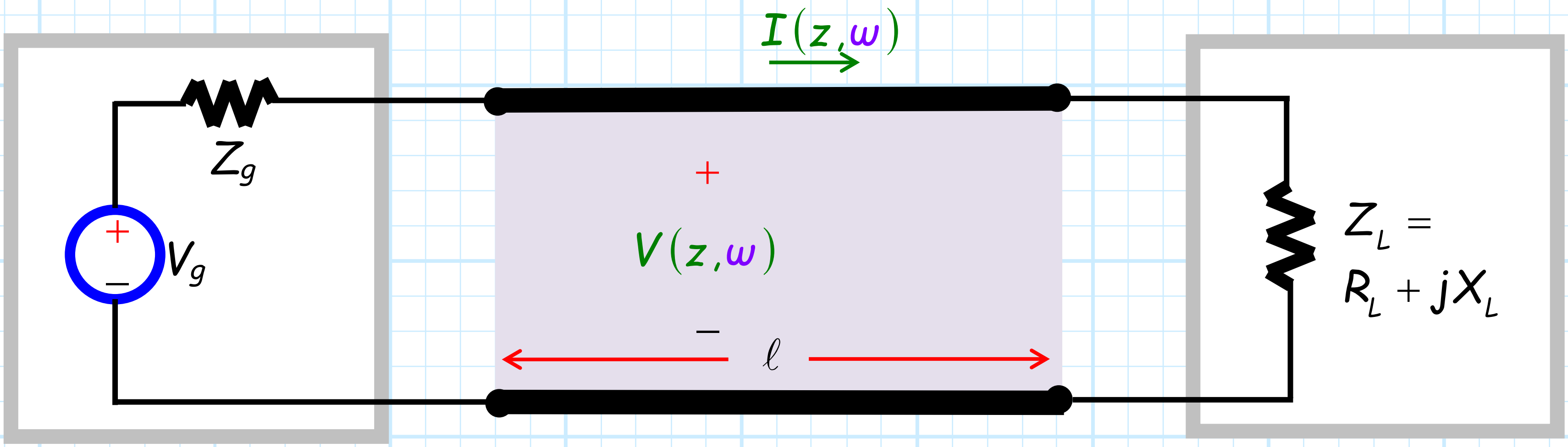
Therefore, a complex **function** of position will have a **magnitude** and **phase** that are **functions** of position as well!

$$\boxed{V(z,\omega) = v(z,\omega)\, e^{-j\varphi_v(z,\omega)}}$$

Transmission Lines—they're LTI

Q: *Yes, but what does this mean **physically** about the current and voltage of a transmission line?*

A: A transmission line is a linear, time invariant **(LTI)** device, and so time-harmonic (i.e., **sinusoidal**) functions are likewise the **eigen functions** of transmission lines!



Thus, if the excitation **source** of a transmission line is sinusoidal with frequency ω, then the voltages and currents at each and **every** location z along the line will **likewise be sinusoidal**—with frequency ω!

Magnitude and relative phase of the sinusoidal oscillation —it depends on position z

The complex voltage function $V(z,\omega)$ thus describes the **magnitude** of the sinusoidal oscillation at each and every location of the transmission line.

$$\left| V(z,\omega) \right| = v(z,\omega)$$

Function $V(z)$ also provides the relative **phase** of the oscillation at each

$$arg\left\{ V(z,\omega) \right\} = \varphi_v(z,\omega)$$

The real-valued voltage

Specifically, the real-valued expression for voltage on transmission line is a function of **both time** t and **distance** z:

$$v(z,t) = v(z)\cos(\omega t + \varphi_v(z))$$

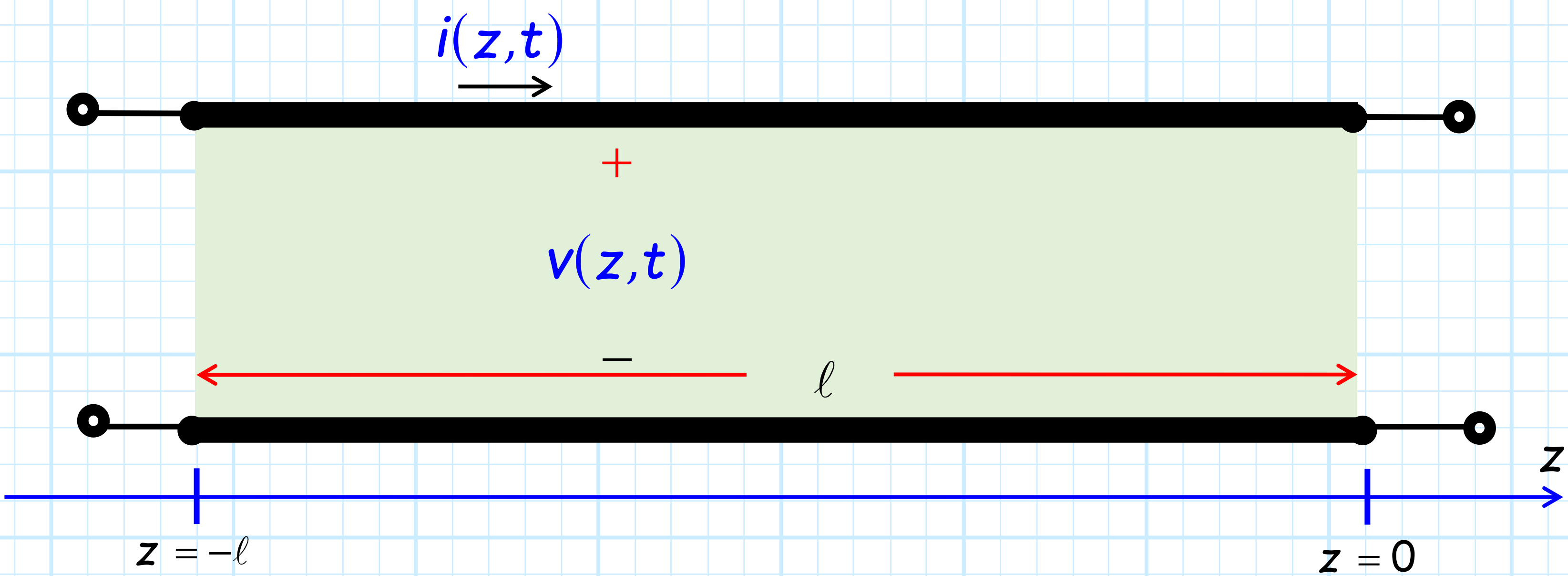
Recall this real-valued function can be determined directly from the complex voltage function $V(z)$ by:

$$\begin{aligned} v(z,t) &= Re\left\{ V(z,\omega)\, e^{j\omega t} \right\} \\ &= Re\left\{ v(z,\omega)\, e^{+j\varphi_v(z,\omega)}\; e^{j\omega t} \right\} \\ &= Re\left\{ v(z,\omega)\, e^{j(\,\omega t \,+\, \varphi_v(z,\omega)\,)} \right\} \\ &= v(z,\omega)\cos(\,\omega t + \varphi_v(z,\omega)\,) \end{aligned}$$

And now for current

Likewise for the **current**:

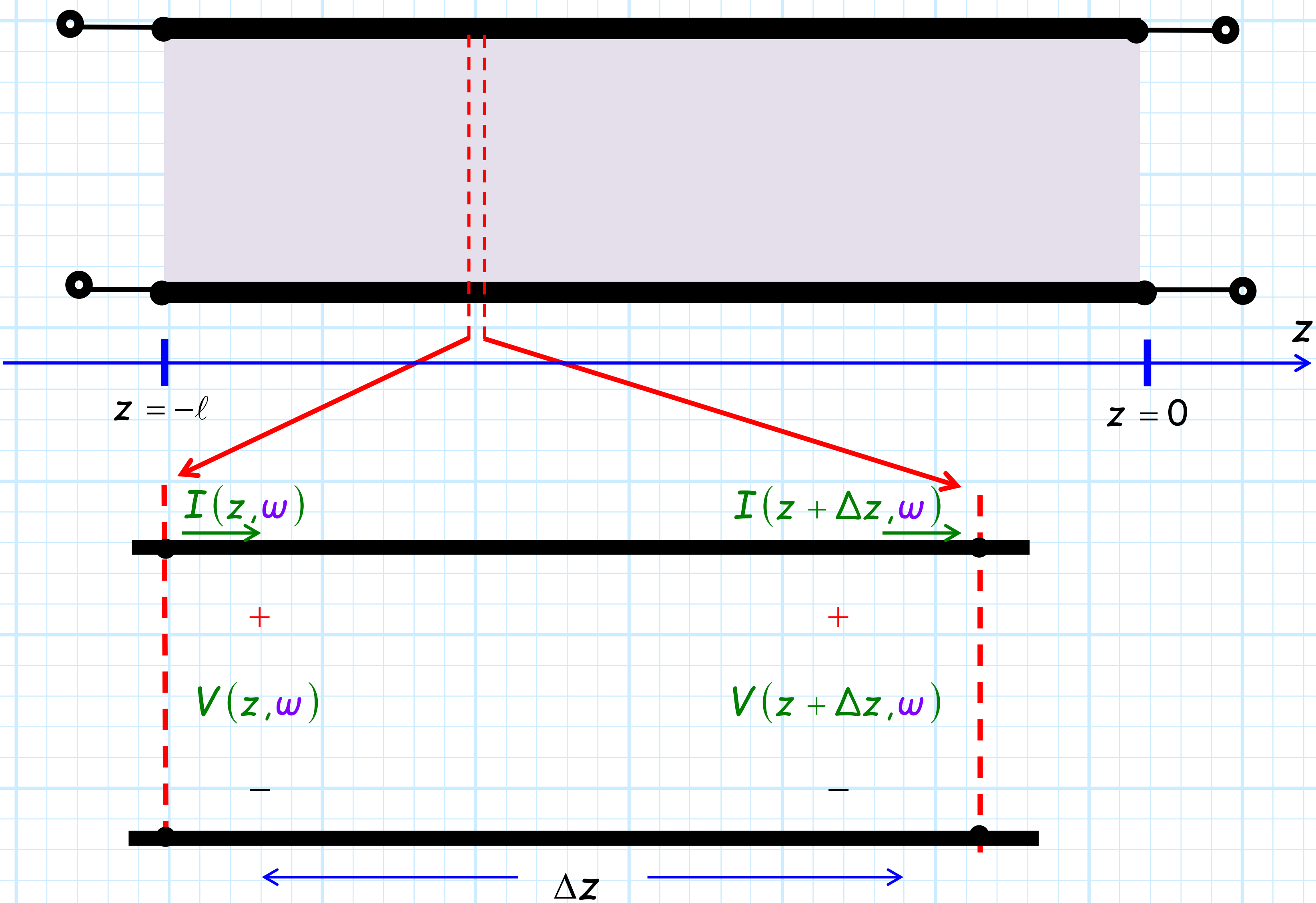
$$i(z,t) = Re\left\{ I(z,\omega)\, e^{j\omega t} \right\} = i(z,\omega)\cos\left(\omega t + \varphi_i(z) \right)$$



The magnitude and relative phase functions of **current** and **voltage** are **quite different**!!

An incremental transmission line

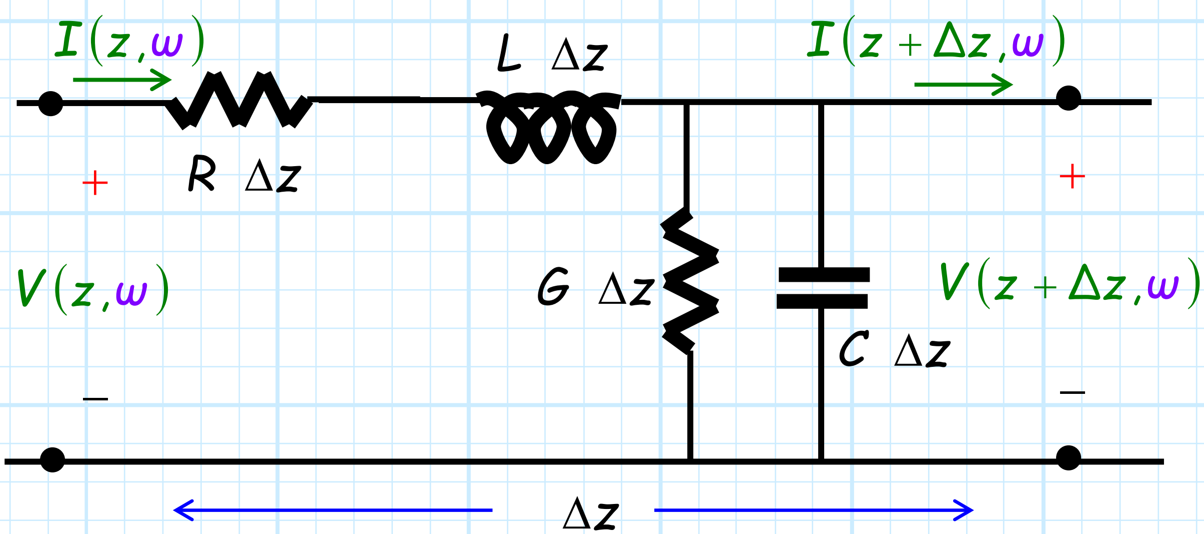
Now, let's examine a **tiny sliver** of a transmission line:



An equivalent circuit

This little slice of our transmission line has **length** Δz, and so the resistance, inductance, capacitance, and conductance of this **tiny section** is respectively $R\,\Delta z$, $L\,\Delta z$, $C\,\Delta z$, and $G\,\Delta z$.

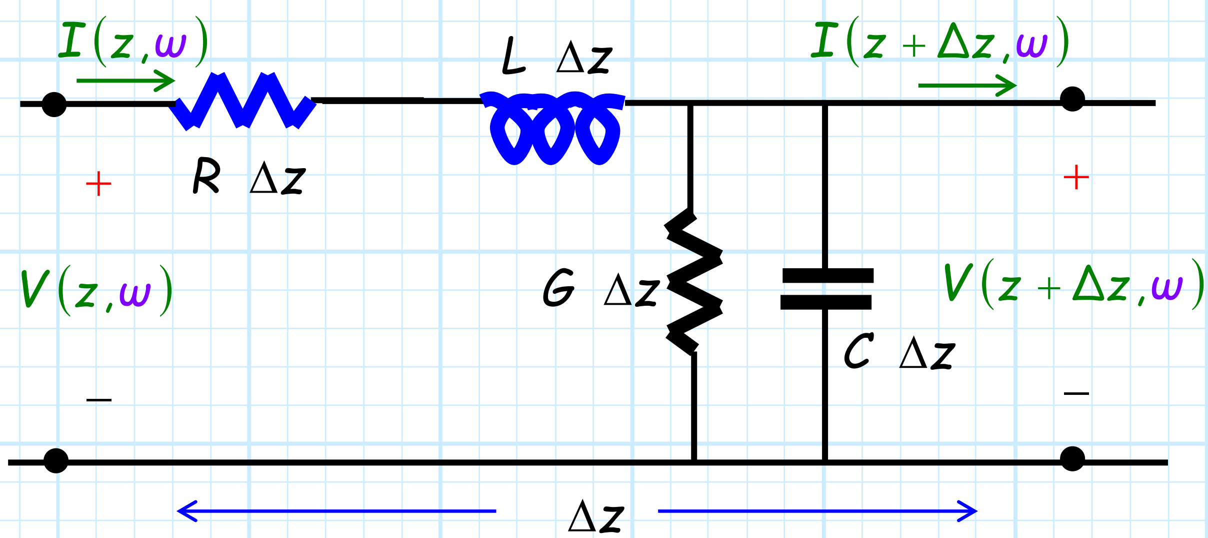
Thus, this sliver has an **equivalent circuit** that represents these **four** characteristics:



Using KVL

Now using **KVL**, to evaluate this circuit, we find:

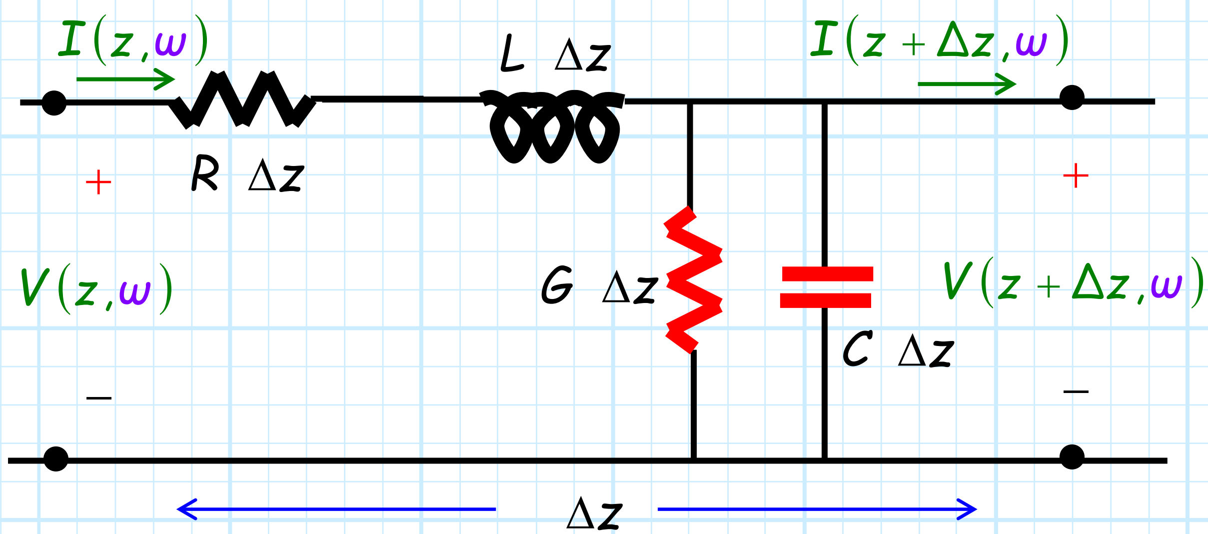
$$V(z+\Delta z,\omega) - V(z,\omega) = -(R\Delta z,\omega)\ I(z,\omega) - j\omega(L\Delta z)\ I(z,\omega)$$



Using KCL

And now from **KCL**:

$$I(z+\Delta z,\omega) - I(z,\omega) = -(G\Delta z)\, V(z+\Delta z,\omega) - j\omega(C\Delta z)\, V(z+\Delta z,\omega)$$



How I entertain myself

Now, let's (just for fun!) **divide** these equations by small distance Δz:

$$\frac{V(z+\Delta z,\omega) - V(z,\omega)}{\Delta z} = -(R+j\omega L)\, I(z,\omega)$$

$$\frac{I(z+\Delta z,\omega) - I(z,\omega)}{\Delta z} = -(G+j\omega C)\, V(z+\Delta z,\omega)$$

Now, this "lumped" element circuit model (and their resulting equations above) is accurate only for **very** small Δz and becomes increasingly more accurate as Δz **approaches zero**.

Let's make Δz really, really small

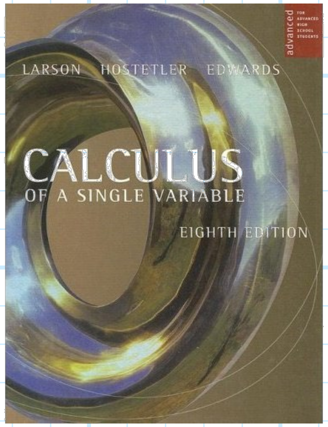
Therefore, let's take this **limit** for our two expressions, as Δz approaches zero:

$$\lim_{\Delta z \to 0} \left\{ \frac{V(z+\Delta z,\omega) - V(z,\omega)}{\Delta z} = -(R+j\omega L)\, I(z,\omega) \right\}$$

$$\lim_{\Delta z \to 0} \left\{ \frac{I(z+\Delta z,\omega) - I(z,\omega)}{\Delta z} = -(G+j\omega C)\, V(z+\Delta z,\omega) \right\}$$

→ Look at the **left side** of these two equations!

Remember?



Remember the definition of a **derivate** operator:

$$\frac{d\,f(x)}{dx} \doteq \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

The left side of each equation has this **same form**, thus they are **derivative** operators.

$$\lim_{\Delta z \to 0} \frac{V(z+\Delta z,\omega) - V(z,\omega)}{\Delta z} = \frac{d\,V(z,\omega)}{dz}$$

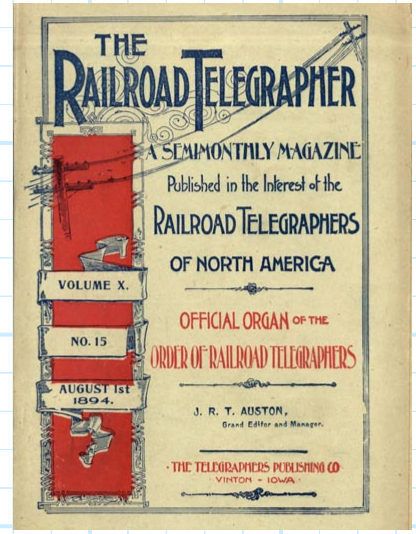
$$\lim_{\Delta z \to 0} \frac{I(z+\Delta z,\omega) - I(z,\omega)}{\Delta z} = \frac{d\,I(z,\omega)}{dz}$$

Inserting these into our previous results, and we find that we have created a coupled set of **differential equations**!

Behold, the telegrapher's equations

$$\frac{\partial V(z,\omega)}{\partial z} = -(R + j\omega L)\; I(z,\omega)$$

$$\frac{\partial I(z,\omega)}{\partial z} = -(G + j\omega C)\; V(z,\omega)$$



These equations are the complex form of the **telegrapher's equations**.

Derived by **Oliver Heaviside**, the telegrapher's equations are essentially the "Maxwell's equations" of transmission lines.

Although **mathematically** the functions $V(z,\omega)$ and $I(z,\omega)$ could take any form, they can **physically** exist **only if** they satisfy the both of the differential equations shown above!