PHSX 521: Homework #6

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Problem 1

A massless spring has unstretched length l_0 and force constant k. One end is now attached to the ceiling and a mass m is hung from the other. The equilibrium length of the spring is now l_1.

(a) Write down the condition that determines l_1.

Solution:

The condition for equilibrium is simply that

$$m\mathbf{F} = \mathbf{a} = 0, \quad \xRightarrow{\mathbf{F}=-kx+mg} \quad 0 = -k(l_1 - l_0) + mg$$

Where l_1 would be the new equilibrium position.

$$l_1 = l_0 + \frac{mg}{k}$$

(b) Suppose now the spring is stretched a further distance x beyond its new equilibrium length. Show that the net force (spring plus gravity) on the mass is $F = -kx$. That is, the net force obeys Hooke's law when x is the distance from the equilibrium position—a very useful result, which lets us treat a mass on a vertical spring just as if it were horizontal.

Solution:

If we consider the motion to be one dimensional in a perfectly vertical axis, we have

$$F = 0 = -k(l_1 - l_0) - mg$$

Now if we find equilibrium position by setting net force to zero,

$$l_0 - \frac{mg}{k} = l_1$$

If we consider some displacement from this position x, we expect mg to cancel:

$$F = -k(x + l_1 - l_0) - mg = -kx$$

(c) Prove the same result by showing that the net potential energy (spring plus gravity) has the form $U(x) = \text{const} + \frac{1}{2}kx^2$.

Solution:

$$U(x) = mgh + \frac{1}{2}k(\alpha - l_0)^2$$

where h is measured from some arbitrary reference level. If we choose our coordinate system such that $x = 0$ at equilibrium, then:

$$h = h_0 + x, \quad \alpha = l_1 + x$$

where h_0 and l_1 correspond to the equilibrium position. Substituting:

$$U = mg(h_0 + x) + \frac{1}{2}k(l_1 + x - l_0)^2$$

At equilibrium,

$$\frac{mg}{k} = l_1 - l_0.$$

Using this and expanding the squared term:

$$\begin{aligned} U &= mgh_0 + mgx + \frac{1}{2}k\left[(l_1 - l_0)^2 + 2(l_1 - l_0)x + x^2\right] \\ &= mgh_0 + mgx + \frac{1}{2}k\left(\frac{mg}{k}\right)^2 + mgx + \frac{1}{2}kx^2 \\ &= \left[mgh_0 + \frac{1}{2}k\left(\frac{mg}{k}\right)^2\right] + \frac{1}{2}kx^2 \end{aligned}$$

The term in square brackets is constant, giving us:

$$U = \text{const} + \frac{1}{2}kx^2$$

This quadratic form of the potential energy immediately implies that the restoring force

$$F = -\frac{dU}{dx} = -kx,$$

Problem 2

The potential energy of two atoms in a molecule can sometimes be approximated by the Morse function,

$$U(r) = A\left[\left(e^{(R-r)/S} - 1\right)^2 - 1\right],$$

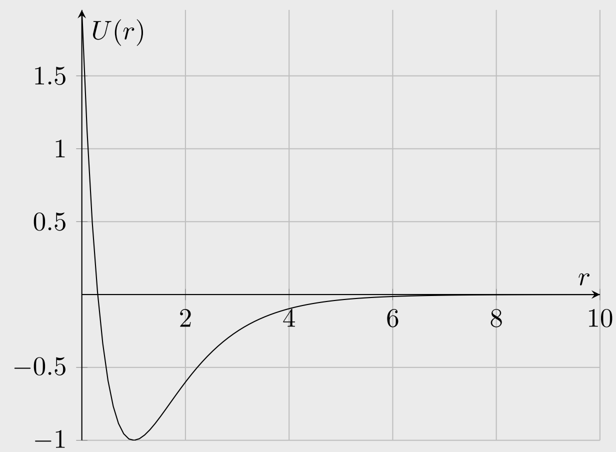
where r is the distance between the two atoms and $A, R,$ and S are positive constants with $S \ll R$.

1. Sketch this function for $0 < r < \infty$.

Solution:

Problem 2 continued on next page…

$$U(r) = A\left[\left(e^{(R-r)/S} - 1\right)^2 - 1\right]$$



2. Find the equilibrium separation r_0, at which $U(r)$ is minimum.

Solution:

$$U(r) = A\left(e^{2(R-r)/S} - 2e^{(R-r)/S}\right)$$
$$\nabla U(r) = \frac{A}{S}\left(2e^{(R-r)/S} - 2e^{2(R-r)/S}\right)$$

Equilibrium for a potential function is when the function is at a minima or maxima

$$\nabla U(r) = 0 = \frac{A}{S}\left(2e^{(R-r)/S} - 2e^{2(R-r)/S}\right)$$
$$r = R$$

3. Now write $r = r_0 + x$ so that x is the displacement from equilibrium, and show that, for small displacements, U has the approximate form $U = \text{const} + \frac{1}{2}kx^2$. That is, Hooke's law applies. What is the force constant k?

Solution:

$$e^{x/S} \approx 1 + \frac{x}{S} + \frac{x^2}{2S^2} + \ldots$$
$$U(r) = A\left[\left(1 + \frac{[R-(r_0+x)]}{S} - 1\right)^2 - 1\right]$$
$$= A\left(\frac{[R-(r_0+x)]^2}{S^2} - 1\right)$$

Which is a quadratic potential with some vertical offset. If $S \ll R$, then the leading term should dominate. We could say that $k = \frac{2A}{S^2}$ such that we have an approximate potential where x is the displacement from the equilibrium

$$U(r) = \frac{1}{2}kx^2$$

Problem 3

The maximum displacement of a mass oscillating about its equilibrium position is $0.2\,\mathrm{m}$, and its maximum speed is $1.2\,\mathrm{m/s}$. What is the period τ of its oscillations?

Solution:

$$\begin{aligned}
x(t) &= A\cos(\omega t + \delta) \\
\dot{x}(t) &= -A\omega\sin(\omega t + \delta) \\
\dot{x}_{max} &= A\omega && \text{maximized when } \sin(\omega t + \delta) = \pm 1 \\
\omega &= \frac{\dot{x}_{max}}{A} \\
\tau &= \frac{2\pi A}{\pi \dot{x}_{max}} && \tau = \frac{2\pi}{\omega} \text{ for sinusoidal motion} \\
&= \frac{\pi}{3} \approx 1.05
\end{aligned}$$

Problem 4

Consider a particle in two dimensions, subject to a restoring force of the form (5.21). (The two constants k_x and k_y may or may not be equal; if they are, the oscillator is isotropic.) Prove that its potential energy is

$$U = \frac{1}{2}\left(k_x x^2 + k_y y^2\right).$$

Solution: We have forces

$$\mathbf{F} = \begin{pmatrix} -k_x x \\ -k_y y \end{pmatrix}$$

We can use familiar techniques to get to a potential function

$$\begin{aligned}
U &= \int -k_x x \; dx = -\frac{1}{2}k_x x^2 - k_y c(y) \\
F_y &= \frac{dU}{dy} \implies -k_y y = -k_y c_y(y) \\
c(y) &= \frac{1}{2}k_y y \\
\mathbf{F} = -\nabla U \qquad \implies \qquad U &= \frac{1}{2}k_x x^2 + \frac{1}{2}k_y y^2
\end{aligned}$$

Problem 5

A damped oscillator satisfies the equation (5.24), where $F_{\text{dmp}} = -b\dot{x}$ is the damping force. Find the rate of change of the energy $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$ (by straightforward differentiation), and, with the help of (5.24), show that dE/dt is (minus) the rate at which energy is dissipated by F_{dmp} .

Solution: Differentiating non-damped energy, we get

$$\frac{dE}{dt} = m\dot{x}\ddot{x} + kx\dot{x} = \dot{x}(m\ddot{x} + kx)$$

Meanwhile

$$F = m\mathbf{a} = F_{dmp} \quad \rightarrow \quad m\ddot{x} = -b\dot{x} - kx$$

where we have $F_{dmp} = -bdx/dt$.

$$E = \frac{m}{2}\left(\frac{dx}{dt}\right)^2 + \frac{k}{2}x^2.$$

$$\frac{dE}{dt} = \left(m\frac{d^2x}{dt^2} + kx\right)\frac{dx}{dt} = -b\frac{dx}{dt}\left(\frac{dx}{dt}\right) = F_{dmp}\frac{dx}{dt},$$