Homework

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1 Math 590 HW9

1.1 Problem 1.

Let T be the following transformation $\mathbf{R}^2 \to \mathbf{R}^2$: projecting onto the line $y = x$, then projecting onto the line $y = 2x$.

(a) What is the matrix A for T, using standard basis?

Solution:

Recall that $proj_v(u) = \dfrac{u \cdot v}{|v|^2} v$. Second, because we are projecting onto the line $y = x$ first and the line $y = 2x$ second, the direction vectors are $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$ respectively. Therefore:

(a) $y = x$:

$$\vec{e}_1 = \frac{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] \cdot \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]}{\sqrt{2}} \begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}\sqrt{2}\\\sqrt{2}\end{bmatrix}$$

$$\vec{e}_2 = \frac{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] \cdot \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]}{\sqrt{2}} \begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}\sqrt{2}\\\sqrt{2}\end{bmatrix}$$

(b) $y = 2x$:

$$\vec{e}_1 = \frac{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] \cdot \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]}{\sqrt{2}} \begin{bmatrix}1\\1\end{bmatrix} = \frac{\sqrt{5}}{2} \begin{bmatrix}1\\2\end{bmatrix}$$

$$\vec{e}_2 = \frac{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] \cdot \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]}{\sqrt{2}} \begin{bmatrix}1\\1\end{bmatrix} = \frac{2\sqrt{5}}{5} \begin{bmatrix}1\\2\end{bmatrix}$$

Finally, the composition of these transformations is the same as applying them sequentially:

$$\begin{bmatrix} \frac{\sqrt{5}}{2} & \frac{2\sqrt{5}}{2} \\ \frac{2\sqrt{5}}{2} & \frac{4\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{3\sqrt{10}}{2} & \frac{3\sqrt{10}}{2} \\ 3\sqrt{10} & 3\sqrt{10} \end{bmatrix}$$

(b) What are the eigenvalues and eigenspaces of T, using algebraic method and the matrix A?

Solution:

(a) Eigenvalues:

$$\det\left(\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{3\sqrt{10}}{2} & \frac{3\sqrt{10}}{2} \\ 3\sqrt{10} & 3\sqrt{10} \end{bmatrix}\right) = 0$$

$$\to \left(\lambda - \frac{3\sqrt{10}}{2}\right)(\lambda - 3\sqrt{10}) - \left(\frac{3\sqrt{10}}{2}\right)(3\sqrt{10}) = 0$$

$$\to \lambda^2 - \lambda\frac{9\sqrt{5}}{\sqrt{2}} = 0 \to \lambda\left(\lambda - \frac{9\sqrt{5}}{\sqrt{2}}\right) = 0$$

$$\implies \lambda = \left\{0, \frac{9\sqrt{5}}{\sqrt{2}}\right\}$$

(b) Eigenvectors:

$$\left(\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{3\sqrt{10}}{2} & \frac{3\sqrt{10}}{2} \\ 3\sqrt{10} & 3\sqrt{10} \end{bmatrix}\right) x = 0$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \implies \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} \to \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{1}$$

$$\left[\begin{array}{cc|c} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right] \implies \begin{bmatrix} \frac{x_2}{2} \\ x_2 \end{bmatrix} \to \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \tag{2}$$

(c) Explain part (b) using geometry.

Solution: Geometrically eigenvectors represent vectors which do not change direction under the map. We can see that $\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$ is only scaled (because we project onto the line $y = x$ first), since it already lies on the line $y = 2x$. The other eigenvector, $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is mapped to zero, because it is perpendicular to the first map to the line $y = x$.

1.2 Problem 2.

Let A be the matrix in Problem 1. Find A^{2023} by:

(a) Using geometry, using the description of T, and results from Problem 1.

Solution:

The final transformation represents a projection onto the line $y = 2x$, so the vectors will not change direction after applying the transformation repeatedly. However, because it is projected onto the line $y = x$ before projecting onto $y = 2x$, the vector will be scaled after each iteration.

(b) Using algebra, via diagonalization.

Solution:

The diagonal matrix is $\begin{bmatrix} 0 & 0 \\ 0 & \frac{9\sqrt{5}}{\sqrt{2}} \end{bmatrix}$. Therefore, A^{2023} is equal to:

$$\begin{bmatrix} 0 & 0 \\ 0 & \left(\frac{9\sqrt{5}}{\sqrt{2}}\right)^{2023} \end{bmatrix}$$