PHSX 531: Homework #7

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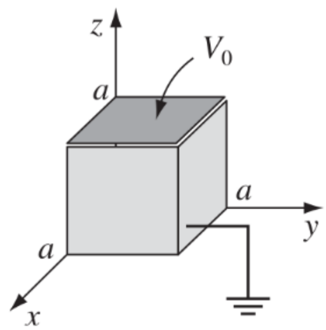
Problem 1

(3 pts) A cubical box (sides of length a) consists of five metal plates, which are welded together and grounded. The top is made of a separate sheet of metal, insulated from the others, and held at a constant potential V_0. Find the potential inside the box.



Solution:
Much like example 3.5 in Griffiths, this is a true 3-D problem.

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x} = k, \quad \frac{1}{Y}\frac{\partial^2 Y}{\partial y} = l, \quad \frac{1}{Z}\frac{\partial^2 Z}{\partial z} = C_3, \quad k + l + C_3 = 0$$

$$\frac{d^2 X}{x^2} = l^2 X, \quad \frac{d^2 Y}{dy^2} = -k^2 Y, \quad \frac{d^2 Z}{dz^2} = (k^2 + l^2) Z$$

$$\begin{cases} X(x) = A\sin(lx) + B\cos(lx) \\ Y(y) = C\sin(ky) + D\cos(ky) \\ Z(z) = Ee^{\sqrt{k^2+l^2}z} + Fe^{-\sqrt{k^2+l^2}z} \end{cases}$$

$$V(x, y, a) = \sum_n \sum_m C_{n,m} \Big(A\sin(lx) + B\cos(lx)\Big)\Big(C\sin(ky) + D\cos(ky)\Big)\Big(Ee^{\sqrt{k^2+l^2}z} + Fe^{-\sqrt{k^2+l^2}z}\Big)$$

Subject to:

$$\begin{cases} V = 0 \text{ when } x = 0, \\ V = 0 \text{ when } x = a, \\ V = 0 \text{ when } y = 0, \\ V = 0 \text{ when } y = a, \\ V = 0 \text{ when } z = 0, \\ V = V_0(x, y, a) \text{ when } z = a. \end{cases}$$

The easy boundary conditions are:

$$\begin{aligned} x(0) &= 0 \implies B = 0 \\ x(a) &= A \sin la = 0 \implies ka = n\pi \\ y(0) &= 0 \implies D = 0 \\ y(a) &= C \sin ka = 0 \implies ka = m\pi \\ z(0) &= 0 \implies E + F = 0 \\ z(z) &= Ee^{\sqrt{k^2+l^2}z} - Ee^{-\sqrt{k^2+l^2}z} = 2E \sinh\left(z\sqrt{k^2+l^2}\right) \end{aligned}$$

$$\boxed{V(x, y, z) = \sum_n \sum_m C_{n,m} \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{a}y\right) \sinh\left(z\sqrt{\frac{n\pi}{a}^2 + \frac{m\pi}{a}^2}\right)}$$

The coefficients can be pulled out with Fourier's trick

$$\begin{aligned} V(x, y, z = a) &= V_0 \\ V_0 \int_{x=0}^{a} \int_{y=0}^{a} \sin \frac{n\pi}{a}x \sin \frac{n\pi}{a}y \; dx \; dy &= C_{n,m} \frac{a^2}{4} \sinh\left(\pi\sqrt{n^2 + m^2}\right) \\ \left(\frac{a}{n\pi} \frac{a}{m\pi} \cos \frac{n\pi}{a}x \cos \frac{m\pi}{a}y\right)\Big|_{x,y=0}^{x,y=a} &= C_{n,m} \frac{a^2}{4} \sinh\left(\pi\sqrt{n^2 + m^2}\right) \\ \frac{a}{n\pi} \frac{a}{m\pi} [\cos n\pi - 1][\cos m\pi - 1] &= C_{n,m} \frac{a^2}{4} \sinh\left(\pi\sqrt{n^2 + m^2}\right) \\ \frac{1}{nm\pi^2} \frac{4V_0}{\sinh\left(\pi\sqrt{n^2 + m^2}\right)} [\cos n\pi - 1][\cos m\pi - 1] &= C_{n,m} \end{aligned}$$

When n or m are even everything goes to zero, meanwhile we get a multiple of 4 when n and m are odd. Therefore only odd solutions exist giving

$$\boxed{C_{n,m=1,3,5,...} = \frac{1}{nm\pi^2} \frac{16V_0}{\sinh\left(\pi\sqrt{n^2 + m^2}\right)}}$$

Problem 2

(3 pts) Suppose the potential is a constant V_0 over the surface of a sphere of radius R. Find the potential inside and outside of the sphere. (Use our solutions to Laplace's equations in spherical coordinates).

Solution:
In spherical coordinates the general solution for Laplace's equation is

$$V(r,\theta) = \sum_{l}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

Inside, A_l must be nonzero meanwhile B_l must be zero to constitute finiteness. The opposite is true if we consider the outside.

i. For the insidse, we have the condition

$$V(R,\theta) = \sum_{l=0}^{R} A_l r^l P_l(\cos\theta) = V_0(\theta)$$

ii. For the outside, we have the condition

$$V(R,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) = V_0(\theta)$$

Following the concepts discussed in Griffiths Example 3.7, we arrive at:

$$\begin{aligned} A_l &= \frac{2l+1}{2R^l} \int_0^{\pi} V_0(\theta) P_l(\cos\theta) \sin\theta \, d\theta \\ B_l &= \frac{2l+1}{2} R^{l+1} \int_{\pi}^{\infty} V_0(\theta) P_l(\cos\theta) \sin\theta \, d\theta \end{aligned}$$

Now with $V_0(\theta) = V_0$, we can insert $P_0(\cos\theta) = 1$.

$$\begin{aligned} A_l &= \frac{2l+1}{2R^l} \int_0^{\pi} P_0(\cos\theta) P_l(\cos\theta) \sin\theta \, d\theta \\ B_l &= \frac{2l+1}{2} R^{l+1} \int_{\pi}^{\infty} P_0(\cos\theta) P_l(\cos\theta) \sin\theta \, d\theta \end{aligned}$$

Now by orthagonality of Legendre Polynomials we can reason that the only nonzero result happens when $l = 0$. The integral terms give

$$\int_0^{\pi} \sin\theta \; d\theta = -\cos\theta|_0^{\pi} = 2$$

$$\begin{cases} A_l = \frac{1}{2} V_0 \cdot 2 = V_0 \\ B_l = \frac{1}{2} R V_0 \cdot 2 = R V_0 \end{cases}$$

Therefore:

$$\boxed{V(r,\theta) = \begin{cases} V_0, & \text{inside} \\ \frac{V_0 R}{r}, & \text{outside} \end{cases}}$$

Problem 3

(3 pts) Derive the most general solution to Laplace's equation in cylindrical coordinates, assuming cylindrical symmetry (no dependence on z). [Make sure you find all the solutions to the radial equation; in particular, your result must accommodate the case of an infinite line charge.]

Solution:

$$\nabla^2 V(r,\theta,z) = r\frac{\partial^2 V}{\partial r^2} + \frac{\partial V}{\partial r} + \frac{1}{r}\frac{\partial^2 V}{\partial \theta^2} = 0$$
$$\frac{1}{\Theta}\frac{d^2\Theta}{d\theta} = -k^2, \qquad \frac{1}{R}\frac{1}{r}\frac{d}{dr}\left(\frac{dR}{dr}\right) = k^2$$

Because these are second order we get two solutions. The first solution for the radial part is a power series, while the second is logarithmic:

$$\begin{aligned}
r\frac{d}{dr}\left(r\frac{dr^n}{dr}\right) &= r\frac{d}{dr}(rnr^{n-1}) \\
&= r\frac{d}{dr}(nr^n) \\
&= rn^2r^{n-1} \\
&= n^2r^n \\
&= k^2R = k^2r^n \\
&\implies n = k
\end{aligned}$$

$$\begin{aligned}
r\frac{d}{dr}\left(r\frac{dR}{dr}\right) &= const \\
\frac{dR}{dr} &= \frac{const}{r} \\
dR &= \frac{dr}{r}\cdot const \\
R &= A\ln r + B
\end{aligned}$$

The angular solutions are the same as before.

$$\begin{aligned}
R_1(r) &= Ar^k + Br^{-k} \\
R_2(r) &= A\ln r + B \\
\Theta_1(\theta) &= C\cos(k\theta) + D\sin(k\theta) \\
\Theta_2(\theta) &= C\theta + D \quad \text{This one is non-physical, gives discontinuous solutions}
\end{aligned}$$

$$V(r,\theta) = A\ln r + B + \sum_{k=1}^{\infty}\left[r^k(a_k\cos k\theta + b_k\cos k\theta) + r^{-k}(a_k\cos k\theta + b_k\sin k\theta)\right]$$