PHSX 531: Homework #8

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Grant Saggars

Problem 1

(3 pts) A physical electric dipole consists of two equal and opposite charges ($\pm q$) separated by a distance d. Find the quadrupole and octopole terms in the potential.

Solution:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathbf{r}_+} - \frac{q}{\mathbf{r}_-}\right)$$

The binomial theorem states that for any real number n:

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n-1} x^1 y^{n-1} + \binom{n}{n} x^0 y^n$$

or for the special case of the dipole expansion:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

$$\frac{1}{\mathbf{r}_\pm} = \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-1/2} = \frac{1}{r}\left(1 \mp x\right)^{-1/2} \approx \cdots + \underbrace{\frac{3}{8}x^2}_{\text{quadrupole}} - \underbrace{\frac{5}{16}x^3}_{\text{octopole}} + \mathcal{O}^4$$

how do we continue the expansion Substituting back, we get

$$\begin{aligned} V(\mathbf{r}) &= \frac{q}{4\pi\epsilon_0}\frac{1}{r}\left(\frac{3d^2}{8r^2}\cancelto{0}{[1-1]}\cos^2\theta + \frac{5d^3}{16r^3}[1-(-1)]\cos^3\theta\right) \\ &= \frac{q}{4\pi\epsilon_0}\frac{5d^3}{8r^4}\cos^3\theta \end{aligned}$$

Problem 2

(3 pts) A sphere of radius R, centered at the origin, carries a charge density

$$\rho(r,\theta) = \frac{R}{r^2}(R-2r)\sin\theta,$$

where k is a constant, and r, θ are the usual spherical coordinates. Find the approximate potential for points on the z axis, far from the sphere. (That is, the first non-zero term in the multipole expansion.)

Solution:

$$
\begin{aligned}
Q_{\text{enc}} &= \frac{k}{a}\int_0^{\pi} \sin^2\theta \, d\theta \int_0^R (R-2r)\, dr \int_0^{2\pi} d\phi \\
&= 4\pi \frac{k}{R}(R^2 - R^2) = 0
\end{aligned}
$$

Since total charge is zero, there can be no monopole. I can try to calculate the dipole:

$$
\begin{aligned}
V_{\text{dip}}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\mathbf{p} \\
\mathbf{p} &= 2\pi \int_0^{\pi}\int_0^R r\cos\theta\rho(r) r^2 \sin\theta \, dr\, d\theta \\
&= 2\pi \underbrace{\int_0^{\pi} \cos\theta \sin^2\theta \, d\theta}_{=0} \int_0^R r\left[\frac{R}{\cancel{r^2}}(R-2r)\cancel{r^2}\right] dr \\
&= 0
\end{aligned}
$$

This is zero so I will try a quadrupole:

$$
\begin{aligned}
V_{\text{quad}}(\mathbf{r}) &= \frac{1}{r^3}\int (r)^2 \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)\rho(\mathbf{r})\, d\tau \\
&= \frac{1}{4\pi\epsilon_0}\frac{R}{r^3}\int_0^{\pi} \frac{1}{2}(3\cos^2 - 1)\theta \sin^2\theta \, d\theta \int_0^R r^2(R-2r)\, dr \int_0^{2\pi} d\phi \\
&= \frac{1}{4\pi\epsilon_0}\left(\frac{R}{r^3}\right)\left(-\frac{R^4}{6}\right)\left(-\frac{\pi}{16}\right)(2\pi) \\
&= \frac{1}{4\pi\epsilon_0}\left(\frac{\pi^2 R^5}{48 r^3}\right)
\end{aligned}
$$

Problem 3

(3 pts) Calculate the dipole moment of a spherical shell of radius R with a surface charge density of $\sigma = k\cos\theta$.

Solution:
assuming the sphere is centered at the origin.

$$
\mathbf{p} \equiv \int \mathbf{r}'\rho(\mathbf{r}')\, d\tau'
$$

Which for a fixed radius R is just the surface integral

$$
\begin{aligned}
\mathbf{P} &= R\int_0^{\pi}\int_0^{2\pi} \sigma k \cos\theta \, d\phi\, d\theta = 2\pi R \int_0^{\pi} \sigma k \cos\theta \, d\theta \\
&= 2\pi R[-\sigma k \cos\pi + \sigma k \cos 0] \\
&= 4\pi\sigma k R
\end{aligned}
$$

Problem 4

(3 pts) A "pure" dipole $\mathbf{p}$ is situated at the origin, pointing in the z direction. How much work does it take to move a charge q from $(a, 0, 0)$ to $(0, 0, a)$?

Solution:
The work done to move a charge from any two points is given by:

$$W = \int_a^b \mathbf{F} \cdot d\ell = -Q \int_a^b E \cdot d\ell = Q[V(b) - V(a)]$$

The potential for this dipole is then

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$

When the charge is at $(0, 0, a)$, the angle between the charge and dipole is $0°$, and when it is at (a,0,0) the angle is $90°$. Therefore we get a force

$$\mathbf{F} = \frac{qp}{4a^2\pi\epsilon_0} (\cos(0) - \cos(\pi/2)) = \frac{qp}{4a^2\pi\epsilon_0}$$