Impedance

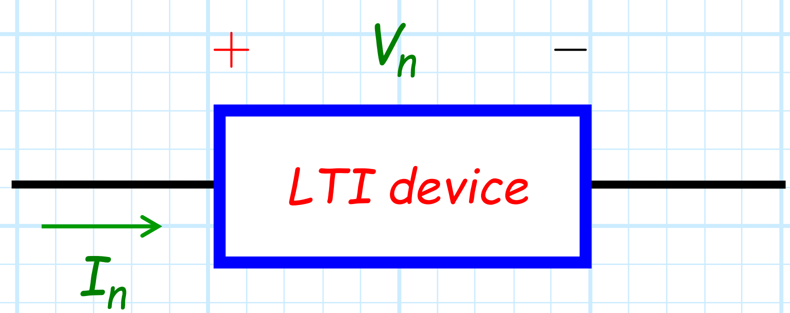
Q: *How are **complex** voltages V_n and **complex** currents I_n related?*

A: We can relate a complex voltage and a complex current **if (and only if!)** they represent:

a) the current I_n **through** and,

b) voltage V_n **across**,

a **linear**, **two-terminal** (i.e., one port) **device:**



Pay particular attention to this

The ratio of the **complex** voltage and **complex** current is a **complex** value that depends on:

1. The **physical characteristics** of the two-terminal device (i.e., resistance and reactance).

2. The **frequency** ω of the time-harmonic (i.e., sinusoidal) excitation source.

→ More importantly, the ratio V_n/I_n depends **only** on these two things!

Complex impedance

This ratio is of course known as the **complex impedance** of the two-terminal, LTI (Linear Time-Invariant) device:

$$\boxed{Z_n \doteq \frac{V_n}{I_n}}$$

Q: *Impedance is a **complex** value—what does this complex number actually tell us?*

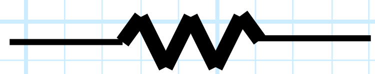
A: Quite a **number of things**!

Real and imaginary…

First, we find that:

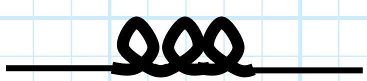
1. the **real part** of the impedance is due to **resistive** elements in the two-terminal device—**no reactive** elements mean the impedance is **purely real** (i.e., the **imaginary** part is **zero)**.

$$Z_R = R + j0 = Re^{j0}$$

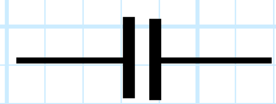


2. the **imaginary part** of Z_n is due to the **reactive** elements within the two-terminal device—**no resistive** elements mean the impedance is **purely imaginary** (i.e., the real part is **zero)**.

$$Z_L(\omega) = 0 + j\omega L = \omega L e^{j(\pi/2)}$$



$$Z_C(\omega) = 0 - j\tfrac{1}{\omega C} = \tfrac{1}{\omega C} e^{j(3\pi/2)}$$



…is resistance and reactance

Thus, we can write a complex **impedance** as:

$$Z_n = R_n + jX_n$$

where:

$R_n = Re\{Z_n\}$ is the element **resistance** and,

$X_n = Re\{Z_n\}$ is element **reactance**.

Q: *I **guess** the magnitude and phase of impedance tells us nothing important?*

A: Actually, the magnitude and phase of Z_n tells us **much**!

Magnitude is important…

Since:

$$Z_n = \frac{V_n}{I_n} = \frac{v_{0n}\, e^{-j\varphi_n^v}}{i_{0n}\, e^{-j\varphi_n^i}} = \frac{v_{0n}}{i_{0n}}\, e^{-j\left(\varphi_n^v - \varphi_n^i\right)}$$

the **magnitude** of Z_n is therefore:

$$\left|Z_n\right| = \left|\frac{v_{0n}}{i_{0n}}\right| \left|e^{-j\left(\varphi_n^v - \varphi_n^i\right)}\right| = \frac{v_{0n}}{i_{0n}}$$

→ The **impedance magnitude** tells us the ratio of the **magnitudes** of the **voltage** sinusoid and the **current** sinusoid!

...as is phase!

Now, the **phase** of Z_n is:

$$arg\{Z_n\} = arg\left\{\frac{V_{0n}}{I_{0n}}\, e^{-j\left(\varphi_n^v - \varphi_n^i\right)}\right\} = \varphi_n^v - \varphi_n^i$$

→ The phase of the **complex** value Z_n tells us the relative **phase difference** between the voltage sinusoid and the current sinusoid!

If Z(w) is known, then everything is known

Remember, complex impedance depends of the **frequency** ω of the time-harmonic oscillation (i.e., the frequency of the **sinusoid**)!

As a result, we find that for most **LTI elements**, the impedance is a **function of frequency**:

$$Z_n(\omega) = R(\omega) + jX(\omega)$$

This impedance function depends on the element—and the element **only**.

→ As a matter of fact, the impedance function $Z_n(\omega)$ **completely characterizes** the LTI element!!!!!!!

Can the real part depend on frequency?

Q: *Wait just a **dog-gone** second! On the **previous page** you wrote:*

$$Z_n(\omega) = R(\omega) + jX(\omega)$$

*Obviously, the imaginary (i.e., **reactive**) component $X(\omega)$ **is** a function of **frequency**, but I **don't** see why the real (i.e., **resistive**) component $R(\omega)$ should depend on ω!*

*Perhaps you made a **mistake**?*

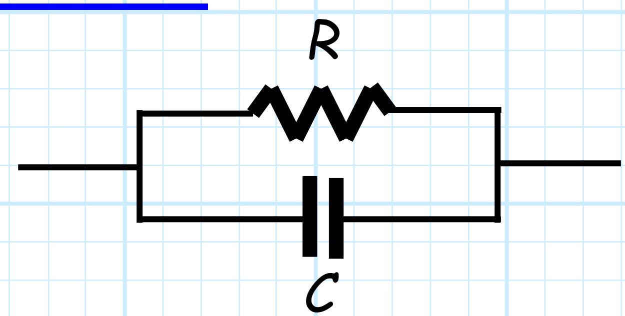
A: Nope, no mistake!

Generally speaking, **both** the imaginary **and** real components of impedance indeed vary with frequency ω.

→ Maybe an **example** would help…

...and here it is

Consider **this** simple LTI device:



The two **parallel** circuit-elements result in **this** impedance:

$$Z(\omega) = R \| \frac{-j}{\omega C} = -j\left(\frac{R}{\omega C}\right)\left(\frac{1}{R - \frac{j}{\omega C}}\right) = \left(\frac{R}{1 + j\omega RC}\right)$$

Q: *Yikes! Just how do we determine the real and imaginary component of* ***that****?*

A: Multiply **both** the numerator **and** denominator by the complex **conjugate of the denominator**, i.e., by:

$$(1 + j\omega RC)^{*} = 1 - j\omega RC$$

The resistive component is frequency dependent!

This impedance is therefore:

$$
\begin{aligned}
Z(\omega) &= \left(\frac{R}{1+j\omega RC}\right)\left(\frac{1-j\omega RC}{1-j\omega RC}\right) \\
&= \frac{R-j\omega R^2C}{1+\omega^2(RC)^2} \\
&= \left(\frac{R}{1+\omega^2(RC)^2}\right) - j\left(\frac{\omega R^2C}{1+\omega^2(RC)^2}\right)
\end{aligned}
$$

And so the real (**resistive**) component is **indeed** a function of frequency ω:

$$R(\omega) = \frac{R}{1+\omega^2(RC)^2} \quad !!!$$