HW2

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Problem 1

Let $S = \left\{ \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}7\\8\\9\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\}$. Show that $\text{Span}(S)$ has dimension two.

$$\begin{bmatrix}1&4&7&1\\2&5&8&1\\3&6&9&1\end{bmatrix} \to \begin{bmatrix}1&4&7&1\\0&-3&-6&-1\\0&-6&-12&-2\end{bmatrix} \to \begin{bmatrix}1&4&7&1\\0&1&2&1/3\\0&6&12&2\end{bmatrix} \to \begin{bmatrix}1&0&-1&-1/3\\0&1&2&1/3\\0&0&0&0\end{bmatrix}$$

Simplifying these vectors by row reduction shows that they are linearly dependent since the third vector can be expressed as a linear combination of the other two, therefore the dimension of this set cannot be $\mathbb{R}^3$ An example of a basis with span S is the first two vectors:

$$\left\{ \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix} \right\}$$

Problem 2

Let $H \subset \mathbb{R}^4$ be the set of solutions to the equation $x_1 + 2x_2 + 3x_3 + 5x_4 = 0$. Show that H is a subspace of $\mathbb{R}^4$ and find a basis for H.

To prove that H is a subspace of $\mathbb{R}^4$, 3 things need to be shown:

1. The set is closed under addition:

$$\begin{aligned} \vec{u}, \vec{v} \in H &\implies \vec{u} + \vec{v} \in H \\ (u_1 + v_1) + 2(u_2 + v_2) + 3(u_3 + v_3) + 5(u_4 + v_4) &= 0 \\ \implies (u_1 + 2u_2 + 3u_3 + 5u_4) + (v_1 + 2v_2 + 3v_3 + 5v_4) &= 0 + 0 = 0 \end{aligned}$$

2. The set is closed under scalar multiplication:

$$\begin{aligned} \vec{v} \in V &\implies \alpha\vec{v} \in V \,\forall\, \alpha \\ \alpha(v_1 + 2v_2 + 3v_3 + 5v_4) &= c \cdot 0 = 0 \end{aligned}$$

3. The set includes zero:

$$0 + 2(0) + 3(0) + 5(0) = 0$$

Problem 3

Let U, V be subspaces of $\mathbb{R}^n$. Let W be the collection of all sums $\vec{u} + \vec{v}$ with $\vec{u} \in U, \vec{v} \in V$. Prove that W is also a subspace.

1. Since u_1 is in U and v_1 is in V, it follows that $u_1 + v_1$ is in W. In class, we discussed that the same would be true for u_2 and v_2. This means that by induction $\vec{u}$ and $\vec{v}$ are in W.

2. Let $\vec{w} = \vec{u} + \vec{v}$, where $\vec{w}$ is in W. To show that the set is closed under scalar multiplication, the following must be true:

   $$\alpha\vec{w} = \alpha(\vec{u} + \vec{v})$$

   Because $\vec{u}$ is in U and $\vec{v}$ is in V, by the definition of a subspace $\vec{u}$ and $\vec{v}$ are independently closed under scalar multiplication. This does not change under addition, so the set must be closed under scalar multiplcation.

3. Finally, the set must include the zero vector, which is trivial since U and V are subspaces, so we know they must contain the zero vector.