Homework 11

Grant Saggars

November 12, 2023

9. Find the general solution of $x' = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} x$:

Solution:

$$\begin{aligned} 0 &= \begin{vmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} \\ &= (1-\lambda)((2-\lambda)(-1-\lambda)-(-1)(1)) + (3)(-1-\lambda) + 2 + 4(3-(2-\lambda)(2)) \\ &= (-\lambda+1)(\lambda+2)(\lambda-3) \implies \lambda = \{1, -2, 3\} \end{aligned}$$

Therefore: $x' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix} x$, and it follows that the solutions to the differential equation are:

$$\begin{matrix} x_1 = c_1 e^{t} \\ x_2 = c_2 e^{-2t} \\ x_3 = c_3 e^{3t} \end{matrix} = \begin{bmatrix} c_1 e^{t} & 0 & 0 \\ 0 & c_2 e^{-2t} & 0 \\ 0 & 0 & c_3 e^{3t} \end{bmatrix}$$

11. Find the general solution of $x' = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} x$:

Solution:

$$\begin{aligned} 0 &= \begin{vmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{vmatrix} \\ &= (5-\lambda)(1-\lambda) + 3 = (\lambda-2)(\lambda-4) \\ \implies& \lambda = \{2, 4\} \end{aligned}$$

Therefore: $x' = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} x$, and it follows that the solutions to the differential equation are:

$$\begin{matrix} x_1 = c_1 e^{2t} \\ x_2 = c_2 e^{4t} \end{matrix} = \begin{bmatrix} c_1 e^{2t} & 0 \\ 0 & c_2 e^{4t} \end{bmatrix}$$

Now, in order to fulfill the boundary conditions:

$$\left[\begin{array}{cc|c} e^0 & 0 & 2 \\ 0 & e^0 & -1 \end{array}\right] \implies \begin{matrix} c_1 = 2 \\ c_2 = -1 \end{matrix}$$

Solution:

$$0 = \begin{vmatrix} 1-\lambda & -5 \\ 1 & -3-\lambda \end{vmatrix} = (1-\lambda)(-3-\lambda) + 5 = 0 \implies \lambda = \{-1+i, -1-i\}$$