PHSX 621: Homework #6

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Problem 1

What is the Poynting vector $\mathbf{S}(R, \theta)$ in a spherical coordinate frame whose center is at the current time (t) location of a charge moving at constant velocity. How does this vary with R? What is $\mathbf{S}$ for $\theta = 0$ and $\theta = 90°$? See pages 461–462 of the text.

Solution:
A moving charge in polar coordinates has fields

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1 - v^2/c^2}{(1 - v^2 \sin^2\theta/c^2)^{\frac{3}{2}}} \frac{\hat{\mathbf{R}}}{R^2} \tag{10.75}$$

$$\mathbf{B} = \frac{1}{c^2}(\mathbf{v} \times \mathbf{E}) \tag{10.76}$$

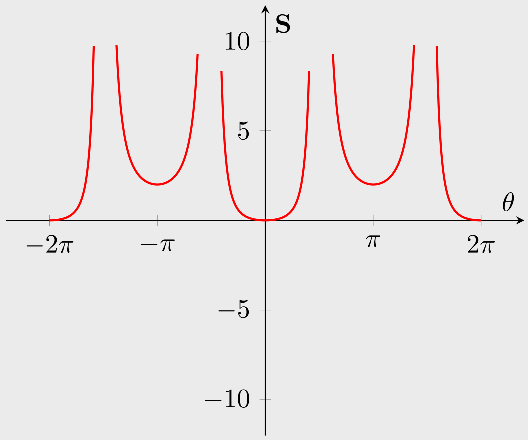
The cross product between the two is needed:

$$\begin{aligned}
\mathbf{E} \times \mathbf{B} &= \frac{1}{c^2}\mathbf{E} \times \mathbf{v} \times \mathbf{E} \\
&= \frac{1}{c^2}\left[\mathbf{v}(\mathbf{E} \cdot \mathbf{E}) - \mathbf{E}(\mathbf{v} \cdot \mathbf{E})\right] && \text{(Triple Product Identity)} \\
&= \frac{E^2}{c^2}\left[\mathbf{v} - \mathbf{R}v\cos\theta\right]
\end{aligned}$$

Therefore,

$$\mathbf{S}(\mathbf{R}, \theta) = \frac{E^2}{\mu_0 c^2}\left[\mathbf{v} - \mathbf{R}v\cos\theta\right] = \left(\frac{q}{4\pi\epsilon_0 R^2} \frac{1 - v^2/c^2}{(1 - v^2\sin^2\theta/c^2)^{3/2}}\right)^2 \frac{1}{\mu_0 c^2}\left[\mathbf{v} - \mathbf{R}v\cos\theta\right]$$

In this case, $\mathbf{S} \propto \frac{1}{R^4}$. The plot of the function's angular dependence goes as $\sec(\theta)$, and therefore has minima at $\{0, 180°, 360°, \dots\}$. Maxima occur at $\{0, 90°, 270°, \dots\}$.Specifically, at $0°$ and $90°$, $\mathbf{S} = 0$ and $\mathbf{S} = \infty$, respectively.



Problem 2

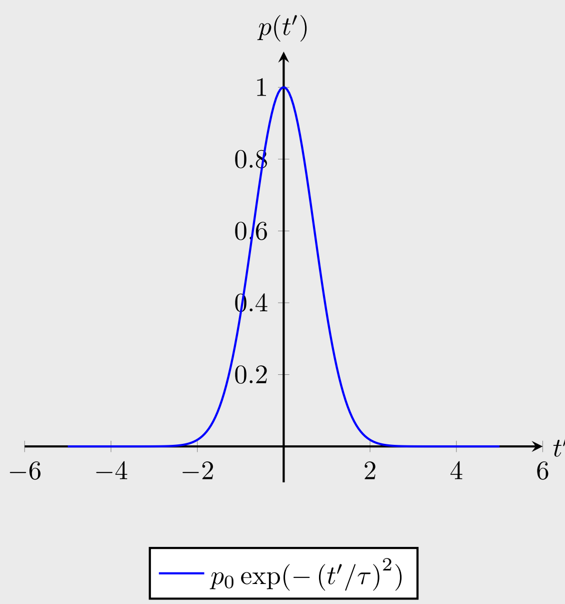
A small dipole has a dipole moment as a function of time:

$$\mathbf{p}(t') = p_0 \exp\left(-\left(\frac{t'}{\tau}\right)^2\right)\hat{\mathbf{z}}$$

where τ is a time constant.

(a) Sketch the function $\mathbf{p}(t')$ versus t'/τ. It is a Gaussian centered at $t' = 0$.

Solution:



(b) What is the radiation electric field at a location $\mathbf{r}$ (far from the dipole) and as a function of time?

Solution:

Taking $\mathbf{E}$ and $\mathbf{B}$ and then the second derivative of the dipole moment,

$$\ddot{\mathbf{p}} = \frac{\partial^2 \mathbf{p}}{\partial t'^2} = \frac{\partial \mathbf{p}}{\partial t}\left(-\frac{2t'}{\tau^2}\mathbf{p}(t')\right) = p_0\left[\left(\frac{t'}{\tau}\right)^2 - 2\right]\cdot\left(\frac{1}{\tau}\right)^2 e^{-t'^2/\tau^2}\hat{\mathbf{z}}$$

$$\hat{\mathbf{r}} \times \ddot{\mathbf{p}} = \frac{4t'^2 - 2\tau^2}{\tau^4} p_0 \exp\left(-\left(\frac{t'}{\tau}\right)^2\right)\hat{\boldsymbol{\phi}}$$

$$\hat{\mathbf{r}} \times \hat{\mathbf{r}} \times \ddot{\mathbf{p}} = \frac{4t'^2 - 2\tau^2}{\tau^4} p_0 \exp\left(-\left(\frac{t'}{\tau}\right)^2\right)\hat{\mathbf{z}}$$

Therefore, we will have fields:

$$\mathbf{E}(\mathbf{r}, t_0) \approx \frac{\mu_0}{4\pi r}[\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \ddot{\mathbf{p}})] = \frac{\mu_0 p_0}{4\pi r}\left[\left(\frac{t'}{\tau}\right)^2 - 2\right]\cdot\left(\frac{1}{\tau}\right)^2 e^{-t'^2/\tau^2}\hat{\mathbf{z}}$$

$$\mathbf{B}(\mathbf{r}, t_0) \approx -\frac{\mu_0}{4\pi rc}[\hat{\mathbf{r}} \times \ddot{\mathbf{p}}] = -\frac{\mu_0 p_0}{4\pi rc}\left[\left(\frac{t'}{\tau}\right)^2 - 2\right]\cdot\left(\frac{1}{\tau}\right)^2 e^{-t'^2/\tau^2}\hat{\phi}$$

$$\mathbf{S}(\mathbf{r}, t_0) \approx \frac{\mu_0}{16\pi^2}[\ddot{\mathbf{p}}(t_0)]^2\frac{\sin^2\theta}{r^2}\hat{r}$$

(c) What is the power detected at $\mathbf{r}$ and as a function of time? Use spherical coordinates.

Solution:

$$P(r,t) = \oint \mathbf{S}(\mathbf{r}, t)\, d\mathbf{a} = \frac{\mu_0}{6\pi c}\left(\ddot{p}\left(t - \frac{r}{c}\right)\right)^2$$

Substituting in our $\ddot{p}$,

$$\boxed{P = \frac{\mu_0}{6\pi c}\left(p_0\left[\left(\frac{t_0}{\tau}\right)^2 - 2\right]\cdot\left(\frac{1}{\tau}\right)^2 e^{-t_0^2/\tau^2}\hat{\mathbf{z}}\right)^2}$$

where $t_0 = t - \frac{r}{c}$.

(d) *Qualitatively*, what would the Fourier transform of the electric field at $\mathbf{r}$ look like versus frequency? If you were designing a detector/receiver what is the rough frequency you would design it for?

Solution:

The characteristic frequency is given by the fourier transform, and is roughly $1/\tau$, so you would need electronics that can handle signals that oscillate at that frequency. If we had a nanosecond pulse, we'd be working with a gigahertz.