MATH 526: Homework #4

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Problem 4

A coin is biased such that a head is three times as likely to occur as a tail. Find the expected number of tails when this coin is tossed twice.

Solution:
Let $3P(H) = P(T)$, and $P(H) + P(T) = 1$. We can solve for the probability of each event:

$$\left(\begin{array}{cc|c} 3 & -1 & 0 \\ 1 & 1 & 1 \end{array}\right) \implies P(H) = 3/4,\ P(T) = 1/4$$

The expectation asked for in this problem is given by the linear rule for expectation values: $E(X) = E_1(X) + E_2(X)$.

$$\begin{aligned} E_1(X) &= 1 \times \frac{1}{4} + 0 \times \frac{3}{4} = 1 \implies E_1(X) = \frac{1}{4} \\ E_2(X) &= 1 \times \frac{1}{4} + 0 \times \frac{3}{4} = 1 \implies E_2(X) = \frac{1}{4} \\ E(X) &= \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \end{aligned}$$

Problem 10

Two tire-quality experts examine stacks of tires and assign a quality rating to each tire on a 3-point scale. Let X denote the rating given by expert A and Y denote the rating given by B. The following table gives the joint distribution for X and Y:

	$y = 1$	$y = 2$	$y = 3$
$x = 1$	0.10	0.05	0.02
$x = 2$	0.10	0.35	0.05
$x = 3$	0.03	0.10	0.20

Find μ_X and μ_Y.

Solution:

(I) Marginal probabilities (the sum of each element in each desired row/column):

$$P_x(1) = (0.10) + (0.05) + (0.02) = 0.17$$
$$P_x(2) = (0.10) + (0.35) + (0.05) = 0.50$$
$$P_x(3) = (0.03) + (0.10) + (0.20) = 0.33$$

Similarly,

$$P_y(1) = (0.10) + (0.10) + (0.03) = 0.23$$
$$P_y(2) = (0.05) + (0.35) + (0.10) = 0.50$$
$$P_y(3) = (0.02) + (0.05) + (0.20) = 0.27$$

(II) The expectation value for a discrete PDF is $\sum x_n P(x_n)$:

$$\mu_x = 1(0.17) + 2(0.50) + 3(0.33) = 2.16$$
$$\mu_y = 1(0.23) + 2(0.50) + 3(0.27) = 2.04$$

Problem 12

If a dealer's profit, in units of \$5000, on a new automobile can be looked upon as a random variable X having the density function $f(x) = 2(1-x)$, $0 < x < 1$, 0, elsewhere, find the average profit per automobile.

Solution:
The mean for a continuous PDF is $\int_{-\infty}^{\infty} xP(x)\, dx$

$$\begin{aligned} \mu_x &= \int_0^1 2x - 2x^2 \, dx \\ &= \frac{1}{3} \end{aligned}$$

Putting this in terms of dollars gives \$1667.

Problem 20

A continuous random variable X has the density function

$$f(x) = \begin{cases} e^{-x}, & \text{if } x > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the expected value of $g(X) = e^{\frac{2X}{3}}$

Problem 20 continued on next page. . .

Solution:

$$E(g(X)) = \int_{-\infty}^{\infty} g(X)f(x)\,dx = \int_{0}^{\infty} \exp\left(-\frac{x}{3}\right)\,dx = 3$$

Problem 34

Let X be a random variable with the following probability distribution:

x	-2	3	5
$f(x)$	0.3	0.2	0.5

Find the standard deviation of X.

Solution:

$$\begin{aligned}\sigma^2 &= \sum P(x)(x-\overline{x})^2 = 0.3(-2-2)^2 + 0.2(3-2)^2 + 0.5(5-2)^2 \\ &= 1.127 \\ \sigma &= 1.062\end{aligned}$$

Problem 36

Suppose that the probabilities are 0.4, 0.3, 0.2, and 0.1, respectively, that 0, 1, 2, or 3 power failures will strike a certain subdivision in any given year. Find the mean and variance of the random variable X representing the number of power failures striking this subdivision.

Solution:

$$\begin{aligned}\overline{x} &= \frac{1}{4}(0+1+2+3) = 1.5 \\ \sigma^2 &= \sum f(x)(x-\overline{x})^2 = 0.4(0-1.5)^2 + 0.3(1-1.5)^2 + 0.2(2-1.5)^2 + 0.1(3-1.5)^2 \\ &= 1.25 \\ \sigma &= 1.12\end{aligned}$$

Problem 39

The total number of hours, in units of 100 hours, that a family runs a vacuum cleaner over a period of one year is a random variable X having the density function given in Exercise 4.13 on page 117. Find the variance of X.

Problem 39 continued on next page...

Solution:

$$f(x) = \begin{cases} x, & 0 < x < 1, \\ 2 - x, & 1 \leq x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

$$\begin{aligned} \langle x \rangle &= \int_0^1 x^2 \; dx + \int_1^2 2x - x^2 \; dx = 1 \\ \langle x^2 \rangle &= \int_0^1 x^3 \; dx + \int_1^2 2x^2 - x^3 \; dx = \frac{7}{6} \\ \sigma^2 &= \langle x^2 \rangle - \langle x \rangle^2 = \frac{7}{6} - 1^2 \approx 0.167 \end{aligned}$$

Problem 50

For a laboratory assignment, if the equipment is working, the density function of the observed outcome X is $f(x) = 2(1 - x)$, $0 < x < 1$, 0 otherwise. Find the variance and standard deviation of X.

Solution:

$$\begin{aligned} \langle x \rangle &= \int_0^1 2x(1 - x) \; dx = \frac{1}{3} \\ \langle x^2 \rangle &= \int_0^1 2x^2(1 - x) \; dx = \frac{1}{6} \\ \sigma^2 &= \frac{1}{6} - \frac{1}{9} \approx 0.0556 \\ \sigma &= 0.236 \end{aligned}$$

Problem 52

Random variables X and Y follow a joint distribution $f(x, y) = 2$, $0 < x \leq y < 1$, 0 otherwise. Determine the correlation coefficient between X and Y.

Problem 52 continued on next page. . .

Solution:

(I) Calculating Covariance:

$$\text{Cov}(X, Y) = \langle XY \rangle - \langle X \rangle \langle Y \rangle$$

$$\langle XY \rangle = \int_0^1 \int_x^1 2xy \; dy \; dx = \frac{1}{4}$$
$$\langle X \rangle = \int_0^1 \int_x^1 2x \; dy \; dx = \frac{1}{3}$$
$$\langle Y \rangle = \int_0^1 \int_x^1 2y \; dy \; dx = \frac{2}{3}$$
$$\implies \text{Cov}(X, Y) = \frac{1}{4} - \frac{2}{9} = \frac{1}{36}$$

(II) Calculating Correlation:

$$\rho = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var(X)Var(Y)}}}$$

$$\langle X^2 \rangle = \int_0^1 \int_x^1 2x^2 \; dy \; dx = \frac{1}{6}$$
$$\langle Y^2 \rangle = \int_0^1 \int_x^1 2y^2 \; dy \; dx = \frac{1}{2}$$

$$\rho = \frac{\frac{1}{36}}{\sqrt{(1/6 - 1/9)(1/2 - 4/9)}} = \frac{1}{2}$$

Problem 57

Let X be a random variable with the following probability distribution:

x	-3	6	9
$f(x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$

Find $E(X)$ and $E(X^2)$ and then, using these values, evaluate $E[(2X + 1)^2]$.

Solution:

$$E(X) = -3\frac{1}{6} + 6\frac{1}{2} + 9\frac{1}{3} = \frac{24}{3}$$
$$E(X^2) = 9\frac{1}{6} + 36\frac{1}{2} + 81\frac{1}{3} = 127$$
$$E[(2X + 1)^2] = E(4X^2 + 4X + 1) = 4(127) + 4\left(\frac{24}{3}\right) + 1 = 541$$

Problem 62

If X and Y are independent random variables with variances $\sigma_X^2 = 5$ and $\sigma_Y^2 = 3$, find the variance of the random variable $Z = -2X + 4Y - 3$.

Solution:
Variance for independent variables has linear properties:

$$\text{Var}(-2X + 4Y - 3) = -2^2\text{Var}(X) + 4^2\text{Var}(y) + \text{Var(-3)}$$

Which works out to:

$$4(5) + 16(3) + 9(0) = 68$$