The Characteristic Impedance of a Transmission Line

Q: *Is the complex* ***current wave amplitude*** $I_0^+(\omega)$ *in any way* ***related*** *to the complex* ***voltage wave amplitude*** $V_0^+(\omega)$*??*

A: Let's see what the **telegrapher's equations** have to say about it!

Since the two **wave functions** are:

$$V^+(z,\omega) = V_0^+(\omega)\, e^{-j\beta z} \qquad\qquad I^+(z,\omega) = I_0^+(\omega)\, e^{-j\beta z}$$

we know that these functions must satisfy **both** telegrapher's equations.

Some interesting math…

Recall the **first** telegrapher's equation is:

$$\frac{d\,V^{+}(z,\omega)}{dz} = -j\omega L\, I^{+}(z,\omega)$$

So, we take the first **derivative** of $V^{+}(z)$:

$$\frac{d\,V^{+}(z,\omega)}{dz} = V_0^{+}(\omega)\frac{d\,e^{-j\beta z}}{dz} = -j\beta V_0^{+}(\omega)\,e^{-j\beta z} = -j\beta V^{+}(z,\omega)$$

and **inserting** this back into the first of the telegrapher's:

$$-j\beta\, V^{+}(z,\omega) = -j\omega L\, I^{+}(z,\omega)$$

from which we conclude:

$$\frac{V^{+}(z,\omega)}{I^{+}(z,\omega)} = \frac{\omega L}{\beta}$$

...leads to an interesting result!

Moreover, we can simplify this **further** by recalling the definition of β (for a lossless line):

$$\beta = \omega\sqrt{LC}$$

Therefore:

$$\frac{\omega L}{\beta} = \frac{\omega L}{\omega\sqrt{LC}} = \sqrt{\frac{L}{C}}$$

And so, we conclude:

$$\boxed{\frac{V^{+}(z,\omega)}{I^{+}(z,\omega)} = \sqrt{\frac{L}{C}}}$$

Q: *Boring!* ***So what****; who the heck cares?*

A: **Look** at this result—**think** about what this result means!

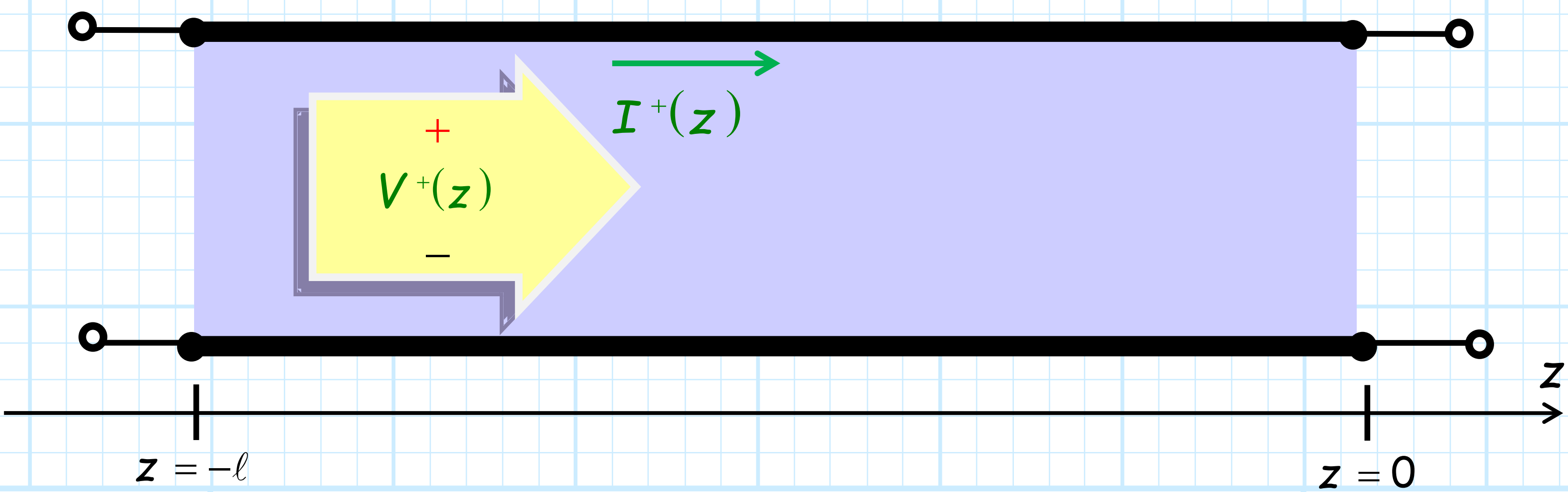
What this result means

Although each propagating plus-wave (i.e., $V^+(z,\omega)$ and $I^+(z,\omega)$), **is a** function of transmission line **position *z*,** the **ratio** of the voltage and current of **each** plus-wave is **independent** of position.

→A **constant** with respect to position z !!!!

Q: *Yikes! How can **that** be possible?*

A: The plus-wave current $I^+(z,\omega)$ and the plus-wave voltage $V^+(z,\omega)$ are manifestations of the **same** single physical phenomenon—an electromagnetic wave propagating in **direction of increasing** z !

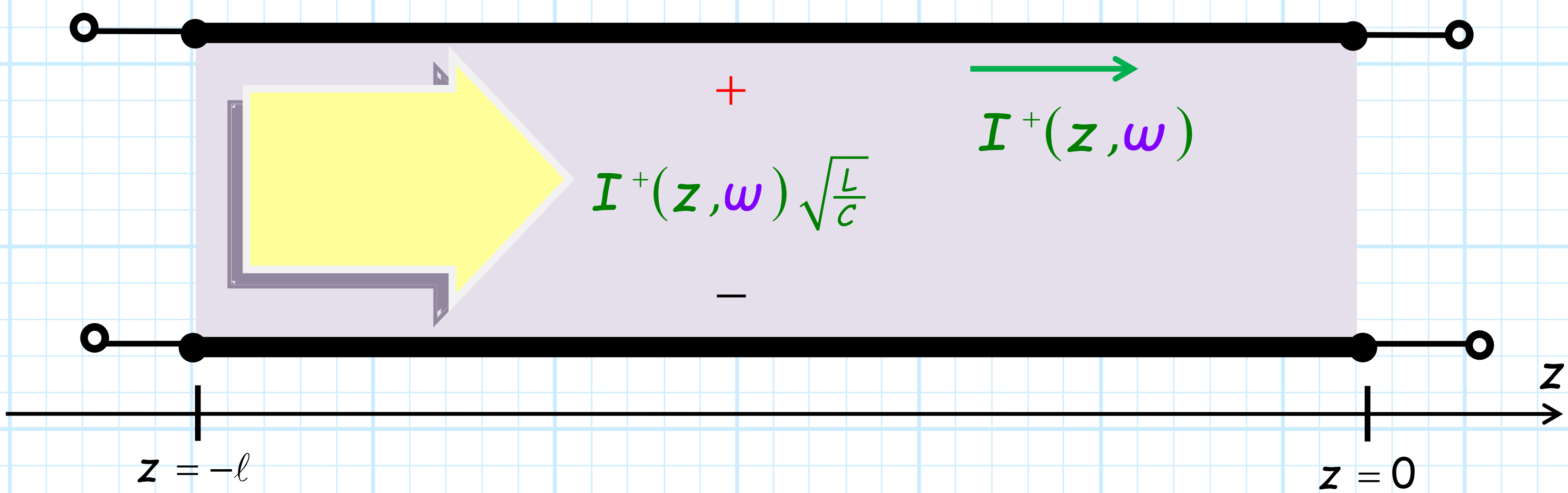


Different manifestations of the same thing

This **single** electromagnetic wave creates both the **current** plus-wave $I^+(z,\omega)$ and the **voltage** plus-wave $V^+(z,\omega)$.

→ It probably should be no surprise that these two functions are **simply** and **directly** related!

$$V^+(z,\omega) = \sqrt{\frac{L}{C}}\; I^+(z,\omega)$$



Now for the minus-wave

Q: *What about the **minus**-wave ?*

A: Inserting the **minus**-wave into the **first** telegrapher's equation, we find:

$$+j\beta\, V^{-}(z,\omega) = -j\omega L\, I^{-}(z,\omega)$$

And from this we can conclude:

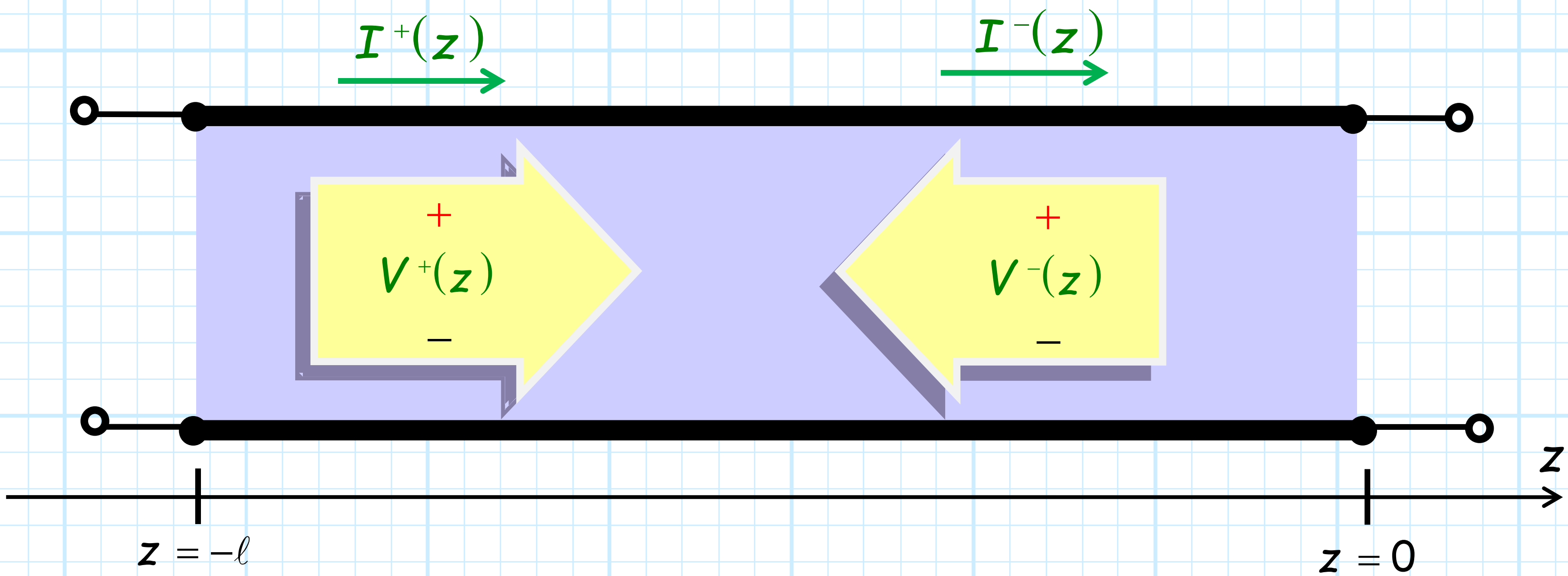
$$\frac{V^{-}(z,\omega)}{I^{-}(z,\omega)} = -\sqrt{\frac{L}{C}} \qquad \text{or} \qquad V^{-}(z,\omega) = -\sqrt{\frac{L}{C}}\; I^{-}(z,\omega)$$

The **same** result as for the plus-wave—only with a **minus sign**!

Remember: $-1 = \exp[j\pi]$

Q: ***Why*** *the minus sign?*

A: The minus-wave propagates in the **opposite** direction of the plus-wave.



For a **given** plus-wave or minus-wave **voltage**, the corresponding current will be the same magnitude—but will flow in **opposite** directions (e.g., a π-radian phase shift).

Characteristic Impedance

Q: *Gee, the value $\sqrt{L/C}$ seems to show up a lot; what's its* ***significance****?*

A: The value $\sqrt{L/C}$ is a **fundamental** parameter of a lossless transmission line—it is called the **characteristic impedance** Z_0 of the lossless line.

$$\boxed{Z_0(\omega) \doteq \sqrt{\frac{L}{C}}}$$

Note the characteristic impedance depends **only** on the **transmission line parameters** L (inductance/unit length) and C (capacitance/unit length).

Thus, characteristic impedance is likewise a **transmission line parameter**—Z_0 is independent and unaffected by the things **attached** to the ends of the transmission line!!!

That's a lot of exclamation points!

Some things to **note** about characteristic impedance $Z_0 = \sqrt{L/C}$ of a **lossless** transmission line:

1. It has units of impedance; i.e., **Ohms.**

2. It is a purely **real** value.

3. It is **independent** of frequency ω.

Q: *A **real**-valued impedance that's independent of frequency—that sounds like a **resistor**!*

*May I conclude that $Z_0 = \sqrt{L/C}$ somehow characterizes the **resistance** of the transmission line?*

A: NO!!
Remember, the characteristic impedance $Z_0 = \sqrt{L/C}$ is a characteristic of a **lossless** transmission line!!!!

A lossless transmission line is a purely reactive device!!!!!!!!

For a **lossless** line, both resistance R and conductance G to **zero**:

$$R = G = 0$$

And, just **look** at the expression for characteristic impedance:

$$Z_0 \doteq \sqrt{\frac{L}{C}}$$

Characteristic impedance depends on **inductance** L and **capacitance** C only—**no** resistance R **nor** conductance G is anywhere to be found!

Two different real-valued characteristics

The **two** parameters β and Z_0 completely **characterize** a lossless **transmission line**.

Q: *But wait! I thought **L** and **C** characterized a transmission line???*

A: They **do**!

The values β and Z_0 are simply an **alternate way** of expressing L and C:

$$\beta = \omega\sqrt{LC} \qquad \text{and} \qquad Z_0 = \sqrt{\frac{L}{C}}$$

or,

$$\omega L = \beta Z_0 \qquad \text{and} \qquad \omega C = \frac{\beta}{Z_0}$$

Note that β is a **function** of time-harmonic **frequency** ω , but characteristic impedance $Z_0 = \sqrt{L/C}$ **independent** of frequency !

The wave amplitudes are simply related

Now we can state that the **ratio** of the plus-wave **voltage** to plus-wave **current** is equal to **characteristic impedance** Z_0 at each and **every** point on the transmission line:

$$\frac{V^+(z,\omega)}{I^+(z,\omega)} = Z_0$$

Of course, the location $z=0$ **is** a point on the transmission line, and so we can conclude:

$$\frac{V^+(z=0,\omega)}{I^+(z=0,\omega)} = \frac{V_0^+(\omega)}{I_0^+(\omega)} = Z_0$$

Likewise for the **minus-wave**:

$$\frac{V^-(z=0,\omega)}{I^-(z=0,\omega)} = \frac{V_0^-(\omega)}{I_0^-(\omega)} = -Z_0$$

Now we need just two values!!!!!

Thus, the **complex wave amplitudes** are simply **related** as:

$$I_0^+(\omega) = \frac{V_0^+(\omega)}{Z_0} \qquad \text{and} \qquad I_0^-(\omega) = -\frac{V_0^-(\omega)}{Z_0}$$

We can therefore **alternatively describe** the total current and total voltage along a transmission line with just $V_0^+(\omega)$ and $V_0^-(\omega)$:

$$V(z,\omega) = V_0^+(\omega)\, e^{-j\beta z} + V_0^-(\omega)\, e^{+j\beta z} = V^+(z,\omega) + V^-(z,\omega)$$

$$I(z,\omega) = \frac{V_0^+(\omega)}{Z_0}\, e^{-j\beta z} - \frac{V_0^-(\omega)}{Z_0}\, e^{+j\beta z} = \frac{V^+(z,\omega)}{Z_0} - \frac{V^-(z,\omega)}{Z_0}$$

Any two wave amplitudes will work

Or **equivalently**, we describe total current and voltage in terms of the **current** wave amplitudes $I_0^+(\omega)$ and $I_0^-(\omega)$:

$$V(z,\omega) = Z_0\, I_0^+(\omega)\, e^{-j\beta z} - Z_0\, I_0^-(\omega)\, e^{+j\beta z}$$

$$I(z,\omega) = I_0^+(\omega)\, e^{-j\beta z} + I_0^-(\omega)\, e^{+j\beta z}$$

→ Note that **instead** of characterizing a transmission line with **real** parameters L **and** C, we can (and typically do!) describe a **lossless** transmission line using real parameters Z_0 **and** β.