PHSX 711: Homework #7

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Problem 1

Exercise 11.2.2 (Shankar) – 5 points Using $T^\dagger(\epsilon)T(\epsilon) = I$ to order ϵ, deduce that $G^\dagger = G$.

Solution:

$$\begin{aligned}
\hat{T}\hat{T}^\dagger = \left(\hat{I} - \frac{i\epsilon}{\hbar}\hat{G}\right)\left(I - \frac{i\epsilon}{\hbar}\hat{G}\right)^\dagger &= \hat{I} && \left(\hat{T} = \hat{I} - \frac{i\epsilon}{\hbar}\hat{G}\right) \\
\left(\hat{I} - \frac{i\epsilon}{\hbar}\hat{G}\right)\left(I + \frac{i\epsilon}{\hbar}\hat{G}^\dagger\right) &= \hat{I} \\
\left(\hat{I} - \frac{i\epsilon}{\hbar}\hat{G}\right)\hat{I}\left(I + \frac{i\epsilon}{\hbar}\hat{G}^\dagger\right) &= \hat{I} && \text{(introduce identity to expand expression)} \\
\cancel{\hat{I}} + \frac{i\epsilon}{\hbar}\left(\hat{G}\hat{I} - \hat{I}\hat{G}\right) &= \cancel{\hat{I}} \\
\frac{i\epsilon}{\hbar}\left(\hat{G}\hat{I} - \hat{G}^\dagger\hat{I}\right) &= 0 \\
\frac{i\epsilon}{\hbar}\left(\hat{G} - \hat{G}^\dagger\right) &= 0
\end{aligned}$$

Therefore the term $\hat{G} - \hat{G}^\dagger$ must equal zero.

Problem 2

Exercise 11.4.1 (Shankar) – 5 points Prove that if $[\Pi, H] = 0$, a system that starts out in a state of even/odd parity maintains its parity. (Note that since parity is a discrete operation, it has no associated conservation law in classical mechanics.)

Solution:

$$
\begin{aligned}
[\Pi, H] = 0 &\implies H(x,p) = H(-x,-p) \\
\Pi\,|\psi(x)\rangle &= |\psi(-x)\rangle && \text{(where } |\psi(x)\rangle = \pm\,|\psi(x)\rangle\text{)} \\
\Pi\,|\psi(x)\rangle &= \pm\,|\psi(x)\rangle
\end{aligned}
$$

$$
\begin{aligned}
\Pi\,|\psi(x,\ t)\rangle &= \Pi e^{iHt/\hbar}\,|\psi(x)\rangle \\
&= e^{iH(\pm x,\pm t)t/\hbar}\,|\psi(\pm x)\rangle && (H(x,p) = H(-x,-p)) \\
&= e^{iHt/\hbar} \pm |\psi(x)\rangle \\
&= \pm\,|\psi(x,t)\rangle
\end{aligned}
$$

Problem 3

Exercise 11.4.2 (Shankar) – 5 points A particle is in a potential

$$V(x) = V_0 \sin\left(\frac{2\pi x}{a}\right)$$

which is invariant under the translations $x \to x + ma$, where m is an integer. Is momentum conserved? Why not?

Hint: You can start this problem with:

$$\frac{d}{dt}\langle\hat{p}\rangle = -\frac{i}{\hbar}\langle[\hat{p}, \hat{H}]\rangle$$

Whether it goes to zero or not depends on the range you used for averaging $\langle \dots \rangle$.

Solution: Shankar tells directly tells us that via Ehrenfest's theorem,

$$\langle[\hat{p}, \hat{H}]\rangle = 0 \implies \dot{p} = 0 \tag{11.2.16}$$

Expanding the commutator, we have in general that

$$
\begin{aligned}
[\widehat{H}, \hat{p}] = [\widehat{T} + \widehat{V}, \hat{p}] &= \left[\frac{\hat{p}^2}{2m}, \hat{p}\right] + [\widehat{V}, \hat{p}] \\
&= \frac{1}{2m}(\hat{p}[\hat{p}, \hat{p}] - [\hat{p}, \hat{p}]\hat{p}) + [\widehat{V}, \hat{p}] \\
&= 0 + V(-i\hbar)\nabla(f) - (-i\hbar)\nabla(Vf) \\
&= -i\hbar(V\nabla(f) - V\nabla(f) - f\nabla(V)) \\
&= i\hbar\nabla V
\end{aligned}
$$

Problem 3 continued on next page…

Where $\nabla V = \frac{2\pi V_0}{a} \cos\left(\frac{2\pi x}{a}\right)$. It follows that we will then have some expectation value for x. Assuming the particle is in a bound state, we would have the time derivative of momentum:

$$\frac{d}{dt}\langle\hat{p}\rangle = \left\langle \frac{2\pi V_0}{a} \cos\left(\frac{2\pi x}{a}\right)\right\rangle = \frac{2\pi V_0}{a} \cos\left(\frac{2\pi x}{a}\right) \neq 0$$

Therefore momentum is not conserved.