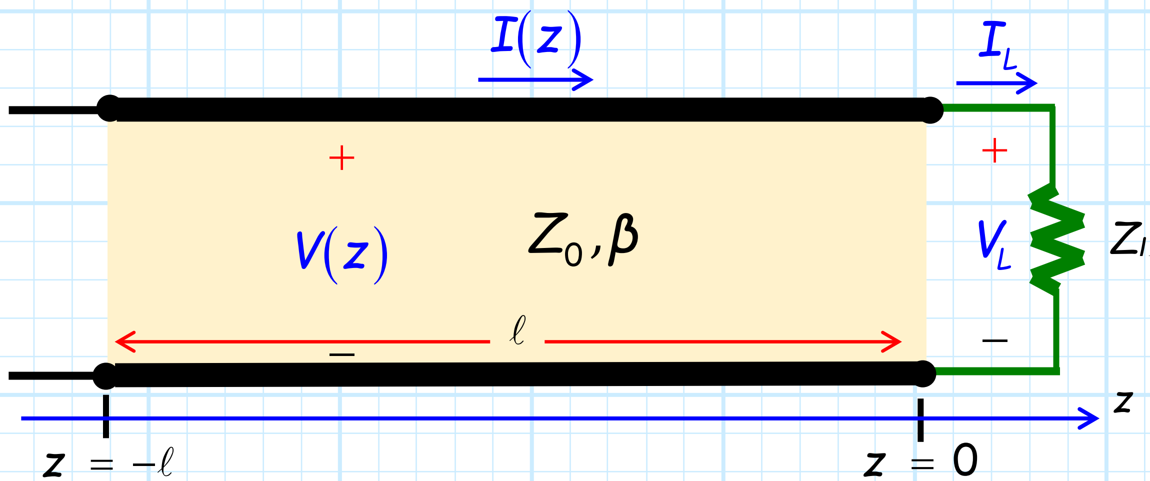
The Terminated, Lossless Transmission Line

Now let's **attach** something to our transmission line. Consider a **lossless** line, length ℓ, terminated with a **load** Z_L:



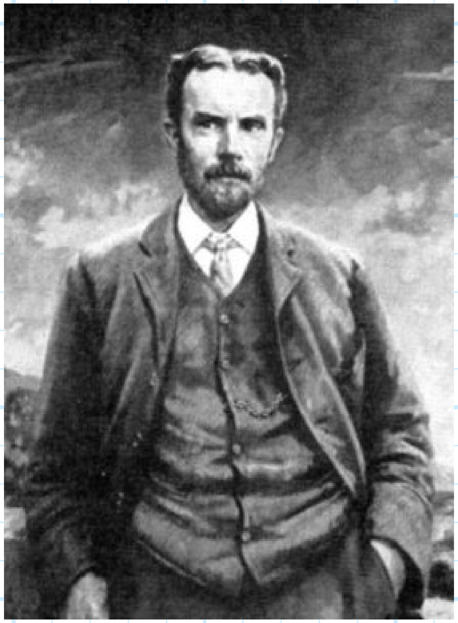
Q: *What is the **current** and **voltage** at each and **every** point on the transmission line (i.e., what is $I(z)$ and $V(z)$ for **all** points z where $-\ell \leq z \leq 0$?)?*

A: To find out, we must apply **boundary conditions**!

First, there's Heaviside

In other words, at the **end** of the transmission line ($z=0$)—where the load is **attached**—we have **many** requirements that **all** must be satisfied!

Requirement 1. To begin with, the voltage and current ($I(z=0)$ and $V(z=0)$) must be consistent with a valid **transmission line solution** (i.e., satisfy the **telegraphers equations**):



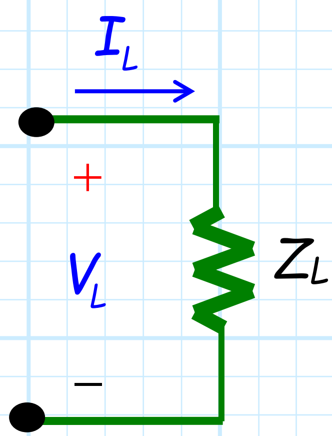
$$\begin{aligned} V(z=0) &= V^{+}(z=0) + V^{-}(z=0) \\ &= V_0^{+}\, e^{-j\beta(0)} + V_0^{-}\, e^{+j\beta(0)} \\ &= V_0^{+} + V_0^{-} \end{aligned}$$

$$\begin{aligned} I(z=0) &= \frac{V^{+}(z=0)}{Z_0} - \frac{V^{-}(z=0)}{Z_0} \\ &= \frac{V_0^{+}}{Z_0} e^{-j\beta(0)} - \frac{V_0^{-}}{Z_0} e^{+j\beta(0)} \\ &= \frac{V_0^{+} - V_0^{-}}{Z_0} \end{aligned}$$

Now for Ohm

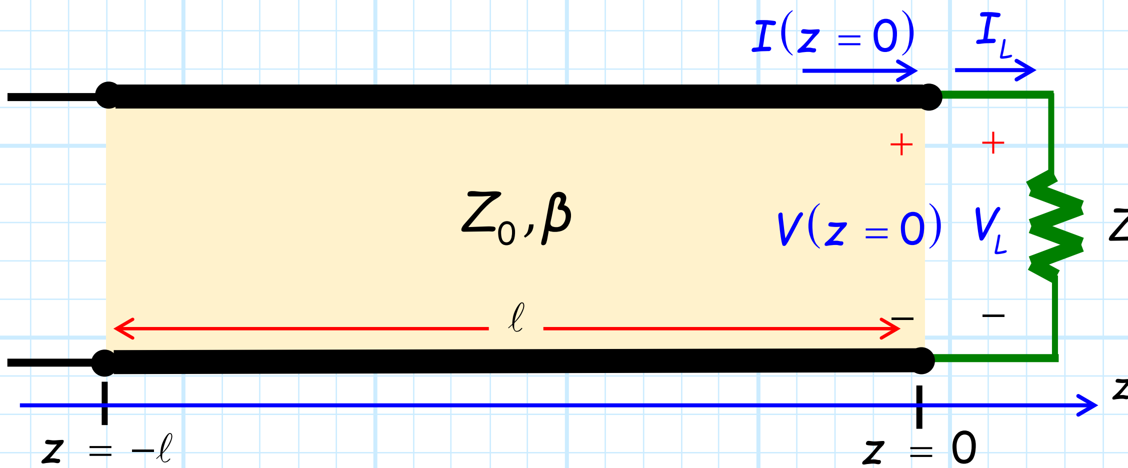
Requirement 2. Likewise, the load voltage and current must be related by the complex form of **Ohm's law**:

$$\frac{V_L}{I_L} = Z_L$$



Finally, we have Kirchoff

Requirement 3. Most importantly, we recognize that the values $I(z=0)$, $V(z=0)$ and I_L, V_L are **not** independent, but in fact are strictly related by **Kirchoff's Laws**!



From KVL and KCL we find these requirements:

$$V(z=0) = V_L \quad \text{and} \quad I(z=0) = I_L$$

→ These are our **boundary conditions!**

The result of some algebra

Combining these results, we find that the current and voltage at the **end** of the line must be related to the **load** impedance:

$$\frac{V(z=0)}{I(z=0)} = Z_L$$

Q: *Hey isn't the **ratio** of total line voltage $V(z)$ and total line current $I(z)$ the **line impedance** $Z(z)$??*

A: It sure is!

Both intuitive and satisfying

Thus, we know for $z=0$:

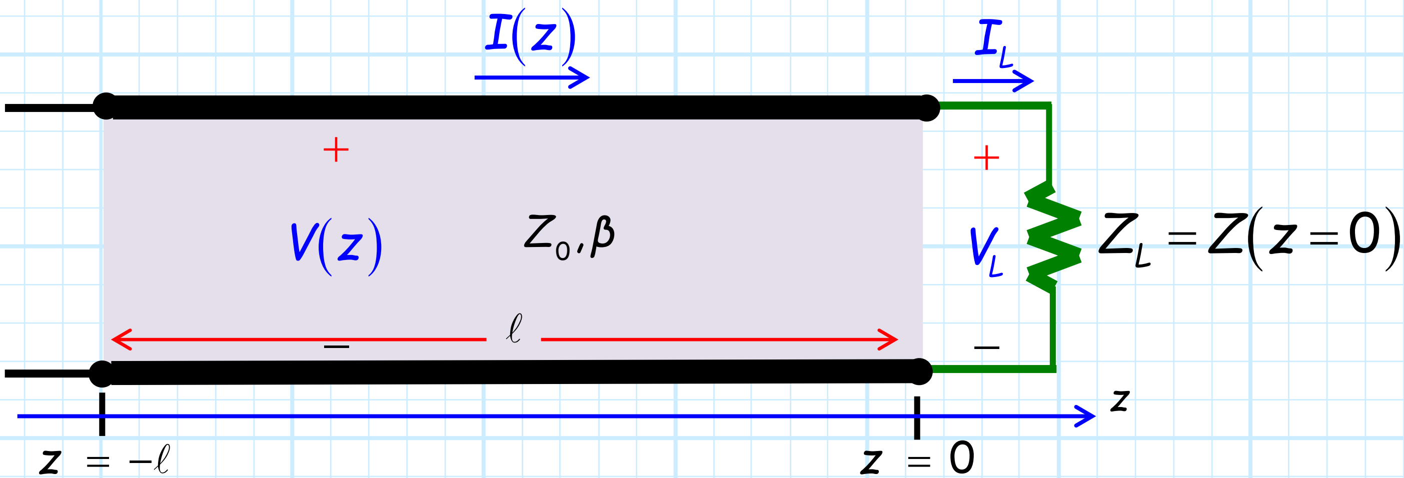
$$Z(z=0) \;=\; \frac{V(z=0)}{I(z=0)}$$

And now combining these last two expressions, we find a very **intuitive** and **satisfying** boundary condition:

$$\boxed{Z(z=0) \;=\; Z_L}$$

→ The **line impedance** at the end of the transmission line (i.e., at $z=0$) **must** be equal to the **load impedance** attached to that end!

More than just one point!



Q: *So what? Who **cares** about the line impedance at **just** the **one** point* $z=0$ *?*

A: It turns out that the **boundary condition** determines the values of line impedance **at all points** z !

Here's where we prove it

To **see** this, we first consider the expression for **line impedance**:

$$Z(z) = Z_0 \left[\frac{V_0^+ \, e^{-j\beta z} + V_0^- \, e^{+j\beta z}}{V_0^+ \, e^{-j\beta z} - V_0^- \, e^{+j\beta z}} \right] = Z_0 \left[\frac{e^{-j\beta z} + \Gamma_0 \, e^{+j\beta z}}{e^{-j\beta z} - \Gamma_0 \, e^{+j\beta z}} \right]$$

Evaluating this **at** $z = 0$:

$$Z(z=0) = Z_0 \left[\frac{e^{-j\beta(0)} + \Gamma_0 \, e^{+j\beta(0)}}{e^{-j\beta(0)} - \Gamma_0 \, e^{+j\beta(0)}} \right] = Z_0 \left[\frac{1 + \Gamma_0}{1 - \Gamma_0} \right]$$

And so, an **equivalent boundary condition** statement is

$$Z(z=0) = Z_0 \left[\frac{1 + \Gamma_0}{1 - \Gamma_0} \right] = Z_L$$

A stunning result!!!!

With some **algebraic** elbow grease we rearrange this last expression to find:

$$\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Now **inserting** this into the general expression for line impedance:

$$\begin{aligned} Z(z) &= Z_0 \left[\frac{e^{-j\beta z} + \Gamma_0 \, e^{+j\beta z}}{e^{-j\beta z} - \Gamma_0 \, e^{+j\beta z}} \right] \\ &= Z_0 \left[\frac{e^{-j\beta z} + \left(\frac{Z_L - Z_0}{Z_L + Z_0}\right) e^{+j\beta z}}{e^{-j\beta z} - \left(\frac{Z_L - Z_0}{Z_L + Z_0}\right) e^{+j\beta z}} \right] \\ &= Z_0 \left[\frac{(Z_L + Z_0)\, e^{-j\beta z} + (Z_L - Z_0)\, e^{+j\beta z}}{(Z_L + Z_0)\, e^{-j\beta z} - (Z_L - Z_0)\, e^{+j\beta z}} \right] \end{aligned}$$

→ **How. About. That.**

Q: *????*

Make sure you see why this is stunning

A: **Look** carefully at this result:

$$Z(z) = Z_0 \left[\frac{(Z_L + Z_0)\, e^{-j\beta z} + (Z_L - Z_0)\, e^{+j\beta z}}{(Z_L + Z_0)\, e^{-j\beta z} - (Z_L - Z_0)\, e^{+j\beta z}} \right]$$

Notice the unknown **wave amplitudes** V_0^+ and V_0^- are **not** in this result!

Q: ????

A: The load impedance Z_L not **only** determines the value of line impedance $Z(z)$ at the **end** of the transmission line (i.e., at $z = 0$).

It **likewise** sets the value of line impedance **at all locations** z!

What about the other viewpoint?

Q: *So, this is the boundary condition result for the* ***voltage*** $V(z)$*,* ***current*** $I(z)$*,* ***impedance*** $Z(z)$ *viewpoint.*

What about the ***plus-wave*** $V^{+}(z)$*,* ***minus-wave*** $V^{-}(z)$*,* ***reflection coefficient*** $\Gamma(z)$*?*

A: First, recall the **reflection coefficient** function is:

$$\Gamma(z) = \Gamma_0 \, e^{+j2\beta z}$$

where the **complex constant** Γ_0 is:

$$\Gamma_0 \doteq \Gamma(z=0) = \frac{V_0^-}{V_0^+}$$

The resulting reflection coefficient function

Now, recall that we **just** determined for a transmission line **terminated** in a **load** Z_L that:

$$\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Meaning the **reflection coefficient function** for a lossless **terminated** transmission line is:

$$\Gamma(z) = \Gamma_0 \, e^{+j2\beta z} = \left[\frac{Z_L - Z_0}{Z_L + Z_0}\right] e^{+j2\beta z}$$

Now we know the value of Γ everywhere

As with line impedance, we see that the load not **only** determines the value of the **reflection coefficient function** at the **end** of the line:

$$\Gamma_0 = \Gamma(z=0) = \left[\frac{Z_L - Z_0}{Z_L + Z_0}\right]$$

but **also** determines the reflection coefficient **at all points *z*** along the line:

$$\Gamma(z) = \left[\frac{Z_L - Z_0}{Z_L + Z_0}\right] e^{+j2\beta z}$$

This shows up a lot

Q: *This* ***complex*** *value:*

$$\frac{Z_L - Z_0}{Z_L + Z_0}$$

sure seems to show up ***a lot****. Does* ***it*** *have any significance?*

A: Note that **it** is the value of the reflection coefficient at the **end** of the transmission line (i.e., where the **load** is attached).

$$\Gamma(z=0) = \left[\frac{Z_L - Z_0}{Z_L + Z_0}\right]$$

Thus, this complex value is known as the **load reflection coefficient** Γ_L:

$$\boxed{\Gamma_L \doteq \frac{Z_L - Z_0}{Z_L + Z_0}}$$

Both intuitive and satisfying

Thus, (**if** the load is located at $z=0$) we can recast our reflection coefficient math as:

$$\Gamma_0 = \frac{V_0^-}{V_0^+} = \Gamma_L$$

and:

$$\Gamma(z) = \Gamma_L \, e^{+j2\beta z}$$

Perhaps most importantly, the **boundary condition** for reflection coefficient is again very **intuitive** and **satisfying**:

$$\boxed{\Gamma(z=0) = \Gamma_L}$$

Now for Kirchoff

Now, one **important** result of this boundary condition is that the value of **wave amplitude** V_0^- and **wave amplitude** V_0^+ are **coupled**.

→ If ya' know **one**, ya' **know** the other.

$$\Gamma_L = \Gamma_0 = \frac{V_0^-}{V_0^+} \qquad \Rightarrow \qquad V_0^- = V_0^+ \, \Gamma_L$$

And thus, the propagating **minus**-wave can be expressed in terms of the **plus-wave amplitude**:

$$V^-(z) = V_0^- \, e^{+j\beta z} = V_0^+ \, \Gamma_L \, e^{+j\beta z}$$

The Bottom Line

And so finally, the **voltage** and **current** along the terminated transmission line can be expressed in terms of **load reflection coefficient** Γ_L:

$$V(z) = V_0^+ \left[e^{-j\beta z} + \Gamma_L \, e^{+j\beta z} \right]$$

$$I(z) = \frac{V_0^+}{Z_0} \left[e^{-j\beta z} - \Gamma_L \, e^{+j\beta z} \right]$$

$$Z(z) = Z_0 \left(\frac{e^{-j\beta z} + \Gamma_L \, e^{+j\beta z}}{e^{-j\beta z} - \Gamma_L \, e^{+j\beta z}} \right)$$

Note the above expressions are accurate **ONLY** if the load Z_L is located at position $z = 0$.

Two waves and a gamma

We can **alternatively** express the solutions as:

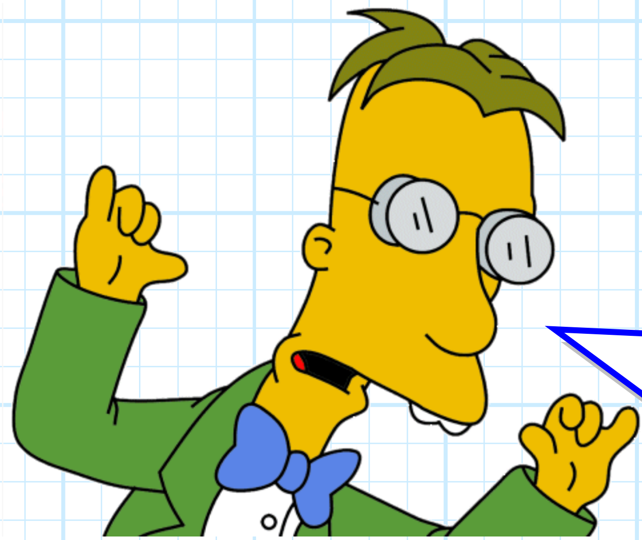
$$\boxed{\begin{aligned} V^{+}(z) &= V_0^{+}\, e^{-j\beta z} \\ V^{-}(z) &= V_0^{+}\, \Gamma_L\, e^{+j\beta z} \\ \Gamma(z) &= \frac{V^{-}(z)}{V^{+}(z)} = \Gamma_L\, e^{+j2\beta z} \end{aligned}}$$

Where:

$$\Gamma_L \doteq \frac{Z_L - Z_0}{Z_L + Z_0}$$

What about V_0^+ ??

Q: *But, how do we determine V_0^+ ??*



A: We require a **second** boundary condition to determine V_0^+. The only boundary left is at the **other end** of the transmission line.

Typically, a **source** of some sort is located there. This makes physical sense, as something must generate the **incident** wave !