Homework 5

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Math 590 HW5 Problem 1. Let $A = \begin{bmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{bmatrix}$. Find a basis for Image$(A)$ and a basis for Kernel(A).

Definition 0.1 (Range and Image are interchangeable terms) For T a function from V to W, the range of T is the subset of W consisting of those vectors that are the form Tv for some $v \in V$.

$$\mathrm{range(T) = \{Tv \colon v \in V\}}$$

Axler, S. (2015). Linear Algebra Done Right (3rd Ed.). Springer.

To find the basis vectors of A, it must be first reduced to RREF:

$$\mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We can see that there are pivots in row 1 & 2, so the basis must be:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$$

Because the third and fourth variables are free, a basis for Kernel(A) is

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Problem 2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation by projecting a vector in $\mathbb{R}^2$ to the line $y = 2x$.

a) Find the algebraic formula for T. Namely the matrix A such that $T(\vec{v}) = A\vec{v}$.

1. Since we are projecting across the line $y = 2x$, the direction vector is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

2. Additionally, the formula for projection is: $\frac{\vec{e}_1 \cdot \vec{d}}{\|\vec{d}\|^2} \cdot \vec{d}$

$$e_1 = \frac{\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}}{1 + 2^2} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 2/5 \end{bmatrix}$$
$$e_1 = \frac{\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}}{1 + 2^2} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2/5 \\ 4/5 \end{bmatrix}$$

Using these vectors as a basis for T:

$$= \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix}$$

b) Show that $A^{100} = A$.

1. Given that $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$:

$$A^2 = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} (1)(1) + (0)(2) & (0)(1) + (0)(0) \\ (2)(1) + (0)(2) & (0)(2) + (0)(0) \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$$

Because after squaring A, A does not change, by induction we can say that $A^{100} = A$.

Problem 3. Give a (total of) one-page summary of Sections $3.1, 3.2$ in the textbook.

Matrix Transformations

Matrices can be understood as functions, since when written in the form $A\vec{x} = b$, we can understand that the independent variable is x, and the dependent variable is b.

Often, we do not call such an operation a function; instead, it is normally called a transformation.

Definition 0.2 A transformation from $\mathbf{R}^n$ to $\mathbf{R}^m$ is a rule T that assigns to each vector x in $\mathbf{R}^n$ a vector $T(x)$ in $\mathbf{R}^m$.

- $\mathbf{R}^n$ is called the domain of T.
- $\mathbf{R}^m$ is called the codomain of T.
- For x in $\mathbf{R}^n$, the vector $T(x)$ in $\mathbf{R}^m$ is the image of x under T.
- The set of all images $\{T(x) \mid x \text{ in } \mathbf{R}^n\}$ is the range of T.

The notation $T : \mathbf{R}^n \longrightarrow \mathbf{R}^m$ means " T is a transformation from $\mathbf{R}^n$ to $\mathbf{R}^m$."

Matrix transformations are defined by matrices and involve transforming vectors from one space to another. Given an $m \times n$ matrix A, the associated matrix transformation T maps vectors from $\mathbb{R}^n$ to $\mathbb{R}^m$ by multiplying them with A. The domain of T is $\mathbb{R}^n$ because A has n columns, and the codomain is $\mathbb{R}^m$ because A results in vectors with m entries when multiplied by a vector. For example, if A is a specific matrix, and $T(x)$ is the associated transformation, T applied to a vector like $[-1, -2, -3]$ would yield $[-14, -32]$ as the result.

Definition 0.3 Let A be an $m \times n$ matrix, and let $T(x) = Ax$ be the associated matrix transformation.

- The domain of T is $\mathbf{R}^n$, where n is the number of columns of A.
- The codomain of T is $\mathbf{R}^m$, where m is the number of rows of A.
- The range of T is the column space of A.

One-to-one and Onto Transformations

Definition 0.4 A transformation $T : \mathbf{R}^n \rightarrow \mathbf{R}^m$ is oneto-one if, for every vector b in $\mathbf{R}^m$, the equation $T(x) = b$ has at most one solution x in $\mathbf{R}^n$.

This essentially means that for every vector, there is zero or one solution. Onto (surjective) solutions are similar, but they have *at least* one solution for every vector b.

T is one-to-one

- $T(x) = b$ has at most one solution for every b.
- The columns of A are linearly independent.
- A has a pivot in every column.

T is onto (Surjective)

- $T(x) = b$ has at least one solution for every b.
- The columns of A span $\mathbf{R}^m$.
- A has a pivot in every row.

Put simply, the difference between the two is that there may be "unmapped" regions for onto (surjective) transformations, but one-to-one projections are not "compressed". This means that every vector gets a single new vector.

- An onto transformation might be a rotation by 90 degrees counterclockwise. Every point in the target space can be reached by rotating some point in the source space.
- A one-to-one transformation might be a scaling transformation that scales all vectors by a factor of 2. No two distinct vectors in the source space will map to the same vector in the target space because the scaling operation preserves distinctness.