PHSX 711: Homework #3

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Problem 1

(15 pts) Exercise 4.2.1 (Shankar) Consider the following operators on a Hilbert space $\mathcal{V}^3(\mathbb{C})$:

$$L_x = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad L_y = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad L_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

1. What are the possible values one can obtain if L_z is measured?
2. Take the state in which $L_z = 1$. In this state, what are $\langle L_x \rangle$, $\langle L_x^2 \rangle$, and ΔL_x?
3. Find the normalized eigenstates and the eigenvalues of L_x in the L_z basis.
4. If the particle is in the state with $L_z = -1$, and L_x is measured, what are the possible outcomes and their probabilities?
5. Consider the state
$$|\psi\rangle = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{bmatrix}$$
in the L_z basis. If L_z^2 is measured in this state and a result $+1$ is obtained, what is the state after the measurement? How probable was this result? If L_x is measured, what are the outcomes and respective probabilities?
6. A particle is in a state for which the probabilities are $P(L_z = 1) = 1/4$, $P(L_z = 0) = 1/2$, and $P(L_z = -1) = 1/4$. Convince yourself that the most general, normalized state with this property is
$$|\psi\rangle = \frac{e^{i\delta_1}}{2}|L_z = 1\rangle + \frac{e^{i\delta_2}}{\sqrt{2}}|L_z = 0\rangle + \frac{e^{i\delta_3}}{2}|L_z = -1\rangle$$
It was stated earlier that if $|\psi\rangle$ is a normalized state, then the state $e^{i\theta}|\psi\rangle$ is a physically equivalent normalized state. Does this mean that the factors $e^{i\theta}$ multiplying the L_z eigenstates are irrelevant? [Calculate for example $P(L_x = 0)$.]

Part 1: The possible values that L_z can take when measured are the eigenvalues. L_z is diagonal so the eigenvalues lie on its diagonal.

$$\{-1, 0, 1\}$$

Part 2: For $L_z = 1$ the eigenket is $|\psi\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. The expectation value of an operator A in state $|\psi\rangle$ is given by:

$$\langle A \rangle = \langle \psi | A | \psi \rangle$$

$$\langle L_x \rangle = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (0)$$

$$\langle L_x^2 \rangle = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (1/2)$$

$$\Delta L_x = \sqrt{\langle L_x^2 \rangle + \langle L_x \rangle^2} = (1/\sqrt{2})$$

Part 3: For this we simply need solve for eigenvalues and eigenstates of the given L_x as it is already in the L_z basis. If it were not already in this basis, say if we wanted it for the L_y basis, we would need to find its eigenvectors and project onto those. This would use diagonalization matrices for L_y on L_x:

$$L_x' = U^{-1} L_x U$$

Regardless, the eigenvalues and eigenvectors for L_x are given by

$$0 = \det(L_x - \lambda)$$

And eigenvectors:

$$0 = \det(L_x - \lambda) \, |\lambda\rangle$$

We have $\lambda = \{0, 1, -1\}$, which then gives eigenvectors (non-normalized):

$$|\lambda = 0\rangle = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \qquad |\lambda = 1\rangle = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}, \qquad |\lambda = -1\rangle = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

Normalizing gives:

$$|\lambda = 0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \qquad |\lambda = 1\rangle = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}, \qquad |\lambda = -1\rangle = \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

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Part 4: We already know the possible outcomes of L_x are $\{1, 0, -1\}$ as we found them in part 1. Additionally, the eigenstate corresponding to $L_z = -1$ is

$$|\psi\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The probability amplitude is now given by the projection of $|\psi\rangle$ onto the basis for L_x squared

$$\langle \lambda = n|\psi\rangle^2$$

$$P(L_x = 0) = \langle \lambda = 0|\psi\rangle^2 = \left| \frac{1}{\sqrt{2}}(-1\ 0\ 1) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right|^2 = \frac{1}{2}$$

$$P(L_x = 1) = \langle \lambda = 1|\psi\rangle^2 = \left| \frac{1}{2}(1\ \sqrt{2}\ 1) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right|^2 = \frac{1}{4}$$

$$P(L_x = -1) = \langle \lambda = -1|\psi\rangle^2 = \left| \frac{1}{2}(1\ -\sqrt{2}\ 1) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right|^2 = \frac{1}{4}$$

Part 5: We have to first project $|\psi\rangle$ on the $L_z^2 = 1$ basis.

$$L_z^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This has eigenvalues $\lambda = \{0, 1\}$ and eigenvectors

$$|\lambda = 0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad |\lambda = 1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Projecting- we recall from the first homework that projection is

$$|\psi'\rangle = |n\rangle \langle n|\psi\rangle$$

$$\begin{aligned}
|\psi'\rangle &= \frac{1}{A}\left(|\lambda=0\rangle\langle\lambda=0| + |\lambda=1\rangle\langle\lambda=1|\right)|\psi\rangle \\
&= \frac{1}{A}\left(\begin{pmatrix}0\\1\\0\end{pmatrix}(0\ 1\ 0) + \begin{pmatrix}1\\0\\0\end{pmatrix}(1\ 0\ 0)\right)\begin{pmatrix}1/2\\1/2\\1/\sqrt{2}\end{pmatrix} \\
&= \frac{1}{A}\begin{pmatrix}1/2\\0\\\sqrt{2}/2\end{pmatrix} \\
&= \frac{1}{\sqrt{3}}\begin{pmatrix}1\\0\\\sqrt{2}\end{pmatrix}
\end{aligned}$$

Where A is the normalization constant and equals $\frac{\sqrt{3}}{2}$.
That is the state after measurement, but what is the probability of that outcome? We find it the same way we find any other probability amplitude:

$$\langle\psi|\,|\lambda=1\rangle\langle\lambda=1| + |\lambda=0\rangle\langle\lambda=0|\,|\psi\rangle^2 = \langle\psi|\psi'\rangle^2 = \frac{3}{4}$$

If we measured L_z the posible outcomes are the eigenvalues $L_z, \{0, \pm 1\}$, with probabilities

$$P(L_z=1) = \left|\begin{pmatrix}1 & 0 & 0\end{pmatrix}\psi'\right|^2 = \left|\frac{1}{\sqrt{3}}\begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\0\\\sqrt{2}\end{pmatrix}\right|^2 = \frac{1}{3},$$

$$P(L_z=0) = \left|\begin{pmatrix}0 & 1 & 0\end{pmatrix}\psi'\right|^2 = \left|\frac{1}{\sqrt{3}}\begin{pmatrix}0 & 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\\\sqrt{2}\end{pmatrix}\right|^2 = 0,$$

$$P(L_z=-1) = \left|\begin{pmatrix}0 & 0 & 1\end{pmatrix}\psi'\right|^2 = \left|\frac{1}{\sqrt{3}}\begin{pmatrix}0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\0\\\sqrt{2}\end{pmatrix}\right|^2 = \frac{2}{3}.$$

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Part 6:
Shankar implies we should try calculating $P(L_x = 0)$ for this problem, which has the corresponding eigenvector

$$\left(\langle \lambda = 0|\psi\rangle^2\right) = \left|\begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{2}e^{i\delta_1} \\ \frac{1}{\sqrt{2}}e^{i\delta_2} \\ \frac{1}{2}e^{i\delta_3} \end{pmatrix}\right|^2 = \left|\frac{e^{i\delta_1}}{2\sqrt{2}} - \frac{e^{i\delta_3}}{2\sqrt{2}}\right|^2 = \frac{1}{2\sqrt{2}}\left|e^{i\delta_1} - e^{i\delta_3}\right|^2$$

Where $|\lambda = 0\rangle$ was found to be the following eigenket in part 3:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

This implies that the probability of finding L_x in state 0 depends on some difference in the phase vectors on the L_z basis! As these are complex exponential vectors, it is likely that you could express this as some sort of rotation where the final difference is between some angles δ_1 and δ_3.

Problem 2

(5 pts) Exercise 4.2.2 (Shankar)
More hint: Evaluate the probability:

$$P(p) = |\langle\psi|p\rangle|^2$$

in the x-basis.

Solution:
The given probability is equivalent to

$$\langle P\rangle = \langle\psi|P|\psi\rangle$$

$$\begin{aligned}
\langle P\rangle &= \int_{-\infty}^{\infty} dx\ \langle\psi|x\rangle\langle x|P|p\rangle \\
&= \int_{-\infty}^{\infty} dx\ \psi^*(x)\left(-i\hbar\frac{d}{dx}\right)\psi(x) \\
&= -i\hbar\int_{-\infty}^{\infty} dx\ \psi^*(x)\frac{d\psi(x)}{dx}
\end{aligned}$$

Problem 3

(5 pts) Exercise 4.2.3 (Shankar) Show that if $\psi(x)$ has mean momentum $\langle \boldsymbol{P} \rangle$, $e^{ip_0 x/\hbar}\psi(x)$ has mean momentum $\langle P \rangle + p_0$.

Solution:

$$\langle \mathbf{P} \rangle = \langle \psi | P | \psi \rangle = -i\hbar \psi^* \frac{d\psi}{dx}$$

$\langle \exp(ip_0 x/\hbar)\psi(x) | P | \exp(ip_0 x/\hbar)\psi(x) \rangle$:

$$\begin{aligned}
&= \int_{-\infty}^{\infty} dx \exp(-ip_0 x/\hbar)\psi^*(x)(-i\hbar) \qquad \underbrace{\frac{d}{dx}\left[\exp(ip_0 x/\hbar)\psi(x)\right]}_{\frac{ip_0}{\hbar}\exp(ip_0 x/\hbar)\psi(x) + \exp(ip_0 x/\hbar)\frac{d\psi(x)}{dx}} \\
&= -i\hbar \int_{-\infty}^{\infty} dx \; \left[\cancel{\exp(-ip_0 x/\hbar)}\psi^*(x)\right]\left[\frac{ip_0}{\hbar}\cancel{\exp(ip_0 x/\hbar)}\psi(x)\right] \\
&\qquad + \int_{-\infty}^{\infty} dx \left[\psi^*(x)\cancel{\exp(-ip_0 x/\hbar)}\cancel{\exp(ip_0 x/\hbar)}\frac{d\psi(x)}{dx}\right] \\
&= p_0 \underbrace{\int_{-\infty}^{\infty} dx \psi^* \psi}_{\langle \psi | \psi \rangle = 1} - \underbrace{i\hbar \int_{-\infty}^{\infty} dx \psi^* \frac{d\psi}{dx}}_{\langle \psi | \mathbf{P} | \psi \rangle = \langle P \rangle} \\
&= p_0 + \langle \mathbf{P} \rangle
\end{aligned}$$