PHSX 711: Homework #8

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Problem 1

Exercise 12.2.1 (Shankar) – 5 points
Provide the steps linking Eq. (12.2.8) to Eq. (12.2.9). [Hint: Recall the derivation of Eq. (11.2.8) from Eq. (11.2.6).]

$$U[R]\,|x, y\rangle = |x - y\varepsilon_z, x\varepsilon_z + y\rangle$$
$$\langle x, y|I - \frac{i\varepsilon_z L_z}{\hbar}|\psi\rangle = \psi(x + y\varepsilon_z, y - x\varepsilon_z)$$

Solution: In the derivation of rotations, we initially define infinitesimal rotations about $\epsilon_z\mathbf{k}$:

$$U[R(\epsilon_z\mathbf{k})] = I - \frac{i\epsilon_z L_z}{\hbar}$$

We can first examine the hermitian conjugate, where we just end up inverting the sign on the parts multiplied by ϵ_z. Then, we can consider the application of the rotation operator such that $\langle x, y|U[R]|\psi\rangle$ to rotate the coordinates which we project ψ into. This leaves us with a modified projection of ψ onto x' and y'.

$$(U[R]\,|x, y\rangle)^\dagger = \langle x, y|\left(I + \frac{i\epsilon_z L_z}{\hbar}\right) = \langle x + y\epsilon_z,\ -x\epsilon_z + y|$$
$$\begin{aligned}\langle x, y|I - \frac{i\varepsilon_z L_z}{\hbar}|\psi\rangle &= \langle x + y\epsilon_z,\ -x\epsilon_z + y|\psi\rangle \\ &= \psi(x + y\epsilon_z,\ y - x\epsilon_z)\end{aligned}$$

Problem 2

Exercise 12.2.3 (Shankar) – 5 points
Derive Eq. (12.2.19) by doing a coordinate transformation on Eq. (12.2.10), and also by the direct method mentioned above.

$$L_z \underset{\text{coordinate basis}}{=} -i\hbar \frac{\partial}{\partial \phi}$$

*You can choose either one of the methods.

Solution:
Equation 12.2.10 says

$$L_z = x\left(-i\hbar\frac{\partial}{\partial y}\right) - y\left(-i\hbar\frac{\partial}{\partial x}\right)$$

Directly doing the coordinate transformation to this is probably more straightforward. The only tricky part is working out the transformation of the partial derivatives to polar coordinates for $r = 1$ (since we don't want to modify length of our vectors).

$$\begin{aligned}
&= \cos\phi\left(-i\hbar\frac{\partial}{\partial \phi}\frac{\partial \phi}{\partial y}\right) - \sin\phi\left(-i\hbar\frac{\partial}{\partial \phi}\frac{\partial \phi}{\partial x}\right) \\
&= \cos\phi\left(-i\hbar\cos\phi\frac{\partial}{\partial \phi}\right) - \sin\phi\left(-i\hbar\sin\phi\frac{\partial}{\partial \phi}\right) \\
&= -i\hbar\left(\cos^2\phi + \sin^2\phi\right)\frac{\partial}{\partial \phi} \\
&= -i\hbar\frac{\partial}{\partial \phi}
\end{aligned}$$

Problem 3

Exercise 12.5.3 (Shankar) – 10 points
(1) Show that $\langle J_x\rangle = \langle J_y\rangle = 0$ in a state $|jm\rangle$.

Solution: We have

$$J_i = L_i + S_i$$

Which in the basis of ladder operators is:

$$J_x = \frac{J_+ + J_-}{2}, \qquad J_y = \frac{J_+ - J_-}{2i}$$

Which we do know how to do calculations with:

$$J_\pm |jm\rangle = \hbar[(j \mp m)(j \pm m + 1)]^{1/2} |j, m \pm 1\rangle$$

Problem 3 continued on next page...

It is relatively simple to express expectation values:

$$
\begin{aligned}
\langle jm|\frac{J_+ + J_-}{2}|jm\rangle &= \frac{1}{2}\left[\langle jm|J_+|jm\rangle + \langle jm|J_-|jm\rangle\right] \\
&= \frac{\hbar}{2}\left\{[(j-m)(j+m+1)]^{1/2}\,\cancelto{0}{\langle jm|j,m+1\rangle} + [(j+m)(j-m+1)]^{1/2}\,\cancelto{0}{\langle jm|j,m-1\rangle}\right\} \\
&= 0 \\
\langle jm|\frac{J_+ - J_-}{2i}|jm\rangle &= \frac{1}{2i}\left[\langle jm|J_+|jm\rangle - \langle jm|J_-|jm\rangle\right] \\
&= \frac{\hbar}{2i}\left\{[(j-m)(j+m+1)]^{1/2}\,\cancelto{0}{\langle jm|j,m+1\rangle} - [(j+m)(j-m+1)]^{1/2}\,\cancelto{0}{\langle jm|j,m-1\rangle}\right\} \\
&= 0
\end{aligned}
$$

We run into the same situation we do with harmonic oscillator ladder operators, in that our off-diagonal (block diagonal) matrix form results in expectation values of zero when a value gets raised but not lowered, and vise versa due to the delta function.

(2) Show that in these states

$$\langle J_x^2\rangle = \langle J_y^2\rangle = \frac{1}{2}\hbar^2[j(j+1) - m^2]$$

(use symmetry arguments to relate $\langle J_x^2\rangle$ to $\langle J_y^2\rangle$).

Solution: As with the harmonic oscillator ladder operators, squaring the sum of these operators results in two terms that have equal amounts raising and lowering.

$$
\begin{aligned}
\frac{1}{2}\langle jm|\left(J_+ + J_-\right)^2|jm\rangle &= \frac{1}{2}\langle jm|\cancelto{0}{J_+^2} + \cancelto{0}{J_-^2} + J_-J_+ + J_+J_-|jm\rangle \\
-\frac{1}{2}\langle jm|\left(J_+ - J_-\right)^2 jm\rangle &= \frac{1}{2}\langle jm|\cancelto{0}{-J_+^2} - \cancelto{0}{J_-^2} + J_-J_+ + J_+J_-|jm\rangle
\end{aligned}
$$

Where the sequence of the one lowering-raising operator and vise-versa is:

$$
\begin{aligned}
\langle jm|J_-J_+|jm\rangle &= \hbar[(j-m)(j+m+1)]^{1/2}\,\langle jm|J_-|jm+1\rangle \\
&= \hbar^2[(j-m)(j+m+1)(j+(m+1))(j-(m+1)+1)]^{1/2}\cancelto{1}{\langle jm|jm\rangle} \\
&= \hbar^2\left(j+m+1\right)\left(j-m\right) \\
\langle jm|J_+J_-|jm\rangle &= \hbar[(j+m)(j-m+1)]^{1/2}\,\langle jm|J_+|jm-1\rangle \\
&= \hbar^2[(j+m)(j-m+1)(j-(m-1))(j+(m-1)+1)]^{1/2}\cancelto{1}{\langle jm|jm\rangle} \\
&= \hbar^2\left(j+m\right)\left(j-m+1\right)
\end{aligned}
$$

Substituting these back in,

$$\langle J_x^2\rangle = \frac{1}{2}\langle jm|\left(J_+ + J_-\right)^2|jm\rangle = \frac{\hbar^2}{2}\left[(j+m+1)(j-m) + (j+m)(j-m+1)\right] = \frac{\hbar^2}{2}\left[j(j+1) - m^2\right]$$

It happens that $\langle J_y^2\rangle$ is the same, since that extra i in the demoninator when squared makes it identical to $\langle J_x^2\rangle$.

Problem 3 continued on next page…

Solution #2: A bit more compactly, we have rotational invariance about the $z - axis$, so $\langle J_x^2 \rangle = \langle J_y^2 \rangle$. Now we can just work out expectation values for:

$$J^2 - J_z^2 = j_x^2 + J_y^2$$

which is the same:

$$\langle J_x^2 \rangle = \langle J_y^2 \rangle = \frac{\hbar^2}{2}\left[j(j+1) - m^2\right]$$

(3) Check that $\Delta J_x \cdot \Delta J_y$ from part (2) satisfies the inequality imposed by the uncertainty principle [Eq. (9.2.9)].

Solution: We have equation (9.2.9):

$$(\Delta\Omega)^2(\Delta\Lambda)^2 \geq |\langle\psi|\Omega\Lambda|\psi\rangle|^2$$

where $\Omega\Lambda$ can also be expressed in terms of commutators

$$\frac{[\Omega, \Lambda]_+ + [\Omega, \Lambda]}{2}$$

I think that it will be easier to just directly compute their product, since it is just

$$\langle J_x J_y \rangle = \frac{1}{4i}\langle J_- J_+ - J_+ J_- \rangle = \frac{m\hbar^2}{2}$$

Now the left hand side, we have by rotational invariance about the z-axis:

$$\begin{aligned} \Delta J_x = \Delta J_y &= \sqrt{\langle J_x^2 \rangle - \langle J_x \rangle^2} \\ &= \sqrt{\frac{\hbar^2}{2}\left[j(j+1) - m^2\right]} \end{aligned}$$

So to verify, we have

$$\frac{\hbar^2}{2}\left[j(j+1) - m^2\right] \geq \frac{m\hbar^2}{2}$$

We must realize that $j \leq |m| \implies j(j+1) \leq m(m+1)$, so then the equality is met.

(4) Show that the uncertainty bound is saturated in the state $|j, \pm j\rangle$.

Solution: The inequality is only saturated when $j = |m| \implies m = \pm j$. This is the state $|j, \pm j\rangle$

Problem 4

Exercise 12.5.12 (Shankar) – 6 points
Since L^2 and L_z commute with Π, they should share a basis with it. Verify that $Y_l^m \underset{\Pi}{\rightarrow} Y_{l\Pi}^m = (-1)^l Y_l^m$.

Hint: (I expanded the hint from the book a little bit)

1. $Y_{l\Pi}^m$ in this question represents the function generated from ΠY_l^m.
2. Recall (from Chap. 11) that Π is an operation that changes (x, y, z) to $(-x, -y, -z)$. For spherical coordinates, $\Pi Y_l^m = Y_{l\Pi}^m(\theta, \phi) = Y_l^m(\pi - \theta, \phi + \pi)$
3. Show the proof for the Y_l^l first.
4. Then, show that the lower operator L_- (projected in the X-basis) is not altered by the parity operation $(\theta \to \pi - \theta, \phi \to \phi + \pi)$

Solution:
We have the harmonic function

$$Y_l^m(\theta, \phi) = \left[\frac{(2l+1)(l-m)!}{4\pi(l+m)!}\right]^{1/2} (-1)^m e^{im\phi} P_l^m(\cos\theta)$$

Starting with (3), we can apply the parity operator to Y_l^l.

$$Y_l^l(\theta, \phi) = \left[\frac{2l+1}{4\pi(2l)!}\right]^{1/2} (-1)^l e^{il\phi} P_l^l(\cos\theta)$$
$$\Pi Y_l^l(\theta, \phi) = \left[\frac{2l+1}{4\pi(2l)!}\right]^{1/2} (-1)^l e^{il(\phi+\pi)} P_l^l(\cos(\pi-\theta))$$

A key thing to see here is that P_l^l obeys a nice rule, where

$$P_\ell^\ell = \sin^\ell\theta = \sin^\ell(\pi - \theta)$$

We don't really care about the normalization term since it doesn't change under parity, so under parity just need to know what happens to the exponential ϕ term.

$$e^{il(\phi+\pi)} = \left(e^{i\phi}\right)^l e^{il\phi} = (-1)^l e^{il\phi}$$

Hence,

$$\Pi Y_l^l = (-1)^l Y_l^l$$

We can now lower the harmonic function using L_- to access the values of $m \neq l$. Recall that the operator $L_\pm$ is defined as

$$L_\pm = \pm\hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\phi}\right).$$

Under the transformations $\theta \to \pi - \theta$ and $\phi \to \phi + \pi$, the differentials remain unchanged except that the differential with respect to θ picks up a negative sign.
To verify the behavior of $\cot\theta$ under these transformations, we only need to check whether

$$\cot\theta \stackrel{?}{=} \cot(\pi - \theta),$$

since it was already shown that $e^{i\phi} \to e^{i(\phi-\pi)} = (-1)^l e^{i\phi}$.
Using trigonometric identities, we compute:

$$\cot(\pi - \theta) = \frac{\cos(\pi-\theta)}{\sin(\pi-\theta)} = \frac{-\cos\theta}{\sin\theta} = -\cot\theta.$$

Thus, $\cot(\pi-\theta) = -\cot\theta$, confirming the behavior of L_- under the transformation.

$$L_- = \Pi L_- \implies [L_-, \Pi] = 0$$

Therefore

$$\Pi Y_l^m = (-1)^l Y_l^m$$

Problem 5

Exercise 12.6.1 (Shankar) – 9 points
A particle is described by the wave function

$$\psi_E(r,\theta,\phi) = Ae^{-r/a_0} \quad (a_0 = \text{const})$$

(1) What is the angular momentum content of the state?

Solution: No ϕ or θ dependence implies it has no angular momentum. Furthermore, $\psi_E \propto Y_0^0$.

(2) Assuming ψ_E is an eigenstate in a potential that vanishes as $r \to \infty$, find E. (Match leading terms in Schrödinger's equation.)

Solution:

$$\hat{H}\psi_E = \left[-\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + V(r)\right]\psi_E = E\psi_E$$
$$\left[-\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + V(r)\right]Ae^{-r/a_0} = EAe^{-r/a_0}$$
$$\left[-\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(Ae^{-r/a_0}\right)\right) + V(r)Ae^{-r/a_0}\right] = EAe^{-r/a_0}$$
$$-\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{A}{a_0^2}\left(2a_0 r - r^2\right)e^{-r/a_0} + V(r)Ae^{-r/a_0} = EAe^{-r/a_0}$$
$$-\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{1}{a_0^2}\left(2a_0 r - r^2\right) + V(r) = E$$
$$-\frac{\hbar^2}{\mu a_0} + \frac{\hbar^2}{2\mu a_0 r} + V(r) = E$$

If potential vanishes as $r \to \infty$, then we are left with one term

$$E = -\frac{\hbar^2}{\mu a_0}$$

Problem 5 continued on next page...

(3) Having found E, consider finite r and find $V(r)$.

Solution: Substituting E in, we have

$$-\frac{\hbar^2}{2\mu a_0 r} = V(r)$$