1 Math 590 HW8

1.1 Problem 1.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Let C_{ij} be the matrix obtained from A by deleting row i, column j and $D_{ij} = \det(C_{ij})$. Show by direct computations that:

(a) $a_{11}D_{11} - a_{12}D_{12} + a_{13}D_{13} = \det(A)$.

Solution:

For a 3x3 matrix, we can show that the cofactor method of determining the determinant is equivalent to the triple product method of finding the determinant:

1. By Cofactor Expansion:

$$a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

2. By Triple Product:

$$\begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} \times \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = (a_{22}a_{33} - a_{23}a_{32}) - (a_{21}a_{33} - a_{31}a_{23}) + (a_{21}a_{32} - a_{22}a_{31})$$

$$\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} \cdot \left(\begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} \times \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} \right) =$$

$$a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Given that these two are equivalent, we've symbolically computed that this is a valid method to find the determinant.

(b) $a_{21}D_{11} - a_{22}D_{12} + a_{23}D_{13} = 0$.

Solution:

We can see that the following would cancel to zero below:

$$a_{21}(a_{22}a_{33} - a_{23}a_{32}) - a_{22}(a_{21}a_{33} - a_{31}a_{23}) + a_{23}(a_{21}a_{32} - a_{22}a_{31})$$

1.2 Problem 2.

Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

(a) Write A and A^{-1} as product of elementary matrices.

Solution:

1. A:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

2. A^{-1}:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

(b) Find A^{-1} using row reduction.

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 2 & -1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to \frac{R_2}{2}} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \\ 0 & 1 & 0 & 1/2 & -1/2 & 1/2 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 1 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{array}\right]$$

(c) Find A^{-1} using cofactor matrices.

Solution:

1. Cofactor Matrix:

$$\begin{bmatrix} -1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$$

2. Adjoint Matrix:
   The adjoint matrix is the transpose of the cofactor matrix:

$$\left(\begin{bmatrix} -1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix}\right)^T = \begin{bmatrix} -1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$$

3. Multiply by Inverse Determinant:
   Given that the determinant is -2:

$$-\frac{1}{2}\begin{bmatrix} -1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \end{bmatrix}$$