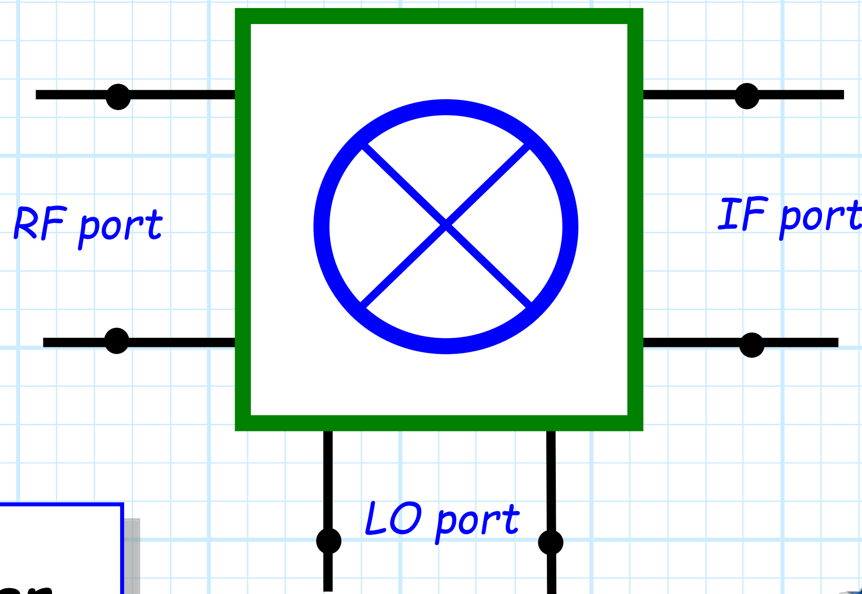
Mixers

A **mixer** is a **three**-port, **non-linear** microwave device.



The Mixer

Usually mixers are **passive devices**, but **active** mixers also exist, and are becoming **more** and more prevalent.

It's a multiplier

The three ports of a mixer are **distinct** and **unique**, and are typically referred to as:

1) The **RF** (Radio Frequency) port

2) The **IF** (Intermediate Frequency) port

3) The **LO** (Local Oscillator) port

Q: *So just what does a mixer **do**??*

A: A **clue** is in its **symbol**: $\otimes$

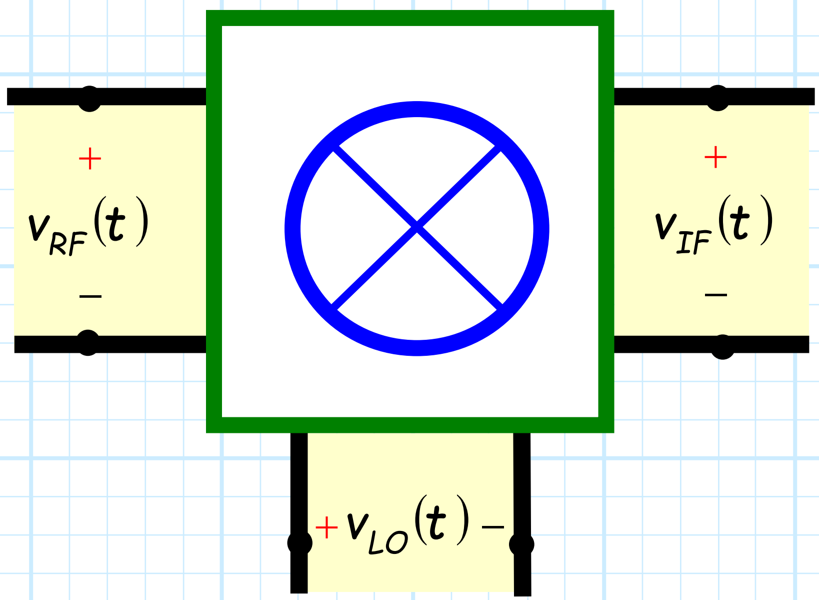
→ A mixer is a **multiplier** ($\times$) !!

An IDEAL Mixer

Say there is a real-valued signal $v_{RF}(t)$ at the **RF mixer** port, and a signal $v_{LO}(t)$ at the **LO mixer** port.

An **ideal** mixer would then produce at the **IF port**, a signal $v_{IF}(t)$, where:

$$\boxed{v_{IF}(t) = v_{RF}(t)\, v_{LO}(t) \qquad \text{(an \textbf{ideal} mixer)}}$$



REAL-valued multiplication

Note the **multiplied voltages** in this case (i.e.,$v_{RF}(t)$ and $v_{LO}(t)$) are:

a) **real**-valued, and

b) some **arbitrary** function of time!

In other words, a mixer does **NOT** multiply two **complex** voltages (e.g., $V_{RF} = j$ and $V_{LO} = e^{j\pi/4}$)!

Instead, it multiplies two **arbitrary real-valued function of time**.

For **example**, if:

$$v_{RF}(t) = 2t - 3t^2 \qquad \text{and} \qquad v_{LO}(t) = 4t^2$$

then:

$$v_{IF}(t) = v_{RF}(t)\, v_{LO}(t) = 8t^3 - 12t^4$$

How and why?

Q: *The **only** things I don't understand are:*

*a) **how** this is **possible**, and*

*b) **why** would this would be **useful?***

A: Let's answer the **second** part (i.e., why would this be **useful**?) first.

To see why multiplication might be **useful**, consider a case where both the RF and LO port signals are **time-harmonic**:

$$v_{RF}(t) = \cos\left[\omega_{RF} t\right] \quad \text{and} \quad v_{LO}(t) = \cos\left[\omega_{LO} t\right]$$

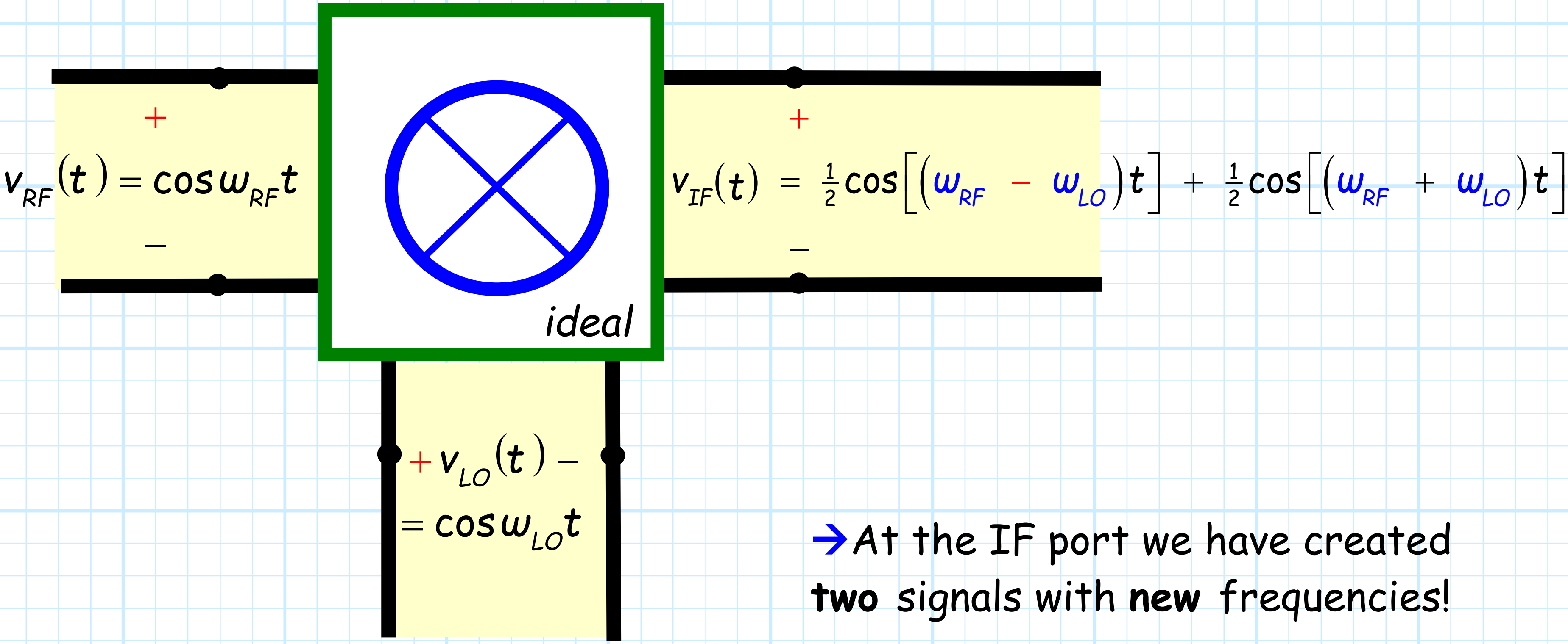
Note again that these are **NOT** **complex** values—they are instead **real-valued** functions of **time**!

The output of an IDEAL mixer

Multiplying these signals, we get (remember your **trig identities**!):

$$
\begin{aligned}
v_{IF}(t) &= v_{RF}(t)\, v_{LO}(t) \\
&= \cos[\omega_{RF} t]\, \cos[\omega_{LO} t] \\
&= \tfrac{1}{2}\cos[(\omega_{RF} - \omega_{LO})t] + \tfrac{1}{2}\cos[(\omega_{RF} + \omega_{LO})t]
\end{aligned}
$$

New signals are created!



→ At the IF port we have created **two** signals with **new** frequencies!

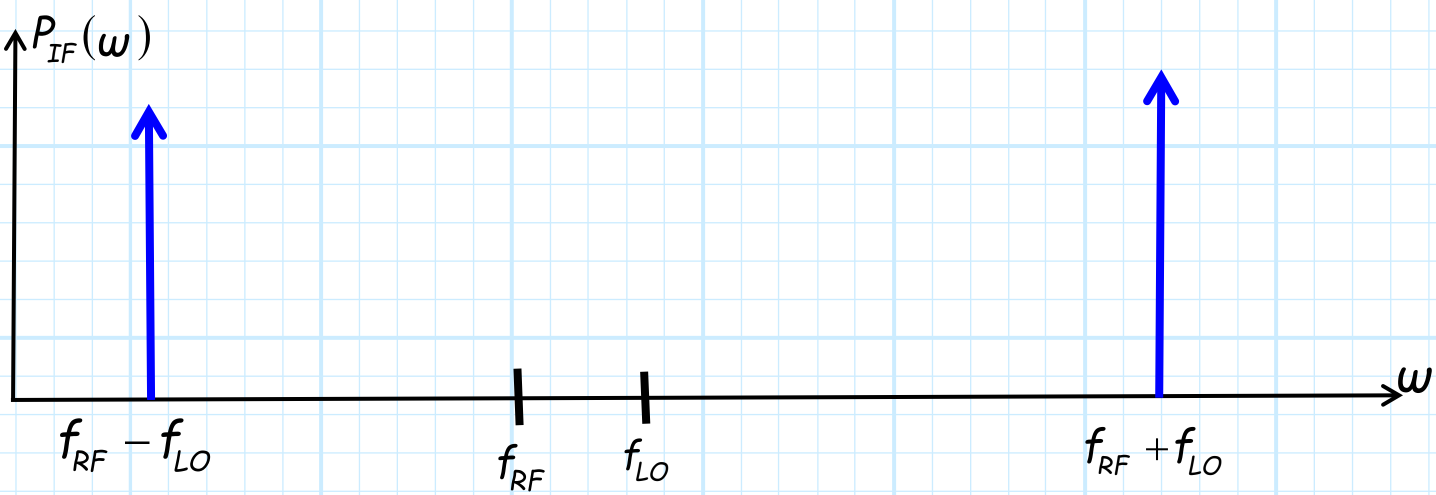
The IDEAL Mixer Output

One new signal has a frequency ω_Δ, that is the **difference** of the LO and RF signal frequencies:

$$\frac{1}{2}\cos\left[\left(\omega_{RF} - \omega_{LO}\right)t\right] \doteq \frac{1}{2}\cos\left[\omega_\Delta t\right]$$

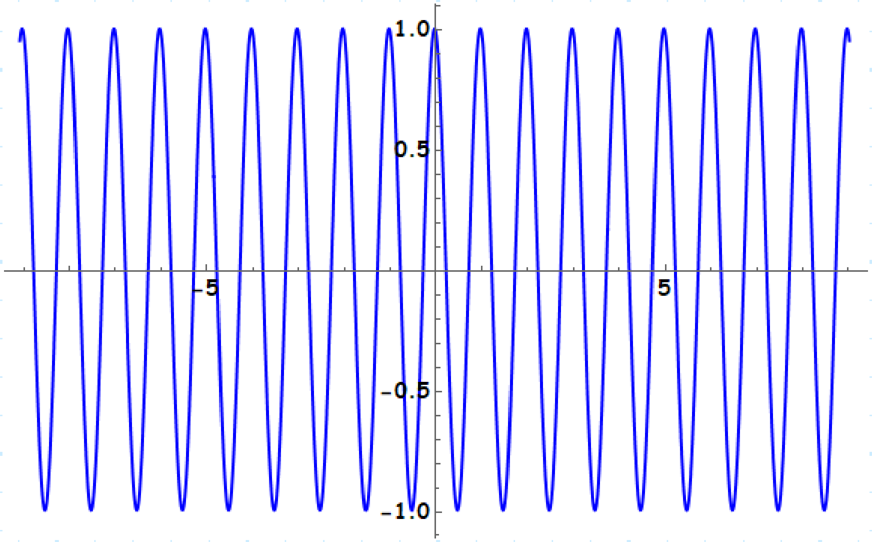
While the **other** new signal has a frequency ω_Σ, that is the **sum** of the LO and RF signal frequencies:

$$\frac{1}{2}\cos\left[\left(\omega_{RF} + \omega_{LO}\right)t\right] \doteq \frac{1}{2}\cos\left[\omega_\Sigma t\right]$$



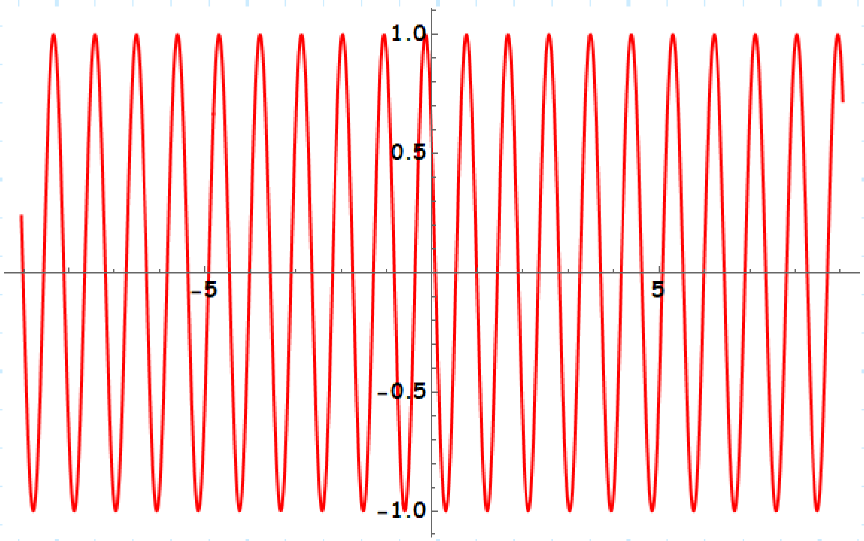
An example…

For example, consider **this** sinusoid with frequency $\omega_{LO} = 2.0\pi$:



$$v_{LO}(t) = cos(2.0\pi\, t)$$

Along with **this** sinusoid, with **higher** frequency $\omega_{RF} = 2.2\pi$:

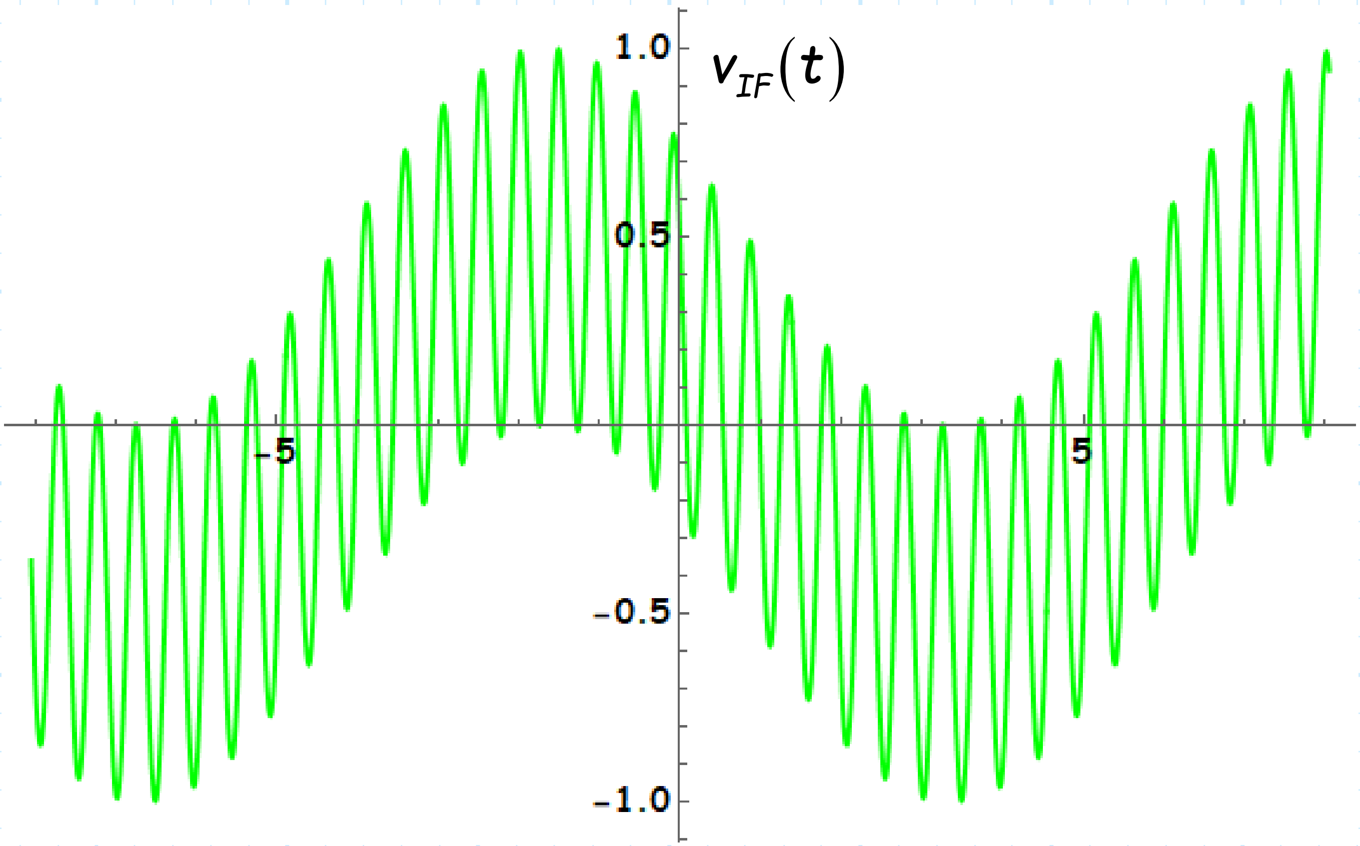


$$v_{RF}(t) = cos(2.2\pi\, t + \pi/3)$$

The example output signal

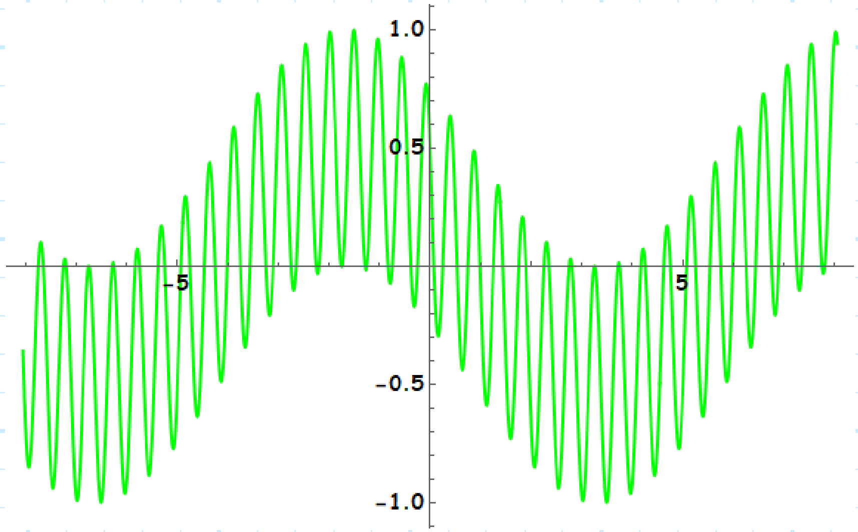
Multiplying these two signals together, we get:

$$
\begin{aligned}
v_{IF}(t) &= v_{RF}(t)\, v_{LO}(t) \\
&= \cos(2.2\pi\, t + \pi/3)\ \cos(2.0\pi\, t) \\
&= \tfrac{1}{2}\cos(0.2\pi\, t + \pi/3) + \tfrac{1}{2}\cos(4.2\pi\, t + \pi/3)
\end{aligned}
$$

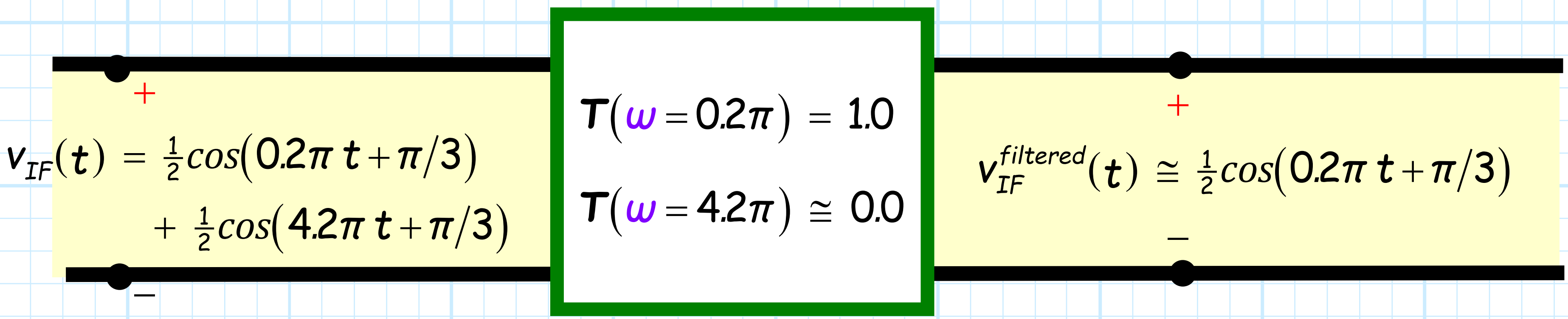


Not the mess you think it is

Q: *What a* ***mess****! I don't see* ***two sinusoids*** *in there at all!*



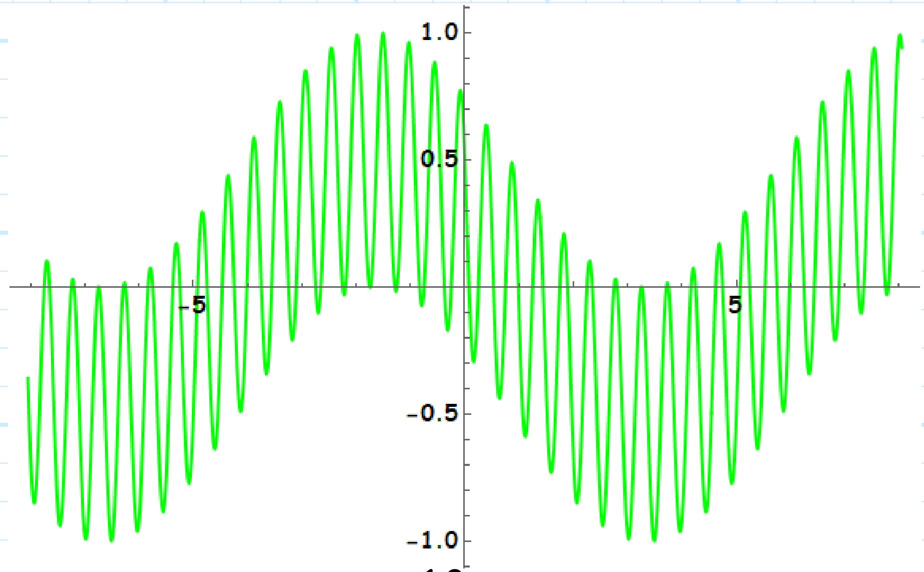
A: Say we pass the above signal through a **bandpass filter**—one which **rejects** the **higher frequency** signal.



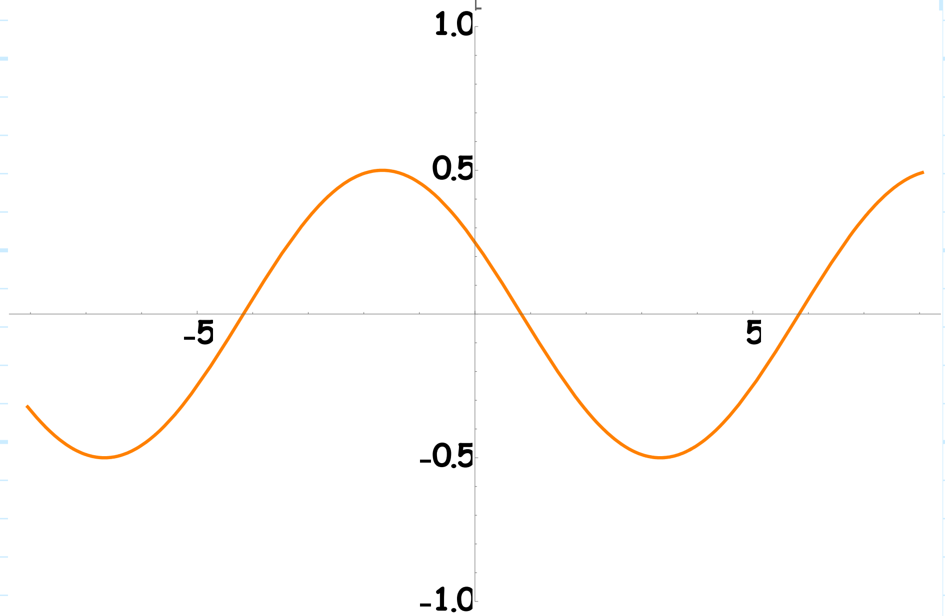
A low-frequency component

The output of the **filter** is thus simply the **low-frequency component** of $v_{IF}(t)$.

$$v_{IF}(t) = \tfrac{1}{2}\cos\left(0.2\pi\, t + \pi/3\right) + \tfrac{1}{2}\cos\left(4.2\pi\, t + \pi/3\right)$$

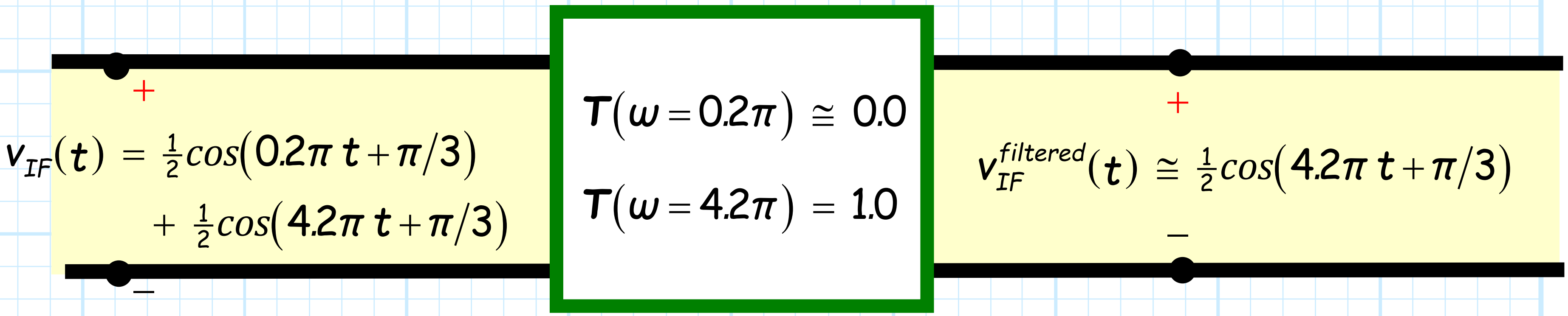


$$v_{IF}^{filtered}(t) \cong \tfrac{1}{2}\cos\left(0.2\pi\, t + \pi/3\right)$$



Or, the high-frequency component

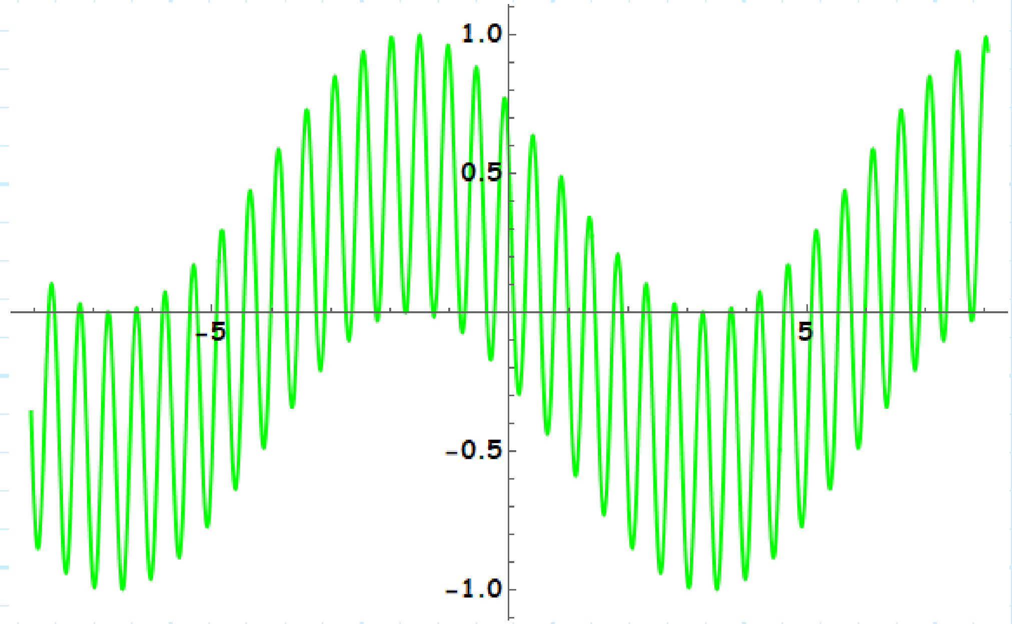
Alternatively, we could pass the signal $v_{IF}(t)$ through a bandpass filter that rejects the **low-frequency** signal.



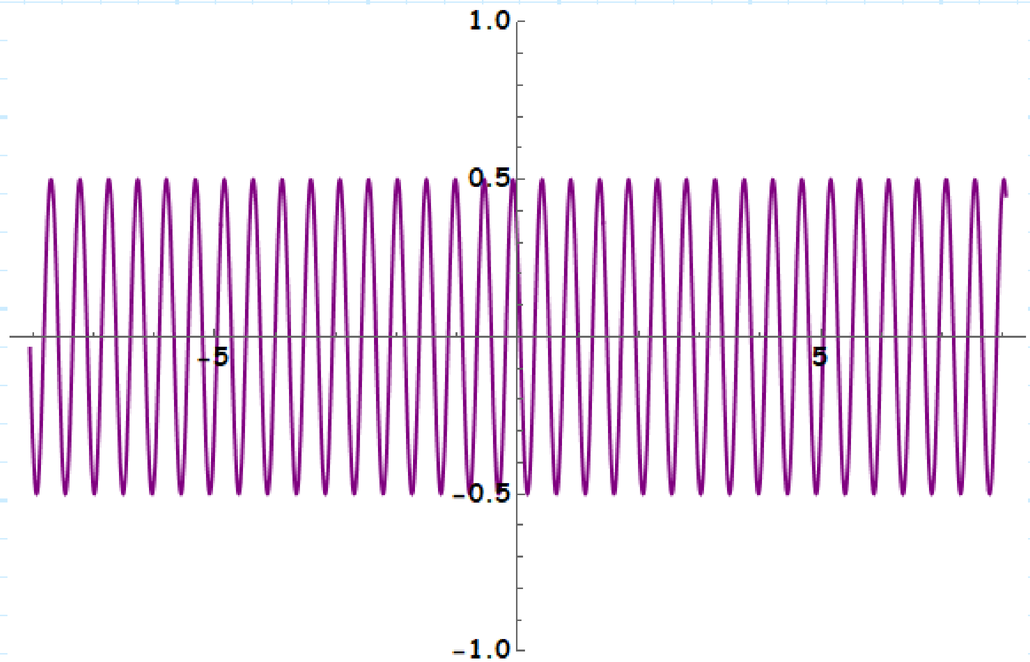
The output of the **filter** is thus simply the **high-frequency component** of $v_{IF}(t)$.

For example…

$$v_{IF}(t) = \frac{1}{2}cos\left(0.2\pi\, t + \pi/3\right) + \frac{1}{2}cos\left(4.2\pi\, t + \pi/3\right)$$



$$v_{IF}^{filtered}(t) \cong \frac{1}{2}cos\left(4.2\pi\, t + \pi/3\right)$$



These don't behave like eigen functions!

Q: *Wait!*

You *made a big, dramatic deal about* ***sinusoids*** *being "****eigen functions****".*

You *said that a sinusoid with a frequency* ω *at* ***one*** *place in a circuit would mean that* ***everywhere*** *in the circuit one would see the* ***same*** *sinusoid with the* ***exact same frequency*** ω.

But.

You *now provide an* ***example*** *where the input frequencies are* $\omega = 2.0\pi$ *and* $\omega = 2.2\pi$ *—yet the output frequencies are a* ***remarkably different*** $\omega = 0.2\pi$ *and* $\omega = 4.2\pi$*!*

You *must be* ***really confused*** *about this "eigen function" thing?*

A mixer is NOT a linear device!

A: No confusion.

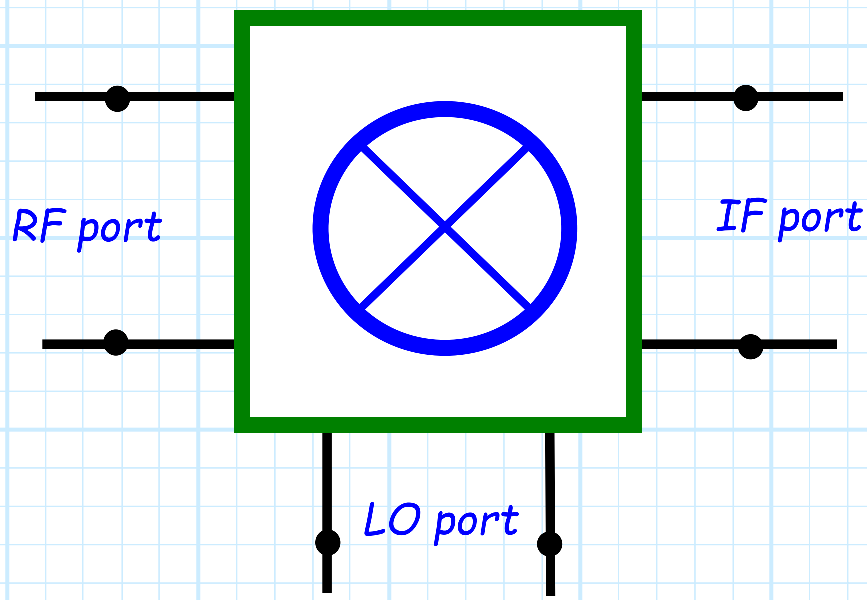
Sinusoids are the eigen functions of **ALL** linear, time-invariant circuits—but **ONLY** of linear, time-invariant circuits.

The mathematical operation:

$$v_{IF}(t) = v_{RF}(t)\, v_{LO}(t)$$

is decidedly **NON-linear**.

This means that an **ideal mixer** is likewise a **non-linear circuit**—the **frequencies** of a sinusoid **can** thus **be altered**!



A switch is a multiplier

Q: *OK, so I'll return to my **first question**:*

*a) **how** is this **possible?***

A: Initially, it indeed appears the math associated with "**signal multiplication**":

$$v_{IF}(t) = v_{RF}(t)\, v_{LO}(t)$$

is **inconsistent** with any electrical engineering **circuit element** (e.g., resistor, capacitors, inductors, transistors, diodes), **nor** with any **microwave** components that we have **previously studied**.

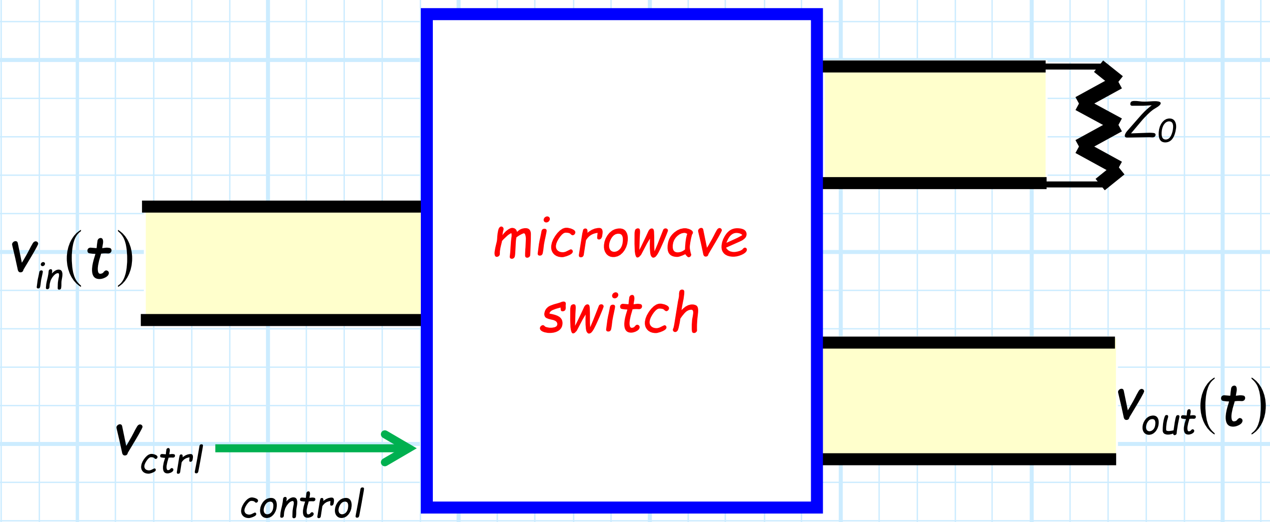
But, this **initial** appearance is **not** correct!

We in fact can achieve **signal multiplication** with the microwave device we **just** examined—the **microwave switch**!

Q: *?????*

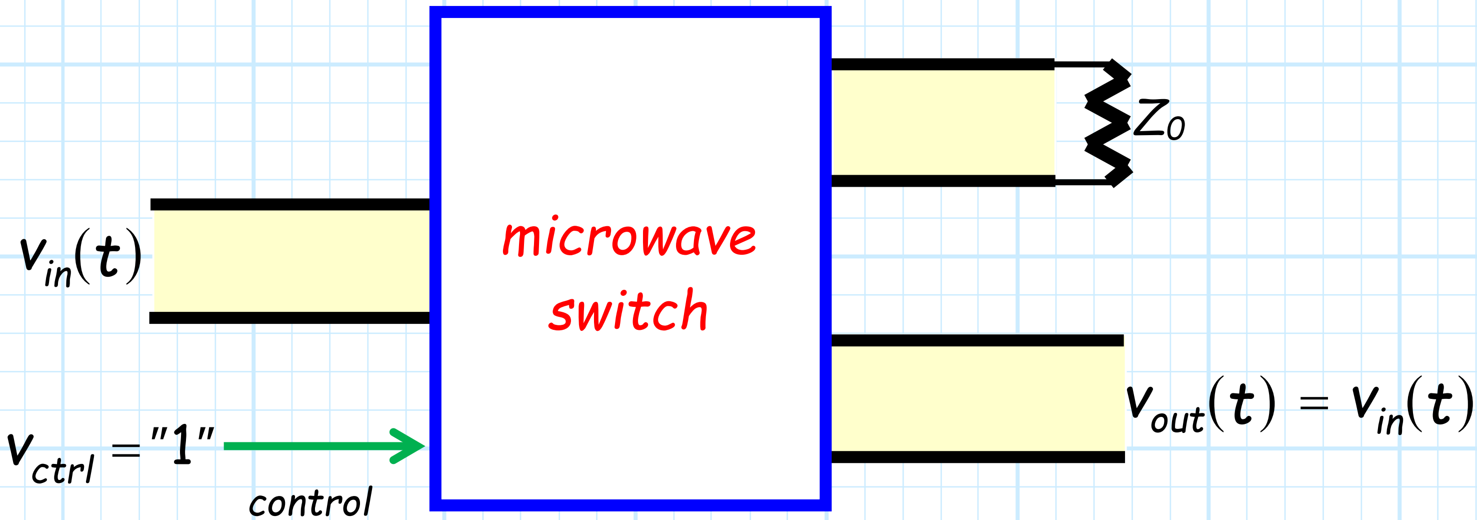
Again, a microwave switch

A: Consider this **switch** with an **input** voltage $v_{in}(t)$ and an output **voltage** $v_{out}(t)$:



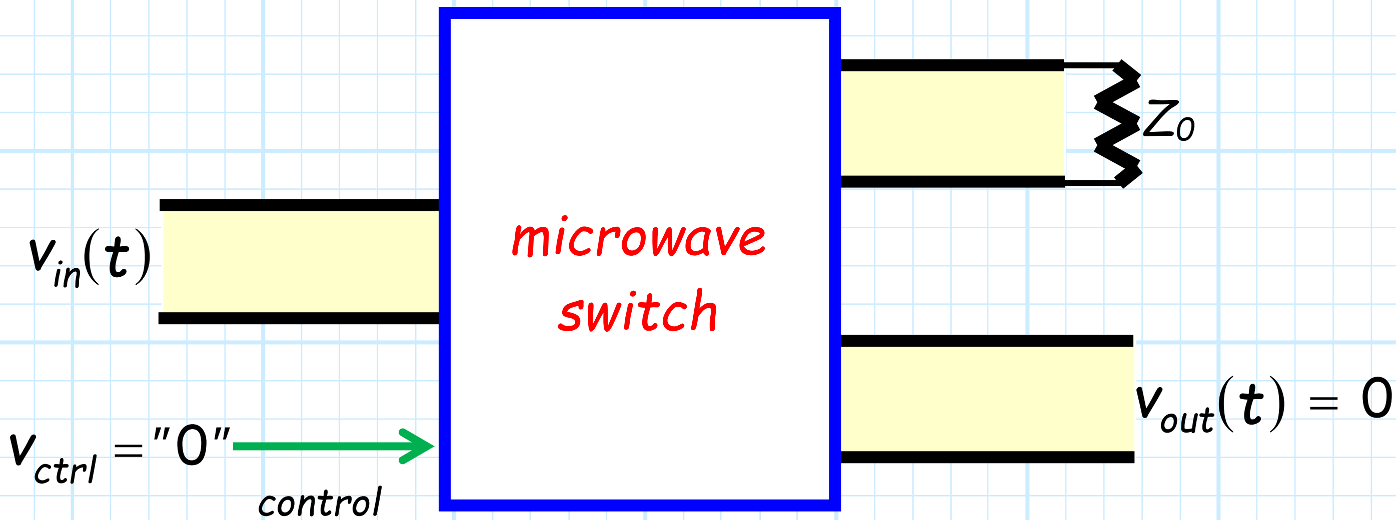
This state: the output is equal to the input

For **one switch state**, the output voltage (at least ideally) is **equal** to the the input voltage:



This state: the output is equal to zero

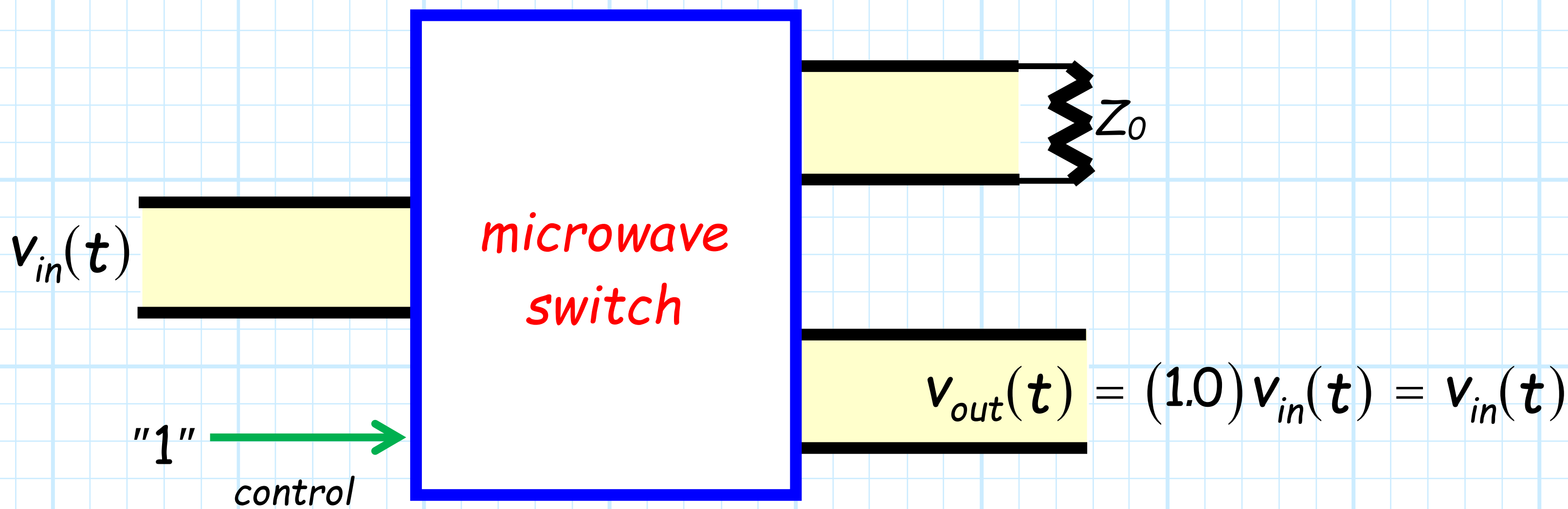
While for the **other** state, the output voltage is **zero**:



It's like it multiplies by one…

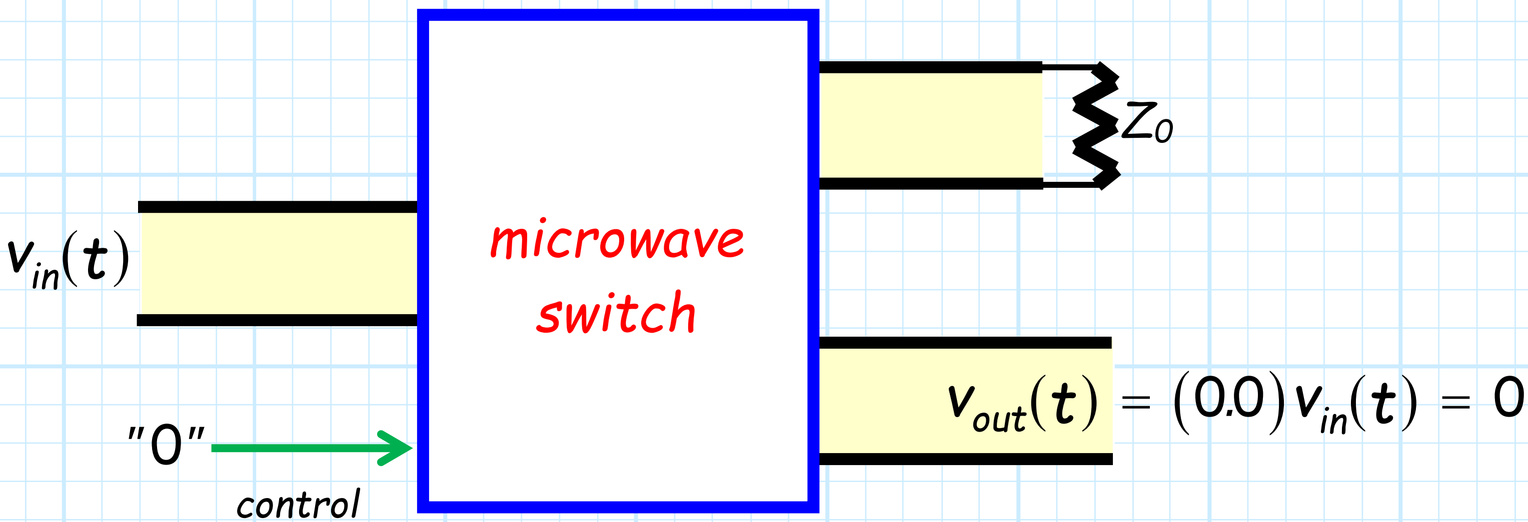
Although we **know** these results are determined from the internal circuitry of the switch (e.g., diodes or transistors), we **could** interpret these results in **another way**.

We **could** argue that for one state, the switch **multiplies** $v_{in}(t)$ with the value **1.0**:



...or it multiples by zero

While for the **other** state, the switch **multiplies** $v_{in}(t)$ with the value **0.0**:



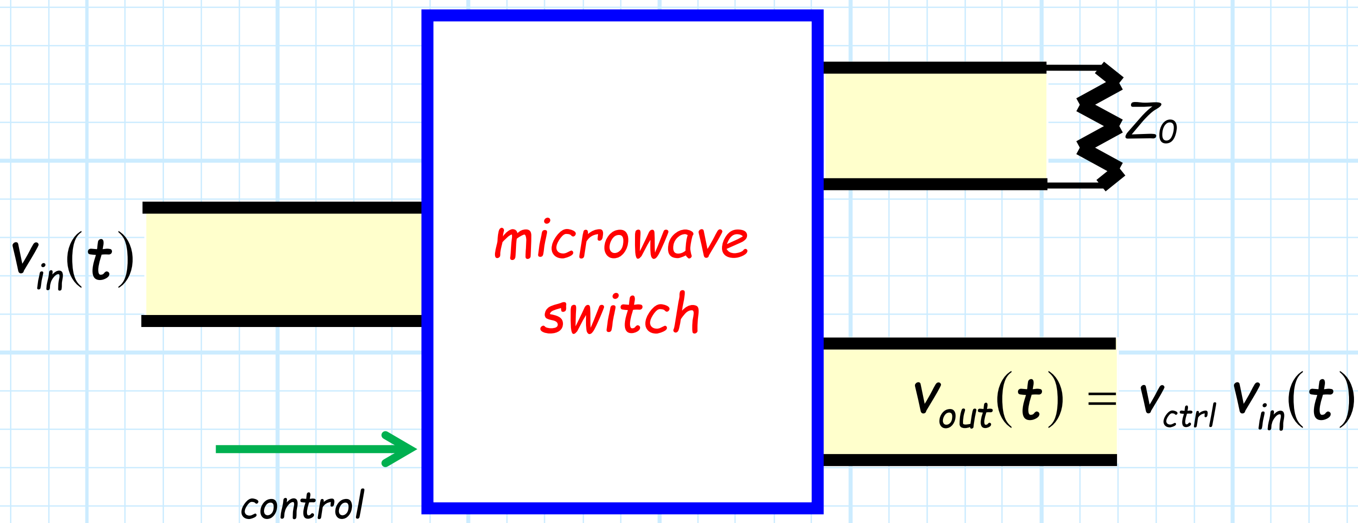
It's multiplied by a control voltage

Say now we define a **control** voltage v_{ctrl}, which can have **either one of two** states:

$$v_{ctrl} = 0.0 \qquad \text{or} \qquad v_{ctrl} = 1.0$$

we can write the switch output as:

$$v_{out}(t) = v_{ctrl}\, v_{in}(t)$$



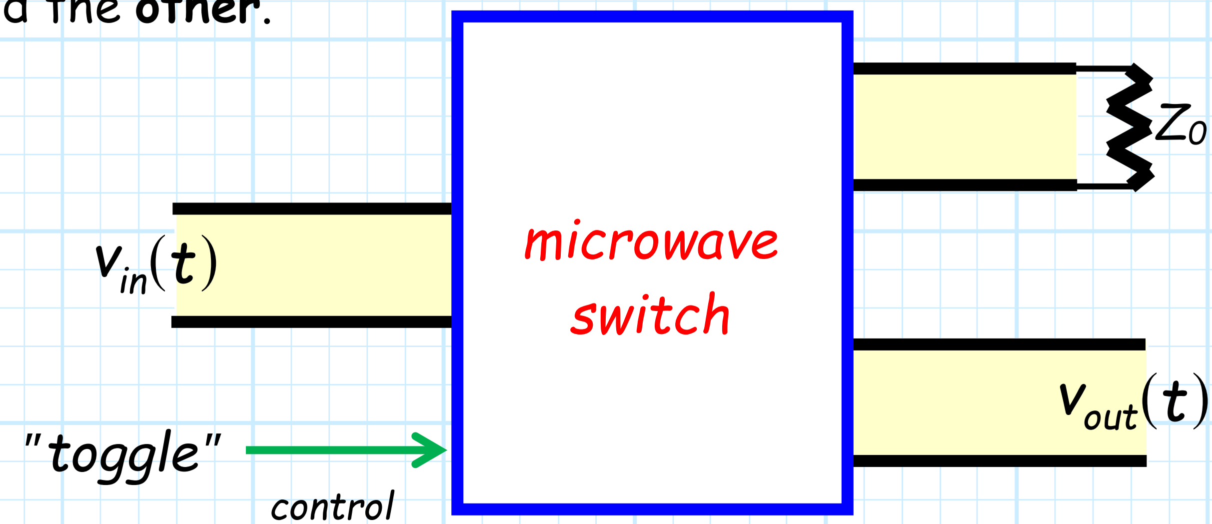
The switch *appears* to multiply voltages v_{ctrl} and $v_{in}(t)$!

Toggle the control

Q: *This "multiplier" seems quite **useless**; nothing at all like:*

$$\begin{aligned} v_{IF}(t) &= cos(2.2\pi\, t + \pi/3)\ cos(2.0\pi\, t) \\ &= \tfrac{1}{2} cos(0.2\pi\, t + \pi/3) + \tfrac{1}{2} cos(4.2\pi\, t + \pi/3) \end{aligned}$$

A: But, say we **continuously "toggle"** the switch control, between **one** state and the **other**.

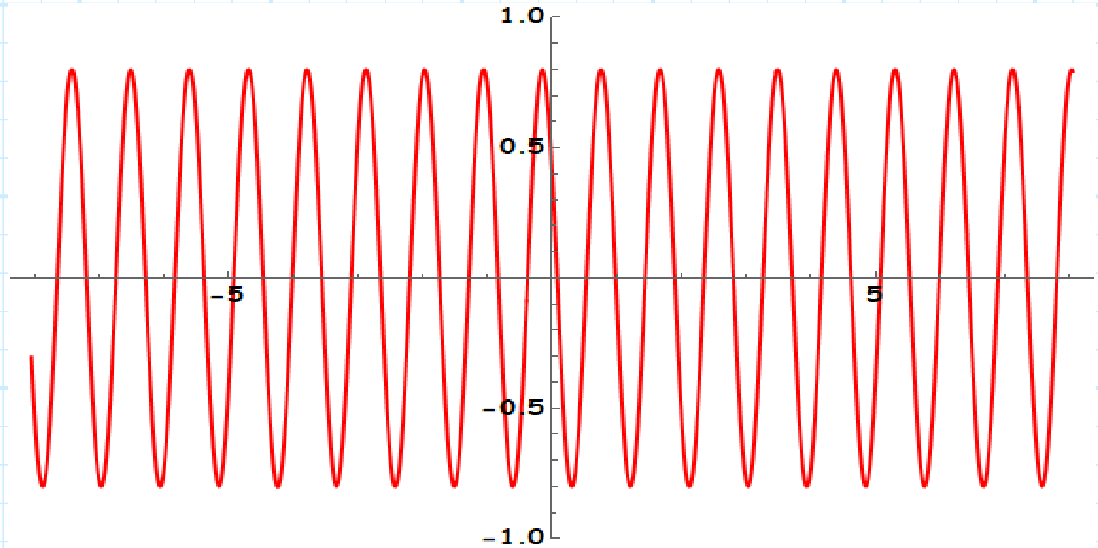


The output will thus **also toggle** between a value of $v_{in}(t)=0$ and $v_{in}(t)=v_{out}(t)$.

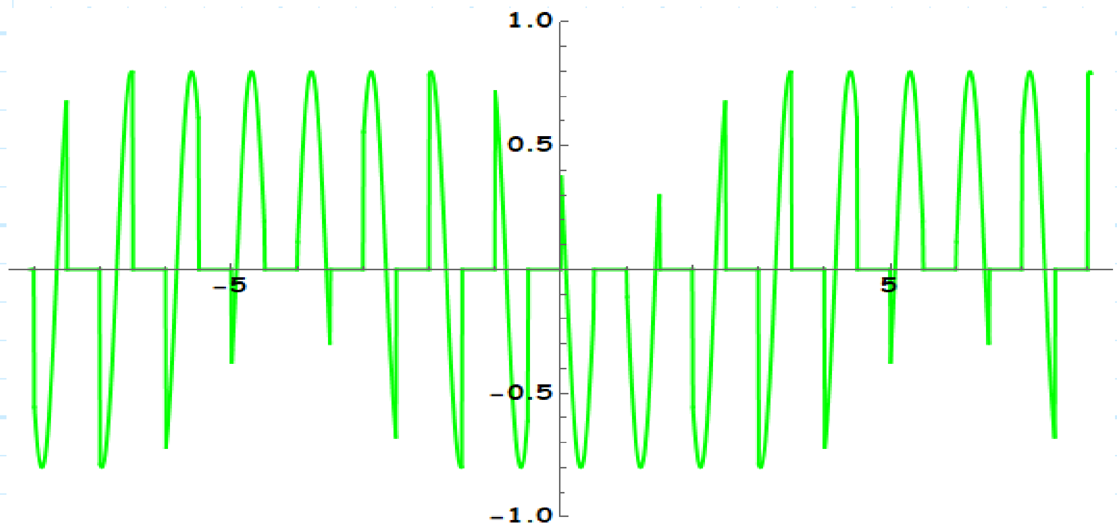
We "chop" the input

Say also that we toggle at a **constant rate**, and with a **50% duty cycle**. The **output** will thus be a "**chopped**" version of the **input**:

$$v_{in}(t) =$$
$$0.8\cos(2.2\pi t + \pi/3)$$



$v_{out}(t)$



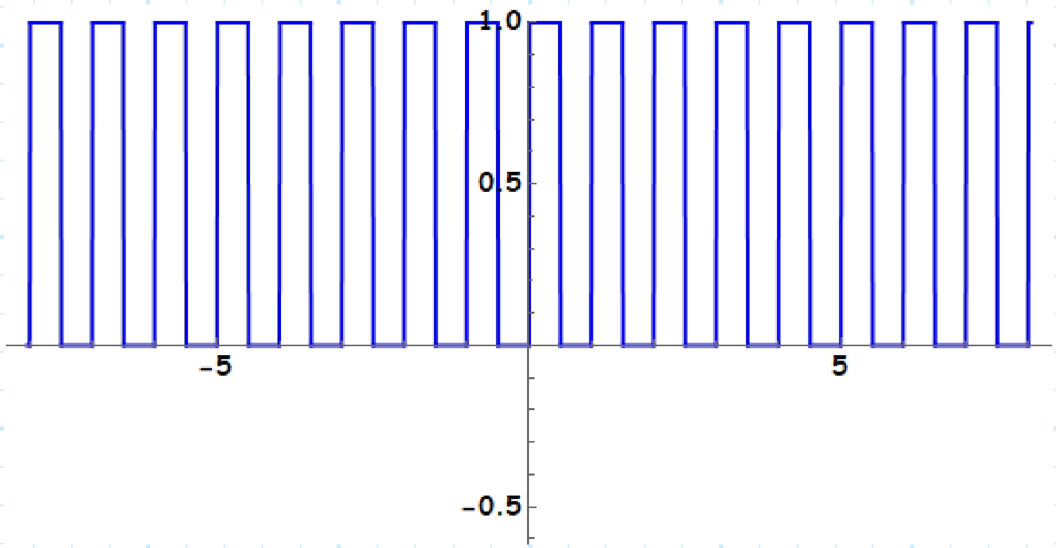
Q: *??????*

"Chopping" is multiplying by a square wave

A: This "**chopped**" result can likewise be **interpreted as multiplication**!

The switch multiplies the **input** voltage $v_{in}(t)$ with a **control voltage** v_{ctrl} —a control voltage that **toggles** between a value of $v_{ctrl}=0.0$ and $v_{ctrl}=1.0$!

$v_{ctrl}(t)$



The control voltage is a "**square wave**"!

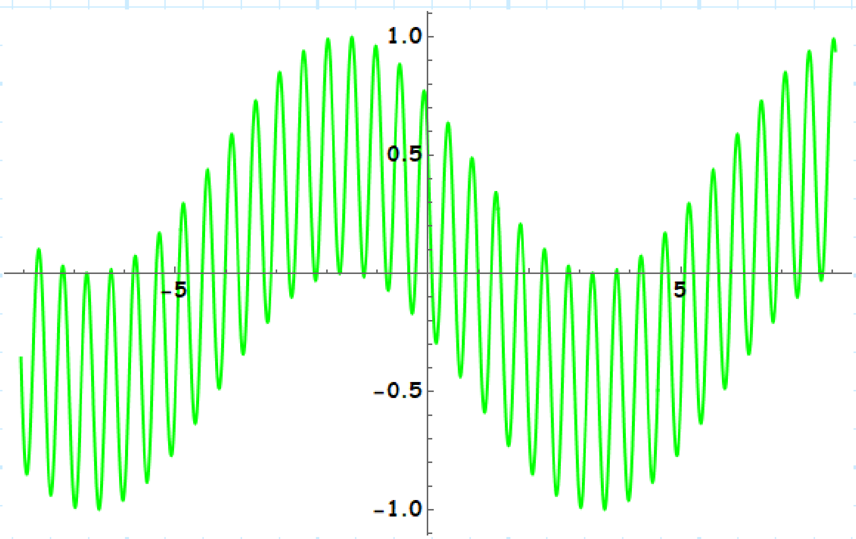
One signal must be a square wave

Thus, a **microwave switch** can be used as a device that **multiplies** some arbitrary **input** signal $v_{in}(t)$ with some arbitrary **square wave** $v_{ctrl}(t)$!

Q: *I don't understand.*

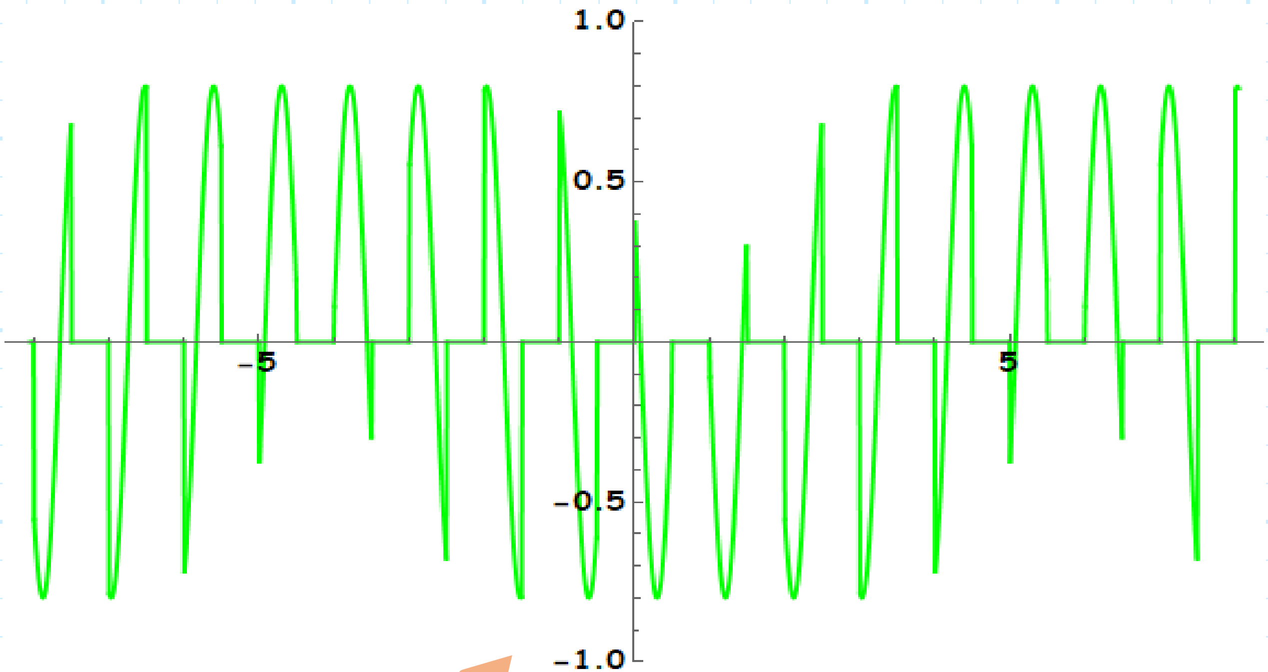
You *said a mixer can be used to multiply* ***two sinusoids****, giving the quite useful output:*

$$\begin{aligned} v_{IF}(t) &= v_{RF}(t)\, v_{LO}(t) \\ &= \cos(2.2\pi t + \pi/3)\ \cos(2.0\pi t) \\ &= \tfrac{1}{2}\cos(0.2\pi t + \pi/3) + \tfrac{1}{2}\cos(4.2\pi t + \pi/3) \end{aligned}$$



Apparently, both ugly and useless

*But, this **switch "multiplier"** can **only** multiply a sinusoid and a **square wave**, giving a result **both ugly** and apparently **useless**:*



A: But **this** result is **not** useless!

As usual, the Fourier transform reveals all

The **usefulness** of this **switch multiplier output** is **revealed** by examining it in the **frequency domain**—i.e., by evaluating its **Fourier Transform**:

$$V_{out}(\omega) = \int_{-\infty}^{\infty} v_{out}(t)\, e^{+j\omega t}\, dt = \int_{-\infty}^{\infty} v_{in}(t)\, v_{ctrl}(t)\, e^{+j\omega t}\, dt$$

Q: *I'm slightly* ***embarrassed*** *to admit that I'm not really sure* ***how*** *to* ***evaluate*** *this integral.*

A: ***YOU*** of course recall that:

> the Fourier Transform of the **product of two functions** is simply the **convolution** of the **Fourier Transform** of each separate function!

Trust me, it makes things WAY easier

In other "words", for our **multiplier**:

$$V_{out}(\omega) = V_{in}(\omega) * V_{ctrl}(\omega)$$

where:

$$V_{in}(\omega) = \int_{-\infty}^{\infty} v_{in}(t)\, e^{+j\omega t}\, dt \qquad \text{and} \qquad V_{ctrl}(\omega) = \int_{-\infty}^{\infty} v_{ctrl}(t)\, e^{+j\omega t}\, dt$$

This makes things **much easier**.

Q: *Umm…it **appears** to me to instead to make things **much more** difficult.*

A: Not for **this** problem! (**You can skip ahead to page 50 if you want**).

Consider arbitrary sinusoid at the input

Consider an **arbitrary sinusoidal** input function of the form:

$$v_{in}(t) = A_{in} \cos(\omega_{in} t + \varphi_{in})$$

From **Euler's identity**, this input can be expressed as:

$$\begin{aligned} v_{in}(t) &= A_{in} \cos(\omega_{in} t + \varphi_{in}) \\ &= \tfrac{1}{2} A_{in} \left(e^{+j(\omega_{in} t + \varphi_{in})} + e^{-j(\omega_{in} t + \varphi_{in})} \right) \\ &= \tfrac{1}{2} A_{in}\, e^{+j\varphi_{in}}\; e^{+j\omega_{in} t} + \tfrac{1}{2} A_{in}\, e^{-j\varphi_{in}}\; e^{-j\omega_{in} t} \end{aligned}$$

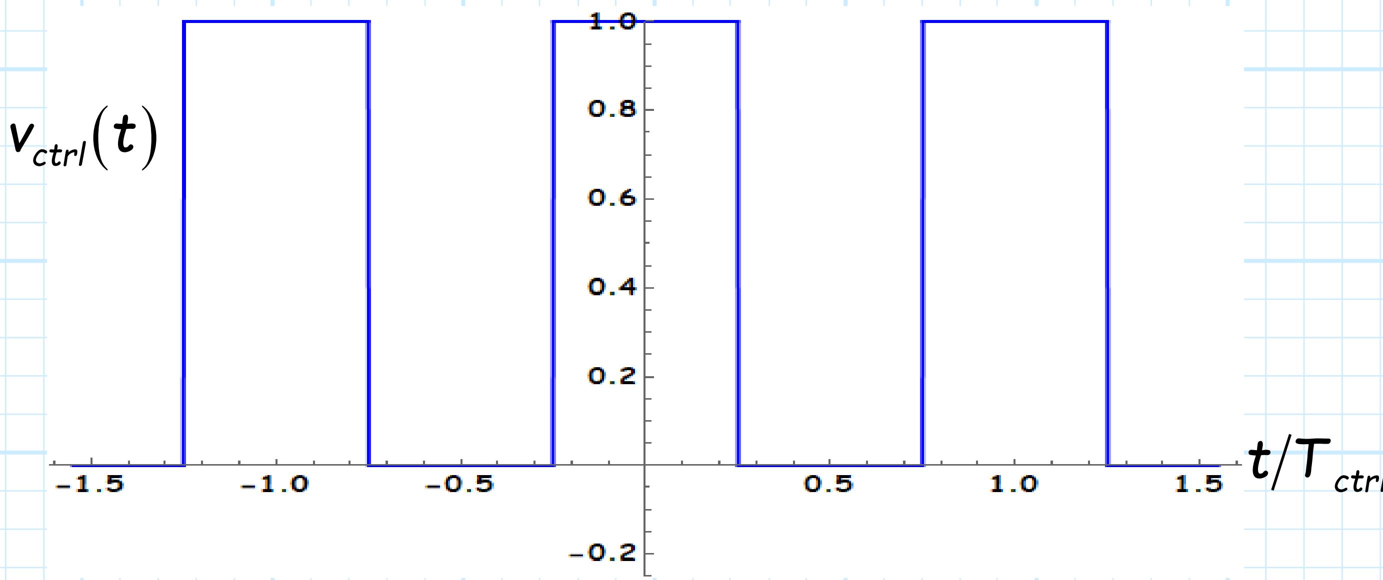
Impulse functions—make sure you understand them!

The **Fourier Transform** of this arbitrary sinusoid is thus:

$$\begin{aligned}
V_{in}(\omega) &= \int_{-\infty}^{\infty} v_{in}(t)\, e^{+j\omega t}\, dt \\
&= \int_{-\infty}^{\infty} \left(\tfrac{1}{2} A_{in}\, e^{+j\varphi_{in}}\ e^{+j\omega_{in} t} + \tfrac{1}{2} A_{in}\, e^{-j\varphi_{in}}\ e^{-j\omega_{in} t} \right) e^{+j\omega t}\, dt \\
&= \tfrac{1}{2} A_{in}\, e^{+j\varphi_{in}} \int_{-\infty}^{\infty} e^{+j(\omega + \omega_{in})t}\, dt + \tfrac{1}{2} A_{in}\, e^{-j\varphi_{in}} \int_{-\infty}^{\infty} e^{+j(\omega - \omega_{in})t}\, dt \\
&= \tfrac{1}{2} A_{in} e^{-j\varphi_{in}}\, 2\pi\, \delta(\omega + \omega_{in}) + \tfrac{1}{2} A_{in} e^{+j\varphi_{in}}\, 2\pi\, \delta(\omega - \omega_{in}) \\
&= \pi\, A_{in} e^{-j\varphi_{in}}\, \delta(\omega + \omega_{in}) + \pi\, A_{in} e^{+j\varphi_{in}}\, \delta(\omega - \omega_{in})
\end{aligned}$$

A periodic square wave

Now the **square-wave** control voltage, which is **periodic** with time T_{ctrl}:



Mathematically, this is described as:

$$v_{LO}(t) = \begin{cases} 0 & (n-\tfrac{1}{2})T_{ctrl} < t < (n-\tfrac{1}{4})T_{ctrl} \\ 1 & (n-\tfrac{1}{4})T_{ctrl} < t < (n+\tfrac{1}{4})T_{ctrl} \\ 0 & (n+\tfrac{1}{4})T_{ctrl} < t < (n+\tfrac{1}{2})T_{ctrl} \end{cases}$$

where n is any arbitrary **integer**.

Its Fourier transform

The **Fourier transform** of this **square-wave** is therefore:

$$
\begin{aligned}
V_{ctrl}(\omega) &= \int_{-\infty}^{\infty} v_{ctrl}(t)\, e^{+j\omega t}\, dt \\
&= \sum_n \left(\int_{(n-\frac{1}{2})T_{ctrl}}^{(n-\frac{1}{4})T_{ctrl}} 0\, e^{+j\omega t}\, dt + \int_{(n-\frac{1}{4})T_{ctrl}}^{(n+\frac{1}{4})T_{ctrl}} 1\, e^{+j\omega t}\, dt + \int_{(n+\frac{1}{4})T_{ctrl}}^{(n+\frac{1}{2})T_{ctrl}} 0\, e^{+j\omega t}\, dt \right) \\
&= \sum_n \int_{(n-\frac{1}{4})T_{ctrl}}^{(n+\frac{1}{4})T_{ctrl}} e^{+j\omega t}\, dt
\end{aligned}
$$

Evaluating that integral

Evaluating the **integral** of this last result:

$$
\begin{aligned}
\int_{(n-\frac{1}{4})T_{ctrl}}^{(n+\frac{1}{4})T_{ctrl}} e^{+j\omega t}\, dt &= \frac{1}{j\omega}\, e^{+j\omega t}\Big|_{(n-\frac{1}{4})T_{ctrl}}^{(n+\frac{1}{4})T_{ctrl}} \\
&= \frac{1}{j\omega}\left(e^{+j\omega(n+\frac{1}{4})T_{ctrl}} - e^{+j\omega(n-\frac{1}{4})T_{ctrl}}\right) \\
&= \frac{e^{+jn\omega T_{ctrl}}}{j\omega}\left(e^{+j\omega T_{ctrl}/4} - e^{-j\omega T_{ctrl}/4}\right) \\
&= \frac{e^{+jn\omega T_{ctrl}}}{j\omega}\left(j2\sin\left[\omega T_{ctrl}/4\right]\right) \\
&= \frac{T_{ctrl}}{2}\left(\frac{\sin\left[\omega T_{ctrl}/4\right]}{\omega T_{ctrl}/4}\right) e^{+jn\omega T_{ctrl}}
\end{aligned}
$$

I'm kinda embarrassed

And so the **Fourier transform** of the **square wave** is:

$$
\begin{aligned}
V_{ctrl}(\omega) &= \int_{-\infty}^{\infty} v_{ctrl}(t)\, e^{+j\omega t}\, dt \\
&= \sum_n \int_{(n-\frac{1}{4})T_{ctrl}}^{(n+\frac{1}{4})T_{ctrl}} e^{+j\omega t}\, dt \\
&= \sum_n \frac{T_{ctrl}}{2} \left(\frac{\sin[\omega T_{ctrl}/4]}{\omega T_{ctrl}/4} \right) e^{+jn\omega T_{ctrl}} \\
&= \frac{T_{ctrl}}{2} \left(\frac{\sin[\omega T_{ctrl}/4]}{\omega T_{ctrl}/4} \right) \sum_n e^{+jn\omega T_{ctrl}}
\end{aligned}
$$

The **summation** can be shown (hand-waving time!) to be:

$$
\sum_{n=-\infty}^{\infty} e^{+jn\omega T_{ctrl}} = \frac{1}{T_{ctrl}} \sum_{n=-\infty}^{n=\infty} \delta\left(\omega - n\left(\frac{2\pi}{T_{ctrl}}\right)\right)
$$

The result

Defining:

$$f_{ctrl} \doteq \frac{1}{T_{ctrl}} \quad \text{and} \quad \omega_{ctrl} \doteq 2\pi f_{ctrl} = \frac{2\pi}{T_{ctrl}}$$

this **summation** is then:

$$\sum_n e^{+jn\omega T_{ctrl}} = \frac{1}{T_{ctrl}} \sum_n \delta\left(\omega - n\left(\tfrac{2\pi}{T_{ctrl}}\right)\right) = \frac{1}{T_{ctrl}} \sum_n \delta(\omega - n\,\omega_{ctrl})$$

And so **finally**:

$$\begin{aligned} V_{ctrl}(\omega) &= \int_{-\infty}^{\infty} v_{ctrl}(t)\, e^{+j\omega t}\, dt \\ &= \frac{T_{ctrl}}{2}\left(\frac{\sin[\omega T_{ctrl}/4]}{\omega T_{ctrl}/4}\right) \sum_n e^{+jn\omega T_{ctrl}} \\ &= \frac{1}{2}\left(\frac{\sin[\omega T_{ctrl}/4]}{\omega T_{ctrl}/4}\right) \sum_n \delta(\omega - n\,\omega_{ctrl}) \end{aligned}$$

In summary

So, we have determined the Fourier transform of the **input sinusoid**:

$$V_{in}(\omega) = \pi A_{in} e^{-j\varphi_{in}} \, \delta(\omega + \omega_{in}) + \pi A_{in} e^{+j\varphi_{in}} \, \delta(\omega - \omega_{in})$$

And also for the **control square wave**:

$$V_{ctrl}(\omega) = \frac{1}{2}\left(\frac{\sin[\omega T_{ctrl}/4]}{\omega T_{ctrl}/4}\right)\sum_{n}\delta(\omega - n\,\omega_{ctrl})$$

Not as difficult as it looks

The Fourier transform of the **output** signal $v_{out}(t)=v_{in}(t)\,v_{ctrl}(t)$ is therefore the **convolution** of these two spectra:

$$\begin{aligned} V_{out}(\omega) &= \int_{-\infty}^{\infty} v_{out}(t)\; e^{+j\omega t}\, dt \\ &= V_{in}(\omega) * V_{ctrl}(\omega) \\ &= \int_{-\infty}^{\infty} V_{in}(\omega-\omega')\,V_{ctrl}(\omega')\,d\omega' \end{aligned}$$

Q: *I thought you said this would be **easy**?*

A: It **is** easy!

The Dirac delta function!

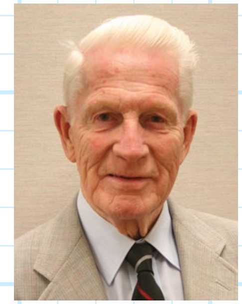
$$
\begin{aligned}
V_{out}(\omega) &= \int_{-\infty}^{\infty} V_{in}(\omega-\omega')\, V_{ctrl}(\omega')\, d\omega' \\
&= \int_{-\infty}^{\infty} \pi V_{in} e^{-j\varphi}\, \delta(\omega-\omega'-\omega_{in}) \sum_n \frac{1}{2}\left(\frac{\sin[\omega' T_{ctrl}/4]}{\omega' T_{ctrl}/4}\right) \delta(\omega'-n\omega_{ctrl})\, d\omega' \\
&\quad + \int_{-\infty}^{\infty} \pi V_{in} e^{+j\varphi}\, \delta(\omega-\omega'+\omega_{in}) \sum_n \frac{1}{2}\left(\frac{\sin[\omega' T_{ctrl}/4]}{\omega' T_{ctrl}/4}\right) \delta(\omega'-n\omega_{ctrl})\, d\omega' \\
&= V_{in} e^{-j\varphi} \frac{\pi}{2} \sum_n \int_{-\infty}^{\infty} \delta(\omega-\omega'-\omega_{in}) \left(\frac{\sin[\omega' T_{ctrl}/4]}{\omega' T_{ctrl}/4}\right) \delta(\omega'-n\omega_{ctrl})\, d\omega' \\
&\quad + V_{in} e^{+j\varphi} \frac{\pi}{2} \sum_n \int_{-\infty}^{\infty} \delta(\omega-\omega'+\omega_{in}) \left(\frac{\sin[\omega' T_{ctrl}/4]}{\omega' T_{ctrl}/4}\right) \delta(\omega'-n\omega_{ctrl})\, d\omega'
\end{aligned}
$$

Q: *I'm still **not** seeing easy **anywhere**.*

A: Look closely at the **integrals**—they involve **Dirac delta function** $\delta(x)$!

As a professor once said to me

And, integrating with the Dirac delta function is "**joy itself**"!



Recall that:

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(x=0)$$

In other words, we evaluate the **integral** simply by evaluating the **function** $f(x)$ at $x=0$ (the result $f(x=0)$ is thus a **number**).

A corollary to this result is the "**sampling integral**":

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-c)\,dx = f(x=c)$$

The integral returns the value of function $f(x)$, evaluated at some **arbitrary** value $x=c$ (e.g., $x=2.7$).

Cake simple

Happily, our integrals involve the Dirac delta function:

$$\delta(\omega' - n\,\omega_{ctrl})$$

Thus, we **simply** evaluate the remainder of the integrand at $\omega' = n\,\omega_{ctrl}$:

$$\int_{-\infty}^{\infty} \delta(\omega - \omega' - \omega_{in}) \left(\frac{\sin[\omega' T_{ctrl}/4]}{\omega' T_{ctrl}/4} \right) \delta(\omega' - n\omega_{ctrl})\; d\omega'$$

$$= \;\; \delta(\omega - n\omega_{ctrl} - \omega_{in}) \left(\frac{\sin[n\omega_{ctrl} T_{ctrl}/4]}{n\omega_{ctrl} T_{ctrl}/4} \right)$$

Wait! It gets even better!

And so:

$$V_{out}(\omega) = V_{in} e^{-j\varphi} \frac{\pi}{2} \sum_n \delta(\omega - n\omega_{ctrl} - \omega_{in}) \left(\frac{\sin[n\omega_{ctrl} T_{ctrl}/4]}{n\omega_{ctrl} T_{ctrl}/4} \right)$$

$$+ \; V_{in} e^{+j\varphi} \frac{\pi}{2} \sum_n \delta(\omega - n\omega_{ctrl} + \omega_{in}) \left(\frac{\sin[n\omega_{ctrl} T_{ctrl}/4]}{n\omega_{ctrl} T_{ctrl}/4} \right)$$

Now, recalling that:

$$\omega_{ctrl} \doteq 2\pi f_{ctrl} = \frac{2\pi}{T_{ctrl}}$$

So that:

$$n\,\omega_{ctrl} T_{ctrl} = n\,2\pi$$

Simple

We can express:

$$\frac{\sin\left[n\omega_{ctrl}\,T_{ctrl}/4\right]}{n\omega_{ctrl}\,T_{ctrl}/4} = \frac{\sin\left[n2\pi/4\right]}{n2\pi/4n\omega_{ctrl}} = \frac{\sin\left[n(\pi/2)\right]}{n(\pi/2)}$$

So that finally:

$$\begin{aligned} V_{out}(\omega) &= V_{in}e^{-j\varphi}\frac{\pi}{2}\sum_{n}\delta\left(\omega - n\omega_{ctrl} - \omega_{in}\right)\frac{\sin\left[n(\pi/2)\right]}{n(\pi/2)} \\ &+ V_{in}e^{+j\varphi}\frac{\pi}{2}\sum_{n}\delta\left(\omega - n\omega_{ctrl} + \omega_{in}\right)\frac{\sin\left[n(\pi/2)\right]}{n(\pi/2)} \end{aligned}$$

Even values result in zero

Evaluating over different **integers *n***:

n	$\dfrac{\sin\left[n(\pi/2)\right]}{n(\pi/2)}$
-4	0
-3	$2/(3\pi)$
-2	0
-1	$2/\pi$
0	1.0
+1	$2/\pi$
+2	0
+3	$2/(3\pi)$
+4	0

Even simpler

It is apparent that all (non-zero) **even** values of integer n result in a **zero** value, and that odd values of n result in $\left(2/|n|\pi\right)$.

Therefore:

$$\begin{aligned} V_{out}(\omega) \;=\; & V_{in} e^{-j\varphi} \frac{\pi}{2}\left(\delta(\omega-\omega_{in}) + \sum_{n\in odd} \frac{2}{\pi|n|}\delta(\omega - n\omega_{ctrl} - \omega_{in})\right) \\ & + V_{in} e^{+j\varphi} \frac{\pi}{2}\left(\delta(\omega+\omega_{in}) + \sum_{n\in odd} \frac{2}{\pi|n|}\delta(\omega - n\omega_{ctrl} + \omega_{in})\right) \end{aligned}$$

Rearrange the delta functions

Q: *I'm not sure I understand what those* ***Dirac delta functions*** *mean.*

A: Remember, the value of $\delta(x)$ is zero at all $x \neq 0$.

Thus, the Dirac delta functions in the above Fourier transform indicate **zero** energy for all frequencies—**except** those **specific few** which make the argument of the delta function **zero**!

Q: *Just what "specific few" frequencies* ***are*** *those?*

A: Let's **rewrite** the Dirac delta function to see them **clearly**.

$$\delta(\omega - n\omega_{ctrl} - \omega_{in}) = \delta(\omega - (n\omega_{ctrl} + \omega_{in}))$$

$$\delta(\omega - n\omega_{ctrl} + \omega_{in}) = \delta(\omega - (n\omega_{ctrl} - \omega_{in}))$$

Zero almost everywhere

$$
\begin{aligned}
V_{out}(\omega) \;=\; & V_{in} e^{-j\varphi} \frac{\pi}{2} \left(\delta(\omega - \omega_{in}) + \sum_{n \in odd} \frac{2}{\pi |n|} \delta(\omega - (n\omega_{ctrl} + \omega_{in})) \right) \\
& + V_{in} e^{+j\varphi} \frac{\pi}{2} \left(\delta(\omega + \omega_{in}) + \sum_{n \in odd} \frac{2}{\pi |n|} \delta(\omega - (n\omega_{ctrl} - \omega_{in})) \right)
\end{aligned}
$$

The Fourier transform of $v_{out}(t)$ is therefore **non-zero only** at frequencies:

$$\omega = n\omega_{ctrl} + \omega_{in}$$

and also:

$$\omega = n\omega_{ctrl} - \omega_{in}$$

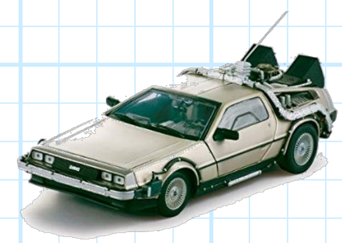
Or equivalently, they are non-zero at **these** "specific few" frequencies:

$$\omega = n\omega_{ctrl} \pm \omega_{in}$$

Back to the time domain!

To see this more explicitly, let's take the **Inverse Fourier transform** of this result, returning again to a **real-valued time domain function**:

$$
\begin{aligned}
v_{out}(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} V_{out}(\omega)\, e^{-j\omega t}\; d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} V_{in} e^{-j\varphi}\, \frac{\pi}{2}\left(\delta(\omega-\omega_{in}) + \sum_{n\in odd}\frac{2}{\pi|n|}\,\delta\big(\omega-(n\,\omega_{ctrl}+\omega_{in})\big)\right) e^{-j\omega t} d\omega \\
&\quad + \frac{1}{2\pi}\int_{-\infty}^{\infty} V_{in} e^{+j\varphi}\, \frac{\pi}{2}\left(\delta(\omega+\omega_{in}) + \sum_{n\in odd}\frac{2}{\pi|n|}\,\delta\big(\omega-(n\,\omega_{ctrl}-\omega_{in})\big)\right) e^{-j\omega t} d\omega \\
&= \frac{V_{in}}{4} e^{+j\varphi}\left(e^{-j\omega_{in}t} + \sum_{n\in odd}\frac{2}{\pi|n|}\, e^{-j(n\omega_{ctrl}+\omega_{in})t}\right) \\
&\quad + \frac{V_{in}}{4} e^{-j\varphi}\left(e^{+j\omega_{in}t} + \sum_{n\in odd}\frac{2}{\pi|n|}\, e^{+j(n\omega_{ctrl}+\omega_{in})t}\right) \\
&= \frac{V_{in}}{4}\left(e^{-j[\omega_{in}t-\varphi]} + \sum_{n\in odd}\frac{2}{\pi|n|}\, e^{-j[(n\omega_{ctrl}+\omega_{in})t-\varphi]}\right) \\
&\quad + \frac{V_{in}}{4}\left(e^{+j[\omega_{in}t-\varphi]} + \sum_{n\in odd}\frac{2}{\pi|n|}\, e^{+j[(n\omega_{ctrl}+\omega_{in})t-\varphi]}\right)
\end{aligned}
$$



Sinusoids!

Now applying **Euler's equation** to this output signal:

$$
\begin{aligned}
v_{out}(t) &= \frac{V_{in}}{4}\left(e^{-j[\omega_{in}t-\varphi]} + \sum_{n\in odd} \frac{2}{\pi|n|} e^{-j[(n\omega_{ctrl}+\omega_{in})t-\varphi]} \right) \\
&\quad + \frac{V_{in}}{4}\left(e^{+j[\omega_{in}t-\varphi]} + \sum_{n\in odd} \frac{2}{\pi|n|} e^{+j[(n\omega_{ctrl}+\omega_{in})t-\varphi]} \right) \\
&= \frac{V_{in}}{4}\left(e^{-j[\omega_{in}t-\varphi]} + e^{+j[\omega_{in}t-\varphi]} \right) \\
&\quad + \frac{V_{in}}{4}\left(\sum_{n\in odd} \frac{2}{\pi|n|} e^{-j[(n\omega_{ctrl}+\omega_{in})t-\varphi]} + \sum_{n\in odd} \frac{2}{\pi|n|} e^{+j[(n\omega_{ctrl}+\omega_{in})t-\varphi]} \right) \\
&= \frac{V_{in}}{2}\cos[\omega_{in}t-\varphi] + V_{in}\sum_{n\in odd} \frac{1}{\pi|n|}\cos[(n\omega_{ctrl}+\omega_{in})t-\varphi]
\end{aligned}
$$

We've seen these before!

Among the most "**energetic**" of these sinusoidal output terms are for $n \in \{-1,0,1\}$, such that the output can be approximated as:

$$v_{out}(t) \cong \frac{V_{in}}{2} cos[\omega_{in} t - \varphi] + \frac{V_{in}}{\pi} cos[(\omega_{in} - \omega_{ctrl})t - \varphi] + \frac{V_{in}}{\pi} cos[(\omega_{in} + \omega_{ctrl})t - \varphi]$$

Compare these last two terms with the output of an **ideal mixer**:

$$v_{IF}(t) = \tfrac{1}{2} cos[(\omega_{RF} - \omega_{LO})t] + \tfrac{1}{2} cos[(\omega_{RF} + \omega_{LO})t] \,!$$

Input is RF, output is IF, control is LO

If we equate the **input** signal with the RF signal:

$$v_{in}(t) = v_{RF}(t)$$

and the **control** signal with the **LO** signal:

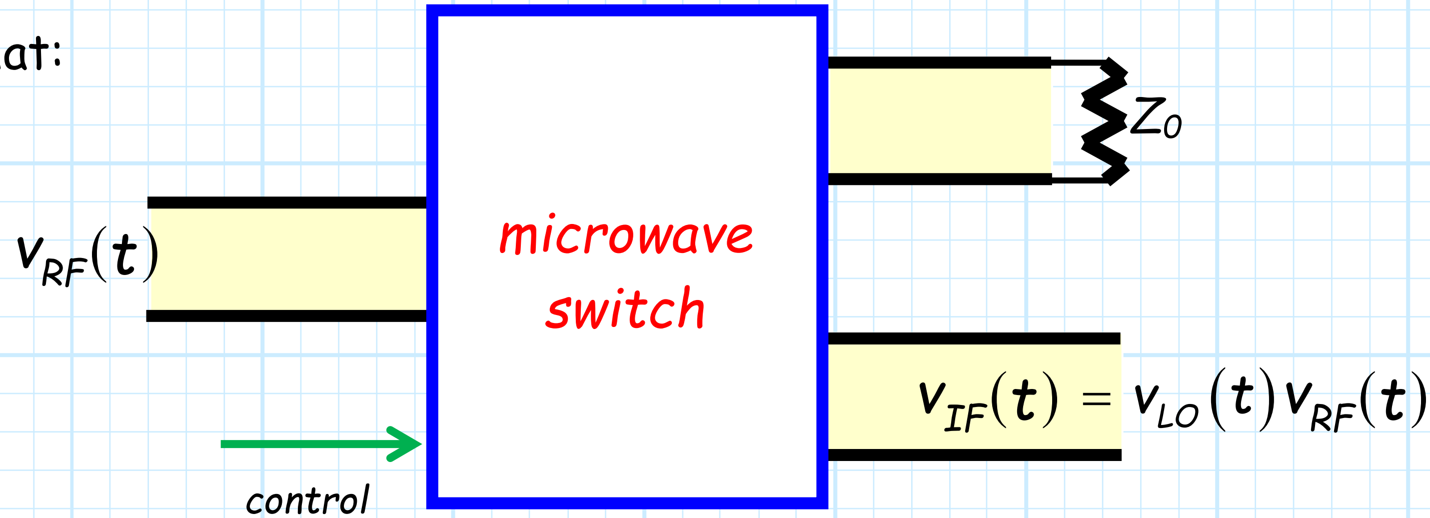
$$v_{ctrl}(t) = v_{LO}(t)$$

and the **output** signal with the **IF** signal:

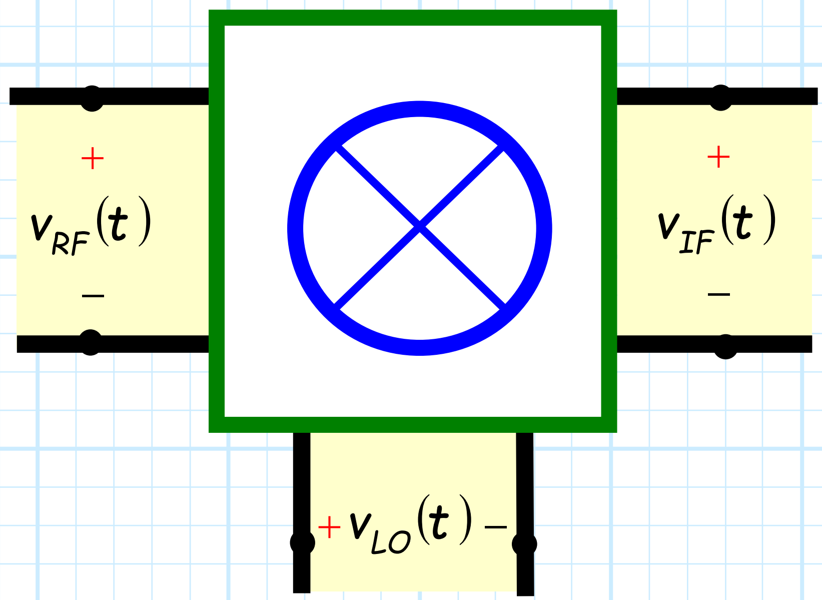
$$v_{out}(t) = v_{IF}(t)$$

A microwave mixer is a toggled switch

Such that:



We see the output of our "toggled" switch is **similar** to an ideal mixer:



The two signals we need

Each device produces a sinusoid that is the **sum** (ω_Σ) of the two frequencies:

$$\cos\left[\left(\omega_{RF}+\omega_{LO}\right)t\right] = \cos\left[\left(\omega_{in}+\omega_{ctrl}\right)t\right]$$

and a sinusoid that is the **difference** (ω_Δ) of the two frequencies:

$$\cos\left[\left(\omega_{RF}-\omega_{LO}\right)t\right] = \cos\left[\left(\omega_{in}-\omega_{ctrl}\right)t\right]$$

A **"toggled"** switch can be used to create similar signals as an **ideal mixer!**

NOT an ideal mixer term

Q: *Yes, but this toggled switch mixer seems also to generate some* ***other*** *signals as well. Most notably a one at* ***input frequency*** $\omega_{in}\left(\omega_{RF}\right)$:

$$\frac{V_{in}}{2}\cos\left[\omega_{in}t-\varphi\right] \;=\; \frac{V_{RF}}{2}\cos\left[\omega_{RF}t-\varphi\right]$$

A: Yes it does.

Q: *Is that a* ***bad*** *thing?*

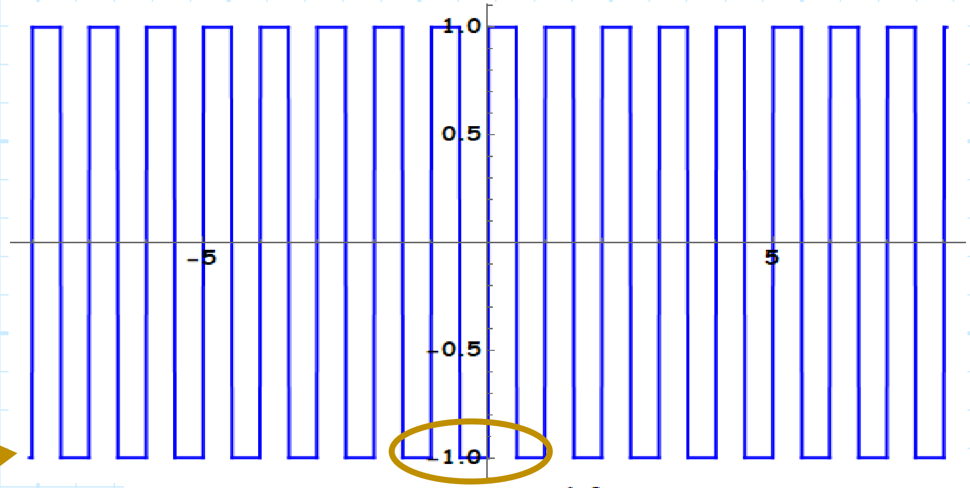
A: **Yes** it is.

Among other things, this "extra" signal **reduces the power** of the two desired signals (i.e., **conservation of energy** at work):

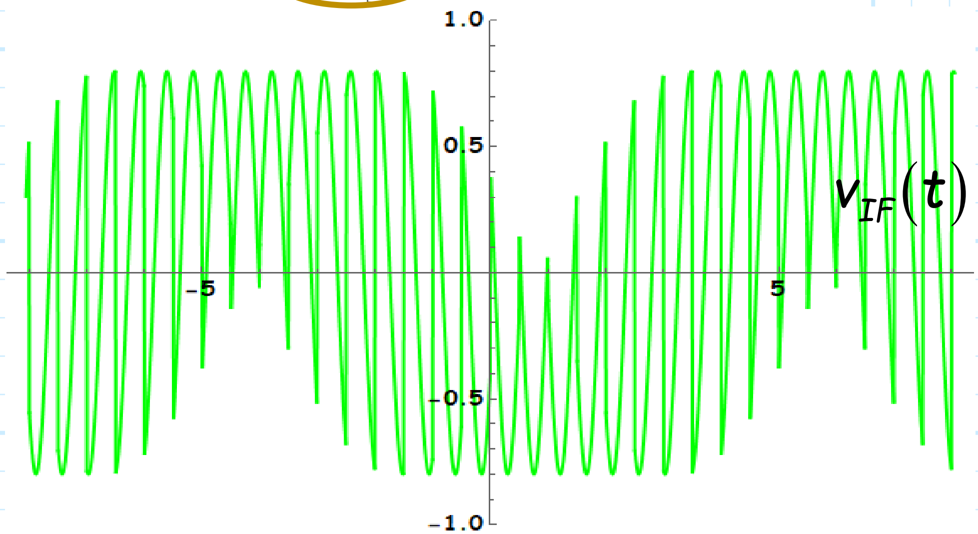
$$\frac{V_{RF}}{\pi}\cos\left[\left(\omega_{RF}-\omega_{LO}\right)t-\varphi\right] \qquad \text{and} \qquad \frac{V_{in}}{\pi}\cos\left[\left(\omega_{RF}+\omega_{LO}\right)t-\varphi\right]$$

A different square wave

To “fix” this problem, we can **instead** “multiply” the input with a **different** square wave—one instead that toggles between a value of 1.0 and **-1.0**.



$$v_{LO}(t)$$



$$v_{IF}(t) = v_{RF}(t)v_{LO}(t)$$

I'll let you verify this

Reworking all the math, **you** will find that the IF (i.e., the output) is now:

$$v_{IF}(t) = V_{RF} \sum_{n \in odd} \frac{2}{\pi |n|} \cos\left[\left(n\omega_{LO} + \omega_{RF}\right)t - \varphi\right]$$

So that the IF (i.e., the output) is **approximately**:

$$v_{IF}(t) \cong V_{RF} \frac{2}{\pi} \cos\left[\left(\omega_{RF} - \omega_{LO}\right)t - \varphi\right] + V_{RF} \frac{2}{\pi} \cos\left[\left(\omega_{RF} + \omega_{LO}\right)t - \varphi\right]$$

Note the ω_{RF} sinusoid is **missing**, and the remaining two terms are **twice as large**!

Balanced and unbalanced

The first result (where the LO toggles between 1.0 and **0.0**), is known as an **unbalanced mixer**:

$$
\begin{aligned}
v_{IF}^{unbal}(t) \cong\ & \frac{V_{RF}}{2}\cos\left[\omega_{RF}t-\varphi_{RF}\right] \\
& + \frac{V_{RF}}{\pi}\cos\left[\left(\omega_{RF}-\omega_{LO}\right)t-\varphi_{RF}\right] + \frac{V_{RF}}{\pi}\cos\left[\left(\omega_{in}+\omega_{ctrl}\right)t-\varphi_{RF}\right]
\end{aligned}
$$

This last result (where the LO toggles between 1.0 and **-1.0**), is known as a **balanced mixer**:

$$
v_{IF}^{bal}(t) \cong V_{RF}\frac{2}{\pi}\cos\left[\left(\omega_{RF}-\omega_{LO}\right)t-\varphi_{RF}\right] + V_{RF}\frac{2}{\pi}\cos\left[\left(\omega_{in}+\omega_{ctrl}\right)t-\varphi_{RF}\right]
$$

The vast majority of microwave mixers are "**balanced**"!

A balanced mixer!

Q: *But how can we **make** these balanced mixers? **How** can we multiply by -1.0?*

A: Remember, multiplying a sinusoid by -1 is the same thing as **phase shifting** it by π radians (i.e., $-1 = e^{j\pi}$).

Thus, we can **equivalently** say that a **balanced mixer** toggles between a 0° phase shift and a 180° phase shift!