PHSX 531: Homework #10

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Problem 1

A thick spherical shell (inner radius a, outer radius b) is made of dielectric material with a "frozen-in" polarization:

$$\mathbf{P}(r) = \frac{k}{r}\hat{r},$$

where k is a constant and r is the distance from the center. (There is no free charge in the problem). Find the electric field in all three regions by two different methods:

(a) (3 pts) Calculate the bound charges. Use Gauss's Law for electric fields to calculate the field the bound charges produce.

Solution: We will have bound charges

$$\rho_b = -\nabla \cdot \mathbf{P} = -\frac{k}{r^2}, \qquad \sigma_b = \mathbf{P} \cdot \hat{n} = \begin{cases} -k/r & r = b \\ k/r & r = a \end{cases}$$

Gauss's law says

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q_{enc}}{r^2}\hat{r}$$

Where Q_{enc} is given by integrals over the surfaces and volumes. It happens that enclosed charge equals zero inside and outside. Total charge vanishes in a neutral conductor, and this is proven in problem 4. This leaves me with the charge between the two boundaries:

$$\begin{aligned} Q_{enc} &= -4\pi\frac{k}{a}a^2 + \int_a^r 4\pi(-\frac{k}{r^2})r^2\, dr \\ &= -ka - k(r-a) \\ &= -kr \end{aligned}$$

Therefore we have electric field for each region

$$\mathbf{E} = \begin{cases} 0 & r < a \\ -\frac{k}{\epsilon_0 r} & a < r < b \\ 0 & r > b \end{cases}$$

(b) (3 pts) Use Gauss's Law in the presence of dielectrics ($\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f,enc}$) to first calculate the displacement $\mathbf{D}$. Then use the relationship between electric field, displacement, and polarization to find the E-field.

Solution:

Gauss's law in the presence of dielectrics is

$$\nabla \cdot \underbrace{(\epsilon_0 \mathbf{E} + \mathbf{P})}_{=\mathbf{D}} = \rho_f$$

$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{enc}}$$

The integral form is more useful for this problem, and it tells me that $\mathbf{D} = 0$, since there are no free charges. We can only have polarized material for $a < r < b$, so:

$$\mathbf{E} = -\frac{\mathbf{P}}{\epsilon_0} = \begin{cases} 0 & r < a \\ -\frac{k}{\epsilon_0 r} & a < r < b \\ 0 & r > b \end{cases}$$

Problem 2

(3 pts) A very long cylinder of linear dielectric material is placed in an otherwise uniform electric field $\mathbf{E}_0$. Find the resulting electric field within the cylinder. (The radius is a, the susceptibility χ_e, and the axis is perpendicular to $\mathbf{E}_0$)

Solution: We have the same case as with our sphere example 4.7, so

(i) $V_{in} = V_{out}$ at $r = a$

(ii) $\epsilon \frac{\partial V_{in}}{\partial r} = \epsilon_0 \frac{\partial V_{out}}{\partial r}$ at $r = a$

(iii) $V_{out} = -E_0 r \cos\theta$ for $r >> a$

I'll immediately eliminate constants A_0 and B_0 outside the sums, since we can't have them with these boundary conditions. Also, by the principle of superposition, I will just add that constant electric field. This gives potentials in both regions:

$$V_{out} = -E_0 a \cos\theta \sum_{k=1}^{\infty} [r^{-k}(a_k \cos k\theta + b_k \sin k\theta)]$$

$$V_{in} = \sum_{k=1}^{\infty} [r^k (c_k \cos k\theta + d_k \sin k\theta)]$$

(i) Requires that

$$\sum_{k=1}^{\infty} [r^k (c_k \cos k\theta + d_k \sin\theta)] = -E_0 r \cos\theta + \sum_{k=1}^{\infty} [r^{-k}(a_k \cos k\theta + b_k \sin k\theta)]$$

(ii) Requires that

$$\epsilon_r \sum_{k=1}^{\infty} [k r^{k-1} (c_k \cos k\theta + d_k \sin k\theta)] = -E_0 \cos\theta - \sum_{k=1}^{\infty} [k r^{-k-1}(a_k \cos k\theta + b_k \sin k\theta)]$$

The contribution from the external electric field vanishes for $k \neq 1$, which sends $c_k = a_k = 0$. Also, for all k $d_k = b_k = 0$, since this would violate (iii). This means that $k = 1$ is the only allowed value.
For $k = 1$:

$$\begin{aligned} a c_1 \cos\theta &= -E_0 a \cos\theta + a^{-1} a_1 \cos\theta \\ \epsilon_a c_1 \cos\theta &= -E_0 \cos\theta - a^{-2} a_1 \cos\theta \\ \implies a_1 &= a(a c_1 + E_0 a) \\ \implies c_1 &= \frac{-E_0 - a^{-2} a_1}{\epsilon_0} \end{aligned}$$

Where if we substitute a_1 into c_1, we get

$$c_1 = -\frac{2E_0}{\epsilon_0 + 1}, \qquad a_1 = E_0 a^2 - \frac{2a^2 E_0}{\epsilon_0 + 1}$$

Then,

$$V_{in} = -\frac{2E_0}{\epsilon_0 + 1} (\underbrace{r\cos\theta}_{=x}), \quad E = -\frac{\partial V}{\partial x}\hat{x} = \frac{2E_0}{\epsilon_0 + 1}\hat{x}$$

Problem 2 continued on next page…

Problem 3

(4 pts) An uncharged conducting sphere of radius a is coated with a thick insulating linear dielectric shell (dielectric constant ε_r) out to radius b. This object is now placed in an otherwise uniform electric field $\mathbf{E}_0$ (which you can assume is in the z direction). Find the electric field in the insulator.

Solution: Unlike in example 4.5 we do not have free charge Q nor do we know bound charges or polarization or potential, so we ahve to use laplace's equation. We have boundary conditions:

(i) $\epsilon \frac{\partial V_D}{\partial r} = \epsilon_0 \frac{\partial V'}{\partial r}$ at $r = b$

(ii) $V_D = v'$ at $r = b$

(iii) $V_D = 0$ at $r = a$

And potentials

(i) $V_D = \sum_{\ell=0}^{\infty} [A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$ inside the insulator.

(ii) $V' = -E_0 r \cos\theta + \sum_{\ell=0}^{\infty} [A'_\ell r^\ell + \frac{B'_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$ outside the insulator.

(iii) $V = 0$ inside the sphere.

Immediately $A'_\ell = 0$ so we don't have infinite potential at infinity.

(i) requires that:

$$\epsilon_r \sum_{\ell=0}^{\infty} \left[A_\ell \ell b^{\ell-1} - (\ell+1)\frac{B_\ell}{b^{\ell+2}} \right] P_\ell(\cos\theta) = -E_0 \cos\theta + \sum_{\ell=0}^{\infty} \left[-(\ell+1)\frac{B'_\ell}{b^{\ell+2}} \right] P_\ell(\cos\theta)$$

(ii) requires that:

$$\sum_{\ell=0}^{\infty} \left[A_\ell b^\ell + \frac{B_\ell}{b^{\ell+1}} \right] P_\ell(\cos\theta) = -E_0 \cos\theta + \sum_{\ell=0}^{\infty} \left[\frac{B'_\ell}{b^{\ell+1}} \right] P_\ell(\cos\theta)$$

Condition (iii) lets us say

$$A_\ell a^\ell + \frac{B'_\ell}{r^{\ell+1}} = 0 \qquad \Longrightarrow \qquad B'_\ell = -A_\ell a^{2\ell+1}$$

For $\ell \neq 1$ the external field term vanishes, since it is equivalent to $P_1(\cos\theta)$. Then we can drop the summations and write

$$A_\ell b^\ell + \frac{B_\ell}{b^{\ell+1}} = \frac{B'_\ell}{b^{\ell+1}}$$
$$\epsilon_r \left[A_\ell \ell b^{\ell-1} - (\ell+1)\frac{B_\ell}{b^{\ell+2}} \right] = -(\ell+1)\frac{B'_\ell}{b^{\ell+2}}$$

And for $\ell = 1$:

$$\left[A_1 b^\ell + \frac{B_1}{b^2} \right] \cos\theta = -E_0 b \cos\theta + \frac{B'_1}{b^2} \cos\theta$$
$$\epsilon_r \left[A_1 + \frac{-2B_1}{b^2} \right] = -E_0 - \frac{2B'_1}{b^2}$$

Need to quickly finish this by combining the boundary conditions

Problem 3 continued on next page…

Problem 4

(Extra credit, 3 pts): When you polarize a neutral dielectric, the charge moves a bit, but the total remains zero. This fact should be reflected in the bound charges σ_b and ρ_b. Use the definition of bound charges to show that the total bound charge is zero. (Hint: Assume some finite dielectric, draw Gaussian surface around it, and calculate the total bound charge inside)

Solution: We have the equations

$$\sigma_b \equiv \mathbf{P} \cdot \hat{n}, \qquad \rho_b \equiv -\nabla \cdot \mathbf{P}$$

In deriving the bound charges, we found the total charge:

$$Q = \oint_S \sigma_b \, d\mathbf{a} + \int_V \rho_b \, dV = \oint_S \mathbf{P} \cdot d\mathbf{a} + \oint_V -\nabla \cdot \mathbf{P} d\tau$$

By the divergence theorem, these should be equal:

$$\oint_S \mathbf{P} \cdot d\mathbf{a} = \oint_V \nabla \cdot \mathbf{P} d\tau$$

Therefore Q due to the bound charges is zero. If there are no free charges, then total charge is zero.