Homework 1

Grant Saggars

February 2, 2024

1 Propose two functions and sketch them (0.5 pt)

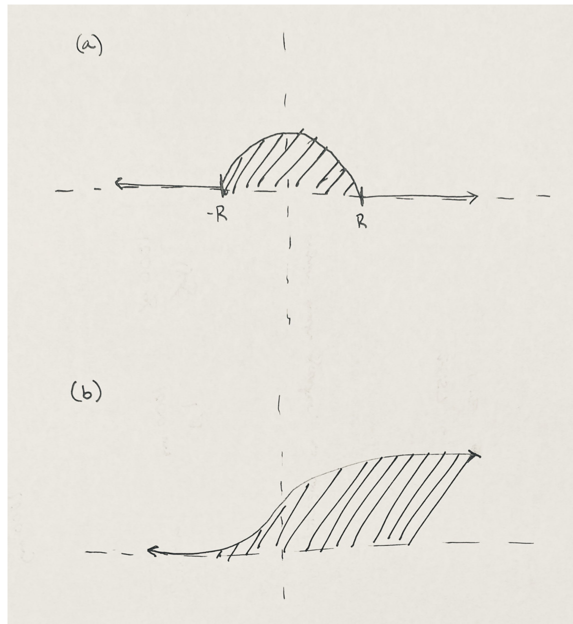
(a) The function $f(x)$ which can serve as a PDF.

Solution:

The Wigner semicircle distribution is a really odd distribution defined as:

$$f(x) = \begin{cases} \frac{2}{\pi R^2}\sqrt{R^2 - x^2} & [-R, R] \\ 0 & (-\infty, -R) \cup (R, \infty) \end{cases}$$

Although this could not technically work as a wavefunction, as it does not have a continuous derivative, it is finite and continuous and is therefore a PDF.



(b) The function $g(x)$ which can't serve as a PDF.

Solution:

A logistic curve is an example of something which cannot be a PDF, as it is not finite:

$$g(x) = \frac{1}{1 + e^{-x}}$$

2 Prove that if a wavefunction is normalized at one point in time it preserves its normaliation over future time (0.75 pt)

Solution:

Because the wave function, once normalized, is only a function of time, it would be a good idea to take the time derivative and show that it must equal zero for the normalization to remain constant for all time (it does not change). We can start by moving the time derivative under the integral sign and expanding the wave function:

$$\frac{d}{dt}\int_{-\infty}^{\infty} |\Psi(x,t)|^2 \, dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial t}|\Psi(x,t)|^2 \, dx$$

$$\frac{\partial}{\partial t}|\Psi|^2 = \frac{\partial}{\partial t}(\Psi^*\Psi) = \Psi^*\frac{\partial \Psi}{\partial t} + \frac{\partial \Psi^*}{\partial t}\Psi \qquad \text{(by the product rule)}$$

Expressions for $\frac{\partial \Psi}{\partial t}$ and its complex conjugate can be expanded using the definition of the Shrödinger equation:

$$\frac{\partial}{\partial t}|\Psi|^2 = \frac{i\hbar}{2m}\left(\Psi^*\frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial^2 \Psi^*}{\partial x^2}\Psi\right) = \frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\left(\Psi^*\frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\right)\right]$$

$$\frac{d}{dt}\int_{-\infty}^{\infty} |\Psi(x,t)|^2 \, dx = \frac{i\hbar}{2m}\left(\Psi^*\frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\right)\Bigg|_{-\infty}^{\infty}$$

Now, because the wave function goes to zero as x goes to $\pm\infty$:

$$\frac{d}{dt}\int_{-\infty}^{\infty} |\Psi(x,t)|^2 \, dx = 0$$

3 Consider the gaussian distribution (0.75 pt)

$$\rho(x) = Ae^{-\lambda(x-a)^2}$$

(a) Determine A.

Solution: (Polar Transformation)

$$\rho = \int_{-\infty}^{\infty} \exp(-a(x-\mu)^2)\ dx \to \int_{-\infty}^{\infty} \exp(-a(u)^2)\ dx \tag{1}$$

Transformation (1) has a jacobian of one, which should make sense since $x - \mu$ does not scale the function. Now, we can temporarily square ρ to convert to polar coordinates.

$$\rho^2 = \int_{-\infty}^{\infty} \exp(-a(u^2+v^2))\ dx \to \int_0^{2\pi} \int_0^{\infty} \rho \exp(-a\rho^2)\ d\rho\ d\theta$$

Finally, a substitution of $a\rho^2 = w$ allows the integral to be solved to completion:

$$\begin{aligned}\rho^2 &= \frac{\cancel{2}\pi}{\cancel{2}a} \int_0^{\infty} e^{-w}\ dx \\ &= \frac{\pi}{a}\left(-e^{-w}\Big|_0^{\infty}\right) \\ &= \sqrt{\frac{\pi}{a}}\end{aligned}$$

(b) Calculate $\langle x \rangle$, $\langle x^2 \rangle$, and σ.

$\langle x \rangle$**:**

The average position is defined as $\int_{-\infty}^{\infty} x|\psi(x)|^2\ dx$.

$$\sqrt{\frac{\pi}{a}}^{-2} \int_{-\infty}^{\infty} x \exp^2(-a(x-\mu)^2)\ dx \to \frac{a}{\pi} \int_{-\infty}^{\infty} x \exp(-2a(x-\mu)^2)\ dx$$

(1) make the substitution: $x - \mu \to u$, $dx = du$

$$\begin{aligned}&\frac{a}{\pi} \int_{-\infty}^{\infty} (u+\mu) \exp(-2au^2)\ du \\ \to &\frac{a}{\pi} \left(\int_{-\infty}^{\infty} u \exp(-2au^2)\ du + \int_{-\infty}^{\infty} \mu \exp(-2au^2)\ du \right)\end{aligned}$$

(2) The first integral can be solved with a single substitution

$$\int_{-\infty}^{\infty} u \exp(-2au^2)\ du = \left(-\exp(-2au^2) \cdot \frac{1}{4a}\right)\Bigg|_{-\infty}^{\infty} = 0$$

(3) The second integral is a gaussian, and can be solved like in part (a).

$$\begin{aligned}
\int_{-\infty}^{\infty} \mu \exp(-2au^2)\ du &= \left(\mu^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(-2a(x^2+y^2))\ dy\ dx\right)^{\frac{1}{2}} \\
&= \left(\mu^2 \int_{0}^{2\pi}\int_{0}^{\infty} \rho \exp(-2a\rho^2)\ d\rho\ d\theta\right)^{\frac{1}{2}} \\
&= \left(\mu^2 \frac{2\pi}{2a} \int_{0}^{\infty} e^{-w}\ dw\right)^{\frac{1}{2}} \\
&= \mu\sqrt{\frac{\pi}{a}}
\end{aligned}$$

(4) Substituting these into (1):

$$\frac{a}{\pi}\left(0 + \mu\sqrt{\frac{\pi}{a}}\right) = \frac{\mu a}{\pi}\sqrt{\frac{\pi}{a}}$$

$\langle x^2 \rangle$ and Standard Deviation:

x^2_{ave} is:

$$\begin{aligned}
\frac{a}{\pi}\int_{-\infty}^{\infty} x^2 \exp(-2a(x-\mu)^2)\ dx &= -2\frac{d}{da}\int_{-\infty}^{\infty} \exp(-2a(x-\mu)^2)\ dx \\
&= -2\frac{d}{da}\sqrt{\frac{\pi}{a}} \\
&= \frac{4}{a}\sqrt{\frac{\pi}{a}}
\end{aligned}$$

Therefore, the standard deviation σ is:

$$\sqrt{\frac{4}{a}\sqrt{\frac{\pi}{a}} - \left(\frac{\mu a}{\pi}\sqrt{\frac{\pi}{a}}\right)^2}$$

(c) Sketch the graph of $\rho(x)$.