EECS622: Homework #13

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Problem 1

A microwave filter has the transfer function:

$$H(\omega) = \left(\frac{4\pi^2 \times 10^{12}}{\omega^2 + 4\pi^2 \times 10^{12}} \right)^{1/2} e^{-j\left(10^{-10}\omega^2 + 10^{-6}\omega\right)}$$

Determine the phase delay of this filter at signal frequency $\omega = 10^4 \text{rad/sec}$.

Solution:

$$H(\omega) = \frac{V_{out}(\omega)}{V_{in}(\omega)} \qquad \text{(Filter Response)}$$

$$\arg[V_{out}(\omega)] = \arg[H(\omega)] + \arg[V_{in}(\omega)] \qquad \text{(Phase)}$$

In an ideal filter, filter phase will be of delay τ and frequency ω, however in general τ is frequency-dependent due to dispersive effects, so we instead use:

$$\tau(\omega) = -\frac{\partial \arg[H(\omega)]}{\partial \omega}$$

Given the exponential form provided,

$$\arg[H(\omega)] = -10^{-10}\omega^2 - 10^{-6}\omega$$

Therefore delay will be,

$$\tau = -\frac{\partial \arg[H(\omega)]}{\partial \omega}\bigg|_{\omega=10^4} = \left(2 \cdot 10^{-10}\omega + 10^{-6}\right)\bigg|_{\omega=10^4}$$

There's some unit weirdness going on here, so I want to stop and consider it for a second. ω is given in rad/sec, while there are dimensions $\frac{\partial \arg[H(\omega)]}{\partial \omega} = \frac{\text{unitless}}{\text{rad/sec}}$. We then have

$$\begin{aligned} \tau &= \left(2 \cdot 10^{-10} \ \frac{\text{sec}^2}{\text{rad}^2}\right)\left(10^4 \ \frac{\text{rad}}{\text{sec}}\right) + 10^{-6} \ \frac{\text{sec}}{\text{rad}} \\ &= \boxed{3 \ \frac{\mu\text{s}}{\text{rad}}} \end{aligned}$$

I suppose that angular units are sort of unitless depending on who is asked, so I could see someone giving this units μs, but I think this is a neat result.

Problem 2

The output of a low-pass microwave filter is terminated in a matched load.
The reflection coefficient resulting from this filter's input impedance is then:

$$\Gamma_{\text{in}}(\omega) = \frac{1}{1 + j\left(\frac{\omega}{1000}\right)}$$

A matched source creates a time-harmonic (i.e., sinusoidal) signal with frequency ω.

This matched source has an available power of 10 mW. This matched source is connected to the input of the filter. As a result, the matched load at the filter output absorbs energy at a rate of 9.0 mW.

Determine the value of signal frequency w.

Solution:
In our previous work we have used the relation:

$$P_{\text{del}} = P_{\text{avl}}\left(1 - |\Gamma_{\text{in}}|^2\right)$$
$$|\Gamma_{\text{in}}|^2 = 1 - \frac{P_{\text{del}}}{P_{\text{avl}}} = 0.1 \text{ mW}$$

Then we can simply plug in our given expression and solve:

$$\begin{aligned}
0.1 &= \frac{1}{1 + j\left(\frac{\omega}{1000}\right)} \frac{1}{1 - j\left(\frac{\omega}{1000}\right)} \\
&= \frac{1}{1 \cancel{- j\left(\frac{\omega}{1000}\right)} + \cancel{j\left(\frac{\omega}{1000}\right)} + \left(\frac{\omega}{1000}\right)^2} \\
0.1 \cdot \left[1 + \left(\frac{\omega}{1000}\right)^2\right] &= 1 \\
0.1 &= \frac{1}{1 + \left(\frac{\omega}{1000}\right)^2} \\
0.1\left[1 + \left(\frac{\omega}{1000}\right)^2\right] &= 1 \\
1 + \left(\frac{\omega}{1000}\right)^2 &= 10 \\
\left(\frac{\omega}{1000}\right)^2 &= 9 \\
\omega &= 3000 \text{ rad/s}
\end{aligned}$$