MATH 220 Cheat Sheet

Grant Saggars

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Homogeneous Equations:

$$y'' + y' - 6y = 0 \rightarrow r^2 + r - 6 = 0$$
$$\implies (r+3)(r-2) = 0$$
$$y_c = Ae^{-3t} + Be^{2t}$$

The solution depends on the roots:

$$\begin{aligned} r_1 = r_2 : &\quad y_c = c_1 e^r + c_2 t e^r \\ r_1 \neq r_2 : &\quad y_c = c_1 e^{r_1} + c_2 e^{r_2} \\ \{r_1, r_2\} \in \mathbb{C} : &\quad y_c = e^{\alpha t} c_1 \cos(\beta t) + c_2 \sin(\beta t) \end{aligned}$$

For matrices the solution is of the form:

$$x_1(t) = c_1$$

Wronskian:

Suppose y_1 and y_2 are solutions. If given initial conditions to find constants c_1 and c_2 $\in y = c_1 y_1(t) + c_2 y_2(t)$ which satisfy the differential equation if and only if the Wronskian is nonzero.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Variation of Parameters:

$$y_p = u_1 y_1 + u_2 y_2$$

- Variation of parameters is a method to find a particular solution. The first equation of the system is the characteristic (homogeneous) solution: $u_1 y_1 + u_2 y_2 = 0$.
- Substitute solutions 1 and 2 from characteristic solution, then differentiate and substitute in equation to obtain the second equation of the system.

When it is not necessary to solve a system of equations (which depends on the function of t), the method is called undetermined coefficients.

1 Practice Problems

1. Compute the determinant of the matrix $A = \begin{pmatrix} -1 & 1 & 2 \\ 4 & 3 & 5 \\ 8 & 6 & 7 \end{pmatrix}$

2. Consider the following differential equation for $t > 0$, $y'' - 2y' + y = te^t$ with $y(0) = 1, y'(0) = 1$

   (a) Find the general solution

Solution:
Using undetermined coefficients:

$$y'' - 2y' + y = 0 \implies (r-1)(r-1) = 0$$
$$y_c = c_1 e^t + c_2 te^t$$

$$\begin{aligned} y_p &= t^2(At + B)e^t \to (At^2 + Bt^2)e^t \\ y_p' &= (3At^2 + 2Bt)e^t + (At^3 + Bt^2)e^t \\ y_p'' &= e^t(At^3 + t^2(3A + B) + 2Bt) + te^t(3At^2 + 2t(3A + B) + 2B) \end{aligned}$$

substituting and simplifying:

$$\cancel{e^t}(At^3 + t^2(6A + \cancelto{0}{B}) + 2t(\cancelto{0}{2B} + 3A) + \cancelto{0}{2B}) - 2\cancel{e^t}(At^3 + t^2(3A + B) + \cancelto{0}{2Bt})$$
$$+ \cancel{e^t}(At^3 + \cancelto{0}{Bt^2}) = t\cancel{e^t}$$

$$\implies 2B = 0 \quad \text{(because 2B is the only constant)}$$
$$\implies A = \frac{1}{6}$$
$$y_p = \frac{1}{6}t^3 e^t$$

Therefore the general solution is:

$$y_c + y_p = c_1 e^t + c_2 te^t + \frac{1}{6}t^3 e^t$$

   (b) Compute the Wronskian of a fundamental set of solutions

Solution:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t} \neq 0$$

3. Find the general solution of the given system $x' = \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix} x$

Solution:

1. Eigenvalues:

$$\begin{vmatrix} 3-\lambda & 6 \\ -1 & -2-\lambda \end{vmatrix} \implies \lambda = \{0, 1\}$$

2. Eigenvectors:

$$\vec{\lambda} = \left\{ \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$$

3. Solution:

$$x(t) = c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^0 + c_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^t$$

4. Find the general solution of the given system $x' = \begin{pmatrix} 4 & 2 \\ 8 & -4 \end{pmatrix} x$

5. Given the system $x' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} x$

   (a) Find the fundamental matrix for the system

Solution:

$$\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_1 = (2+i)m_2 \implies m_2 = 1, \quad m_1 = 2+i$$

$$m_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies x_1 = \left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) e^{it}$$

I would like to try solving this my own way.

(b) Find the fundamental matrix $\Phi(t)$ satisfying $\Phi(0)$

Solution:
Given that the general solution is:

$$x(t) = \begin{pmatrix} 2\cos t - \sin t & 2\sin t + \cos t \\ \cos t & \sin t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \Psi(t)$$

$$\Psi(0) = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \implies \Psi^{-1}(0) = -\begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}$$

Now, it follows that $\Phi = \Psi(t)\Psi^{-1}(0)$:

$$= \begin{pmatrix} 2\cos t - \sin t & 2\sin t + \cos t \\ \cos t & \sin t \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$$

$$\Phi = \begin{pmatrix} 2\sin t + \cos t & -t\sin t \\ \sin t & \cos t - 2\sin t \end{pmatrix}$$

$$\Phi(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

6. Find the solution of the initial value problem $x' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$