MATH 648: Homework #4

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Math 648 S24 Homework 4 (Thursday 04/11/24)

- Section 7.1: #2, #4.
  - Hint for #2: Note that the boundary condition $M(0)$ at $t = 0$ is fixed but the boundary condition $M(T)$ at $t = T$ is not fixed so that one needs the natural boundary condition (NBC) at $t = T$.
- Section 7.3: #1, #2.
  - Hint for #1: Use the result from Example 2.3.4 for a parametric form of the solution of the E-L:

$$x(\psi) = \kappa_2 - \kappa_1(2\psi + \sin(2\psi)), \quad y(\psi) = \kappa_1(1 + \cos(2\psi)).$$

Suppose the angle parameter ψ varies from ψ_0 to ψ_1. So that $\psi = \psi_0$ corresponds to the end point $(x, y) = (0, 0)$, that is $x(\psi_0) = y(\psi_0) = 0$, and $\psi = \psi_1$ corresponds to the other end point on the curve given by $y = x - 1$. Show that, the natural boundary condition at $x = x_1$ gives $y'(x_1) = -1$ or, in terms of ψ, $\tan\psi = -1$ (since $y' = \tan\psi$ from Example 2.3.4). Show that the condition $x(\psi_0) = y(\psi_0) = 0$ gives $\psi_0 = (n + 1/2)\pi$ for some integer n (we can take $n = 0$), and hence, $\kappa_2 = \kappa_1\pi$. Use $y'(x_1) = -1$ and $y \geq 0$ to show that $x_1 > 0$, and hence, $\psi_1 < \psi_0 = \pi/2$. Use again $y'(x_1) = \tan\psi_1 = -1$ to get $\psi_1 = -\pi/4$. Finally, use the condition that $y(\psi_1) = x(\psi_1) - 1$ to get $\kappa_1 = \frac{2}{3\pi} > 0$, and hence, $\kappa_2 = \frac{2}{3}$. Therefore, an extremal of the problem is parameterized by

$$x(\psi) = \frac{2}{3} - \frac{2}{3\pi}(2\psi + \sin(2\psi)), \quad y(\psi) = \frac{2}{3\pi}(1 + \cos(2\psi))$$

for ψ from $\pi/2$ to $-\pi/4$.

Problem 1

A simplified version of the Ramsey growth model in economics concerns a functional of the form

$$J(M) = \int_0^T \left(c_1 \left(c_2 M(t) - M'(t) - c_3\right)^2\right) \, dt.$$

Here, J corresponds to the "total product," M is the capital, and the c_k are positive constants. The problem is to find the best use of capital such that J is minimized in a given planning period $[0, T]$. Now, the initial capital $M(0) = M_0$ is known, but the final capital $M(T)$ is not prescribed. Use the natural boundary conditions to find the extremal for J and the final capital $M(T)$.

Solution:
The E-L for this is:

$$\left[2c_1 c_2^2 M - \cancel{2c_1 c_2 M'} - 2c_1 c_2 c_3\right] - \frac{d}{dt}\left[\cancel{-2c_1 c_2 M} - 2c_1 M' + \cancel{2c_1 c_3}\right]$$

Simplifying to:

$$M'' - c_2^2 M = -c_2 c_3$$

Characteristic equation:

$$\lambda^2 - c_2^2 = 0 \implies y_c = Ae^{c_2 t} + Be^{-c_2 t}$$

Particular equation:

$$y_p = y_c'' - c_2^2 y_c = -c_2 c_3$$

Problem 2

Let $\mathbf{q} = (q_1, \dots, q_n)$ and $J(\mathbf{q}) = \int_{t_0}^{t_1} L(t, \mathbf{q}, \dot{\mathbf{q}})\,dt$. Derive the natural boundary conditions that an extremal must satisfy if neither $\mathbf{q}(t_0)$ nor $\mathbf{q}(t_1)$ are prescribed.

Solution: Consider the conditions for a functional $J = \int_{x_0}^{x_1} f(x, y, y')\,dx$ without any boundary conditions. We need to first consider whether this is even well posed.
It is obvious that we ought to express both the x-coordinate and y-coordinate as things which can vary.
Let

$$\hat{x}_0 = x_0 + \epsilon\bar{x}_0 + O(\epsilon^2), \quad \hat{x}_1 = x_1 + \epsilon\bar{x}_1 + O(\epsilon^2)$$

Consider

$$\begin{aligned}
\hat{y}(x) &= y(x) + \epsilon\eta(x), \quad \text{for } x \in [x_0, x_1] \cup [\hat{x}_0, \hat{x}_1]; \\
\hat{y}(\hat{x}_0) &= y(x_0) + \epsilon\bar{y}_0 + O(\epsilon^2), \quad \hat{y}(\hat{x}_1) = y(x_1) + \epsilon\bar{y}_1 + O(\epsilon^2), \text{ and:} \\
\hat{y}'(\hat{x}_{0,1}) &= y'(x_{0,1}) + O(\epsilon)
\end{aligned}$$

Expressing our functional now:

$$\begin{aligned}
J(\hat{y}) - J(y) &= \int_{\hat{x}_0}^{\hat{x}_1} f(x, \hat{y}, \hat{y}')\,dx - \int_{x_0}^{x_1} f(x, y, y')\,dx \\
&= \int_{x_0}^{x_1} f(x, y + \epsilon\eta, y' + \epsilon\eta')\,dx + \int_{\hat{x}_0}^{x_1} f(x, y + \epsilon\eta, y' + \epsilon\eta')\,dx \\
&\quad + \int_{x_1}^{\hat{x}_1} f(x, y + \epsilon\eta, y' + \epsilon\eta')\,dx - \int_{x_0}^{x_1} f(x, y, y')\,dx
\end{aligned}$$

By integration by parts and mean value theorem (MVT approximating the middle two terms):

$$\begin{aligned}
\epsilon\left\{\int_{x_0}^{x_1} \eta\left[f_y(x, y, y') - \frac{d}{dx}f)y'(x, y, y')\right]\,dx + f_{y'}\eta\mid_{x_0}^{x_1} + O(\epsilon^2)\right. \\
-\epsilon f(x_0, y(x_0), y'(x_0)) + O(\epsilon^2) \\
+\epsilon f(x_1, y(x_1), y'(x_1))\bar{x}_1 + O(\epsilon^2)
\end{aligned}$$

Now,

$$\begin{aligned}
J(\hat{y}) - J(y) &= \epsilon\delta J(\eta, y) + O(\epsilon^2) \\
&= f_{y'}\eta|_{x_0}^{x_1} \int_{x_0}^{x_1}\left[f_y - \frac{d}{dx}f_{y'}\right]\eta\,dx + f|_{x_1}\bar{x}_1 - f|_{x_0}\bar{x}_0
\end{aligned}$$

This is the first variation of a functional at y in the direction of η in the general case.
From equations 1 and 2:

$$\begin{aligned}
\epsilon\eta(x_0) &= \hat{y}(x_0) - y(x_0) \\
&= \hat{y}(x_0) - \hat{y}(\hat{x}_0) + \hat{y}(\hat{x}_0) - y(x_0) \\
&= \hat{y}'(\langle x_0\rangle)(x_0 - \hat{x}_0) + \epsilon y_0 + O(\epsilon^2) \qquad \text{(MVT?)} \\
&= -\epsilon y'(x_0)\bar{x}_0 + \epsilon\bar{y}_0 + O(\epsilon^2) \\
\implies \eta(x_0) &= -y'(x_0)\bar{x}_0 + \bar{y}_0 \\
\implies \eta(x_1) &= -y'(x_1)\bar{x}_1 + \bar{y}_1
\end{aligned}$$

Problem 2 continued on next page…

Substituting that back into our functional derivative:

$$\delta J(\eta, y) = \int_{x_0}^{x_1} \left[f_y - \frac{d}{dx} f_{y'} \right] \eta \, dx + f_{y'}|_{x_1}(-y'(x_1)\bar{x}_1 + \bar{y}_1) - f_{y'}|_{x_0}(-y'(x_0)\bar{x}_0 + \bar{y}_0)$$

Again, we conclude that for any extremal our functional has to equal zero no matter what endpoints. If we fix endpoints the extra terms drop out and we recover the original E-L equation.
We can pull out conditions for our extremal:

$$\begin{aligned} [f_{y'}y' - f]_{x_0} &= 0 \\ [f_{y'}y' - f]_{x_1} &= 0 \\ f_{y'}|_{x_0} &= 0 \\ f_{y'}|_{x_1} &= 0 \end{aligned}$$

Because we don't have enough conditions, this problem is not well posed in general.
One needs two relations among $\bar{x}_0,\ \bar{x}_1,\ \bar{y}_0,\ \bar{y}_1$ to reduce the degrees of freedom. A possibility is that we require $(x_0,\ y_0)$ lies on one given curve $y = \varphi(x)$.
Let

$$\hat{x}_0 + \epsilon \bar{x}_0 + \dots, \quad \hat{y}(\hat{x}_0) = y(x_0) + \epsilon \bar{y}_0 + \dots$$

$$\begin{aligned} (\hat{x}_0, \hat{y}(\hat{x}_0)) &= (x_0 + \epsilon \bar{x}_0, y(x_0) + \epsilon \bar{y}_0) + O(\epsilon^2) \\ y(x_0) + \epsilon \bar{y}_0 &= \varphi(x_0 + \epsilon \bar{x}_0) = \varphi(x_0) + \epsilon \varphi'(x_0)\bar{x}_0 + O(\epsilon^2) \\ &\implies \bar{y}_0 = \varphi'(x_0)\bar{x}_0 \end{aligned}$$

End points on the curves $(x_0, y(x_0))$ on $\varphi_0(x_0)$, and $(x_1, y(x_1))$ on the curve $y = \varphi_1(x_1)$. Recall the basic functional derivative:

$$\begin{aligned} \delta J(y, \eta) = \int_{x_0}^{x_1} &\left[f_y - \frac{d}{dx} f_{y'} \right] dx + f_{y'}|_{x_1}\bar{y}_1 - f_{y'}|_{x_0}\bar{y}_0 \\ &+ \bar{x}_1(f - y'f_{y'})_{x_1} - \bar{x}_0(f - y'f_{y'})_{x_0} \end{aligned}$$

We can simplify it to

$$\begin{aligned} \delta J(\eta, y) = \int_{x_0}^{x_1} &\eta \left(f_y - \frac{d}{dx} f_{y'} \right) dx \\ &+ (f_{y_1}(\varphi_1' - y') + f)|_{x_1}\bar{x}_1 - (f_{y_1}(\varphi_0' - y') + f)|_{x_0}\bar{x}_0 \end{aligned}$$

Where our extra two boundary conditions are

$$[f_{y_1}(\varphi_1' - y') + f]|_{x_1} = 0, \qquad [f_{y'}(\varphi_0' - y') + f]|_{x_0} = 0$$

Problem 3

The functional for the brachystochrone is

$$J(y) = \int_0^{x_1} \sqrt{\frac{1+y'^2}{y}}\,dx$$

Find an extremal for J subject to the condition that $y(0) = 0$ and $(x_1, y(x_1))$ lies on the curve $y = x - 1$.

Solution:
As previously found, the solutions to this functional are parametric of form:

$$x(\psi) = \kappa_2 - \kappa_1(2\psi + \sin(2\psi)), \quad y(\psi) = \kappa_1(1 + \cos(2\psi)).$$

(I)

$$x(\psi_0) = y(\psi_0) = 0 \implies 0 = \kappa_2 - \kappa_1(2\psi_0 + \sin(\psi_0)) = \kappa(1 + \cos(2\psi))$$

This gives valid values of ψ_0 equal to $\pi(n + \frac{1}{2})$, which if we let n equal zero we get $\pi\kappa_1 = \kappa_2$

(II)

$$r(\xi) = (\xi, \xi - 1)$$
$$\left(\frac{dx_\Gamma}{d\xi} \cdot \frac{dy_\Gamma}{d\xi}\right) \cdot \left(1, \frac{dy}{dx}\right) = 1 + \frac{dy}{dx} = 0$$
$$\frac{dy}{dx} \to \frac{dy/d\psi}{dx/d\psi} = \frac{-2\kappa_1 \sin(2\psi)}{-2\kappa(1 + \cos(2\psi))} = \tan(\psi)$$

Evaluating this gives $\frac{dy}{dx} = -1$ which gives values of $\psi_1 = -\frac{\pi}{4} + \pi n$, which we can again use $n = 0$ for.

(III) Because this is a brachystochrone curve, we can automatically assume $x_1 < x_0$, as that is the point of the problem. I am curious how you would prove that mathematically though. We can substitute into our second boundary condition now to find κ_1 and κ_2:

$$y(\psi_1) = x(\psi_1) - 1$$
$$\kappa_1\left(1 + \cos\left(-\frac{\pi}{2}\right)\right) = \kappa_2 - \kappa_1\left(-\frac{\pi}{2} + \sin\left(-\frac{\pi}{2}\right)\right) - 1$$
$$-\kappa_1\frac{\pi}{2} + 1 = \kappa_2 = \pi\kappa_1$$
$$1 = \frac{3\pi\kappa_1}{2}$$
$$\kappa_1 = \frac{2}{3\pi}$$
$$\kappa_2 = \frac{2}{3}$$

Substiting these gives the desired:

$$x(\psi) = \frac{2}{3} - \frac{2}{3\pi}(2\psi + \sin(2\psi)), \quad y(\psi) = \frac{2}{3\pi}(1 + \cos(2\psi))$$

Problem 4

Let

$$J(y) = \int_0^{x_1} (y'^2 + y^2)\, dx.$$

Find an extremal for J subject to the condition that $y(0) = 0$ and $(x_1, y(x_1))$ lies on the curve $y = 1 - x$. Determine the appropriate constants in terms of implicit relations.

Solution: This functional has E-L:

$$\frac{d}{dx} 2y' - 2y = 0$$

is a homogeneous ODE with solutions:

$$y(x) = c_1 e^x + c_2 e^{-x}$$

(I) It is obvious that $c_1 = -c_2$.

(II)

$$\begin{aligned} y'(x_1) &= -1 \\ c_1 e^{x_1} + c_1 e^{-x_1} &= -1 \\ 2\sinh(x_1) &= \frac{1}{c_1} \\ \frac{1}{2\sinh(x_1)} &= c_1 \end{aligned}$$

(III)

$$x_1 - 1 = \frac{1}{2\sinh(x_1)} e^{x_1} - \frac{1}{2\sinh x_1} e^{-x_1}$$

Using a graphing calculator to numerically solve this gives $x_1 = 2$.

(IV) Therefore the solution is:

$$y(x) = \frac{1}{2\sinh(2)} e^x - \frac{1}{2\sinh(2)} e^{-x}$$