Math 590 HW5

Grant Saggars

Problem 1. Let $A = \begin{bmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & x \end{bmatrix}$. Find all x so that the transformation $T : \mathbb{R}^4 \to \mathbb{R}^3, T(\vec{v}) = A\vec{v}$ is:

a) injective.

Solution:

1. **Injectivity is equivalent to null space equals 0** (Proof taken from Axler 61)

   First suppose T is injective. We want to prove that null $T = \{0\}$. We already know that $0 \subset$ null T, since any subspace contains the zero vector, and the null space is a subspace. To prove the inclusion in the other direction, suppose $v \in \operatorname{null} T$. Then:

   $$T(v) = 0 = T(0)$$

   Because T is injective, the equation above implies that $v = 0$. Thus we can conclude that null $T = 0$, as desired.

   To prove the implication in the other direction, now suppose null T $= \{0\}$. We want to prove that T is injective. To do this, suppose u, $v \in V$ and $Tu = Tv$. Then

   $$0 = Tu - Tv = T(u - v)$$

   Thus $u - v$ is in null T, which equals $\{0\}$. Hence $u - v = 0$, which implies that $u = v$; hence, T is injective.

2. To make A injective, we must therefore find x such that null A equals $\{0\}$:

   Because A is not square, there must be a free variable. Consequently, this means that there will always be a nonzero nullity, regardless of the value chosen for x. **Additionally, this implies that a map to a smaller dimensional space is not injective.**

b) surjective.

Solution:

1. A function $T \colon V \to W$ is surjective if its range equals W.

   Given that the range of A is equivalent to the number of pivots, and W equals 3, range A equals 3 for all $x \neq 12$, since if x equals 12, there will be no pivot in row 3.

Problem 2. Prove carefully that if a linear transformation T is bijective, then it is invertible. (Don't forget to show that T^{-1} is also a linear transformation).

Solution: To show that T is invertible if and only if it is injective and surjective, we must show that: (1) $Tu = Tv \implies u = v$ (injective), (2) range T equals W (surjective), and (3) When T is injective and surjective it is invertible.

1. Suppose T is invertible, and suppose $u,\ v \in V$:

   $$u = T^{-1}(Tu) = T^{-1}(Tv) = v$$

2. Again, suppose T is invertible, and let $w \in W$:

   $$w = T(T^{-1}w)$$

   This shows that w is still in the range of T. Therefore, range T equals W.

3. Finally, assume T is injective and surjective: let S be the inverse of T, as in $T(Sw)$ for $w \in W$ equals w. Note that this also implies w is in the range of T. Clearly, $T \circ S$ must be the identity matrix. Now:

   $$T((S \circ T)v) = (T \circ S)(Tv) = I(Tv) = Tv$$

   This implies that $(S \circ T)v = v$, proving that $S \circ T$ is the identity map.

   Finally, it must be shown that S is linear. To do this, suppose $w_1,\ w_2 \in W$. Then:

   $$T(Sw_1 + Sw_2) = T(Sw_1) + T(Sw_2) = w_1 + w_2$$

   Therefore, $Sw_1 + Sw_2$ is a unique element which T maps to $w_1 + w_2$. This indicates that S satisfies the additive property. Similarly:

   $$T(\lambda Sw) = \lambda T(Sw) = \lambda w$$

   implies the multiplicative property required for linearity. As a consequence, this must mean that S is linear.

Problem 3. a) Show that if T, S are invertible linear transformations $\mathbb{R}^n \to \mathbb{R}^n$, then so is the composition $T \circ S$.

Solution: To prove the composition of two invertible matrices is still invertible, it suffices to show that:

$$(ST)(T^{-1}S^{-1}) = S(TT^{-1})S^{-1} = S(I)S^{-1} = SS^{-1} = I$$

b) Alice claims that if $T,\ S$ are invertible linear transformations $\mathbb{R}^2 \to \mathbb{R}^2$, then there are infinitely many $\alpha \in \mathbb{R}$ such that $T + \alpha S$ is invertible. Is she correct? Justify your claim fully.

Solution:

1. Before considering whether $T + \alpha S$ is generally invertible, it would make sense to first consider whether $T + S$ is in general invertible. Consider the situation:

   $$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

   We can see that this gives $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, which is not invertible. As a result, we can see that because the sum of two invertible matrices is not invertible always, the original statement is not true by counterexample.