PHSX 611: Homework # 3

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Problem 1

We learned in Section 2.3 what a commutator of operators $[\hat{A}, \hat{B}]$ is. Derive commutators of the following:

a) Potential energy $V(x)$ and momentum operator $\hat{p}$;

Solution:

$$\widehat{V} = V(x)$$
$$\hat{p} = -i\hbar \frac{\partial}{\partial x}$$

$$\begin{aligned}
[\widehat{V}, \hat{p}]f &= \widehat{V}\hat{p} - \hat{p}\widehat{V} \\
&= V(x)(-i\hbar)\frac{d}{dx}(f) - (-i\hbar)\frac{d}{dx}(V(x)f) \\
&= -i\hbar \left(\frac{df}{dx}V(x) - \frac{df}{dx}V(x) - \frac{dV(x)}{dx}f \right) \\
&= i\hbar f \frac{dV(x)}{dx} \\
\implies [\widehat{V}, \hat{p}] &= i\hbar \frac{dV}{dx}
\end{aligned}$$

b) Kinetic energy operator $\widehat{T}$ and momentum operator $\hat{p}$;

Solution:

$$\widehat{T} = \frac{\hat{p}^2}{2m}$$
$$\hat{p} = -i\hbar\frac{\partial}{\partial x}$$

$$\begin{aligned}
[\widehat{T}, \hat{p}] &= \widehat{T}\hat{p} - \hat{p}\widehat{T} \\
&= \left(\frac{\hat{p}^2}{2m}\right)(\hat{p}) - (\hat{p})\left(\frac{\hat{p}^2}{2m}\right) \\
&= \frac{1}{2m}\left(\hat{p}^2(\hat{p}) - (\hat{p})\,\hat{p}^2\right) \\
&= \frac{1}{2m}\left(\hat{p}[\hat{p},\hat{p}] - [\hat{p},\hat{p}]\hat{p}\right)
\end{aligned}$$

Because operators commute with themselves, $[\hat{p},\hat{p}]$ equals zero.

$$[\widehat{T}, \hat{p}] = 0$$

c) Hamiltonian operator $\widehat{H}$ and momentum operator $\hat{p}$;

Solution:

$$\widehat{H} = \widehat{T} + \widehat{V} = \frac{\hat{p}^2}{2m} + V(x)$$
$$\hat{p} = -i\hbar\frac{\partial}{\partial x}$$

$$\begin{aligned}
[\widehat{H}, \hat{p}] &= [\widehat{T} + \widehat{V}, \hat{p}] = \cancelto{0}{[\widehat{T}, \hat{p}]} + \cancelto{ih\frac{dV}{dx}}{[\widehat{V}, \hat{p}]} \\
&= i\hbar\frac{dV}{dx}
\end{aligned}$$

d) Define the condition at which $[\widehat{H}, \hat{p}] = 0$. (0.25 pt.)

Solution: If $V(x)$ is constant or independent of position, $\frac{dV}{dx}$ goes to zero, making the operator commute.

Problem 2

Calculate the commutator of raising and lowering operators: $[\widehat{a_+}, \widehat{a_-}]$.(0.25pt.)

Solution:

$$a_{\pm} = \frac{1}{\sqrt{2m\hbar\omega}} (\mp i\hat{p} + m\omega\hat{x})$$

$$\begin{aligned}[\widehat{a_+}, \widehat{a_-}] &= \frac{1}{2m\hbar\omega} (-i\hat{p} + m\omega\hat{x})(i\hat{p} + m\omega\hat{x}) - \frac{1}{2m\hbar\omega} (i\hat{p} + m\omega\hat{x})(-i\hat{p} + m\omega\hat{x}) \\ &= 0\end{aligned}$$

Problem 3

(Problem 2.7) Infinite square well. A particle in the infinite square well has the initial wave function:

$$\Psi(x, 0) = \begin{cases} Ax, 0 \leq x \leq a/2 \\ A(a - x), a/2 \leq x \leq a \end{cases}$$

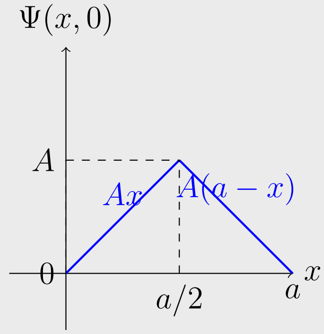
a) Sketch $\Psi(x, 0)$, and determine the constant A (normalize wavefunction).

Solution:

$$\begin{aligned}1 &= \int_0^{a/2} A^2x^2 \, dx + \int_{a/2}^{a} A^2(a - x)^2 \, dx \\ 1 &= \frac{A^2a^3}{24} + \frac{A^2a^3}{24} \to \frac{2A^2a^3}{24} \\ A^2 &= \frac{24}{2a^3} \\ A &= \sqrt{\frac{12}{a^3}}\end{aligned}$$

Problem 3 continued on next page...

A plot of the above wavefunction is:



b) Find $\Psi(x,t)$.

Solution:

Begin by finding c_n:

$$c_n = \int \psi_n^* \Psi(x,0)\, dx$$
$$= \begin{cases} \frac{2}{a}\int_0^{a/2} \sqrt{\frac{12}{a^3}} x \sin\left(\frac{n\pi x}{a}\right)\, dx, & 0 \leq x \leq a/2 \\ \frac{2}{a}\int_{a/2}^{a} \sqrt{\frac{12}{a^3}} (a-x) \sin\left(\frac{n\pi x}{a}\right)\, dx, & a/2 \leq x \leq a \end{cases}$$
$$= \begin{cases} \frac{2\sqrt{3}\left(2\sin\left(\frac{\pi n}{2}\right)-\pi n\cos\left(\frac{\pi n}{2}\right)\right)}{\pi^2 n^2 \sqrt{a}}, & 0 \leq x \leq a/2 \\ \frac{2\sqrt{3}\left(\pi n\cos\left(\frac{\pi n}{2}\right)+2\sin\left(\frac{\pi n}{2}\right)\right)}{\pi^2 n^2 \sqrt{a}}, & a/2 \leq x \leq a \end{cases}$$

The general solution is therefore (leaving the c_n's symbolic to save space on the page):

$$\Psi(x,t) = \begin{cases} \sqrt{\frac{12}{a^3}} \sum c_n x \exp\left(\frac{-iE_n}{\hbar}t\right), & 0 \leq x \leq a/2 \\ \sqrt{\frac{12}{a^3}} \sum c_n (a-x) \exp\left(\frac{-iE_n}{\hbar}t\right), & a/2 \leq x \leq a \end{cases}$$

c) What is the probability that a measurement of the energy would yield the value E_1.

Solution:

The expectation value for energy is defined as $\sum c_n^2 E_n$, where the square of c_n goes to 1 at infinity; consequently, c_n^2 is the probability to measure a given energy, so:

$$P(E_1) = c_{n1}c_{n1}^*$$
$$= \begin{cases} \frac{12\left(2\sin\left(\frac{\pi(1)}{2}\right)-\pi(1)\cos\left(\frac{\pi(1)}{2}\right)\right)^2}{\pi^4(1)^4 a}, & 0 \le x \le a/2 \\ \frac{12\left(\pi(1)\cos\left(\frac{\pi(1)}{2}\right)+2\sin\left(\frac{\pi(1)}{2}\right)\right)^2}{\pi^4(1)^4 a}, & a/2 \le x \le a \end{cases}$$

This of course depends on the barrier width, a.

d) Find the expectation value of the energy (it doesn't matter how you will achieve that). (0.5 pt.)

Solution:

Using the expectation value for the Hamiltonian defined in terms of $\hat{T}, \hat{V}$:

(I believe this only involves the time-independent equation, not the generalized form)

$$\langle \hat{H} \rangle = \int_0^{a/2} \psi_I^* \hat{H} \psi_I \; dx + \int_{a/2}^{a} \psi_{II}^* \hat{H} \psi_{II} \; dx$$
$$= \int_0^{a/2} -\frac{\hbar^2}{2m}\overset{\frac{24}{a^3}}{\cancel{\frac{\partial^2}{\partial x^2}\frac{12}{a^3}x^2}} + \frac{12}{a^3}(x)^2 V \; dx + \int_{a/2}^{a} -\frac{\hbar^2}{2m}\overset{\frac{24}{a^3}}{\cancel{\frac{\partial^2}{\partial x^2}\frac{12}{a^3}(a-x)^2}} + \frac{12}{a^3}(a-x)^2 V \; dx$$
$$= \int_0^{a/2} -\frac{\hbar^2}{2m}\frac{24}{a^3} + \frac{12}{a^3}x^2 V \; dx + \int_{a/2}^{a} \frac{-\hbar^2}{2m}\frac{24}{a^3} + \frac{12}{a^3}(a-x)V \; dx$$
$$= \frac{\hbar^2}{2m}\left(\frac{24}{a^2}\right) + V$$

Problem 4

(Problem 2.10) Harmonic oscillator.

a) Construct $\psi_2(x)$.

Solution:

$$\psi_n = A_n \hat{a}_+^n \psi_0$$
$$\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right)$$

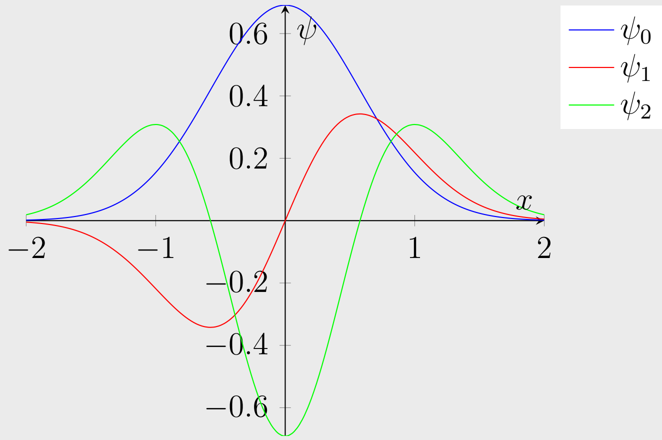
$$\begin{aligned}
\psi_1 = A_1 \hat{a}_+ \psi_0 &= \frac{A_1}{\sqrt{2\hbar m\omega}} \left(-\hbar\frac{d}{dx} + m\omega x\right) \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x} \\
&= A_1 \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{\frac{2m\omega}{\hbar}} x e^{-m\omega x^2/2\hbar}
\end{aligned}$$

$$\begin{aligned}
\psi_2 &= \frac{A_1}{\sqrt{2m\hbar\omega}} \left(-\hbar\frac{d}{dx} + m\omega x\right) \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{\frac{2m\omega}{\hbar}} x e^{-m\omega x^2/2\hbar} \\
&= \frac{A_1}{\sqrt{2m\hbar\omega}} \left(-\hbar \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{\frac{2m\omega}{\hbar}} \left(e^{-m\omega x^2/2\hbar} + \left(-\frac{m\omega}{\hbar}\right) x^2 e^{-m\omega x^2/2\hbar}\right)\right) \\
&+ (m\omega x) \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{\frac{2m\omega}{\hbar}} x e^{-m\omega x^2/2\hbar} \\
&= A_2 \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \left(\frac{2m\omega}{\hbar}x^2 - 1\right) \exp\left(-\frac{m\omega}{2\hbar}x^2\right)
\end{aligned}$$

Problem 4 continued on next page. . .

b) Sketch $\psi_0(x), \psi_1(x), \psi_2(x)$.

Solution:



c) Check the orthogonality of $\psi_0(x), \psi_1(x), \psi_2(x)$ by explicit integration. (0.5pt.)

Solution:

(1) $\psi_0(x)$:

$$\begin{aligned}\int_{-\infty}^{\infty} \psi_1^* \psi_0 \, dx &= \int_{-\infty}^{\infty} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{\frac{2m\omega}{\hbar}} x e^{-m\omega x^2/2\hbar} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar} x^2\right) \, dx \\ &= \int_{-\infty}^{\infty} \left(\frac{2m^2\omega^2}{\pi\hbar^2}\right)^{\frac{1}{2}} x e^{-\frac{m\omega x^2}{\hbar}} \, dx \\ &= -\frac{1}{\sqrt{2\pi}} e^{-\frac{\omega m x^2}{\hbar}} \Big|_{-\infty}^{\infty} \\ &= 0\end{aligned}$$

(2) $\psi_1(x)$: Because there is no complex component in these, the integration is identical as with ψ_0 above.

(3) $\psi_2(x)$:

$$\begin{aligned}\int_{-\infty}^{\infty} \psi_0^* \psi_2 \, dx &= \int_{-\infty}^{\infty} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{\frac{2m\omega}{\hbar}} x e^{-m\omega x^2/2\hbar} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \left(\frac{2m\omega}{\hbar} x^2 - 1\right) \exp\left(-\frac{m\omega}{2\hbar} x^2\right) \, dx \\ &= \int_{-\infty}^{\infty} \left(\frac{2m^2\omega^2}{\pi\hbar^2}\right)^{\frac{1}{2}} x \exp\left(-\frac{m\omega x^2}{\hbar}\right) \, dx \\ &= -\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{mwx^2}{h}\right)\Big|_{-\infty}^{\infty} \qquad = 0\end{aligned}$$

Problem 5

(Problem 2.13) A particle in the harmonic oscillator potential starts out in the state:

$$\Psi(x, 0) = A\left[3\psi_0(x) + 4\psi_1(x)\right].$$

a) Find A.

Solution:

As a note to myself, here I attempted the normalization, moved on, then remembered the properties of orthonormal wave functions. The normalization can be greatly simplified by applying them:

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} A^2[3\psi_0 + 4\psi_1][3\psi_0^* + 4\psi_1^*]\ dx \\
&= A^2 \int_{-\infty}^{\infty} \cancelto{9}{9\psi_0\psi_0^*} + \cancelto{0}{12\psi_0\psi_1^*} + \cancelto{0}{12\psi_1\psi_0^*} + \cancelto{16}{16\psi_1\psi_1^*}\ dx \\
A &= \sqrt{\frac{1}{25}} = \frac{1}{5}
\end{aligned}$$

b) Construct $\Psi(x, t)$ and $|\Psi(x, t)|^2$.

Solution:

$$\begin{aligned}
\Psi(x, t) &= \frac{3}{5}\psi_0(x) + \frac{4}{5}\psi_1(x) \\
&= \begin{bmatrix} \frac{3}{5}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right) \exp\left(\frac{-iE_0 t}{\hbar}\right) \\ +\frac{4}{5}\left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4} x \exp\left(-\frac{m\omega}{2\hbar}x^2\right) \exp\left(\frac{-iE_1 t}{\hbar}\right) \end{bmatrix} \\
&= \begin{bmatrix} \frac{3}{5}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right) \exp\left(\frac{-i\omega}{2}t\right) \\ +\frac{4}{5}\left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4} x \exp\left(-\frac{m\omega}{2\hbar}x^2\right) \exp\left(\frac{-3i\omega}{2}t\right) \end{bmatrix} \qquad (1)
\end{aligned}$$

Problem 5 continued on next page...

$$|\Psi(x,t)|^2 = \Psi(x,t)\Psi^*(x,t)$$

$$= \left[\begin{array}{c} \frac{3}{5}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right)\exp\left(\frac{-i\omega}{2}t\right) \\ +\frac{4}{5}\left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4} x\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\exp\left(\frac{-3i\omega}{2}t\right) \end{array}\right]$$

$$\times \left[\begin{array}{c} \frac{3}{5}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right)\exp\left(\frac{i\omega}{2}t\right) \\ +\frac{4}{5}\left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4} x\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\exp\left(\frac{3i\omega}{2}t\right) \end{array}\right]$$

Out of respect for myself, as I am writing this in LaTeX, I will just skip to the simplified answer:

$$\frac{1}{25}\sqrt{\frac{m\omega}{\pi\hbar}}\left[9+\frac{32m\omega}{\hbar}x^2+12\sqrt{\frac{2m\omega}{\hbar}}x\cancelto{\cos(\omega t)}{(e^{i\omega t}+e^{-i\omega t})}\right]\exp\left(-\frac{m\omega}{\hbar}x^2\right) \tag{2}$$

c) Find $\langle x\rangle$ and $\langle p\rangle$. (0.5 pt.)

Note: you don't need to check if Ehrenfest's theorem holds as it is asked in Griffiths. We will come back to this theorem later in the course.

Solution:

(1) $\langle x\rangle$:

$$\int_{-\infty}^{\infty} x\frac{1}{25}\sqrt{\frac{m\omega}{\pi\hbar}}\left[9+\frac{32m\omega}{\hbar}x^2+12\sqrt{\frac{2m\omega}{\hbar}}x\cos(\omega t)\right]\exp\left(-\frac{m\omega}{\hbar}x^2\right)\,dx$$
$$=\frac{1}{25}\sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty} A+B+C\,dx$$

Let:

$$A = 9x\exp\left(\frac{-m\omega}{\hbar}x^2\right)$$
$$B = \frac{32m\omega}{\hbar}x^3\exp\left(\frac{-m\omega}{\hbar}x^2\right)$$
$$C = 12\sqrt{\frac{2m\omega}{\hbar}}x^2\cos(\omega t)\exp\left(\frac{-m\omega}{\hbar}x^2\right)$$

Integrating these separately gives:

$$A = \frac{9\hbar}{m\omega}\exp\left(-\frac{m\omega}{\hbar}x^2\right)\Big|_{-\infty}^{\infty} = 0$$
$$B = 0$$
$$C = \frac{1}{4}\sqrt{\pi}$$

A_2 threw me off and some research indicated x to an odd power outside the exponent should make the integral go to zero due to "odd symmetry". Recombining these gives:

$$\frac{6}{25}\sqrt{\frac{2\hbar}{\pi m\omega}}\cos(\omega t)$$

(2) $\langle p\rangle$:

$$\int_{-\infty}^{\infty} -i\hbar\frac{\partial}{\partial x}|\Psi|^2\,dx$$

(this is a terrible derivative and integral which makes me question my correctness, I should bring this up at office hours to inquire about an easier way to do this)

I'll make the substitution $\alpha = \frac{\sqrt{\frac{2}{\pi}}m^2\omega^2\cos(\omega t)}{\hbar^2}$, as this is a constant in the eyes of the integral. I'll also substitute $\beta = \frac{m\omega\sqrt{\frac{m\omega}{\hbar}}}{\sqrt{\pi}\hbar}$. Third, I'll substitute $\gamma = \frac{m\omega}{\hbar}$.

$$i\hbar\int_{-\infty}^{\infty}\frac{24}{25}\alpha x^2 e^{-\gamma x^2} + \cancelto{0}{\frac{64}{25\hbar}\beta x^3 e^{-\gamma x^2}} - \frac{12\hbar}{25}\alpha e^{-\gamma x^2} - \cancelto{0}{\frac{46\hbar}{25}\beta x e^{-\gamma x^2}}$$
$$= i\hbar\left(\sqrt{\frac{\pi}{4}}\frac{\alpha}{(\gamma)^{3/2}} - \sqrt{\pi}\frac{\alpha}{\gamma}\right)$$

Making the substitutions gives:

$$= i\hbar\left(\sqrt{\frac{\pi}{4}}\frac{\sqrt{\frac{2}{\pi}}m^2\omega^2\cos(\omega t)}{\hbar^2(\frac{m\omega}{\hbar})^{3/2}} - \sqrt{\pi}\frac{\sqrt{\frac{2}{\pi}}m^2\omega^2\cos(\omega t)}{\hbar^2\frac{m\omega}{\hbar}}\right)$$