PHSX 611: Homework #2

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Problem 1

Exercise 1.8.2 (Shankar) Consider the matrix

$$\Omega = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

(1) Is it Hermitian?

Solution: Yes, this is Hermitian. Ω is equal to its complex conjugate transpose:

$$\Omega^\dagger = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \Omega$$

(2) Find its eigenvalues and eigenvectors.

Solution:

i. Eigenvalues:

$$\begin{vmatrix} 0-\lambda & 0 & 1 \\ 0 & 0-\lambda & 0 \\ 1 & 0 & 0-\lambda \end{vmatrix} = 0$$

$$\implies -\lambda(\lambda^2) + (\lambda) = 0$$

$$-\lambda(\lambda+1)(\lambda-1) = 0$$

$$\implies \lambda = 0, \pm 1$$

ii. Eigenvectors:

$$\left(\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right) \implies |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right) \implies |2\rangle = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \implies |3\rangle = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

(3) Verify that $U^\dagger \Omega U$ is diagonal, U being the matrix of eigenvectors of Ω.

Solution:

Ordinarily, a diagonalized matrix is given by:

$$\Omega = U\Lambda U^{-1}$$

Where it can be rearranged to give

$$U^{-1}\Omega U = \Lambda$$

However, if Ω is Hermitian, $U^\dagger \Omega U$ is also diagonal:

$$U^\dagger \Omega U = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Notice that this is a non-normalized (solely because I did not normalize my eigenvectors) matrix of eigenvalues along the diagonal (Λ). Note that we can also express Ω as:

$$\Omega = U\Lambda U^\dagger = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

(again I was lazy with my normalization, apologies)

Problem 2

Exercise 1.13 (Sakurai) A two-state system is characterized by the Hamiltonian

$$\hat{H} = H_{11} |1\rangle \langle 1| + H_{22} |2\rangle \langle 2| + H_{12}[|1\rangle \langle 2| + |2\rangle \langle 1|]$$

where H_{11}, H_{22} and H_{12} are real numbers with the dimension of energy, and $|1\rangle$ and $|2\rangle$ are eigenkets of some observable ($\neq H$). Find the energy eigenkets and their corresponding energy eigenvalues. Make sure that your answer makes good sense for $H_{12} = 0$.

(Hint: you can treat $|1\rangle$ and $|2\rangle$ as an orthonormal basis, i.e. $\langle i|j\rangle = \delta_{ij}$ where $i, j = 1$ or 2)

Solution:

$$\langle 1|\hat{H}|1\rangle = H_{11} \quad \langle 1|\hat{H}|2\rangle = H_{12}$$
$$\langle 2|\hat{H}|1\rangle = H_{12} \quad \langle 2|\hat{H}|2\rangle = H_{22}$$

i. Finding eigenvalues:

$$\begin{vmatrix} H_{11} - \lambda & H_{12} \\ H_{12} & H_{22} - \lambda \end{vmatrix} = 0$$
$$(H_{11} - \lambda)(H_{22} - \lambda) - 2H_{12} = 0$$

The solution to this quadratic gives

$$\lambda = \frac{1}{2}\left[H_{11} + H_{22} \pm \sqrt{H_{11}^2 - 2H_{11}H_{22} + H_{22}^2 + 8H_{12}}\right]$$

ii. Finding eigenvectors: Solving for a general eigenvalue

$$\left(\begin{array}{cc|c} H_{11} - \lambda & H_{12} & 0 \\ H_{12} & H_{22} - \lambda & 0 \end{array}\right)$$
$$\implies H_{11} = \frac{\lambda - H_{22}}{H_{12}}$$
$$|\xi\rangle = \begin{pmatrix} 1 \\ \frac{\lambda - H_{22}}{H_{12}} \end{pmatrix}$$

Substitution of λ into the eigenket gives:

$$|1\rangle = \begin{pmatrix} 1 \\ \frac{H_{11} + \sqrt{8H_{12} + (H_{11} - H_{22})^2}}{2H_{12}} \end{pmatrix}$$
$$|2\rangle = \begin{pmatrix} 1 \\ \frac{H_{11} - \sqrt{8H_{12} + (H_{11} - H_{22})^2}}{2H_{12}} \end{pmatrix}$$

iii. Alternatively, if $H_{12} = 0$, we get eigenvalues $\lambda = H_{11}$ & H_{22}. Substitution into the matrix and solving like before gives

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$|2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Problem 3

Exercise 1.8.5 (Shankar) Consider the matrix

$$\Omega = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

(1) Show that it is unitary.

Solution:

A matrix is unitary if

$$UU^\dagger = I$$

This is indeed the case!

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \cos\theta\sin\theta \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(2) Show that its eigenvalues are $e^{i\theta}$ and $e^{i\theta}$.

Solution:

$$\left| \begin{array}{cc|c} \cos\theta - \lambda & \sin\theta & 0 \\ -\sin\theta & \cos\theta - \lambda & 0 \end{array} \right|$$

$$(\cos\theta - \lambda)^2 + \sin^2\theta = 0$$
$$\lambda^2 + 2\lambda\cos\theta + \cancelto{1}{\cos^2\theta + \sin^2\theta} = 0$$
$$\frac{-2\cos\theta \pm \sqrt{(\cos^2\theta) - 4}}{2} = \lambda$$

$$\lambda = \cos\theta \pm \sqrt{\cos^2\theta - 1} = \cos\theta \pm i\sin\theta = e^{\pm i\theta}$$

Problem 3 continued on next page...

(3) Find the corresponding eigenvectors; show that they are orthagonal.

Solution:

i. Eigenvalue 1.

$$\left(\begin{array}{cc|c} \cos\theta - e^{i\theta} & \sin\theta & 0 \\ -\sin\theta & \cos\theta - e^{i\theta} & 0 \end{array}\right) = \left(\begin{array}{cc|c} -i\sin\theta & \sin\theta & 0 \\ -\sin\theta & -i\sin\theta & 0 \end{array}\right)$$
$$= \left(\begin{array}{cc|c} 1 & -i & 0 \\ 1 & -i & 0 \end{array}\right)$$

ii. Eigenvalue 2.

$$\left(\begin{array}{cc|c} \cos\theta + e^{i\theta} & \sin\theta & 0 \\ -\sin\theta & \cos\theta + e^{i\theta} & 0 \end{array}\right) = \left(\begin{array}{cc|c} i\sin\theta & \sin\theta & 0 \\ -\sin\theta & i\sin\theta & 0 \end{array}\right)$$
$$= \left(\begin{array}{cc|c} 1 & i & 0 \\ 1 & i & 0 \end{array}\right)$$

This gives eigenvectors:

$$|1\rangle = \begin{pmatrix} i \\ 1 \end{pmatrix}$$
$$|2\rangle = \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

Notice that these are conjugate. They are also clearly orthagonal by geometry. Just to be sure, though

$$\langle 1|2\rangle = (i)(-i) + 1 = -1 + 1 = 0$$

(4) Verify that $U^\dagger \Omega U$ = (diagonal matrix), where U is the matrix of eigenvectors of Ω

Solution:

$$\begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix} \begin{pmatrix} i\cos\theta + \sin\theta & -i\cos\theta + \sin\theta \\ -i\sin\theta + \cos\theta & i\sin\theta + \cos\theta \end{pmatrix}$$
$$= \begin{pmatrix} 2\cos\theta - 2i\sin\theta & 0 \\ 0 & 2i\sin\theta + 2\cos\theta \end{pmatrix}$$
$$= \begin{pmatrix} -2e^{i\theta} & \\ 0 & 2e^{i\theta} \end{pmatrix}$$

Problem 4

Exercise 1.10.1 (Shankar) Show that $\delta(ax) = \delta(x)/|a|$. [Consider $\int \delta(ax)\ d(ax)$. Remember that $\delta(x) = \delta(-x)$.]

Solution #1:
My intuition tells me that as Shankar relates the delta function to the gaussian, $\delta(ax)$ corresponds to stretching along the x. Because it is not normalized in such a case, this "flattens" the curve. This has the effect of scaling the delta function inversely by a. To prove this more rigorously, though-

$$\lim_{\Delta \to 0} \int_{-\infty}^{\infty} \frac{1}{4\pi\Delta^2}^{1/2} \exp\left[-\frac{(ax)^2}{\Delta^2}\right]\ dx$$

We can see the effects a has on the delta function by integrating over all space. I have done such integrals by changing to polar coordinates in the past, but in this case we can check just about any book on QM (even wikipedia has a page on "common integrals in QFT" with a similar integral) to see that this is a well known integral. The effect of this a^2 term does what we expect, scaling by $\frac{1}{|a|}$.

Solution #2:
Another way to show frame this is by utilizing a change of $dx \to d(ax)$ alongisde the properties of delta functions. To get to the integral with respect ax, we need to introduce the substitution

$$dx = \frac{1}{a} d(ax)$$

And since $\delta(x) = \delta(-x)$, $a = |a|$, by the properties of delta functions:

$$\frac{1}{|a|} \int_{-\infty}^{\infty} \delta(|a|x)\ d(|a|x) = \frac{1}{|a|}\delta(x)$$

Problem 5

Exercise 1.10.3 (Shankar) Consider the theta function $\theta(x-x')$ which vanishes if $x-x'$ is negative and equals 1 if $x-x'$ is positive. Show that $\delta(x-x') = d/dx\theta(x-x')$.

Solution:

$$\begin{aligned}
\int_{-\infty}^{\infty} g(x)\frac{d}{\cancel{dx}}\theta(x-x')\,\cancel{dx} \\
[\theta(x-x')g(x)]|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} g'(x)\theta(x-x')dx \\
g(\infty) - [g(\infty) - g(0)] \\
= g(0) = \int_{-\infty}^{\infty} g(x)\delta(x)\,dx
\end{aligned}$$