

Chapter 9

Quantum Statistical Mechanics

9.1 Black-body Radiation

We can model the electromagnetic radiation emitted from a Black-body radiator as resulting from a gas of photons contained within the radiator. If the accessible energy states for the photons in the gas are $\epsilon_1, \epsilon_2, \dots$, then the energy of the gas is

$$E = n_1\epsilon_1 + n_2\epsilon_2 + \dots$$

The variable n_i in this equation denotes the number of photons with energy ϵ_i . In other words, n_i is the number of photons in the state with energy ϵ_i . The partition function for this gas is found by summing over all possible values of n_i .

$$Z = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \dots e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \dots)} \rightarrow Z = \prod_{i=1}^{\infty} \sum_{n_i=0}^{\infty} e^{-\beta n_i \epsilon_i}$$

The summation over the number of photons in each energy state is a power series.

$$\sum_{n_i=0}^{\infty} e^{-\beta n_i \epsilon_i} = \frac{1}{1 - e^{-\beta \epsilon_i}} \rightarrow Z = \prod_{i=1}^{\infty} \frac{1}{1 - e^{-\beta \epsilon_i}}$$

We can convert the product in this equation to a sum by taking the natural log of both sides of the equation.

$$\ln Z = - \sum_{i=1}^{\infty} \ln(1 - e^{-\beta \epsilon_i})$$

Lastly, we can express the partition function in terms of the angular frequency of the photons rather than their energies.

$$\epsilon_i = \hbar\omega_i \quad \rightarrow \quad \ln Z = - \sum_{i=1}^{\infty} \ln(1 - e^{-\beta\hbar\omega_i})$$

Density of States

In order to proceed with calculating the partition function for the photon gas, we need to determine the corresponding density of states for the photons in this system. For a 3-dimensional gas, we have

$$\Gamma(k) = 2 \left[\frac{1}{8} \frac{\frac{4}{3}\pi k^3}{(\frac{\pi}{L})^3} \right] \quad \rightarrow \quad \Gamma(k) = \frac{L^3 k^3}{3\pi^2}$$

The factor of 2 in this equation accounts for the two possible polarizations of the photons. The density of states for this system is thus

$$D(k)dk = \frac{L^3 k^2}{\pi^2} dk \quad \rightarrow \quad D(k)dk = \frac{V k^2}{\pi^2} dk$$

The variable $V = L^3$ denotes the volume of the gas. Writing this expression in terms of ω rather than k gives us

$$k = \frac{\omega}{c} \quad \rightarrow \quad D(\omega)d\omega = \frac{V \left(\frac{\omega}{c}\right)^2}{\pi^2} \frac{d\omega}{c}$$

$$D(\omega)d\omega = \frac{V\omega^2}{\pi^2 c^3} d\omega$$

Energy and Energy Density

We can now determine the partition function through integration.

$$\ln Z = - \int_0^{\infty} \ln(1 - e^{-\beta\hbar\omega}) \frac{V\omega^2}{\pi^2 c^3} d\omega$$

$$\ln Z = - \left(\frac{V}{\pi^2 c^3} \right) \left(-\frac{\pi^4}{45\beta^3 \hbar^3} \right) \quad \rightarrow \quad \ln Z = \frac{V\pi^2}{45(\beta\hbar c)^3}$$

The mean energy of the gas is determined from differentiation of this partition function.

$$\overline{E} = - \frac{\partial}{\partial \beta} \left[\frac{V\pi^2}{45(\beta\hbar c)^3} \right] \quad \rightarrow \quad \overline{E} = \frac{V\pi^2}{15(\hbar c)^3 \beta^4}$$

It is customary to express this equation in terms of the constant a .

$$a = \frac{\pi^2 k_B^4}{15(\hbar c)^3} \quad \rightarrow \quad \overline{E} = a V T^4$$

Thus, the energy density (*i.e.*, energy per volume) of the gas is

$$u(\beta) = \frac{\bar{E}}{V} \rightarrow u(T) = aT^4$$

This equation relating the intensity of the Black-body's photons to the Black-body's temperature is known as the Stefan-Boltzmann Law. However, this law is more commonly written as a relationship between the total energy radiated per unit surface area per unit time, known as the radiant exitance and denoted as M .

$$M = \frac{c}{4}u \rightarrow M = \sigma T^4$$

The variable σ in this equation is the Stefan-Boltzmann constant.

$$\sigma = \frac{c}{4}a \rightarrow \sigma = \frac{\pi^2 k_B^4}{60 \hbar^3 c^2}$$

Pressure

The pressure of the photon gas can also be determined from the partition function of the gas.

$$P = \frac{1}{\beta} \frac{\partial}{\partial V} \left[\frac{V\pi^2}{45(\beta\hbar c)^3} \right] \rightarrow P = \frac{\pi^2}{45(\hbar c)^3 \beta^4} \rightarrow P = \frac{1}{3} a T^4$$

$$P = \frac{1}{3} \frac{\bar{E}}{V}$$

As expected, the pressure of the gas is a measure of the energy density of the gas

Entropy

The entropy of the gas can also be determined from the partition function of the gas

$$\ln Z = \frac{aVT^3}{3k_B} \rightarrow F = -\frac{1}{3}aVT^4$$

$$S = - \left(\frac{\partial F}{\partial T} \right)_V = \frac{4}{3}aVT^3$$

As expected, the entropy is a function of volume and temperature (*i.e.*, volume and energy).

Number of Particles

The number of particles in each accessible energy state can also be found from the partition function of the gas.

$$\bar{n}_i = -\frac{1}{\beta} \frac{\partial}{\partial \epsilon_i} \ln Z = \frac{e^{-\beta \epsilon_i}}{1 - e^{-\beta \epsilon_i}} = \frac{1}{e^{\beta \epsilon_i} - 1}$$

As before, we can write this equation in terms of the angular frequency of the photon rather than the energy of the photon.

$$\bar{n}_i = \frac{1}{e^{\beta \hbar \omega_i} - 1}$$

The number of photons with angular frequency between ω and $\omega + d\omega$, denoted as dN_ω , is the product of \bar{n}_i and the density of states determined earlier.

$$dN_\omega = \frac{1}{e^{\beta \hbar \omega} - 1} \frac{V \omega^2}{\pi^2 c^3} d\omega$$

The energy of the photons with angular frequency between ω and $\omega + d\omega$, denoted as dE_ω , is the product of dN_ω and the photon energy $\epsilon = \hbar \omega$.

$$dE_\omega = \hbar \omega dN_\omega = \left(\frac{V}{\pi^2 c^3} \right) \frac{\hbar \omega^3}{e^{\beta \hbar \omega} - 1} d\omega$$

The total energy density is thus

$$u(\beta, \omega) d\omega = \frac{dE_\omega}{V} = \left(\frac{1}{\pi^2 c^3} \right) \frac{\hbar \omega^3}{e^{\beta \hbar \omega} - 1} d\omega$$

This expression is known as Planck's law. For long wavelengths (*i.e.*, small values of ω) we can approximate this equation as

$$u(\beta, \omega) d\omega = \left(\frac{1}{\pi^2 c^3} \right) \frac{\hbar \omega^3}{1 + \beta \hbar \omega} d\omega$$

$$u(\beta, \omega) d\omega = \left(\frac{1}{\pi^2 c^3} \right) \frac{\hbar \omega^3}{\beta \hbar \omega} d\omega \rightarrow u(\beta, \omega) d\omega = \left(\frac{\omega^2}{\pi^2 c^3 \beta} \right) d\omega$$

The expression $u(\beta, \omega) \approx \omega^2$ is known as the Rayleigh-Jeans Law. The total energy density, denoted as $u(\beta)$, is found through integration of $u(\beta, \omega)$ over all values of ω .

$$u(\beta) = \int_0^\infty u(\beta, \omega) d\omega$$

$$u(\beta) = \int_0^\infty \left(\frac{1}{\pi^2 c^3} \right) \frac{\hbar \omega^3}{e^{\beta \hbar \omega} - 1} d\omega$$

$$u(\beta) = \left(\frac{\hbar}{\pi^2 c^3} \right) \left(\frac{\pi^4}{15\beta^4 \hbar^4} \right) = \frac{\pi^2}{15\beta^4 (\hbar c)^3}$$

Fortunately, this is identical to what we calculated before. The final calculation we perform is to determine the angular frequency associated with the maximum energy density.

$$\frac{d}{d\omega} u(\beta, \omega) = 0 \rightarrow \frac{d}{d\omega} \left[\left(\frac{1}{\pi^2 c^3} \right) \frac{\hbar \omega^3}{e^{\beta \hbar \omega} - 1} \right] = 0$$

$$\left(\frac{1}{\pi^2 c^3} \right) \left[\frac{3\hbar \omega_{max}^2}{e^{\beta \hbar \omega_{max}} - 1} - \frac{\hbar^2 \omega_{max}^3 \beta e^{\beta \hbar \omega_{max}}}{(e^{\beta \hbar \omega_{max}} - 1)^2} \right] = 0$$

$$3(e^{\beta \hbar \omega_{max}} - 1) - \hbar \omega_{max} \beta e^{\beta \hbar \omega_{max}} = 0$$

$$\beta \hbar \omega_{max} = 2.8214 \dots \rightarrow \frac{\hbar \omega_{max}}{k_B T} = 2.8214 \dots$$

This inverse relationship between the maximum frequency and temperature is known as Wien's Displacement Law. Thus, a total of three laws - Wien's Displacement Law, Planck's Law, and the Stefan-Boltzmann's Law - are all obtained from the same partition function. That's a lot of heavy lifting by a single summation over states.

9.2 Fermi Energy

Unsurprisingly, the behavior of fermions are described by Fermi-Dirac statistics. When describing a system of fermions, Equation 3.17 is written as the following continuous distribution

$$f(\epsilon) = \frac{1}{e^{\beta(\epsilon - \mu)} + 1} \quad (9.1)$$

This distribution is often referred to as the Fermi function. The chemical potential μ in this equation is a function of temperature and its value at $T = 0$ is called the Fermi Energy, denoted as ϵ_F . At $T = 0$, the Fermi function becomes

$$f(\epsilon) = \begin{cases} 1 & \epsilon < \epsilon_F \\ 0 & \epsilon > \epsilon_F \end{cases}$$

In other words, at $T = 0$, all states with energy less than the Fermi energy are occupied and all states with energy above the Fermi energy are unoccupied. We can now determine an expression for the Fermi energy using the density of states of the system. For a three dimensional system, the density of states is

$$D(k)dk = 2 \left(\frac{1}{8} \right) \left(\frac{4V k^2}{\pi^2} \right) dk \rightarrow D(k)dk = \left(\frac{V k^2}{\pi^2} \right) dk$$

The factor of 2 accounts for the two possible spin states of the fermions. Expressing this density of states in terms of the energy of the particle gives us

$$\begin{aligned}\epsilon &= \frac{\hbar^2 k^2}{2m} \rightarrow k = \left(\frac{2m\epsilon}{\hbar^2}\right)^{\frac{1}{2}} \rightarrow dk = \left(\frac{2m}{\hbar^2}\right)^{\frac{1}{2}} \frac{1}{2}\epsilon^{-\frac{1}{2}}d\epsilon \\ D(\epsilon)d\epsilon &= \left(\frac{V}{\pi^2}\right) \left(\frac{2m\epsilon}{\hbar^2}\right) \left(\frac{2m}{\hbar^2}\right)^{\frac{1}{2}} \frac{1}{2}\epsilon^{-\frac{1}{2}}d\epsilon \\ D(\epsilon)d\epsilon &= \left(\frac{V}{2\pi^2}\right) \left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \epsilon^{\frac{1}{2}}d\epsilon\end{aligned}$$

Thus, at $T = 0$ we have

$$\begin{aligned}N &= \int_0^{\infty} f(\epsilon)D(\epsilon)d\epsilon = \int_0^{\epsilon_F} \left(\frac{V}{2\pi^2}\right) \left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \epsilon^{\frac{1}{2}}d\epsilon \\ N &= \left(\frac{V}{2\pi^2}\right) \left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \frac{2}{3}\epsilon_F^{\frac{3}{2}} \\ \epsilon_F &= \frac{\hbar^2}{2m} \left(\frac{3N\pi^2}{V}\right)^{\frac{2}{3}}\end{aligned}\tag{9.2}$$

Electrons in metals

We can model the free electrons in a metal as being contained within a 3-dimensional infinite square well potential. These electrons occupy energy states up to the level of the chemical potential of the metal, which is called the Fermi level. The Fermi level of electrons in a metal at room temperature is approximately equal to the Fermi energy of the metal. For example, a typical density of free electrons in a metal is on the order of $\frac{N}{V} = 10^4 \frac{\text{kg}}{\text{m}^3}$, which would correspond to a Fermi energy of $\epsilon_F \approx 10^{-18} \text{J}$. The corresponding Fermi temperature (T_F), obtained by dividing the Fermi energy by the Boltzmann constant, would thus be $T_F \approx 10^5 \text{K}$.

The internal energy of a gas of N fermions is

$$\begin{aligned}\overline{E}_{FD} &= \int_0^{\infty} \epsilon f(\epsilon)D(\epsilon)d\epsilon \\ \overline{E}_{FD} &= \int_0^{\infty} \left(\frac{4V}{\pi^2}\right) \left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \frac{\epsilon^{\frac{3}{2}}}{e^{\beta(\epsilon-\mu)} + 1} d\epsilon \\ \overline{E}_{FD} &= \frac{3}{5}N\epsilon_F \left[1 + \frac{5\pi^2}{12} \left(\frac{T}{T_F}\right)^2 - \frac{\pi^4}{16} \left(\frac{T}{T_F}\right)^4 + \dots\right]\end{aligned}$$

Thus, the average energy of an electron in a metal at $T = 0$ is $\frac{3}{5}\epsilon_F$. This is about four orders of magnitude larger than the average energy of a particle in a classical gas $\frac{3}{2}k_B T$

The heat capacity of the system can be determined from the mean energy of the system.

$$C_{FD} = \frac{d\bar{E}_F}{dT}$$

$$C_{FD} = \frac{\pi^2}{2} N k_B \left[\frac{T}{T_F} - \frac{3\pi^2}{10} \left(\frac{T}{T_F} \right)^3 + \dots \right]$$

Thus, for most temperatures

$$C_{FD} \approx \frac{\pi^2}{2} N k_B \left(\frac{T}{T_F} \right) \rightarrow C_{FD} = \frac{\pi^2}{2} N k_B \left(\frac{k_B T}{\epsilon_F} \right)$$

This is quite a bit smaller than the heat capacity of a particle in a classical gas. While we might reflexively find this surprising since the mean energy of the fermions is so much larger than the mean energy of a classical gas, we must remind ourselves that heat capacity is a measure of how temperature changes with energy changes, rather than being reflective of absolute values of energy or temperature. Indeed, even though the average energy of the electrons in a metal is large, that average energy changes only slightly with changes in temperature since only the electrons in the highest energy quantum states are capable of being excited into yet higher energy states.

The pressure of a gas of fermions can be determined from the Helmholtz free energy of the gas, which, in turn, can be determined from the entropy of the gas, which we can calculate from the heat capacity of the gas.

$$\int_0^S dS_{FD} = \int_0^T \frac{C_e}{T} dT$$

$$S_{FD} = \int_0^T \frac{\pi^2}{2} N k_B \left[\frac{1}{T_F} - \frac{3\pi^2}{10} \left(\frac{T^2}{T_F^3} \right) + \dots \right] dT$$

$$S_{FD} = \frac{\pi^2}{2} N k_B \left[\frac{T}{T_F} - \frac{\pi^2}{10} \left(\frac{T}{T_F} \right)^3 + \dots \right]$$

The Helmholtz free energy is thus

$$F_{FD} = N k_B \left[\frac{3}{5} T_F - \frac{\pi^2}{4} \frac{T^2}{T_F} + \frac{\pi^4}{80} \frac{T^4}{T_F^3} \right]$$

$$F_{FD} = N k_B T_F \left[\frac{3}{5} - \frac{\pi^2}{4} \left(\frac{T}{T_F} \right)^2 + \frac{\pi^4}{80} \left(\frac{T}{T_F} \right)^4 \right]$$

The pressure is found from the derivative of the Helmholtz free energy with respect to volume. Calculating this derivative requires us to differentiate the Fermi temperature with respect to volume.

$$P = - \left(\frac{\partial F}{\partial V} \right)_{T,N} = - \left(\frac{\partial F}{\partial T_F} \right)_{=T,N} \left(\frac{\partial T_F}{\partial V} \right)_{T,N}$$

$$T_F = \frac{\hbar^2}{2mk_B} \left(\frac{3N}{8\pi V} \right)^{\frac{2}{3}} \rightarrow \left(\frac{\partial T_F}{\partial V} \right)_{T,N} = \frac{\hbar^2}{2mk_B} \left(\frac{3N}{8\pi} \right)^{\frac{2}{3}} \left(-\frac{2}{3} \right) \frac{1}{V^{\frac{5}{3}}}$$

$$\left(\frac{\partial T_F}{\partial V} \right)_{T,N} = -\frac{2T_F}{3V}$$

Thus,

$$P_{FD} = - \left[Nk_B \left(\frac{3}{5} + \frac{\pi^2}{4} \frac{T^2}{T_F^2} - \frac{3\pi^4}{80} \frac{T^4}{T_F^4} \right) \right] \left(-\frac{2T_F}{3V} \right)$$

$$P_{FD} = \frac{2Nk_B T_F}{5V} \left[1 + \frac{5\pi^2}{12} \left(\frac{T}{T_F} \right)^2 + \frac{\pi^4}{16} \left(\frac{T}{T_F} \right)^4 \right]$$

$$P_{FD} = \frac{2N\epsilon_F}{5V} \left[1 + \frac{5\pi^2}{12} \left(\frac{T}{T_F} \right)^2 + \frac{\pi^4}{16} \left(\frac{T}{T_F} \right)^4 \right] \quad (9.3)$$

Thus, $P_{FD} = \frac{2}{3} \frac{\bar{E}_{FD}}{V}$, which is the same relationship we have for a classical gas.

9.3 Neutron Stars

Let's consider a neutron star consisting of N neutrons. The average kinetic energy of these neutrons at $T = 0$ is

$$K = \frac{3}{5} N \epsilon_F \rightarrow K = \frac{3}{5} N \frac{\hbar^2}{2m_n} \left(\frac{3\pi^2 N}{V} \right)^{\frac{2}{3}}$$

$$K = \frac{3^{\frac{5}{3}} \hbar^2 \pi^{\frac{4}{3}}}{10m_n} \frac{N^{\frac{5}{3}}}{V^{\frac{2}{3}}}$$

The variable m_n in this equation is the mass of a neutron. We can also express this kinetic energy in terms of the total mass of the neutron star, denoted as M , and the total volume of the star, denoted as V .

$$M = Nm_n \rightarrow K = \frac{3^{\frac{5}{3}} \hbar^2 \pi^{\frac{4}{3}}}{10m_n^{\frac{8}{3}}} \frac{M^{\frac{5}{3}}}{V^{\frac{2}{3}}}$$

$$V = \frac{4}{3}\pi R^3 \rightarrow K = \frac{3^{\frac{5}{3}} \hbar^2 \pi^{\frac{4}{3}}}{10m_n^{\frac{8}{3}}} \frac{M^{\frac{5}{3}}}{\left(\frac{4}{3}\pi R^3\right)^{\frac{2}{3}}}$$

$$K = \frac{3^{\frac{7}{3}} \hbar^2 \pi^{\frac{2}{3}} M^{\frac{5}{3}}}{2^{\frac{7}{3}} m_n^{\frac{8}{3}} 5} \frac{1}{R^2} \rightarrow K = \frac{A}{R^2}$$

The variable A has been introduced for algebraic simplicity. Let's now calculate the gravitational potential energy associated with the interactions between the neutrons in the star. We can express this as a differential equation involving the mass of a thin shell of the neutron star and the mass of the star contained within that shell.

$$dU_g = -G \frac{m_{\text{shell}} m_{\text{interior}}}{r}$$

$$dU_g = -G \frac{(\rho 4\pi r^2 dr) (\frac{4}{3}\pi r^3 \rho)}{r}$$

The variable ρ in this equation is the mass density of the neutron star. We can assume that this density is constant.

$$U_g = \int dU \rightarrow U_g = \int_0^R -G \frac{(\rho 4\pi r^2 dr) (\frac{4}{3}\pi r^3 \rho)}{r}$$

The total gravitational potential energy is found through integration.

$$U_g = -\frac{16}{3} G \rho^2 \pi^2 \int_0^R r^4 dr \rightarrow U_g = -\frac{16}{15} G \rho^2 \pi^2 R^5$$

Let's rewrite this expression in terms of the total mass of the star.

$$U_g = -\frac{16}{15} G \left(\frac{3M}{4\pi R^3} \right)^2 \pi^2 R^5$$

$$U_g = -\frac{3GM^2}{5R} \rightarrow U_g = -\frac{B}{R}$$

The variable B has been introduced for algebraic simplicity. The total energy of the neutron star is the sum of its kinetic energy and gravitational potential energy.

$$E_{\text{tot}} = K + U_g \rightarrow E_{\text{tot}} = \frac{A}{R^2} - \frac{B}{R}$$

At equilibrium, the energy of the neutron star is at a minimum. This allows us to determine the radius of a neutron star corresponding to this equilibrium state.

$$\frac{d}{dR} E_{\text{tot}} = 0 \rightarrow -2 \frac{A}{R_{eq}^2} + \frac{B}{R_{eq}^2} = 0 \rightarrow R_{eq} = 2 \frac{A}{B}$$

$$R_{eq} = 2 \left(\frac{3^{\frac{7}{3}} \hbar^2 \pi^{\frac{2}{3}} M^{\frac{5}{3}}}{2^{\frac{7}{3}} m_n^{\frac{8}{3}} 5} \right) \left(\frac{5}{3GM^2} \right)$$

$$R_{eq} = \frac{3^{\frac{4}{3}} \hbar^2 \pi^{\frac{2}{3}}}{G 2^{\frac{4}{3}} m_n^{\frac{8}{3}} M^{\frac{1}{3}}} \rightarrow R_{eq} = \left(\frac{9\pi}{4}\right)^{\frac{2}{3}} \frac{\hbar^2}{G m_n^{\frac{8}{3}} M^{\frac{1}{3}}}$$

Simplifying this equation through substitution of the values of the constants then give us.

$$R_{eq} = \left(\frac{9\pi}{4}\right)^{\frac{2}{3}} \frac{\left(1.05 \times 10^{-34} \frac{\text{m}^2 \text{kg}}{\text{s}}\right)^2}{\left(6.67 \times 10^{-11} \frac{\text{m}^3}{\text{s}^2 \text{kg}}\right) \left(1.67 \times 10^{-27} \text{kg}\right)^{\frac{8}{3}} M^{\frac{1}{3}}}$$

$$R_{eq} = \frac{1.55 \times 10^{14} \text{kg}^{\frac{1}{3}} \text{m}}{M^{\frac{1}{3}}}$$

Substitution of the typical mass of a neutron star then gives us

$$M = 2 \times 10^{30} \text{kg} \rightarrow R_{eq} = \frac{1.55 \times 10^{14} \text{kg}^{\frac{1}{3}} \text{m}}{(2 \times 10^{30} \text{kg})^{\frac{1}{3}}} \rightarrow R_{eq} = 12.3 \times 10^3 \text{m}$$

9.4 White Dwarfs

A white dwarf is a star is the core left behind after a star has run out of its nuclear fuel and has expelled its outer layers. This core consists mostly of oxygen and carbon, with the associated electrons forming a Fermi gas. The average energy of which at $T = 0$ is

$$K = \frac{3}{5} N \epsilon_F \rightarrow K = \frac{3}{5} N \frac{\hbar^2}{2m_e} \left(\frac{3\pi^2 N}{V}\right)^{\frac{2}{3}}$$

$$K = \frac{3^{\frac{5}{3}} \hbar^2 \pi^{\frac{4}{3}}}{10m_e} \frac{N^{\frac{5}{3}}}{V^{\frac{2}{3}}}$$

Let's assume that each of the N electrons is associated with one proton and one neutron. If we also assume that the mass of a neutron is equal to the mass of a proton, the total mass of the white dwarf is $M = 2Nm_p$. The kinetic energy of the Fermi gas is thus

$$M = 2Nm_p \rightarrow K = \frac{3^{\frac{5}{3}} \hbar^2 \pi^{\frac{4}{3}}}{10m_e m_p^{\frac{5}{3}} 2^{\frac{5}{3}}} \frac{M^{\frac{5}{3}}}{V^{\frac{2}{3}}}$$

Let's next express this kinetic energy in terms of the volume of the star.

$$V = \frac{4}{3} \pi R^3 \rightarrow K = \frac{3^{\frac{5}{3}} \hbar^2 \pi^{\frac{4}{3}}}{10m_e m_p^{\frac{5}{3}} 2^{\frac{5}{3}}} \frac{M^{\frac{5}{3}}}{\left(\frac{4}{3} \pi R^3\right)^{\frac{2}{3}}}$$

$$K = \frac{3^{\frac{7}{3}} \hbar^2 \pi^{\frac{2}{3}}}{5m_e m_p^{\frac{5}{3}} 16} \frac{M^{\frac{5}{3}}}{R^2} \rightarrow K = \frac{A}{R^2}$$

We have introduced the variable A for algebraic simplicity. Building off the derivation above, we see that the equilibrium radius of a white dwarf is

$$R_{eq} = 2 \left(\frac{3^{\frac{7}{3}} \hbar^2 \pi^{\frac{2}{3}} M^{\frac{5}{3}}}{5 m_e m_p^{\frac{5}{3}} 16} \right) \left(\frac{5}{3GM^2} \right)$$

$$R_{eq} = \frac{3^{\frac{4}{3}} \hbar^2 \pi^{\frac{2}{3}}}{GM^{\frac{1}{3}} m_e m_p^{\frac{5}{3}} 8} \rightarrow R_{eq} = \frac{(9\pi)^{\frac{2}{3}}}{8} \frac{\hbar^2}{GM^{\frac{1}{3}} m_e m_p^{\frac{5}{3}}} \quad (9.4)$$

Simplifying this equation through substitution of the values of the constants then give us.

$$R_{eq} = \frac{(9\pi)^{\frac{2}{3}}}{8} \frac{\left(1.05 \times 10^{-34} \frac{\text{m}^2 \text{kg}}{\text{s}}\right)^2}{\left(6.67 \times 10^{-11} \frac{\text{m}^3}{\text{s}^2 \text{kg}}\right) M^{\frac{1}{3}} (9.11 \times 10^{-31} \text{kg}) (1.67 \times 10^{-27} \text{kg})^{\frac{5}{3}}}$$

$$R_{eq} = \frac{8.95 \times 10^{16} \text{kg}^{\frac{1}{3}} m}{M^{\frac{1}{3}}}$$

Substitution of the typical mass of a white dwarf star then gives us

$$M = 2 \times 10^{30} \text{kg} \rightarrow R_{eq} = \frac{8.95 \times 10^{16} \text{kg}^{\frac{1}{3}} m}{(2 \times 10^{30} \text{kg})^{\frac{1}{3}}} \rightarrow R_{eq} = 7.1 \times 10^6 \text{m}$$

9.5 Special Relativity Review

We begin with the relativistic definition of linear momentum for a particle

$$\vec{p} = \gamma m \vec{v}$$

Where $\gamma = \left(\frac{1}{1 - \frac{v^2}{c^2}} \right)^{\frac{1}{2}}$. We can use this definition to write an expression for the net force acting on the particle using Newton's second law.

$$\vec{F}_{net} = \frac{d(\gamma m \vec{v})}{dt} \rightarrow \vec{F}_{net} = \left(\frac{d\gamma}{dt} \right) m v + \gamma m \left(\frac{dv}{dt} \right)$$

$$\vec{F}_{net} = m v \frac{d}{dt} \left[\left(1 - \frac{v^2}{c^2} \right)^{-\frac{1}{2}} \right] + \gamma m a$$

Let's now deploy our good friend the chain rule.

$$\vec{F}_{net} = m v \left[-\frac{1}{2} \left(1 - \frac{v^2}{c^2} \right)^{-\frac{3}{2}} \left(-\frac{2v}{c^2} \right) \left(\frac{dv}{dt} \right) \right] + \gamma m a$$

On we go

$$\begin{aligned}\vec{F}_{net} &= mv \left[\frac{v}{c^2} \left(1 - \frac{v^2}{c^2} \right)^{-\frac{3}{2}} a \right] + \gamma ma \rightarrow \vec{F}_{net} = m \frac{v^2}{c^2} \gamma^3 a + \gamma ma \\ \vec{F}_{net} &= \gamma ma \left[\frac{v^2}{c^2} \gamma^2 + 1 \right] \rightarrow \vec{F}_{net} = \gamma ma \left[\frac{\frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}} + 1 \right] \\ \vec{F}_{net} &= \gamma ma \left[\frac{1}{1 - \frac{v^2}{c^2}} \right] \rightarrow \vec{F}_{net} = \gamma^3 ma\end{aligned}$$

We can now use this expression for the net force to derive an expression for the kinetic energy of the particle.

$$\begin{aligned}K &= \int_0^v (\gamma^3 ma) dx \rightarrow K = \int_0^v \gamma^3 m \left(\frac{dv}{dt} \right) dx \rightarrow K = \int_0^v \gamma^3 m \left(\frac{dx}{dt} \right) dv \\ K &= \int_0^v \gamma^3 mv dv \rightarrow K = \int_0^v \frac{mv}{\left(1 - \frac{v^2}{c^2} \right)^{\frac{3}{2}}} dv\end{aligned}$$

Now for a little variable substitution

$$\begin{aligned}x &= 1 - \frac{v^2}{c^2} \rightarrow \int \frac{v}{\left(1 - \frac{v^2}{c^2} \right)^{\frac{3}{2}}} dv = - \left(\frac{2}{c^2} \right) \int \frac{dx}{x^{\frac{3}{2}}} = - \left(\frac{2}{c^2} \right) \left(-2x^{\frac{1}{2}} \right) \\ \int_0^v \gamma^3 m \left(\frac{dx}{dt} \right) dv &= mc^2 \left(\frac{1}{1 - \frac{v^2}{c^2}} \right)^{\frac{1}{2}} \Big|_0^v \\ K &= mc^2 \left(\left(\frac{1}{1 - \frac{v^2}{c^2}} \right)^{\frac{1}{2}} - 1 \right) \rightarrow K = \gamma mc^2 - mc^2 \\ \gamma mc^2 &= K + mc^2 \rightarrow E_{total} = K + mc^2\end{aligned}$$

The term $E_{total} = \gamma mc^2$ in this expression is the total energy of the particle and mc^2 is the rest energy of the particle. We can show that this expression gives the expected result when $v \ll c$.

$$K = mc^2 \left(\left(\frac{1}{1 - \frac{v^2}{c^2}} \right)^{\frac{1}{2}} - 1 \right) = mc^2 \left(\left(1 - \frac{v^2}{c^2} \right)^{-\frac{1}{2}} - 1 \right)$$

Now is a good time for a series expansion

$$\begin{aligned}
K &= mc^2 \left(1 - \frac{1}{2} \left(-\frac{v^2}{c^2} \right) + \frac{3}{8} \left(-\frac{v^2}{c^2} \right)^2 - \frac{5}{6} \left(-\frac{v^2}{c^2} \right)^3 + \dots - 1 \right) \\
K &= mc^2 \left(\frac{v^2}{2c^2} - \frac{3v^4}{8c^4} + \frac{5v^6}{6c^6} + \dots \right) \\
K &= \frac{1}{2}mv^2 - \frac{3}{8}mv^2 \left(\frac{v}{c} \right)^2 + \frac{5}{6}mv^2 \left(\frac{v}{c} \right)^4 + \dots
\end{aligned}$$

The leading term is the classical expression for the linear momentum of the particle and will dominate the series when $v \ll c$.

Finally, let's determine an equation for the total energy of the particle in terms of the particle's linear momentum.

$$\begin{aligned}
p = \gamma mv &\rightarrow pc = \gamma mc^2 \left(\frac{v}{c} \right) \rightarrow pc = \gamma mc^2 \left(1 - \frac{1}{\gamma^2} \right)^{\frac{1}{2}} \\
(p c)^2 = (\gamma mc^2)^2 \left(1 - \frac{1}{\gamma^2} \right) &\rightarrow (pc)^2 = (\gamma mc^2)^2 - (mc^2)^2 \\
E_{total}^2 &= (pc)^2 + (mc^2)^2
\end{aligned}$$

Pressure

Let's consider a system consisting of a single particle bouncing back and forth between two walls separated by a length L . Let's denote the axis along which the particle moves as the x -axis. If the walls are infinitely massive compared to the particle, the magnitude of the momentum change associated with a collision between the wall and the particle is $2p_x$. The speed of the particle can be expressed in terms of the total energy and momentum of the particle.

$$\frac{p_x = \gamma mv_x}{E_{total} = \gamma mc^2} \rightarrow v_x = \frac{c^2 p_x}{E_{total}}$$

Since the average time between collisions is $\Delta t = \frac{2L_x}{v_x}$, the magnitude of the average force exerted on the wall associated with a collision is

$$\begin{aligned}
F_{avg} &= \frac{\Delta p_x}{\Delta t} \rightarrow F_{avg} = \frac{2p_x}{\frac{2L_x E_{total}}{p_x c^2}} \\
F_{avg} &= \frac{c^2 p_x^2}{L_x E_{total}}
\end{aligned}$$

The pressure associated with this force is found by dividing by the area of the surface over which the force is applied.

$$P_{avg} = \frac{F_{avg}}{A} \rightarrow P_{avg} = \frac{c^2 p_x^2}{AL_x E_{total}}$$

The product AL_x is the volume of the container.

$$P_{avg} = \frac{c^2 p_x^2}{V E_{total}}$$

Let's consider an isotropic gas where the momenta vectors of the gas particles have equal probability to point in any direction. In that case,

$$\overline{p^2} = \overline{p_x^2} + \overline{p_y^2} + \overline{p_z^2} = 3\overline{p_x^2}$$

Thus, the pressure for the gas is

$$P_{avg} = \frac{c^2 p_x^2}{3V E_{total}}$$

Integrating over all possible values of the momentum then yields

$$P = 2 \int \frac{p^2 c^2}{3V \sqrt{p^2 c^2 + m^2 c^4}} \frac{d^3 \vec{p} d^3 \vec{q}}{h^3}$$

The factor of 2 reflects the two possible spin states associated with each momentum state. Simplifying this expression further gives us

$$\begin{aligned} \int d^3 \vec{q} = V &\rightarrow P = \frac{2}{3h^3} \int_0^{p_F} \frac{N p^2 c^2}{3\sqrt{p^2 c^2 + m^2 c^4}} 4\pi p^2 dp \\ P &= \frac{8\pi}{3h^3} \int_0^{p_F} \frac{p^2 c^2}{\sqrt{p^2 c^2 + m^2 c^4}} p^2 dp \end{aligned}$$

In the non-relativistic limit, $p^2 c^2 \ll m^2 c^4$. In this case,

$$\begin{aligned} P_{NR} &= \frac{8\pi}{3h^3} \int_0^{p_F} \frac{p^2 c^2}{\sqrt{m^2 c^4}} p^2 dp \rightarrow P_{NR} = \frac{8\pi}{3h^3} \int_0^{p_F} \frac{p^4}{m} dp \\ P_{NR} &= \frac{8\pi}{15mh^3} p_F^5 \end{aligned}$$

In the ultra-relativistic limit, $p^2 c^2 \gg m^2 c^4$. In this case,

$$\begin{aligned} P_{UR} &= \frac{8\pi}{3h^3} \int_0^{p_F} \frac{p^2 c^2}{\sqrt{p^2 c^2}} p^2 dp \rightarrow P_{UR} = \frac{8\pi}{3h^3} \int_0^{p_F} c p^3 dp \\ P_{UR} &= \frac{2\pi c}{3h^3} p_F^4 \end{aligned}$$

Substitution of the definition of p_F then gives

$$\epsilon_F = \frac{p_F^2}{2m} \rightarrow p_F = \hbar \left(\frac{3N\pi^2}{V} \right)^{\frac{1}{3}}$$

$$P_{NR} = \frac{8\pi}{15m\hbar^3} \left(\frac{3\hbar^3 N \pi^2}{V} \right)^{\frac{5}{3}} \rightarrow P_{NR} = \frac{\hbar^2}{5m} (3\pi^2)^{\frac{2}{3}} \left(\frac{N}{V} \right)^{\frac{5}{3}}$$

$$P_{UR} = \frac{2\pi c}{3\hbar^3} \left(\frac{3\hbar^3 N \pi^2}{V} \right)^{\frac{4}{3}} \rightarrow P_{UR} = \frac{\hbar c}{4} (3\pi^2)^{\frac{1}{3}} \left(\frac{N}{V} \right)^{\frac{4}{3}}$$

We can also rewrite these equations in terms of the density of the gas, $\rho = m \frac{N}{V}$.

$$P_{NR} = \frac{\hbar^2}{5} \left(\frac{3\pi^2}{m^4} \right)^{\frac{2}{3}} \rho^{\frac{5}{3}}$$

$$P_{UR} = \frac{\hbar c}{4} \left(\frac{3\pi^2}{m^4} \right)^{\frac{1}{3}} \rho^{\frac{4}{3}}$$

9.6 Black Holes

The pressure a macroscopic spherical object must exert in order to resist collapsing under the force of gravity acting on its constituent bits of mass can be determined from the equation of hydrostatic equilibrium.

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2}$$

The variables M , r , and ρ in this equation are the mass, radius, and density, respectively, of the sphere. The mass of the sphere is a function of the density of the sphere, denoted by ρ , and the radius of the sphere.

$$\frac{dP}{dr} = -\frac{G(\frac{4}{3}\pi r^3 \rho)}{r^2} \rightarrow \frac{dP}{dr} = -\frac{4G\pi\rho^2}{3}r$$

The pressure can be found through integration. If the radius of the sphere is R , we have

$$dP = -\frac{4G\pi\rho^2}{3}r dr \rightarrow \int_P^0 dP = -\frac{4G\pi\rho^2}{3} \int_0^R r dr$$

The limits of integration for the pressure reflect the fact that the pressure will be largest at the center of the sphere and smallest (equal to zero) at the surface of the sphere. Completing the integration gives us

$$P = \frac{2}{3}\pi G \rho^2 R^2$$

Let's now express the density in terms of the mass and radius of the sphere.

$$P = \frac{2}{3}\pi G \left(\frac{3M}{4\pi R^3} \right)^2 R^2 \rightarrow P_g = \frac{3GM^2}{8\pi R^4}$$

The subscript g has been added to denote that this is gravitational pressure. Now let's first consider the case of a non-relativistic black hole.

$$P_{NR} = P_g \rightarrow \frac{\hbar^2}{5m} (3\pi^2)^{\frac{2}{3}} \left(\frac{N}{V}\right)^{\frac{5}{3}} = \frac{3GM^2}{8\pi R^4}$$

The number of particles in the sphere is related to the total mass of the sphere by the number of particles in the sphere N and the mass of each particle m by the equation $M = Nm$. If the radius of the sphere is R , then our equilibrium condition becomes

$$\begin{aligned} \frac{\hbar^2}{5m} (3\pi^2)^{\frac{2}{3}} \left(\frac{M}{m(\frac{4}{3}\pi R^3)}\right)^{\frac{5}{3}} &= \frac{3GM^2}{8\pi R^4} \\ \frac{\hbar^2 M^{\frac{5}{3}} 3^{\frac{7}{3}}}{5m^{\frac{8}{3}} \pi^{\frac{1}{3}} 4^{\frac{5}{3}} R^5} &= \frac{3GM^2}{8\pi R^4} \end{aligned}$$

Solving for the radius yields

$$R = \frac{8\pi^{\frac{2}{3}} \hbar^2 3^{\frac{4}{3}}}{5Gm_n^{\frac{8}{3}} 4^{\frac{5}{3}} M^{\frac{1}{3}}} = \left(\frac{2}{5}\right) \left[\left(\frac{9\pi}{4}\right)^{\frac{2}{3}} \frac{\hbar^2}{Gm^{\frac{8}{3}} M^{\frac{1}{3}}} \right]$$

We recognize the bracketed term as R_{eq} in Equation 9.4 with $m_n = m$ and $m_e = 0$.

$$R = \left(\frac{2}{5}\right) R_{eq}$$

Next, let's consider the case of an ultra-relativistic Black Hole

$$P_{UR} = P_g \rightarrow \frac{\hbar c}{4} (3\pi^2)^{\frac{1}{3}} \left(\frac{N}{V}\right)^{\frac{4}{3}} = \frac{3GM^2}{8\pi R^4}$$

Substitution of $M = Nm$ and $V = \frac{4}{3}\pi R^3$ gives us

$$\begin{aligned} \frac{\hbar c}{4} (3\pi^2)^{\frac{1}{3}} \left(\frac{M}{m(\frac{4}{3}\pi R^3)}\right)^{\frac{4}{3}} &= \frac{3GM^2}{8\pi R^4} \\ \frac{\hbar c}{4} (3\pi^2)^{\frac{1}{3}} \left(\frac{3M}{4m\pi}\right)^{\frac{4}{3}} &= \frac{3GM^2}{8\pi} \end{aligned}$$

Solving for M yields

$$\frac{\hbar c \pi^{\frac{1}{3}}}{2G} \left(\frac{3}{2m^2}\right)^{\frac{2}{3}} = M^{\frac{2}{3}} \rightarrow M = \left(\frac{3}{2m^2}\right) \left(\frac{\hbar c}{2G}\right)^{\frac{3}{2}} \pi^{\frac{1}{2}}$$

If we use the mass of a proton as m , we have

$$M = \left(\frac{3}{2(1.67 \times 10^{-27} \text{kg})^2} \right) \left(\frac{\left(1.05 \times 10^{-34} \frac{\text{m}^2 \text{kg}}{\text{s}} \right) \left(3 \times 10^8 \frac{\text{m}}{\text{s}} \right)}{2 \left(6.67 \times 10^{-11} \frac{\text{m}^3}{\text{s}^2 \text{kg}} \right)} \right)^{\frac{3}{2}} \pi^{\frac{1}{2}}$$

$$M = 6.92 \times 10^{30} \text{kg}$$

This is about twice the mass of our sun. If the mass of the star is greater than this, the gravitational pressure will overcome the nucleon degeneracy pressure and the star will collapse, likely forming a black hole. All of this must be corrected for general relativity, however.

9.7 Bose-Einstein Condensation

Let's now turn our attention to a system of bosons. The number of particles in this system, denoted by N_{BE} , can be determined using Equation 3.18 and the following density of states

$$\begin{aligned} D(k) dk &= \left(\frac{1}{8} \right) \left(\frac{4V k^2}{\pi^2} \right) dk \quad \rightarrow \quad D(k) dk = \frac{V k^2}{2\pi^2} dk \\ D(\epsilon) d\epsilon &= \frac{V}{2\pi^2} \left(\frac{2m\epsilon}{\hbar^2} \right) \left(\frac{2m}{\hbar^2} \right)^{\frac{1}{2}} \frac{1}{2} d\epsilon \\ D(\epsilon) d\epsilon &= \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{\frac{3}{2}} \epsilon^{\frac{1}{2}} d\epsilon \end{aligned}$$

Note that this differs from the density of state for a system of fermions by a factor of 2 since the bosons possess integer spin. The number of particles in this system is thus

$$N_{BE} = \int_0^\infty \left(\frac{V}{4\pi^2} \right) \left(\frac{2m}{\hbar^2} \right)^{\frac{3}{2}} \frac{\epsilon^{\frac{1}{2}}}{e^{\beta(\epsilon-\mu)} - 1} d\epsilon$$

The density of states cannot account for the ability of multiple bosons to occupy the $\epsilon = 0$ state; recall that the Pauli Exclusion principle does not apply to bosons. To account for bosons in the $\epsilon = 0$ state, we can rewrite the equation for N_{BE} as

$$N_{BE} = \frac{1}{e^{-\beta\mu} - 1} + \left(\frac{V}{4\pi^2} \right) \left(\frac{2m}{\hbar^2} \right)^{\frac{3}{2}} \int_0^\infty \frac{\epsilon^{\frac{1}{2}}}{e^{\beta(\epsilon-\mu)} - 1} d\epsilon$$

The first term in this expression is the number of bosons in the $\epsilon = 0$ state, which we can denote as N_0 , while the integral is the number of bosons in all other states, which we will denote as N_{ex} ; the subscript *ex* reflects that these are bosons in the excited (*i.e.*, not $\epsilon = 0$) states.

$$N_0 = \frac{1}{e^{-\beta\mu} - 1} \quad N_{ex} = \left(\frac{V}{4\pi^2}\right) \left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \int_0^{\infty} \frac{\epsilon^{\frac{1}{2}}}{e^{\beta(\epsilon-\mu)} - 1} d\epsilon$$

At $T = 0$, all bosons will be in the ground state

$$N_{BE} = \frac{1}{e^{-\beta\mu} - 1} \quad \rightarrow \quad e^{-\beta\mu} = 1 - \frac{1}{N_{BE}}$$

Thus, when $T = 0$ and N_{BE} is large

$$e^{-\beta\mu} \approx 1 \quad \rightarrow \quad \mu \approx 0$$

The chemical potential of a system of bosons must approach zero as the temperature of the system is lowered. We can determine the critical temperature, denoted as T_B , for when $\mu = 0$ from the equation for N_{ex} .

$$\begin{aligned} N_{BE} &= \left(\frac{V}{4\pi^2}\right) \left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \int_0^{\infty} \frac{\epsilon^{\frac{1}{2}}}{e^{\beta_B \epsilon} - 1} d\epsilon \\ N_{BE} &= \left(\frac{V}{4\pi^2}\right) \left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \frac{2.3152}{\beta_B^{\frac{3}{2}}} \\ \beta_B &= \left(\frac{2m}{\hbar^2}\right) \left(\frac{0.05864V}{N_{BE}}\right)^{\frac{2}{3}} \quad \rightarrow \quad T_B = \left(\frac{\hbar^2}{2mk_B}\right) \left(\frac{0.2422N_{BE}}{V}\right)^{\frac{2}{3}} \end{aligned}$$

For temperatures above this critical temperature, all bosons are in the excited states and $\mu \neq 0$. As T decreases below this temperature, the number of bosons in the ground state increases and μ approaches zero. We can represent these populations in terms of their fraction of the total number of bosons in the system.

$$\frac{N_0}{N_{BE}} = 1 - \frac{N_{ex}}{N_{BE}}$$

$$\frac{N_0}{N_{BE}} = 1 - \left(\frac{T}{T_B}\right)^{\frac{3}{2}}$$

The equations above are valid for $T < T_B$ as $N_0 \approx 0$ for $T > T_B$. The particles in the N_0 state possess zero energy and zero momentum, and thus do not contribute to the pressure or the viscosity of the system. The process of concentrating particles in the N_0 state is referred to as *Bose-Einstein Condensation*. The energy of this system at $T < T_B$ is

$$E \approx N_{ex}k_B T \quad \rightarrow \quad E \approx Nk_B \frac{T^{\frac{5}{2}}}{T_B^{\frac{3}{2}}}$$

Thus the heat capacity of the system at $T < T_B$ is

$$C_V \approx \frac{5}{2} N k_B \left(\frac{T}{T_B} \right)^{\frac{5}{2}}$$

We now also have the pieces we need to calculate the pressure of the system.

$$S = \int_0^{T_B} \frac{C_V}{T} dT \rightarrow S = \frac{5}{3} \frac{E}{T}$$

Thus,

$$F = E - T \left(\frac{5}{3} \frac{E}{T} \right) \rightarrow F = -\frac{2}{3} E \rightarrow P = \frac{2}{3} \left(\frac{\partial E}{\partial V} \right)_T$$

$$P = \frac{2}{3} \left(\frac{\partial E}{\partial T_B} \right)_T \left(\frac{\partial T_B}{\partial V} \right)_T$$

$$P = \frac{2}{3} \left(-\frac{3}{2} N k_B \left(\frac{T}{T_B} \right)^{\frac{5}{2}} \right) \left(\frac{h^2}{2m k_B} \right) \left(-\frac{2}{3} \right) (0.2422 N_{BE})^{\frac{2}{3}} V^{-\frac{5}{3}}$$

$$P = \frac{2}{3} \left(N k_B \left(\frac{T}{T_B} \right)^{\frac{5}{2}} \right) \frac{T_B}{V}$$

$$P = \frac{2}{3} \frac{N k_B}{V} \frac{T^{\frac{5}{2}}}{T_B^{\frac{3}{2}}}$$

The situation is different in two dimensions, however. In that case, the density of states is

$$D(\epsilon) d\epsilon = \frac{Am}{2\pi\hbar^2} d\epsilon$$

The number of particles in energy states above the ground state is therefore

$$N_{\epsilon>0} = \int_0^{\infty} \frac{1}{e^{\beta(\epsilon-\mu)}} \frac{Am}{2\pi\hbar^2} d\epsilon \rightarrow N_{\epsilon>0} = \left(\frac{Am}{2\pi\hbar^2} \right) \int_0^{\infty} \frac{d\epsilon}{e^{\beta(\epsilon-\mu)}}$$

This integral diverges as $T \rightarrow 0$, indicating that there are always an infinite number of particles in energy states above the ground state. Thus, there is no condensation into the ground state.