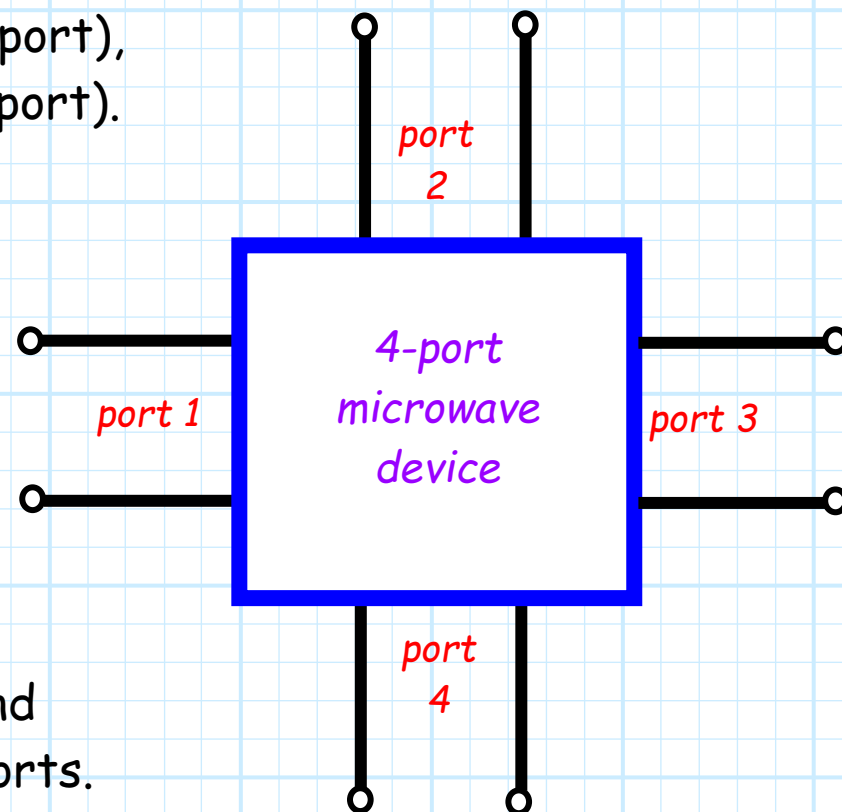


Multiport Networks and the Impedance Matrix

Most microwave components are **neither** strictly a source (with one output port), **nor** strictly a load (with one input port).



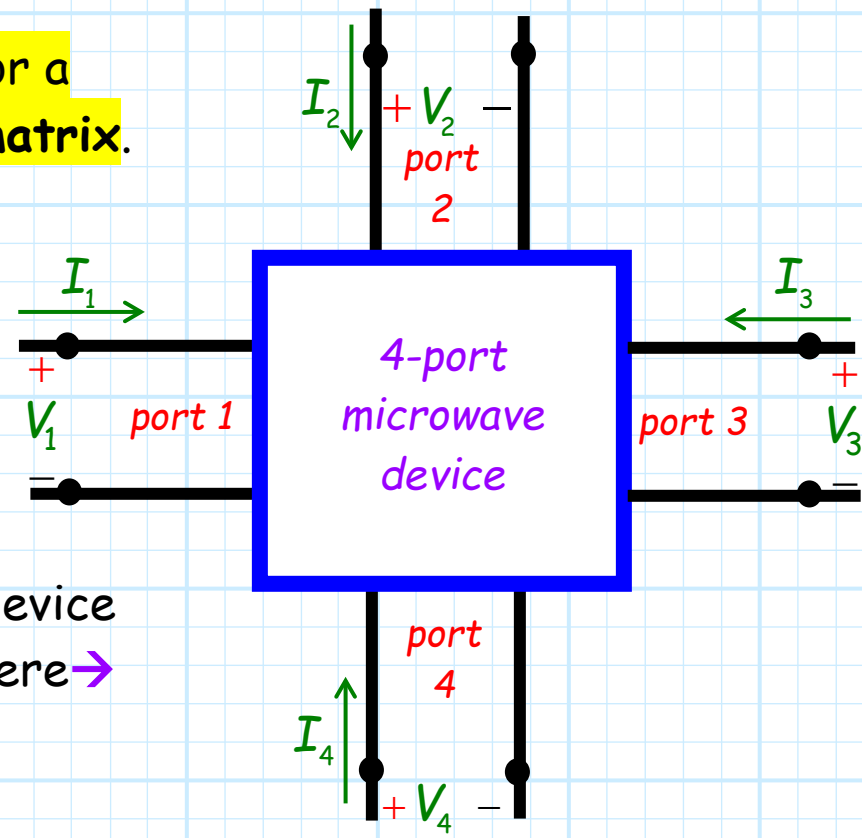
Instead, microwave components and networks typically have **multiple** ports.

Is there a multi-port equivalent of impedance?

Q: We use input *impedance* to characterize a *single-port* network; is there some way to *characterize* a *multiport* component?

A: Absolutely!

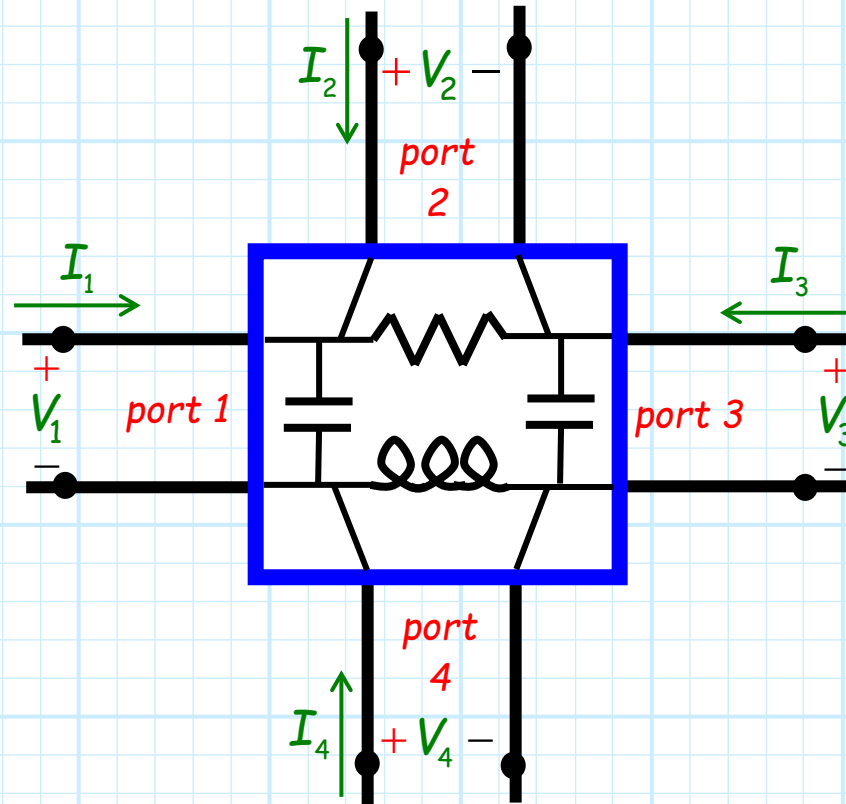
The **equivalent** to input impedance for a **multi-port** device is its **impedance matrix**.



Consider the **4-port** microwave device shown here →

Trans-Impedance parameters

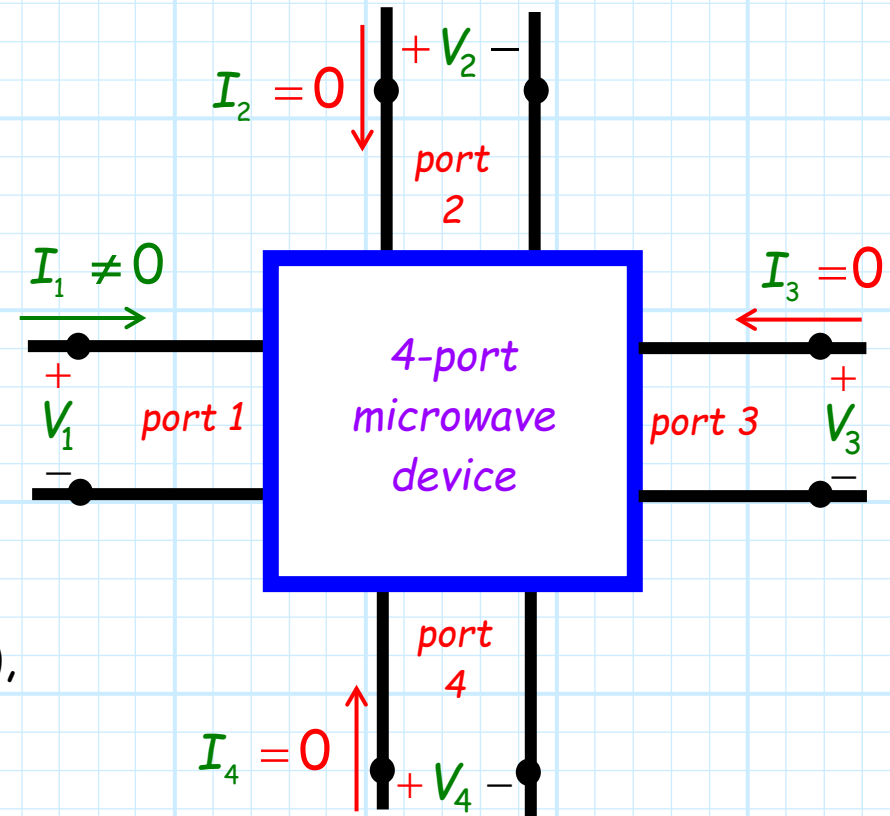
Inside the blue box there may be a very **simple linear** device/circuit, or it might contain a very large and **complex linear** microwave system.



→ Either way, this linear "box" can be fully characterized by its **trans-impedance parameters!**

Note that all currents but one are zero!

Now; say there exists a non-zero current at **port 1** (i.e., $I_1 \neq 0$), while the current at all other ports are known to be **zero** (i.e., $I_2 = I_3 = I_4 = 0$).



Say we **measure/determine** the **current** at port 1 (i.e., determine I_1), and we then **measure/determine** the **voltage** at port 2 (i.e., determine V_2).

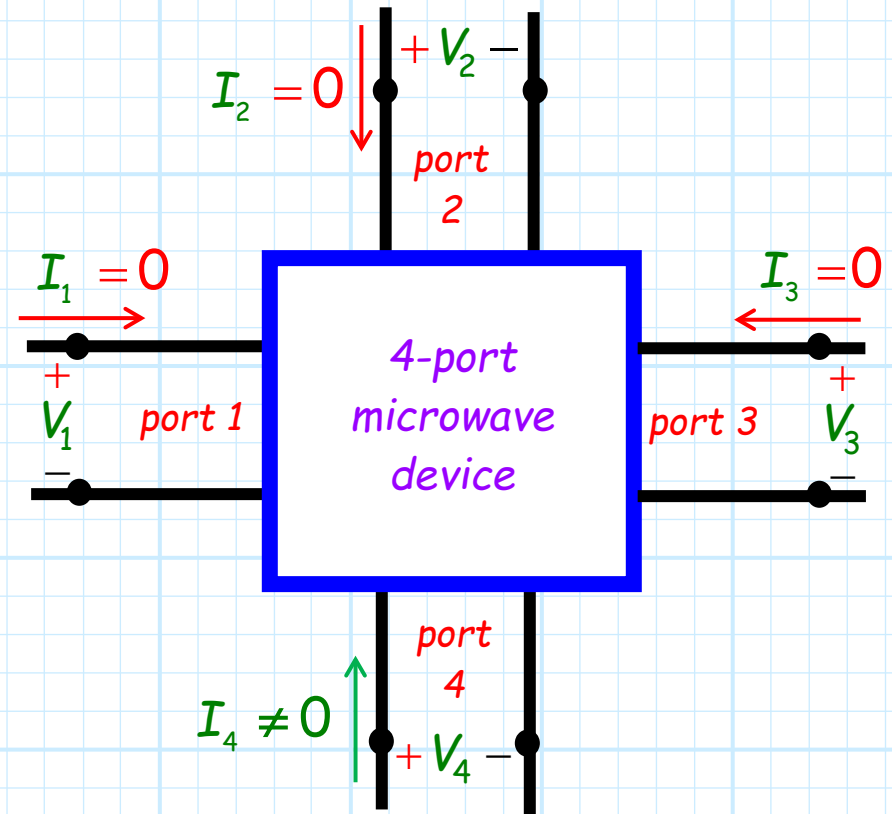
The complex ratio between V_2 and I_1 is known as the **trans-impedance parameter Z_{21}** :

$$Z_{21} = \left. \frac{V_2}{I_1} \right|_{I_2=I_3=I_4=0}$$

Many, many parameters

We could **also** define, say, trans-impedance parameter Z_{34} as the **ratio** between:

- a) the complex value I_4 (current into port 4) and,
- b) V_3 (the voltage at port 3) **given that,**
- c) **the current at all other ports (1, 2, and 3) are zero.**



$$Z_{34} = \left. \frac{V_3}{I_4} \right|_{I_1=I_2=I_3=0}$$

A 4-port device has 16 parameters

Or, we could determine, for example:

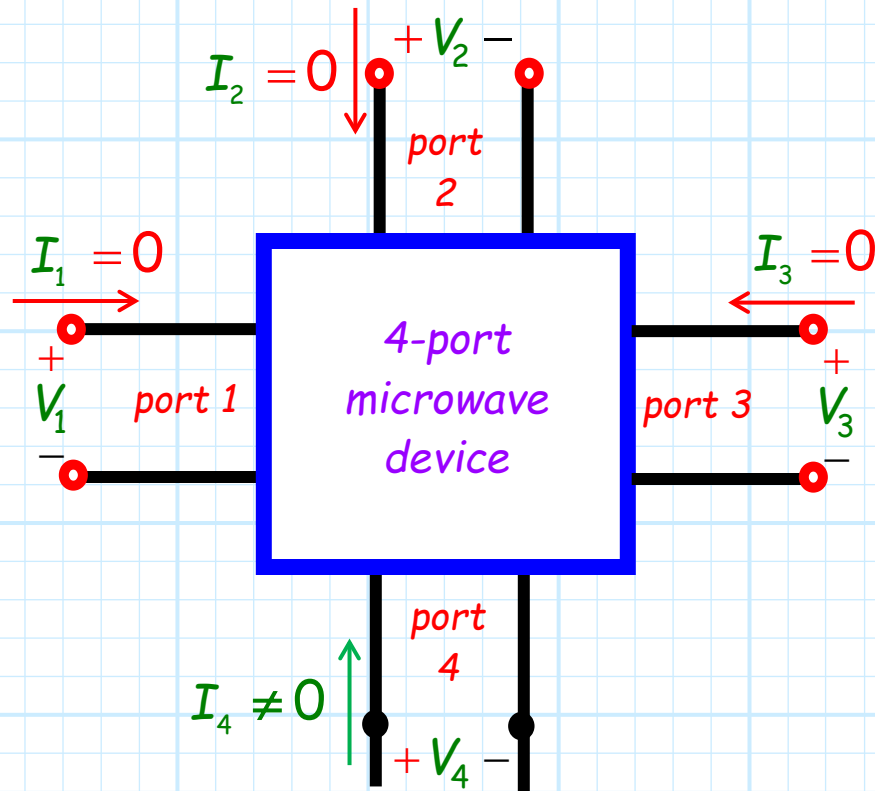
$$Z_{33} = \left. \frac{V_3}{I_3} \right|_{I_1=I_2=I_4=0} \quad Z_{24} = \left. \frac{V_2}{I_4} \right|_{I_1=I_2=I_3=0} \quad Z_{12} = \left. \frac{V_1}{I_2} \right|_{I_1=I_3=I_4=0}$$

Thus, more **generally**, the ratio of the current into port n and the voltage at port m is:

$$Z_{mn} = \left. \frac{V_m}{I_n} \right|_{I_{m \neq n} = 0} \quad (\text{when } I_k = 0 \text{ for all } m \neq n)$$

Note for an N -port device, there are N^2 trans-impedance parameters!

An open enforces $I=0$



Q: But how do we ensure that all but **one** port current is zero?



A: Place an **open-circuit** at those ports!

Placing an **open** at a port **enforces** the condition that $I = 0$.

Now, we can thus **equivalently** state the definition of trans-impedance as:

$$Z_{mn} = \left. \frac{V_m}{I_n} \right|_{I_{m \neq n} = 0} \quad (\text{when all ports } m \neq n \text{ are open-circuited})$$

And ONLY when all ports are open-circuited!

Note that the ratio of voltage V_3 current I_1 (say) is **equal** to trans-impedance parameter $Z_{31}...$

...if, and only if, all ports—except port 1—are open circuited

or, stated another way:

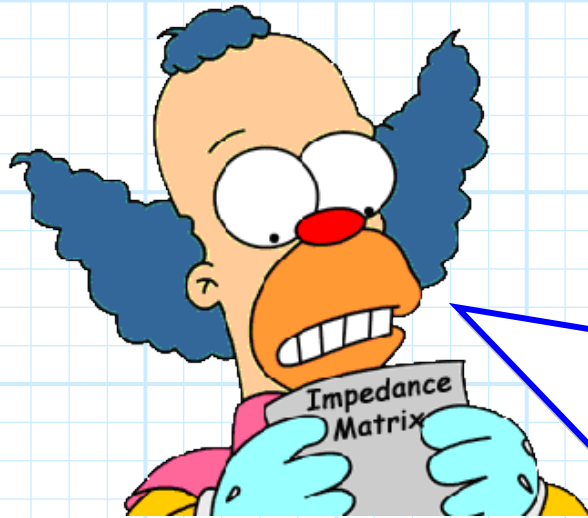
...if, and only if, all port currents I_m —except I_1 —are zero.

Thus, when the above statements are **not** valid, the ratio of V_3 and I_1 is **not numerically equal** to the trans-impedance parameter Z_{31} :

$$\frac{V_3}{I_1} \neq Z_{31}$$



Useful devices are not commonly connected to open-circuits



*Q: As impossible as it may sound, this presentation is **even more** boring and **pointless** than all of your previous efforts.*

Why are we studying this?

*After all, what is the likelihood that **useless open-circuits** will be placed on **all** but one port of an otherwise **useful** device?*

A: Of course, a multi-port device **will** generally be connected at **every** port to some other **useful** devices (e.g., amplifier, filter, antenna).

→ **And so generally, a non-zero current will exist at all ports!**

Yet in this case, the trans-impedance parameters of the device are still **very important** and **very helpful**.

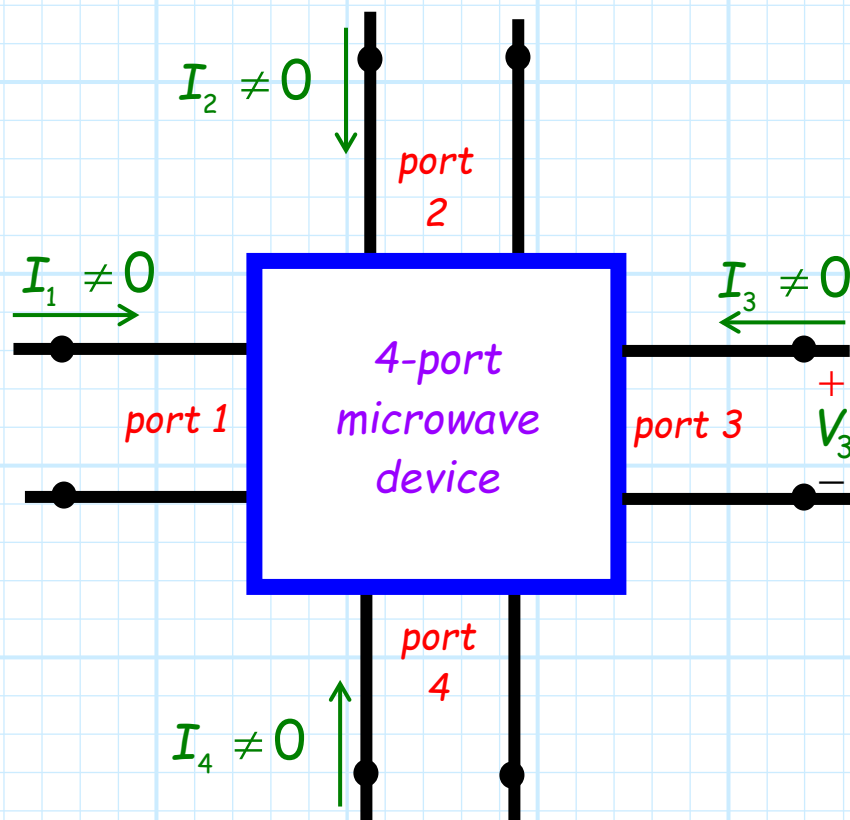
It's linear—let's use superposition!

For a **linear** device, the voltage at any **one** port is a **superposition** of the voltages resulting from each and **every** individual port **current**.

For **example**, the voltage at **port-3** is a superposition of **four terms**:

$$V_3 = V_{31} + V_{32} + V_{33} + V_{34}$$

where each term represents the voltage at port-3 **due to the current** at one—and **only one**—specific-port.



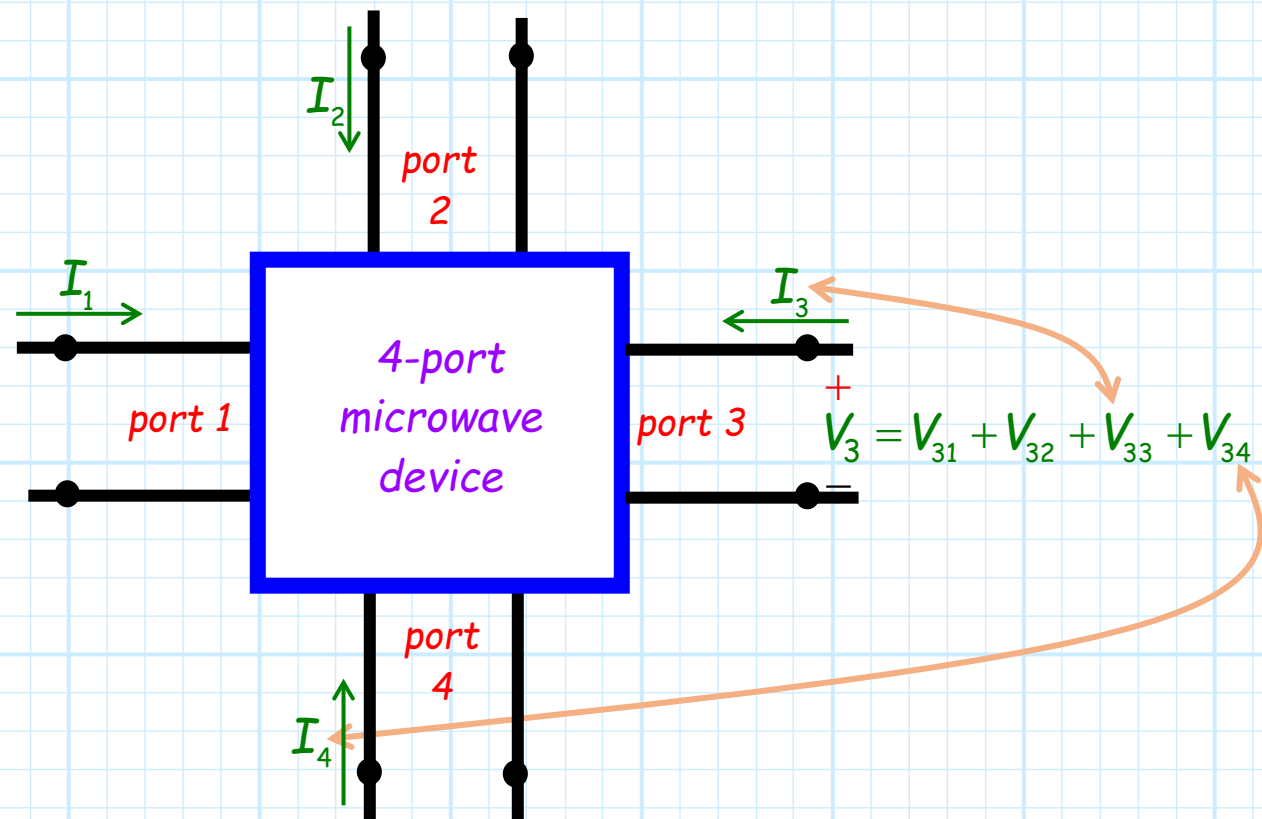
Voltage components

For this **example**:

V_{34} is the voltage at port-3 due to the current at port 4

and

V_{33} is the voltage at port-3 due to the current at port 3.



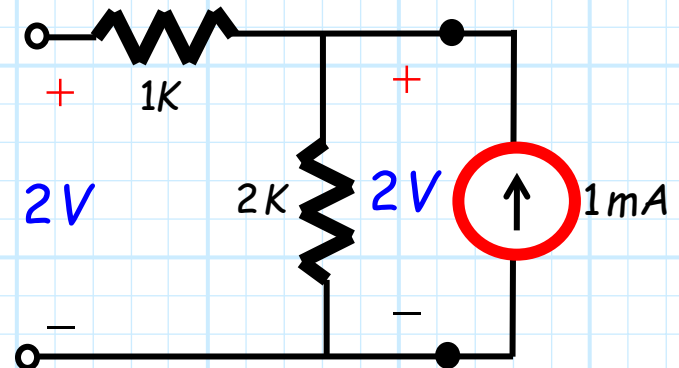
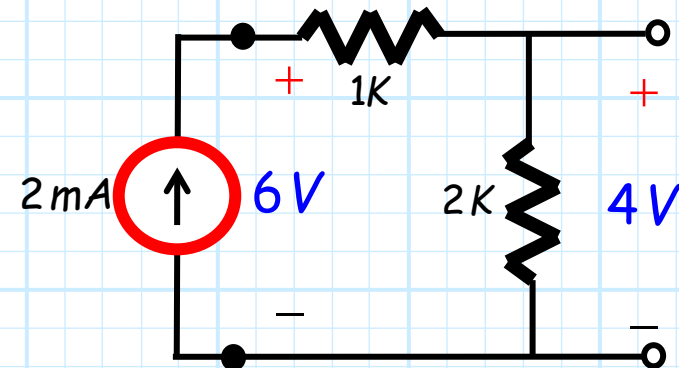
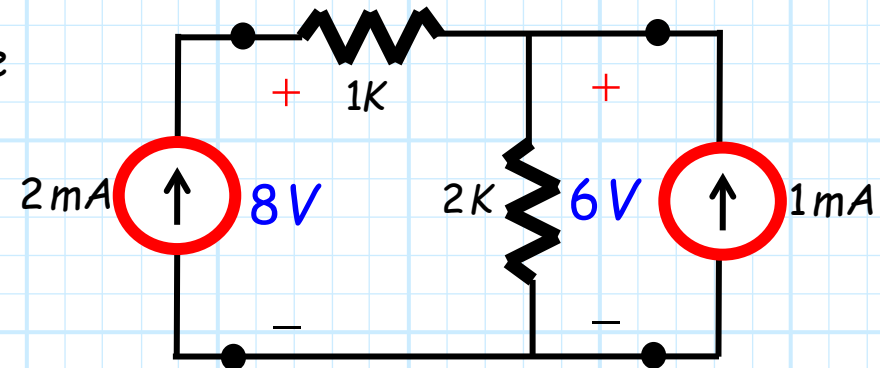
Remember this?

Q: But how do we **determine** these terms? How can we **separate** a voltage into its components?

A: Think about how we use **superposition** to analyze a linear circuit—we first turn **off** all sources, **except one**, and then analyze.

The result are the currents and voltages **due to that one source**.

We do this for **each source** in the circuit, and then the **sum** of all resulting analyses is the correct **total result**!



Deja-vu all over again

Thus, we can likewise determine the voltage **due to** a specific port current by “turning off” all **other** port currents—setting all other port currents to **zero**!

For **example**:

$$V_{33} = V_3|_{I_1=I_2=I_4=0} \qquad V_{31} = V_3|_{I_2=I_3=I_4=0}$$

Q: Wait a second—setting all other port **currents to zero**—don't we also do that when determining **trans-impedance** parameters?

A: Exactly. In fact, the components of each port voltage can be **directly determined** from knowledge of the device's trans-impedance parameters. E.G.:

$$V_{33} = V_3|_{I_1=I_2=I_4=0} = Z_{33}I_3 \qquad V_{31} = V_3|_{I_2=I_3=I_4=0} = Z_{31}I_1$$

Superposition! Superposition! Superposition!

Therefore if **all currents** are known, the voltage at port-3 can be determined by as:

$$\begin{aligned} V_3 &= V_{31} + V_{32} + V_{33} + V_{34} \\ &= Z_{34} I_4 + Z_{33} I_3 + Z_{32} I_2 + Z_{31} I_1 \end{aligned}$$

More **generally**, the voltage at port m of an N -port device is:

$$V_m = \sum_{n=1}^N Z_{mn} I_n$$

If a device is linear, we can apply superposition!

See? What did I tell you

To emphasize an **earlier point**, let's take the **ratio** of voltage V_3 and current I_1 :

$$\begin{aligned}\frac{V_3}{I_1} &= \frac{Z_{34} I_4 + Z_{33} I_3 + Z_{32} I_2 + Z_{31} I_1}{I_1} \\ &= Z_{34} \left(\frac{I_4}{I_1} \right) + Z_{33} \left(\frac{I_3}{I_1} \right) + Z_{32} \left(\frac{I_2}{I_1} \right) + Z_{31}\end{aligned}$$



Note that this ratio is **NOT** equal to Z_{31} (!)—unless of course, **all other currents are zero** (i.e., if $I_2 = I_3 = I_4 = 0$):

$$\begin{aligned}\left. \frac{V_3}{I_1} \right|_{I_2=I_3=I_4=0} &= \left[Z_{34} \left(\frac{I_4}{I_1} \right) + Z_{33} \left(\frac{I_3}{I_1} \right) + Z_{32} \left(\frac{I_2}{I_1} \right) + Z_{31} \right]_{I_2=I_3=I_4=0} \\ &= Z_{34} \left(\frac{0}{I_1} \right) + Z_{33} \left(\frac{0}{I_1} \right) + Z_{32} \left(\frac{0}{I_1} \right) + Z_{31} \\ &= Z_{31}\end{aligned}$$

The impedance matrix

This expression can be written in **matrix** form as:

$$\mathbf{V} = \mathbf{Z} \mathbf{I}$$

Where \mathbf{I} is the **vector**:

$$\mathbf{I} = [I_1, I_2, I_3, \dots, I_N]^T$$

and \mathbf{V} is the vector:

$$\mathbf{V} = [V_1, V_2, V_3, \dots, V_N]^T$$



And the matrix \mathbf{Z} is called the **impedance matrix**:

$$\mathbf{Z} = \begin{bmatrix} Z_{11} & \dots & Z_{1n} \\ \vdots & \ddots & \vdots \\ Z_{m1} & \dots & Z_{mn} \end{bmatrix}$$

The impedance matrix is frequency dependent

The impedance matrix is a N by N matrix that **completely characterizes** a linear, N -port device.

Effectively, the impedance matrix describes a **multi-port** device the way that Z_L describes a **single-port** device (e.g., a load)!

→ But **beware!**



The values of the impedance matrix for a particular device or network, just like Z_L , are **frequency dependent!**

Thus, it may be more instructive to **explicitly** write:

$$\mathbf{Z}(\omega) = \begin{bmatrix} Z_{11}(\omega) & \cdots & Z_{1n}(\omega) \\ \vdots & \ddots & \vdots \\ Z_{m1}(\omega) & \cdots & Z_{mn}(\omega) \end{bmatrix}$$