PHSX 711: Homework #1

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Problem 1

Exercise 1.3.2 (Shankar) Show how to go from the basis

$$|I\rangle = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \qquad |II\rangle = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \qquad |III\rangle = \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix}$$

to the orthonormal basis

$$|I\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad |II\rangle = \begin{pmatrix} 0 \\ 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} \qquad |III\rangle = \begin{pmatrix} 0 \\ -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$$

Solution:

i. Normalize the first vector, I will use $|I\rangle$ since it only has one nonzero component it becomes $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. If we want to write this more formally,

$$|I'\rangle = \frac{|I\rangle}{|I|}, \text{ where } |I| = \sqrt{\langle I|I\rangle}$$

ii. Subtract from the second vector its projection along the first, leaving behind only the part perpendicular to the first. Of course it should also be normalized.

$$\begin{aligned} |II'\rangle &= |II\rangle - |I'\rangle \langle I'|II\rangle \\ &= \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (0) \end{aligned}$$

We find $|I'\rangle$ and $|II\rangle$ are orthagonal, so we just normalize:

$$\begin{pmatrix} 0 \\ 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$$

iii. Finally for the third vector:

$$
\begin{aligned}
|III'\rangle &= |III\rangle - \cancelto{0}{|I'\rangle \langle I'|III\rangle} - |II'\rangle \langle II'|III\rangle \\
&= |III\rangle - |II'\rangle \, (12/\sqrt{5}) \\
&= \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix} - \begin{pmatrix} 0 \\ 12/5 \\ 24/5 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}
\end{aligned}
$$

This is already normalized.

Problem 2

Prove $(\Omega\Lambda)^\dagger = \Lambda^\dagger\Omega^\dagger$

Solution:
Let f and g be functions (in this case of x) for the operators to act on. Expanding back to the integral form can help demonstrate how things move around:

$$\begin{aligned}
\langle f|\Omega\Lambda|g\rangle &= \langle f|\Omega\Lambda|g\rangle \\
&= \int_{-\infty}^{\infty} f^*\Omega\Lambda g \; dx \\
&= \int_{-\infty}^{\infty} f^* \left\{\Omega\left(\Lambda g\right)\right\} \; dx \\
&= \int_{-\infty}^{\infty} \left(\Omega^\dagger f\right)^* \left(\Lambda g\right) \; dx \\
&= \int_{-\infty}^{\infty} (\Lambda^\dagger\{\Omega^\dagger f\})^* g \, dx \\
&= \int_{-\infty}^{\infty} (\Lambda^\dagger\Omega^\dagger f)^* g \, dx \\
&= \langle \Lambda^\dagger\Omega^\dagger f|g\rangle
\end{aligned}$$

Problem 3

Show that for any operator Ω, $\frac{\Omega+\Omega^\dagger}{2}$ is a Hermitian operator $\frac{\Omega-\Omega^\dagger}{2}$ is an anti-Hermitian operator.

Solution:
A Hermitian operator is one which obeys the following:

$$Q^\dagger = Q$$

Conversely, an anti-Hermitian operator picks up a negative sign under its Hermite conjugate:

$$Q^\dagger = -Q$$

Additionally, for the sake of my future self, I will leave the additional properties of Hermitian conjugates here:

$$(A+B)^\dagger = A^\dagger + B^\dagger \tag{1}$$
$$(cA)^\dagger = c^* A^\dagger \qquad \text{where c is a scalar} \tag{2}$$
$$(A^\dagger)^\dagger = A \tag{3}$$

i. For the first proof, let me begin by defining $O = \frac{\Omega+\Omega^\dagger}{2}$.

$$\begin{aligned} O &\stackrel{?}{=} O^\dagger \\ &\stackrel{?}{=} \frac{(\Omega+\Omega^\dagger)}{2} \\ &\stackrel{?}{=} \frac{(\Omega+\Omega^\dagger)^\dagger}{2} \\ &\stackrel{?}{=} \frac{\Omega^\dagger+(\Omega^\dagger)^\dagger}{2} \\ &\stackrel{?}{=} \frac{(\Omega^\dagger+\Omega)}{2} \end{aligned}$$

This is indeed equivalent to the original operator O. Therefore any such combination of operators will be Hermitian due to property (1) of Hermitian operators.

ii. The second proof takes the exact same approach, let's use $P = \frac{\Omega-\Omega^\dagger}{2}$ instead:

$$\begin{aligned} P &\stackrel{?}{=} P^\dagger \\ &\stackrel{?}{=} \frac{(\Omega-\Omega^\dagger)}{2} \\ &\stackrel{?}{=} \frac{(\Omega-\Omega^\dagger)^\dagger}{2} \\ &\stackrel{?}{=} \frac{\Omega^\dagger-(\Omega^\dagger)^\dagger}{2} \\ &\stackrel{?}{=} \frac{(\Omega^\dagger-\Omega)}{2} \end{aligned}$$

This shows that it is indeed anti-Hermitian.

Problem 3 continued on next page...

Problem 4

Exercise 1.6.2 (Shankar) Given Ω and Λ are Hermitian, what can you say about (1) $\Omega\Lambda$; (2) $\Omega\Lambda + \Lambda\Omega$; (3) $[\Omega, \Lambda]$; and (4) $i[\Omega, \Lambda]$

(Hint: Whether they are Hermitian or anti-Hermitian.)

Solution:

(1) The product of two Hermitian operators may not necessarily be Hermitian. Because Hermitian operators are symmetric, the product only remains symmetric if Ω and Λ commute.

$$(\Omega\Lambda)^\dagger = \Lambda\Omega$$

We can see that $\Omega\Lambda = \Lambda\Omega$ only if they commute. Otherwise this is neither.

(2) This is Hermitian:

$$\begin{aligned}(\Omega\Lambda + \Lambda\Omega)^\dagger &= (\Omega\Lambda)^\dagger + (\Lambda\Omega)^\dagger \\ &= \Lambda\Omega + \Omega\Lambda \\ &= \Omega\Lambda + \Lambda\Omega\end{aligned}$$

(3) This is anti-Hermitian:

$$\begin{aligned}(\Omega\Lambda - \Lambda\Omega)^\dagger &= (\Omega\Lambda)^\dagger - (\Lambda\Omega)^\dagger \\ &= \Lambda\Omega - \Omega\Lambda \\ &= -(\Omega\Lambda + \Lambda\Omega)\end{aligned}$$

(4) This is hermitian:

$$\begin{aligned}[i(\Omega\Lambda - \Lambda\Omega)]^\dagger &= (i\Omega\Lambda - i\Lambda\Omega)^\dagger \\ &= (i\Omega\Lambda)^\dagger - (i\Lambda\Omega)^\dagger \\ &= (\Lambda\Omega)(-i) - (\Omega\Lambda)(-i) \\ &= i\Omega\Lambda - i\Lambda\Omega\end{aligned}$$

Problem 5

Exercise 1.6.5 (Shankar) Verify that $R(\frac{1}{2}\pi i)$ is unitary (orthagonal) by examining its matrix.

Note that $R(\frac{1}{2}\pi i)$ designates the rotation about unit vector i by $\frac{1}{2}\pi$.

Solution:

i. **Unitary Matrices:** By definition, a unitary matrix is one which obeys the following

$$\Omega^\dagger \Omega = \Omega \Omega^\dagger = I$$

ii. **Matrix form of $R(\frac{1}{2}\pi i)$:** In the $|1\rangle, |2\rangle, |3\rangle$ basis,

$$R\left(\frac{1}{2}\pi i\right)|1\rangle = |1\rangle$$
$$R\left(\frac{1}{2}\pi i\right)|2\rangle = |3\rangle$$
$$R\left(\frac{1}{2}\pi i\right)|3\rangle = -|2\rangle$$

Which in matrix form is given by:

$$R\left(\frac{1}{2}\pi i\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Where its hermitian conjugate is

$$\left[R\left(\frac{1}{2}\pi i\right)\right]^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

(which is really just the transpose, since it is real)

iii. Their product gives the identity matrix, therefore it is unitary:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$