The Wave Interpretation

We can **alternatively** write the solution to the telegrapher's equations as:

$$V(z,\omega) = V^{+}(z,\omega) + V^{-}(z,\omega)$$

$$I(z,\omega) = I^{+}(z,\omega) + I^{-}(z,\omega)$$

where:

$$V^{+}(z,\omega) = V_0^{+}(\omega)\, e^{-j\beta z} \qquad V^{-}(z,\omega) = V_0^{-}(\omega)\, e^{+j\beta z}$$

$$I^{+}(z,\omega) = I_0^{+}(\omega)\, e^{-j\beta z} \qquad I^{-}(z,\omega) = I_0^{-}(\omega)\, e^{+j\beta z}$$

Waves

Q: *Just what do these functions $V^{+}(z,\omega)$ and $V^{-}(z,\omega)$* ***tell*** *us?*

Do they have any ***physical*** *meaning?*

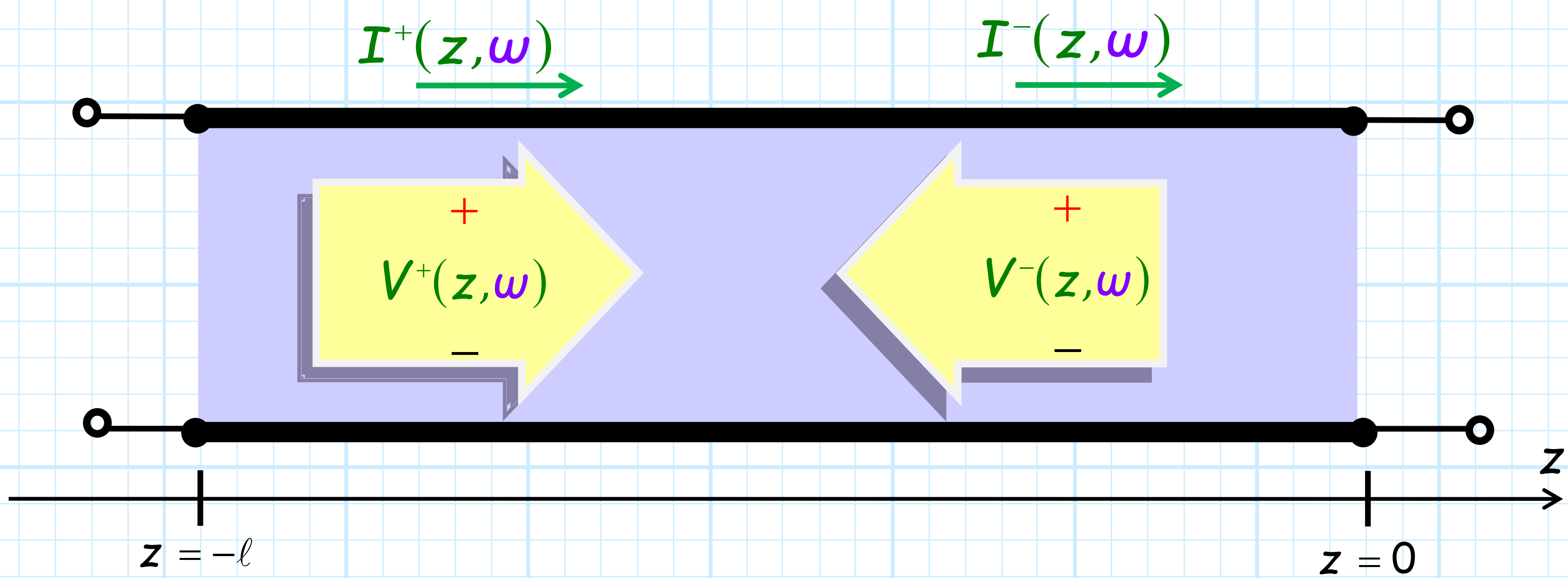
A: Yes! An **incredibly important** physical meaning!

Plus wave goes right, minus wave left

The **two terms** in each solution are the result of **two electromagnetic waves** propagating along the transmission line.

1. **One** wave ($V^{+}(z,\omega)$ and $I^{+}(z,\omega)$) propagates in the direction of **increasing** z.

2. The **other** wave ($V^{-}(z,\omega)$ and $I^{-}(z,\omega)$) propagates in the direction of **decreasing** z.



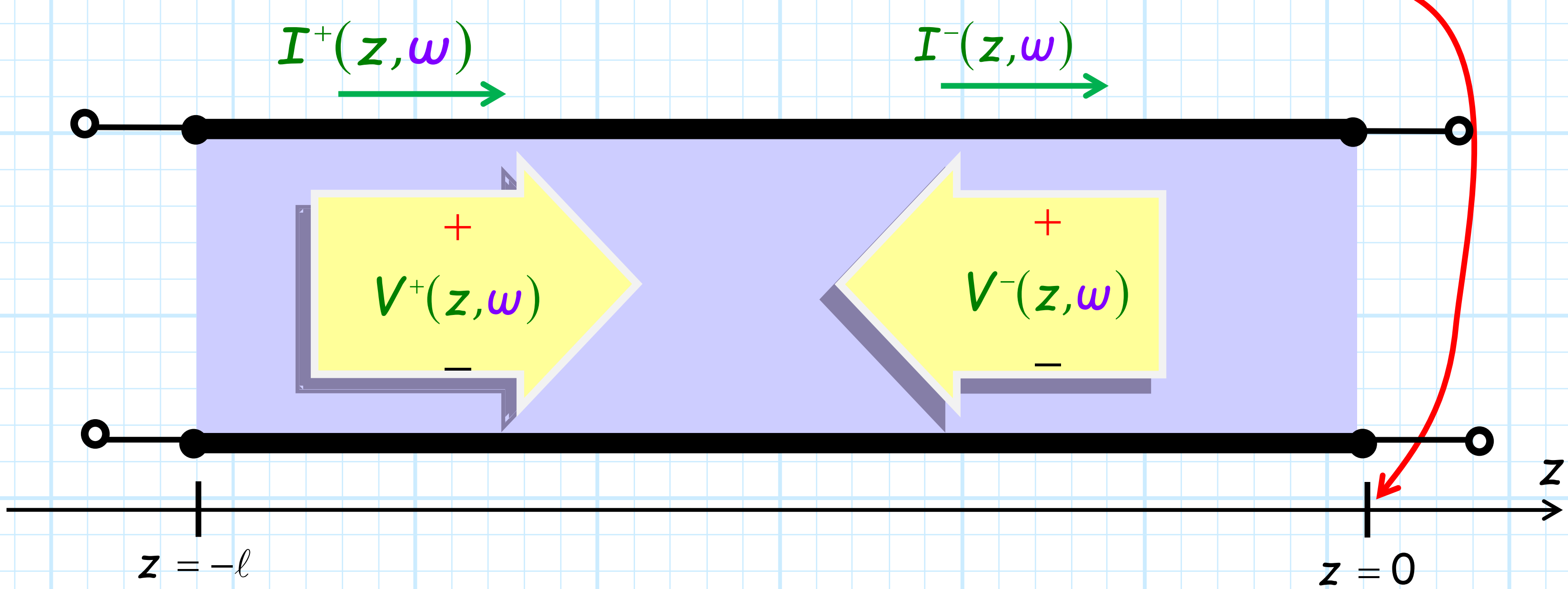
Complex wave amplitudes

Q: *So just what **are** the complex values* $V_0^+(\omega), V_0^-(\omega), I_0^+(\omega), I_0^-(\omega)$?

A: They are called the **complex amplitudes** of each propagating wave.

Q: *Do they have any physical meaning?*

A: Consider the wave solutions at **one specific point** on the transmission line—the point where $z=0$.



The zero subscript means "at z = 0"

We find that the **complex** value of the **voltage wave** at that point is:

$$\begin{aligned} V^{+}(z=0,\omega) &= V_0^{+}(\omega)\, e^{-j\beta(z=0)} \\ &= V_0^{+}(\omega)\, e^{-(0)} \\ &= V_0^{+}(\omega)\,(1) \\ &= V_0^{+}(\omega) \end{aligned}$$

So, the **complex wave amplitude** $V_0^{+}(\omega)$ is simply the **complex** value of the wave function $V^{+}(z,\omega)$—when evaluated **at the point** $z=0$.

That's what the **subscript 0** means—the value at $z=0$!

Likewise:

$$V_0^{-}(\omega) = V^{-}(z=0,\omega) \qquad I_0^{+}(\omega) = I^{+}(z=0,\omega) \qquad I_0^{-}(\omega) = I^{-}(z=0,\omega)$$

Uncompress the Compression Algorithm

Now let's evaluate the **complex wave** function $V^{+}(z,\omega) = V_0^{+}(\omega)\, e^{-j\beta z}$ to determine the **magnitude** and **phase** of the time-harmonic oscillations!

First, the wave **magnitude** (Recall $|ab| = |a|\,|b|$ and $|e^{j\phi}| = 1$):

$$v^{+}(z,\omega) = \left|V^{+}(z,\omega)\right| = \left|V_0^{+}(\omega)\, e^{-j\beta z}\right| = \left|V_0^{+}(\omega)\right|\left|e^{-j\beta z}\right| = \left|V_0^{+}(\omega)\right|$$

Although $V^{+}(z,\omega)$ is a function of position z, the value $\left|V_0^{+}(\omega)\right|$ is just a **constant**—the **magnitude** of this wave is the **same** at **all** points along the transmission line!!

Likewise:

$$v^{-}(z,\omega) = \left|V^{-}(z,\omega)\right| = \left|V_0^{-}(\omega)\right|$$

$$i^{+}(z,\omega) = \left|I^{+}(z,\omega)\right| = \left|I_0^{+}(\omega)\right|$$

$$i^{-}(z,\omega) = \left|I^{-}(z,\omega)\right| = \left|I_0^{-}(\omega)\right|$$

The relative phase is not constant!

Q: *What about **phase**?*

*Is it a **constant** with respect to position z as well?*

A: **Nope**. Instead, we find that:

$$\begin{aligned}
\varphi^{+}(z,\omega) &= arg\left\{V^{+}(z,\omega)\right\} \\
&= arg\left\{V_0^{+}(\omega)\, e^{-j\beta z}\right\} \\
&= arg\left\{\left|V_0^{+}(\omega)\right| e^{+j\varphi_0}\, e^{-j\beta z}\right\} \\
&= \varphi_0^{+}(\omega) - \beta z
\end{aligned}$$

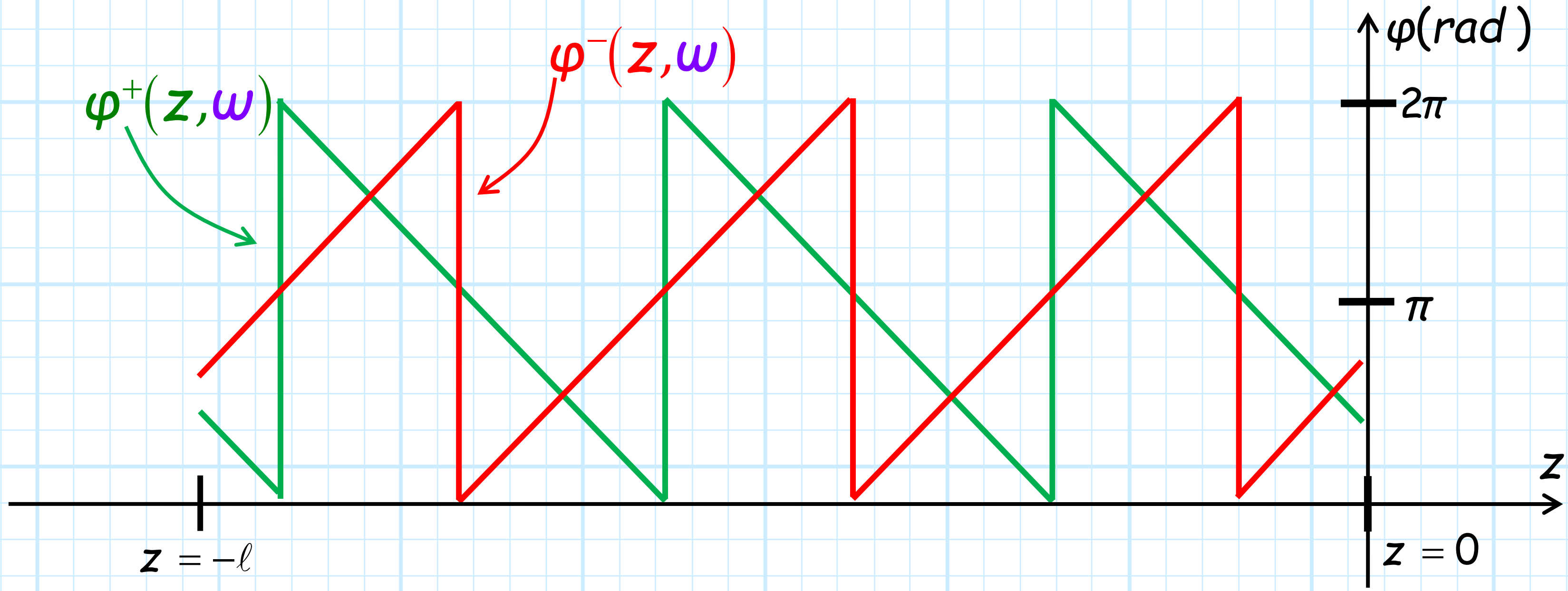
→ The relative phase of the voltage wave is **directly proportional** to line position *z*!

One slope is positive, the other negative

For the **minus**-wave $V^-(z,\omega) = V_0^- e^{+j\beta z}$, the relative phase is likewise proportional:

$$\varphi^-(z,\omega) = arg\{V^-(z,\omega)\} = \varphi_0^-(\omega) + \beta z$$

However, note the relative phase of the **plus**-wave has **negative** proportionality constant $-\beta$, while the **minus**-wave has the **opposite** (i.e., positive) value $+\beta$



Careful!

Q: *So, the **magnitudes** of wave $V^{+}(z,\omega)$ and wave $V^{-}(z,\omega)$ are **constant** with respect to position z:*

$$|V^{+}(z,\omega)| = |V_0^{+}(\omega)| \qquad\qquad |V^{-}(z,\omega)| = |V_0^{-}(\omega)|$$

*and the voltage $V(z,\omega)$ is the **sum** of these two waves:*

$$V(z,\omega) = V^{+}(z,\omega) + V^{-}(z,\omega)$$

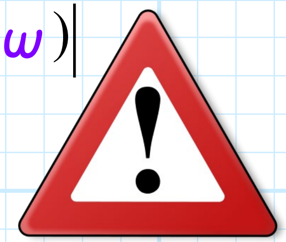
*Isn't it then evident that the **magnitude** of $V(z,\omega)$ is likewise **constant**?*

$$|V(z,\omega)| \overset{?}{=} |V^{+}(z,\omega)| + |V^{-}(z,\omega)| = |V_0^{+}(\omega)| + |V_0^{-}(\omega)| \quad ??$$

Do this and lose about a million exam points

A: **NOOO!!** Do **not** make this mistake!

$$\left|V(z,\omega)\right| \neq \left|V^{+}(z,\omega)\right| + \left|V^{-}(z,\omega)\right| = \left|V_0^{+}(\omega)\right| + \left|V_0^{-}(\omega)\right|$$

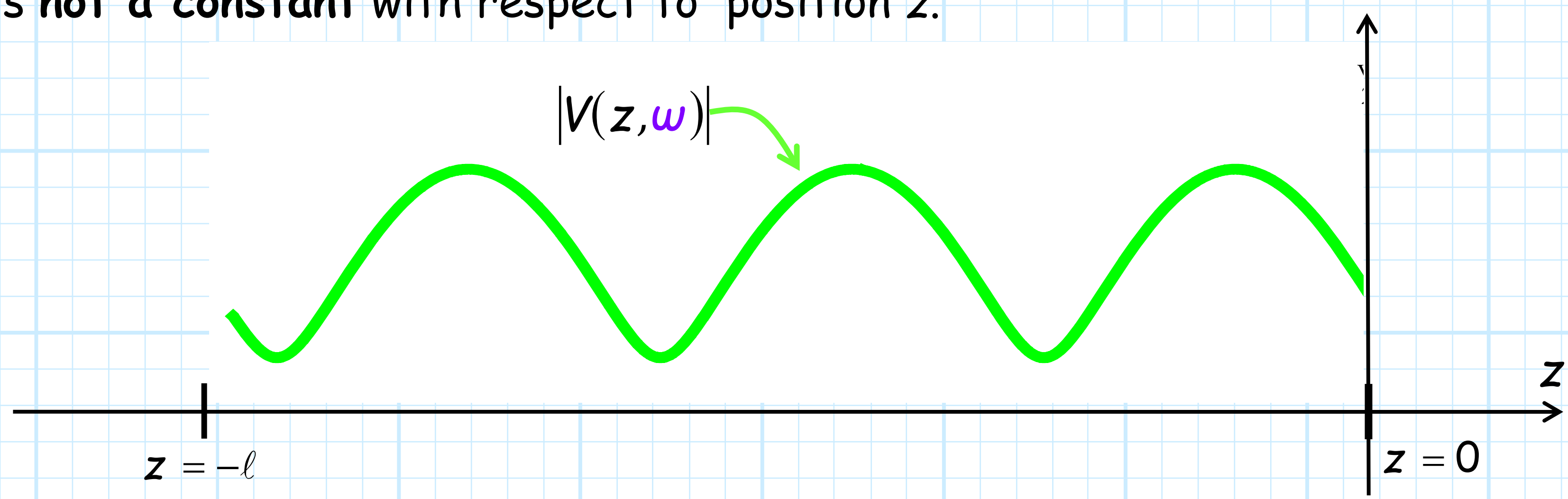


Remember, $|a+b| \neq |a|+|b|$.

Therefore, the **magnitude** of the transmission line voltage:

$$\left|V(z,\omega)\right| = \left|V^{+}(z,\omega) + V^{-}(z,\omega)\right|$$

is **not a constant** with respect to position z.



Catch a wave

Q: *But **why** do you refer to complex functions $V^+(z,\omega)$ and $V^-(z,\omega)$ as **waves**?*

*They **don't** seem very wave-like!*

A: To **see** their wave behavior, we first must express each term as a **real-valued** voltage.

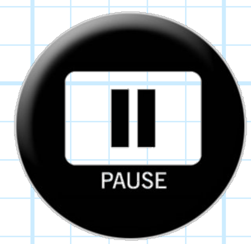
For **example**, the plus-wave:

$$v^+(z,t) = Re\left\{V_0^+(\omega)\,e^{-j\beta z}\,e^{j\omega t}\right\} = \left|V_0^+(\omega)\right|\,cos\left(\omega t - \beta z + \varphi_0^+\right)$$

Now, let's consider this function at **one particular time**—time $t=0$.

$$v^+(z,t=0) = \left|V_0^+(\omega)\right|\,cos\left(-\beta z + \varphi_0^+\right)$$

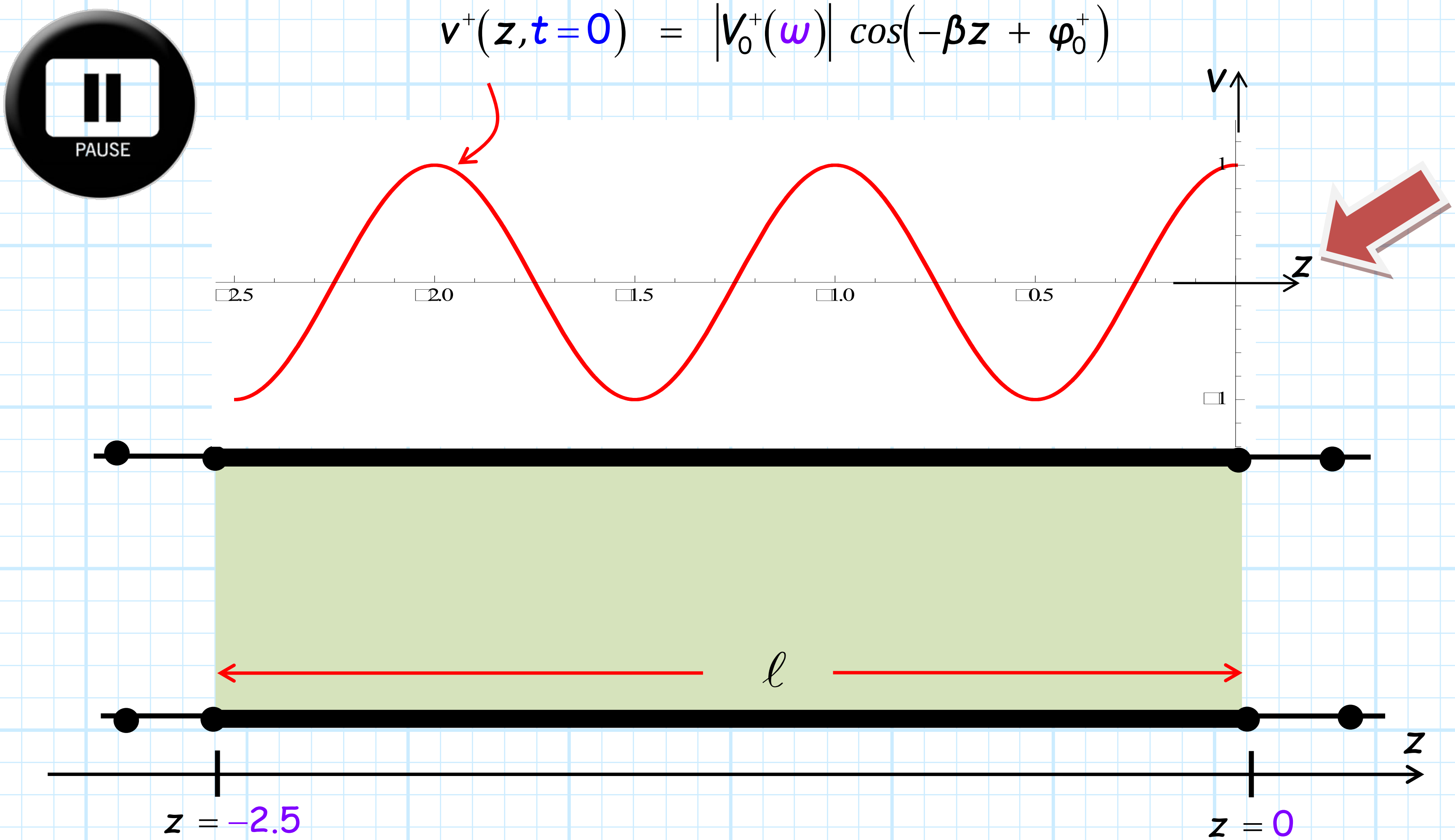
Note this is now a function of **position *z* only** (since $t=0$).



Sinusoids can be a function of z as well as t

Plotting this function with respect to z, we find yet another sinusoid—**but this one is a function of** position!

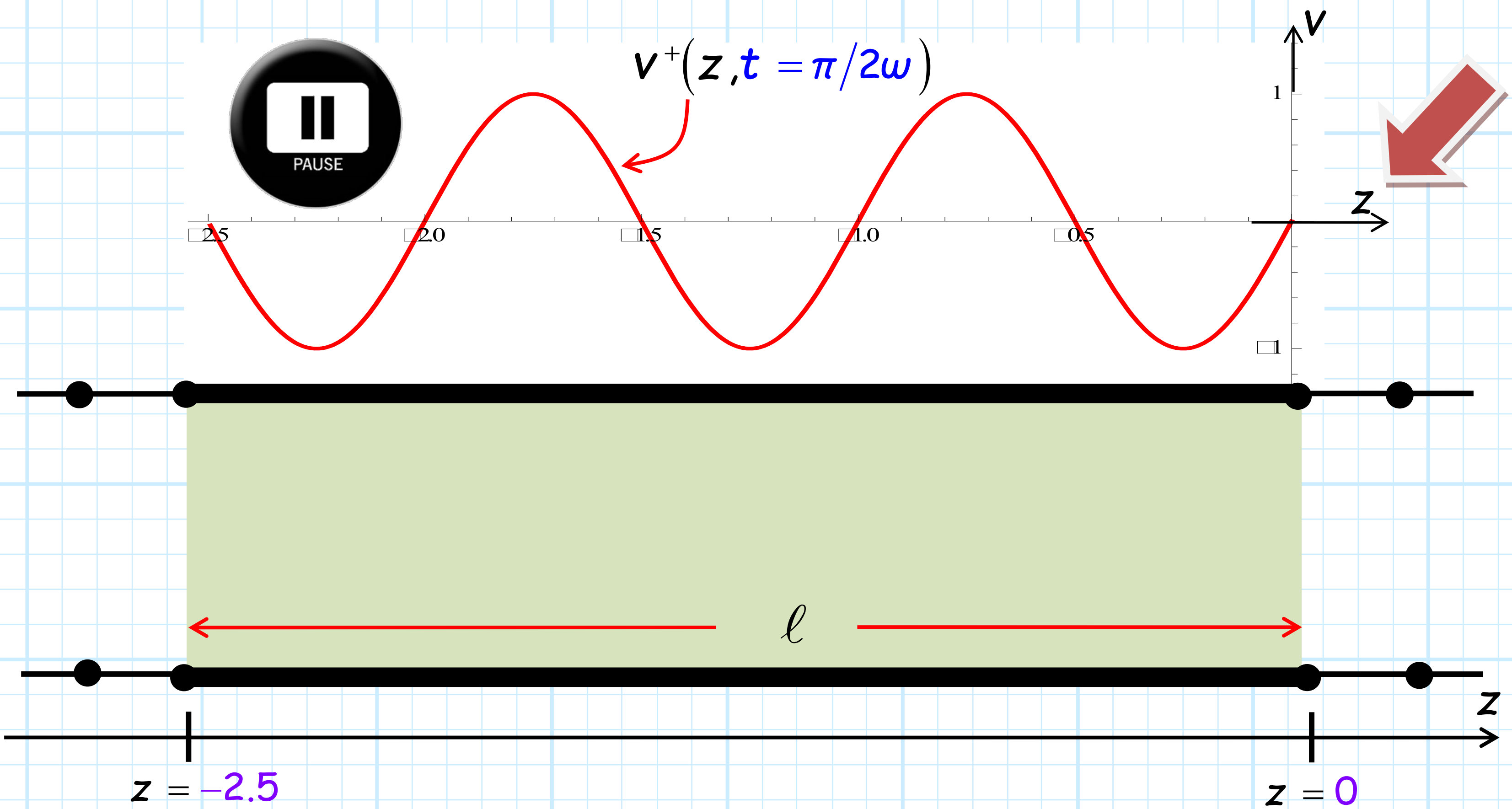
$$v^{+}(z,t=0) = \left|V_0^{+}(\omega)\right| \cos\left(-\beta z + \varphi_0^{+}\right)$$



Fast forward just a little

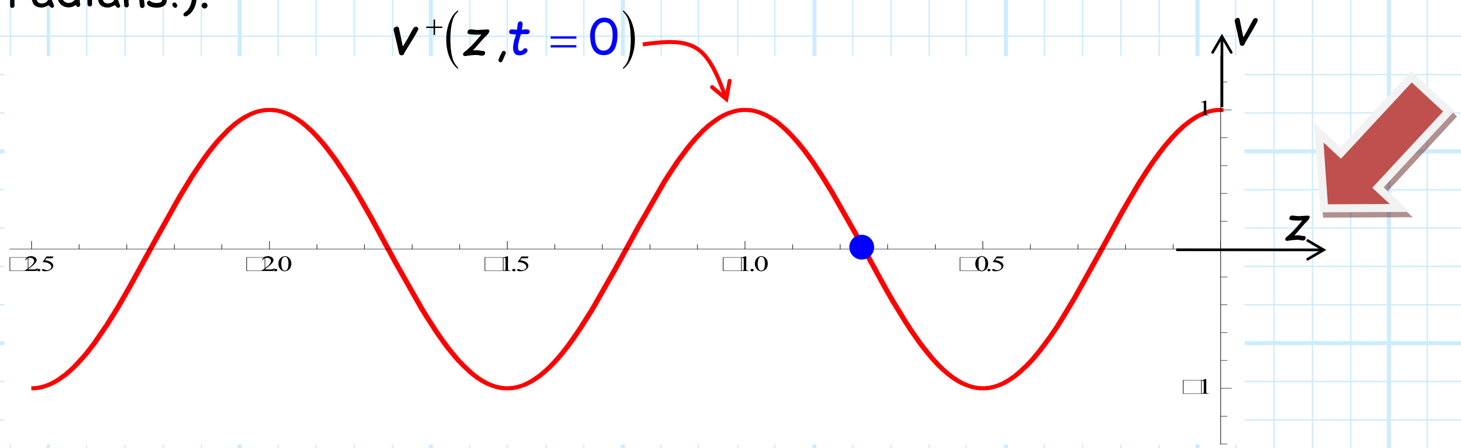
Now, we **again** evaluate $v^+(z,t)$, this time at time $t=\pi/2\omega$, such that $\omega t=\pi/2$:

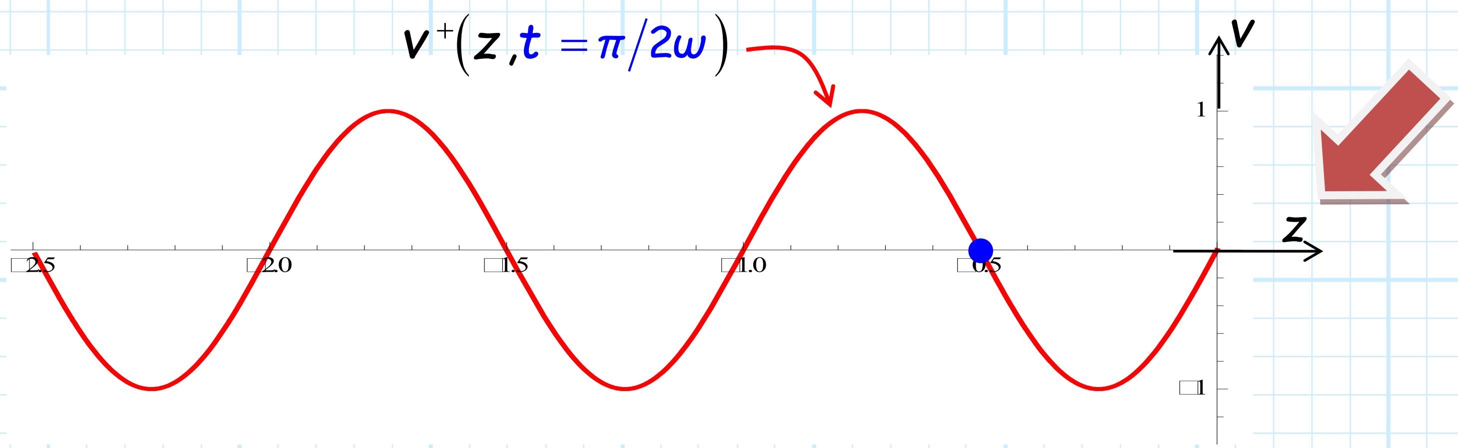
$$
\begin{aligned}
v^+(z,t=\pi/2\omega) &= |V_0^+(\omega)|\ \cos\left(\omega(\pi/2\omega) - \beta z + \varphi_0^+\right) \\
&= |V_0^+(\omega)|\ \cos\left(\pi/2 - \beta z + \varphi_0^+\right)
\end{aligned}
$$



The plus wave propagates right

What we see is that from time $t = 0$, to time $t = \pi/2\omega$, the sinusoidal function **"slides" to the right** by an amount equal to one-quarter cycle (i.e. $\pi/2$ radians!).



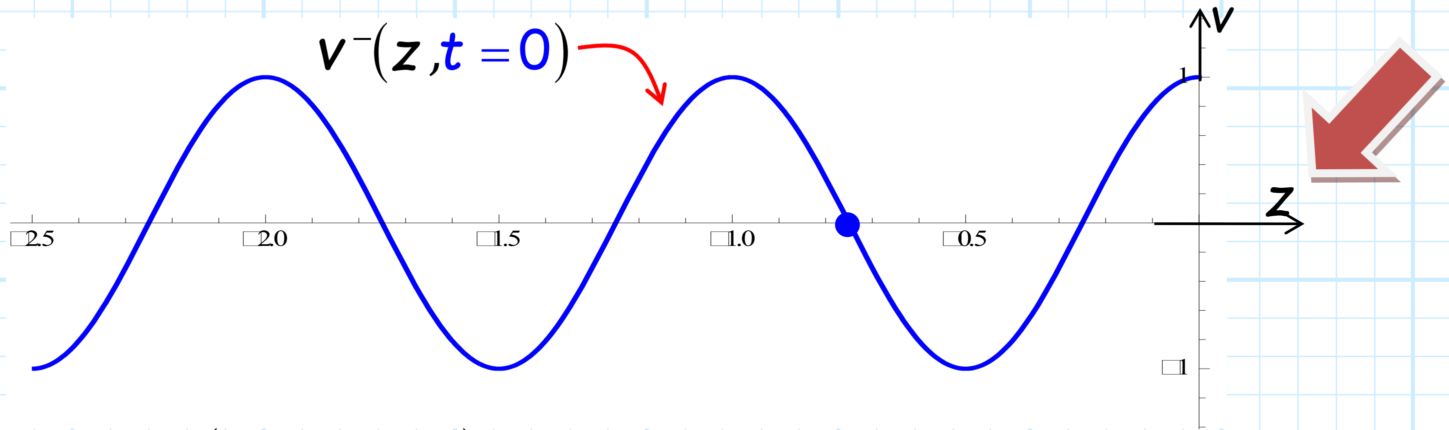


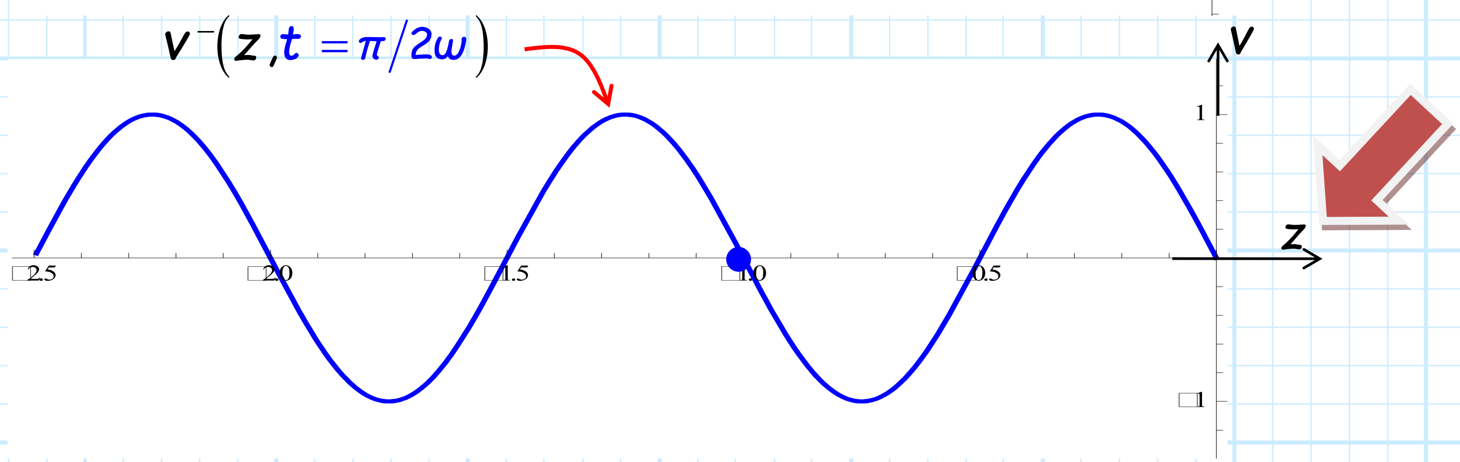
→ The **plus**-wave is propagating in the direction of ***increasing z*** !

The minus wave propagates left

Likewise, the real-valued voltage of the **minus-wave** is:

$$v^-(z,t) \;=\; Re\left\{V_0^-(\omega)\, e^{+j\beta z}\, e^{j\omega t}\right\} \;=\; \left|V_0^-(\omega)\right| cos\left(\omega t + \beta z + \varphi_0^-\right)$$





→ The **minus**-wave propagates in the direction of **decreasing *z*** !