MATH 648: Homework 2

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- Section 2.5, #2, #3.
- Section 3.1, Page 59: #1, #2, #3.
- Section 3.2, Page 64: #1, #2, #3.

[Hint for §3.1 problem #3: This problem is tricky. Note that the integrand $F = y'\sqrt{1+(y'')^2}$ does not depend on y neither on x. Thus, $\frac{d}{dx}Fy'' - Fy' = c_1$ and $H = y''\frac{F}{y''} - y'\left(\frac{d}{dx}Fy'' - Fy'\right) - F = c_2$. Together, they give $y''\frac{F}{y''} - c_1 y' - F = c_2$ or $y'\left(\frac{y''^2}{\sqrt{1+y''^2}} - c_1 - \sqrt{1+y''^2}\right) = c_2$. (1) Normally, one would solve for y'' form (1) to get $y'' = g(y', c_1, c_2)$, find a general solution for y, then, apply the boundary conditions at the last moment. For this problem, this general approach would be extremely complicated. Instead, the observation is that, the boundary condition $y'(0) = 0$ allows you to conclude, from equation (1), that $c_2 = 0$.]
[Hint for §3.2 problem #3: For $k \neq 0$, apply formula (3.17) first and then use the E-L for q_1-component. For $k = 0$, make the change of variable $q_0 = \frac{q_2^2}{2}$ to simplify the functional.]

Section 2.5 (Invariance/Substitution)

Problem 2

Let J be the functional defined by

$$J(r) = \int_{\pi/2}^{\pi} \sqrt{r^2 + \dot{r}^2} d\phi.$$

Find an extremal for J satisfying the boundary conditions $r(\pi/2) = 1$ and $r(\pi) = -1$.

Solution:
The integrand F does not explicitly depend on ϕ, so the E-L equation goes to:

$$\begin{aligned}
\dot{r}\frac{dF}{d\dot{r}} - F &= C \\
\dot{r}\frac{\dot{r}}{\sqrt{r^2+\dot{r}^2}} - \sqrt{r^2+\dot{r}^2} &= C \\
\frac{\cancel{\dot{r}^2} - (r^2 + \cancel{\dot{r}^2})}{\sqrt{r^2+\dot{r}^2}} &= C \\
\frac{r^2}{C^2} &= r^2 + \dot{r}^2 \\
\dot{r} = \frac{dr}{d\phi} &= \frac{r}{C}\sqrt{r^2 - C^2} \\
\int \frac{C}{r\sqrt{r^2-C^2}}\, dr &= \int d\phi \\
\arctan\left(\frac{\sqrt{r^2-C^2}}{C}\right) + c_1 &= \phi
\end{aligned}$$

Solving this for $r(\phi)$ gives (Applying the trig identity $\sec^2 - 1 = \tan^2$):

$$\begin{aligned}
\frac{\sqrt{r^2-C^2}}{C^2} &= \tan(\phi - c_1) \\
r^2 &= C^2\tan^2\phi + C^2 = C^2\sec^2(\phi - c_1) \\
r &= C\sec(\phi - c_1)
\end{aligned}$$

For the boundary conditions:

$$\begin{aligned}
r(\pi/2) = 1 &\implies 1 = C\sec(\pi/2 - c_1) \\
r(\pi) = -1 &\implies -1 = C\sec(\pi - c_1)
\end{aligned}$$

$$A = \frac{1}{\sqrt{2}}, \quad B = \left(\pi n + \frac{\pi}{8}\right)$$

Problem 3

Let J be a functional of the form

$$J(y) = \int_{x_0}^{x_1} g\left(x^2 + y^2\right) \sqrt{1 + y'^2} dx$$

where g is some function of $x^2 + y^2$. Use the polar coördinate transformation to find the general form of the extremals in terms of $g, r,$ and ϕ.

Solution:
Let

$$g(x^2 + y^2) \to g(r^2) \tag{1}$$
$$\sqrt{1 + y'^2}\, dx \to \sqrt{r^2 + \dot{r}^2}\, d\phi \tag{2}$$

(these substitutions are made in the textbook example 2.5.1)

$$J(y) = \int_{\phi_0}^{\phi_1} g(r^2)\sqrt{r^2 + \dot{r}^2}\, d\phi$$

The E-L equation for this is:

$$g(r^2)\frac{\dot{r}^2}{\sqrt{r^2 + \dot{r}^2}} - g(r^2)\sqrt{r^2 + \dot{r}^2} = C$$

The final solution to this differential equation is:

$$g(r^2)r(\phi) = C\sqrt{\tan^2(c_1 C + \phi) + 1}$$

Section 3.1 (Second Variation)

Problem 1

Find the general solution for the extremals to the functional J defined by

$$J(y) = \int_{x_0}^{x_1} \left((y'')^2 - y^2 + 2yx^3\right) dx.$$

Solution:
Because there is no explicit y' dependence, the E-L equation goes to:

$$\frac{d^2}{dx^2} f_{y''} - f_y = C$$
$$\frac{d^2}{dx^2} 2y'' - \left(2y + 2x^3\right) = C$$
$$2y^{(4)} - 2y = C + 2x^3$$

The characteristic polynomial for this is $2\lambda^4 - 2 = 0$, which gives solutions $\lambda = \{1, -1, i, -i\}$.

$$
\begin{aligned}
y_c(x) &= c_1e^x + c_2e^{-x} + c_3e^{ix} + c_4e^{-ix} \\
y_c'(x) &= c_1e^x - c_2e^{-x} + c_3ie^{ix} - c_4ie^{-ix} \\
y_c''(x) &= c_1e^x + c_2e^{-x} + c_3\cancelto{-1}{i^2}e^{ix} + c_4\cancelto{-1}{i^2}e^{-ix} \\
y_c'''(x) &= c_1e^x - c_2e^{-x} - c_3ie^{ix} + c_4ie^{-ix} \\
y_c''''(x) &= c_1e^x + c_2e^{-x} - c_3\cancelto{-1}{i^2}e^{ix} - c_4\cancelto{-1}{i^2}e^{-ix}
\end{aligned}
$$

Substituting these:

$$
\begin{aligned}
2\left[\cancel{c_1e^x} + c_2e^{-x} + c_3e^{ix} + c_4e^{-ix} - \cancel{c_1e^x} + c_2e^{-x} + c_3e^{ix} + c_4e^{-ix}\right] &= C + 2x^3 \\
\cancel{2}[2c_2e^{-x} + 2c_3e^{ix} + 2c_4e^{-ix}] &= \frac{C}{2} + \cancel{2}x^3 \\
y_p = 2c_2e^{-x} + 2c_3e^{ix} + 2c_4e^{-ix} - \frac{C}{2} - x^3
\end{aligned}
$$

General solution:

$$
y(x) = c_1e^x + 3c_2e^{-x} + 3c_3e^{ix} + 3c_4e^{-ix} - \frac{C}{2} - x^3
$$

Problem 2

Conservation Law: Suppose the integrand f defining the functional J does not depend on x explicitly. Prove that equation (3.4) is satisfied along any extremal.

Solution:

$$
y''f_{y''} - y'\left(\frac{d}{dx}f_{y''} - f_{y'}\right) - f = const \tag{3.4}
$$

Suppose that y is an extremal for J. Now,

$$
\begin{aligned}
&\frac{d}{dx}\left(y''f_{y''} - y'\left(\frac{d}{dx}f_{y''} - f_{y'}\right) - f\right) \\
&= y'''\frac{\partial f}{\partial y''} + y''\frac{d}{dx}\frac{\partial f}{\partial y''} - y''\frac{d}{dx}\frac{\partial f}{\partial y''} - y'\frac{d^2}{dx^2}\frac{\partial f}{\partial y''} + y'\frac{\partial f}{\partial y'} + y'\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{d}{dx}(f(y, y', y'')) \\
&= \cancel{y'''\frac{\partial f}{\partial y''}} + \cancel{y''\frac{d}{dx}\frac{\partial f}{\partial y''}} - \cancel{y''\frac{d}{dx}\frac{\partial f}{\partial y''}} - y'\frac{d^2}{dx^2}\frac{\partial f}{\partial y''} + \cancel{y''\frac{\partial f}{\partial y'}} + y'\frac{d}{dx}\frac{\partial f}{\partial y'} - y'\frac{\partial f}{\partial y} - \cancel{y''\frac{\partial f}{\partial y'}} - \cancel{y'''\frac{\partial f}{\partial y''}} \\
&= -y'\left(f_y - \frac{d}{dx}f_{y'} - \frac{d^2}{dx^2}f_{y''}\right)
\end{aligned}
$$

Since y is an extremal, the E-L equation is satisfied.

Problem 3

For the functional J defined by

$$J(y) = \int_0^1 y' \sqrt{1 + (y'')^2} dx$$

find an extremal satisfying the conditions $y(0) = 0, y'(0) = 0, y(1) = 1$, and $y'(1) = 2$.

Solution:

$$f_{y'} - \frac{d}{dx} f_{y''} = 0$$

$$\begin{aligned}
\implies \sqrt{1+y''^2} - \frac{d}{dx}\left(y' \frac{y''}{\sqrt{1+y''^2}}\right) &= 0 \\
\sqrt{1+y''^2} - \frac{y''^4 + y''^2 + y^{(3)}y'}{(1+y''^2)^{3/2}} &= 0 \\
\frac{\cancel{y''^4} + \cancel{2y''^2} + 1}{(1+y''^2)^{3/2}} - \frac{\cancel{y''^4} + \cancel{y''^2} + y^{(3)}y'}{(1+y''^2)^{3/2}} &= 0 \\
\frac{1}{\cancel{(1+y''^2)^{3/2}}} &= \frac{y^{(3)}y'}{\cancel{(1+y''^2)^{3/2}}} \\
y^{(3)}y' &= 1
\end{aligned}$$

Unfortunately we never worked with higher order differential equations in MATH 220, so I have no idea how to solve this directly, however, I am very curious if the special form of the functional briefly discussed in class could be applied: Let $y''' = z''$, $y'' = z'$, and $y' = z$, subject to the constraint $\int_0^1 z(x)\,dx = y_1 - y_0$. This may be an inappropriate time to apply this substitution, however, I will try continuing regardless.

$$y''y' = 1$$

(can a laplace transformation be used here?)

$$\begin{aligned}
z(x) = y'(x) &= c_2 + \frac{2}{3}\sqrt{2}(c_1 + x)^{3/2} \\
y(x) &= \int c_2 + \frac{2}{3}\sqrt{2}(c_1 + x)^{3/2}\,dx \\
&= c_2 x + \int \frac{2}{3}\sqrt{2}(c_1 + x)^{3/2}\,dx \\
&= c_2 x + \frac{2\sqrt{2}}{15}\left(2x\,(c_1 + x)^{\frac{3}{2}} + 2c_1\,(x + c_1)^{\frac{3}{2}}\right) + c_3 \\
&= \frac{4\sqrt{2}}{15}(c_1 + x)^{5/2} + c_2 x + c_3
\end{aligned}$$

Now, applying the constraints:

$$\begin{aligned}
y_1 - y_0 = 1 &= \frac{4\sqrt{2}}{15}(c_1 + x)^{5/2} + c_2 x + c_3 \\
0 &= c_2 + \frac{3\sqrt{2}}{2}(c_1)^{3/2} \\
2 &= c_2 + \frac{3\sqrt{2}}{2}(c_1 + 1)^{3/2}
\end{aligned}$$

Problem 3 continued on next page. . .

Section 3.2

Problem 1

Let

$$L(t, \mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right) - gq_2,$$

where g is a constant.

(a) Find the extremals for the functional J defined by

$$J(\mathbf{q}) = \int_{t_0}^{t_1} L(t, \mathbf{q}, \dot{\mathbf{q}})dt.$$

Solution:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$$

The special case could be applied here, however it gives a horrible differential equation whereas the standard form is much kinder:

$$\begin{aligned} \frac{d}{dt}\left(\dot{q}_1\right) &= \ddot{q}_1 = 0 \\ \frac{d}{dt}\left(\dot{q}_2\right) - g = \ddot{q}_2 - g &= 0 \end{aligned}$$

This gives a beautifully independent system of linear differential equations with solutions:

$$\begin{aligned} q_1(t) &= c_2(t) + c_1 \\ q_2(t) &= \frac{1}{2}gt^2 + c_2 t + c_1 \end{aligned}$$

(b) Verify that equation (3.17) is satisfied.

Solution:

Showing that equation 3.17 is satisfied implies showing that energy is conserved along an extremal. Following the Hamiltonian definition of energy: $E = T(\dot{q}) + V(q)$ and applying equation 3.17:

$$\begin{aligned} H &= \dot{q}\frac{\partial L}{\partial \dot{q}} - L \\ &= \dot{q}\frac{\partial T}{\partial \dot{q}} - L \\ &= \overset{2T}{\cancel{2m\dot{q}^2}} - T + V \\ &= T + V \end{aligned}$$

Therefore, energy is conserved along any extremal.

Problem 2

Prove equation (3.17).

Solution:
Suppose that there is an etremal q_j for L, given that L is a functional: $L(q, \dot{q})$ Now,

$$\begin{aligned} \frac{d}{dt}\left(\dot{q}\frac{\partial L}{\partial \dot{q}} - L\right) &= \ddot{q}\frac{\partial L}{\partial \dot{q}} + \dot{q}\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\left(L(q, \dot{q})\right) \\ &= \ddot{q}\frac{\partial L}{\partial \dot{q}} + \dot{q}\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \left(\dot{q}\frac{\partial L}{\partial q} + \ddot{q}\frac{\partial L}{\partial \dot{q}}\right) \\ &= \dot{q}\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}\right) \end{aligned}$$

Problem 3

Let

$$L(t, \mathbf{q}, \dot{\mathbf{q}}) = \sqrt{\dot{q}_1^2 + q_2^2\dot{q}_2^2} - kq_2,$$

where k is a constant. Find the extremals for the functional J defined by

$$J(\mathbf{q}) = \int_{t_0}^{t_1} L(t, \mathbf{q}, \dot{\mathbf{q}})dt.$$

Solution:

$$\sum_{j=1}^{n} q_j L_{\dot{q}_j} - L = C$$

$$\begin{aligned} q_1\frac{\dot{q}_1}{\sqrt{\dot{q}_1^2 + q_2^2\dot{q}_2^2}} - \sqrt{\dot{q}_1^2 + q_2^2\dot{q}_2^2} - kq_2 = C \\ q_2\frac{\dot{q}_2}{\sqrt{\dot{q}_1^2 + q_2^2\dot{q}_2^2}} - \sqrt{\dot{q}_1^2 + q_2^2\dot{q}_2^2} - kq_2 = C \end{aligned}$$

I unfortunately have no idea how to even begin to approach these differential equations, and attempting the typical E-L equation would likely produce an even more difficult system of equations with higher order derivatives.