MATH 648: Homework #1

February 8, 2024

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- Section 2.2, Pages 35-36: #1, #2, #4, #5; [Hints: For 5(a), a general solution is not smooth at $x = 0$ except the trivial solutions $y(x) =$ constants.

  For 5(b), it is clear that $J(y) \geq 0$. Next, you may assume that there is a function $\varphi \in C^2([-1, 1])$ so that $\varphi(-1) = -1$, $\varphi(1) = 1$, and $\varphi'(-1) = \varphi'(1) = \varphi''(-1) = \varphi''(1) = 0$.

  (Can you construct such a function φ?)

  Now, for any $\varepsilon > 0$, define $y_\varepsilon(x) = \begin{cases} -1, & \text{if } x \in [-1, -\varepsilon) \\ \varphi(x/\varepsilon), & \text{if } x \in [-\varepsilon, \varepsilon] \\ 1, & \text{if } x \in (\varepsilon, 1]. \end{cases}$

  You should be able to show that $y_\varepsilon \in S$ and $J(y_\varepsilon) \to 0$ as $\varepsilon \to 0$. But there is no $y \in S$ so that $J(y) = 0$.

- Section 2.3, Page 41: #1, #3.

- An extra problem not from the textbook: Find the extrema $y = y(x)$ of

$$J(y) = \int_0^1 (y')^2 \sqrt{1 + (y')^2}, dx, \ y(0) = 1 = y(1).$$

  Which one is a minimum or maximum? Why?

Section 2.2

Problem 1

Let J be a functional. Prove that for $\frac{dJ}{d\epsilon} = 0$ as ϵ approaches 0 for $J(y + \epsilon\eta)$ leads to condition (2.6).

$$\frac{dJ}{d\epsilon} = \int_{x_0}^{x_1} \left(\frac{dJ}{dx}\frac{dx}{d\epsilon} + \frac{dJ}{d\hat{y}}\frac{d\hat{y}}{d\epsilon} + \frac{dJ}{d\hat{y}'}\frac{d\hat{y}'}{d\epsilon} \right)\bigg|_{\epsilon=0} dx \tag{1}$$

$$\hat{y} = y + \epsilon\eta \implies \frac{d\hat{y}}{d\epsilon} = \eta, \quad \hat{y}' = y' + \epsilon\eta \implies \frac{d\hat{y}'}{d\epsilon} = \eta' \tag{2}$$

$$\frac{dJ}{d\epsilon} = \int_{x_0}^{x_1} \left(\frac{dJ}{d\hat{y}}\eta + \frac{dJ}{d\hat{y}'}\eta' \right)\bigg|_{\epsilon=0} dx$$

The final equation above is condition 2.6 from the textbook.

Problem 2

The First Variation: Let $J : S \to \Omega$ and $K : S \to \Omega$, be functionals defined by

$$J = \int_{x_0}^{x_1} f(x, y, y')\, dx, \quad K = \int_{x_0}^{x_1} g(x, y, y')\, dx$$

where f and g are smooth functions of the indicated arguments and $\Omega \subset \mathbb{R}$.

(a) Show that for any real numbers A and B,

$$\delta\left(AJ + BK\right)(\eta, y) = A\delta J(\eta, y) + B\delta K(\eta, y)$$

(linear)

(b)

$$\delta(JK)(\eta, y) = K(\eta, y)\delta J(\eta, y) + J(\eta, y)\delta K(\eta, y)$$

(product rule)

(c) Suppose that $G : \Omega \times \Omega \to \mathbb{R}$ is differentiable on $\Omega \times \Omega$. Show that

$$\delta G(J, K)(\eta, y) = \frac{\partial G}{\partial J}\delta J(\eta, y) + \frac{\partial G}{\partial K}\delta K(\eta, y)$$

(chain rule)

(a)

$$\begin{aligned}
\delta(AJ + BK)(\eta, y) &= \lim_{\epsilon \to 0} \frac{AJ(y + \epsilon\eta) + BK(y + \epsilon\eta) - (AJ(y) + BK(y))}{\epsilon} \\
&= \lim_{\epsilon \to 0} \left[\frac{AJ(y + \epsilon\eta) - AJ(y)}{\epsilon} + \frac{BK(y + \epsilon\eta) - BK(y)}{\epsilon} \right] \\
&= A\delta J(\eta, y) + B\delta K(\eta, y)
\end{aligned}$$

Problem 2 continued on next page. . .

(b) Let $H(y) = J(y)K(y)$:

$$\begin{aligned}
\delta H(\eta, y) &= \lim_{\epsilon \to 0} \left[\frac{H(y+\epsilon\eta) - H(y)}{\epsilon} \right] \\
&= \lim_{\epsilon \to 0} \left[\frac{J(y+\epsilon\eta)K(y+\epsilon\eta) - J(y)K(y)}{\epsilon} \right] \\
&= \lim_{\epsilon \to 0} \left[\frac{J(y+\epsilon\eta)K(y+\epsilon\eta) - J(y)K(y+\epsilon\eta) + J(y)K(y+\epsilon\eta) - J(y)K(y)}{\epsilon} \right] \\
&= \lim_{\epsilon \to 0} \left[\frac{(J(y+\epsilon\eta) - J(y)) \times K(y+\epsilon\eta) + J(y) \times (K(y+\epsilon\eta) - K(y)}{\epsilon} \right] \\
&= \lim_{\epsilon \to 0} \frac{J(y+\epsilon\eta) - J(y)}{\epsilon} \times \lim_{\epsilon \to 0} K(y+\epsilon\eta) + \lim_{\epsilon \to 0} J(y) \times \lim_{\epsilon \to 0} \frac{K(y+\epsilon\eta) - K(y)}{\epsilon} \\
&= K(\eta, y)\delta J(\eta, y) + J(\eta, y)\delta K(\eta, y)
\end{aligned}$$

(c) …?

Problem 4

Let J be the functional defined by

$$J = \int_0^1 (y'^2 + y^2 + 4y2^x)\ dx$$

with boundary conditions $y(0) = 0$ and y(1)=1. Find the extremals for J.

$$\begin{aligned} \frac{\partial J}{\partial y} &= 2y + 4e^x \\ \frac{d}{dx}\left(\frac{\partial J}{\partial y'}\right) &= \frac{d}{dx}(2y') = 2y'' \\ \implies 0 &= 2y'' - 2y - 4e^x \end{aligned} \tag{1}$$

$$\begin{aligned} 2y'' - 2y = 4e^x &\to y' - y = 2e^x \\ e^{-x}y'' - e^{-x}y = 2 & \\ (e^{-x}y')' = 2 &\to \int (e^{-x}y')'\ dx = \int 2\ dx \\ e^{-x}y' = 2x + c_1 &\to y' = 2xe^x + c_1e^x \end{aligned} \tag{2}$$

$$\begin{aligned} \int y'\ dx &= \int 2xe^x + c_1e^x\ dx \\ y &= 2(xe^x - e^x) + c_1e^x + c_2 \end{aligned} \tag{3}$$

Now, to fulfill our boundary conditions:

$$\begin{aligned} y(0) = 0 &\implies 0 = -2 + c_1 + c_2 \\ y(1) = 1 &\implies 1 = c_1e + c_2 \\ c_1 &= \frac{1}{1-e} \\ c_2 &= \frac{2e-1}{e-1} \end{aligned}$$

Problem 5

Consider the functional defined by:

$$J = \int_{-1}^1 x^4y'^2\ dx$$

(a) Show that no extremals in $C^2[-1,1]$ exist which satisfy the boundary conditions $y(-1) = -1,\ y(1) = 1$.

(b) Without resorting to the Euler-Lagrange equation, prove that J cannot have a local minimum in the set:

$$S = \{y \in C^2[-1,1] : y(-1) = -1 \text{ and } y(1) = 1\}$$

(a)

$$\begin{aligned} \frac{d}{dx}\left(\frac{\partial J}{\partial y'}\right) &= \frac{d}{dx}\left(2y'x^4\right) = 2y''x^4 + 8y'x^3 \\ \frac{\partial J}{\partial y} &= 0 \end{aligned}$$

$$\implies 2y''x^4 + 8y'x^3 = 0$$
$$\dots$$
$$y(x) = \frac{c_1}{x^3} + c_2$$

For the boundary conditions:

$$-1 = -c_1 + c_2$$
$$1 = c_1 + c_2$$

it is clear that these are linearly dependent, and therefore no solutions exist for these boundary conditions.

(b) Let φ equal the step function $\begin{cases} -1, & -1 \le x \le 0 \\ 1, & 0 < x \le 1 \end{cases}$.

$$y_\epsilon(x) = \begin{cases} -1 & \text{if } x \in [-1, -\epsilon) \\ \varphi\left(\frac{x}{\epsilon}\right) & \text{if } x \in [-\epsilon, \epsilon] \\ 1 & \text{if } x \in (\epsilon, 1] \end{cases}$$

As ϵ approaches zero, $y_\epsilon(x)$ approaches the piecewise function $\varphi(x)$, and $J(y_\epsilon)$ approaches 0. However there is no function in $y(x)$ such that $J(y) = 0$ since that would violate the boundary conditions.

Section 2.3

Problem 1

Find the general solution to the Euler-Lagrange equation corresponding to the functional

$$J = \int_{x_0}^{x_1} f(x)\sqrt{1 + y'^2}\, dx$$

where $x_0 > 0$, and investigate the special cases: (i) $f(x) = \sqrt{x}$, (ii) $f(x) = x$.

General Case

$$J = \int_{x_0}^{x_1} y\sqrt{1 + y'^2}\, dx$$
$$\frac{d}{dy}(J) = \sqrt{1 + y'^2}$$
$$\frac{d}{dx}\left(\frac{yy'}{\sqrt{1 + y'^2}}\right) = \frac{yy'' + y'^4 + y'^2}{(1 + y'^2)^{3/2}}$$
$$\implies \frac{yy'' + y'^4 + y'^2}{(1 + y'^2)^{3/2}} = 0$$
$$\dots$$
$$y(x) = ?$$

Problem 1 continued on next page...

Special Cases

(i)

$$J = \int_{x_0}^{x_1} \sqrt{x + xy'^2}\ dx$$

$$\begin{aligned}
\frac{d}{dx}\left(\frac{xy}{\sqrt{x+xy'^2}}\right) &= \frac{x(2xy'' + y'^3 + y')}{2(xy'^2 + x)^{3/2}} \\
\implies \frac{x(2xy'' + y'^3 + y')}{2(xy'^2 + x)^{3/2}} &= 0 \\
&\dots \\
y(x) = c_2 \pm 2ie^{c_1}\sqrt{e^{2c_1} - x}
\end{aligned}$$

(ii)

$$J = \int_{x_0}^{x_1} x\sqrt{1 + y'^2}\ dx$$

$$\begin{aligned}
\frac{d}{dx}\left(\frac{xy'}{\sqrt{y'^2+1}}\right) &= \frac{xy'' + y'^3 + y'}{(y'^2 + 1)^{3/2}} \\
\implies \frac{xy'' + y'^3 + y'}{(y'^2 + 1)^{3/2}} &= 0 \\
&\dots \\
y(x) = c_2 \pm ie^{c_1}\tan^{-1}\left(\frac{x}{\sqrt{e^{2c_1} - x^2}}\right)
\end{aligned}$$

Problem 3

Find a smooth extremal for J satisfying the boundary conditions $y(2) = 1$ and $y(3) = \sqrt{3}$.

$$J = \int_2^3 y^2(1 - y')^2\ dx$$

$$\begin{aligned}
\frac{d}{dx}\left(\frac{\partial J}{\partial y'}\right) &= -2y^2(1 - y') = 2y\left(yy'' + 2y'^2 - 2y'\right) \\
\frac{\partial J}{\partial y} &= 2y(1 - y')^2
\end{aligned}$$

$$\implies 2y\left(yy'' + 2y'^2 - 2y'\right) - 2y(1 - y')^2 = 0 \tag{1}$$

$$\to 2yy'' + \overset{2}{\cancel{4}}yy'^2 - \cancel{4yy'} - 2y + \cancel{4yy'} - \cancel{2yy'^2} = 0$$

$$\to 2y(yy'' + y'^2 - 1) = 0 \tag{2}$$

$$\dots$$

$$y(x) = \pm\sqrt{2c_2 x + c_2^2 - c_1 + x^2}$$

Problem 3 continued on next page...

For the boundary conditions:

$$\begin{aligned}
1 &= \sqrt{4c_2 + c_2^2 - c_1 + 4} \\
\sqrt{3} &= \sqrt{6c_2 + c_2^2 - c_1 + 9} \\
c_1 &= -3/2 \\
c_2 &= -3/4
\end{aligned}$$

Problem 4

An extra problem not from the textbook: Find the extrema $y = y(x)$ of

$$J(y) = \int_0^1 (y')^2 \sqrt{1 + (y')^2}, dx,\ y(0) = 1 = y(1).$$

Which one is a minimum or maximum? Why?

$$\begin{aligned}
\frac{d}{dx}\left(\frac{\partial J}{\partial y'}\right) &= \frac{d}{dx}\left(2y'\sqrt{1+y'^2} + y'^2 \frac{y'}{\sqrt{1+y'^2}}\right) \\
&= \frac{d}{dx}\left(\frac{2y'(1+y'^2) + y'^3}{\sqrt{1+y'^2}}\right) \\
&= \frac{d}{dx}\left(\frac{3y'^3 + 2y'}{\sqrt{1+y'^2}}\right)
\end{aligned}$$

let

$$\begin{aligned}
u &= 3y'^3 + 2y' \\
u' &= 9y'^2 y'' + 2y'' \\
v &= \sqrt{1+y'^2} \\
v' &= \frac{y'y''}{\sqrt{1+y'^2}}
\end{aligned}$$

apply the quotient rule:

$$\begin{aligned}
\frac{d}{dx}\left(\frac{3y'^3 + 2y'}{\sqrt{1+y'^2}}\right) &= \frac{uv' - vu'}{v^2} \\
&= \frac{(3y'^3 + 2y')y'y'' - 9y'^2 y'' + 2y'}{\sqrt{1+y'^2}^{3/2}} \\
&= \frac{y''(6y'^4 + 9y'^2 + 2)}{(1+y'^2)^{3/2}}
\end{aligned}$$

By the Euler-Lagrange equation the following gives the extrema of the functional:

$$\frac{y''(6y'^4 + 9y'^2 + 2)}{(1+y'^2)^{3/2}} = 0$$

Problem 4 continued on next page...

I was not capable of directly solving this, wolfram alpha tells me the solution is:

$$y(x) = c_1 \pm \frac{1}{2} i x \sqrt{3 \pm \sqrt{11/3}}$$

In order to determine what is a minima and maxima, I would need to apply the second variation to check the sign at each extremal. A positive sign would indicate a minimal and a negative a maximal.