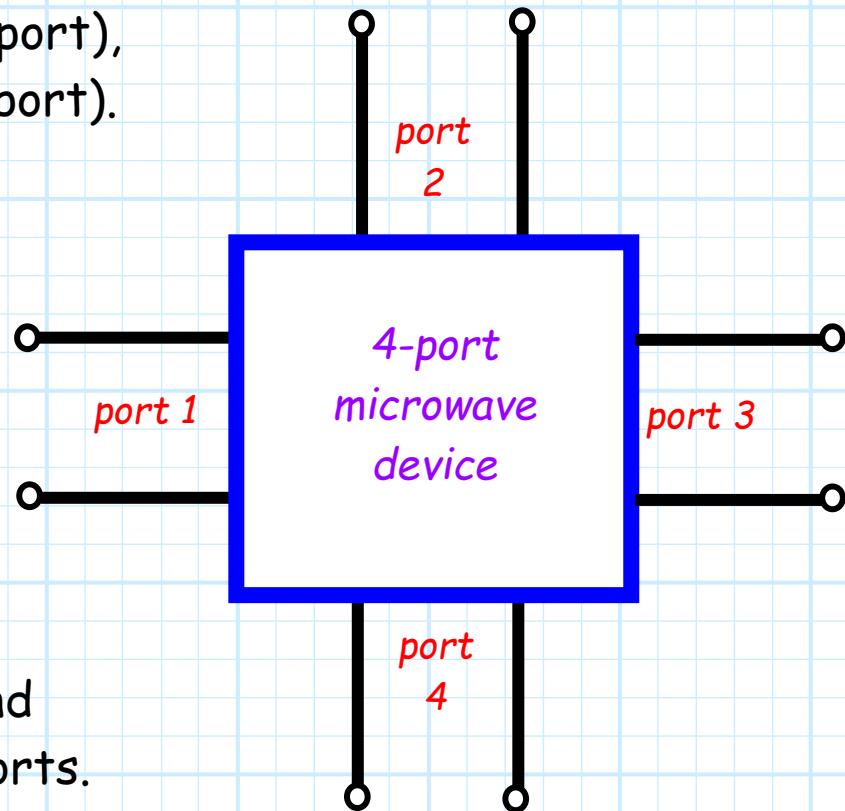


Multiport Networks and the Impedance Matrix

Most microwave components are **neither** strictly a source (with one output port), **nor** strictly a load (with one input port).



Instead, microwave components and networks typically have **multiple** ports.

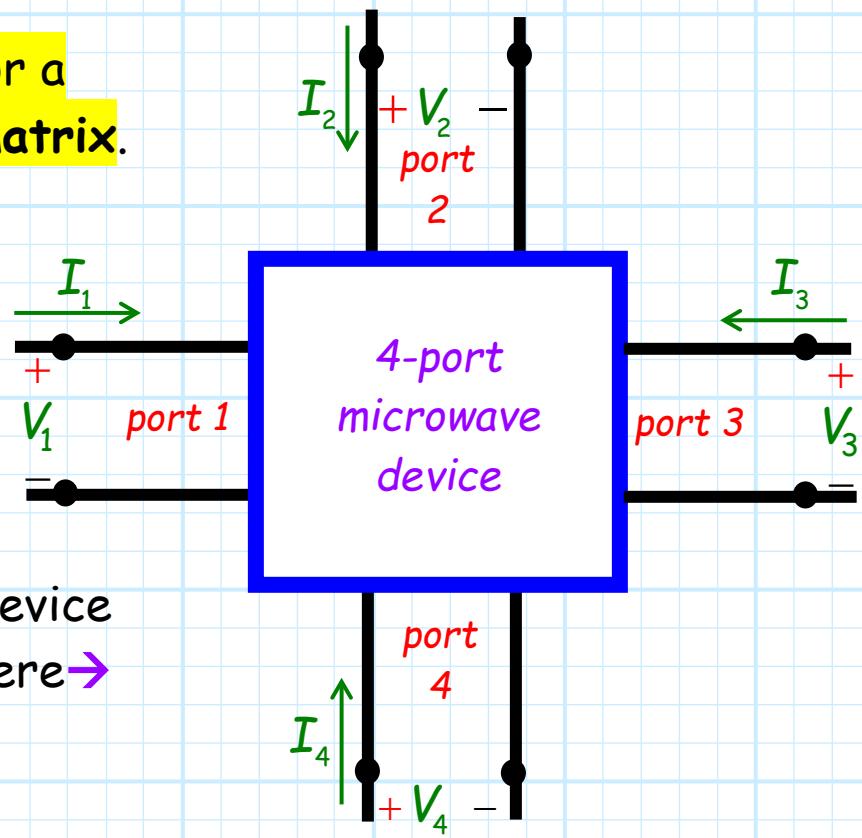
Is there a multi-port equivalent of impedance?

Q: We use input impedance to characterize a *single-port* network; is there some way to characterize a *multiport* component?

A: Absolutely!

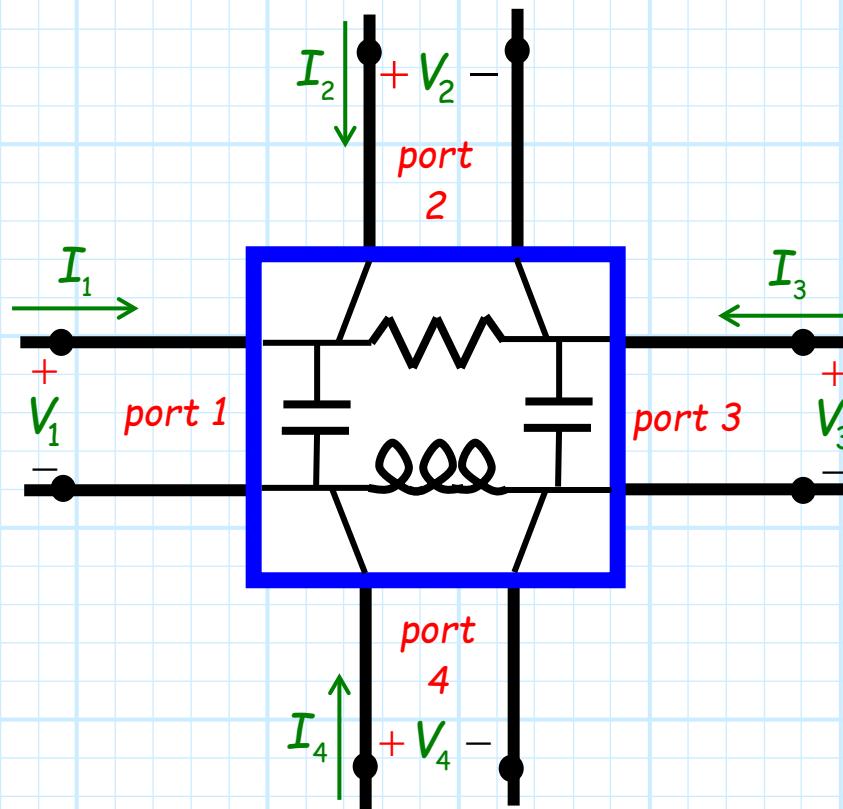
The **equivalent** to input impedance for a multi-port device is its **impedance matrix**.

Consider the **4-port microwave device** shown here →



Trans-Impedance parameters

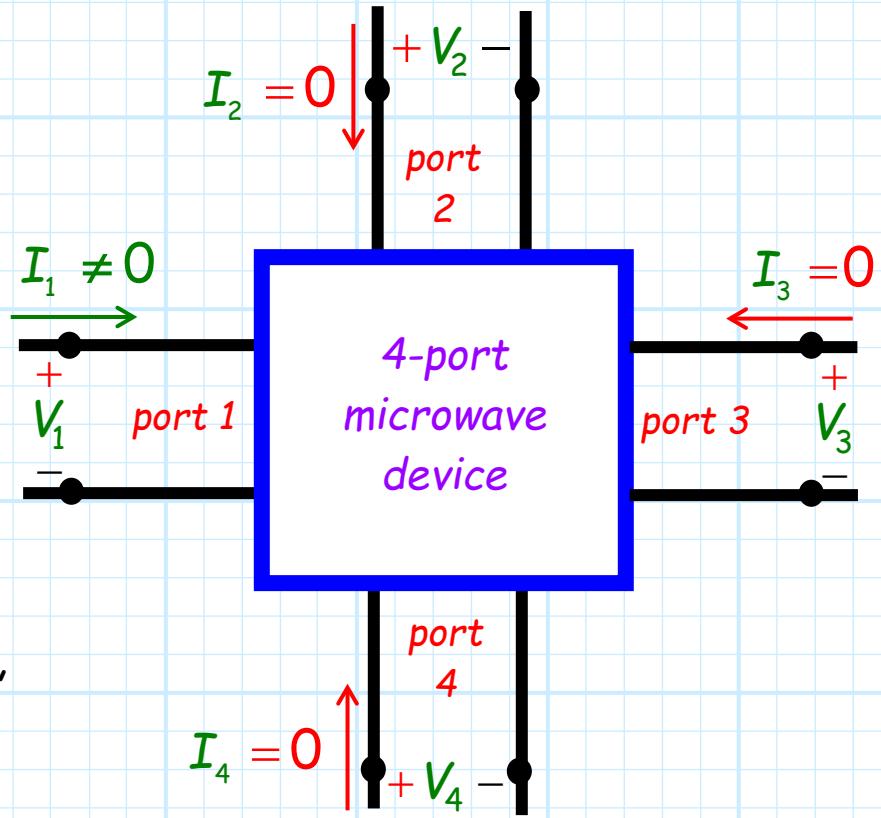
Inside the blue box there may be a very **simple linear device/circuit**, or it might contain a very large and **complex linear microwave system**.



→ Either way, this linear "box" can be fully characterized by its **trans-impedance parameters!**

Note that all currents but one are zero!

Now; say there exists a non-zero current at port 1 (i.e., $I_1 \neq 0$), while the current at all other ports are known to be **zero** (i.e., $I_2 = I_3 = I_4 = 0$).



Say we measure/determine the current at port 1 (i.e., determine I_1), and we then measure/determine the voltage at port 2 (i.e., determine V_2).

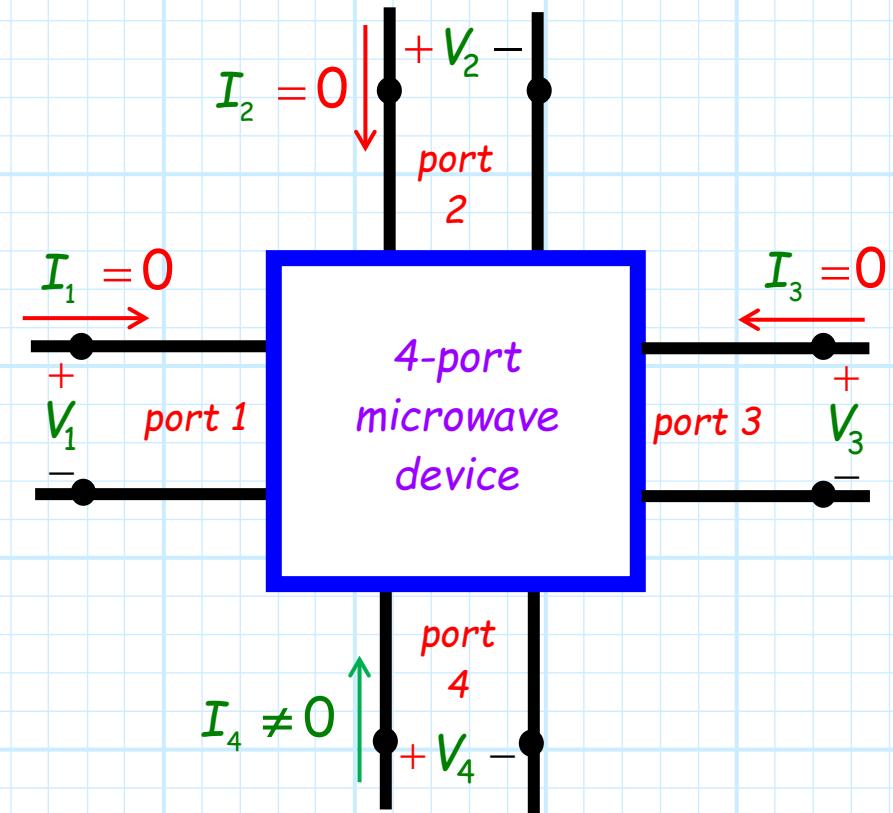
The complex ratio between V_2 and I_1 is known as the trans-impedance parameter Z_{21} :

$$Z_{21} = \left. \frac{V_2}{I_1} \right|_{I_2=I_3=I_4=0}$$

Many, many parameters

We could also define, say, trans-impedance parameter Z_{34} as the ratio between:

- a) the complex value I_4 (current into port 4) and,
- b) V_3 (the voltage at port 3) given that,
- c) the current at all other ports (1, 2, and 3) are zero.



$$Z_{34} = \left. \frac{V_3}{I_4} \right|_{I_1=I_2=I_3=0}$$

A 4-port device has 16 parameters

Or, we could determine, for example:

$$Z_{33} = \left. \frac{V_3}{I_3} \right|_{I_1=I_2=I_4=0}$$

$$Z_{24} = \left. \frac{V_2}{I_4} \right|_{I_1=I_2=I_3=0}$$

$$Z_{12} = \left. \frac{V_1}{I_2} \right|_{I_1=I_3=I_4=0}$$

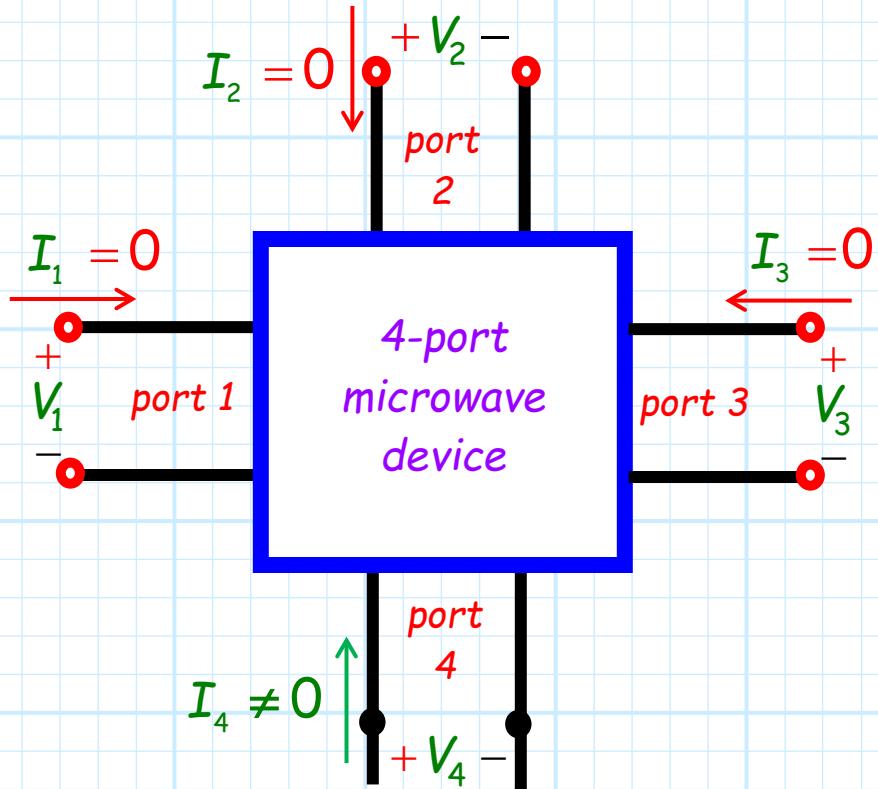
Thus, more generally, the ratio of the current into port n and the voltage at port m is:

$$Z_{mn} = \left. \frac{V_m}{I_n} \right|_{I_{m \neq n}=0}$$

(when $I_k = 0$ for all $m \neq n$)

Note for an N -port device, there are N^2 trans-impedance parameters!

An open enforces $I=0$



Q: But how do we ensure that all but one port current is zero?



A: Place an open-circuit at those ports!

Placing an open at a port enforces the condition that $I = 0$.

Now, we can thus equivalently state the definition of trans-impedance as:

$$Z_{mn} = \left. \frac{V_m}{I_n} \right|_{I_{m \neq n}=0} \quad (\text{when all ports } m \neq n \text{ are open-circuited})$$

And ONLY when all ports are open-circuited!

Note that the ratio of voltage V_3 current I_1 (say) is equal to trans-impedance parameter Z_{31} ...

...if, and only if, all ports—except port 1—are open circuited

or, stated another way:

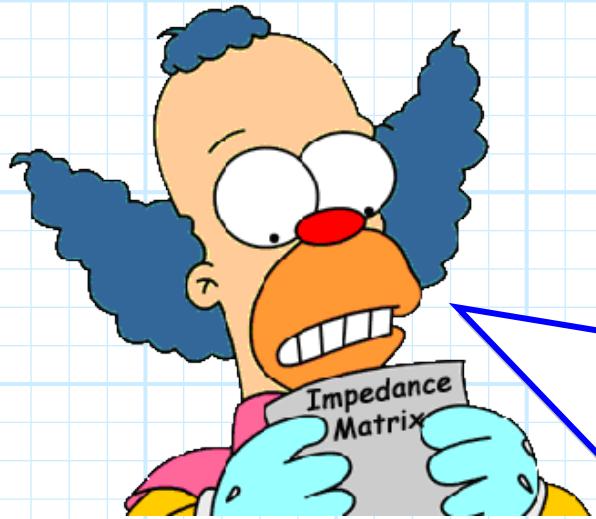
...if, and only if, all port currents I_m —except I_1 —are zero.

Thus, when the above statements are not valid, the ratio of V_3 and I_1 is not numerically equal to the trans-impedance parameter Z_{31} :

$$\frac{V_3}{I_1} \neq Z_{31} \text{ !!!!!!!}$$



Useful devices are not commonly connected to open-circuits



Q: As impossible as it may sound, this presentation is even more boring and pointless than all of your previous efforts.

Why are we studying this?

After all, what is the likelihood that **useless open-circuits** will be placed on **all** but one port of an otherwise **useful** device?

A: Of course, a multi-port device **will** generally be connected at **every** port to some other **useful** devices (e.g., amplifier, filter, antenna).

→ And so generally, a **non-zero** current will exist at **all ports!**

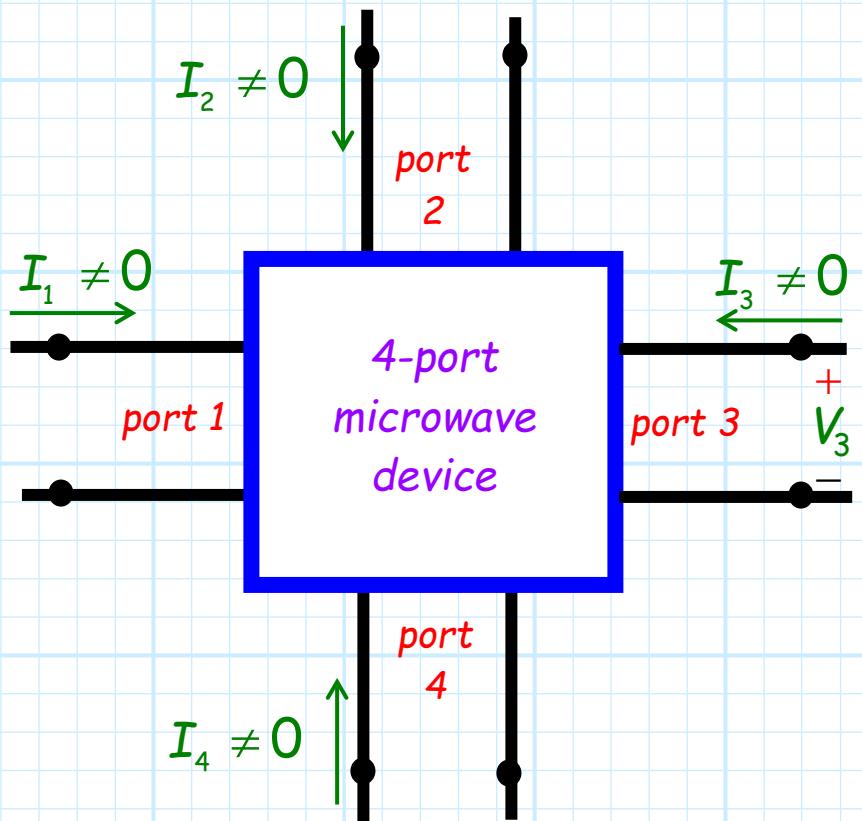
Yet in this case, the trans-impedance parameters of the device are still **very important** and **very helpful**.

It's linear—let's use superposition!

For a linear device, the voltage at any **one** port is a **superposition** of the voltages resulting from each and **every** individual port current.

For example, the voltage at **port-3** is a superposition of **four terms**:

$$V_3 = V_{31} + V_{32} + V_{33} + V_{34}$$



where each term represents the voltage at port-3 due to the current at one—and only one—specific-port.

$$V_3 = V_{31} + V_{32} + V_{33} + V_{34}$$

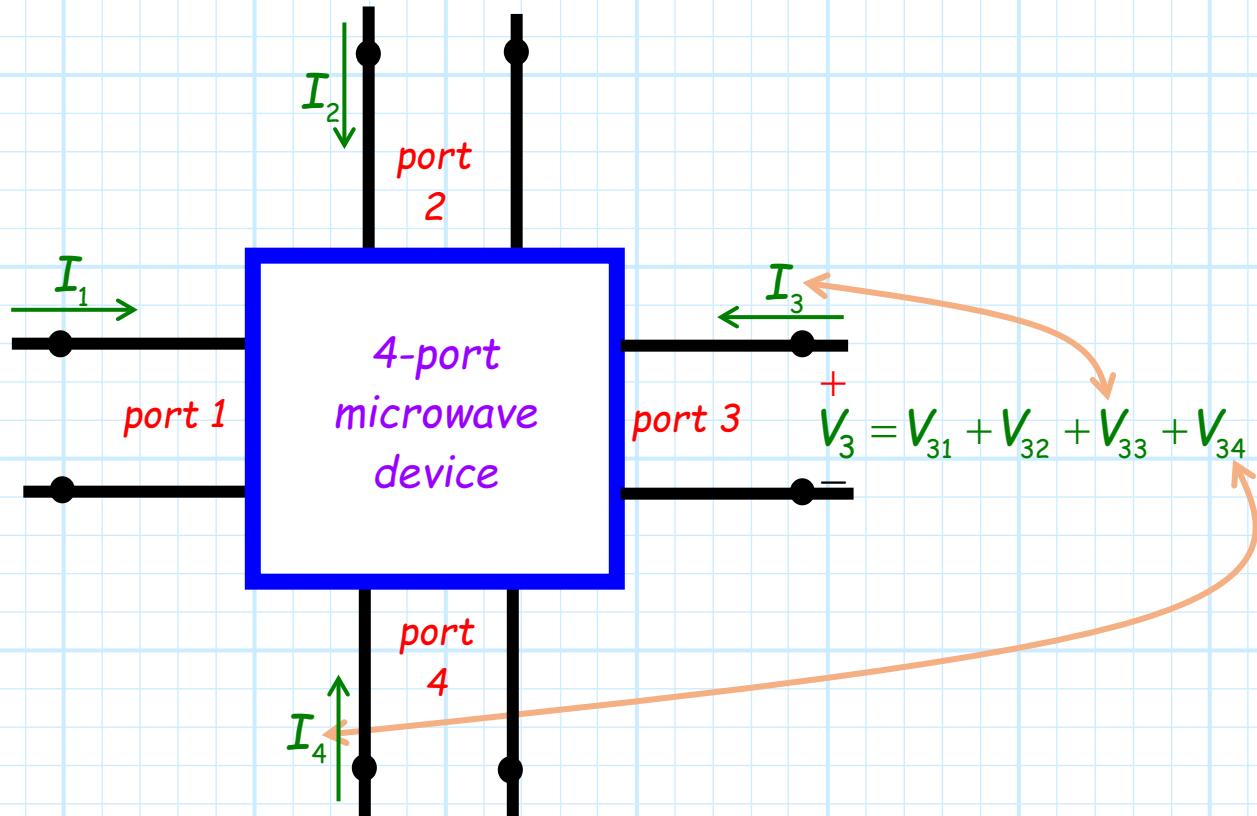
Voltage components

For this example:

V_{34} is the voltage at port-3 due to the current at port 4

and

V_{33} is the voltage at port-3 due to the current at port 3.



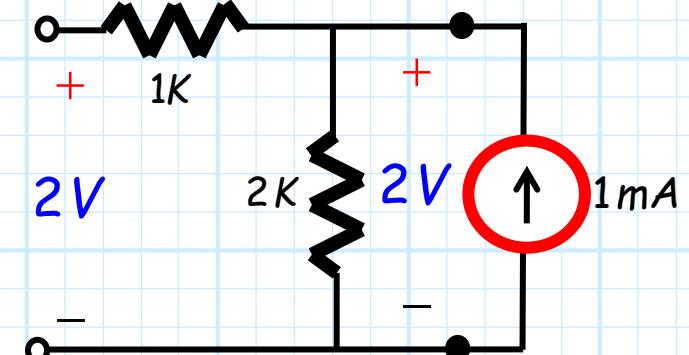
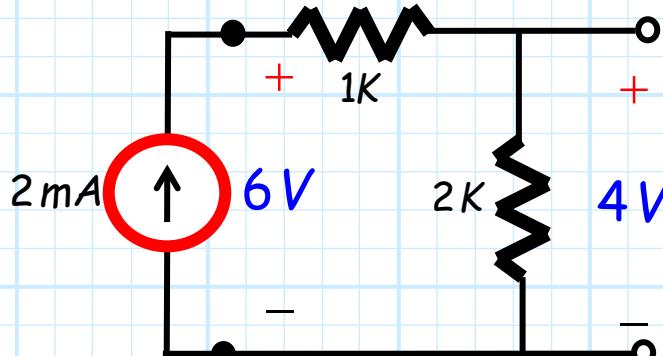
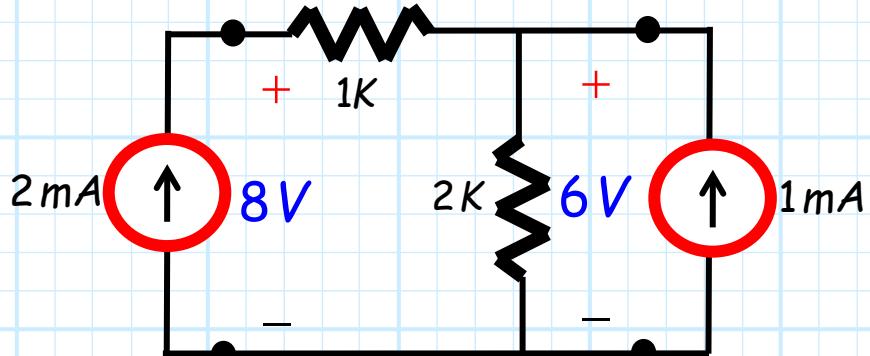
Remember this?

Q: But how do we determine these terms? How can we separate a voltage into its components?

A: Think about how we use superposition to analyze a linear circuit—we first turn off all sources, **except one**, and then analyze.

The result are the currents and voltages due to that one source.

We do this for **each source** in the circuit, and then the **sum** of all resulting analyses is the correct total result!



Deja-vu all over again

Thus, we can likewise determine the voltage due to a specific port current by "turning off" all other port currents—setting all other port currents to zero!

For example:

$$V_{33} = V_3 \Big|_{I_1=I_2=I_4=0}$$

$$V_{31} = V_3 \Big|_{I_2=I_3=I_4=0}$$

Q: Wait a second—setting all other port currents to zero—don't we also do that when determining trans-impedance parameters?

A: Exactly. In fact, the components of each port voltage can be directly determined from knowledge of the device's trans-impedance parameters. E.G.:

$$V_{33} = V_3 \Big|_{I_1=I_2=I_4=0} = Z_{33} I_3$$

$$V_{31} = V_3 \Big|_{I_2=I_3=I_4=0} = Z_{31} I_1$$

Superposition! Superposition! Superposition!

Therefore if all currents are known, the voltage at port-3 can be determined by as:

$$\begin{aligned}V_3 &= V_{31} + V_{32} + V_{33} + V_{34} \\&= Z_{34} I_4 + Z_{33} I_3 + Z_{32} I_2 + Z_{31} I_1\end{aligned}$$

More generally, the voltage at port m of an N -port device is:

$$V_m = \sum_{n=1}^N Z_{mn} I_n$$

If a device is linear, we can apply superposition!

See? What did I tell you

To emphasize an **earlier point**, let's take the **ratio** of voltage V_3 and current I_1 :

$$\begin{aligned}\frac{V_3}{I_1} &= \frac{Z_{34} I_4 + Z_{33} I_3 + Z_{32} I_2 + Z_{31} I_1}{I_1} \\ &= Z_{34} \left(\frac{I_4}{I_1} \right) + Z_{33} \left(\frac{I_3}{I_1} \right) + Z_{32} \left(\frac{I_2}{I_1} \right) + Z_{31}\end{aligned}$$



Note that this ratio is **NOT** equal to Z_{31} (!)—unless of course, **all other currents are zero** (i.e., if $I_2 = I_3 = I_4 = 0$):

$$\begin{aligned}\left. \frac{V_3}{I_1} \right|_{I_2=I_3=I_4=0} &= \left[Z_{34} \left(\frac{I_4}{I_1} \right) + Z_{33} \left(\frac{I_3}{I_1} \right) + Z_{32} \left(\frac{I_2}{I_1} \right) + Z_{31} \right]_{I_2=I_3=I_4=0} \\ &= Z_{34} \left(\frac{0}{I_1} \right) + Z_{33} \left(\frac{0}{I_1} \right) + Z_{32} \left(\frac{0}{I_1} \right) + Z_{31} \\ &= Z_{31}\end{aligned}$$

The impedance matrix

This expression can be written in **matrix** form as:

$$\mathbf{V} = \mathcal{Z}\mathbf{I}$$

Where **I** is the **vector**:

$$\mathbf{I} = [\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \dots, \mathcal{I}_N]^T$$

and **V** is the **vector**:

$$\mathbf{V} = [V_1, V_2, V_3, \dots, V_N]^T$$



And the matrix \mathcal{Z} is called the **impedance matrix**:

$$\mathcal{Z} = \begin{bmatrix} Z_{11} & \dots & Z_{1n} \\ \vdots & \ddots & \vdots \\ Z_{m1} & \dots & Z_{mn} \end{bmatrix}$$

The impedance matrix is frequency dependent

The impedance matrix is a N by N matrix that **completely characterizes** a linear, N -port device.

Effectively, the impedance matrix describes a **multi-port** device the way that Z_L describes a **single-port** device (e.g., a load)!

→ But beware!



The values of the impedance matrix for a particular device or network, just like Z_L , are **frequency dependent**!

Thus, it may be more instructive to **explicitly** write:

$$\mathcal{Z}(w) = \begin{bmatrix} Z_{11}(w) & \dots & Z_{1n}(w) \\ \vdots & \ddots & \vdots \\ Z_{m1}(w) & \dots & Z_{mn}(w) \end{bmatrix}$$