Homework 7 (Variation of Parameters)

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5 $y'' + 9y = 9\sec^2(3t), \quad 0 < t < \pi/6$

Find the general solution of the given differential equation.

Solution:
Because t is a function of sec, the method of undetermined coefficients cannot be used. Instead, we should apply the method of variation of parameters to find the particular solution.

1. Homogeneous Solution:

$$r^2 + 9 = 0 \implies r = \pm 3i$$
$$y_c = c_1 \cos(3t) + c_2 \sin(3t)$$

2. Particular Solution:

   The particular solution when using variation of parameters is defined as $y_p = u_1 y_1 + u_2 y_2$, where u_1 and u_2 are functions of t. Additionally, $0 = u_1' y_1 + u_2' y_2$ (1).

$$\begin{aligned} y_p &= u_1 \cos(3t) + u_2 \sin(3t) \\ y_p' &= \cancel{u_1' \cos(3t)} - 3u_1 \sin(3t) + \cancel{u_2' \sin(3t)} + 3u_2 \cos(3t) \\ y_p'' &= -3u_1' \sin(3t) - 9u_1 \cos(3t) + 3u_2' \cos(3t) - 9u_2 \sin(3t) \end{aligned}$$

   Substituting and simplifying:

$$\begin{aligned} \begin{matrix} -3u_1' \sin(3t) - \cancel{9u_1 \cos(3t)} + 3u_2' \cos(3t) \\ -\cancel{9u_2 \sin(3t)} + \cancel{9u_1 \cos(3t)} + \cancel{9u_2 \sin(3t))} \end{matrix} &= 9\sec^2(3t) \\ \implies 3u_2' \cos(3t) - 3u_1' \sin(3t) &= 9\sec^2(3t) \qquad (2) \end{aligned}$$

Now we can multiply equation one by $\sin^2(3t)$ and equation two by $\cos^2(3t)$ to eliminate the $\sec^2(3t)$.

$$u_1' \cos(3t)\sin^2(3t) + u_2' \sin^3(3t) = 0 \tag{1}$$

$$3u_2' \cos^3(3t) - 3u_1' \sin(3t)\cos^2(3t) = 9\cancelto{9}{\sec^2(3t)\cos^2(3t)} \tag{2}$$

7 $\quad 4y'' + y = 2\sec(t/2), \quad -\pi < t < \pi$

Find the general solution of the given differential equation.

Solution:

1. Homogeneous Solution:

$$4r^2 + 1 = 0 \implies r = \pm\frac{1}{2}i$$
$$y_c = c_1 \cos\left(\frac{1}{2}t\right) + c_2 \sin\left(\frac{1}{2}t\right)$$

2. Particular Solution:
   Let $y_1 = \cos\left(\frac{1}{2}t\right)$ and $y_2 = \sin\left(\frac{1}{2}t\right)$;

$$\begin{aligned}
y_p &= u_1 \cos\left(\tfrac{1}{2}t\right) + u_2 \sin\left(\tfrac{1}{2}t\right)\\
y_p' &= \cancel{u_1' \cos\left(\tfrac{1}{2}t\right)} - \tfrac{1}{2}u_1 \sin\left(\tfrac{1}{2}t\right) + \cancel{u_2' \sin\left(\tfrac{1}{2}t\right)} + \tfrac{1}{2}u_2 \cos\left(\tfrac{1}{2}t\right)\\
y_p'' &= -\tfrac{1}{2}u_1' \sin\left(\tfrac{1}{2}t\right) - \tfrac{1}{4}u_1 \cos\left(\tfrac{1}{2}t\right) + \tfrac{1}{2}u_2' \cos\left(\tfrac{1}{2}t\right) - \tfrac{1}{4}u_2 \sin\left(\tfrac{1}{2}t\right)
\end{aligned}$$

   Substituting these back into the initial differential equation, we get:

$$\begin{aligned}
-2u_1' \sin\left(\frac{1}{2}t\right) - \cancel{u_1 \cos\left(\frac{1}{2}t\right)} + 2u_2' \cos\left(\frac{1}{2}t\right) - \cancel{u_2 \sin\left(\frac{1}{2}t\right)}&\\
+\cancel{u_1 \cos\left(\frac{1}{2}t\right)} + \cancel{u_2 \sin\left(\frac{1}{2}t\right)} &= 2\sec(t/2)
\end{aligned}$$

   Now we are left with the equations:

$$u_1' \cos\left(\frac{1}{2}t\right) + u_2' \sin\left(\frac{1}{2}t\right) = 0$$
$$-2u_1' \sin\left(\frac{1}{2}t\right) + 2u_2' \cos\left(\frac{1}{2}t\right) = 2\sec(t/2)$$

Multiplying the first by $\sin(t/2)$ and the second by $\cos(t/2)$ and adding to simplify:

$$u_1' \cos\left(\frac{1}{2}t\right) \sin\left(\frac{1}{2}t\right) + u_2' \sin\left(\frac{1}{2}t\right) \cos\left(\frac{1}{2}t\right)$$
$$-2u_1' \sin\left(\frac{1}{2}t\right) \cos\left(\frac{1}{2}t\right) + 2u_2' \cos\left(\frac{1}{2}t\right) \sin\left(\frac{1}{2}t\right) = 2$$

I need to solve my system now and im done

14 $x^2y'' - xy' + (x^2 - \frac{1}{4})y = 3x^{3/2}\sin x, \quad x > 0;$ $y_1 = x^{-1/2}\sin(x),\ y_2 = x^{-1/2}\cos(x)$

Verify that the given functions y_1 and y_2 satisfy the corresponding homogeneous equation; then find a particular solution of the given nonhomogeneous equation.

Solution:

1. Verifying y_1 and y_2:

$$\begin{aligned} y_1 &= x^{-1/2}\sin(x) \\ y_1' &= -\frac{\sin x}{2x^{3/2}} + \frac{\cos x}{\sqrt{x}} \\ y_1'' &= \frac{-4x^2\sin x - 4x\cos x + 3\sin x}{4x^{5/2}} \end{aligned}$$

2. Particular Solution:

   (a) System:

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0; \\ u_1y_1 + u_2y_2 &= y_p \end{aligned}$$

   (b)