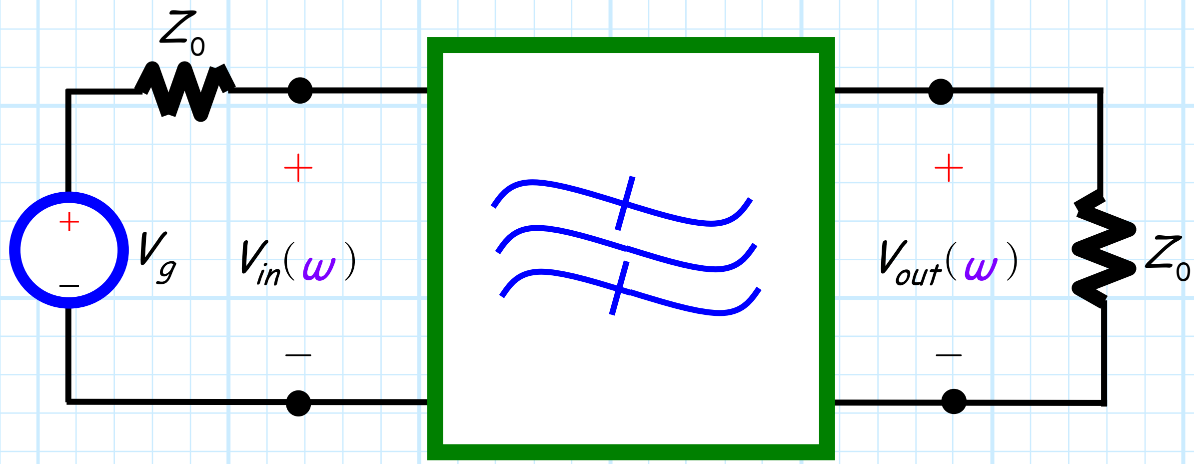
The Filter Phase Function



Recall that the **complex** voltages at the input and output of a two-port network (e.g., a filter) are related by the network's **frequency response** $H(\omega)$:

$$H(\omega) = \frac{V_{out}(\omega)}{V_{in}(\omega)}$$

This result provides us with the **magnitude** of the (sinusoidal!) output:

$$|V_{out}(\omega)| = |H(\omega)|\ |V_{in}(\omega)|$$

Phase—it matters too!

We can also determine the **relative phase** of the (sinusoidal!) output:

$$\arg\left[V_{out}(\omega)\right] = \arg\left[H(\omega)\right] + \arg\left[V_{in}(\omega)\right]$$

Q: ***Phase!?***

***Why** are you mentioning **phase**?*

*The definitions of filter **pass band** and **stop band** did **not** depend on phase.*

*Shouldn't we just **ignore phase** if we are considering **filters**?*

A: Hardly! This phase response is **very** important!

Phase shift as a function of frequency

Remember, since $H(\omega)$ is complex, it can be expressed in terms of its magnitude and **phase**:

$$H(\omega) = \text{Re}\{H(\omega)\} + j\,\text{Im}\{H(\omega)\} = |H(\omega)|\, e^{j\arg[H(\omega)]}$$

where the "phase" is denoted as $\arg[H(\omega)]$:

$$\arg[H(\omega)] = \tan^{-1}\left[\frac{\text{Im}\{H(\omega)\}}{\text{Re}\{H(\omega)\}}\right]$$

→ We likewise care **very** much about this phase function!

Phase shift is a result of delay

Q: *Just what does this "phase" **tell** us?*

A: It describes the relative phase difference **between** the (sinusoidal) input and the (sinusoidal) output of the filter.

We say there has been a "**phase shift**" of $\arg\left[H(\omega)\right]$ between the input and output.

Q: *What **causes** this phase shift?*

A: Propagation **delay**.

It takes some non-zero amount of **time** for signal energy to **propagate** from the input of the filter to the output.

Linear systems theory!!!

Q: *Can we tell from $H(\omega)$ how **long** this delay is?*

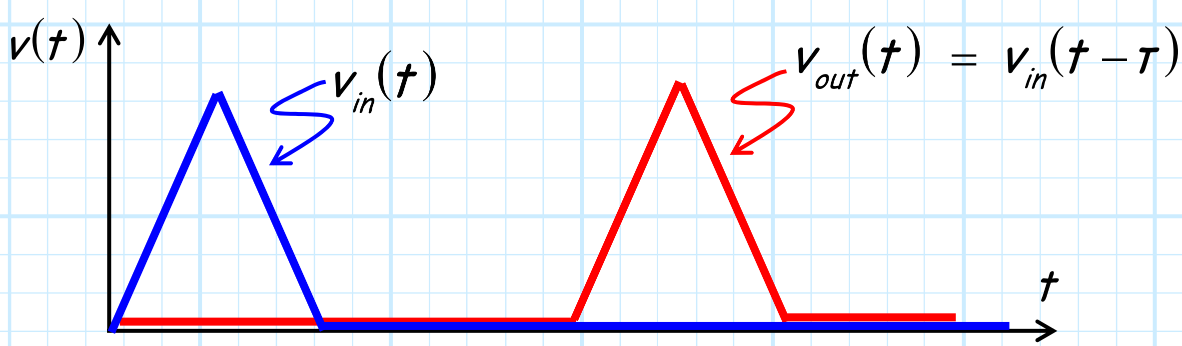
A: Yes!

To see how, consider an **example** two-port network with the impulse response:

$$h(t) = \delta(t-\tau)$$

This device would merely **delay** and input signal by some amount τ:

$$v_{out}(t) = \int_{-\infty}^{\infty} h(t-t')\, v_{in}(t')\, dt' = \int_{-\infty}^{\infty} \delta(t-t'-\tau)\, v_{in}(t')\, dt' = v_{in}(t'-\tau)$$



The phase shift increases with frequency!

Taking the **Fourier transform** of this impulse response, we find the **frequency response** of this two-port network is:

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\; e^{-j\omega t} dt = \int_{-\infty}^{\infty} \delta(t-\tau)\; e^{-j\omega t} dt = e^{-j\omega\tau}$$

In other words:

$$|H(\omega)| = 1 \qquad \text{and} \qquad \arg\left[H(\omega)\right] = -\omega\tau$$

The interesting result here is the **phase**.

→The result means that a **delay** of τ seconds results in an output "**phase shift**" of $-\omega\tau$ radians!

See?

Note that although the **delay** of device is a **constant** τ, the **phase shift** is a **function** of ω—in fact, it is **directly proportional** to frequency ω.

Thus, if the **input** signal for this device is of the form:

$$v_{in}(t) = \cos \omega t$$

Then the **output** would be:

$$\begin{aligned} v_{out}(t) &= \cos\left[\omega(t-\tau)\right] \\ &= \cos\left[\omega t - \omega\tau\right] \\ &= \left|H(\omega)\right| \cos\left[\omega t + \arg\left[H(\omega)\right]\right] \end{aligned}$$

Thus, we could **either** view the signal $v_{in}(t) = \cos \omega t$ as:

1. being **delayed** by an amount τ seconds, or

2. **phase shifted** by an amount $-\omega\tau$ radians.

It's not as easy as it looks

Q: *So, by **measuring** the output signal phase shift* $\arg[H(\omega)]$, *we could determine the delay* τ *through the device with the equation:*

$$\tau = -\frac{\arg[H(\omega)]}{\omega} \qquad \textit{right?}$$

A: Not exactly.

The problem is that we cannot **unambiguously** determine the phase shift $\arg[H(\omega)] = -\omega\tau$ by **looking** at the output signal!

The reason for this is of course that:

$$\cos[\omega t + \arg H(\omega)] = \cos[\omega t + \arg H(\omega) + 2\pi] = \cos[\omega t + \arg H(\omega) - 4\pi]$$

, etc.

Phase shift *measurements* are ambiguous!

More specifically:

$$\cos\left[\omega t + \arg H(\omega)\right] = \cos\left[\omega t + \arg H(\omega) + n2\pi\right]$$

where n is **any integer**—positive or negative.

→ We can't tell at **which** of these output signals we are looking!

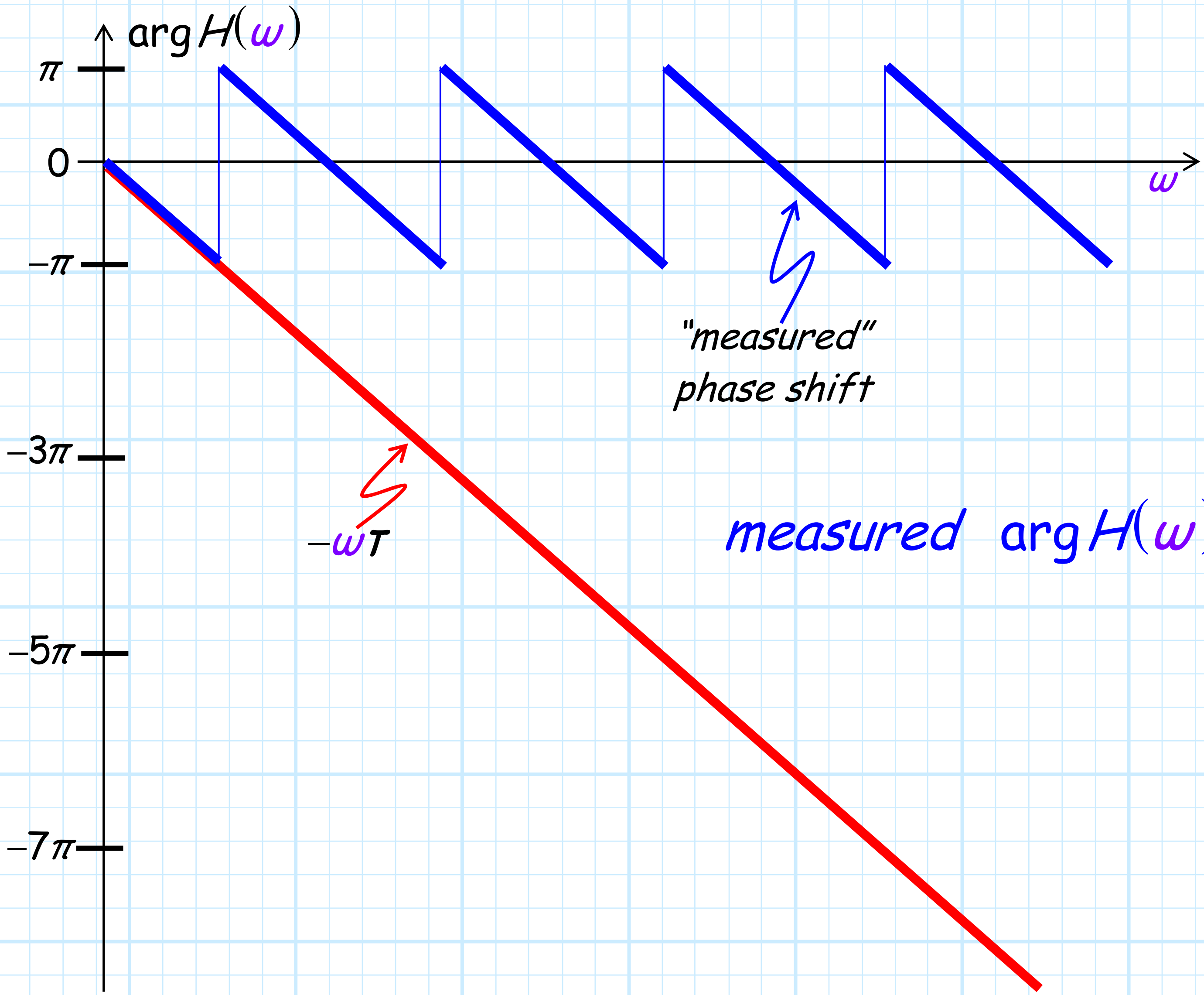
Thus, any phase shift **measurement** has an inherent **ambiguity**.

Typically, we **interpret** a phase **measurement** (in radians) such that:

$$-\pi < \arg H(\omega) \leq \pi \qquad \text{or} \qquad 0 \leq \arg H(\omega) < 2\pi$$

But almost certainly the **actual** value of $\arg H(\omega) = -\omega\tau$ is **nowhere near** these interpretations!

What we *measure* is not what it is!



$$\text{measured } \arg H(\omega) \neq -\omega\tau$$

Using this equation could provide negative delay!

Clearly, using the equation:

$$\tau = -\frac{\arg H(\omega)}{\omega}$$

would **NOT get us the correct result in this case**—after all, there will be **several** frequencies ω with exactly the **same measured** phase $\arg H(\omega)$!

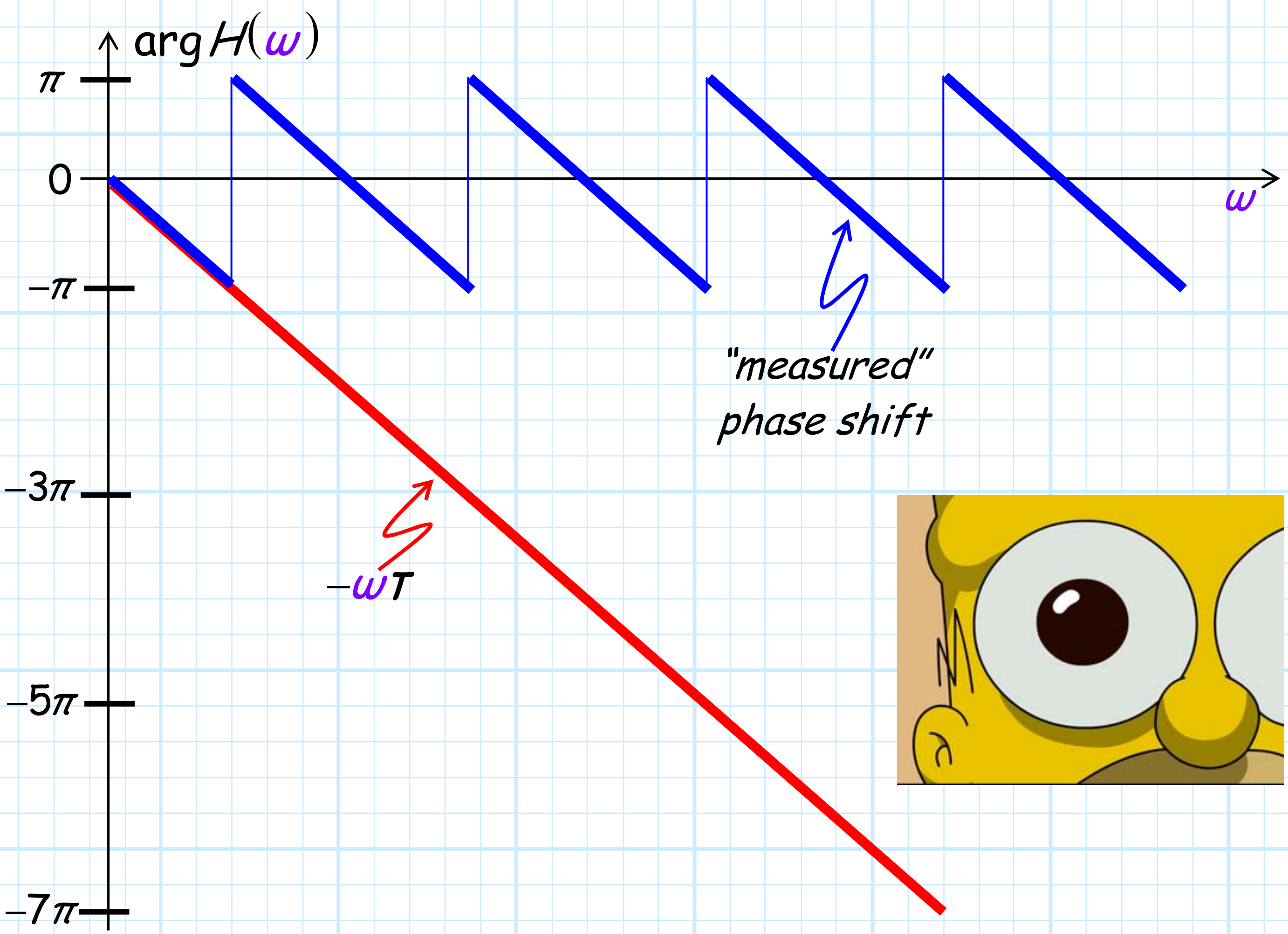
Q: *So, determining the delay τ is **impossible**?*

A: NO!

→It is **entirely** possible—we simply must find the correct **method**.

So different…yet in one way so similar

Looking at the plot of the previous page, this method should become **apparent**.



Their derivatives are the same at every point!

Note that although the **measured phase** function is definitely **not** equal to the phase function $-\omega T$, the **slope** of these two functions **are identical** at **every** point!

Q: *What good is knowing the **slope** of these functions?*

A: Just **look**!

Recall that we can determine the slope by taking the first **derivative**:

$$\frac{\partial(-\omega T)}{\partial \omega} = -T$$

→ The **slope** of this function directly tells us the **propagation delay**!

A network analyzer measures phase at many, many frequencies

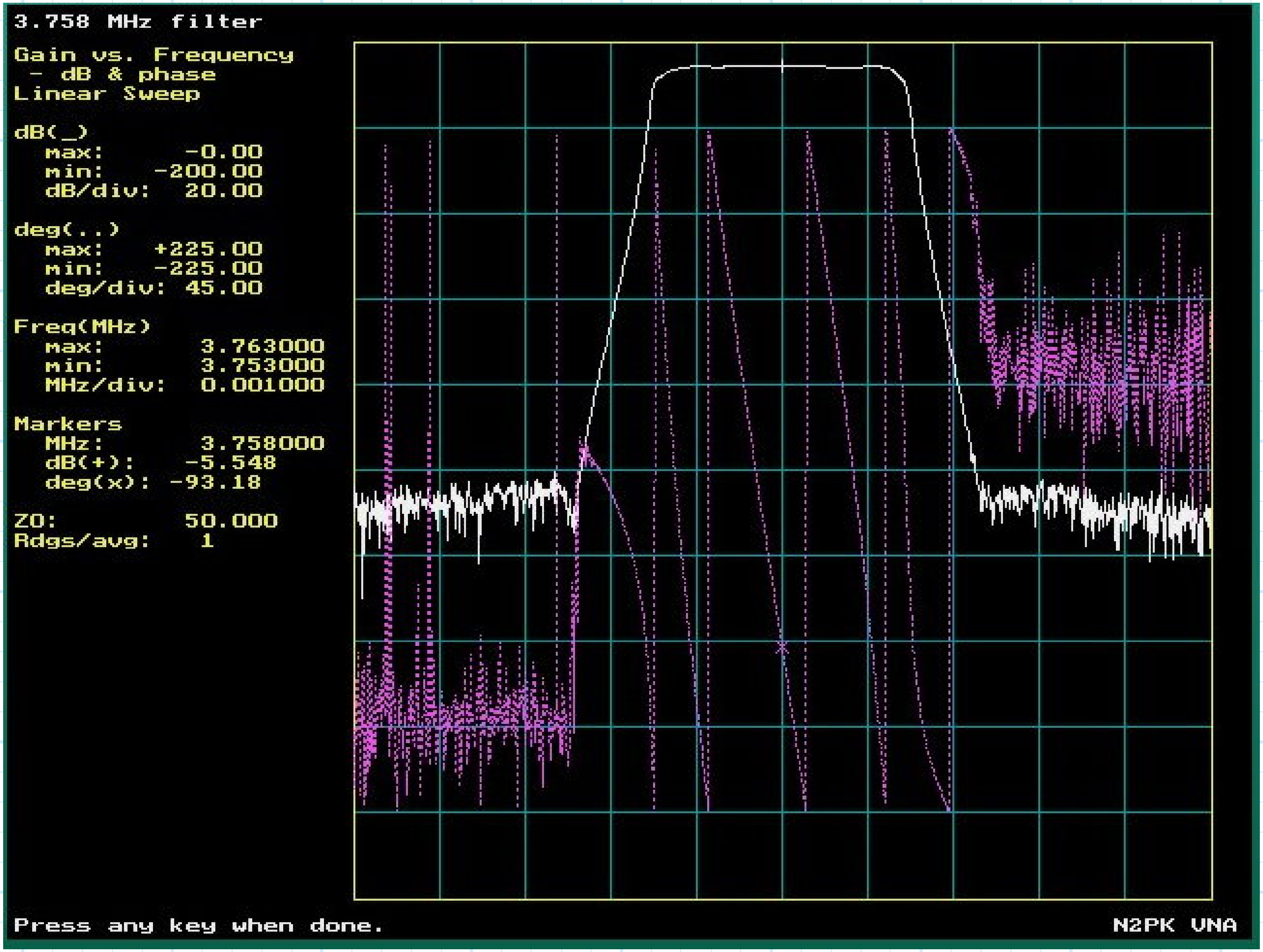
Thus, we can determine the **propagation delay** of this device by:

$$\tau = -\frac{\partial \arg\left[H(\omega)\right]}{\partial \omega}$$

where $\arg\left[H(\omega)\right]$ can be the **measured phase**.

Of course, the method requires us to **measure** $\arg\left[H(\omega)\right]$ as a **function** of frequency (i.e., to make measurements at **many signal frequencies**).

A Network analyzer filter measurement



From: n2pk.com

Microwave filters are nothing like our example (Doh!)

Q: ***Now*** *I see! If we wish to* ***determine*** *the propagation* ***delay*** τ *through some* ***filter****, we simply need to take the* ***derivative*** *(with respect to frequency* ω*) of measured function* $\arg[H(\omega)]$*.* ***Right****?*

A: Well, sort of.

Recall for the **example** case:

$$h(t) = \delta(t-\tau) \qquad \text{and} \qquad \arg[H(\omega)] = -\omega\tau$$

where τ is a **constant**.

→ But, for a microwave filter:

$$h(t) \neq \delta(t-\tau) \qquad \text{and so} \qquad \arg[H(\omega)] \neq -\omega\tau \text{!!!!}$$

Delay changes with frequency!

Specifically, the phase function $\arg H(\omega)$ will typically be some **arbitrary function** of frequency ($\arg\left[H(\omega)\right] \neq -\omega T$).

Q: *How could this be true?*

I thought you said that phase shift was ***due*** *to filter* ***delay*** *τ!*

A: Phase shift **is** due to device delay, it's just that the propagation delay of most devices (such as filters) is **not a constant**!

→Instead, delay depends on the **frequency** of the signal.

In other words, the propagation **delay** of a filter is typically some **arbitrary function** of frequency (i.e., $\tau(\omega)$).

That's why the phase $\arg H(\omega)$ is **likewise** an arbitrary function of frequency.

Phase delay is the derivative

Q: *Yikes! Is there* ***any*** *way to determine the relationship between these two arbitrary functions (i.e.,* $\arg H(\omega)$ *and* $\tau(\omega)$*).*

A: **Yes** there is!

Just as before, the two can be related by a first **derivative**:

$$\boxed{\tau(\omega) = -\frac{\partial \arg\left[H(\omega)\right]}{\partial \omega}}$$

This result $\tau(\omega)$ is also known as **phase delay**, and is a **very important** function to consider when designing/specifying/ selecting a microwave **filter**.

Horrible, grotesque, and ugly (and dispersion is not good either).

Q: *Why is phase delay* $\tau(\omega)$ ***so important****; what might* ***happen*** *if it's* ***"bad"****?*

A: If you get a filter with the wrong $\tau(\omega)$, your **output** signal could be **horribly distorted**—distorted by the evil effects of **signal dispersion**!