PHSX 521: Homework #3

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Problem 1

(Taylor 2.6)

(a) Equation (2.33) gives the velocity of an object dropped from rest. At first, when v_y is small, air resistance should be unimportant and (2.33) should agree with the elementary result $v_y = gt$ for free fall in a vacuum. Prove that this is the case. [Hint: Remember the Taylor series for $e^x = 1+x+x^2/2!+x^3/3!+\cdots$, for which the first two or three terms are certainly a good approximation when x is small.]

Solution:

We have previously derived that for something in free fall when using linear drag, it obeys:

$$\dot{q}_y(t) = v_{\text{ter}}(1 - e^{-t/\tau}) \tag{2.33}$$

Taylor expansion around the right hand side of the equation gives

$$\begin{aligned}\dot{q}_y(t) &= v_{\text{ter}}\left[1 - 1 + \left(\frac{t}{\tau}\right) + \mathcal{O}^2\right]\\ &= \frac{\cancel{m}g}{\cancel{b}}\left[0 + \left(\frac{t}{\cancel{(m/b)}}\right) + \mathcal{O}^2\right]\end{aligned}$$

Ignoring the higher order terms, we in fact have proved that for small t we get the desired expression:

$$\dot{q}_y(t) = gt$$

(b) The position of the dropped object is given by (2.35) with $v_{y0} = 0$. Show similarly that this reduces to the familiar $y = \frac{1}{2}gt^2$ when t is small

Solution:

If instead we want position for such drag we would have the expression

$$q_y(t) = v_{\text{ter}}t + (\dot{q}_{\text{y0}} - v_{\text{ter}}\tau)(1 - e^{-t/\tau}) \tag{2.35}$$

Since we have $\dot{q}_y$ we do not have to taylor expand equation (2.35), and instead we can easily see that we get the desired expression by integrating with respect to t in the previous solution:

$$\begin{aligned}\int_0^t \dot{q}_y(t)\ dt &= \int_0^t gt\ dt\\ q_y(t) &= \frac{1}{2}gt^2\end{aligned}$$

Problem 2

(Taylor 2.8) A mass m has velocity v_o at time $t = 0$ and coasts along the x axis in a medium where the drag force is $F(v) = -cv^{3/2}$. Use the method of Problem 2.7 to find v in terms of the time t and the other given parameters. At what time (if any) will it come to rest?

Solution:
Problem 2.7 has us solve such nonlinear differential equations by getting our differential equation into the form

$$dt = m\frac{d\dot{q}}{F(\dot{q})} \implies t = m\int_{v_0}^{v} \frac{d\dot{q}'}{F(\dot{q}')}$$

(Taylor's notation bothers me a bit here, it took a depressing amount of time to realize that the prime does not refer to a derivative. I rewrote it here with dot notation out of personal preference and clarity.)

It is worth noting this happens to take care of our boundary condition for $t = 0$ since we just handle it with our integration bounds!

$$\begin{aligned}
dt &= m\frac{d\dot{q}}{F(\dot{q})} \\
\int_0^t dt &= -\frac{m}{c}\int_{v_0}^{v} \dot{q}'^{-3/2}\, d\dot{q}' \\
-\frac{c}{m}t &= -2\,(v')^{-1/2}\Big|_{v_0}^{v} \\
v(t) &= \frac{4m^2 v_0}{(ct\sqrt{v_0} + 2m)^2}
\end{aligned}$$

Problem 3

(Taylor 2.9) We solved the differential equation (2.29), $m\dot{v}_y = -b(v_y - v_{\text{ter}})$, for the velocity of an object falling through air, by inspection - a most respectable way of solving differential equations. Nevertheless, one would sometimes like a more systematic method, and here is one. Rewrite the equation in the "separated" form

$$\frac{m dv_y}{v_y - v_{\text{ter}}} = -b dt$$

and integrate both sides from time 0 to t to find v_y as a function of t. Compare with (2.30).

Solution:
Integration over the separated differential equation:

$$\int_{\dot{q}_{y0}}^{\dot{q}_y} \frac{1}{\dot{q}'_y - v_{\text{ter}}}\, d\dot{q}'_y = -\frac{b}{m}\int_0^t dt$$

Let $u = \dot{q}'_y - v_{\text{ter}}$, $du = d\dot{q}_y$;

$$\begin{aligned}
(\ln|u|)\Big|_{\dot{q}_{y0}-v_{\text{ter}}}^{\dot{q}-v_{\text{ter}}} &= -\frac{b}{m}(t)\Big|_0^t \\
\ln\left|\frac{\dot{q}_y - v_{\text{ter}}}{\dot{q}_{y0} - v_{\text{ter}}}\right| &= -\frac{b}{m}t \\
\left|\frac{\dot{q}_y - v_{\text{ter}}}{\dot{q}_{y0} - v_{\text{ter}}}\right| &= e^{-t/\tau} \\
\dot{q}_y(t) &= v_{\text{ter}} + (\dot{q}_{y0} - v_{\text{ter}})e^{-t/\tau}
\end{aligned}$$

Problem 4

(Taylor 2.10, a) For a steel ball bearing (diameter 2 mm and density $7.8\,\mathrm{g/cm^3}$) dropped in glycerin (density $1.3\,\mathrm{g/cm^3}$ and viscosity $12\,\mathrm{N\cdot s/m^2}$ at STP), the dominant drag force is the linear drag given by (2.82) of Problem 2.2. (a) Find the characteristic time τ and the terminal speed v_{ter}. In finding the latter, you should include the buoyant force of Archimedes. This just adds a third force on the right side of Equation (2.25).

Solution:
The sum of forces given by the problem is

$$\begin{aligned}
m_{\text{s}}g - f_{\text{lin}} - f_{\text{b}} &= m\ddot{q} \\
\rho_{\text{s}}Vg - 3\pi\eta D\dot{q} - \rho_{\text{g}}Vg &= \rho_{\text{s}}V\ddot{q} \\
\rho_{\text{s}}V\ddot{q} - 3\pi\eta D\dot{q} + \rho_{\text{s}}Vg - \rho_{\text{g}}Vg &= 0 \\
\ddot{q} + \frac{3\pi\eta D}{\rho_{\text{s}}V}\dot{q} - g + \frac{\rho_{\text{g}}}{\rho_{\text{s}}}g &= 0 \\
\ddot{q} + \frac{3\pi\eta D}{\rho_{\text{s}}V}\dot{q} &= g - \frac{\rho_{\text{g}}}{\rho_{\text{s}}}g \\
\ddot{q} + \frac{3\pi\eta D}{\rho_{\text{s}}V}\dot{q} &= -\frac{\rho_{\text{s}}g - \rho_{\text{g}}g}{\rho_{\text{s}}}
\end{aligned}$$

Problem 4 continued on next page...

I will define κ and γ to make things easier to read:

$$\kappa \equiv \frac{3\pi\eta D}{\rho_s V} = \frac{18\eta}{\rho_s D^2}$$
$$\gamma \equiv \frac{\rho_s g - \rho_g g}{\rho_s} = g\left(1 - \frac{\rho_g}{\rho_s}\right)$$

This is a nice non-homogeneous linear differential equation with characteristic polynomial that implies exponential solutions:

$$\lambda^2 + \kappa\lambda = 0$$

Which has solutions $\lambda = \{0, \kappa\}$. This gives us exponential solutions of form:

$$q_c = \overset{1}{\cancel{e^{0t}}} C_1 + C_2 e^{-\kappa t}$$

We assume the solution to the particular part is linear:

$$\begin{cases} q_p = ut \\ \dot{q}_p = u \\ \ddot{q}_p = 0 \end{cases}$$

$$0 + \kappa u = \gamma \qquad \Longrightarrow \qquad u = \frac{\gamma}{\kappa}$$

Now we write the general solution

$$q(t) = q_c + q_p \implies \begin{cases} q(t) & = C_1 + C_2 e^{-\kappa t} + \frac{\gamma}{\kappa} t \\ \dot{q}(t) & = -\kappa C_2 e^{-\kappa t} + \frac{\gamma}{\kappa} \\ \ddot{q}(t) & = \kappa^2 C_2 e^{-\kappa t} \end{cases}$$

Based on the equation, we can define characteristic time

$$\tau \equiv \frac{1}{\kappa} = \frac{\rho_s D^2}{18\eta}$$

Before back-substituting to be in terms of the original variables, we can fit the boundary conditions:

$$\begin{cases} q(t_0) = \text{undef.} & \implies q(t_0) = C_1 \\ \dot{q}(0) = 0 & \implies \quad 0 = -\kappa C_2 + \frac{\gamma}{k} \quad \implies \quad C_2 = \frac{\gamma}{\kappa^2} \end{cases}$$

Therefore we have simplified solutions, where we notice $\gamma\tau = v_{\text{ter}}$

$$\begin{cases} q(t) & = C_1 + \tau(v_{\text{ter}})e^{-t/\tau} + v_{\text{ter}} t \\ \dot{q}(t) & = v_{\text{ter}}\left(1 - e^{-t/\tau}\right) \\ \ddot{q}(t) & = \gamma e^{-t/\tau} \end{cases}$$

Problem 5

(Taylor 2.12) Problem 2.7 is about a class of one-dimensional problems that can always be reduced to doing an integral. Here is another. Show that if the net force on a one-dimensional particle depends only on position, $F = F(x)$, then Newton's second law can be solved to find v as a function of x given by

$$v^2 = v_0^2 + \frac{2}{m}\int_{x_0}^{x} F(x')\,dx'. \tag{2.85}$$

[Hint: Use the chain rule to prove the following handy relation, which we could call the "v dv/dx rule": If you regard v as a function of x, then]

$$\dot{v} = v\frac{dv}{dx} = \frac{1}{2}\frac{dv^2}{dx}. \tag{2.86}$$

Solution:

$$\begin{aligned}
F(x) &= m\frac{dv}{dt} \\
&= m\frac{dv}{dx}\frac{dx}{dt} \\
&= \frac{m}{2}\left(2v\frac{dv}{dt}\right) \\
&= \frac{m}{2}\frac{d}{dx}\left(v^2\right) \\
\frac{2}{m}\int_{x_0}^{x} F(x')\ dx' &= \int_{x_0}^{x}\frac{d}{dx'}(v^2)\ dx' \\
\frac{2}{m}W &= (v)^2 - (v_0)^2 \\
W &= \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2
\end{aligned}$$

Use this to rewrite Newton's second law in the separated form $m\,d(v^2) = 2F(x)dx$ and then integrate from x_0 to x. Comment on your result for the case that $F(x)$ is actually a constant. (You may recognize your solution as a statement about kinetic energy and work, both of which we shall be discussing in Chapter 4.)

Problem 6

(Taylor 2.38, a, b) A projectile that is subject to quadratic air resistance is thrown vertically up with initial speed v_0.

(a) Write down the equation of motion for the upward motion and solve it to give v as a function of t.

(b) Show that the time to reach the top of the trajectory is

$$t_{\text{top}} = \left(\frac{v_{\text{ter}}}{g}\right)\arctan\left(\frac{v_0}{v_{\text{ter}}}\right).$$

Problem 6 continued on next page…

Solution:

We have a system of differential equations given by:

$$\begin{pmatrix} m\ddot{q}_1 \\ m\ddot{q}_2 \end{pmatrix} = \begin{pmatrix} -c\dot{q}_1^2 \\ -mg - c\dot{q}_2^2 \end{pmatrix}$$

These are independent and we only really care about the q_2 (y) component. We also only care about the velocity, so we can reduce the order by 1 while still complying with the boundary conditions. Let $\dot{q}_2 = v$ and $\ddot{q}_2 = \frac{dv}{dt}$.

$$\frac{1}{-g - \frac{c}{m}v^2} dv = dt$$
$$-\frac{1}{g}\frac{1}{1 + \frac{c}{mg}v^2} = \int_0^t gdt$$

Apply the substitution $u = \sqrt{\frac{c}{mg}}v$:

$$-\frac{\sqrt{\frac{mg}{c}}}{g}\frac{1}{1+u^2}\, dv = t$$
$$\sqrt{\frac{m}{cg}}\int_{v_0\sqrt{\frac{c}{mg}}}^{v\sqrt{\frac{c}{mg}}}\frac{1}{1+u^2}\, du = t$$

This gives $\tan(u)$. After integration, we get:

$$\sqrt{\frac{m}{cg}}\left(\tan^{-1}\left(v\sqrt{\frac{c}{mg}}\right) - \tan^{-1}\left(v_0\sqrt{\frac{c}{mg}}\right)\right) = t$$
$$\tan^{-1}\left(v\sqrt{\frac{c}{mg}}\right) = \sqrt{\frac{cg}{m}}t + \tan^{-1}\left(v_0\sqrt{\frac{c}{mg}}\right)$$
$$v\sqrt{\frac{c}{mg}} = \tan\left(\sqrt{\frac{cg}{m}}t + \tan^{-1}\left(v_0\sqrt{\frac{c}{mg}}\right)\right)$$
$$v(t) = \sqrt{\frac{mg}{c}}\tan\left(\sqrt{\frac{cg}{m}}t + \tan^{-1}\left(v_0\sqrt{\frac{c}{mg}}\right)\right)$$
$$v(t) = \sqrt{\frac{mg}{c}}\tan\left(\sqrt{\frac{cg}{m}}t + v_0\sqrt{\frac{c}{mg}}\right)$$

when $v = 0$ we have

$$t = -\sqrt{\frac{m}{gc}}\arctan\left(\sqrt{\frac{c}{mg}}v_0\right)$$

Which gives a negative sign probably because of how I defined my coordinate system, but this should be equivalent to the desired time to reach the top.