1 $y'' - 2y' - 3y = 3e^{2t}$

Solution:

1. Homogeneous Solution:

$$r^2 - 2r - 3 = 0 \implies r_1 = 3,\, r_2 = -1$$
$$y_c = c_1 e^{3t} + c_2 e^{-t}$$

2. Particular Solution:

$$\cancel{4Ae^{2t}} - \cancel{2(2Ae^{2t})} - 3Ae^{2t} \implies A = 1 \qquad \begin{aligned} y_p &= Ae^{2t} \\ y_p' &= 2Ae^{2t} \\ y_p'' &= 4Ae^{2t} \end{aligned}$$

3. Final Solution:

$$y(t) = y_c(t) + y_p(t) \quad = \quad c_1 e^{3t} + c_2 e^{-t} + e^{2t}$$

2 $y'' + y' - 6y = 12e^{3t} + 12e^{-2t}$

Solution:

1. Homogeneous Solution:

$$r^2 + r - 6 = 0 \implies r = -3,\, 2$$
$$y_c = c_1 e^{-3t} + c_2 e^{2t}$$

2. Particular Solution:

$$Ae^{3t} + 3Ae^{3t} - 6Ae^{3t} = 12e^{3t} \implies A = -6 \qquad \begin{aligned} y_{p1} &= Ae^{3t} \\ y_{p1}' &= 3Ae^{3t} \\ y_{p1}'' &= 9Ae^{3t} \end{aligned}$$

$$Be^{-2t} - 2Be^{-2t} - 4Be^{-2t} = 12e^{-2t} \implies B = -12/5 \qquad \begin{aligned} y_{p2} &= Be^{-2t} \\ y_{p2}' &= -2Be^{-2t} \\ y_{p2}'' &= 4Ae^{-2t} \end{aligned}$$

3. Final Solution:

$$c_1 e^{-3t} + c_2 e^{2t} - 6e^3 t - \frac{12}{5} e^{-2t}$$

7 $y'' + y = 3\sin(2t) + t\cos(2t)$

Solution:

1. Homogeneous Solution:

$$r^2 + r = 0 \implies r = -i$$
$$y_c = c_1 \cos t + c_2 \sin t$$

2. Particular Solution:

$$-3A\sin(2t) - 3B\cos(2t) = 3\sin(2t) \qquad \begin{aligned} y_{p1} &= A\sin(2t) + B\cos(2t) \\ y'_{p1} &= 2A\cos(2t) - 2B\sin(2t) \\ y''_{p1} &= -4A\sin(2t) - 4B\cos(2t) \end{aligned}$$
$$\implies A = -1,\ B = 0$$

$$\begin{aligned} y_{p2} &= Ct\cos(2t) + Dt\sin(2t) \\ y'_{p2} &= C\cos(2t) - 2Ct\sin(2t) + D\sin(2t) + 2Dt\cos(2t) \\ y''_{p2} &= -4C\sin(2t) - 4Ct\cos(2t) + 4D\cos(2t) - 4Dt\sin(2t) \end{aligned}$$

$$\begin{aligned} -4C\sin(2t) - 4Ct\cos(2t) &+ 4D\cos(2t) \\ &- 4Dt\sin(2t) + Ct\cos(2t) + Dt\sin(2t) = t\cos(2t) \\ \implies &- 4C\sin(2t) + 4D\cos(2t) \\ &- 3Dt\sin(2t) - 3Ct\cos(2t) = t\cos(2t) \\ \implies &C = -\frac{t\cos(2t) - 3Dt\sin(2t) - 4D\cos(2t)}{4\sin(2t) + 3t\cos(2t)}, \\ \implies &D = \frac{t\cos(2t) + 3Ct\cos(2t) + 4C\sin(2t)}{-3t\sin(2t) + 4\cos(2t)} \end{aligned}$$

3. Final Solution:

$$y(t) = y_c + y_{p1} + y_{p2}$$

12 $y'' - 2y' + y = te^t + 4, \qquad y(0) = 0, y'(0) = 2$

Solution:

1. Homogeneous Solution:

$$
\begin{aligned}
r^2 - 2r + 1 = 0 &\implies r = 1 \\
y_c = c_1 e^t + c_2 te^t, &\qquad y_c' = c_1 e^t + c_2(e^t + te^t) \\
y_c(0) = 0 &\implies c_1 = 0 \\
y_c'(0) = 2 &\implies c_2 = 2 \\
\implies y_c = 2te^t &
\end{aligned}
$$

2. Particular Solution:

$$
A(2e^t + te^t) - 2A(e^t + te^t) + Ate^t = te^t \qquad
\begin{aligned}
y_{p1} &= Ate^t \\
y_{p1}' &= A(e^t + te^t) \\
y_{p1}'' &= A(2e^t + te^t)
\end{aligned}
$$

$$\implies \text{True for all } A.$$

$$
B = 4 \qquad
\begin{aligned}
y_{p2} &= B \\
y_{p2}' &= 0 \\
y_{p2}'' &= 0
\end{aligned}
$$

3. Final Solution:

$$y(t) = (2te^t) + (A(2e^t + te^t) - 2A(e^t + te^t) + Ate^t) + (4)$$