Math 590 HW10

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1. Let f_n be the sequence given by $a_0 = 0$, $a_1 = 5$, $a_{n+1} = a_n + 6a_{n-1}$ for each $n \geq 1$. Use diagonalization to find the general formula for a_n.

Solution:

(a) The sequence f_n is given by $f_n = 6f_{n-2} + f_{n-1}$ Define $T \in \mathcal{L}(\mathbf{R}^2)$ by $T(x, y) = (y, 6x + y)$. It follows that $T^n(0, 5) = (f_n, f_{n-1})$ for each positive integer n:

$$\begin{pmatrix} 0 & 5 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 6 & 1 \end{pmatrix} = \begin{pmatrix} 30 & 5 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 5 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 6 & 1 \end{pmatrix}^2 = \begin{pmatrix} 30 & 35 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 5 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 6 & 1 \end{pmatrix}^3 = \begin{pmatrix} 210 & 65 \end{pmatrix}$$

(b) By computing the value of the matrix raised to an arbitrary power n, we can compute an explicit formula for the sequence. This can be done by diagonalizing:

$$-\lambda(1-\lambda) - 6 = 0 \implies \lambda = \{-2, 3\}$$
$$\xi_1 = \left(\begin{array}{cc|c} 2 & 1 & 0 \\ 6 & 3 & 0 \end{array}\right) \implies \xi_1 = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$$
$$\xi_2 = \left(\begin{array}{cc|c} -3 & 1 & 0 \\ 6 & -2 & 0 \end{array}\right) \implies \xi_2 = \begin{pmatrix} -1/3 \\ 1 \end{pmatrix}$$

Therefore:

$$P = \begin{pmatrix} 1/2 & -1/3 \\ 1 & 1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} -6/5 & 2/5 \\ 6/5 & 3/5 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}$$

(c) Finally, because $A = P\Lambda P^{-1}$, and $A^n = P\Lambda^n P^{-1}$, a direct formula can be computed:

$$\begin{aligned}
\begin{pmatrix} 0 & 5 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 6 & 1 \end{pmatrix}^n &= \begin{pmatrix} 0 & 5 \end{pmatrix} \begin{pmatrix} 1/2 & -1/3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}^n \begin{pmatrix} 6 & -2 \\ -6 & 3 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 5 \end{pmatrix} \begin{pmatrix} 1/2 & -1/3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (-2)^n & 0 \\ 0 & (3)^n \end{pmatrix} \begin{pmatrix} -6/5 & 2/5 \\ 6/5 & 3/5 \end{pmatrix}
\end{aligned}$$

Which, after multiplying out the matrices, gives:

$$\begin{aligned}
f_n &= -6(-2)^n + 2(3)^{1+n} \\
f_{n-1} &= 2(-2)^n + 3^{1+n}
\end{aligned}$$

2. Let W be the span of $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$ in $\mathbb{R}^4$.

   (a) Find $W^\perp$.

Solution:

$$\begin{aligned}
A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix} &\to A^T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} \\
W^\perp = \ker(A^T) &= A^T = 0
\end{aligned}$$

The RREF form of A is:

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

Therefore the kernel is:

$$\begin{pmatrix} x_3 + 2x_4 \\ -2x_3 - 3x_4 \\ x_3 \\ x_4 \end{pmatrix} \to \begin{pmatrix} s + 2t \\ -2s - 3t \\ s \\ t \end{pmatrix}$$

(b) Find the point in W closest to the point $(1, 1, -1, -1)$.

Solution:

The orthogonal projection of the given vector into W will give the nearest point in W to the given vector. Such a projection is:

$$\begin{aligned} v - P_W v &= v - \frac{W \cdot v}{|v|^2} v \\ &= v - \begin{pmatrix} \frac{-2s-2t}{4} \\ \frac{-2s-2t}{4} \\ \frac{2s+2t}{4} \\ \frac{2s+2t}{4} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} - \begin{pmatrix} \frac{-2s-2t}{4} \\ \frac{-2s-2t}{4} \\ \frac{2s+2t}{4} \\ \frac{2s+2t}{4} \end{pmatrix} \end{aligned}$$

3. Find the best-fit parabola through the point $(1, 1)$, $(2, 2)$, $(3, 3)$, $(4, 5)$. (See section 6.5 of the text).

Solution:

The textbook describes the general method for fitting a polynomial curve to a set of points:

$$\beta = \left(X^T X\right)^{-1} X^T Y$$

Where β is the coefficients (a, b, c), X the matrix of x-values arranged in row-major order. Finally, matrix Y is the column vector of y-values.

I used Julia to optimize the curve:

```
julia> X = [1 1 1; 1 2 4; 1 3 9; 1 5 25]
4×3 Matrix{Int64}:
 1  1   1
 1  2   4
 1  3   9
 1  5  25

julia> Y = [1; 2; 3; 5]
4-element Vector{Int64}:
 1
 2
 3
 5

julia> (transpose(X)*X)^-1*transpose(X)*Y
3-element Vector{Float64}:
 -2.842170943040401e-14
  1.0
  0.0
```

The final curve is effectively linear, since the coefficient a is quite small.