1 Problem 1

Let $\mathbf{A} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{bmatrix}$. Compute the determinant of $\mathbf{A}$ using the following methods:

(a) Cofactor expansion.

(b) Row reduction.

(c) Using the "exterior" product $(\mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3)(4\mathbf{e}_1 + 5\mathbf{e}_2 + 6\mathbf{e}_3)(7\mathbf{e}_1 + 8\mathbf{e}_2 + 10\mathbf{e}_3)$.

(d) By calculating directly the volume of the tetrahedron or parallelepiped formed by the column vectors of $\mathbf{A}$.

Solution:

1. The determinant in terms of cofactors is $a_{11}C_{11} - a_{12}C_{12} + a_{13}C_{13}$.

$$\begin{vmatrix} 5 & 8 \\ 6 & 10 \end{vmatrix} - 4\begin{vmatrix} 2 & 8 \\ 3 & 10 \end{vmatrix} + 7\begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix} = -3$$

2. The determinant can be found by row reduction by first row reducing, then computing the determinant in the opposite order.

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & -6 & -12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$= -3$ Because from step 3 to 2, **R2** is multiplied by -3

3. The determinant can be found using the exterior product, which is defined as: $\alpha \wedge \beta \wedge \gamma = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1$. Because the matrix $\mathbf{A}$ is constructed from three column vectors, we can apply the wedge product to find the determinant.

$$(1)(5)(10) + (2)(6)(7) + (3)(4)(8) - (1)(6)(8) - (2)(4)(10) - (3)(5)(7) = -3$$

4. The volume of a parallelpiped can be found with the triple product (this is only applicable on a 3x3 matrix): $a \cdot (b \times c)$.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \left(\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \times \begin{bmatrix} 7 \\ 8 \\ 10 \end{bmatrix} \right) = -3$$

2 Problem 2

Use Cramer's rule to solve the system $\mathbf{A}x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ with the matrix $\mathbf{A}$ in Problem 1.

Solution:

1. We have already found the determinant of $\mathbf{A}$ to be -3.
2. Now we can write the solutions x_1, x_2, and x_3 in terms of the determinants:

$$\frac{\begin{vmatrix} 1 & 4 & 7 \\ 1 & 5 & 8 \\ 1 & 6 & 10 \end{vmatrix}}{-3} = 1 \tag{1}$$

$$\frac{\begin{vmatrix} 1 & 1 & 7 \\ 2 & 1 & 8 \\ 3 & 1 & 10 \end{vmatrix}}{-3} = -1 \tag{2}$$

$$\frac{\begin{vmatrix} 1 & 4 & 1 \\ 2 & 5 & 1 \\ 3 & 6 & 1 \end{vmatrix}}{-3} = 0 \tag{3}$$