Assignment 3 Solution

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Calculating π with Darts

The accompanying code, output files, and embedded figures illustrate the measurement of π using random numbers, both by sampling from a 3-d cube, and from a 2-d square.

Firstly, let's review the 2-d method. With a square of side-length 2, centered on the origin, the square's area is 4, and the area of the inscribed circle with radius of 1 is just π. Therefore one expects that the probability is $\pi/4$ for a point from the 2-d uniform probability distribution on the square to fall within the circle.

Similarly, by extension, in 3-d, with a cube of side-length 2, centered on the origin, and volume of 8, the inscribed sphere centered on the origin with radius of 1 has a volume of $4\pi/3$. Therefore one expects that the probability is $(4\pi/3)/8 = \pi/6$ for a point from the 3-d uniform probability distribution on the square to fall within the circle.

The code simulates one million random points in either 2-d or 3-d and keeps track of whether the measured point is within 1 unit of the origin, reporting a value of 1 if it is (a success), and a value of 0 (failure) if not. These results are then averaged to form estimates of the probability parameter of the underlying Bernouilli distribution, $\hat{p}$, but averaged over one million sampled points. The probability parameters are then transformed into measurements of π denoted $\hat{\pi}$ using

$$\hat{\pi} = 6\hat{p}$$

for 3-d, and

$$\hat{\pi} = 4\hat{p}$$

for 2-d. Results are shown in Table 1 and Figure 1 in 3-d and in Table 2 and Figure 2 for 2-d.

The estimated uncertainty for the 3-d method is 0.0952% and for the 2-d method it is 0.0523%. The 3-d method is therefore about a factor of 1.8 less precise than the 2-d method when using the same number of points. The source of this difference is two-fold. Firstly the measured probability error is larger, and this gets amplified by 50% (6/4) when converting the average probability to an estimate of π using the 3-d method rather than the 2-d method. The variance of the underlying Bernouilli distribution where $X = 1$ with probability, p and $X = 0$ with probability, $1 - p$, is given by

$$V(X) = p(1 - p)$$

which is either $(\pi/6)(1 - \pi/6)$ or $(\pi/4)(1 - \pi/4)$ namely 0.2494 or 0.1685 in the two cases. The measured variances bear this out.

N_{trials}	1 000 000
$N_{\text{successes}}$	524 349
$p = \pi/6$	0.523599
$\hat{p}$	0.524349 ± 0.000499
Measured variance of $\hat{p}$	0.24941
$\hat{\pi}$	3.14609 ± 0.00300
$\sigma(\hat{\pi})/\hat{\pi}$	0.0952 %
Standardized deviation	$+1.50\ \sigma$

Table 1: 3-d results

N_{trials}	1 000 000
$N_{\text{successes}}$	785 025
$p = \pi/4$	0.785398
$\hat{p}$	0.785025 ± 0.000411
Measured variance of $\hat{p}$	0.16876
$\hat{\pi}$	3.14010 ± 0.00164
$\sigma(\hat{\pi})/\hat{\pi}$	0.0523 %
Standardized deviation	$-0.91\ \sigma$

Table 2: 2-d results

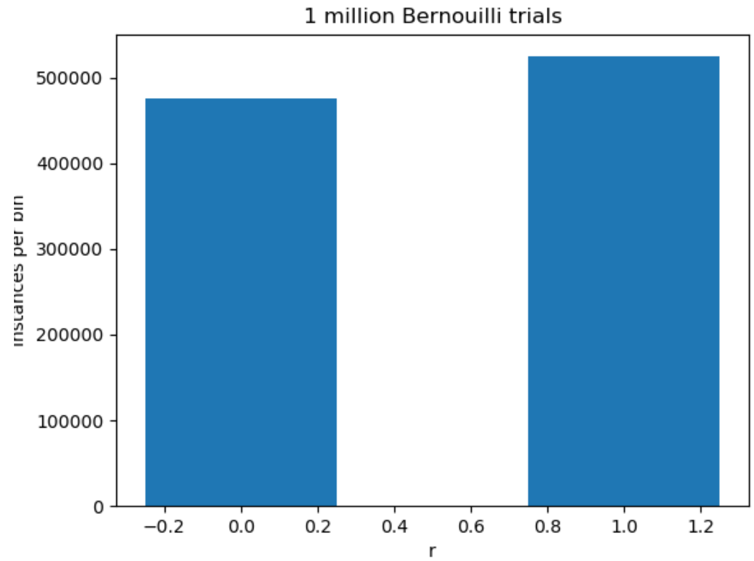

Figure 1: Bernouilli distribution for one million 3-d trials with Bernouilli trial success probability of $\pi/6$.

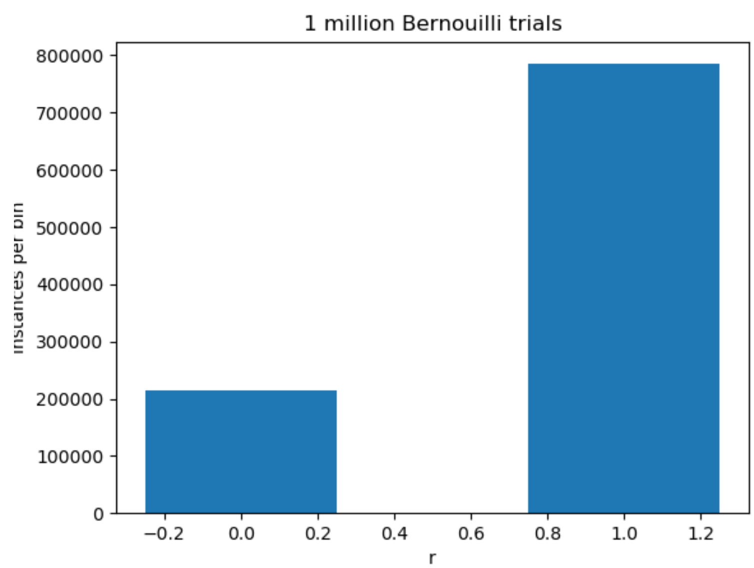

Figure 2: Bernouilli distribution for one million 2-d trials with Bernouilli trial success probability of $\pi/4$.