Assignment 3 Solution

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Calculating π with Darts

The accompanying code, output files, and figures included in this writeup illustrate the measurement of π using random numbers, both by sampling from a 3-d cube, and from a 2-d square.

Firstly, let's review the 2-d method. With a square of side-length 2, centered on the origin, the square's area is 4, and the area of the inscribed circle with radius of 1 is just π. Therefore one expects that the probability is $\pi/4$ for a point from the 2-d uniform probability distribution on the properly centered square to fall within the circle.

Similarly, by extension, in 3-d, with a cube of side-length 2, centered on the origin, and volume of 8, the inscribed sphere centered on the origin with radius of 1 has a volume of $4\pi/3$. Therefore one expects that the probability is $(4\pi/3)/8 = \pi/6$ for a point from the 3-d uniform probability distribution on the properly centered cube to fall within the sphere.

The code simulates one million random points in either 2-d or 3-d and keeps track of whether the measured point is within a 2-d or 3-d distance of 1 unit of the origin, reporting a value of 1 if it is (a success), and a value of 0 (failure) if not. These results are then averaged to form an estimate of the probability parameter of the underlying Bernouilli distribution, $\hat{p}$, but averaged over one million sampled points. An alternative (and very much related) way to look at this is that one is measuring simply the total number of successes, and that this number should follow the Binomial distribution with parameter, $n = N_{\text{trials}}$, and success probability, p. In this case our estimate of the probability is simply,

$$\hat{p} = N_{\text{successes}}/N_{\text{trials}} \quad .$$

The probability parameter estimate is then transformed into an estimate of π denoted $\hat{\pi}$ using either

$$\hat{\pi} = 6\hat{p}$$

for 3-d, and

$$\hat{\pi} = 4\hat{p}$$

for 2-d. Results are shown in Table 1 and Figure 1 for 3-d and in Table 2 and Figure 2 for 2-d.

The estimated uncertainty from the data samples for the 3-d method is 0.0952% and for the 2-d method it is 0.0523%. These are calculated by measuring the variance of the sampled Bernouilli distributions, $\hat{V}$, and scaling appropriately, to find the uncertainty on the measured probability,

$$\sigma(\hat{p}) = \sqrt{\hat{V}/N_{\text{trials}}} \quad ,$$

see below for a more detailed derivation of this. The uncertainty on the π estimate is then simply,

$$\sigma(\hat{\pi}) = 6\,\sigma(\hat{p})$$

or

$$\sigma(\hat{\pi}) = 4\,\sigma(\hat{p})$$

in the two cases. The 3-d method is therefore about a factor of 1.8 less precise than the 2-d method when using the same number of points. The source of this difference is two-fold. Firstly the measured probability error is larger, and this gets amplified by 50% (6/4) when converting the average probability to an estimate of π using the 3-d method rather than the 2-d method. The variance of the underlying Bernouilli distribution that describes the result of a single trial, where $X = 1$ with probability, p, and $X = 0$ with probability, $1 - p$, is given by

$$V(X) = \sigma_X^2 = p(1-p) \quad .$$

This gives a standard deviation of $\sqrt{(\pi/6)(1-\pi/6)} = 0.4994428$ in the 3-d case, and $\sqrt{(\pi/4)(1-\pi/4)} = 0.4105458$ in the 2-d case. The measured standard deviations, $\hat{\sigma}_X$, bear this out.

We can write, the average results over n trials as

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i = \hat{p}$$

and can show using the variance on a single Bernouilli trial,

$$V(\hat{p}) = \frac{1}{n^2}\sum_{i=1}^{n} V(X_i) = \frac{\hat{p}(1-\hat{p})}{n} \quad .$$

More explicitly one can show that the expected deterioration factor when using 3-d compared to 2-d is

$$f \equiv \sigma(\hat{\pi}_{3d})/\sigma(\hat{\pi}_{2d}) = \sqrt{\frac{6-\pi}{4-\pi}} = 1.8248 \quad .$$

Another way to approach this problem, more consistent with how I initially discussed this in class, is to think of the experiment as one single binomial experiment with expected number of successes given by $N_{\mathrm{trials}}\, p$ and expected standard deviation on this number of successes of $\sqrt{N_{\mathrm{trials}} p(1-p)}$. The measured probability,

$$\hat{p} = N_{\mathrm{successes}}/N_{\mathrm{trials}}$$

then has an expected uncertainty of $\sqrt{p(1-p)/N_{\mathrm{trials}}}$, identical to the scaled Bernouilli version above. But with only one measurement it is not feasible to come up with a measured or empirical uncertainty from the standard deviation of the measurements (one single number has no dispersion ...). A satisfactory but more cumbersome alternative is to sub-divide into say 1000 experiments each with 1000 trials and measure the standard deviation for the ensemble of experiments).

Conclusion

Random numbers can be used to sample from particular probability distributions and used to perform numerical integration by the Monte Carlo method. In these introductory cases, this is used to measure π. The resulting uncertainties scale as $1/\sqrt{N_{\mathrm{trials}}}$.

N_{trials}	$1\,000\,000$
$p = \pi/6$	0.5235988
$\sigma_X = \sqrt{V_X} = \sqrt{p\,(1-p)}$	0.4994428
$\mu = N_{\text{trials}}\,p$	$523\,598.8$
$\sigma_\mu = \sqrt{V_\mu} = \sqrt{N_{\text{trials}}\,p\,(1-p)}$	499.4
seed	208
$N_{\text{successes}}$	524 349
$\hat{p}$	0.524349 ± 0.000499
$\hat{\sigma}_X$	0.499407 ± 0.000499
$\hat{\pi}$ $(6\hat{p})$	3.14609 ± 0.00300
$\sigma(\hat{\pi})/\hat{\pi}$	$0.0952\,\%$
Standardized deviation	$+1.50\,\sigma$

Table 1: 3-d results. The upper part of the table presents the parameters and expectation values. The lower part shows the results for a particular random number seed.

N_{trials}	$1\,000\,000$
$p = \pi/4$	0.7853982
$\sigma_X = \sqrt{V_X} = \sqrt{p\,(1-p)}$	0.4105458
$\mu = N_{\text{trials}}\,p$	$785\,398.2$
$\sigma_\mu = \sqrt{V_\mu} = \sqrt{N_{\text{trials}}\,p\,(1-p)}$	410.5
seed	400
$N_{\text{successes}}$	785 025
$\hat{p}$	0.785025 ± 0.000411
$\hat{\sigma}_X$	0.410805 ± 0.000411
$\hat{\pi}$ $(4\hat{p})$	3.14010 ± 0.00164
$\sigma(\hat{\pi})/\hat{\pi}$	$0.0523\,\%$
Standardized deviation	$-0.91\,\sigma$

Table 2: 2-d results. The upper part of the table presents the parameters and expectation values. The lower part shows the results for a particular random number seed.

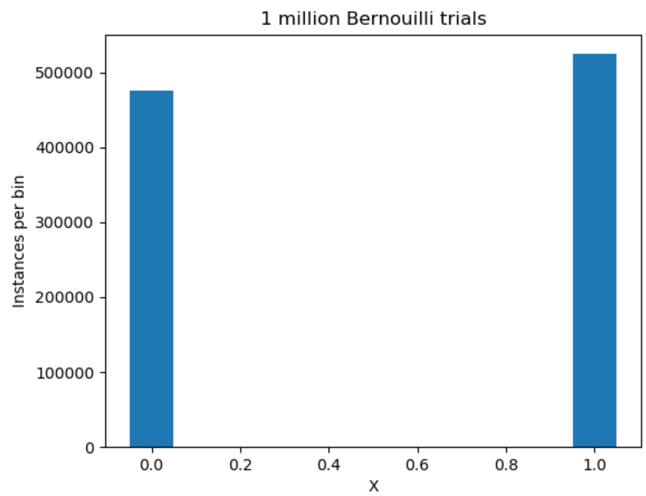

Figure 1: Bernouilli distribution for 1M 3-d trials with Bernouilli trial success probability of $\pi/6$. The mean of this X distribution gives $\hat{p}$ and the standard deviation gives the measured value of the standard deviation, $\hat{\sigma}_X$.

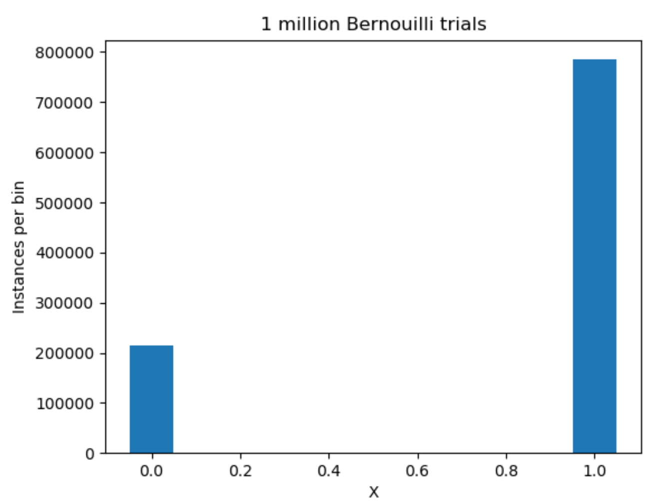

Figure 2: Bernouilli distribution for 1M 2-d trials with Bernouilli trial success probability of $\pi/6$. The mean of this X distribution gives $\hat{p}$ and the standard deviation gives the measured value of the standard deviation, $\hat{\sigma}_X$.