STATISTICAL SIGNAL PROCESSING

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Course Outline

COURSE OUTLINE

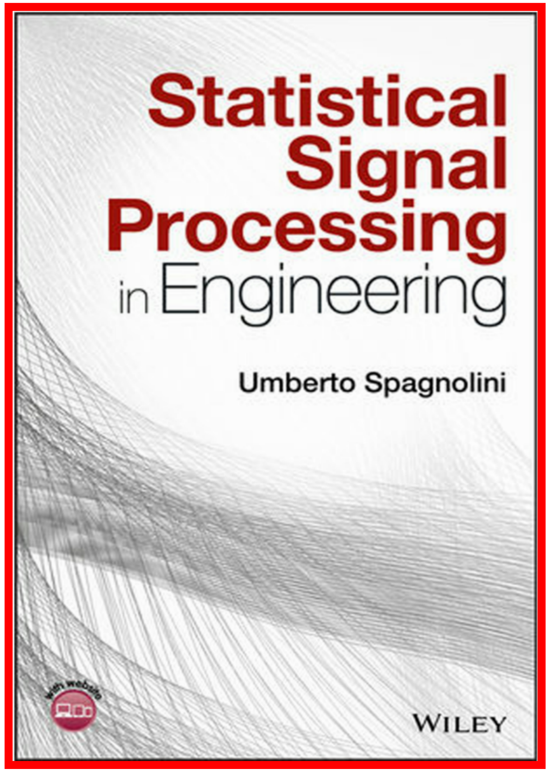
- Discrete representation of continuous-time signals
- Introduction to discrete-time random processes
- Estimation of deterministic parameters
- Estimation of random parameters (Bayesian estimation)
- Modeling and identification of discrete-time random processes
- Smoothing, filtering, and prediction of random signals
- Power spectrum estimation (parametric and nonparametric)
- Basic concepts of Time-Frequency Analysis

EXAM OPERATING METHOD

- Written and oral exam
- Exam enrollment: on-line via the website esami.unipi.it

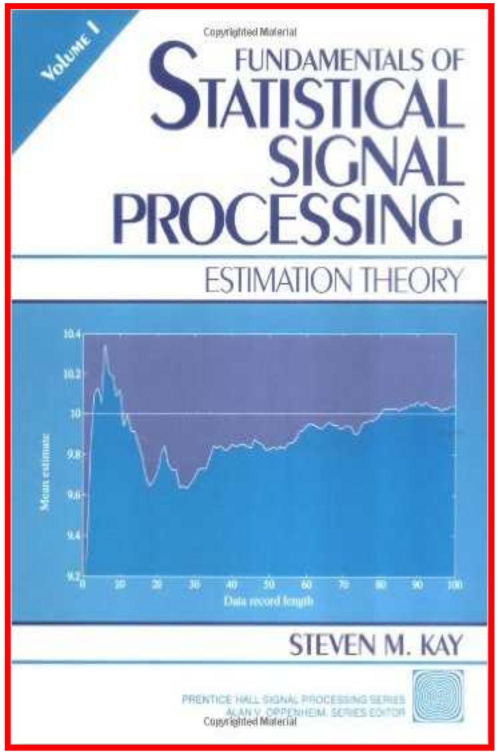
Course Textbooks

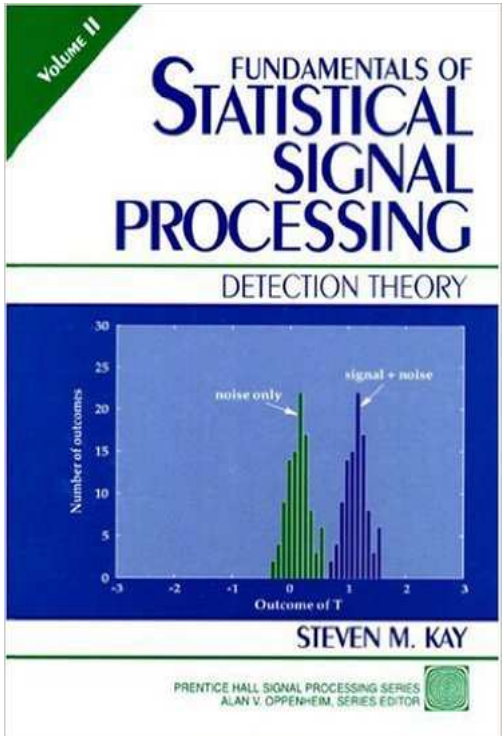
- **<u>COURSE TEXTBOOKS</u>:**
- **U. Spagnolini: *Statistical Signal Processing in Engineering, Wiley.* ISBN: 978-1-119-29397-2. First published:15 December 2017.**
- **Material provided by the instructor (vugraphs + Matlab code)**

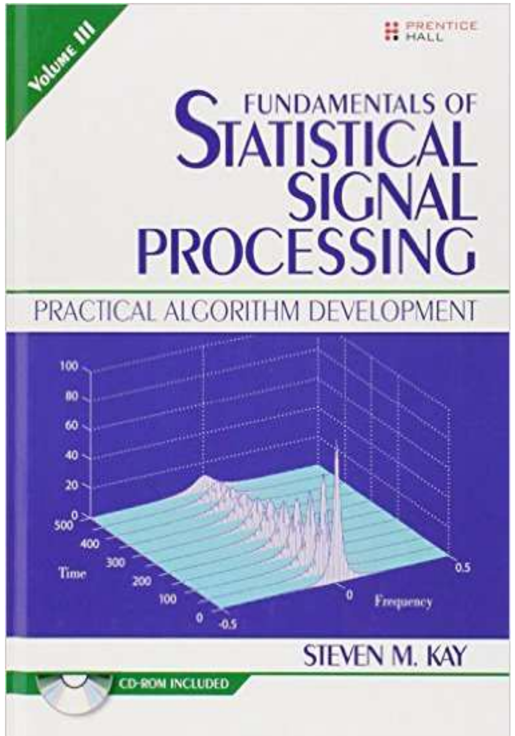


Course Textbooks

- **<u>COURSE TEXTBOOKS</u>:**
- **S. Kay: *Fundamentals of Statistical Signal Processing, Volume 1: Estimation Theory*, Prentice Hall, 1993.**





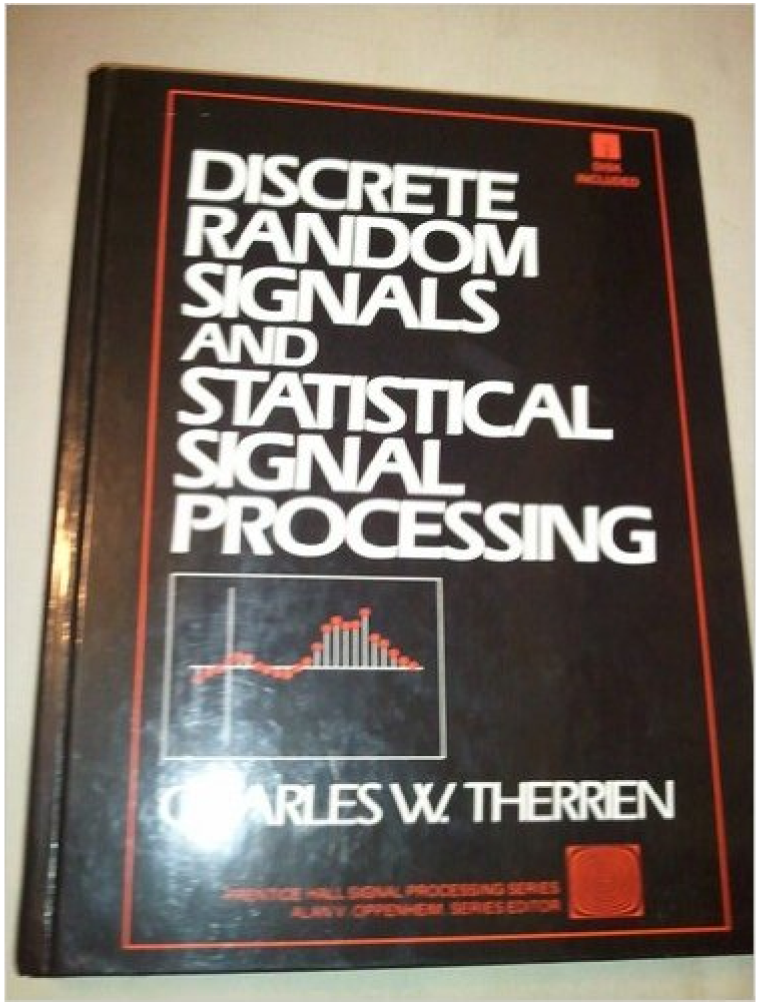


Course Textbooks

- <u>**ANOTHER EXCELLENT BOOK**</u>**:**
- **Charles. W. Therrien:**

Discrete Random Signals and Statistical Signal Processing

Prentice Hall, 1992.



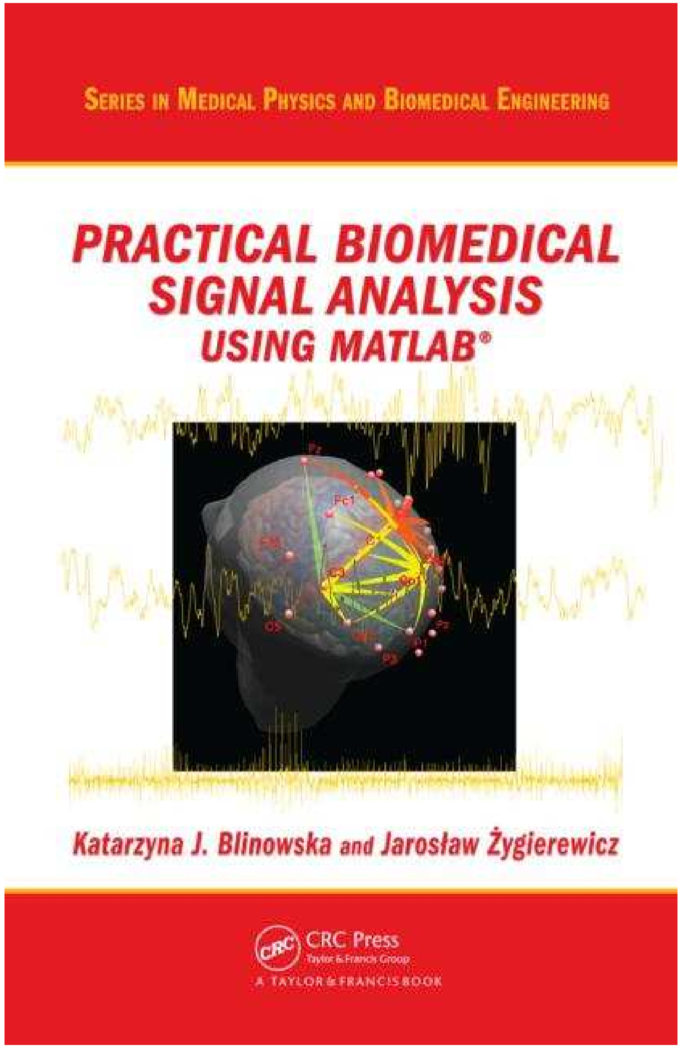
For further studies

- **FOR FURTHER STUDIES:**
- **Katarzyn J. Blinowska, Jaroslaw Zygierewicz**

Practical Biomedical Signal Analysis Using MATLAB®

2011, CRC Press

Series: Series in Medical Physics and Biomedical Engineering



Discrete Representation of Continuous-Time Signals

Discrete Representation of Continuous-Time Signals

- A **signal** is a physical quantity that can carry information.
- Physical and biological signals may be classified as <u>deterministic</u> or <u>stochastic</u>. The stochastic signal contrary to the deterministic one cannot be described by a mathematical function.
- Examples of **biomedical signals**:
  - electroencephalogram (EEG), electrocorticogram (ECoG), event-related potential (ERP), electrocardiogram (ECG), heart rate variability signal (HRV), electromyograms (EMG), electroenterograms (EEnG), and electrogastrograms (EGG).
  - Magnetic fields connected with the activity of brain (MEG) and heart (MCG).
  - Acoustic signals: phonocardiograms (PCG) and otoacoustic emissions (OAE).

Discrete Representation of Continuous-Time Signals

- In nature, most of the signals of interest are some physical values changing in time and/or space.
- These signals are continuous in time and in space.
- On the other hand, we use computers to store and analyze the data.
- To adapt the natural continuous data to the digital computer systems we need to digitize them.
- The most typical way to achieve this is to **sample** the physical values in certain moments in time and places in space and assign them a numeric value with a finite precision.
- **Analog-to-Digital Conversion (ADC)** is made of two processes:
  - **sampling** (selecting discrete moments in time), and
  - **quantization** (assigning a value of finite precision to an amplitude).

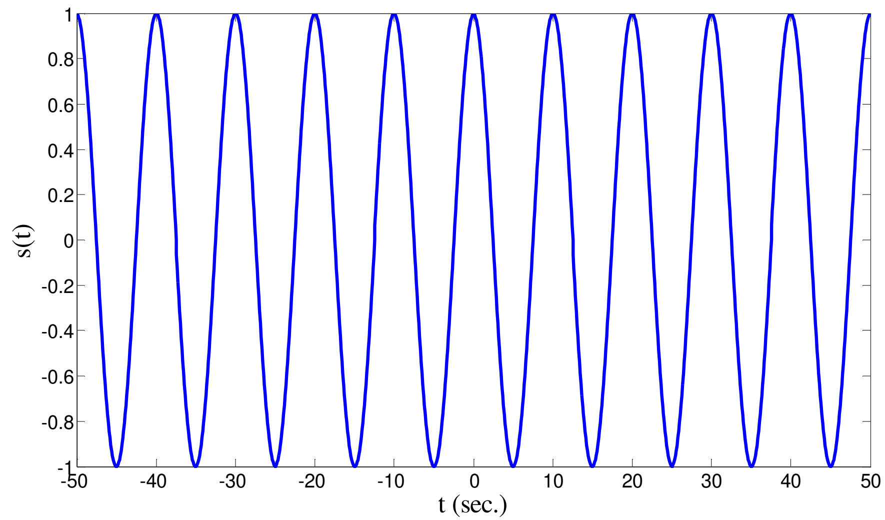
Discrete Representation of Continuous-Time Signals

- **Example**: sinusoidal signal

Analytical expression:

$$s(t) = A\cos\left(2\pi f_0 t + \theta\right), \quad t \in \mathbb{R}$$

Graphical representation:



- **Fourier Analysis:** Discrete representation (in the frequency domain) of a **finite-power periodic continuous-time signal** is possible by resorting to the Fourier Series (FS):

$$s(t) \overset{FS}{\Leftrightarrow} S_k, \qquad k \in \mathbb{Z}$$

Discrete Representation of Continuous-Time Signals

- Discrete representation of a continuous-time signal, periodic of T_0, having finite power P_s, is possible by resorting to the **Fourier Series**:

$$s(t) = s(t - T_0) \quad \forall t, \quad P_s \triangleq \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |s(t)|^2 dt < +\infty$$

Synthesis Equation

Analysis Equation

$$s(t) = \sum_{k=-\infty}^{+\infty} S_k \cdot e^{j 2\pi \frac{k}{T_0} t} \quad \overset{FS}{\Leftrightarrow} \quad S_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} s(t) \cdot e^{-j 2\pi \frac{k}{T_0} t} dt$$

- The **Fourier Series (FS)** S_k provides a (frequency-domain) discrete representation of the continuous-time signal $s(t)$.

Discrete Representation of Continuous-Time Signals

- We can use this discrete representation to calculate some signal parameters of interest (**Bessel-Parseval formula**):

- Power of $s(t)$: $$P_s \triangleq \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} |s(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |S_k|^2$$

- Cross-power between $s(t)$ and $x(t)$: $$P_{sx} \triangleq \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} s(t)x^*(t)dt = \sum_{k=-\infty}^{+\infty} S_k X_k^*$$

- Angle between $s(t)$ and $x(t)$:

$$\theta \triangleq Angle\big(s(t), x(t)\big) \quad \rightarrow \quad \cos\theta \triangleq \frac{\mathrm{Re}\{P_{sx}\}}{\sqrt{P_s P_x}} \in [-1,1]$$

  - $P_{sx}=0 \rightarrow \cos\theta=0 \rightarrow$ $s(t)$ and $x(t)$ are **orthogonal**
  - $\cos\theta=\pm1 \rightarrow$ $s(t)$ and $x(t)$ are **colinear**

Discrete Representation of Continuous-Time Signals

- If the signal is **bandlimited**, the number of non-zero coefficients is finite.
- As an example:

$$s(t) = A\cos\left(2\pi f_0 t + \theta\right) \overset{FS}{\Leftrightarrow} S_k = \begin{cases} \frac{A}{2}e^{j\theta}, & k = 1 \\ \frac{A}{2}e^{-j\theta}, & k = -1 \\ 0, & otherwise \end{cases}$$

- Power of $s(t)$: $$P_s = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \left|s(t)\right|^2 dt = \sum_{k=-\infty}^{+\infty} \left|S_k\right|^2 = 2\left(\frac{A}{2}\right)^2 = \frac{A^2}{2}$$

Bessel-Parseval equality

Discrete Representation of Continuous-Time Signals

- The Fourier Series representation is also useful to find the output *y*(*t*) of a **linear time-invariant (LTI) system** when the input is a finite-power periodic signal *s*(*t*):

$$y(t) = s(t) \otimes h(t) = \int_{-\infty}^{+\infty} s(\tau) h(t-\tau) d\tau \quad \overset{FS}{\Leftrightarrow} \quad Y_k = S_k \cdot H\left(\frac{k}{T_0}\right)$$

convolution integral algebraic product

- *h*(*t*) = impulse response of the LTI system
- *H*(*f*) = FT[*h*(*t*)] frequency response of the LTI system

Discrete Representation of Continuous-Time Signals

- Another example of discrete representation: **bandlimited finite-energy aperiodic signals**.

- We know from the **Sampling Theorem** (**Shannon Theorem, 1949**) that signals in this class can be represented in terms of its temporal samples via the following **Whittaker-Shannon interpolation formula**:

$$s(t) = 2BT_c \sum_{k=-\infty}^{+\infty} s(kT_c)\cdot \mathrm{sinc}\left(2B(t-kT_c)\right), \quad \text{where } \mathrm{sinc}(x) \triangleq \frac{\sin(\pi x)}{\pi x}$$

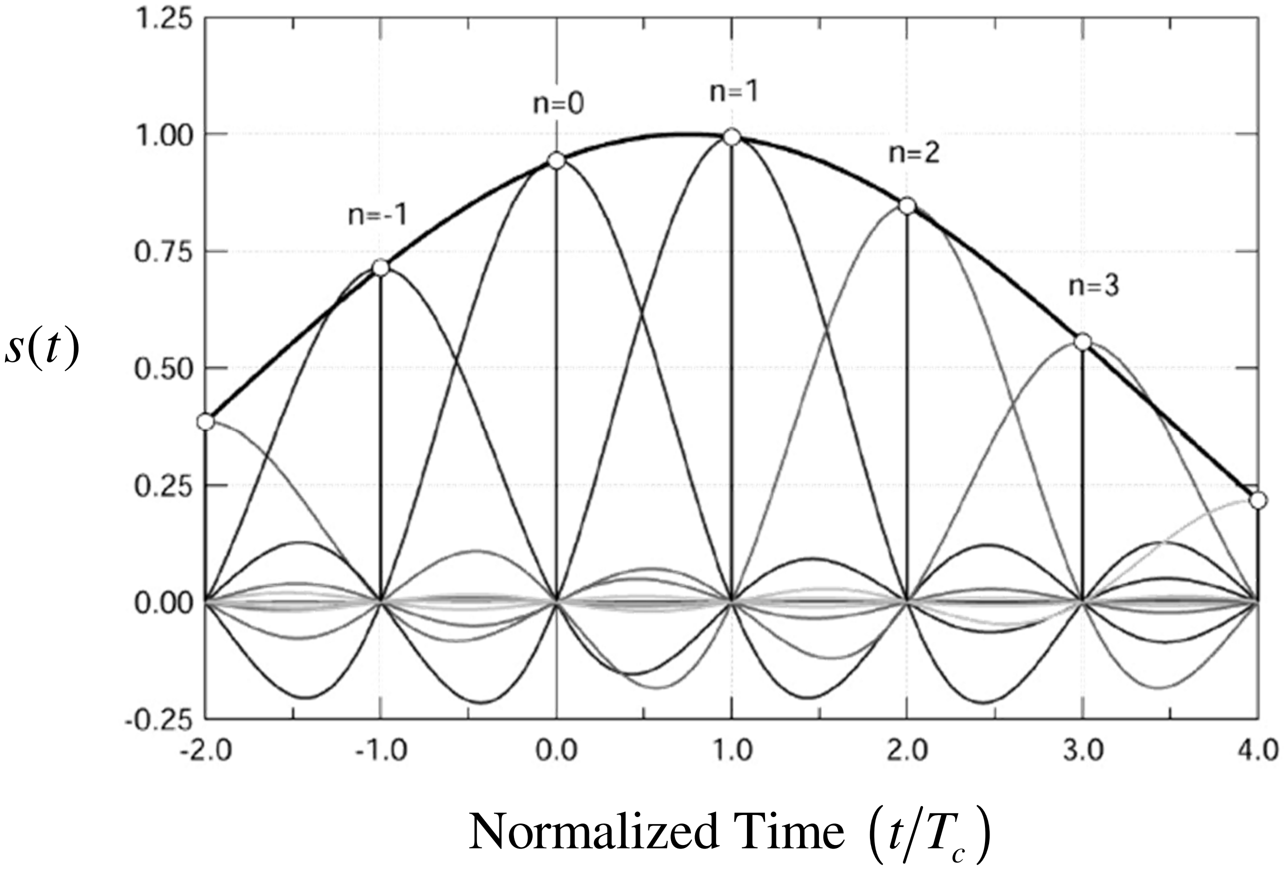
- B is the bandwidth of $s(t)$ and T_c is the sampling interval.

- Perfect reconstruction of the signal $s(t)$ in terms of its samples is possible <u>if and only if</u> sampling satisfies the **Nyquist condition**: $T_c \leq 1/(2B)$.

$$\begin{array}{ccc} s(t) & \overset{sampling}{\Leftrightarrow} & s(kT_c) \\ t \in \mathbb{R} & & k \in \mathbb{Z} \end{array}$$

Discrete Representation of Continuous-Time Signals

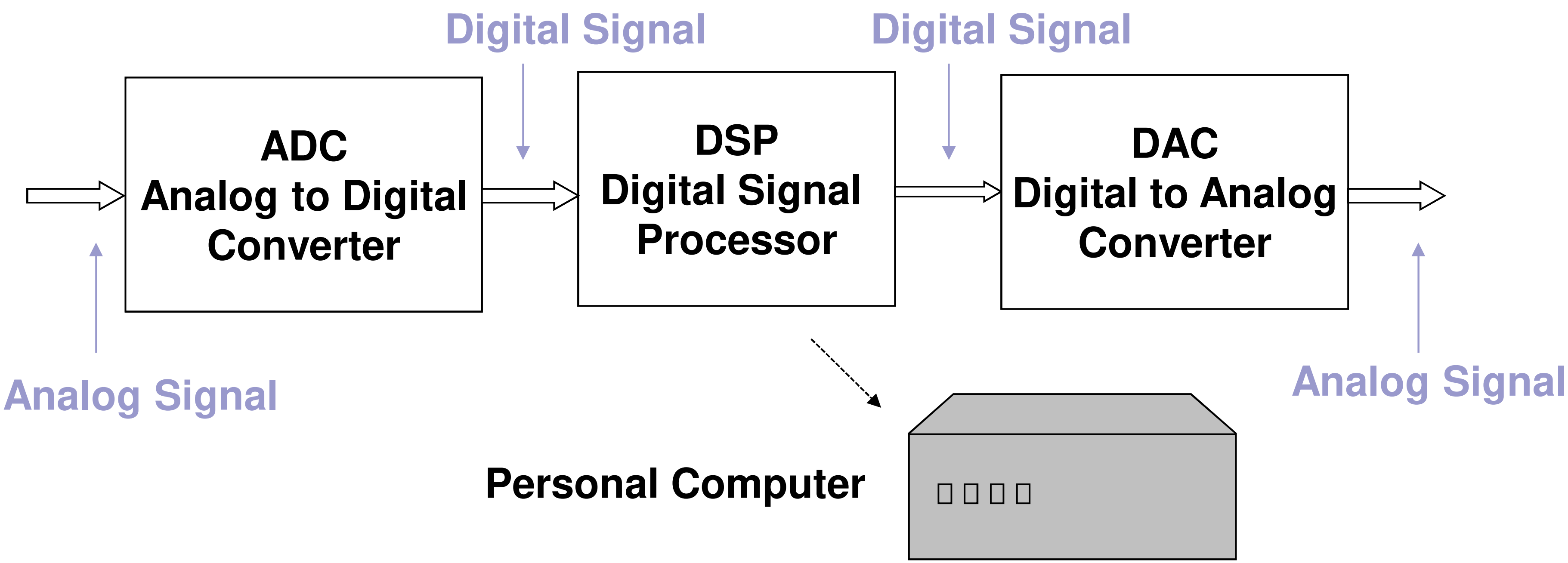
$$s(t) = 2BT_c \sum_{k=-\infty}^{+\infty} s(kT_c)\cdot \mathrm{sinc}\big(2B(t-kT_c)\big), \quad T_c = \frac{1}{2B}$$



Discrete Representation of Continuous-Time Signals

- **Discrete representation of continuous-time signals**.

- Discrete representation is useful for storing and processing the signals of interest $\rightarrow$ processing can be done digitally, by using a dedicated microprocessor, i.e. a **Digital Signal Processor (DSP)**.



Discrete Representation of Continuous-Time Signals

- Note that in both cases, the signal of interest is represented as a linear combination of some predefined continuous-time functions (**signal expansion**):

$$s(t) = \sum_{k=-\infty}^{+\infty} s_k \cdot \varphi_k(t)$$

- **For finite-power periodic signals**:

$$s_k \triangleq S_k = FS\{s(t)\}, \quad \varphi_k(t) \triangleq e^{j2\pi \frac{k}{T_0} t}, \quad k \in \mathbb{Z}$$

- **For bandlimited finite-energy aperiodic signals**:

$$s_k \triangleq \frac{2BT_c}{\sqrt{2B}} s(kT_c), \quad \varphi_k(t) \triangleq \sqrt{2B}\,\mathrm{sinc}\big(2B(t-kT_c)\big), \quad k \in \mathbb{Z}$$

Discrete Representation of Continuous-Time Signals

- This approach can be generalized beyond the FS or the Sampling Theorem results: **Generalized Fourier Analysis** or **Basis Expansion**, sometimes called **Karhunen-Loéve (KL) Expansion** or **Karhunen-Loéve Transform (KLT):**

$$s(t) = \sum_{k=-\infty}^{+\infty} s_k \cdot \varphi_k(t) \quad \overset{Basis}{\Leftrightarrow} \quad s_k$$

$$t \in \mathbb{R} \qquad\qquad\qquad\qquad k \in \mathbb{Z}$$

- The $\{\varphi_k(t)\}$'s are a set of predefined deterministic continuous-time signals called "**basis**" (or **dictionary**). Usually (but not always), they are **orthonormal functions**.

- The discrete representation of the continuous-time signal $s(t)$ is provided by the so-called **image** vector **s** (in general, infinite dimensional), whose components are the discrete values $\{s_k\}$, i.e. the coefficients of the linear combination:

$$s(t) \quad \overset{Basis}{\Leftrightarrow} \quad \mathbf{s} = [\ \cdots \quad s_{-1} \quad s_0 \quad s_1 \quad \cdots\ ]^T \quad \text{image vector}$$

- If the $\{\varphi_k(t)\}$'s are **orthonormal functions,** i.e.

$$\left\|\varphi_k(t)\right\|_2 = \sqrt{\left(\varphi_k(t), \varphi_k(t)\right)} = 1, \qquad \left(\varphi_k(t), \varphi_i(t)\right) = \delta_{k,i} = \begin{cases} 1, & k = i \\ 0, & k \neq i \end{cases}$$

$\rightarrow$ the elements of the **image** vector **s** can be derived as the **scalar product** (**orthogonal projection**) between the signal and the base functions:

$$s_k = \left(s(t), \varphi_k(t)\right), \quad \forall k$$

- For example, for **finite energy signals**, the scalar product can be defined as the cross-energy between the two signals and the Euclidean norm induced by the scalar product is the square root of the energy:

$$\left(s(t), x(t)\right) \triangleq \int_{-\infty}^{+\infty} s(t) x^*(t) dt = E_{sx}, \qquad \left\|s(t)\right\|_2 \triangleq \sqrt{\left(s(t), s(t)\right)} = \sqrt{\int_{-\infty}^{+\infty} \left|s(t)\right|^2 dt} = \sqrt{E_s}$$

Discrete Representation of Continuous-Time Signals

- Once we have defined the **scalar product** and the **norm** (induced by the scalar product), we can also define a **distance** between two signals:

$$d\left(s(t),x(t)\right) \triangleq \left\|s(t)-x(t)\right\|_2 = \sqrt{\left(s(t)-x(t),s(t)-x(t)\right)}$$

- For **finite power periodic signals**, the scalar product can be defined as the cross-power between the two signals and the Euclidean norm induced by the scalar product is the square root of the power:

$$\left(s(t),x(t)\right) \triangleq \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} s(t)x^*(t)dt = P_{sx}$$

$$\left\|s(t)\right\|_2 \triangleq \sqrt{\left(s(t),s(t)\right)} = \sqrt{\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \left|s(t)\right|^2 dt} = \sqrt{P_s}$$

$$d\left(s(t),x(t)\right) \triangleq \left\|s(t)-x(t)\right\|_2 = \sqrt{\left(s(t)-x(t),s(t)-x(t)\right)}$$

Discrete Representation of Continuous-Time Signals

- **Proof** of the result: $s_k = \left(s(t), \varphi_k(t)\right), \quad \forall k$
- **Assumption**: the $\{\varphi_k(t)\}$'s are **orthonormal** functions: $\left(\varphi_i(t), \varphi_k(t)\right) = \delta_{i,k}$

$$\left(s(t), \varphi_k(t)\right) = \left(\sum_{i=-\infty}^{+\infty} s_i \cdot \varphi_i(t), \varphi_k(t)\right)$$

$$= \sum_{i=-\infty}^{+\infty} s_i \cdot \left(\varphi_i(t), \varphi_k(t)\right)$$

$$= \sum_{i=-\infty}^{+\infty} s_i \cdot \delta_{i,k}$$

$$= s_k$$

- This proof makes use of the known property of the scalar product that:

$$\left(a \cdot s(t) + b \cdot q(t), \varphi_k(t)\right) = a \cdot \left(s(t), \varphi_k(t)\right) + b \cdot \left(q(t), \varphi_k(t)\right)$$

Discrete Representation of Continuous-Time Signals

- **Important**: if the signals are expanded on an orthonormal basis, the **scalar product** between two signals and their **norms** can be obtained by processing the image vectors as well.

- *Proof*:

$$s(t)=\sum_{k=-\infty}^{+\infty} s_k \cdot \varphi_k(t), \quad x(t)=\sum_{k=-\infty}^{+\infty} x_k \cdot \varphi_k(t)$$

$$\begin{aligned}
\left(s(t),x(t)\right) &= \left(\sum_{i=-\infty}^{+\infty} s_i \cdot \varphi_i(t), \sum_{k=-\infty}^{+\infty} x_k \cdot \varphi_k(t)\right) \\
&= \sum_{i=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty} s_i x_k^* \cdot \left(\varphi_i(t),\varphi_k(t)\right) \\
&= \sum_{i=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty} s_i x_k^* \delta_{i,k} \\
&= \sum_{i=-\infty}^{+\infty} s_i x_i^* = \mathbf{x}^H \mathbf{s} \equiv \mathbf{s}\cdot\mathbf{x} \equiv \left(\mathbf{s},\mathbf{x}\right)
\end{aligned}$$

Discrete Representation of Continuous-Time Signals

- Hence, whatever is the definition of scalar product we adopt, we have:

$$\left(s(t), x(t)\right) = \left(\mathbf{s}, \mathbf{x}\right)$$

- If we assume *x*(*t*)=*s*(*t*), we also get:

$$\left\|s(t)\right\|_2^2 = \left(s(t), s(t)\right) = \left(\mathbf{s}, \mathbf{s}\right) = \mathbf{s}^H \mathbf{s} = \left\|\mathbf{s}\right\|_2^2$$

- **In summary**: if the signals are expanded on an orthonormal basis, the scalar product between two signals and their norms can be obtained by processing the image vectors instead of processing the continuous-time signals.

- The Bessel-Parseval equality for the Fourier series is a particular case of this result:

$$\left\|s(t)\right\|_2^2 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \left|s(t)\right|^2 dt = \sum_{k=-\infty}^{+\infty} \left|S_k\right|^2 = \left\|\mathbf{s}\right\|_2^2$$

Discrete Representation of Continuous-Time Signals

- In all practical applications, we observe the signal on a limited time interval.

- If we assume as t=0 the initial observation time and we denote by T the length of the observation interval, we have that the signal is of finite duration T and we are interested on the discrete representation of $s(t)$ for $t \in [0, T)$.

- Since the signal has finite duration T, its energy is finite, but the bandwidth is theoretically infinite, e.g.

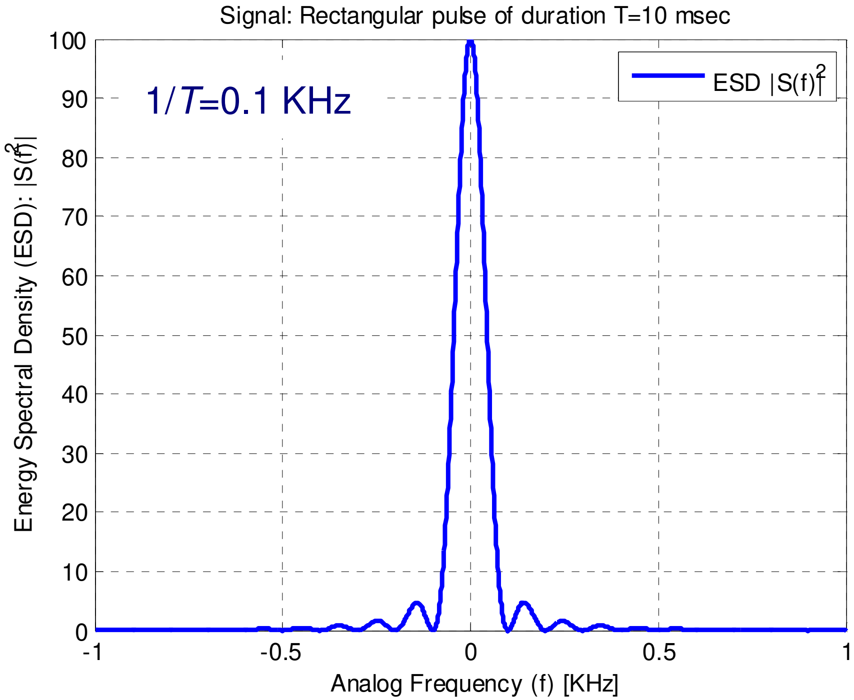
$$s(t) = A \cdot rect\left(\frac{t - T/2}{T}\right)$$

$$\Updownarrow$$

$$S(f) = AT \cdot \text{sinc}\left(fT\right) e^{-j\pi fT}$$

- Energy Spectral Density (ESD):

$$ESD(f) = \left|S(f)\right|^2 = A^2 T^2 \cdot \text{sinc}^2\left(fT\right)$$

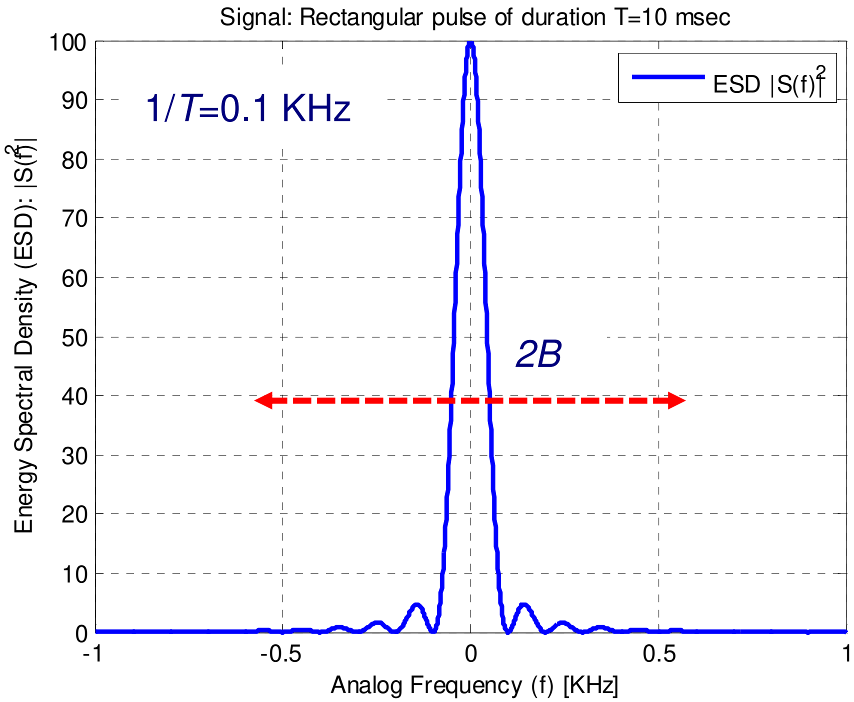


- Energy of the signal:

$$E_s = \int_0^T |s(t)|^2 \, dt = \int_{-\infty}^{+\infty} |S(f)|^2 \, df$$

- However, even if $s(t)$ is theoretically infinite-bandwidth, most of the energy is contained within a limited bandwidth B:

$$E_B = \int_{-B}^{B} |S(f)|^2 \, df = 2\int_0^B |S(f)|^2 \, df$$



- If B is large enough: $E_B \cong E_s$

Discrete Representation of Continuous-Time Signals

- To represent in discrete form a continuous-time signal observed in $[0, T)$ we can apply the following result: It can be proved that if we choose B large enough, such that $\boldsymbol{N=T/T_c=2BT>>1}$, then:

$$\text{for} \quad t \in [0,T) \quad \rightarrow \quad s(t) \cong \sum_{k=0}^{N-1} s\left(\frac{k}{2B}\right) \cdot \operatorname{sinc}\left(2B\left(t-\frac{k}{2B}\right)\right) = \sum_{k=0}^{N-1} s_k \cdot \varphi_k(t)$$

$$\text{where:} \quad \varphi_k(t) \triangleq \sqrt{2B}\,\operatorname{sinc}\left(2B\left(t-\frac{k}{2B}\right)\right), \quad s_k \triangleq \frac{1}{\sqrt{2B}} s\left(\frac{k}{2B}\right), \quad 0 \le k \le N-1$$

$$Basis: \quad \boldsymbol{\Psi} \equiv \left\{\varphi_k(t) = \sqrt{2B}\,\operatorname{sinc}\left(2B\left(t-\frac{k}{2B}\right)\right)\right\}_{k=0}^{N-1}, \quad \left(\varphi_k(t), \varphi_i(t)\right) = \delta_{k,i}$$

$$s(t) \quad \overset{\boldsymbol{\Psi}}{\Leftrightarrow} \quad \mathbf{s} = [\, s_0 \quad s_1 \quad \cdots \quad s_{N-2} \quad s_{N-1} \,]^T$$

Discrete Representation of Continuous-Time Signals

- The *approximation* should be interpreted as follows $\rightarrow$ if we define the "error" signal as:

$$s_\Delta(t) \triangleq s(t) - \hat{s}(t) = s(t) - \sum_{k=0}^{N-1} s\left(\frac{k}{2B}\right) \cdot \mathrm{sinc}\left(2B\left(t - \frac{k}{2B}\right)\right)$$

$$E_s \triangleq \int_0^T |s(t)|^2\, dt \quad [\text{Energy of } s(t)]$$

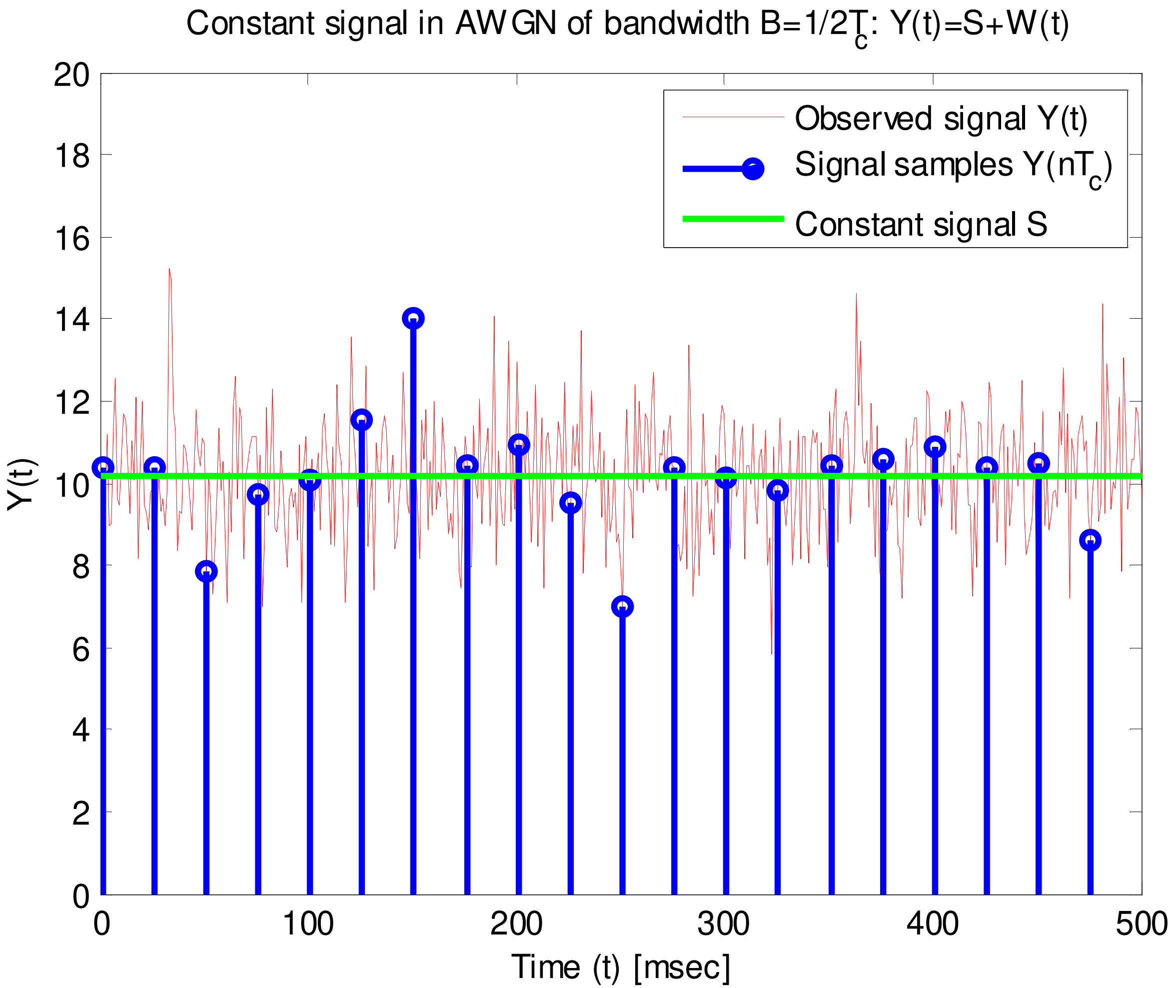
$$E_\Delta \triangleq \int_{-\infty}^{+\infty} |s_\Delta(t)|^2\, dt \quad [\text{Energy of } s_\Delta(t)]$$

If B is large enough, such that $N = \dfrac{T}{T_c} = 2BT \gg 1$

then $E_\Delta \leq \alpha E_s$ with $\alpha \ll 1$

- In this case, we say that *X*(*t*) has *essential* bandwidth *B* at level α and *B* is the <u>*essential* bandwidth at level α</u> of the signal *X*(*t*) .

Discrete Representation of Continuous-Time Signals



$$T = 500\ m\sec$$

$$T_c = \frac{1}{2B} = 25\ m\sec$$

$$N = 2BT = 20$$

The Discrete Karhunen-Loéve Transform (DKLT)

Basis Expansion of Discrete-Time Signals

- The tendency nowadays is to sample the signal as soon as possible in the processing chain and perform all the processing digitally.
- If *T* is the observation interval and *B* the bandwidth of the signal (i.e of the anti-aliasing filter that is at the input of the Analog-to-Digital Converter (ADC)), we have seen that if the sampling interval is such that $T_c \leq 1/(2B)$ *and* $N = T/T_c >> 1$, then we have:

$$X(t) \cong 2BT_c \sum_{k=0}^{N-1} X(kT_c) \cdot \mathrm{sinc}\big(2B(t - kT_c)\big), \quad t \in [0, T)$$

$$\mathbf{\Psi} \equiv \left\{ \varphi_k(t) = \sqrt{2B}\,\mathrm{sinc}\big(2B(t - kT_c)\big) \right\}_{k=0}^{N-1}, \qquad N = \frac{T}{T_c}$$

$$X(t) \overset{\mathbf{\Psi}}{\Leftrightarrow} \mathbf{X} = \frac{2BT_c}{\sqrt{2B}} \left[X(0)\, X(T_c) \cdots X\big((N-1)T_c\big) \right]^T$$

Basis Expansion of Discrete-Time Signals

- If we choose the **Nyquist sampling interval** $T_c=1/(2B)$, we have:

$$\mathbf{\Psi} \equiv \left\{ \varphi_k(t) = \sqrt{2B}\,\mathrm{sinc}\left(2B\left(t - \frac{k}{2B} \right)\right)\right\}_{k=0}^{N-1}, \qquad N = \frac{T}{T_c} = 2BT$$

- The finite-energy (because of the finite duration N) discrete-time signal we obtain by sampling is:

$$\mathbf{X} = \left[X[0]\ X[1]\ \cdots\ X[N-1] \right]^T, \text{ where } X[n] \triangleq \frac{1}{\sqrt{2B}} X\left(\frac{n}{2B} \right)$$

- $N=2BT>>1$ is the dimension of the image vector **X** and we also say that N is the <u>dimension</u> of the continuous-time signal $X(t)$.

Basis Expansion of Discrete-Time Signals

- In many cases, it can be useful to expand the discrete-time signal obtained by sampling into a set of deterministic **discrete-time orthonormal basis** functions as follows:

$$X[n] = \sum_{i=0}^{K-1} \alpha_i \varphi_i[n], \qquad n = 0, 1, \cdots, N-1; \quad K \le N$$

- The basis is orthonormal if : $\left(\varphi_i[n], \varphi_k[n]\right) = \delta_{i,k} = \begin{cases} 1, & k = i \\ 0, & k \ne i \end{cases}$

inner product between two basis functions

- Note that if *K*<*N* the expansion represents a **data compression**:

$$\underset{N\times 1}{\mathbf{X}} \overset{\Psi}{\Leftrightarrow} \underset{K\times 1}{\boldsymbol{\alpha}}$$

Basis Expansion of Discrete-Time Signals

- In the case of finite duration *N, i.e.* finite-energy, discrete-time signals, the **inner product** and the induced **Euclidean norm** can be defined as follows:

$$\left(s[n], x[n]\right) \triangleq \sum_{n=0}^{N-1} s[n] x^*[n] = E_{sx} \quad (cross-energy)$$

$$\left\| s[n] \right\|_2 \triangleq \sqrt{\left(s[n], s[n]\right)} = \sqrt{\sum_{n=0}^{N-1} \left| s[n] \right|^2} = \sqrt{E_s} \quad \left(\sqrt{energy}\right)$$

- The coefficients in the expansion are derived as the inner product between *X*[*n*] and the basis functions:

$$\alpha_i = \left(X[n], \varphi_i[n] \right) = \sum_{n=0}^{N-1} X[n] \varphi_i^*[n] = \boldsymbol{\varphi}_i^H \mathbf{X}, \qquad i = 0, 1, \cdots, K-1.$$

Basis Expansion of Discrete-Time Signals

- **Proof** of the result: $\alpha_i = \left(X[n], \varphi_i[n] \right) \quad \forall k$
- **Assumption**: the $\{\varphi_k[n]\}$'s are **orthonormal** functions: $\left(\varphi_k[n], \varphi_i[n] \right) = \delta_{k,i}$

$$
\begin{aligned}
\left(X[n], \varphi_i[n] \right) &= \sum_{n=0}^{N-1} X[n] \varphi_i^*[n] \\
&= \sum_{n=0}^{N-1} \sum_{k=0}^{K-1} \alpha_k \varphi_k[n] \varphi_i^*[n] \\
&= \sum_{k=0}^{K-1} \alpha_k \sum_{n=0}^{N-1} \varphi_k[n] \varphi_i^*[n] \\
&= \sum_{k=0}^{K-1} \alpha_k \cdot \left(\varphi_k(t), \varphi_i(t) \right) \\
&= \sum_{k=0}^{K-1} \alpha_k \cdot \delta_{k,i} \\
&= \alpha_i
\end{aligned}
$$

Basis Expansion of Discrete-Time Signals

- **Problem**: Which is the choice of the basis that makes the coefficients α_i <u>uncorrelated</u>? Remember that if $X(t)$ is a random process the coefficients are r.v.'s.

$$\underset{N\times 1}{\mathbf{X}} = \left[X[0] \;\; X[1] \;\; \cdots \;\; X[N-1]\right]^T$$

$$\underset{K\times 1}{\boldsymbol{\alpha}} = \left[\alpha_0 \;\; \alpha_1 \;\; \cdots \;\; \alpha_{K-1}\right]^T$$

$$\underset{N\times 1}{\boldsymbol{\varphi}_i} = \left[\varphi_i[0] \;\; \varphi_i[1] \;\; \cdots \;\; \varphi_i[N-1]\right]^T$$

$$\rightarrow \quad \left(\varphi_k[n], \varphi_i[n]\right) = (\boldsymbol{\varphi}_k, \boldsymbol{\varphi}_i) = \boldsymbol{\varphi}_i^H \boldsymbol{\varphi}_k = \delta_{i,k}$$

$$\underset{N\times K}{\boldsymbol{\Phi}} = \left[\boldsymbol{\varphi}_0 \quad \boldsymbol{\varphi}_1 \quad \cdots \quad \boldsymbol{\varphi}_{K-1}\right], \qquad \text{if } K = N \rightarrow \text{unitary matrix} \quad \boldsymbol{\Phi}^{-1} = \boldsymbol{\Phi}^H$$

Basis Expansion of Discrete-Time Signals

- Which is the choice of the basis that makes the coefficients α_i uncorrelated?
- Assume $K=N$ and, without loss of generality, also that $E\{X[n]\}=0 \rightarrow E\{\alpha_i\}=0$.

$$X[n]=\sum_{i=0}^{N-1}\alpha_i\varphi_i[n], \qquad n=0,1,\cdots,N-1$$

$$\mathbf{X}=\sum_{i=0}^{N-1}\alpha_i\,\boldsymbol{\varphi}_i=\boldsymbol{\Phi\alpha} \qquad \rightarrow \qquad \boldsymbol{\Phi}^H\mathbf{X}=\boldsymbol{\Phi}^H\boldsymbol{\Phi\alpha}=\boldsymbol{\alpha}$$

$$\boldsymbol{\Phi}: \quad E\left\{\boldsymbol{\alpha\alpha}^H\right\}=\begin{bmatrix}\sigma_0^2 & 0 & \cdots & 0\\ 0 & \sigma_1^2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \sigma_{N-1}^2\end{bmatrix}=\boldsymbol{\Lambda}, \quad \text{where } \sigma_i^2 \triangleq E\left\{\left|\alpha_i\right|^2\right\}$$

$$\text{and } E\left\{\alpha_i\alpha_k^*\right\}=0 \text{ if } i\neq k$$

Basis Expansion of Discrete-Time Signals

$$\mathbf{\Phi}: \quad E\left\{\boldsymbol{\alpha}\boldsymbol{\alpha}^H\right\} = E\left\{\mathbf{\Phi}^H \mathbf{X}\mathbf{X}^H \mathbf{\Phi}\right\} = \mathbf{\Phi}^H E\left\{\mathbf{X}\mathbf{X}^H\right\}\mathbf{\Phi} = \mathbf{\Phi}^H \mathbf{R}_X \mathbf{\Phi} = \mathbf{\Lambda}$$

$$\mathbf{\Lambda} = \mathbf{\Phi}^H \mathbf{R}_X \mathbf{\Phi} \;\Rightarrow\; \mathbf{\Phi}\mathbf{\Lambda}\mathbf{\Phi}^H = \mathbf{\Phi}\mathbf{\Phi}^H \mathbf{R}_X \mathbf{\Phi}\mathbf{\Phi}^H \;\Rightarrow\; \mathbf{R}_X = \mathbf{\Phi}\mathbf{\Lambda}\mathbf{\Phi}^H$$

- Matrix $\mathbf{\Phi}$ diagonalizes the correlation matrix $\mathbf{R}_X$ $\rightarrow$ the basis that makes the coefficients uncorrelated is obtained from the **Eigenvector-Eigenvalue Decomposition (EVD)** of the correlation matrix $\mathbf{R}_X$ of the random vector **X** $\rightarrow \mathbf{\Phi}$ is formed by the eigenvectors of $\mathbf{R}_X$.

- The expansion on this basis is called **Karhunen-Loève expansion** and the vector of coefficients α is called **discrete Karhunen-Loève transform (DKLT)**.

Basis Expansion of Discrete-Time Signals

- Definition of the correlation matrix $\mathbf{R}_X$ of a random vector $\mathbf{X}$:

$$\mathbf{R}_X \triangleq E\left\{\mathbf{X}\mathbf{X}^H\right\} = E\left\{\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix}\begin{bmatrix} X_1^* & X_2^* & \cdots & X_N^* \end{bmatrix}\right\}$$

$$= E\left\{\begin{bmatrix} |X_1|^2 & X_1X_2^* & \cdots & X_1X_N^* \\ X_2X_1^* & |X_2|^2 & \cdots & X_2X_N^* \\ \vdots & \vdots & \ddots & \vdots \\ X_NX_1^* & X_NX_2^* & \cdots & |X_N|^2 \end{bmatrix}\right\}$$

Basis Expansion of Discrete-Time Signals

$$\mathbf{R}_X = \begin{bmatrix} E\left\{\left|X_1\right|^2\right\} & E\left\{X_1 X_2^*\right\} & \cdots & E\left\{X_1 X_N^*\right\} \\ E\left\{X_2 X_1^*\right\} & E\left\{\left|X_2\right|^2\right\} & \cdots & E\left\{X_2 X_N^*\right\} \\ \vdots & \vdots & \ddots & \vdots \\ E\left\{X_N X_1^*\right\} & E\left\{X_N X_2^*\right\} & \cdots & E\left\{\left|X_N\right|^2\right\} \end{bmatrix}$$

$$= \begin{bmatrix} E\left\{\left|X_1\right|^2\right\} & E\left\{X_1 X_2^*\right\} & \cdots & E\left\{X_1 X_N^*\right\} \\ \left(E\left\{X_1 X_2^*\right\}\right)^* & E\left\{\left|X_2\right|^2\right\} & \cdots & E\left\{X_2 X_N^*\right\} \\ \vdots & \vdots & \ddots & \vdots \\ \left(E\left\{X_1 X_N^*\right\}\right)^* & \left(E\left\{X_2 X_N^*\right\}\right)^* & \cdots & E\left\{\left|X_N\right|^2\right\} \end{bmatrix} = \mathbf{R}_X^H$$

Basis Expansion of Discrete-Time Signals

- If the data vector **X** is obtained by sampling a continuous-time wide-sense stationary (w.s.s.) process *X*(*t*):

$$X_{i+1} \triangleq [\mathbf{X}]_{i+1} = X[i] = \frac{1}{\sqrt{2B}} X\left(\frac{i}{2B}\right), \quad i = 0, 1, 2, \cdots, N-1$$

- ACF of the w.s.s discrete-time process *X*[*n*]

$$\left[\mathbf{R}_X\right]_{i+1, j+1} = E\left\{X_{i+1} X_{j+1}^*\right\} = E\left\{X[i] X^*[j]\right\} = R_X[j-i]$$

$$= \frac{1}{2B} E\left\{X\left(\frac{i}{2B}\right) X^*\left(\frac{j}{2B}\right)\right\} = \frac{1}{2B} R_X\left(\frac{j-i}{2B}\right)$$

$$i, j = 0, 1, 2, \cdots, N-1$$

- ACF of the w.s.s. continuous-time process *X*(*t*)

Basis Expansion of Discrete-Time Signals

- Hence, if the data vector **X** is obtained by sampling a continuous-time w.s.s. process $X(t)$ with Autocorrelation Function (ACF) $R_X(\tau)=E\{X(t)X^*(t+\tau)\}$, in addition to be Hermitian, the correlation matrix of **X** has a Toeplitz structure:

$$\mathbf{R}_X = \frac{1}{2B}\begin{bmatrix} R_X(0) & R_X\left(\frac{1}{2B}\right) & R_X\left(\frac{2}{2B}\right) & \cdots & R_X\left(\frac{N-2}{2B}\right) & R_X\left(\frac{N-1}{2B}\right) \\ R_X^*\left(\frac{1}{2B}\right) & R_X(0) & R_X\left(\frac{1}{2B}\right) & \cdots & R_X\left(\frac{N-3}{2B}\right) & R_X\left(\frac{N-2}{2B}\right) \\ R_X^*\left(\frac{2}{2B}\right) & R_X^*\left(\frac{1}{2B}\right) & R_X(0) & \cdots & R_X\left(\frac{N-4}{2B}\right) & R_X\left(\frac{N-3}{2B}\right) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ R_X^*\left(\frac{N-2}{2B}\right) & R_X^*\left(\frac{N-3}{2B}\right) & R_X^*\left(\frac{N-4}{2B}\right) & \cdots & R_X(0) & R_X\left(\frac{1}{2B}\right) \\ R_X^*\left(\frac{N-1}{2B}\right) & R_X^*\left(\frac{N-2}{2B}\right) & R_X^*\left(\frac{N-3}{2B}\right) & \cdots & R_X^*\left(\frac{1}{2B}\right) & R_X(0) \end{bmatrix}$$

- The **discrete Karhunen-Loève transform (DKLT)** is a **linear whitening (decorrelating) transformation** of the original data vector **X**:

$$\underset{N\times 1}{\mathbf{X}} \overset{DKLT}{\Leftrightarrow} \underset{N\times 1}{\boldsymbol{\alpha}}$$

$$\boldsymbol{\alpha} = \boldsymbol{\Phi}^H \mathbf{X}$$

$$\boldsymbol{\Phi}: \quad E\left\{\boldsymbol{\alpha}\boldsymbol{\alpha}^H\right\} = \begin{bmatrix} \sigma_0^2 & 0 & \cdots & 0 \\ 0 & \sigma_1^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{N-1}^2 \end{bmatrix}, \quad \text{where} \quad \sigma_i^2 \triangleq E\left\{\left|\alpha_i\right|^2\right\}$$

- In many receivers, the first operation that is performed is a **linear whitening transformation** of the data vector **X**.

- The **DKLT** of the original data vector **X** requires knowledge of the correlation matrix $\mathbf{R}_X$ of the random vector **X**, i.e. of the ACF of the w.s.s. process *X*(*t*).

- Φ is obtained from the **Eigenvector-Eigenvalue Decomposition (EVD)** of the correlation matrix $\mathbf{R}_X \rightarrow \Phi$ is formed by the eigenvectors of $\mathbf{R}_X$:

$$\mathbf{R}_X = \mathbf{\Phi}\mathbf{\Lambda}\mathbf{\Phi}^H$$

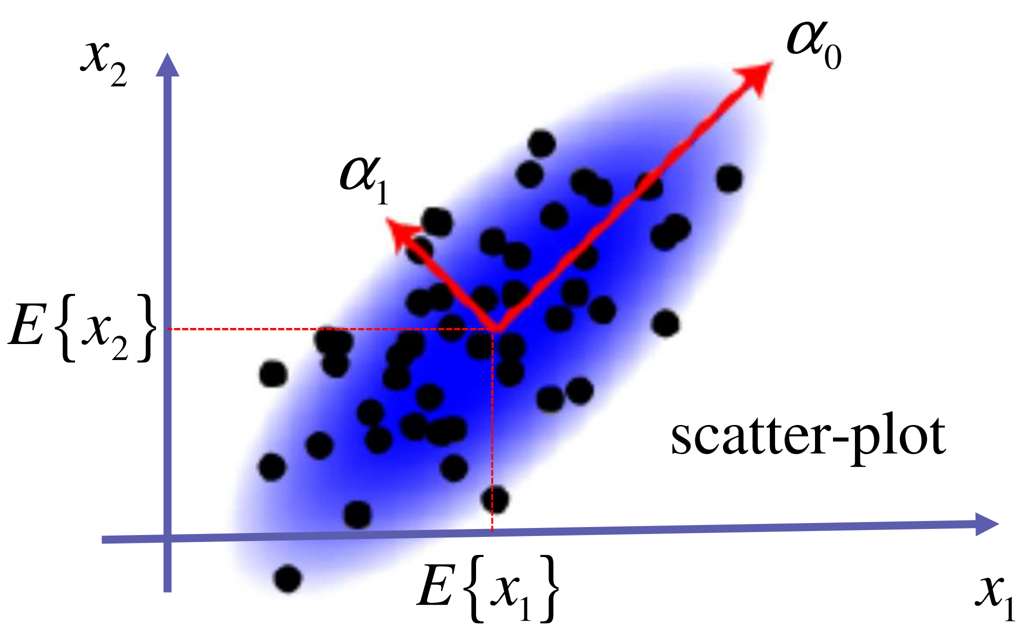
- In many applications, we do not know a priori the correlation matrix $\mathbf{R}_X$, so we have to estimate it from the data:

$$\hat{\mathbf{R}}_X \quad \Rightarrow \quad \left(\hat{\mathbf{\Phi}}, \hat{\mathbf{\Lambda}}\right) \text{ from the EVD of } \hat{\mathbf{R}}_X : \quad \hat{\mathbf{R}}_X = \hat{\mathbf{\Phi}}\hat{\mathbf{\Lambda}}\hat{\mathbf{\Phi}}^H$$

- We will see later on how to estimate the correlation matrix of an *N*-dimensional random vector **X** given a number *M*>*N* of independent realizations of **X**.

Basis Expansion of Discrete-Time Signals

- Φ is a unitary matrix and $\alpha = \Phi^H \mathbf{X}$ is a linear transformation of the data.
- It is basically a rotation of the reference system in such a way that the components in the new reference system are uncorrelated:
- If **X** is non-zero mean, we can remove the mean, and Φ is obtained by the EVD of the covariance matrix of vector **X.**



$$\boldsymbol{\alpha} = \boldsymbol{\Phi}^H \left(\mathbf{X} - \mathbf{m}_X \right)$$

where $\mathbf{m}_X \triangleq E\{\mathbf{X}\}$,

$$\boldsymbol{\Phi}: \quad \mathbf{C}_X \triangleq E\left\{ \left(\mathbf{X} - \mathbf{m}_X \right) \left(\mathbf{X} - \mathbf{m}_X \right)^H \right\} = \boldsymbol{\Phi} \boldsymbol{\Lambda} \boldsymbol{\Phi}^H$$

Basis Expansion of Discrete-Time Signals

- **Important**: It is well-known that if the correlation time *L* of the discrete-time sequence *X*[*n*] is much lower than the vector size *N*, then the matrix Φ that diagonalizes the correlation matrix $\mathbf{R}_X$ is (approximately) the DFT matrix and the DKLT coincides with the (unitary) **Discrete Fourier Transform (DFT)**:

$$L \ll N \quad \rightarrow \quad \varphi_i[n] \cong \frac{1}{\sqrt{N}} e^{j2\pi \frac{i}{N} n}; \quad i, n = 0, 1, \cdots, N-1$$

$$\mathbf{X} = \mathbf{\Phi}\boldsymbol{\alpha} \cong IDFT\{\boldsymbol{\alpha}\} \quad \Leftrightarrow \quad \boldsymbol{\alpha} = \mathbf{\Phi}^H \mathbf{X} \cong DFT\{\mathbf{X}\}$$

$$X[n] \cong \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} \alpha_i e^{j2\pi \frac{i}{N} n} \qquad \alpha_i = \left(X[n], \varphi_i[n]\right) \cong \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} X[n]\, e^{-j2\pi \frac{i}{N} n}$$

Basis Expansion of Discrete-Time Signals

$$\mathbf{\Phi} = \frac{1}{\sqrt{N}} \underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{j\frac{2\pi}{N}} & \cdots & e^{j\frac{2\pi}{N}(N-1)} \\ 1 & e^{j\frac{2\pi}{N}2} & \cdots & e^{j\frac{2\pi}{N}2(N-1)} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & e^{j\frac{2\pi}{N}(N-1)} & \cdots & e^{j\frac{2\pi}{N}(N-1)(N-1)} \end{bmatrix}}_{DFT\ matrix \quad N\times N}$$

$$\mathbf{\Phi}^{-1} = \mathbf{\Phi}^{H}$$

Unitary matrix

- Moreover, if *N*>>*L*, the eigenvalues of $\mathbf{R}_x$ are the samples of the Power Spectral Density (PSD):

$$\sigma_i^2 = E\left\{\left|\alpha_i\right|^2\right\} \overset{N\gg L}{\cong} S_X\left(e^{j2\pi\frac{i}{N}}\right) = DFT\left\{R_X[m]\right\}, \quad i = 0,1,2,\cdots,N-1$$

- *Proof*:

$$\sigma_i^2 = E\left\{\left|\alpha_i\right|^2\right\} = E\left\{\alpha_i \alpha_i^*\right\}$$

$$= E\left\{\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} X[n]\, e^{-j2\pi\frac{i}{N}n}\, \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} X^*[k]\, e^{j2\pi\frac{i}{N}k}\right\}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} E\left\{X[n]X^*[k]\right\} e^{-j2\pi\frac{i}{N}(n-k)}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} R_X[n-k] e^{-j2\pi\frac{i}{N}(n-k)} = \frac{1}{N}\sum_{m=-(N-1)}^{N-1}\left(N-|m|\right) R_X[m] e^{-j2\pi\frac{i}{N}m}$$

$$= \sum_{m=-(N-1)}^{N-1}\left(1-\frac{|m|}{N}\right) R_X[m] e^{-j2\pi\frac{i}{N}m} \overset{N \gg L}{\cong} \sum_{m=-L}^{L} R_X[m] e^{-j2\pi\frac{i}{N}m}$$

$$= DFT\left\{R_X[m]\right\} = S_X\left(e^{j2\pi\frac{i}{N}}\right), \quad i = 0, 1, 2, \cdots, N-1$$

Basis Expansion of Discrete-Time Signals

- Why is it important to come up with a discrete-representation where the components of the image vector are uncorrelated?

- This can be easily understood if for example the process $X(t)$ is Gaussian. In this case, also the discrete-time process $X[n]$ obtained by sampling is **Gaussian** and, as a consequence, also the image vector α obtained as a linear transformation of the image vector $\mathbf{X}$.

- We know that for Gaussian random vectors, uncorrelation implies independence (remember that independence always implies uncorrelation, but the reverse is usually not true). Hence, we can easily derive the joint **probability density function (pdf)** of the image vector α as the product of the marginal pdf's of the components of α:

$$f_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}) = \prod_{i=0}^{N-1} f_{\alpha_i}(\alpha_i), \quad \text{where } N = 2BT$$

Principal Component Analysis (PCA)

- Sometimes the **DKLT** is called **Principal Component Analysis (PCA)** and is used for dimensionality reduction (or redundancy reduction):

$$X[n] = \sum_{i=0}^{N-1} \alpha_i \varphi_i[n] \cong \sum_{i=0}^{K-1} \alpha_i \varphi_i[n]$$

$$\underset{N\times 1}{\mathbf{X}} \overset{PCA}{\Leftrightarrow} \underset{K\times 1}{\boldsymbol{\alpha}}$$

where :

1) $\sigma_i^2 \triangleq E\left\{\left|\alpha_i\right|^2\right\}, \quad \sigma_i^2 \geq \sigma_{i+1}^2$ (decreasing order)

2) $E\left\{\alpha_i \alpha_j^*\right\} = 0$ for $i \neq j$

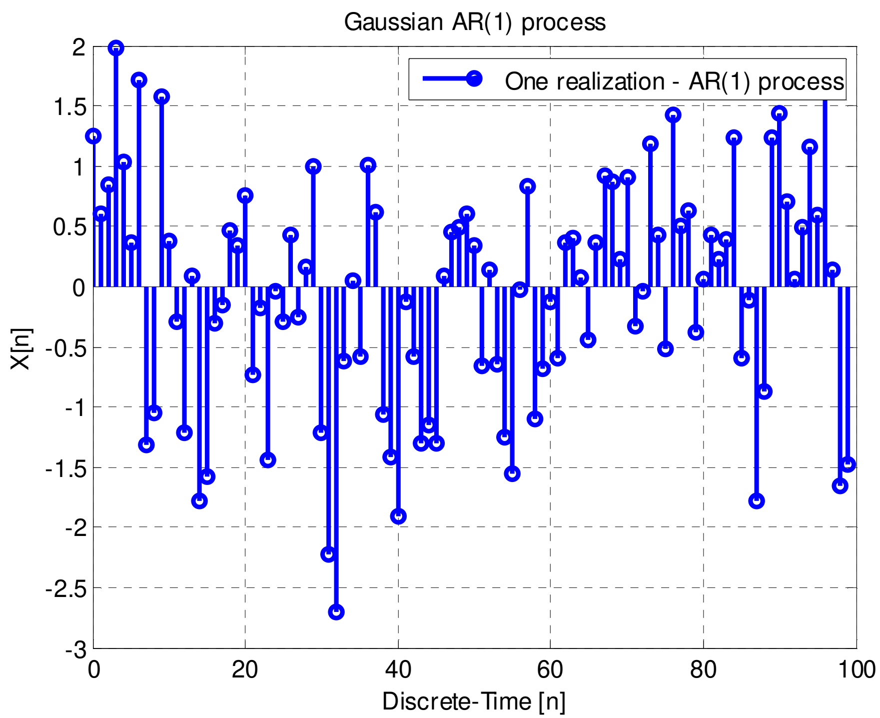
3) $K < N$ such that $\sum_{i=0}^{K-1} \sigma_i^2 \gg \sum_{i=K}^{N-1} \sigma_i^2$

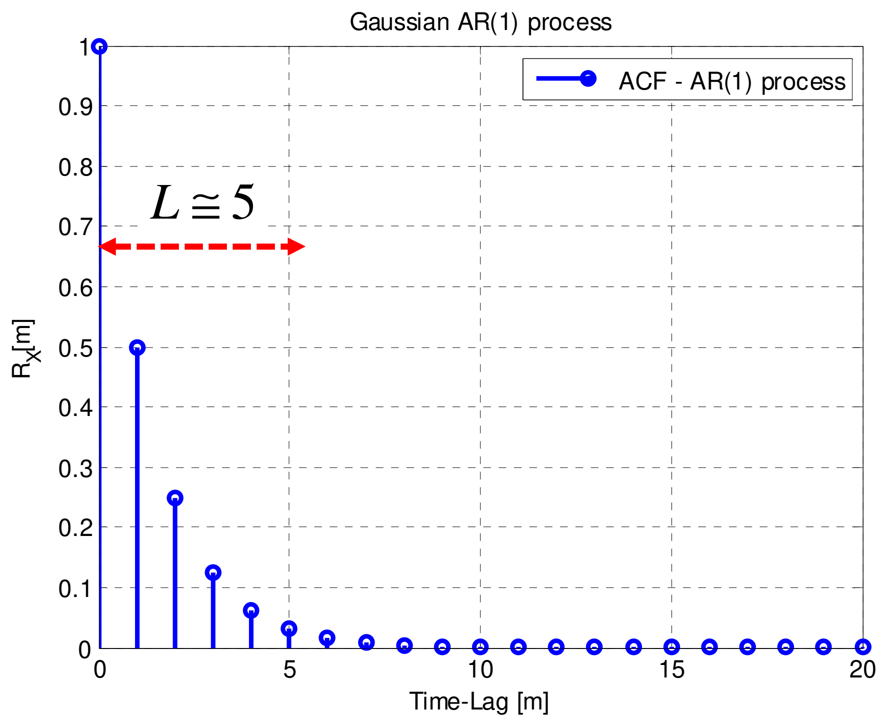
- The best choice of *K* basis functions (dimensionality reduction) corresponds to the *K* eigenvectors of $\mathbf{R}_X$ with the largest eigenvalues (the principal components).

Basis Expansion of Discrete-Time Signals

- *Example*: $X[n]$ random process with autocorrelation function (ACF):

$$R_X[m] = E\left\{X[n]X^*[n+m]\right\} = (0.5)^{|m|} \quad \rightarrow \quad L \cong 5 \ll N = 100$$



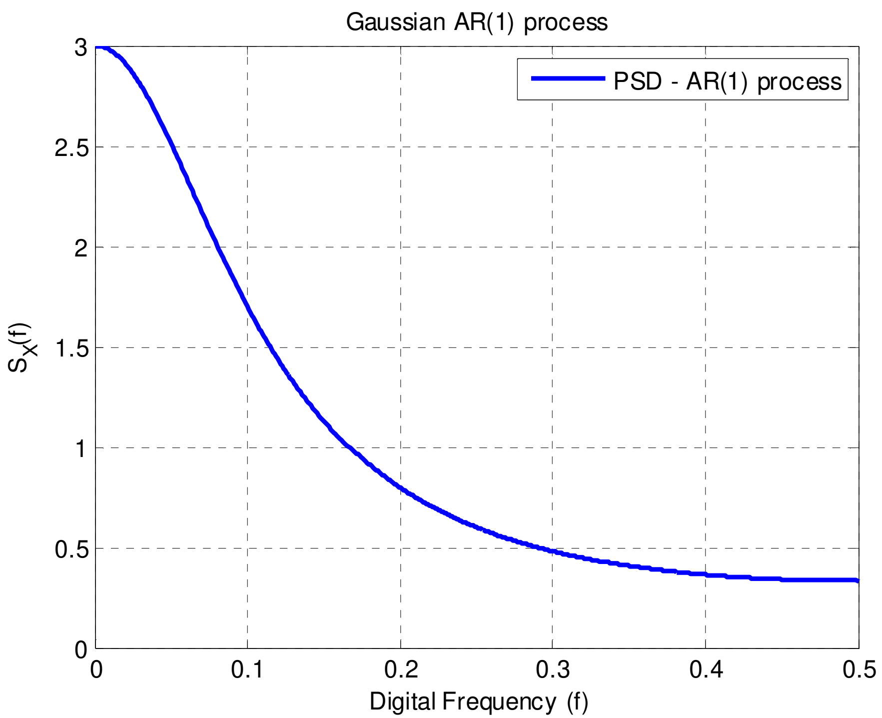


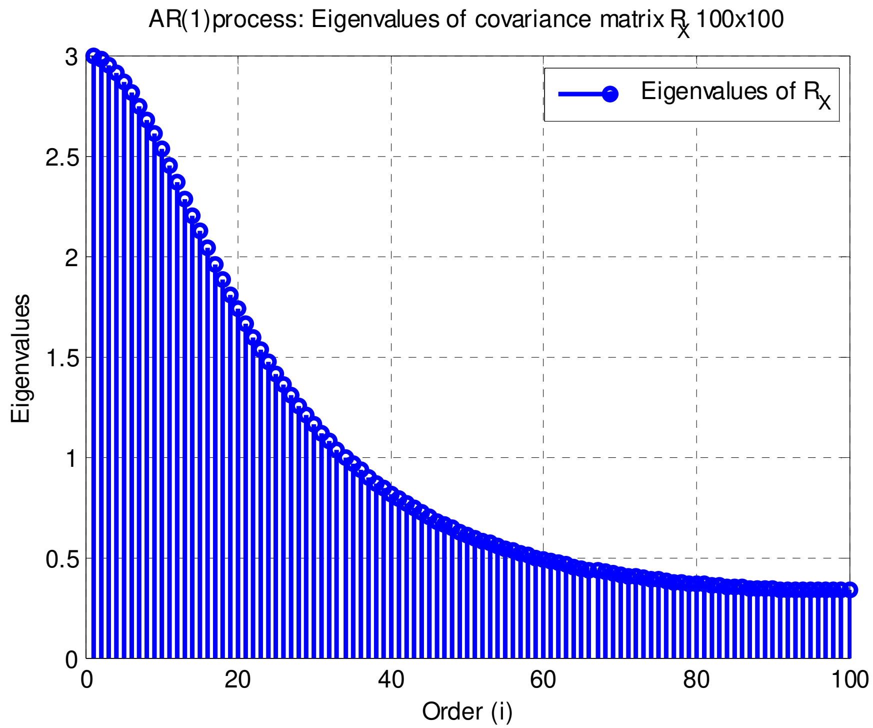
Basis Expansion of Discrete-Time Signals

$$S_X\left(e^{j2\pi f}\right) = DTFT\left\{R_X[m]\right\}$$

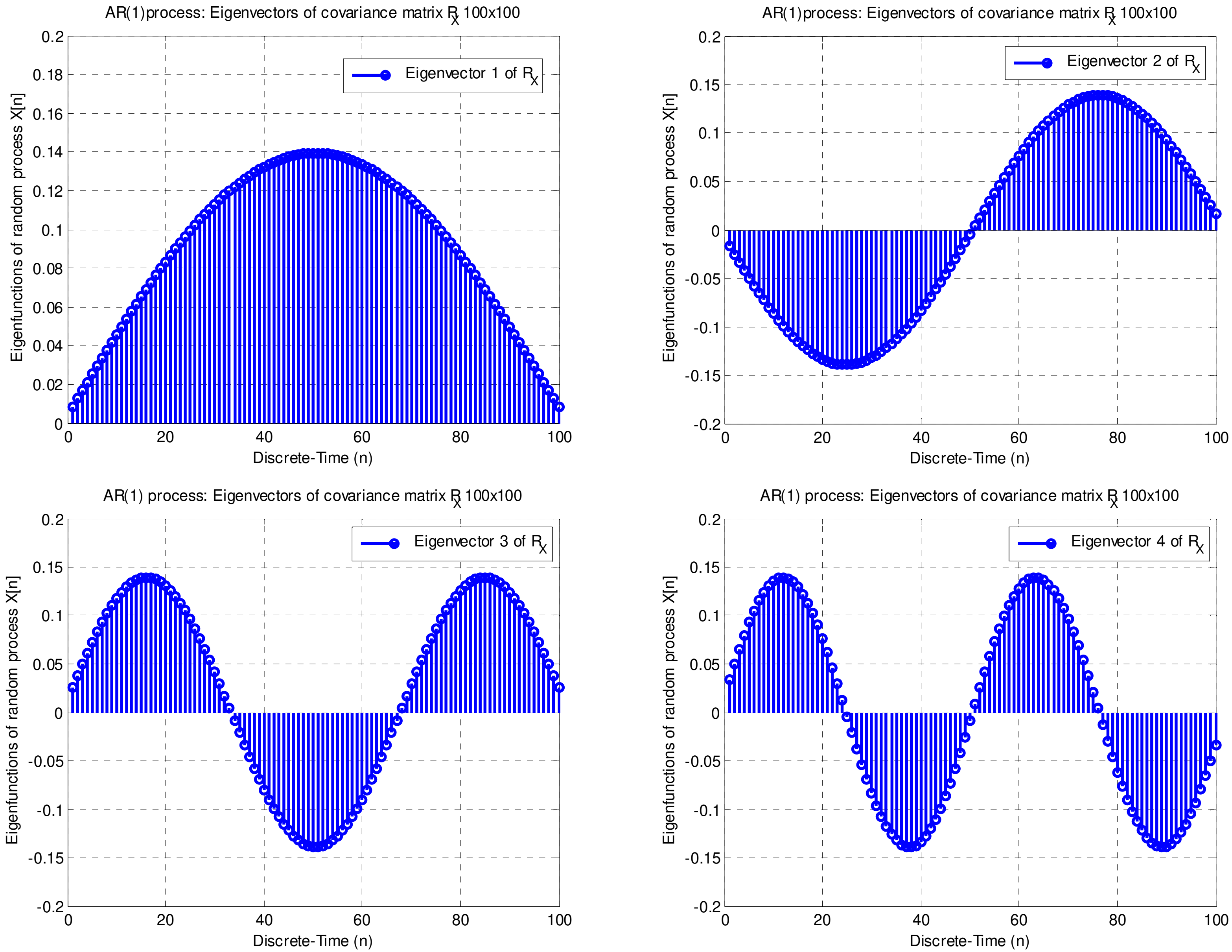
$$Eigenvalues: \quad \sigma_i^2 \triangleq E\left\{\left|\alpha_i\right|^2\right\}$$

in decreasing order





Basis Expansion of Discrete-Time Signals



Basis Expansion of Discrete-Time Signals

- **Example**: $X(t) = A\cos\left(2\pi f_0 t + \theta\right) + W(t), \quad t \in [0, T),$ $\qquad S_W(f) = \dfrac{N_0}{2}$

  signal of interest $s(t)$

  Additive White Gaussian Noise (AWGN)

- A typical problem is to estimate the signal parameters (amplitude and phase) based on the observation of a noise corrupted version of the signal in the interval $[0, T)$.

- The signal frequency f_0 may be a priori known or unknown. Let's assume first that it is <u>a-priori known</u>.

- After anti-aliasing filtering with bandwidth $B=1/(2T_c)$ and ADC:

$$X[n] = A\cos\left(2\pi F_0 n + \theta\right) + W[n], \quad n = 0, 1, \cdots, N-1$$

$$\text{where} \quad F_0 \triangleq f_0 T_c \in (0, 1/2), \qquad N = T/T_c = 2BT, \qquad W[n] \in \mathcal{N}\left(0, \sigma_W^2\right) \quad \text{IID}$$

- The noise samples are Independent and Identically distributed (IID).

Basis Expansion of Discrete-Time Signals

- The discrete-time signal obtained by sampling can be expanded on the orthonormal basis functions as follows:

$$
\begin{aligned}
X[n] &= A\cos\left(2\pi F_0 n+\theta\right)+W[n] \\
&= A\cos\left(\theta\right)\cos\left(2\pi F_0 n\right)-A\sin\left(\theta\right)\sin\left(2\pi F_0 n\right)+W[n] \\
&= \sqrt{\frac{N}{2}}A\cos\left(\theta\right)\varphi_0[n]-\sqrt{\frac{N}{2}}A\sin\left(\theta\right)\varphi_1[n]+W[n]
\end{aligned}
$$

- where:

$$
\varphi_0[n]\triangleq\sqrt{\frac{2}{N}}\cos\left(2\pi F_0 n\right),\quad \varphi_1[n]\triangleq\sqrt{\frac{2}{N}}\sin\left(2\pi F_0 n\right),\quad n=0,1,\cdots,N-1
$$

$$
\left(\varphi_i,\varphi_k\right)\triangleq\sum_{n=0}^{N-1}\varphi_i[n]\varphi_k^*[n]=\delta_{i,k}\quad \text{(Kronecker's delta)}\quad \Leftrightarrow\quad N\gg 1
$$

- In fact, if N>>1:

$$(\varphi_0,\varphi_0)=\sum_{n=0}^{N-1}\sqrt{\frac{2}{N}}\cos(2\pi F_0 n)\sqrt{\frac{2}{N}}\cos(2\pi F_0 n)=\frac{2}{N}\sum_{n=0}^{N-1}\cos^2(2\pi F_0 n)$$

$$=\frac{2}{N}\sum_{n=0}^{N-1}\left(\frac{1}{2}+\frac{1}{2}\cos(4\pi F_0 n)\right)=1+\frac{1}{N}\sum_{n=0}^{N-1}\cos(4\pi F_0 n)\cong 1$$

$$(\varphi_1,\varphi_1)=\sum_{n=0}^{N-1}\sqrt{\frac{2}{N}}\sin(2\pi F_0 n)\sqrt{\frac{2}{N}}\sin(2\pi F_0 n)=\frac{2}{N}\sum_{n=0}^{N-1}\sin^2(2\pi F_0 n)$$

$$=\frac{2}{N}\sum_{n=0}^{N-1}\left(\frac{1}{2}-\frac{1}{2}\cos(4\pi F_0 n)\right)=1-\frac{1}{N}\sum_{n=0}^{N-1}\cos(4\pi F_0 n)\cong 1$$

$$(\varphi_0,\varphi_1)=\sum_{n=0}^{N-1}\sqrt{\frac{2}{N}}\cos(2\pi F_0 n)\sqrt{\frac{2}{N}}\sin(2\pi F_0 n)=\frac{2}{N}\sum_{n=0}^{N-1}\frac{1}{2}\sin(4\pi F_0 n)$$

$$=\frac{1}{N}\sum_{n=0}^{N-1}\sin(4\pi F_0 n)\cong 0$$

Basis Expansion of Discrete-Time Signals

- The other *N*-2 orthonormal basis functions should be chosen to be unit-norm and orthogonal to the first two (we do not need to specify them!).

- The discrete-time signal can be expanded by using an orthonormal basis set and the coefficients of the expansion are derived as follows:

$$\alpha_k = \left(X[n], \varphi_k[n] \right) = \sum_{n=0}^{N-1} X[n] \varphi_k^*[n]$$

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_{N-1} \end{bmatrix}^T$$

$$\begin{aligned}
\alpha_k &= \left(X[n], \varphi_k[n] \right) = \sum_{n=0}^{N-1} X[n] \varphi_k^*[n] \\
&= \sum_{n=0}^{N-1} \left(\sqrt{\frac{N}{2}} A\cos(\theta) \varphi_0[n] - \sqrt{\frac{N}{2}} A\sin(\theta) \varphi_1[n] + W[n] \right) \varphi_k^*[n] \\
&= \sqrt{\frac{N}{2}} A\cos(\theta) \sum_{n=0}^{N-1} \varphi_0[n] \varphi_k^*[n] - \sqrt{\frac{N}{2}} A\sin(\theta) \sum_{n=0}^{N-1} \varphi_1[n] \varphi_k^*[n] + \sum_{n=0}^{N-1} W[n] \varphi_k^*[n]
\end{aligned}$$

Basis Expansion of Discrete-Time Signals

$$\alpha_k = \sqrt{\frac{N}{2}} A\cos(\theta)\,\delta_{0,k} - \sqrt{\frac{N}{2}} A\sin(\theta)\,\delta_{1,k} + W_k, \quad k = 0,1,\cdots,N-1$$

- where: $W_k \triangleq \left(W[n],\varphi_k[n]\right) = \sum_{n=0}^{N-1} W[n]\varphi_k^*[n]$

$$E\{W_k\} = \sum_{n=0}^{N-1} E\{W[n]\}\varphi_k^*[n] = 0$$

$$\boldsymbol{\alpha} = \mathbf{s} + \mathbf{W} = \underbrace{\begin{bmatrix} \sqrt{\frac{N}{2}} A\cos(\theta) \\ -\sqrt{\frac{N}{2}} A\sin(\theta) \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\mathbf{s}} + \underbrace{\begin{bmatrix} W_0 \\ W_1 \\ W_2 \\ \vdots \\ W_{N-1} \end{bmatrix}}_{\mathbf{W}}$$

Basis Expansion of Discrete-Time Signals

$$E\left\{W_k W_i^*\right\} = E\left\{\sum_{n=0}^{N-1} W[n]\varphi_k^*[n] \sum_{l=0}^{N-1} W^*[l]\varphi_i\,[l]\right\}$$

$$= \sum_{n=0}^{N-1}\sum_{l=0}^{N-1} E\left\{W[n]W^*[l]\right\}\varphi_i\,[l]\varphi_k^*[n]$$

$$= \sum_{n=0}^{N-1}\sum_{l=0}^{N-1} \sigma_W^2 \delta[n-l]\varphi_i\,[l]\varphi_k^*[n]$$

$$= \sigma_W^2 \sum_{n=0}^{N-1} \varphi_i\,[n]\varphi_k^*[n] = \sigma_W^2 \delta_{i,k}$$

$$\Rightarrow \quad W_k \in \mathcal{N}\left(0, \sigma_W^2\right) \quad \text{IID,} \qquad \text{where} \quad \sigma_W^2 = \frac{N_0}{2T_c} = 2B\frac{N_0}{2} = N_0 B$$

Basis Expansion of Discrete-Time Signals

- Only the first two components bring information on the amplitude and phase of the useful signal. The other components are **irrelevant**, so we do not need to compute them:

$$\alpha_0 = \left(X[n], \varphi_0[n] \right) = \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} X[n] \cos\left(2\pi F_0 n\right)$$

$$\alpha_1 = \left(X[n], \varphi_1[n] \right) = \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} X[n] \sin\left(2\pi F_0 n\right)$$

- This operation is equivalent to the extraction of the **In-phase (I)** and **Quadrature (Q) components**, typically called "**demodulation**", but in digital fashion, after the ADC.

Basis Expansion of Discrete-Time Signals

- Note that the components of the transformed vector α are uncorrelated:

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{N-1} \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{N}{2}} A\cos(\theta) \\ -\sqrt{\frac{N}{2}} A\sin(\theta) \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} W_0 \\ W_1 \\ W_2 \\ \vdots \\ W_{N-1} \end{bmatrix} \quad \rightarrow \quad E\{\boldsymbol{\alpha}\} = E\{\mathbf{s}+\mathbf{W}\} = \mathbf{s}$$

$$\operatorname{cov}(\alpha_k, \alpha_i) = E\left\{(\alpha_k - s_k)(\alpha_i - s_i)^*\right\} = E\left\{W_k W_i^*\right\} = \sigma_W^2 \delta_{i,k}$$

$$\rightarrow \quad \operatorname{cov}(\alpha_k, \alpha_i) = 0 \quad \text{for} \quad i \neq k$$

Basis Expansion of Discrete-Time Signals

- The correlation matrix of vector α is given by:

$$\mathbf{R}_\alpha = E\left\{\boldsymbol{\alpha}\boldsymbol{\alpha}^H\right\} = \begin{bmatrix} \sigma_0^2 & r_{01} & \cdots & 0 \\ r_{10} & \sigma_1^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{N-1}^2 \end{bmatrix} = \mathbf{C}_\alpha + \boldsymbol{\eta}_\alpha \boldsymbol{\eta}_\alpha^H = \sigma_W^2 \mathbf{I} + \mathbf{s}\mathbf{s}^H$$

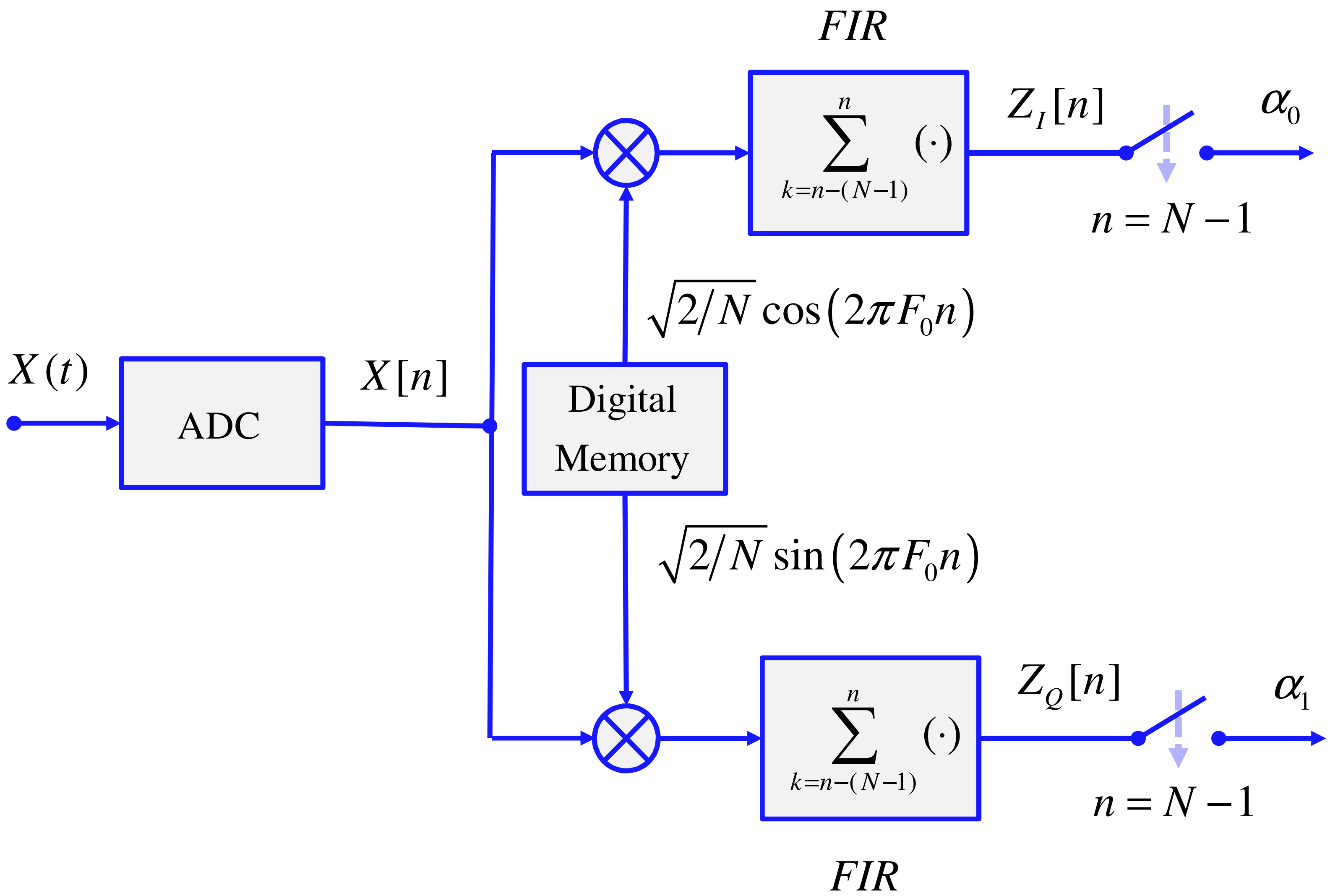
$$r_{01} = r_{10} = E\left\{\alpha_1 \alpha_0^*\right\} = E\left\{\alpha_1\right\} E\left\{\alpha_0^*\right\} = s_1 s_0^* = -\frac{NA^2}{2}\cos(\theta)\sin(\theta) = -\frac{NA^2}{4}\sin(2\theta)$$

$$\sigma_i^2 = E\left\{\left|\alpha_i\right|^2\right\} = E\left\{\left|s_i + W_i\right|^2\right\} = \left|s_i\right|^2 + E\left\{W_i^2\right\} = \left|s_i\right|^2 + \sigma_W^2$$

$$\underbrace{\sigma_0^2 = \frac{NA^2}{2}\cos^2(\theta) + \sigma_W^2, \quad \sigma_1^2 = \frac{NA^2}{2}\sin^2(\theta) + \sigma_W^2,}_{\text{Principal components}} \quad \sigma_i^2 = \sigma_W^2 \ \text{ for } 2 \le i \le N-1$$

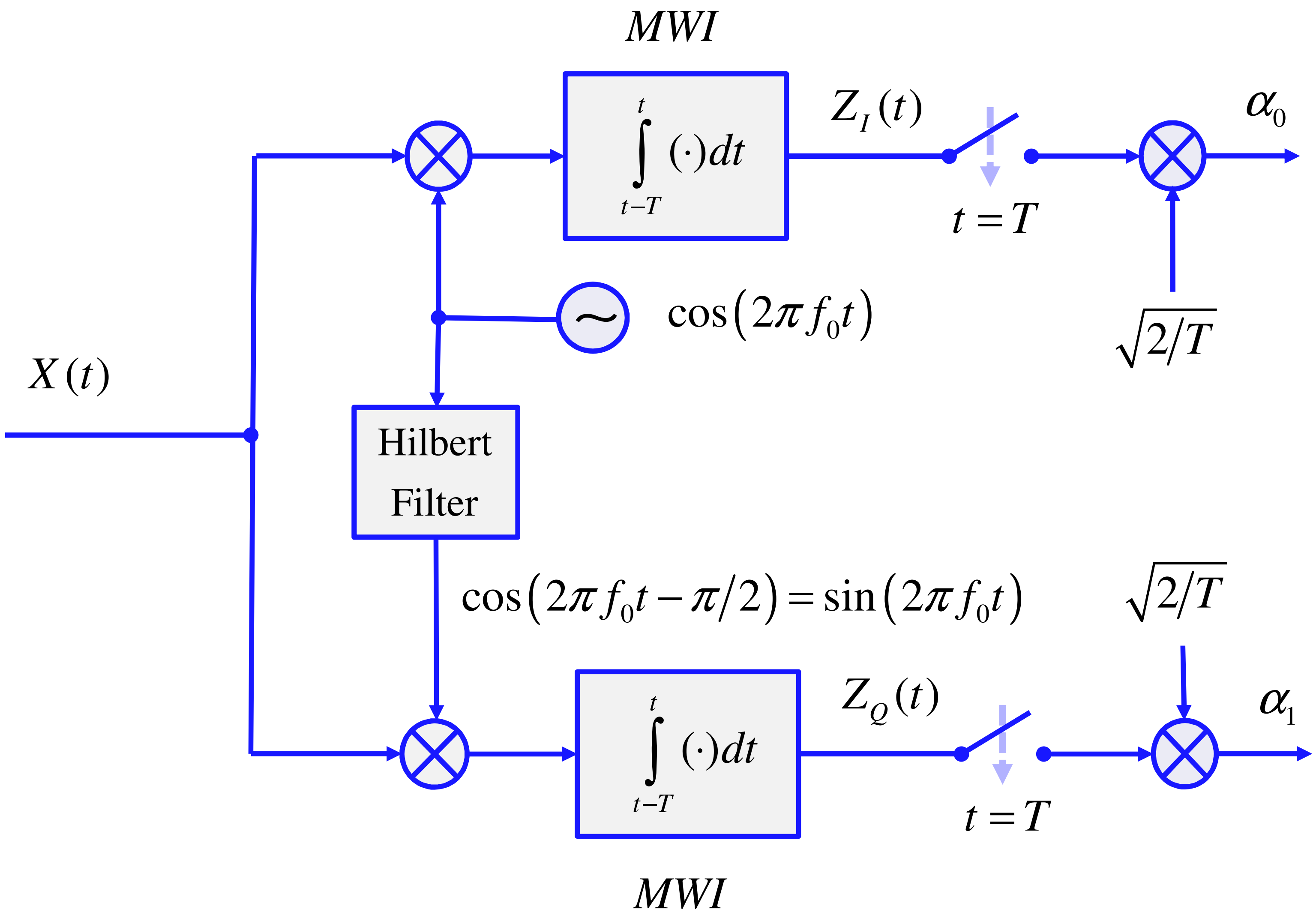
Basis Expansion of Discrete-Time Signals

- Block diagram where the **inner product** (**correlation**) is implemented digitally:



Basis Expansion of Discrete-Time Signals

- Alternative approach: **correlator** structure that uses an analog **moving window integrator (MWI)**:



Basis Expansion of Discrete-Time Signals

- **Typical approach** → first sample (ADC conversion) to get a discrete-time signal and then expand it by using the discrete-time orthonormal basis functions:

$$\varphi_0[n] = \sqrt{\frac{2}{N}}\cos(2\pi F_0 n), \quad \varphi_1[n] = \sqrt{\frac{2}{N}}\sin(2\pi F_0 n), \quad n = 0, 1, \cdots, N-1$$

- Only the first two components of the new image vector are **relevant**, i.e. they are the two **principal components**, so the 2D image vector is:

$$\boldsymbol{\alpha} \triangleq \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{N}{2}}A\cos(\theta) \\ -\sqrt{\frac{N}{2}}A\sin(\theta) \end{bmatrix} + \begin{bmatrix} W_0 \\ W_1 \end{bmatrix} = \mathbf{s} + \mathbf{W}$$

Basis Expansion of Discrete-Time Signals

- The components α_0 and α_1 are typically called the **In-phase (I)** and **Quadrature (Q) components** with respect to the frequency f_0.
- If $N=2BT>>1$ the two approaches to extract the I & Q components are equivalent from the point of view of **Signal to Noise power Ratio (SNR)**.
- The first approach is usually preferred because more efficient in terms of computational complexity (down-conversion to baseband, i.e. extraction of the I & Q components, is done digitally).
- Let us calculate the *Signal to Noise Ratio* (SNR) for the image vector **X** (*Input SNR*) and for the transformed image vector α, after the discrete-time base expansion (*Output SNR*).

Basis Expansion of Discrete-Time Signals

- *Signal to Noise Ratio* (SNR) for the image vector **X** (*Input SNR*):

$$X[n] = \sqrt{\frac{N}{2}} A\cos(\theta)\varphi_0[n] - \sqrt{\frac{N}{2}} A\sin(\theta)\varphi_1[n] + W[n]; \quad n = 0, 1, \cdots, N-1$$

where $W[n] \in \mathcal{N}(0, \sigma_W^2)$ IID, $\sigma_W^2 = N_0 B$, $N = T/T_c = 2BT$

$$\mathbf{X} = \underbrace{\sqrt{\frac{N}{2}} A\cos(\theta)\boldsymbol{\varphi}_0 - \sqrt{\frac{N}{2}} A\sin(\theta)\boldsymbol{\varphi}_1}_{\mathbf{s}_X} + \mathbf{W}_X$$

where $\mathbf{W}_X \triangleq [W[0] \;\; W[1] \;\; \cdots \;\; W[N-1]]^T$

$$SNR_{IN} = \frac{E\left\{\left\|\mathbf{s}_X\right\|_2^2\right\}}{E\left\{\left\|\mathbf{W}_X\right\|_2^2\right\}} = \frac{\left\|\mathbf{s}_X\right\|_2^2}{E\left\{\sum_{n=0}^{N-1} W^2[n]\right\}} = \frac{\left\|\sqrt{\frac{N}{2}}A\cos(\theta)\boldsymbol{\varphi}_0 - \sqrt{\frac{N}{2}}A\sin(\theta)\boldsymbol{\varphi}_1\right\|_2^2}{\sum_{n=0}^{N-1} E\left\{W^2[n]\right\}}$$

$$= \frac{\frac{NA^2}{2}\cos^2(\theta)\left\|\boldsymbol{\varphi}_0\right\|_2^2 + \frac{NA^2}{2}\sin^2(\theta)\left\|\boldsymbol{\varphi}_1\right\|_2^2 - 2\cdot\frac{NA^2}{2}\cos(\theta)\sin(\theta)\left(\boldsymbol{\varphi}_0,\boldsymbol{\varphi}_1\right)}{N\sigma_W^2}$$

$$= \frac{\frac{NA^2}{2}\cos^2(\theta) + \frac{NA^2}{2}\sin^2(\theta)}{N\sigma_W^2} = \frac{A^2}{2\sigma_W^2}$$

$$SNR_{IN} = \frac{A^2}{2\sigma_W^2}$$

Basis Expansion of Discrete-Time Signals

- *Signal to Noise Ratio* (SNR) for the transformed image vector α (*Output SNR*):

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{N}{2}} A\cos(\theta) \\ -\sqrt{\frac{N}{2}} A\sin(\theta) \end{bmatrix} + \begin{bmatrix} W_0 \\ W_1 \end{bmatrix} = \mathbf{s} + \mathbf{W}, \quad \text{where} \quad W_k \in \mathcal{N}\left(0, \sigma_W^2\right), \quad IID$$

$$SNR_{OUT} = \frac{E\left\{\|\mathbf{s}\|_2^2\right\}}{E\left\{\|\mathbf{W}\|_2^2\right\}} = \frac{s_0^2 + s_1^2}{E\left\{W_0^2 + W_1^2\right\}} = \frac{A^2 N/2}{2\sigma_W^2} = \frac{A^2 N/2}{2 \cdot N_0 B} = \frac{A^2 T}{2N_0} = \frac{E_s}{N_0}$$

- The **Processing Gain (PG)** is the gain in SNR that we achieve thanks to the linear transformation (i.e. the discrete-time base expansion):

$$PG = \frac{SNR_{OUT}}{SNR_{IN}} = \frac{A^2 N/2}{2\sigma_W^2} \Big/ \frac{A^2}{2\sigma_W^2} = \frac{N}{2}$$

Basis Expansion of Discrete-Time Signals

- Calculation of these two principal components is equivalent to calculating the **Discrete-Time Fourier Transform (DTFT)** of the observed signal $X[n]$ at the digital frequency F_0, which is the most powerful frequency component present in the *Signal of Interest* (SoI) $s[n]$ observed in $[0, T)$:

$$\alpha_0 = \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} X[n]\cos\left(2\pi F_0 n\right) = \sqrt{\frac{2}{N}} \Re e\left\{ \sum_{n=0}^{N-1} X[n] e^{-j2\pi F_0 n} \right\}$$

$$= \sqrt{\frac{2}{N}} \Re e\left\{ DTFT\{\mathbf{X}\}\Big|_{F=F_0} \right\}$$

$$\alpha_1 = \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} X[n]\sin\left(2\pi F_0 n\right) = -\sqrt{\frac{2}{N}} \Im m\left\{ \sum_{n=0}^{N-1} X[n] e^{-j2\pi F_0 n} \right\}$$

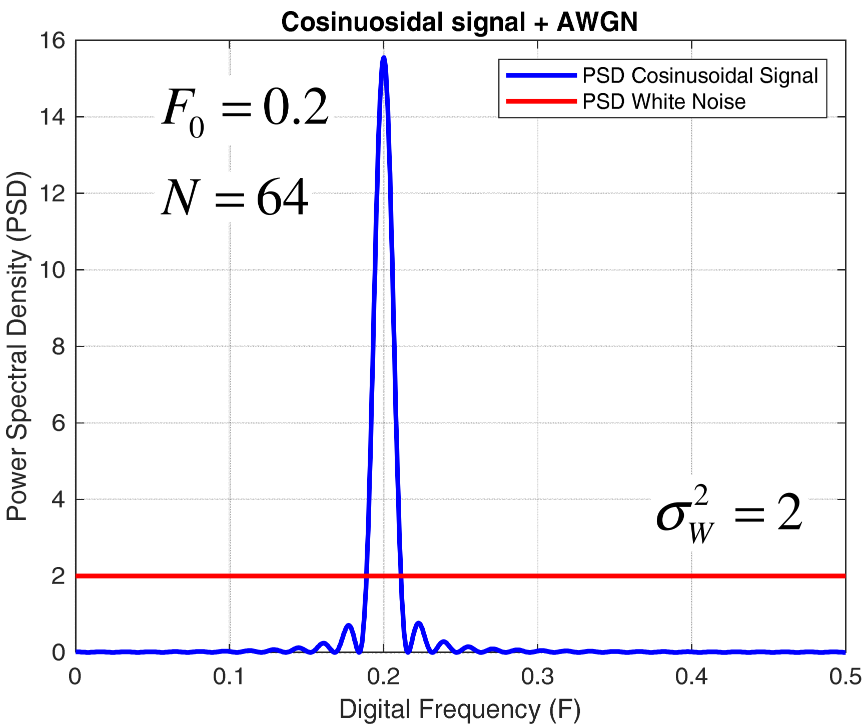
$$= -\sqrt{\frac{2}{N}} \Im m\left\{ DTFT\{\mathbf{X}\}\Big|_{F=F_0} \right\}$$

Basis Expansion of Discrete-Time Signals

$$X[n] = s[n] + W[n] = A\cos\left(2\pi F_0 n + \theta\right) + W[n], \quad n = 0, 1, \cdots, N-1$$

$$S\left(e^{j2\pi F}\right) = DTFT\{s[n]\} = \sum_{n=0}^{N-1} s[n] e^{-j2\pi F}$$

$$PSD_S\left(e^{j2\pi F}\right) = \cancel{\lim_{N\to\infty}} \frac{1}{N}\left|S\left(e^{j2\pi F}\right)\right|^2$$



$$SNR(F) = \frac{PSD_S\left(e^{j2\pi F}\right)}{PSD_W\left(e^{j2\pi F}\right)}$$

$$F_0 = \arg\max_F \left\{SNR(F)\right\}$$

Basis Expansion of Discrete-Time Signals

$$\alpha_0 = \sqrt{\frac{2}{N}} \Re e\left\{ DTFT\{\mathbf{X}\}\Big|_{F=F_0} \right\}$$

$$\alpha_1 = -\sqrt{\frac{2}{N}} \Im m\left\{ DTFT\{\mathbf{X}\}\Big|_{F=F_0} \right\}$$

- The values of the DTFT at frequencies different from F_0 are not relevant, i.e. they do not bring additional useful information about the amplitude and phase of the signal of interest $s[n]$.
- The two principal components α_0 and α_1 contains all the relevant information about the amplitude A and the phase θ of the continuous-time signal of interest $s(t)$. Hence, A and θ can be estimated from α_0 and α_1.

Basis Expansion of Discrete-Time Signals

- This is possible if we know *a priori* the signal frequency F_0 (remember that the basis functions should be deterministic perfectly known).
- What about if the signal frequency F_0 is also unknown?

We use the FFT to calculate the DTFT of the N-dimensional vector $\mathbf{X}$ at all the discrete frequencies $F_k=k/N$ (or $F_k=k/N_{zp}$ if we use zero-padding) and then we select the two principal components, i.e. the real and imaginary parts of the FFT at the frequency for which the $|\text{FFT}|^2$ assumes the greatest value [*we will investigate this problem in detail later on*].