Mechanics

Robotics Engineering

SUMMARY (Lecture 6)

17/10/25

Lecture 6 (summary) 17/10/25

Summary of linearized equilibrium theory for linear elasticity (basic)

$$\begin{cases} M' + T + m = 0 \\ N' + f_1 = 0 \\ T' + f_2 = 0 \end{cases} \qquad \begin{cases} T = -M' - m \\ M'' = -m' + f_2 \\ N' = -f_1 \end{cases} \qquad T' = -M'' - m'$$

$N = EA\, u'$ $\quad N = EA\, \varepsilon$

$M = EI\, w''$ $\quad M = EI\, \kappa$

+ Static BCs (M, N, T) + Kinematic BCs (u, w, w')

5 linear (!) ODEs in the 5 unknown functions M, N, T, u, w.

More compact formulation.

$T = -M' - m = -EI\, w''' - m$

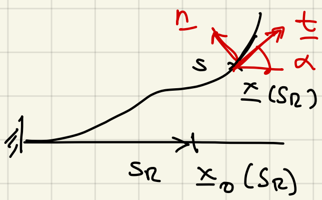
$$\begin{cases} EA\, u'' = -f_1 \\ EI\, w'''' = -m' + f_2 \end{cases} \quad (*) \qquad + \text{ Static \& kinematic BC's on } \begin{matrix} u, u' \\ w, w', w'', w''' \end{matrix}$$

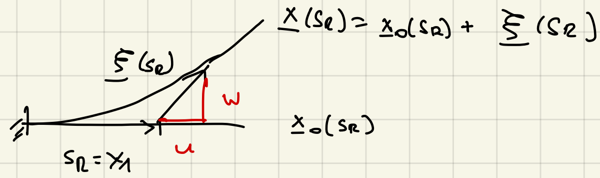
2 linear (!) ODEs in the unknown functions u, w.

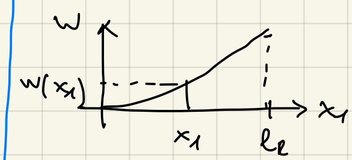
Important Remark

In the linear theory, elastic displacements are small perturbations of the reference configuration. Equilibrium equations are written in the reference configuration.

General scheme of equilibrium problem







small deformations
linearized theory

Kinematics

$$\alpha'(s_R) = \kappa(s_R) + \cancel{\underbrace{c_0(s_R)}_{=0}} \text{ in this case (straight ref. conf.)}$$

$$\underline{x}'(s_R) = (1+\varepsilon(s_R))\underbrace{(\cos\alpha(s_R)\,\underline{e}_1 + \sin\alpha(s_R)\,\underline{e}_2)}_{\underline{t}(s_R)}$$

+ kinematic BC's on $\alpha(0)$ and $\underline{x}(0)$

$\rightarrow$ recover $\alpha, \underline{x}$ by integration from ε, κ

$\hookrightarrow$ placement of the rod's axis

$$\varepsilon = u'$$
$$\alpha = w'$$
$$\kappa = w''$$

+ kinematic BC's on $u(0), w(0), w'(0)$ ......

Equilibrium

$$\begin{cases} M_R' + (1+\varepsilon)T_R + m_R = 0 \\ N_R' - (c_0+\kappa)T_R + f_{Rt} = 0 \\ T_R' + (c_0+\kappa)N_R + f_{Rn} = 0 \end{cases}$$

where

$$\underline{R}_R = N_R\,\underline{t} + T_R\,\underline{n}$$

$$\underline{f}_R = f_{Rt}\,\underline{t} + f_{Rn}\,\underline{n}$$

+ static BC's $M(\ell), T(\ell), N(\ell)$

$$\begin{cases} M' + T + m = 0 \\ N' + f_1 = 0 \\ T' + f_2 = 0 \end{cases}$$

can rewrite as

$$\begin{cases} T = -M' - m \\ M'' = -m' - T' = \\ \quad = -m' + f_2 \\ N' = -f_1 \end{cases}$$

where

$$\underline{R}_R = N_R\,\underbrace{\underline{e}_1}_{=\underline{t}_0} + T_R\,\underbrace{\underline{e}_2}_{=\underline{n}_0}$$

$$\underline{f}_R = f_1\,\underline{e}_1 + f_2\,\underline{e}_2$$

+ static BC's $M(\ell), N(\ell), T(\ell), \ldots$

Constitutive equations

$N_R = \tilde{N}(\varepsilon, \kappa)$, $M_R = \tilde{M}(\varepsilon, \kappa)$

Use for now linear dependence of N_R on ε (resp. M_R on κ) as in linear theory. More later.

$$N_R = EA\,\varepsilon$$

$$M_R = EI\,\kappa$$

$$N = EA\,u'$$

$$M = EI\,w''$$

In summary

$$\underset{\text{}}{8} = \overset{x'}{2} + \overset{\alpha'}{1} + \underset{\text{equil}}{3} + \overset{\text{const.}}{2}$$

ODEs
in two unknown functions

$$5 = \underset{\text{equil.}}{3} + \overset{\text{const.}}{2}$$

$\underbrace{x_1, x_2, \alpha}_{\text{3 geom}}, \underbrace{M_R, N_R, T_R}_{\text{3 int. forces}}, \underbrace{\varepsilon, \kappa}_{\text{2 strains}}$

$\underbrace{M, N, T}_{\text{3 int. forces}}, \underbrace{u, w}_{\text{2 perturbation displacements}}$

[Non-linear equation]

[linear equations]

More compact formulation

$$T = -M' - m = -EI\,w''' - m$$

$$\begin{cases} EA\,u'' = -f_1 \\ EI\,w'''' = -m' + f_2 \end{cases}$$

2 linear ODEs in two unknown functions u, w
Once u, w are known can recover M, N from const. equations and T from equilibrium.