Mechanics

Robotics Engineering

Lecture 7

22/10/25

lecture 7 22/10/25

Summary of linearized equilibrium theory for linear elasticity (basic)

$$\begin{cases} M' + T + m = 0 \\ N' + f_1 = 0 \\ T' + f_2 = 0 \end{cases} \qquad \begin{cases} T = -M' - m \\ M'' = -m' + f_2 \\ N' = -f_1 \end{cases}$$

$N = EA\, u'$

$M = EI\, w''$

$+$ static BCs (M, N, T) $+$ kinematic BCs (u, w, w')

5 linear (!) ODEs in the 5 unknown functions M, N, T, u, w.

More compact formulation.

$T = -M' - m = -EI w''' - m$

$$\begin{cases} EA\, u'' = -f_1 \\ EI\, w'''' = -m' + f_2 \end{cases} \quad (*) \qquad + \text{ static \& kinematic BC's on } \begin{matrix} u, u' \\ w, w', w'', w''' \end{matrix}$$

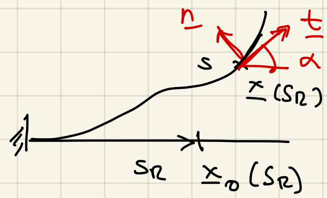
2 linear (!) ODEs in the unknown functions u, w.

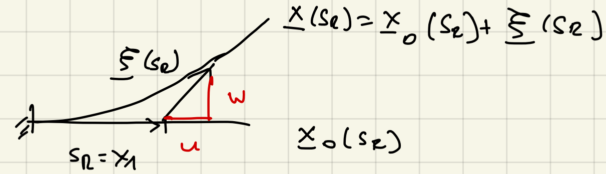
(*) is known as "equazione della linea elastica"

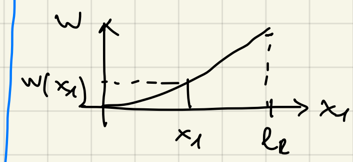
Important Remark

In the linear theory, elastic displacements are small perturbations of the reference configuration. Equilibrium equations are written in the reference configuration.

General scheme of equilibrium problem







small deformation linearized theory

Kinematics

$\alpha'(s_R) = k(s_R) + \cancel{c_0(s_R)}$ $=0$ in this case (straight ref. conf.)

$$\underline{x}'(s_R) = (1+\varepsilon(s_R))\underbrace{(\cos\alpha(s_R)\,\underline{e}_1 + \sin\alpha(s_R)\,\underline{e}_2)}_{\underline{t}(s_R)}$$

+ kinematic BC's on $\alpha(0)$ and $\underline{x}(0)$

→ recover $\alpha, \underline{x}$ by integration from ε, k

↳ placement of the rod's axis

$\varepsilon = u'$

$\alpha = w'$

$k = w''$

+ kinematic BC's on $u(0), w(0), w'(0)$ ......

Equilibrium

$$\begin{cases} M_R' + (1+\varepsilon)T_R + m_R = 0 \\ N_R' - (c_0 + k)T_R + f_{Rt} = 0 \\ T_R' + (c_0 + k)N_R + f_{Rn} = 0 \end{cases}$$

where

$\underline{R}_R = N_R\,\underline{t} + T_R\,\underline{n}$

$\underline{f}_R = f_{Rt}\,\underline{t} + f_{Rn}\,\underline{n}$

+ static BC's $M(\ell), T(\ell), N(\ell)$

$$\begin{cases} M' + T + m = 0 \\ N' + f_1 = 0 \\ T' + f_2 = 0 \end{cases}$$

can rewrite as

$$\begin{cases} T = -M' - m \\ M'' = -m' - T' = \\ \quad = -m' + f_2 \\ N' = -f_1 \end{cases}$$

where

$\underline{R}_R = N_R\,\underbrace{\underline{e}_1}_{=\underline{t}_0} + T_R\,\underbrace{\underline{e}_2}_{=\underline{n}_0}$

$\underline{f}_R = f_1\,\underline{e}_1 + f_2\,\underline{e}_2$

+ static BC's $M(\ell), N(\ell), T(\ell), \dots$

Constitutive equations

$N_R = \tilde{N}(\varepsilon, k)$, $M_R = \tilde{M}(\varepsilon, k)$

Use for now linear dependence of N_R on ε (resp. M_R on k) as in linear theory. More later.

$N_R = EA\,\varepsilon$

$M_R = EI\,k$

$N = EA\,u'$

$M = EI\,w''$

In summary

$$8 = \overset{x'}{2} + \overset{\alpha'}{1} + \underset{\text{equil.}}{3} + \overset{\text{const.}}{2}$$

ODEs
in two unknown functions

$$5 = \underset{\text{equil.}}{3} + \overset{\text{const.}}{2}$$

$$\underbrace{x_1, x_2, \alpha}_{\text{3 geom}},\ \underbrace{M_R, N_R, T_R}_{\text{3 int. forces}},\ \underbrace{\varepsilon, \kappa}_{\text{2 strains}}$$

$$\underbrace{M, N, T}_{\text{3 int. forces}},\ \underbrace{u, w}_{\text{2 perturbation displacements}}$$

[Non-linear equations]

[linear equations]

<u>More compact formulation</u>

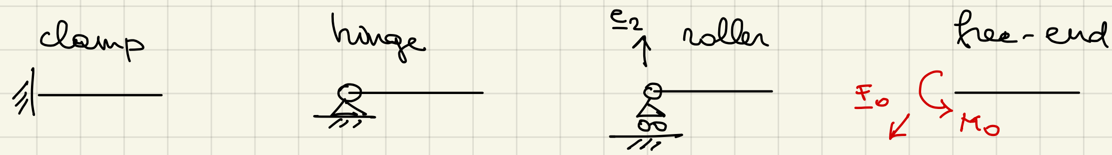
$$T = -M' - m = -EI\,w''' - m$$

$$\begin{cases} EA\,u'' = -f_1 \\ EI\,w'''' = -m' + f_2 \end{cases}$$

2 linear ODEs in the unknown functions u, w
Once u, w are known can recover M, N from const. equations and T from equilibrium.

Solving equilibrium problems in the linear theory (HPP of the initial/reference configuration)

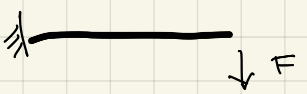
The BCs for M, N, T come from statics. Depending on whether one end is free or constrained we know M, N, T at that end or not. For example, at $s_R = 0$



$u(0)=w(0)=0$	$u(0)=w(0)=0$	$u(0)$? $w(0)=0$	$u(0)$? $w(0)$?
$\alpha(0)=w'(0)=0$	$\alpha(0)=w'(0)$?	$\alpha(0)=w'(0)$?	$\alpha(0)=w'(0)$?
$\underline{F}_0\cdot\underline{e}_1=-N(0)$?	$\underline{F}_0\cdot\underline{e}_1$?	$\underline{F}_0\cdot\underline{e}_1=-N(0)=0$	$\underline{F}_0\cdot\underline{e}_1=-N(0)$
$\underline{F}_0\cdot\underline{e}_2=-T(0)$?	$\underline{F}_0\cdot\underline{e}_2$?	$\underline{F}_0\cdot\underline{e}_2=-T(0)$?	$\underline{F}_0\cdot\underline{e}_2=-T(0)$
$M_0=-M(0)$?	$M(0)=0$	$M(0)=0$	$M_0=-M(0)$

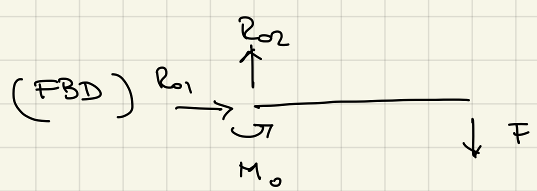
Remark Observe the "complementarity" between kinematic and static conditions in the prescription of constraints. For non-dissipative constraints, this is always the case.

Isostatic case (Free Body Diagrams FBD)



$F>0$

(ie. $\underline{F}_\ell=-F\underline{e}_2$)



3 scalar constraints suppress all rigid DoFs, 3 scalar unknown reactions

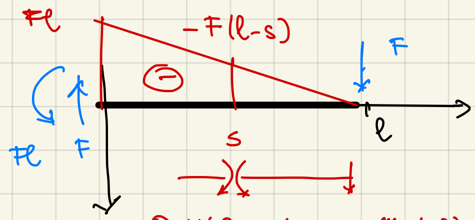
3 global equil. equations
$$\begin{cases} \underline{f}=0 \\ \underline{M}_0=0 \end{cases} \Rightarrow \begin{array}{l} R_{01}=-N(0)=0 \\ R_{02}-F=0 \Rightarrow R_{02}=-T(0)=F \\ M_0-F\ell=0 \Rightarrow -M_0=M(0)=-F\ell \end{array}$$

$\Rightarrow$ M, N, T can be determined by statics alone. Can integrate equil. eqn's directly, using $N(\ell)=0$, $T(\ell)=-F$, $M(\ell)=0$.

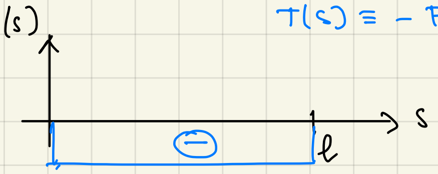
$$\begin{cases} N'(s)=0 \rightarrow 0=\int_s^\ell N'(\sigma)d\sigma = N(\ell)-N(s) \rightarrow N(s)\equiv N(\ell)=0 \\ T'(s)=0 \rightarrow T(s)\equiv T(\ell)=-F \qquad \boxed{T(0)=-F \checkmark} \\ M'(s)=-T(s)=+F \rightarrow \underbrace{M(\ell)}_{=0}-M(s)=F(\ell-s) \rightarrow M(s)=-F(\ell-s) \qquad \boxed{M(0)=-F\ell \checkmark} \end{cases}$$

Diagrams of internal forces (convention for $s \mapsto M(s)$ diagram) ① → ② → ③

$T = -M' - \cancel{m} = -M'$



④



$T(s) \equiv -F$ ①

$M(s)$ ③

① $M(s) = -M^-(s) = m_s(s,\ell)$ by equilibrium of (s,ℓ)

② positive moment put bottom fibers in tension!

$M' = -T - \cancel{m} \Rightarrow M(s)$ affine ②

①+② can be replaced by $M'' = -\cancel{m}' + \cancel{f_2} = 0 \Rightarrow M(s)$ affine

Assuming linear constitutive equations hold, can integrate them to recover the small elastic perturbations of the equilibrium configuration (the initial or reference configuration) caused by the loads

$$u' = \frac{N}{EA} \rightarrow u(s) = u(0) + 0 \equiv 0$$

$$w'' = \frac{M}{EI} \rightarrow w'(s) = \underbrace{w'(0)}_{=0} + \int_0^s \frac{F(\sigma - \ell)}{EI}\, d\sigma = \frac{F}{EI} \left. \frac{(\sigma - \ell)^2}{2} \right|_0^s =$$

$$= \frac{F}{EI}\left(\frac{s^2}{2} - \frac{\cancel{2} s\ell}{\cancel{2}} + \cancel{\frac{\ell^2}{2}} - \cancel{\frac{\ell^2}{2}} \right)$$

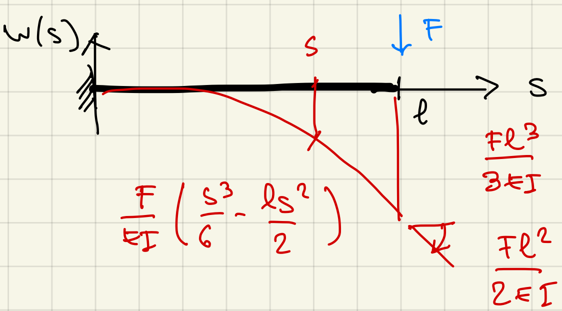
$$w(s) = \underbrace{w(0)}_{=0} + \frac{F}{EI}\int_0^s \left(\frac{\sigma^2}{2} - \ell\sigma\right) d\sigma = \frac{F}{EI}\left(\frac{s^3}{3} - \frac{\ell s^2}{2}\right)$$

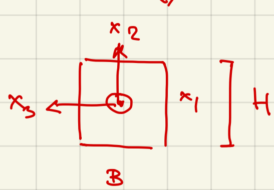
$$w(\ell) = \frac{F}{EI}\left(\frac{\ell^3}{6} - \frac{\ell^3}{2}\right) = -\frac{1}{3}\frac{F\ell^3}{EI}$$

$$w'(\ell) \overset{\sim}{=} \overset{w(\ell)=}{\alpha(\ell)} = \frac{F}{EI}\left(\frac{\ell^2}{2} - \ell^2\right) = -\frac{F\ell^2}{2EI}$$

Important values, check dimensions using $[EI] = FL^{-2}L^4 = FL^2$

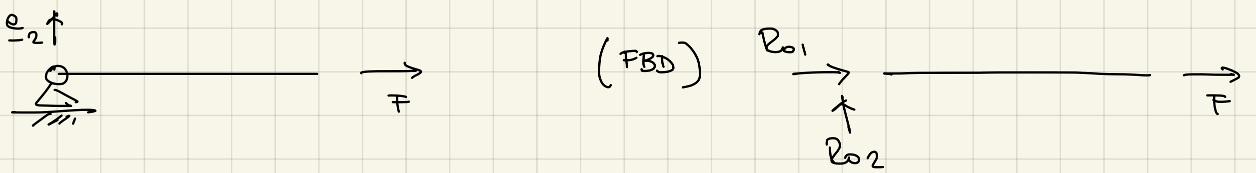
Graph of the transversal displacements (a cubic)





$$I = \int_{-B/2}^{B/2}\int_{-H/2}^{H/2} x_2^2\, dx_3\, dx_2 = \frac{BH^3}{12}$$

Exercise 2 Force $\underline{F} = F\underline{e}_1$ at the end ℓ of a beam supported at $s = 0$



$R_{01} = -F$
$R_{02} = 0$
$M_0 = 0$

Equilibrium $\underline{f} = 0$, $M_0 = 0$ is satisfied in ref. config. and we have enough BC's to integrate equil. equations. We have a free end: $N(\ell) = F$, $T(\ell) = 0$, $M(\ell) = 0$

$$\int_s^\ell N' = 0 \Rightarrow N(\ell) - N(s) = 0 \Rightarrow N(s) \equiv N(\ell) = F$$

$$\int_s^\ell T' = 0 \Rightarrow T(\ell) - T(s) = 0 \Rightarrow T(s) \equiv T(\ell) = 0$$

$$\int_s^\ell M' = 0 \Rightarrow M(\ell) - M(s) = 0 \Rightarrow M(s) \equiv M(\ell) = 0$$

(special affine function, $\equiv 0$)

With linear constitutive equations we can calculate u, w

$$u' = \frac{N}{EA} = \frac{F}{EA} \rightarrow u(s) = \underbrace{u(0)}_{=0} + \frac{F}{EA} s$$

$u(\ell) = \frac{F}{EA} \ell$

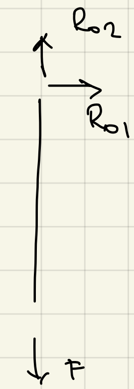
$$w'' = \frac{M}{EI} = 0 \rightarrow w(s) = a + bs$$
$$w(0) = a = 0$$
$$b = w'(0) \text{ arbitrary}$$

$w(s) = bs$, b arbitrary

arbitrary small rotation allowed (!)

Exercise 3   Force $\underline{F}_\ell = -F\underline{e}_2$ at the end ℓ of a beam supported at $s=0$

(FBD)



$R_{01} = 0$

$R_{02} = F$

$M_0 = 0$

[$\underline{f} = 0$, $m_0 = 0$ ✓]

$N' = 0 \Rightarrow N(s) = N(\ell) = F$ (since $N(\ell) = \underline{R}_\ell(\ell) \cdot \underline{t}_0 = \underbrace{\underline{F}_\ell}_{(-F\underline{e}_2)} \cdot \underbrace{\underline{t}_0}_{(-\underline{e}_2)} = F$)

$T' = 0 \Rightarrow T(\ell) = T(s) = 0$

$M' = -T = 0 \Rightarrow M(\ell) = M(s) = 0$

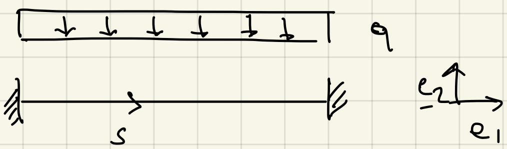
Calculate u, w ...

$$u(s) = \frac{F}{EA} s$$

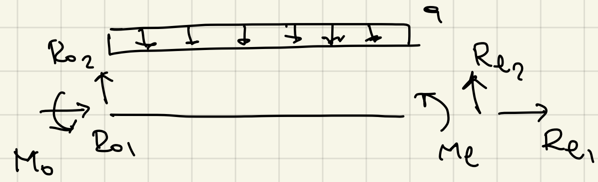
$w(s) = bs$ b arbitrary   (arbitrary small rotation allowed)

Exercise 4

$\underline{f}(s) = -q\,\underline{e}_2\,,\quad f_2(s) \equiv -q$



(FBD)



There exist (∞^3) values of the reaction parameters that guarantee equilibrium in the ref. configuration, but cannot integrate the equil. equ's because static BCs involve unknown reactions. Using constitutive equations we can integrate the reduced system in u, w

$$EA\,u'' = 0 \quad (1) \qquad\qquad EI\,w'''' = -q \quad (2)$$

2 equ's in 2 unknowns u, w

6 BC's of kinematic / static nature

$(1) \Rightarrow u(s) = A + Bs \qquad u(0) = 0 = A, \quad u(\ell) = B\ell = 0 \Rightarrow B = 0 \qquad \Rightarrow u(s) \equiv 0$

$(2)\ w'''' = -\dfrac{q}{EI} \Rightarrow w(s) = a + bs + cs^2 + ds^3 - \dfrac{q}{24EI}s^4$

$w(s)$ = 4th order polynomial

$$\left.\begin{aligned} w(0)&=0\\ w'(0)&=0\\ w(\ell)&=0\\ w'(\ell)&=0 \end{aligned}\right\} \to \left\{\begin{aligned} &a=0\\ &b=0\\ &c\ell^2 + d\ell^3 - \frac{q}{24EI}\ell^4 = 0\\ &2c\ell + 3d\ell^2 - \frac{q}{6EI}\ell^3 = 0 \end{aligned}\right\} \qquad \begin{aligned} c &= -\frac{1}{24}\frac{q}{EI}\ell^2\\ d &= \frac{1}{12}\frac{q}{EI}\ell \end{aligned}$$

$$w''(s) = 2c + 6ds - \frac{q}{2EI}s^2 = \frac{M(s)}{EI}\,;\quad \frac{T(s)}{EI} = -\frac{M'}{EI} = -6d + \frac{q}{EI}s$$

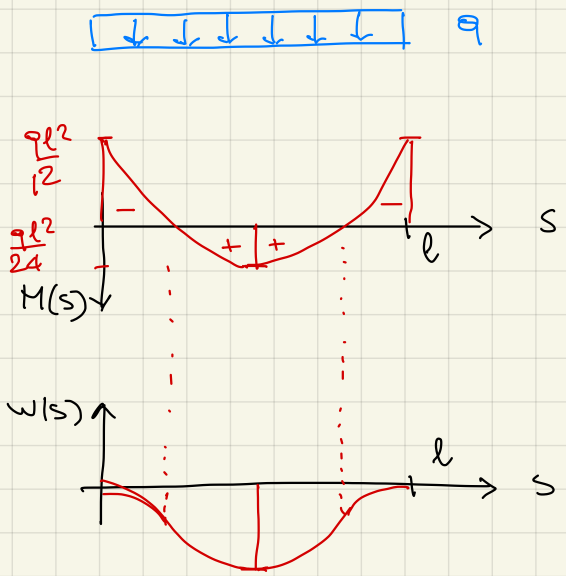
$M(s)$ quadratic $\qquad\qquad T(s)$ affine

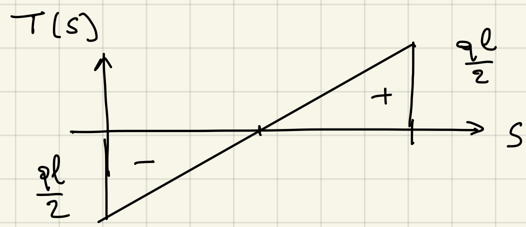
check (symmetry)

$M(0) = 2EIc = -\dfrac{q\ell^2}{12} = M(\ell) \qquad M\left(\dfrac{\ell}{2}\right) = \dfrac{q\ell^2}{24}$

$T(0) = -6EId = -\dfrac{q\ell}{2}\,,\quad T(\ell) = \dfrac{q\ell}{2}$

Diagrams





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Remark

Isostatic: (1) Determine M, N, T

(2) Determine u, w by integrating constitutive equations

Hyperstatic: (1) Determine u, w

(2) Determine M, N, T by computing derivatives in the constitutive equations

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Nonlinear theory: need to solve for the equilibrium configuration.

Accept constitutive relations

$$N_R = EA\,\varepsilon$$

$$M_R = EI\,k$$

Look at general scheme for equilibrium problems again.