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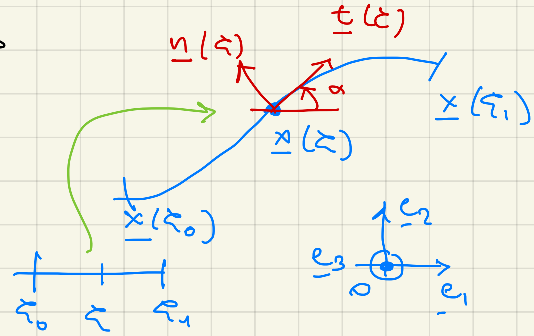
Mechanics, Robotics

Lecture 1

27 / 10/ 2023

<u>Lecture 1</u> (27/10/23) A planar parametric curve is a map $\mathbb{R} \supset [\xi_0, \xi_1] \ni \xi \mapsto \underline{x}(\xi) \in \mathbb{R}^2$



s.t. $\left|\frac{d}{d\xi}\underline{x}(\xi)\right| = |\underline{x}'(\xi)| \neq 0 \quad \forall \xi \in [\xi_0, \xi_1]$

[map = motion of a point along a 2d trajectory, s.t. the point never makes U-turns]

[natural parametrizations: time elapsed, distance travelled]

We define

$\underline{t}(\xi)$ unit tangent vector, $\quad \underline{t}(\xi) = \dfrac{\underline{x}'(\xi)}{|\underline{x}'(\xi)|}$

$\underline{n}(\xi)$ unit normal vector, $\quad \underline{n}(\xi) = \underline{e}_3 \wedge \underline{t}(\xi) = \underline{t}^{\perp}(\xi)$

where $\quad \underline{t}^{\perp}(\xi) = \underline{\underline{R}}(\underline{e}_3, \frac{\pi}{2})\,\underline{t}(\xi)$

[$\underline{\underline{R}}(\underline{e}_3, \theta)$ rotation by θ around $\underline{e}_3$, positive counter-clock-wise]

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

<u>Remark</u> other definition for $\underline{n}$ is possible, see FRENET formula later

$\alpha(\xi)$ angle that $\underline{t}(\xi)$ forms with $\underline{e}_1$, positive counter-clockwise

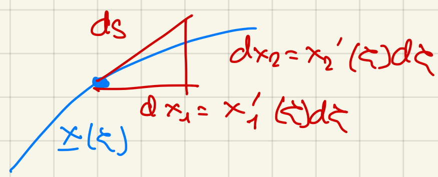
$\underline{t}(\xi) = \cos\alpha(\xi)\,\underline{e}_1 + \sin\alpha(\xi)\,\underline{e}_2 = (\cos\alpha(\xi), \sin\alpha(\xi))$

$\underline{n}(\xi) = \qquad = (-\sin\alpha(\xi), \cos\alpha(\xi))$

<u>Arc-length</u>: $\xi \mapsto s(\xi)$ distance travelled along the curve when parameter goes from ξ_0 to ξ

How to compute it?

ds distance travelled along the curve when parameter increases from ξ to $\xi + d\xi$



$$ds = s'(\xi)d\xi = (x_1'^2(\xi)d\xi^2 + x_2'^2(\xi)d\xi^2)^{1/2} = \underbrace{|\underline{x}'(\xi)|}_{\geq 0,\, \neq 0} d\xi$$

$$\Rightarrow \boxed{s'(\xi) = |\underline{x}'(\xi)| > 0}$$

distance travelled always strictly increases because of no U-turns

$s(\xi) = \int_{\xi_0}^{\xi} |\underline{x}'(\tilde{\xi})|\,d\tilde{\xi}$ arc-length parameter of ξ

$s(\xi_0) = 0$

$s(\xi_1) = \int_{\xi_0}^{s_1} |\underline{x}'(\xi)|\,d\xi$ length of the curve

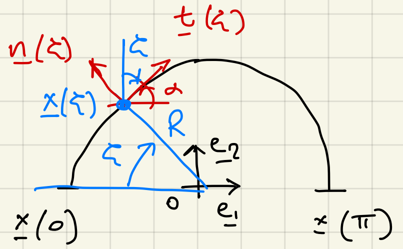
Curvature

$$C = \text{"}\frac{d\alpha}{ds}\text{"}$$

$$c(\xi) = \frac{\frac{d\alpha}{d\xi}}{\frac{ds}{d\xi}} = \frac{\alpha'(\xi)}{s'(\xi)} = \frac{\alpha'(\xi)}{|\underline{x}'(\xi)|} \quad \text{curvature}$$

Remark With this definition, curvature has a sign, and this sign depends on parametrization. Changing the orientation of the curve changes the sign of curvature, see example below

Example 1 (circular arch)



$$\xi \in [\underset{\to \xi_0}{0}, \underset{\to \xi_1}{\pi}]$$

$$\underline{x}(\xi) = -R\cos\xi\, \underline{e}_1 + R\sin\xi\, \underline{e}_2 = (-R\cos\xi, R\sin\xi)$$

$$\underline{x}'(\xi) = R\sin\xi\, \underline{e}_1 + R\cos\xi\, \underline{e}_2, \quad |\underline{x}'(\xi)| = (R^2\sin^2\xi + R^2\cos^2\xi)^{1/2} \overset{=|R|=}{=} R = s'(\xi)$$

$$\underline{t}(\xi) = \frac{\underline{x}'(\xi)}{|\underline{x}'(\xi)|} = \underbrace{\frac{1}{R} R}_{1}(\sin\xi, \cos\xi), \quad \underline{n}(\xi) = (-\cos\xi, \sin\xi)$$

$$\alpha(\xi) = \frac{\pi}{2} - \xi$$

[Check that, indeed, $\underline{t}(\xi) = (\cos\alpha(\xi), \sin\alpha(\xi))$]

$\xi = \frac{\pi}{2} - \alpha; \quad (\sin(\frac{\pi}{2} - \alpha), \cos(\frac{\pi}{2} - \alpha)) = (\cos\alpha, \sin\alpha)$ ✓

[Re: $s'(\xi) = |x'(\xi)|$]

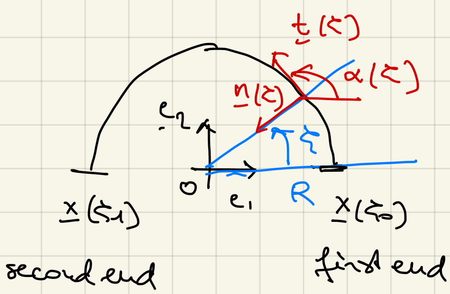
$$c(\xi) = \frac{\alpha'(\xi)}{s'(\xi)} = \frac{-1}{R}$$

[Negative because $\xi \mapsto \alpha(\xi)$ is decreasing]

$$s(\xi) = \int_{\xi_0 = 0}^{\xi} |\underline{x}'(\bar\xi)|\, d\bar\xi = \int_0^{\xi} R\, d\bar\xi = \underbrace{R}_{\text{radius}}\ \underbrace{(\xi - 0)}_{\text{angle}} \Big\} \text{ length of arc}$$

$$\ell = s(\xi_1) = R\pi \qquad \text{length of half-circle}$$

Remark $|c| = \frac{1}{R}$. Changing the orientation of the parametrized curve changes sign of c



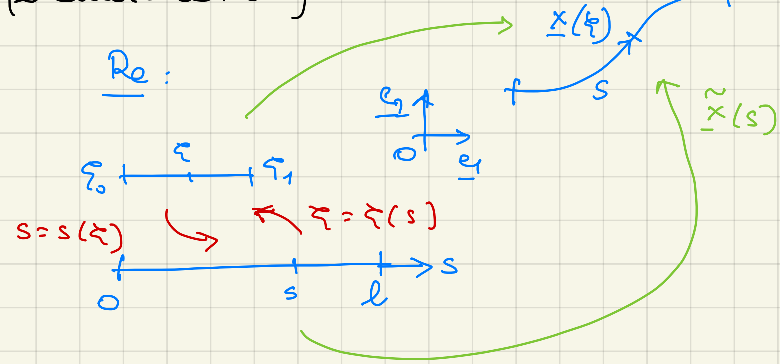
$$\alpha(\xi) = \frac{\pi}{2} + \xi$$

$\xi \mapsto \alpha(\xi)$ increasing

$$c(\xi) = \frac{\alpha'(\xi)}{|\underline{x}'(\xi)|} > 0 \qquad \text{Indeed, } c(\xi) = \frac{1}{R}.$$

Arc-length parametrization (normal parametrization)

Re:



$\xi \mapsto \underline{x}(\xi)$

$s'(\xi) = |\underline{x}'(\xi)| > 0$

$\xi \mapsto s(\xi)$ is strictly increasing

hence it is invertible.

Can define $s \mapsto \xi(s)$ from $\dfrac{d\xi}{ds} = \left(\dfrac{ds}{d\xi}\right)^{-1}$ (*)

and reparametrize curve by arc-length (instead of, say, time)

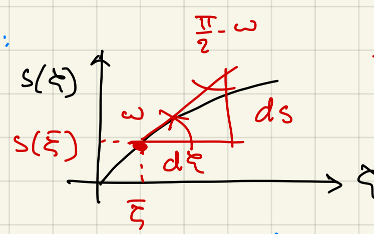
$\Rightarrow$ arc-length parametrization, normal parametrization, unit-speed parametrization

$\tilde{\underline{x}}(s) := \underline{x}(\xi(s))$

$\tilde{\underline{x}}'(s) = \dfrac{d}{ds}\tilde{\underline{x}}(s) = \dfrac{d}{d\xi}\underline{x}(\xi(s))\ \xi'(s)$

$= \underline{x}'(\xi(s))\dfrac{1}{|\underline{x}'(s)|} = \dfrac{\underline{x}'(\xi)}{|\underline{x}'(\xi)|}\Bigg|_{\xi=\xi(s)}$

(*) Re:



$\dfrac{ds}{d\xi} = \tan\omega$

$\dfrac{d\xi}{ds} = \tan\left(\dfrac{\pi}{2}-\omega\right) = \dfrac{1}{\tan\omega} = \left(\dfrac{ds}{d\xi}\right)^{-1}$

More formally

$\dfrac{d}{ds}\xi(s) = \left(\dfrac{d}{d\xi}s(\xi)\right)^{-1}\Bigg|_{\xi=\xi(s)}$; $\dfrac{d}{d\xi}s(\xi) = \left(\dfrac{d}{ds}\xi(s)\right)^{-1}\Bigg|_{s=s(\xi)}$

$\Rightarrow \tilde{\underline{t}}(s) = \tilde{\underline{x}}'(s) = (\cos\tilde{\alpha}(s), \sin\tilde{\alpha}(s))$, where $\tilde{\alpha}(s) = \alpha(\xi(s))$

$\hookrightarrow$ a unit vector, consistent with "unit-speed parametrization"
[no need to "normalize"]

$\tilde{c}(s) = c(\xi(s)) = \dfrac{\frac{d\alpha}{d\xi}}{\frac{ds}{d\xi}}\Bigg|_{\xi=\xi(s)} \overset{(*)}{=} \dfrac{d\alpha}{d\xi}(\xi(s))\,\dfrac{d\xi}{ds}(s) = \dfrac{d}{ds}\tilde{\alpha}(s)$ by chain rule

$\Rightarrow \tilde{c}(s) = \tilde{\alpha}'(s) = \dfrac{d\tilde{\alpha}}{ds}(s)$ [no need to "normalize"]

$\dfrac{d}{ds}\tilde{\underline{t}}(s) = \tilde{\alpha}'(s)\underbrace{(-\sin\tilde{\alpha}(s), \cos\tilde{\alpha}(s))}_{\tilde{\underline{n}}(s)} = \tilde{c}(s)\,\tilde{\underline{n}}(s)$

$\dfrac{d}{ds}\tilde{\underline{n}}(s) = \tilde{\alpha}'(s)(-\cos\tilde{\alpha}(s), -\sin\tilde{\alpha}(s)) = -\tilde{c}(s)\,\tilde{\underline{t}}(s)$

Remark: $s \mapsto \underline{u}(s)$ unit vector

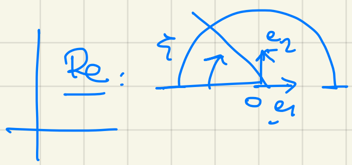
Then $\underline{u}'(s)\cdot\underline{u}(s) \equiv 0$. Indeed

$\underline{u}\cdot\underline{u} \equiv 1 \Rightarrow \underline{u}'\cdot\underline{u} + \underline{u}\cdot\underline{u}' = 0$

or $2\,\underline{u}\cdot\underline{u}' \equiv 0$

Example 1bis (circular arc in arc-length parametrization)

$s'(\xi) = |\underline{x}'(\xi)| = R$



$\xi \in [0, \pi]$
$\underline{x}(\xi) = -R(\cos\xi\, \underline{e}_1 - \sin\xi\, \underline{e}_2)$
$\alpha(\xi) = \frac{\pi}{2} - \xi\,, \quad c(\xi) = -\frac{1}{R}$

$\Rightarrow \frac{ds}{d\xi} = R \Rightarrow ds = R\, d\xi \Rightarrow s(\xi) = R\xi \Rightarrow \xi(s) = \frac{s}{R}$ (inverse)

[Remark: compare with (*) in previous page]

$\tilde{\underline{x}}(s) = \underline{x}(\xi(s)) = -R\left(\cos\frac{s}{R}\,\underline{e}_1 - \sin\frac{s}{R}\,\underline{e}_2\right)$

$\tilde{\underline{t}}(s) = \tilde{\underline{x}}'(s) = -\cancel{R}\left(-\frac{1}{\cancel{R}}\sin\frac{s}{R}\,\underline{e}_1 - \frac{1}{\cancel{R}}\cos\frac{s}{R}\,\underline{e}_2\right) \Rightarrow \tilde{\underline{t}}(s) = \left(\sin\frac{s}{R}, \cos\frac{s}{R}\right)$

$\tilde{\alpha}(s) = \alpha(\xi(s)) = \frac{\pi}{2} - \frac{s}{R}$

Check: $\tilde{t}(s) = (\cos\tilde{\alpha}(s), \sin\tilde{\alpha}(s))$ ✓

$\tilde{c}(s) = c(\xi(s)) = -\frac{1}{R}$

Check: $\tilde{c}(s) = \tilde{\alpha}'(s) = \frac{d}{ds}\left(\frac{\pi}{2} - \frac{s}{R}\right) = -\frac{1}{R}$ ✓

Frenet formula for normal and curvature

$s \mapsto \tilde{\underline{x}}(s)$

$\tilde{\underline{t}}(s) = \tilde{\underline{x}}'(s) = (\cos\tilde{\alpha}(s), \sin\tilde{\alpha}(s))$

When parametrized by arc-length the curve is traced at unit speed.

$\tilde{\underline{t}}'(s) = \tilde{\underline{x}}''(s) \perp \tilde{\underline{t}}(s)$

because for a unit vector $\tilde{\underline{t}}' \cdot \underline{t} \equiv 0$

We have chosen to write $\tilde{\underline{t}}'(s) = \tilde{c}(s)\,\tilde{\underline{n}}(s)$, where $\tilde{\underline{n}}(s) = \tilde{\underline{t}}^{\perp}(s) = \underline{R}(e_3, \frac{\pi}{2})\,\tilde{\underline{t}}(s)$

Alternatively, $\tilde{\underline{x}}''(s)$ is the "(centripetal) acceleration"

$$\tilde{\underline{t}}'(s) = \tilde{\underline{x}}''(s) = \underbrace{|\tilde{\underline{t}}'(s)|}_{=:\,\tilde{\kappa}}\ \underbrace{\frac{\tilde{\underline{t}}'(s)}{|\tilde{\underline{t}}'(s)|}}_{\tilde{\underline{\bar{n}}}}$$

[OK provided that $|\tilde{\underline{t}}'(s)| = |\tilde{c}(s)| \neq 0$!]
$\tilde{\underline{\bar{n}}}$ not defined when $\tilde{\underline{t}}' = 0$: problem for straight rods

$\tilde{\kappa}$: normal curvature > 0

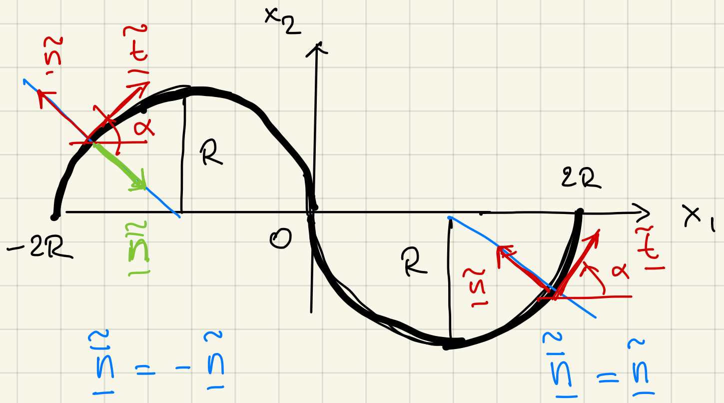
$\tilde{\underline{\bar{n}}}$: a unit vector $\perp$ to the curve, always pointing towards center of osculating circle (centripetal acceleration)

$\tilde{\underline{t}}'(s) = \tilde{c}(s)\,\tilde{\underline{n}}(s) = \tilde{\kappa}(s)\,\tilde{\underline{\bar{n}}}(s) \Rightarrow$ (Frenet)

$|\tilde{c}(s)| = \tilde{\kappa}(s)$
$\tilde{c}(s) = \pm\tilde{\kappa}(s)$, + when $\tilde{c}(s) > 0$
$\tilde{\underline{n}}(s) = \pm\tilde{\underline{\bar{n}}}(s)$

Example 2 — Two arcs of circle of radius R

Parametrize by arc-length, starting from 1st end with coordinates $(-2R, 0)$



At point $O = (0,0)$

jump of $\tilde{c}$ (not of $\tilde{\kappa}$)

jump of $\tilde{\underline{n}}$ (not of $\tilde{\underline{n}}$)

$\tilde{\alpha}'(s) < 0$

$\Rightarrow \tilde{c}(s) < 0$

$\tilde{\kappa}(s) = -\tilde{c}(s) = \frac{1}{R}$

$\tilde{\alpha}'(s) > 0$

$\Rightarrow \tilde{c}(s) > 0$

$\tilde{\kappa}(s) = \tilde{c}(s) = \frac{1}{R}$

EX. Write down arc-length parametrization explicitly for Example 2.