Problem solving session 16/11/2022, Self-evaluation: 150/150

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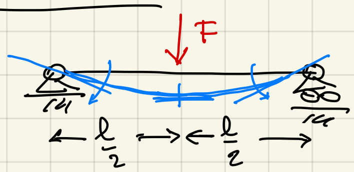
Two exercises on the linear theory

Using linear theory, compute and graph the internal forces, and compute the elastic displacements in the cases sketched below, write answers on this sheet

Exercise 1 (total 70 points)

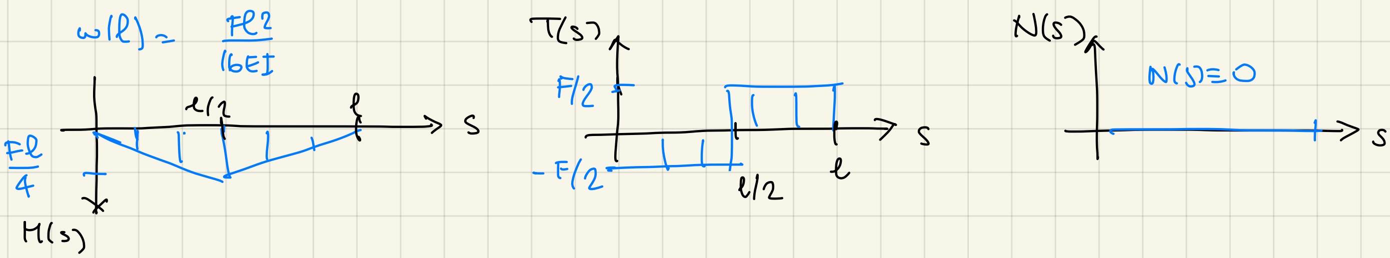
Data: $F = -F e_2$ applied at $s = \frac{\ell}{2}$



Diagrams of $N(s), T(s), M(s)$ (10 points each)

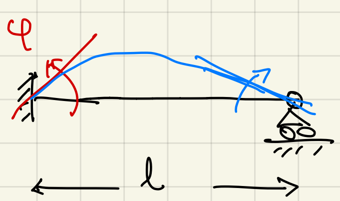
$w(0) = -\frac{F\ell^2}{16EI}$ $\quad w\left(\frac{\ell}{2}\right) = 0$ $\quad w\left(\frac{\ell}{2}\right) = -\frac{F\ell^3}{48EI}$ $\quad u(\ell) = 0$ (10 points each)

$w(\ell) = \frac{F\ell^2}{16EI}$



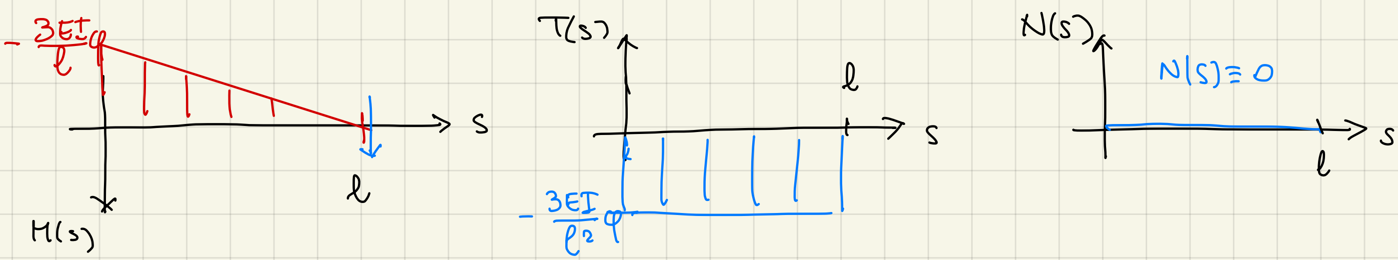
Exercise 2 (total 80 points)

Data: $w'(0) = \varphi$

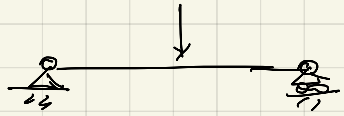


Diagrams of $N(s), T(s), M(s)$ (20 points each)

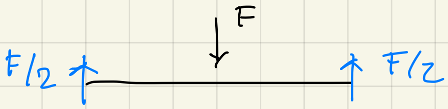
$w(\ell) = -\frac{\varphi}{2}$ (20 points)



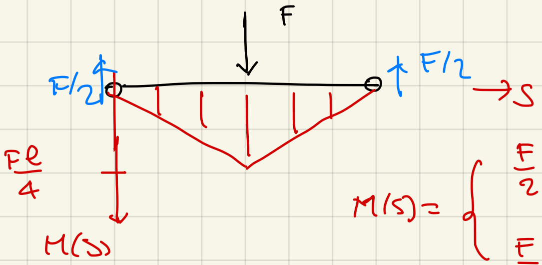
1)

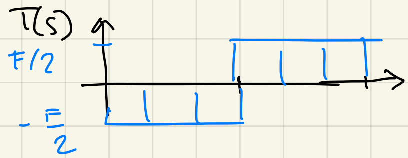


Isostatic : solve for statics first



FBD





$N(s) \equiv 0$

$$M(s) = \begin{cases} \frac{F}{2} s & s \in (0, \frac{\ell}{2}) \\ \frac{F}{2}(\ell - s) & s \in (\frac{\ell}{2}, \ell) \end{cases}$$

$$w''(s) = \begin{cases} \frac{F}{2EI} s, & s \in (0, \frac{\ell}{2}) \\ \frac{F}{2EI}(\ell - s), & s \in [\frac{\ell}{2}, \ell) \end{cases}$$

$\longrightarrow$ integrate between 0 and s

$$w'(s) = w'(0) + \frac{F}{4EI} s^2, \quad w(s) = \underbrace{w(0)}_{=0} + w'(0)s + \frac{F}{12EI} s^3$$

$$= -\frac{F}{2EI}(s - \ell)$$

$$w'(\tfrac{\ell}{2}) = w'(0) + \frac{F}{16EI}\ell^2, \quad w(\tfrac{\ell}{2}) = w'(0)\frac{\ell}{2} + \frac{F}{96EI}\ell^3$$

Integrate between $\frac{\ell}{2}$ and $s \in (\frac{\ell}{2}, \ell)$

$$w'(s) = w'(\tfrac{\ell}{2}) - \frac{F}{2EI} \frac{(\sigma - \ell)^2}{2} \Big|_{\frac{\ell}{2}}^{s} = w'(\tfrac{\ell}{2}) - \frac{F}{2EI}\frac{(s-\ell)^2}{2} + \frac{F}{2EI}\frac{\ell^2}{8}$$

$$w(s) = w(\tfrac{\ell}{2}) + w'(\tfrac{\ell}{2})(s - \tfrac{\ell}{2}) - \frac{F}{2EI}\frac{(s-\ell)^3}{6} + \frac{F}{2EI}\frac{\ell^2}{8}(s - \tfrac{\ell}{2})$$

BC $\quad 0 = w(\ell) = w'(0)\frac{\ell}{2} + \frac{F}{96EI}\ell^3 + w'(\tfrac{\ell}{2})\frac{\ell}{2} - \frac{F}{2EI}\frac{\ell^3}{48} + \frac{F}{16EI}\frac{\ell^3}{2}$

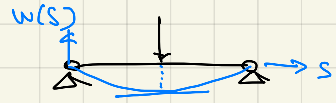
$$= w'(0)\frac{\ell}{2} + \left(w'(0) + \frac{F}{16EI}\ell^2\right)\frac{\ell}{2} + \frac{F\ell^3}{96EI}\left(1 - 1 + 3\right)$$

$$= w'(0)\ell + \frac{F\ell^3}{96EI}(3 + 3) = w'(0)\ell + \frac{F\ell^3}{16EI} \qquad \left[\frac{6}{96} = \frac{1}{16}\right]$$

$$\Rightarrow w'(0) = -\frac{F\ell^2}{16EI}, \quad w'(\tfrac{\ell}{2}) = 0, \quad w'(s) = -\frac{F}{4EI}\left((s-\ell)^2 - \frac{\ell^2}{4}\right), \quad w'(\ell) = +\frac{F\ell^2}{16EI}$$

$$w(s) = -\frac{F\ell^2}{16EI}s + \frac{F}{12EI}s^3, \quad s \in (0, \tfrac{\ell}{2})$$

$$w(\tfrac{\ell}{2}) = -\frac{F\ell^3}{32EI} + \frac{F\ell^3}{96EI} = \frac{F\ell^3}{96EI}(1 - 3) = -\frac{F\ell^3}{48EI}$$

To speed up things can use $w'(\frac{\ell}{2}) = 0$ by symmetry 

and integrate only on $s \in (0, \frac{\ell}{2})$, recovering $w(s)$, $s \in (\frac{\ell}{2}, \ell]$ by symmetry

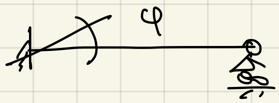
$$w'(s) = w'(0) + \frac{F}{4EI} s^2 \quad , \quad w'(\tfrac{\ell}{2}) = w'(0) + \frac{F\ell^2}{16EI} = 0$$

$$\Rightarrow \quad w'(0) = -\frac{F\ell^2}{16EI} \quad , \quad \left[w'(\ell) = \frac{F\ell^2}{16EI} \text{ by symmetry } (!) \right]$$

$$w(s) = -\frac{F\ell^2}{16EI} s + \frac{F}{12EI} s^3 \quad , \quad w(\tfrac{\ell}{2}) = -\frac{F\ell^3}{32EI} + \frac{F\ell^3}{96EI}$$

$$= \frac{F\ell^3}{EI} \frac{-3+1}{96}$$

$$= -\frac{F\ell^3}{48EI}$$

2) 

Hyperstatic : use $EI\, w'''' = M'' = 0$

$$w(s) = a + bs + cs^2 + ds^3$$

$$\left.\begin{aligned} w(0) &= a = 0 \\ w'(0) &= b = \varphi \end{aligned}\right\} \quad w(s) = \varphi s + cs^2 + ds^3$$

$$w'(s) = \varphi + 2cs + 3ds^2$$
$$w''(s) = 2c + 6ds$$

$$w(\ell) = \varphi \cancel{\ell} + c\ell^{\cancel{2}} + d\ell^{\overset{2}{\cancel{3}}} = 0$$

$$M(\ell) = 0 \Rightarrow w''(\ell) = \cancel{2}c + \overset{3}{\cancel{6}}d\ell = 0$$

Solve
$$c\cancel{\ell} + d\ell^{\cancel{2}} = -\varphi/\ell$$
$$c + 3d\ell = 0$$

det $= 3\ell - \ell = 2\ell > 0$

$$(-) \quad 0 \quad -2d\ell = -\frac{\varphi}{\ell} \Rightarrow d = \frac{\varphi}{2\ell^2}$$
$$c = -3d\ell = -\frac{3\varphi}{2\ell}$$

$$w(s) = \varphi\left(s - \frac{3}{2\ell}s^2 + \frac{1}{2\ell^2}s^3\right)$$

Check : $w(\ell) = \varphi\left(\ell - \frac{3}{2}\ell + \frac{1}{2}\ell\right) = 0$

$$w'(s) = \varphi - \frac{3\varphi}{\ell}s + \frac{3\varphi}{2\ell^2}s^2$$

$$w'(\ell) = \varphi - 3\varphi + \frac{3}{2}\varphi = \frac{2-6+3}{2}\varphi = -\frac{\varphi}{2}$$

$M''(s) = 0 \Rightarrow M(s)$ affine

$$M(s) = EI\left(-\frac{3\varphi}{\ell} + \frac{3\varphi}{\ell^2}s\right) = -\frac{3EI\varphi}{\ell^2}(\ell - s) \le 0$$

$$M(0) = -\frac{3EI}{\ell}\varphi$$