Mechanics, Robotics

Equilibrium of 2D rods

Lectures 3 and 4

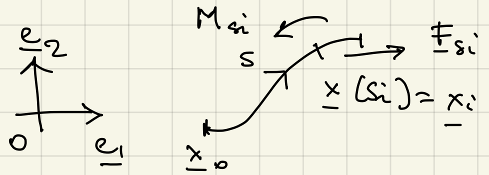
8-10/10/2025

Lecture 3-4 Statics of planar rods

8/10/2025

Start by describing all the loads that may act on the rods.

1. Concentrated forces and couples



$$\underline{F}_{si} = F_{si1}\,\underline{e}_1 + F_{si2}\,\underline{e}_2$$

$$\underline{M}_{si} = M_{si}\,\underline{e}_3$$

positive component along $\underline{e}_3 \Leftrightarrow$ counter-clockwise (thumb) (right hand)

total force $\quad \underline{\mathcal{F}} = \sum_i \underline{F}_{si}$

total moment with respect to $\underline{x}_0$ $\quad \underline{\mathcal{M}}_{x_0} = \sum_i (\underline{x}_i - \underline{x}_0) \wedge \underline{F}_{si} + \sum_i \underline{M}_{si}$

$$\mathcal{M}_{x_0} = \underline{\mathcal{M}}_{x_0} \cdot \underline{e}_3 = \underbrace{\underline{e}_3 \cdot \sum_i (\underline{x}_i - \underline{x}_0) \wedge \underline{F}_{si}}_{=: \sum_i (\underline{x}_i - \underline{x}_0) \wedge \underline{F}_{si}} + \sum_i M_{si}$$

2. Distributed forces

: forces per unit length along the curve (linear densities)

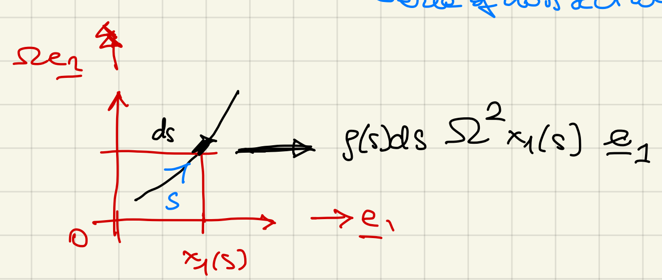
$$s \mapsto \underline{f}(s) = (f_1(s), f_2(s)) = f_t(s)\,\underline{t}(s) + f_n(s)\,\underline{n}(s) \qquad \text{applied at } \underline{x}(s)$$

Examples

- weight $\quad \underline{f}(s) = \rho(s)\,\underline{g} \quad$ where typically $\underline{g} = -g\,\underline{e}_2$ ($\simeq 10\ \text{m s}^{-2}$; vertical, upwards)

  $\rho(s)$ = mass p. unit length = $\rho_v(s)\,A(s)$ (volumetric mass density; area of cross section)

- centrifugal force $\quad \underline{f}(s) = \rho(s)\,\Omega^2 x_1(s)\,\underline{e}_1$



These forces contribute to total force $\underline{\mathcal{F}}$ through integrals: $\int_0^\ell \underline{f}(\sigma)\,d\sigma$

3. Distributed couples

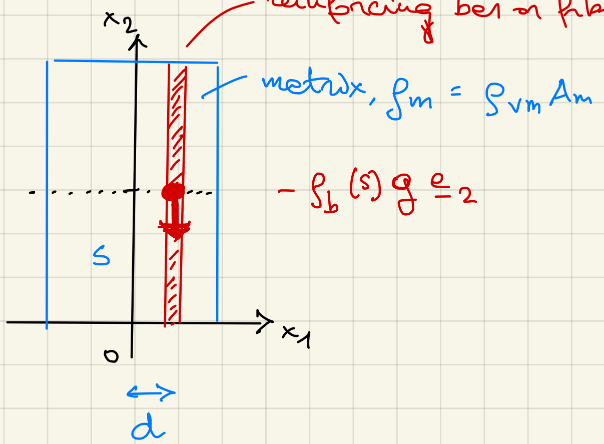
: couples per unit length along the curve (linear densities)

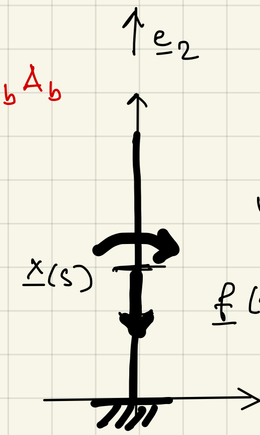
$$s \mapsto \underline{m}(s) = m(s)\,\underline{e}_3 \qquad \text{applied at } \underline{x}(s)$$

These couples do not contribute to total force $\underline{\mathcal{F}}$ and contribute to total moment $\mathcal{M}_{x_0}$ through integrals: $\int_0^\ell m(\sigma)\,d\sigma$

Distributed couples typically arise from distributed forces applied with an offset (eccentrically) with respect to $\underline{x}(s)$ (mid-axis of the rod)

Examples



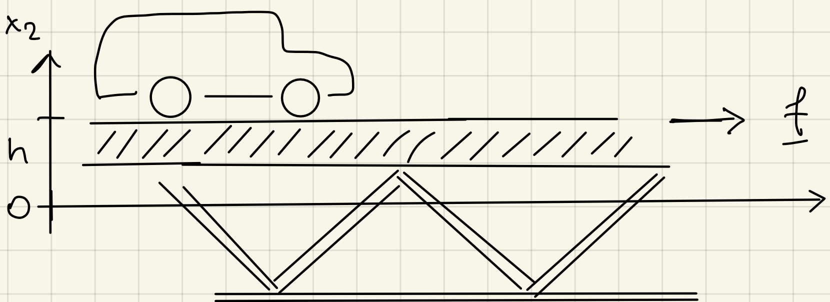


$$\underline{m}(s) = -\rho_b(s)\, g\, d \quad \underline{e}_3$$

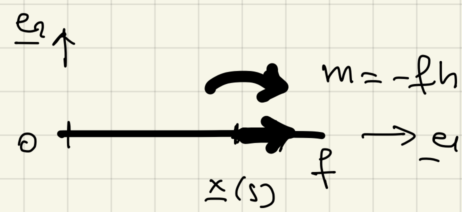
$$m(s) = -\rho_b(s)\, g\, d$$

$$\underline{f}(s) = -\left(\rho_m(s) + \rho_b(s)\right) g\, \underline{e}_2$$

Another example is the load transmitted by a car braking on a bridge

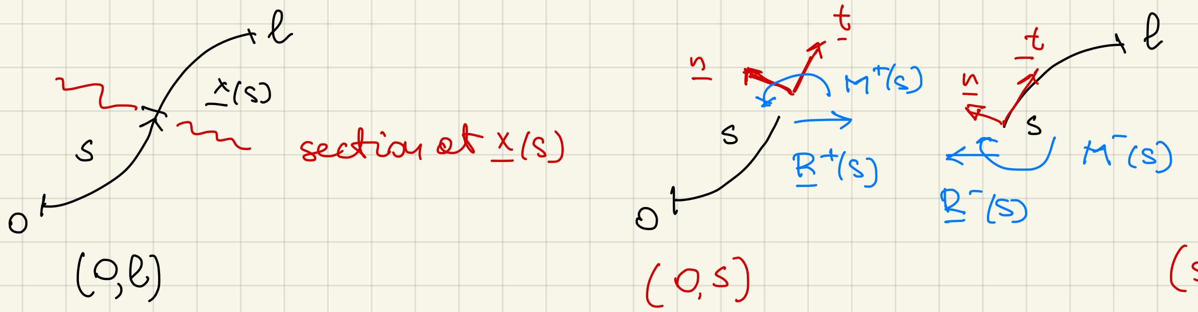


$$\underline{f} = f\, \underline{e}_1$$



4. Internal "forces" (stresses in 3D): "forces" transmitted through internal points

Take $s \in (0,\ell)$ and assume that no concentrated "forces" (forces or couples) act near $\underline{x}(s)$



The action of + part (s,ℓ) on the − part $(0,\ell)$ can be represented as

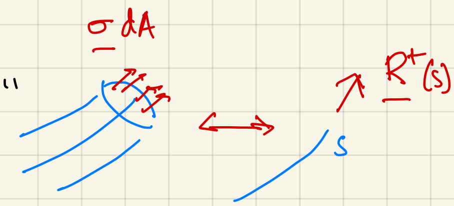
- force $\underline{R}^+(s)$ applied at $\underline{x}(s)$ (total force)
- couple $M^+(s)\underline{e}_3$, or $M^+(s)$ (total moment with respect to $\underline{x}(s)$) applied at $\underline{x}(s)$

similarly

The action of − part $(0,s)$ on the + part (s,ℓ) can be represented as

- force $\underline{R}^-(s)$ applied at $\underline{x}(s)$
- couple $M^-(s)\underline{e}_3$ or $M^-(s)$ applied at $\underline{x}(s)$

represented means "with the same total force and moment"



We can write:

$$\underline{R}^+(s) = N^+(s)\,\underline{t}(s) + T^+(s)\,\underline{n}(s)$$

$N^+(s)$ → axial force (sforzo normale)

$T^+(s)$ → shear force (sforzo di taglio)

$M^+(s)$ bending moment

Remark "action and reaction" principle would suggest $\underline{R}^-(s) = -\underline{R}^+(s)$ and

$M^-(s) = -M^+(s)$.

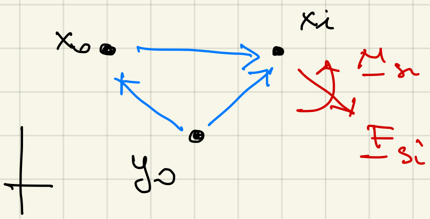
We will see that this is indeed true.

Equilibrium

For a rod to be in equilibrium we'll require that it's in equilibrium as a whole, and also that every one of its parts is in equilibrium. Equilibrium means:

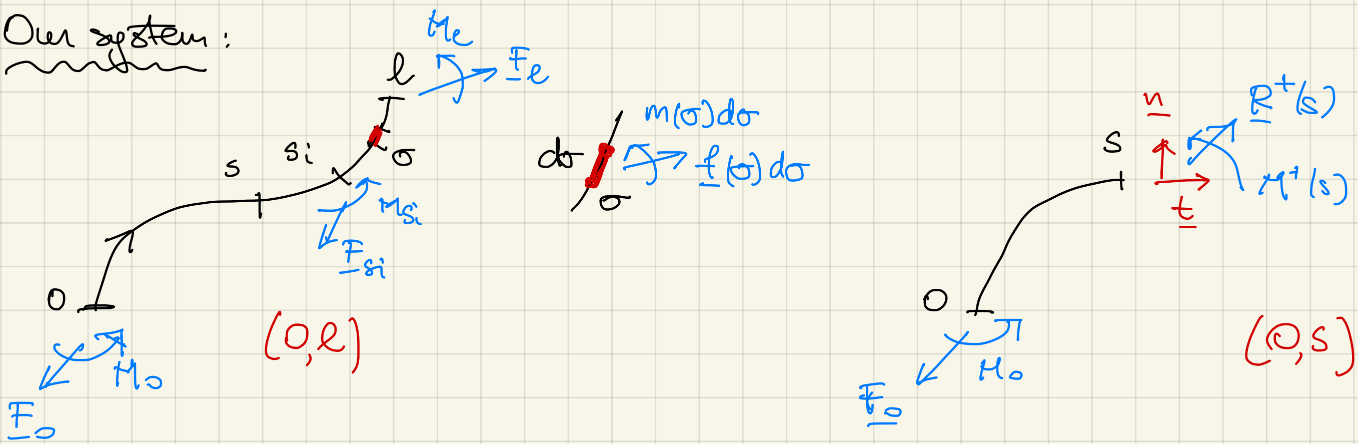
$$\begin{cases} \underline{\mathcal{F}} = \underline{0} & \text{total force} = 0 \\ \mathcal{M}_{x_0} = 0 & \text{total moment w/r to an arbitrary chosen point } x_0 = 0 \end{cases}$$

<u>Remark</u> The reason the choice of x_0 is immaterial is that, if the equations above hold, then also $\mathcal{M}_{y_0} = 0 \ \forall y_0$. Indeed



$$\underline{\mathcal{M}}_{y_0} = \sum_i \underbrace{(x_i - y_0)}_{(x_i - x_0)+(x_0 - y_0)} \wedge \underline{F}_{si} + \sum_i \underline{M}_{si} = \underline{\mathcal{M}}_{x_0} + (x_0 - y_0) \wedge \underbrace{\sum_i \underline{F}_{si}}_{\underline{\mathcal{F}}}$$

Our system:



no external concentrated loads in a neighbourhood of s

<u>Global equilibrium</u> $(0,\ell)$

$$\underline{0} = \underline{F}_0 + \int_0^\ell \underline{f}(\sigma)d\sigma + \sum_{0<s_i<\ell} \underline{F}_{si} + \underline{F}_\ell$$

$$0 = \mathcal{M}_{x(0)} = M_0 + \int_0^\ell m(\sigma)d\sigma + \sum_{0<s_i<\ell} M_{si} + M_\ell$$
$$+ \int_0^\ell (\underline{x}(\sigma) - \underline{x}(0)) \wedge \underline{f}(\sigma)d\sigma + \sum_{0<s_i<\ell} (\underline{x}(s_i) - \underline{x}(0)) \wedge \underline{F}_{si} + (\underline{x}(\ell) - \underline{x}(0)) \wedge \underline{F}_\ell$$

<u>Equilibrium of part $(0,s)$</u>

$$\underline{0} = \underline{F}_0 + \int_0^s \underline{f}(\sigma)d\sigma + \sum_{0<s_i<s} \underline{F}_{si} + \underline{R}^+(s)$$

$$0 = \mathcal{M}_{x(0)} = M_0 + \int_0^s m(\sigma)d\sigma + \sum_{0<s_i<s} M_{si} + M^+(s)$$
$$+ \int_0^s (\underline{x}(\sigma) - \underline{x}(0)) \wedge \underline{f}(\sigma)d\sigma + \sum_{0<s_i<s} (\underline{x}(s_i) - \underline{x}(0)) \wedge \underline{F}_{si} + (\underline{x}(s) - \underline{x}(0)) \wedge \underline{R}^+(s)$$

$$\underline{0} = \underline{F}_0 + \int_0^s \underline{f}(\sigma)\,d\sigma + \sum_{0<s_i<s} \underline{F}_{s_i} + \underline{R}^+(s)$$

$$0 = m_{x(0)} = M_0 + \int_0^s m(\sigma)\,d\sigma + \sum_{0<s_i<s} M_{s_i} + M^+(s)$$
$$+ \int_0^s (\underline{x}(\sigma)-\underline{x}(0))\wedge \underline{f}(\sigma)\,d\sigma + \sum_{0<s_i<s} (\underline{x}(s_i)-\underline{x}(0))\wedge \underline{F}_{s_i} + (\underline{x}(s)-\underline{x}(0))\wedge \underline{R}^+(s)$$

These formulas show that $\underline{R}^+(s)$ and $M^+(s)$ are continuous away from points s_i where concentrated loads are applied. (actually: differentiable, see later)

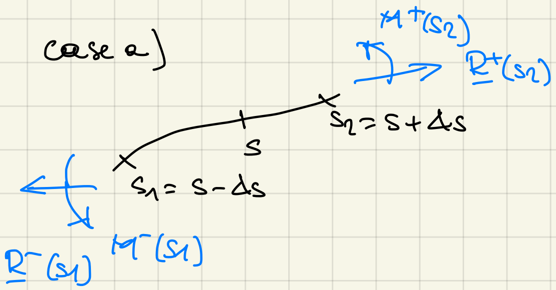
At s_i where $\underline{F}_{s_i}$ (resp. M_{s_i}) is applied, $\underline{R}^+(s)$ (resp., $M^+(s)$) has a jump discontinuity. (see later for jump relations)

Equilibrium of part (s,ℓ)

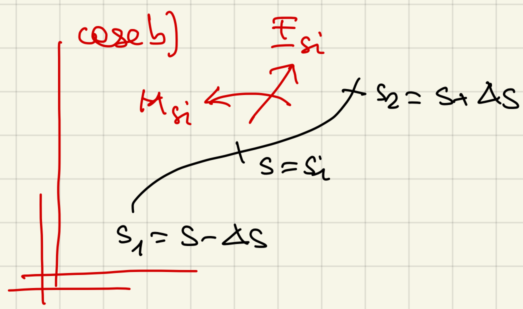
Analogous to case for $(0,s)$, and it shows that $\underline{R}^-(s)$ (respectively, $M^-(s)$) is differentiable away from points s_i where concentrated forces (resp., concentrated couples) are applied

Equilibrium of part $(0 < s_1 = s - \Delta s,\ s_2 = s + \Delta s < \ell)$

case a)



case b)



$$\underline{0} = \underline{R}^-(s_1) + \int_{s_1}^{s_2} \underline{f}(\sigma)\,d\sigma + \sum_{s_1<s<s_2} \underline{F}_{s_i} + \underline{R}^+(s_2)$$

$$0 = m_{x(s)} = M^-(s_1) + \int_{s_1}^{s_2} m(\sigma)\,d\sigma + \sum_{s_1<s_i<s_2} M_{s_i} + M^+(s_2)$$
$$+ (\underline{x}(s_1)-\underline{x}(s))\wedge \underline{R}^-(s_1) + \int_{s_1}^{s_2} (\underline{x}(\sigma)-\underline{x}(s))\wedge \underline{f}(\sigma)\,d\sigma + \sum_{s_1<s_i<s_2} (\underline{x}(s_i)-\underline{x}(s))\wedge \underline{F}_{s_i} + (\underline{x}(s_2)-\underline{x}(s))\wedge \underline{R}^+(s_2) \qquad (*)$$

$$\underline{0} = \underline{R}^-(s_1) + \int_{s_1}^{s_2} \underline{f}(\sigma)\,d\sigma + \underbrace{\sum_{s_1<s<s_2} \underline{F}_{s_i}}_{} + \underline{R}^+(s_2)$$

$$\underline{0} = \underline{M}_{\underline{x}(s)} = \underline{M}^-(s_1) + \int_{s_1}^{s_2} \underline{m}(\sigma)\,d\sigma + \underbrace{\sum_{s_1<s_i<s_2} \underline{M}_{s_i}}_{} + \underline{M}^+(s_2)$$

$$+ (\underline{x}(s_1)-\underline{x}(s)) \wedge \underline{R}^-(s_1) + \int_{s_1}^{s_2} (\underline{x}(\sigma)-\underline{x}(s)) \wedge \underline{f}(\sigma)\,d\sigma + \underbrace{\sum_{s_1<s_i<s_2} (\underline{x}(s_i)-\underline{x}(s)) \wedge \underline{F}_{s_i}}_{} + (\underline{x}(s_2)-\underline{x}(s)) \wedge \underline{R}^+(s_2) \qquad (*)$$

a) Case of no concentrated loads at s, hence on $(s_1 = s-\Delta s,\ s_2 = s+\Delta s)$ for Δs small enough.

$$\underline{0} = \underline{R}^-(s-\Delta s) + \int_{s-\Delta s}^{s+\Delta s} \underline{f}(\sigma)\,d\sigma + \boxed{\underline{0}} + \underline{R}^+(s+\Delta s)$$

Take $\lim_{\Delta s \to 0}$ and recall that $\underline{R}^\pm(s)$, $\underline{M}^\pm(s)$ are continuous away from s_i

$$\lim_{\Delta s\to 0} \underline{0} = \underline{R}^-(s) + \underline{0} + \boxed{\underline{0}} + \underline{R}^+(s)$$

$\#$ $\boxed{+\underline{F}_{s_i}}$

$$\Rightarrow \boxed{\underline{R}^-(s) = -\underline{R}^+(s)} \quad \text{(action/reaction 1)}$$

$$\underline{0} = \underline{M}_{\underline{x}(s)} = \underline{M}^-(s-\Delta s) + \int_{s-\Delta s}^{s+\Delta s} \underline{m}(\sigma)\,d\sigma + \boxed{\underline{0}} + \underline{M}^+(s+\Delta s) + (\underline{x}(s-\Delta s)-\underline{x}(s)) \wedge \underline{R}^-(s-\Delta s)$$

$$+ \int_{s-\Delta s}^{s+\Delta s} (\underline{x}(\sigma)-\underline{x}(s)) \wedge \underline{f}(\sigma)\,d\sigma$$

$$+ \boxed{\underline{0}} + (\underline{x}(s+\Delta s)-\underline{x}(s)) \wedge \underline{R}^+(s+\Delta s)$$

$$\lim_{\Delta s\to 0} \underline{0} = \underline{M}^-(s) + \underline{0} + \boxed{\underline{0}} + \underline{M}^+(s) + \underline{0} + \underline{0} + \boxed{\underline{0}} + \underline{0}$$

$\#$ $+\underline{M}_{s_i} + \underbrace{(\underline{x}(s_i)-\underline{x}(s_i)) \wedge \underline{F}_{s_i}}_{=0}$

$$\Rightarrow \boxed{\underline{M}^-(s) = -\underline{M}^+(s)} \quad \text{(action/reaction 2)}$$

Notation write $\underline{R}^+(s) = \underline{R}(s)$ and $\underline{R}^-(s) = -\underline{R}(s)$
$\underline{M}^+(s) = \underline{M}(s)$ $\underline{M}^-(s) = -\underline{M}(s)$

These are well defined continuous functions away from $s = s_i$. In particular,

$$\underline{R}(s) = \underset{\sim}{N}(s)\,\underline{t}(s) + \underset{\sim}{T}(s)\,\underline{n}(s) \qquad \text{continuous}$$

these are also continuous if α is continuous

b) Case $s = s_i$. Need to add $\underline{F}_{s_i}$ and $\underline{M}_{s_i}$ to the balance equations above, see $\#$

Case b) continued

Know from balances on $(0,s)$ that $\underline{R}(s)$, $M(s)$ are not defined at $s=s_i$ because they jump.

But they have well defined limits from right $(s+)$ and left $(s-)$, and jumps at $s=s_i$

$$0 = \underline{R}^-(s_i-) + \underline{F}_{s_i} + \underline{R}^+(s_i+) \Rightarrow \underline{R}(s_i+) - \underline{R}(s_i-) + \underline{F}_{s_i} = 0 \Rightarrow [\![\underline{R}]\!](s_i) + \underline{F}_{s_i} = \underline{0}$$

$$0 = M^-(s_i-) + M_{s_i} + M^+(s_i+) \Rightarrow M(s_i+) - M(s_i-) + M_{s_i} = 0 \Rightarrow [\![M]\!](s_i) + M_{s_i} = 0$$

Remark [skip]

Assume $s \neq s_i$. Can combine partial equilibria of $(0,s)$ and (s,ℓ) to obtain global $(0,\ell)$ equil. by summing the two partial balances. Or can use global $(0,\ell)$ and partial $(0,s)$ equil. to obtain equil. of (s,ℓ) by subtraction

$$\underline{\mathcal{F}}(0,\ell) = \underline{\mathcal{F}}(0,s) + \underline{\mathcal{F}}(s,\ell)$$

$$\mathcal{M}_{x(0)}(0,\ell) = \mathcal{M}_{x(0)}(0,s) + \mathcal{M}_{x(0)}(s,\ell)$$

$\underline{R}^+(s)$ and $\underline{R}^-(s)$ cancel out

$M^+(s)$ and $M^-(s)$ " "

(by action and reaction)

Differential equations of equilibrium

Take $s \neq s_i$, s.t. there are no concentrated loads in $s-\Delta s,\ s+\Delta s$

Rk: In this case, $\underline{R}$ and M are differentiable at s

Consider force and $M_{x(0)}$ balance for $(0,s]$ and take $\frac{d}{ds}$

$$\underline{0} = \mathcal{F}(0,s) = \underline{F}_0 + \int_0^s \underline{f}(\sigma)\,d\sigma + \sum_{s_i<s} \underline{F}_{s_i} + \underline{R}(s)$$

$$\Rightarrow \boxed{\underline{0} = \underline{f}(s) + \underline{R}'(s)} \quad (*)$$

$$0 = \mathcal{M}_{x(0)}(0,s) = M_0 + \int_0^s m(\sigma)\,d\sigma + \sum_{s_i<s} M_{s_i} + M(s)$$

$$+ \int_0^s (\underline{x}(\sigma)-\underline{x}(0)) \wedge \underline{f}(\sigma)\,d\sigma + \sum_{s_i<s} (\underline{x}(s_i)-\underline{x}(0)) \wedge \underline{F}_{s_i} + (\underline{x}(s)-\underline{x}(0)) \wedge \underline{R}(s)$$

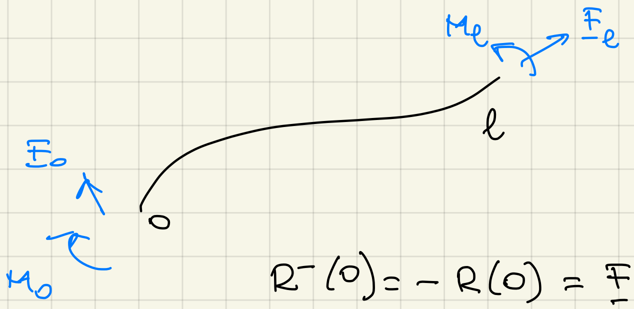
$$\Rightarrow 0 = m(s) + M'(s) + (\underline{x}(s)-\underline{x}(0)) \wedge \underline{f}(s) + \underline{t}(s) \wedge \underline{R}(s) + (\underline{x}(s)-\underline{x}(0)) \wedge \underline{R}'(s)$$

sum to zero by $(*)$

$$\underbrace{N(s)\,\underline{t}(s)}_{e_3 \cdot \underline{t} \wedge \underline{t} = 0} + T(s)\,\underbrace{\underline{n}(s)}_{e_3 \cdot \underline{t} \wedge \underline{n} = 1}$$

$$\Rightarrow \boxed{M'(s) + T(s) + m(s) = 0}$$

BC's for the differential equations of equilibrium



$$\underline{R}^+(\ell) = \underline{R}(\ell) = \underline{F}_\ell$$
$$M^+(\ell) = M(\ell) = M_\ell$$

$$R^-(0) = -R(0) = \underline{F}_0 \qquad\qquad \underline{R}(0) = -\underline{F}_0$$
$$\Rightarrow$$
$$M^-(0) = -M(0) = M_0 \qquad\qquad M(0) = -M_0$$

Summary of Statics of 2d rods

$$\lim_{s\to 0} \underline{R}(s) = -\underline{F}_0 \qquad \lim_{s\to 0} M(s) = -M_0$$

$$\frac{d}{ds}\underline{R}(s) + \underline{f}(s) = 0 \qquad (*)$$

$$\frac{d}{ds}M(s) + T(s) + m(s) = 0$$

$$[\![\underline{R}]\!](s_i) + \underline{F}_{s_i} = \underline{0}$$

$$[\![M]\!](s_i) + M_{s_i} = 0$$

$$\lim_{s\to \ell} \underline{R}(s) = \underline{F}_\ell \qquad \lim_{s\to \ell} M(s) = M_\ell$$

(*) in components, relative to the local frame $\{\underline{x}(s), \underline{t}(s), \underline{n}(s)\}$

$$0 = \underbrace{\frac{d}{ds}\left(N(s)\underline{t}(s) + T(s)\underline{n}(s)\right)}_{N'\underline{t} + N\underline{t}' + T'\underline{n} + T\underline{n}'} + f_t\,\underline{t}(s) + f_n\,\underline{n}(s)$$

$$\underline{t}' \to c\underline{n}, \qquad \underline{n}' \to -c\underline{t}$$

$$(N' - cT)\underline{t} + (T' + cN)\underline{n}$$

$$\Rightarrow \begin{cases} N' - cT + f_t = 0 \\ T' + cN + f_n = 0 \end{cases}$$

Special case straight rod, $c \equiv 0$:

$$\begin{cases} \dfrac{d}{ds}N(s) + f_t(s) = 0 \\[2mm] \dfrac{d}{ds}T(s) + f_n(s) = 0 \end{cases}$$

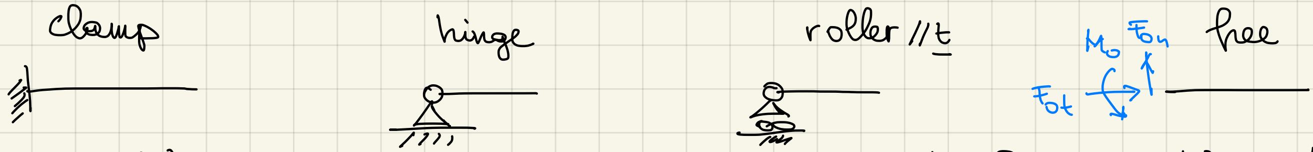
- Summary of equilibrium equations on a straight rod

$$M' + T + m = 0$$
$$T' + f_n = 0$$
$$N' + f_t = 0$$

- Boundary conditions (at $s=0$; as an exercise, work out BC's at $s=\ell$)

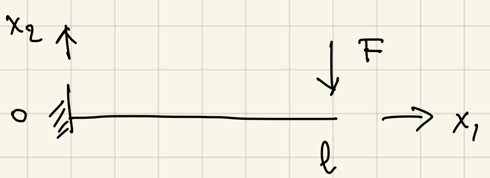
clamp | hinge | roller $//t$ | free

unknown

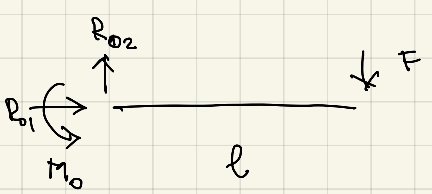


clamp	hinge	roller $//t$	free
$u(0)=w(0)=0$	$u(0)=w(0)=0$	$u(0)$ u $\quad w(0)=0$	$u(0)$, $w(0)$ U
$\alpha(0)=0$	$\alpha(0)$ u	$\alpha(0)$ u	$\alpha(0)$ U
$R_{0t} = -N(0)$ u	$R_{0t}=-N(0)$ u	$R_{0t} = -N(0) = 0$	$F_t = -N(0)$
$R_{0n} = -T(0)$ u	$R_{0n}=-T(0)$ u	$R_{0n} = -T(0)$ u	$F_{0n} = -T(0)$
$M(0)$ u	$M(0)=0$	$M(0)=0$	$M_0 = -M(0)$

- Diagrams of the components of internal forces



Free Body Diagram (FBD): replace constraints with the unknown reactions they can exert



$R_{01} = 0 \qquad\qquad\qquad\qquad = -N(0)$

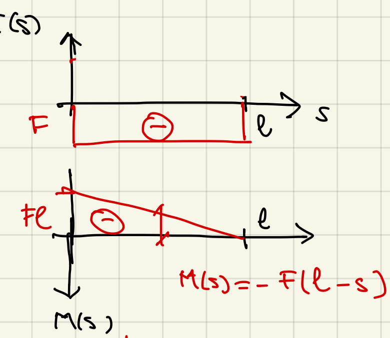
$R_{02} - F = 0 \Rightarrow R_{02} = F \qquad = -T(0)$

$M_0 - F\ell = 0 \Rightarrow M_0 = F\ell \qquad = -M(0)$

$N' = 0, \; N(0) = 0 \Rightarrow N(s) \equiv 0$

$T' = 0, \; T(\ell) = -F \Rightarrow T(s) \equiv -F$

$M' = -T, \; M(0) = -F\ell \Rightarrow M(s) = -F(\ell - s)$



Remark: the M graph is on the side of stretched fibers. Positive M stretches bottom fibers