Mechanics, Bionics

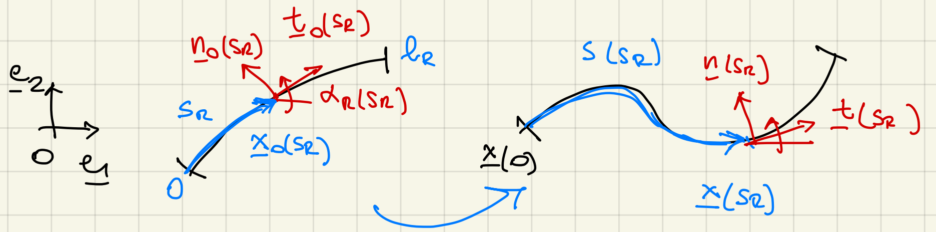
Lectures 3-4

4-6/10/23

Lecture 4 Kinematics of deformable planar rods

Planar curves as models for 1D deformable structures (from robotic arms to elephant trunks)

Deformation: a map that transforms each point of a reference curve (reference configuration) onto a point of a new, deformed curve (deformed configuration)



$[0,\ell_R] \ni s_R \mapsto \underline{x}(s_R) \in \mathbb{R}^2$

s.t. $|\underline{x}'(s_R)| \neq 0 \ \forall s_R \in [0,\ell_R]$

Assume, for simplicity, that the ref. config. is parametrized by arc-length, but drop ~ notation.

$[0,\ell_R] \ni s_R \mapsto \underline{x}_0(s_R)$ reference configuration, s_R ref. arc-length

$\underline{t}_0(s_R) = \underline{x}_0'(s_R) = (\cos\alpha_0(s_R), \sin\alpha_0(s_R))$

$c_0(s_R) = \alpha_0'(s_R)$ etc. .....

$[0,\ell_R] \ni s_R \mapsto \underline{x}(s_R)$ deformed configuration

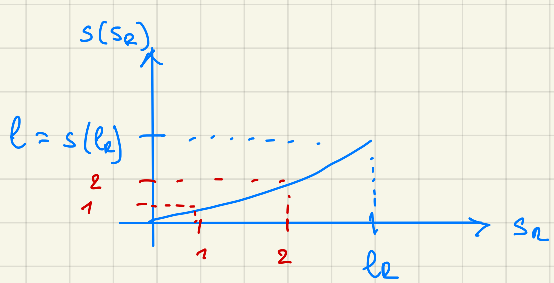
↳ ref. arc-length = marker of a point in the ref. config. (Lagrangian marker)

If the curve is stretched in the deformation, arc-length s will NOT be s_R.

$$\underline{t}(s_R) = \frac{\underline{x}'(s_R)}{|\underline{x}'(s_R)|} = (\cos\alpha(s_R), \sin\alpha(s_R))$$

$$s(s_R) = \int_0^{s_R} |\underline{x}'(\sigma)|\, d\sigma$$

$$c(s_R) = \frac{\alpha'(s_R)}{|\underline{x}'(s_R)|}$$



From $s'(s_R) = |\underline{x}'(s_R)| > 0 \ \forall s_R$, s is monotone increasing: $s_{R1} < s_{R2} \Rightarrow s(s_{R1}) < s(s_{R2})$

This has now a mechanical meaning. Two points along the curve cannot change their ordering when a deformation takes place.

$$s(\ell_R) = \int_0^{\ell_R} |\underline{x}'(\sigma)|\, d\sigma = \ell$$ length of the deformed curve

If $|\underline{x}'(s_R)| = 1 \ \forall s_R$, then s_R is arc-length also for $\underline{x}(s_R)$, and $\ell = \ell_R$

If $|\underline{x}'(s_R)| \not\equiv 1$, then s_R is NOT arc-length for $\underline{x}(s_R)$, and length of the curve changes with the deformation

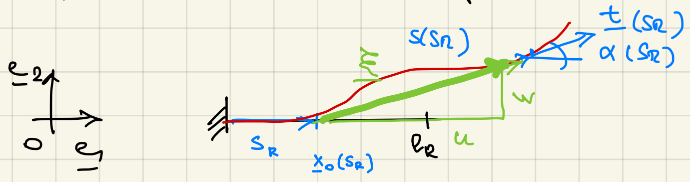
Setting $|\underline{x}'(s_R)| = 1 + \varepsilon(s_R) > 0$ we can define the extensional strain $\varepsilon \in (-1, +\infty)$

$$\varepsilon(s_R) := \left| \frac{d}{ds_R} \underline{x}(s_R) \right| - 1 \begin{cases} > 0 & \text{if the curve is locally lengthened} \quad \dfrac{ds}{ds_R} = |\underline{x}'(s_R)| > 1 \\ < 0 & \text{" " " " " shortened} \quad \dfrac{ds}{ds_R} = |\underline{x}'(s_R)| < 1 \end{cases}$$

$$\underline{\xi}(s_R) = \underline{x}(s_R) - \underline{x}_R(s_R) \quad \text{displacement (vector) at } s_R$$
$$= \xi_1(s_R)\underline{e}_1 + \xi_2(s_R)\underline{e}_2$$
$$= u(s_R)\underline{t}_0(s_R) + w(s_R)\underline{n}_0(s_R)$$

u, w axial and transversal components of the displacement at s_R

If the reference configuration is straight line



$$\underline{\xi}(s_R) = \underline{x}(s_R) - \underline{x}_R(s_R)$$
$$= u(s_R)\underbrace{\underline{t}_0(s_R)}_{\underline{e}_1} + w(s_R)\underbrace{\underline{n}_0(s_R)}_{\underline{e}_2}$$

$$\Rightarrow u(s_R) = \xi_1(s_R), \quad w(s_R) = \xi_2(s_R)$$

$$\underline{t}(s_R) = \frac{\underline{x}'(s_R)}{|\underline{x}'(s_R)|} = \frac{\underline{x}'(s_R)}{1 + \varepsilon(s_R)} = (\cos\alpha(s_R), \sin\alpha(s_R))$$

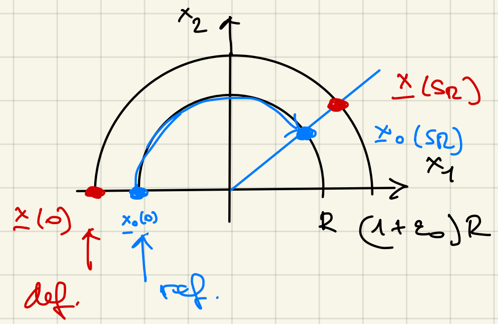
$$\omega(s_R) = \alpha(s_R) - \alpha_R(s_R) \quad \text{rotation at } s_R$$

$$k(s_R) = \omega'(s_R) = \alpha'(s_R) - \alpha_R'(s_R) \quad \text{bending strain at } s_R \quad k \in (-\infty, +\infty)$$
$$= (1 + \varepsilon(s_R))c(s_R) - c_R(s_R)$$

Remark k is NOT the difference of curvatures. It becomes the difference of curvatures ONLY when $\varepsilon = 0$ (inextensible case), so that s_R is arc length and $k = \alpha'(s_R) - \alpha_0'(s_R)$
$$= c(s_R) - c_0(s_R)$$

Extensional and bending strains $\neq 0$ are the descriptors of the change of shape of a planar rod.

Example 1



$$s(s_R) = (1+\varepsilon_0)\, s_R$$

$$\alpha(s_R) = \alpha_0(s_R)$$

$$c_0(s_R) = -\frac{1}{R}, \qquad c(s_R) = -\frac{1}{(1+\varepsilon_0)R}$$

With our definitions: $\varepsilon = \varepsilon_0$, $\kappa = \alpha'(s_R) - \alpha_0'(s_R) = 0$, $c(s_R) - c_0(s_R) = \dfrac{-\cancel{1} + (\cancel{1}+\varepsilon_0)}{(1+\varepsilon_0)R} \neq 0$

So, in this example, ext. strain $= \varepsilon_0$ and no bending strain.

Remark: This is the configuration that would be reached due to a thermal dilation $\varepsilon_0 = \alpha \Delta t$ (purely extensional) of the reference configuration

Special cases

1. Inextensible curve, $\varepsilon \equiv 0$

$|\underline{x}'| = 1+\varepsilon = 1 \Rightarrow s'(s_R) \equiv 1 \Rightarrow s(s_R) = s_R$ — s_R is arc-length on deformed curve as well [drop R from $s_{\cancel{R}}$]

$$c(s_{\cancel{R}}) = \alpha'(s_{\cancel{R}})$$

$$\kappa = \alpha'(s_{\cancel{R}}) - \alpha_0'(s_{\cancel{R}}) = c(s_{\cancel{R}}) - c_0(s_{\cancel{R}})$$

If, in addition, the ref. config. is straight $(\alpha_0'(s_{\cancel{R}}) \equiv 0)$, then $\kappa(s_{\cancel{R}}) = \alpha'(s_{\cancel{R}}) = c(s_{\cancel{R}})$

In this last case, change of shape = change of curvature from zero to $\alpha'(s_{\cancel{R}}) = c(s_{\cancel{R}})$

2. If $\varepsilon \equiv 0$ and $\kappa \equiv 0$, inextensible and inflexible rod $\Rightarrow$ the rod is rigid

$\varepsilon = 0 \Rightarrow s_R$ is arc-length on $\underline{x}$ as well and, in particular, $\underline{x}$ and $\underline{x}_0$ have the same length

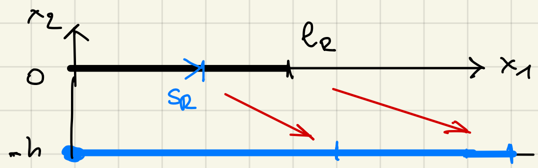
$\kappa = 0 \Rightarrow \underbrace{\alpha'(s_R)}_{=c(s_R)} = \underbrace{\alpha_0'(s_R)}_{=c_0(s_R)}$ so that $\underline{x}$ and $\underline{x}_0$ have the same curvature

The two curves $\underline{x}$ and $\underline{x}_0$ can only differ by the integration constants $\alpha(0), \alpha_0(0)$; $\underline{x}(0), \underline{x}_0(0)$ in the curvature $\rightarrow$ curve integration

$\Rightarrow$ The reference and deformed curve can differ at most by a rigid motion.

Example 2 (pure stretching of a straight rod)

[Change notation: $\underline{x}_R(s_R) \to \underline{x}_0(s_R)$]



$\underline{x}_0(s_R) = 0 + s_R \underline{e}_1$ $\quad$ $\underline{t}_0 = \underline{x}_0' = \underline{e}_1 = \cos\alpha_0 \underline{e}_1 + \sin\alpha_0 \underline{e}_2$

$\underline{x}_R(s_R) = 0 + s_R \underline{e}_1$, $\underline{t}_R = \underline{x}_R' = \underline{e}_1 = \cos\alpha_R \underline{e}_1 + \sin\alpha_R \underline{e}_2$

ref. arc-length

$\Rightarrow \alpha_R \equiv 0$, $c_R(s_R) = \dfrac{d\alpha_R}{ds_R} = 0$

$\alpha_0 \equiv 0$ $\quad c_0(s_R)$ $\quad$ etc.

$\underline{x}_0 = (0,-h) \Rightarrow \underline{x}(0) = \underline{x}^0 = (0,-h)$

$\underline{x}(s_R) = \underline{x}_0 + x_1(s_R)\,\underline{e}_1$

Special case: affine deformation (drawing not enough!)

$x_1'(s_R) \equiv \lambda_0 = 1 + \varepsilon_0$ constant

$\Rightarrow x_1(s_R) - \underbrace{x_1(0)}_{=0} = \lambda_0 s_R = (1+\varepsilon_0)s_R$

$\underline{x}'(s_R) = x_1'(s_R)\,\underline{e}_1 + 0\,\underline{e}_2$

$|\underline{x}'(s_R)| = |x_1'(s_R)| = 1 + \underbrace{\varepsilon(s_R)}_{\equiv \varepsilon_0 \in (-1,+\infty)} > 0$

$s'(s_R) = 1 + \varepsilon(s_R)$

$s(s_R) = \displaystyle\int_0^{s_R} (1 + \varepsilon(\sigma))\,d\sigma$ $\qquad \varepsilon(\sigma) \equiv \varepsilon_0$ (affine def.)

$= \displaystyle\int_0^{s_R} (1+\varepsilon_0)\,d\sigma = (1+\varepsilon_0)s_R$

$\ell = s(\ell_R) = (1+\varepsilon_0)\ell_R = \lambda_0 \ell_R \Rightarrow$

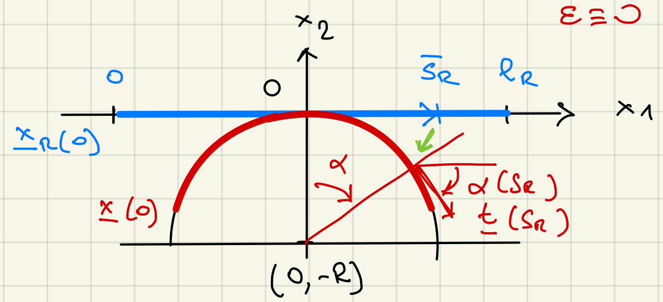
$\lambda_0 = \dfrac{\ell}{\ell_R}$ stretching ratio

$\ell = \ell_R + \varepsilon_0 \ell_R \Rightarrow \varepsilon_0 = \dfrac{\ell - \ell_R}{\ell_R}$ change of length per unit reference length

[If deformation is not affine, same story but at the local/infinitesimal level

$\lambda(s_R) = |\underline{x}'(s_R)| = \dfrac{ds(s_R)}{ds_R}$, $\quad \varepsilon(s_R) = \dfrac{ds(s_R)}{ds_R} - 1 = \dfrac{ds(s_R) - ds_R}{ds_R}$

Example 3 (pure bending of an $\underbrace{\text{inextensible}}_{\varepsilon \equiv 0}$, initially $\underbrace{\text{straight}}_{c_R(s_R) \equiv 0}$ curve)



$\underline{x}_R(s_R) = \left(-\dfrac{\ell_R}{2}, 0\right) + s_R \underline{e}_1$, $\quad s_R \in (0, \ell_R)$

$\underline{x}(s_R)$ wraps the circle, $\underline{x}\left(\dfrac{\ell_R}{2}\right) = \underline{x}_R\left(\dfrac{\ell_R}{2}\right) = (0,0)$

$\hookrightarrow$ arc-length on def. curve as well, since $\varepsilon = 0$

Impose that $s(\bar{s}_R) = \bar{s}_R$:

$$-\underbrace{\alpha(\bar{s}_R)}_{<0} R = \bar{s}_R - \frac{\ell_R}{2} \Rightarrow \boxed{\alpha(s_R) = -\frac{1}{R}\left(s_R - \frac{\ell_R}{2}\right)} \quad (*)$$

[Check signs: $\alpha > 0$ for $s_R < \dfrac{\ell_R}{2}$, $\quad \alpha < 0$ for $s_R > \dfrac{\ell_R}{2}$]

Notice: angular coverage $\alpha(\ell_R) = -\dfrac{1}{R}\left(\ell_R - \dfrac{\ell_R}{2}\right) = -\dfrac{\ell_R}{2R} = -\dfrac{\pi}{2}$ if $\ell_R = \pi R$

[EX: How long should the ref. curve be to cover 20 times a disk of radius R?]

A: $20 \times 2\pi R$

Also: $c(s_R) = \alpha'(s_R) = -\dfrac{1}{R}$, $\quad \kappa(s_R) = \underbrace{\alpha'(s_R)}_{= c(s_R)} - \underbrace{\alpha_R'(s_R)}_{= c_R(s_R) = 0} = -1/R$

arc-length ↑

From (*) giving $s_R \mapsto \alpha(s_R)$ can reconstruct the curve by integrating

$$\underline{x}'(s_R) = \cos\alpha(s_R)\,\underline{e}_1 + \sin\alpha(s_R)\,\underline{e}_2$$

$$\underline{x}\left(\frac{\ell_R}{2}\right) = (0,0)$$

$\rightsquigarrow$ <u>exercise</u> : write the parametrization $s_{\cancel{R}} \mapsto \underline{x}(s_{\cancel{R}})$ explicitly

<u>Example 4</u> (Later)

Roll a ribbon around a spool.

a) circular disk, ghost ribbon (ribbon thickness $= 0$)

b) logarithmic spiral, to take into account the thickness of the ribbon.

Relation between strains ε, κ and displacements

[Change notation $\underline{x}_R(s_R) \to \underline{x}_0(s_R)$
$\underline{t}_R(s_R) \to \underline{t}_0(s_R)$]

Assume s_R is arc-length on x_R and drop $\sim$

$$\underline{x}(s_R) = \underline{x}_0(s_R) + \underline{\xi}(s_R) \qquad \underline{x}'(s_R) = \underline{x}_0'(s_R) + \underline{\xi}'(s_R)$$
$$= \underline{t}_0(s_R) + \underline{\xi}'(s_R)$$

Now compute the various quantities ....

$$|\underline{x}'|^2 = 1 + 2\underline{t}_0 \cdot \underline{\xi}' + \underline{\xi}' \cdot \underline{\xi}'$$

$$|\underline{x}'| = 1 + \varepsilon = \left(1 + 2\underline{t}_0 \cdot \underline{\xi}' + |\underline{\xi}'|^2\right)^{1/2} \Rightarrow \varepsilon = \left(1 + 2\underline{t}_0 \cdot \underline{\xi}' + |\underline{\xi}'|^2\right)^{1/2} - 1$$

$$\underline{\xi} = u\,\underline{t}_0 + w\,\underline{n}_0 \qquad \underline{t}_0' = c_0\,\underline{n}_0 \qquad \underline{n}_0' = -c_0\,\underline{t}_0$$

$$\underline{\xi}' = u'\underline{t}_0 + u\underline{t}_0' + w'\underline{n}_0 + w\underline{n}_0'$$
$$= (u' - c_0 w)\,\underline{t}_0 + (w' + c_0 u)\,\underline{n}_0 \qquad \Rightarrow \underline{\xi}' \cdot \underline{t}_0 = (u' - c_0 w)$$

$$|\underline{\xi}'|^2 = (u' - c_0 w)^2 + (w' + c_0 u)^2$$

$$\varepsilon = \left(1 + 2(u' - c_0 w) + (u' - c_0 w)^2 + (w' + c_0 u)^2\right)^{1/2} - 1$$
$$= \sqrt{(1 + u' - c_0 w)^2 + (w' + c_0 u)^2} - 1$$

Special case : $c_0 \equiv 0$ (straight ref. config.)

$$\boxed{\varepsilon = \sqrt{(1 + u')^2 + w'^2} - 1} \quad (1)$$

Similar calculations for $\sin\omega$, $\cos\omega$, κ as functions of u, w.

General formulas in LNs, here special case $c_R \equiv 0$

$$\cos\omega = \frac{1 + u'}{((1 + u')^2 + w'^2)^{1/2}}, \quad \sin\omega = \frac{w'}{((1 + u')^2 + w'^2)^{1/2}}, \quad \boxed{\kappa = \omega' = \frac{w''(1 + u') - w'u''}{(1 + u')^2 + w'^2}} \quad (\text{Ex}) \quad (2)$$

Complements Proof of $\kappa = \omega' = \dfrac{w''(1+u') - w'u''}{(1+u')^2 + w'^2}$ in the case $c_0 \equiv 0$.

We have $c_0 = \alpha_0' \equiv 0 \Rightarrow \alpha_0(s_R) \equiv \alpha_0^\circ = \text{const}$. Assume $\overset{\text{for simplicity}}{\alpha_0^\circ = 0}$, so that $\omega = \alpha - \underbrace{\cancel{\alpha_0}}_{=0} = \alpha$

$$\underline{x}(s_R) = \underline{x}_0(s_R) + \underline{\xi}(s_R) = s_R\,\underline{e}_1 + u(s_R)\,\underbrace{\underline{t}_0}_{=\underline{e}_1} + w(s_R)\,\underbrace{\underline{n}_0}_{=\underline{e}_2}$$

$$\underline{t} = \cos\alpha\,\underline{e}_1 + \sin\alpha\,\underline{e}_2 = \frac{1}{|\underline{x}'|}\left(\underline{t}_0 + \underline{\xi}'(s_R)\right) = \frac{1}{|\underline{x}'|}\left((1+u')\underline{e}_1 + w'\underline{e}_2\right)$$

$$\cos\alpha = \cos\omega = \frac{1}{\sqrt{(1+u')^2+w'^2}}\,(1+u')$$

$$\sin\alpha = \sin\omega = \frac{\quad}{\sqrt{(1+u')^2+w'^2}}\;w'$$

Re: $|\underline{x}'| = 1+\varepsilon = \sqrt{(1+u')^2+w'^2}$

To compute $\kappa = \omega'$, observe that

$$\frac{d}{ds_R}\cos\omega = -\omega'\sin\omega =$$

$$= \frac{1}{(1+u')^2+w'^2}\left[u''\left((1+u')^2+w'^2\right)^{1/2} - (1+u')\frac{1}{\cancel{2}}\,\frac{\cancel{2}(1+u')u'' + \cancel{2}w'w''}{\left((1+u')^2+w'^2\right)^{1/2}}\right]$$

$$= \frac{1}{\left((1+u')^2+w'^2\right)^{1/2}}\left[u'' - (1+u')\,\frac{(1+u')u'' + w'w''}{(1+u')^2+w'^2}\right]$$

$$= -\frac{1}{\sqrt{(1+u')^2+w'^2}}\left[\frac{\cancel{(1+u')^2u''} + (1+u')w'w'' - u''\cancel{(1+u')^2} - u''w'^2}{(1+u')^2+w'^2}\right]$$

$$= -\frac{1}{\sqrt{(1+u')^2+w'^2}}\left[\frac{(1+u')w'' - u''w'}{(1+u')^2+w'^2}\right]w' = -\frac{(1+u')w'' - u''w'}{(1+u')^2+w'^2}\;\underbrace{\frac{w'}{\sqrt{(1+u')^2+w'^2}}}_{\sin\omega}$$

From this identity we recognize that

$$\omega' = \kappa = \frac{(1+u')w'' - u''w'}{(1+u')^2+w'^2}$$

Case of "small deformations": $\underline{x}(s_R) = \underline{x}_0(s_R) + \eta \bar{u}\, \underline{t}_0 + \eta \bar{w}\, \underline{n}_0$

$\underline{t}_0 = \underline{e}_1$ $\quad$ $\underline{n}_0 = \underline{e}_2$ $\quad$ because $c_0 \equiv 0$

where $u = \|u\| \dfrac{u}{\|u\|} = \eta \bar{u}$, η small : $\eta \ll 1$, $\eta^2 \ll \eta$: neglect η^2 w/r to η

↳ e.g., $\sup |u|$, $\left(\int_0^{\ell_R} |u(\sigma)|^2 d\sigma \right)^{1/2}$, etc., ...

(1)

$$\varepsilon = \left(1 + 2\eta \bar{u}' + \eta^2 \bar{u}'^2 + \eta^2 \bar{w}'^2\right)^{1/2} - 1$$

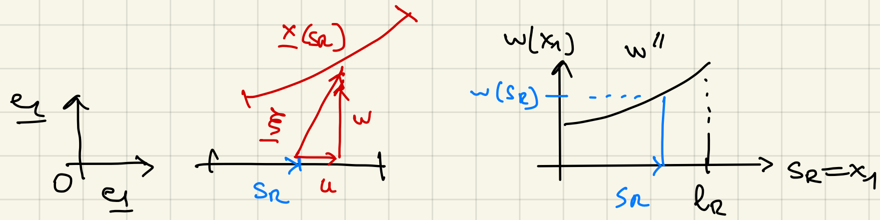
$$\simeq \left(1 + 2\eta \bar{u}'\right)^{1/2} - 1 \simeq \cancel{1} + \frac{1}{2} 2\eta \bar{u}' - \cancel{1} = \eta \bar{u}' = u'$$

$$(1+t)^{1/2} = 1 + \frac{1}{2}(1+t)^{-1/2}\Big|_{t=0} t + \dots \simeq 1 + \frac{1}{2} t + \dots$$

(2)

$$\kappa = \frac{\eta \bar{w}''(1 + \eta \bar{u}') - \eta^2 \bar{w}' \bar{u}''}{1 + 2\eta \bar{u}' + \eta^2 \bar{u}'^2 + \eta^2 \bar{w}'^2}$$

$$\simeq \frac{\eta \bar{w}''}{1} = w''$$



In other words, $\boxed{\varepsilon \simeq u'}$ (1) and $\boxed{\kappa \simeq w''}$ (2) as in Scienza delle Costruzioni.

Remark

In the solution of elasticity pb's in large deformations it is often more convenient to work directly with the geometric variables $\varepsilon(s_R) = s'(s_R) - 1$, $\kappa(s_R) = \omega'(s_R)$ rather than with displacement components u, w

possibly $= c(s_R) - c_0(s_R)$