Mechanics

Bionics Eng.

Lecture 6

17/10/2025

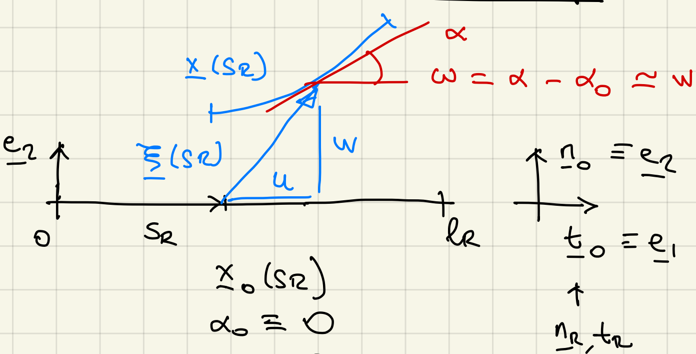
Lecture 10 $\qquad\qquad$ 10/11/23

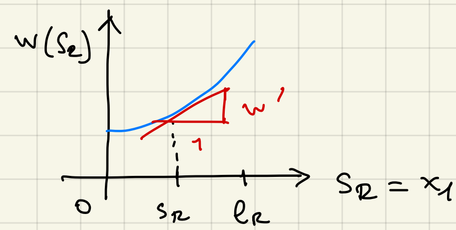
Re: Equilibrium equations in referential form

$$\left[\begin{aligned} & M_R' + (1+\varepsilon)T_R + m_R = 0 \\ & T_R' + (\cancel{c_0}+\kappa)N_R + f_{Rn} = 0 \\ & N_R' - (\underbrace{\cancel{c_0}}_{=0}+\kappa)T_R + f_{Rt} = 0 \end{aligned}\right.$$

[$c_0 \equiv 0$, straight reference configuration]

Small deformation theory





$$\underline{\xi} = u\,\underline{e}_1 + w\,\underline{e}_2 = \eta\bar{u}\,\underline{e}_1 + \eta\bar{w}\,\underline{e}_2, \quad 0<\eta\ll 1$$

$$\varepsilon \simeq u' \ll 1$$
$$\omega \simeq w' \ll 1$$
$$\kappa = \omega' \simeq w'' \ll 1$$

Also N_R, M_R, T_R "small", $N_R = \eta\bar{N}_R$, etc.

as a consequence of "small" loads $\underline{f}_R = \eta\,\bar{\underline{f}}_R$, etc.

$$\begin{cases} M_R' + (1+\varepsilon)T_R + m_R = 0 & 1+\varepsilon \simeq 1 \\ N_R' - (\underbrace{\cancel{c_0}}_{=0}+\kappa)T_R + f_{Rt} = 0 & \kappa \ll 1 \\ T_R' + (\underbrace{\cancel{c_0}}_{=0}+\kappa)N_R + f_{Rn} = 0 & \kappa \ll 1 \end{cases}$$

$$(*)\begin{cases} M_R' + T + m_R = 0 \\ N_R' + f_{Rt} = 0 \\ T_R' + f_{Rn} = 0 \end{cases}$$

where

$$f_{Rt} = \underline{f}_R\cdot(\cos\alpha, \sin\alpha) = \eta\,\bar{\underline{f}}_R\cdot\Big(1-\frac{\alpha^2}{2}+\dots,\ \underbrace{\alpha+\dots}_{\text{of order }\eta}\Big) \simeq \underline{f}_R\cdot(1,0) = f_{R1}$$

$$f_{Rn} = \underline{f}_R\cdot(-\sin\alpha, \cos\alpha) \simeq \underline{f}_R\cdot(0,1) = f_{R2}$$

Similarly

$$\underline{R}_R \simeq N_R\,\underline{e}_1 + T_R\,\underline{e}_2, \text{ where } N_R = \underline{R}_R\cdot(1,0),\ T_R = \underline{R}_R\cdot(0,1)$$

Equations (*) represent the equilibrium equations written in the reference configuration

$$(*)\begin{cases} M'_{\cancel{R}} + T_{\cancel{R}} + m_{\cancel{R}} = 0 \\ N'_{\cancel{R}} + f_{\cancel{R}1} = 0 \\ T'_{\cancel{R}} + f_{\cancel{R}2} = 0 \end{cases}$$

[eventually, drop subscript $_R$ from notation]

$$\underline{R}'_{\cancel{R}} + \underline{f}_{\cancel{R}} = (N_{\cancel{R}}\, \underline{e}_1 + T_R\, \underline{e}_2)' + f_{\cancel{R}1}\, \underline{e}_1 + f_{\cancel{R}2}\, \underline{e}_2 = 0$$

$\underline{e}_1 \hookrightarrow \underline{t}_0$ $\quad$ $\underline{e}_2 \hookrightarrow \underline{n}_0$ $\quad$ $[\underline{t}_R, \underline{n}_R]$

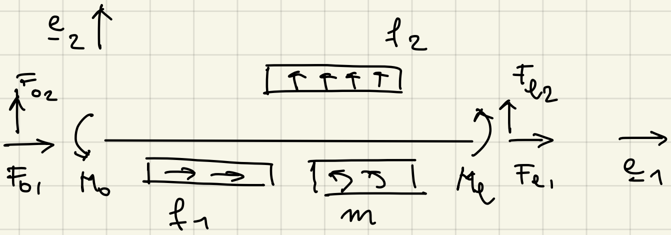
These equations say that the loads are in equilibrium in the reference config.

In fact, they are equivalent to this statement: $(*) \Longleftrightarrow$ equilibrium in ref. conf.

Equations (*) represent equilibrium equations written in the reference configuration.

Necessary condition for (*) to have solution is that the loads are in equilibrium in the reference configuration (implication $\Rightarrow$) For example, for global $(0,\ell)$ equilibrium

[drop subscript R from notation]



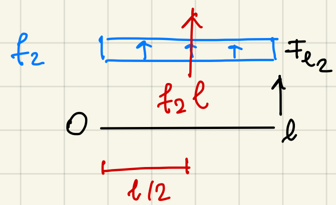
Assume, for simplicity, f_1, f_2, m constant over $(0,\ell)$

$$\left.\begin{aligned} &\int_0^\ell (N' + f_1)\,ds = 0 \Rightarrow N(\ell) - N(0) + f_1 \ell = 0 \Rightarrow F_{\ell 1} + F_{01} + f_1 \ell = 0 \\ &\int_0^\ell (T' + f_2)\,ds = 0 \Rightarrow \ \dots \ \Rightarrow F_{\ell 2} + F_{02} + f_2 \ell = 0 \end{aligned}\right\} \Rightarrow \underline{\mathcal{F}}(0,\ell) = 0$$

In order to recover $T(s)$:

$$0 = \int_s^\ell T' + f_2 \Rightarrow 0 = T(\ell) - T(s) + f_2(\ell - s) \Rightarrow T(s) = F_{\ell 2} - f_2(s-\ell) \Rightarrow \int_0^\ell T(s)\,ds = F_{\ell 2}\,\ell - \frac{f_2(s-\ell)^2}{2}\Bigg|_0^\ell$$

$$= \underbrace{F_{\ell 2}\,\ell + f_2 \frac{\ell^2}{2}}$$



$\leftarrow$ moment w/r to $\underline{x}(0)$ of forces acting on the rod

$$0 = \int_0^\ell M' + T + m = M_\ell + M_0 + F_{\ell 2}\,\ell + f_2 \frac{\ell^2}{2} + m\ell = \mathcal{M}_{x(0)}(0,\ell) \Rightarrow \mathcal{M}_{x(0)}(0,\ell) = 0$$

If these conditions on the external loads ("data") are not satisfied, then there is no solution to the equilibrium equations, see example later.

The opposite implication (*) $\Leftarrow$ the loads are in equilibrium in the ref. config. can be easily proved by doing again the same argument we have followed to derive the differential form of the equilibrium equations in the non linear theory:

$$\left.\begin{aligned} &\underline{\mathcal{F}}(0,s) = 0 \\ &\mathcal{M}_{x(0)}(0,s) = 0 \end{aligned}\right. \quad \text{in the ref. conf.} \Rightarrow (*) \begin{cases} \underline{R}_R' + \underline{f}_R = 0 \\ M_R' + T_R + m_R = 0 \end{cases}$$

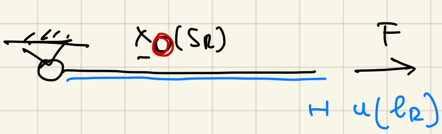
$$(*)\quad \begin{cases} M' + T + m = 0 \\ N' + f_1 = 0 \\ T' + f_2 = 0 \end{cases}$$

These equations imply that the loads are in equilibrium in the reference config.
Therefore, the fact that the loads are in equilibrium in the ref. config. is a necessary condition for the existence of a solution of the equilibrium problem in the linearized theory. In the general nonlinear theory this is NOT the case because one solves for the unknown equilibrium as well.(*)

(*) There are extra unknown functions $\alpha(s_R)$, $x_1(s_R)$, $x_2(s_R)$ that we have to solve for. The equilibrium equations lead to the discovery of equilibrium configs, if such configs exist.

The system can "look" for an equilibrium configuration, a-priori unknown, while in the linear theory the configuration on which equilibrium is sought is the initial/reference one, and the displacements u, w represent small perturbations from that equilibrium configurations due, e.g., to elasticity.

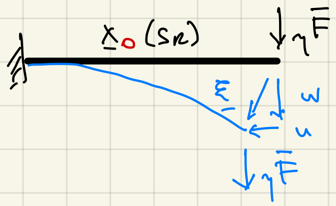
Examples



$\ell = \ell_R + u(\ell_R) = (1+\varepsilon)\ell_R$
$\simeq \ell_R$ if $\varepsilon \ll 1$

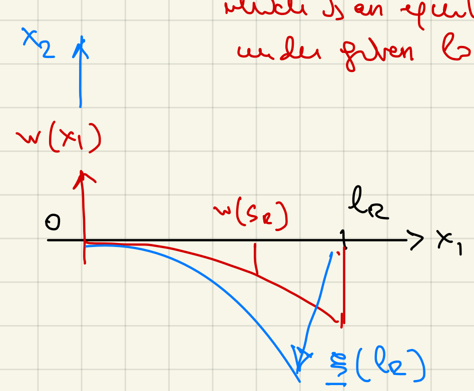
$\boxed{\text{Here ref. and def. config's are NOT very different}}$

(unknown elastic displacements ARE small perturbations of the ref. config. which is an equil. conf. under given loads)



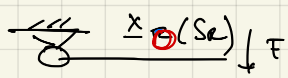
(a weight)

$\underline{\xi} = \eta(\underbrace{\bar{u}}_{=0}\underline{e}_1 + \bar{w}\underline{e}_2)$

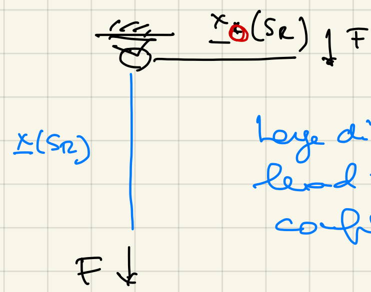


$M_0(0,\ell_R) = \eta\bar{F}(\ell_R - \eta\bar{u}(\ell_R)) \simeq \eta\bar{F}\ell_R$

The situation changes dramatically if the ref. config. is NOT an equilibrium configuration under the given loads



No equilibrium solution in linear theory ($\underline{x}_0$ not an equil. config. under the given loads)



Large displacement (rotations) lead to equilibrium configuration $\underline{x}(S_R)$

$\boxed{\text{Here ref. and def. config's ARE very different}}$

(unknown displacements are NOT small perturbations of the reference configuration which is NOT an equil. config. under the given loads)

There is no equilibrium solution within the linear theory. The linear theory is helpless

There is "always" a solution in the nonlinear theory. The nonlinear theory "looks" for an equilibrium configuration, a-priori unknown, by solving simultaneously for the equilibrium value of the internal "forces" M, N, T and the geometry x_1, x_2, α on which they are in equilibrium.

(extra unknowns, nonlinear equations)

Remark: Different role of the displacements:

small perturbations due to elasticity

Possibly large perturbations due to change of geometry: placement, rotation, shape
x_1, x_2 $\quad \alpha \quad$ ε, κ

Equilibrium equations (*) are 3 ODEs in the three unknown functions M_R, N_R, T_R (loads are now "data")

different sign conventions

$$(*)\begin{cases} M' + T + m = 0 \\ N' + f_1 = 0 \\ T' + f_2 = 0 \end{cases} \Rightarrow \quad T = -M' - m$$

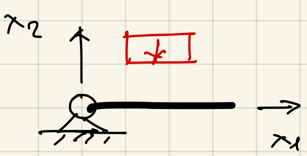
$$\Rightarrow \quad M'' = -m' - T' = -m' + f_2$$

can rewrite (*) as

$$\begin{cases} T = -M' - m \\ M'' = -m' + f_2 \\ N' = -f_1 \end{cases}$$

If enough boundary conditions on M, N, T are known [Static BCs involving $M(0), N(0), T(0)$, $M(\ell), N(\ell), T(\ell)$] then (*) can be integrated and they determine uniquely $M(s), N(s), T(s)$.

Basics of structure analysis: structure types

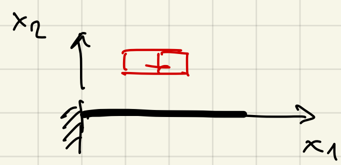


$\underline{x}(0) = 0$

mechanism (labile)

In equilibrium only if

$m_0\ (0,\ell) = 0$



$\underline{x}(0) = 0$

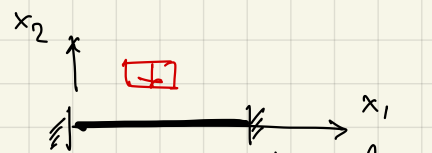
$\alpha(0) = 0$

isostatic

In equilibrium $\forall$ loads

$M(\ell) = N(\ell) = T(\ell) = 0$

Can determine M, N, T from equilibrium alone



$\underline{x}(0) = 0$ $\quad \underline{x}(\ell) = 0 + \ell\, \underline{e}_1$

$\alpha(0) = 0$ $\quad \alpha(\ell) = 0$

hyperstatic

In equilibrium $\forall$ loads

M, N, T at $s = 0, \ell$ are NOT known a-priori. Cannot integrate (*). Cannot determine M, N, T from equilibrium alone.

Can then add kinematic equations (and kinematic compatibility with external constraints: "congruenza" in Italian) and constitutive equations to solve equilibrium problem for hyperstatic structures.

→ Constitutive equations

$$N_R = EA\ \varepsilon \quad \simeq EA\ u'$$

$$M_R = EI\ k \quad \simeq EI\ w''$$

↳ geometrically linear theory

↳ linear constitutive equations

Therefore we have ⟵ SUMMARY

$$\begin{cases} M' + T + m = 0 \\ N' \quad + f_1 = 0 \\ T' \quad + f_2 = 0 \end{cases}$$ equilibrium

5 ODEs in the unknown functions M, N, T, u, w

$$\begin{cases} N = EA\, u' \\ M = EI\, w'' \end{cases}$$ constitutive eqn.

+ Static BCs (M, N, T) + kinematic BCs (u, w, w')

Can always integrate for hyperstatic structures

↳ see later, table with kinematic / static BCs associated with all possible constraints.

STOP HERE : EQUILIBRIO e CONGRUENZA

These concepts are at the root of structural mechanics, which is articulated in two main chapters:

- Static admissibility (Equilibrio, in Italian)
- Kinematic admissibility (Congruenza, in Italian)