Mechanics, Bionics

Lecture 2

29/9/23

From curvature to curve

Up to now: curve $\to$ unit tangent $\to$ curvature, by differentiation

$s \mapsto \tilde{\underline{x}}(s)$

$\tilde{\underline{t}}(s) = \tilde{\underline{x}}'(s) = (\cos\tilde{\alpha}(s), \sin\tilde{\alpha}(s))$

$\tilde{c}(s) = \tilde{\alpha}'(s)$

Integrate: curvature $\to$ angle $\leftrightarrow$ $\tilde{\underline{x}}'(s)$ $\to$ curve

$\tilde{\alpha}'(s)$ $\quad$ $\tilde{\alpha}(s)$ $\quad$ $\tilde{\underline{x}}(s)$

$$\tilde{\alpha}(s) = \tilde{\alpha}(0) + \int_0^s \tilde{c}(\sigma)\,d\sigma$$

$$\tilde{\underline{x}}(s) = \tilde{\underline{x}}(0) + \int_0^s \tilde{\underline{x}}'(\sigma)\,d\sigma = \tilde{\underline{x}}(0) + \int_0^s \cos\tilde{\alpha}(\sigma)\,d\sigma\,\underline{e}_1 + \int_0^s \sin\tilde{\alpha}(\sigma)\,d\sigma\,\underline{e}_2$$

In other words:

change by rigid body "motion"

$$\text{Given}\left\{\begin{array}{ll} \left.\begin{array}{l}\tilde{x}(0) = x_0 \\ \tilde{\alpha}(0) = \alpha_0\end{array}\right\} & \begin{array}{l}\text{position}\\ \text{orientation}\end{array} \\ \left.\begin{array}{l} s \mapsto \tilde{c}(s) \\ s \in [0,\ell]\end{array}\right\} & \begin{array}{l}\text{shape}\\ \text{length}\end{array}\end{array}\right\} \text{ then } s \mapsto \tilde{\underline{x}}(s) \text{ is uniquely determined.}$$

change by bending and stretching

Example 3 Given $s \mapsto \tilde{\varkappa}(s) \equiv c_0 = -\frac{1}{R} < 0$, $s \in [0, \underbrace{\ell = \pi R}_{\text{length of the curve}}]$

find (the) curve with curvature $\tilde{\varkappa}(s)$ and s.t. $\begin{cases} \tilde{\alpha}(0) = \alpha_0 = \frac{\pi}{2} \\ \underline{\tilde{x}}(0) = x_0 = (0,0) \end{cases}$

$$\tilde{\alpha}(s) = \tilde{\alpha}(0) + \int_0^s \left(-\frac{1}{R}\right) d\sigma = \frac{\pi}{2} - \frac{s}{R}$$

$$\underline{\tilde{x}}(s) = \underline{\tilde{x}}(0) + \int_0^s \cos\left(\frac{\pi}{2} - \frac{\sigma}{R}\right) d\sigma \, \underline{e}_1 + \int_0^s \sin\left(\frac{\pi}{2} - \frac{\sigma}{R}\right) d\sigma \, \underline{e}_2$$

$$= 0 + \int_0^s \sin\left(\frac{\sigma}{R}\right) d\sigma \, \underline{e}_1 + \int_0^s \cos\left(\frac{\sigma}{R}\right) d\sigma \, \underline{e}_2 \rightarrow$$

$y = \frac{\sigma}{R} \in \left[\frac{0}{R}, \frac{s}{R}\right]$

$\sigma = Ry$, $d\sigma = R\,dy$

$$\int_0^s \sin\left(\frac{\sigma}{R}\right) d\sigma = \int_0^{s/R} \sin y \, R\,dy = R(-\cos y)\Big|_0^{s/R}$$

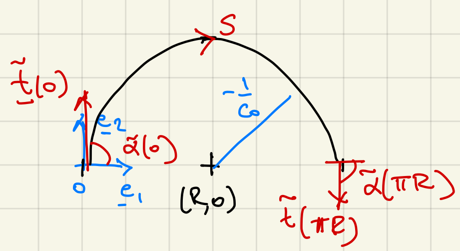
$$\int_0^s \cos\left(\frac{\sigma}{R}\right) d\sigma = \int_0^{s/R} \cos y \, R\,dy = R \sin y\Big|_0^{s/R}$$

$$= 0 + \underbrace{R\left(1 - \cos\frac{s}{R}\right)}_{\substack{\tilde{x}_1(s) \\ = (\underline{\tilde{x}}(s) - 0)\cdot \underline{e}_1}} \underline{e}_1 + \underbrace{R \sin\frac{s}{R}}_{\substack{\tilde{x}_2(s) \\ = (\underline{\tilde{x}}(s) - 0)\cdot \underline{e}_2}} \underline{e}_2$$

Rearrange $\tilde{x}_1(s) = R - R\cos\frac{s}{R}$, $\tilde{x}_2(s) = R \sin\frac{s}{R}$

$$\left(\tilde{x}_1(s) - R\right)^2 = R^2 \cos^2\frac{s}{R}$$

$$\left(\tilde{x}_2(s) - 0\right)^2 = R^2 \sin^2\frac{s}{R}$$



$$\left(\tilde{x}_1(s) - R\right)^2 + \left(\tilde{x}_2(s) - 0\right)^2 = R^2$$

circle centered at $(R, 0)$ of radius $R = \left|\frac{1}{c_0}\right|$

Remark This exercise shows that a curve with constant curvature c_0 is an arc of a circle with radius $\left|\frac{1}{c_0}\right|$

What have we learned?

The curvature governs the shape of a curve.

How this shape sits in space is governed by a rigid motion that fixes the orientation (= tangent at first end, α_0) and position (= position of first end, $\underline{x}_0$) which are integration constants in the process of reconstructing a curve from its curvature.

We can extend this argument to curves with internal "angles changes" $\rightarrow$ articulated arms

(discontinuities in $\tilde{\alpha}(s)$)

The only care that's needed is to use $\int_{s_1}^{s_2} \tilde{\alpha}(s)\,ds = \tilde{\alpha}(s_2) - \tilde{\alpha}(s_1)$ only to intervals

(s_1, s_2) where $\tilde{\alpha}(s)$ does not jump and its derivative is well defined

We will learn to use the following formulas

Let $f(x) = F'(x)$ [e.g., $\tilde{c}(s) = \tilde{\alpha}'(s)$]

If F is continuous, then

$$F(x) = F(0) + \int_0^x f(\bar{x})\,d\bar{x}$$ (Fundamental theorem of calculus)

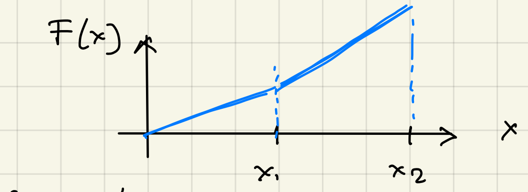
If F is discontinuous and $[\![F]\!](x_i) := F(x_i+) - F(x_i-)$ is a jump at x_i

$$F(x-) = F(0+) + \int_0^x f(\bar{x})\,d\bar{x} + \sum_{0<x_i<x} [\![F]\!](x_i) \qquad (*)$$

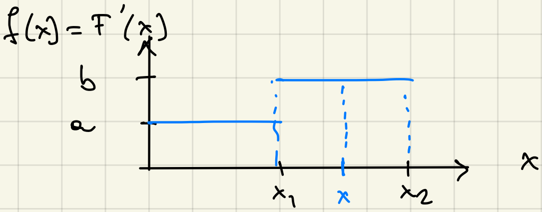
Look at the example of the folding rule contrasted with the tape measure.

Metro a nastro = tape measure = metro flessibile
Metro da falegname = folding rule = metro pieghevole
} insert photos

Recovering a (continuous) function F from its derivative $f = F'$



$$F(x) = \begin{cases} ax & x \in (0, x_1) \\ ax_1 + b(x - x_1) & x \in (x_1, x_2) \end{cases}$$



$$f(x) = F'(x) = \begin{cases} a & x \in (0, x_1) \\ b & x \in (x_1, x_2) \end{cases}$$

$$\int_0^x f(\tilde{x})\,d\tilde{x} = \dots = \begin{cases} ax & x \in (0, x_1) \\ ax_1 + b(x - x_1) & x \in (x_1, x_2) \end{cases}$$

$$\int_0^x f(\tilde{x})\,d\tilde{x} = \int_0^{x_1} f(\tilde{x})\,d\tilde{x} + \int_{x_1}^{x} f(\tilde{x})\,d\tilde{x} = a\,x_1 + \underbrace{\int_{x_1}^{x} b\,d\tilde{x}}_{= b(x - x_1)} \quad ✓$$

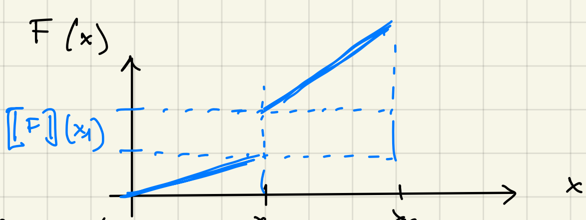
Remark This shows that, when $f = F'$, then

$$F(x) = \underbrace{F(0)}_{= 0 \text{ in our case}} + \int_0^x f(\tilde{x})\,d\tilde{x} \iff \underline{F(x) - F(0) = \int_0^x F'(\tilde{x})\,d\tilde{x}}$$

Fundamental theorem of Calculus (we'll use a lot!)

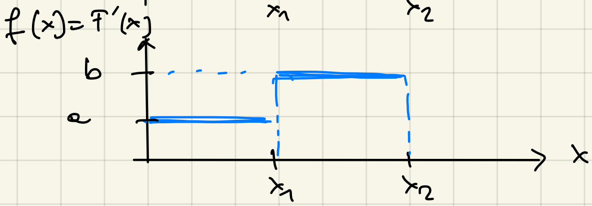
Moreover, $$\frac{d}{dx} F(x) = \underline{\frac{d}{dx} \int_0^x f(\tilde{x})\,d\tilde{x}} = \frac{d}{dx} \begin{cases} ax \\ ax_1 + b(x - x_1) \end{cases} = \begin{cases} a & x \in (0, x_1) \\ b & x \in (x_1, x_2) \end{cases} = \underline{f(x)}$$

What if the function F is discontinuous?



$$F(x) = \begin{cases} ax & x \in (0, x_1) \\ ax_1 + [\![F]\!](x_1) + b(x - x_1) & x \in (x_1, x_2) \end{cases}$$

where $[\![F]\!](x_1) = F(x_1+) - F(x_1-)$ ($\neq$ from before)



$$f(x) = F'(x) = \begin{cases} a & x \in (0, x_1) \\ b & x \in (x_1, x_2) \end{cases}$$ (same as before)

Then $$F(x-) = F(0+) + \int_0^x f(\tilde{x})\,d\tilde{x} + \sum_{0 < x_i < x} [\![F]\!](x_i) \qquad (*)$$

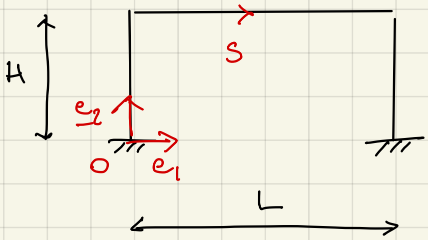
Still true that $F'(x) = f(x) = \begin{cases} a & x \in (0, x_1) \\ b & x \in (x_1, x_2) \end{cases}$

Idea: can recover a continuous function from its derivative

can recover a discontinuous function from its derivative and its jumps

Example 4 Parametrize by arc length a portal

→ Drop ~ from notation, for simplicity



$$\underline{x}(s) = \begin{cases} s\,\underline{e}_2 & s \in [0, H) \\ H\underline{e}_2 + (s-H)\underline{e}_1 & s \in [H, H+L) \\ L\underline{e}_1 + (L+2H-s)\,\underline{e}_2 & s \in [H+L, 2H+L) \end{cases}$$

$$H\underline{e}_2 + (\cancel{H}+L-\cancel{H})\underline{e}_1 + (s-(H+L))(-\underline{e}_2)$$

$$\underline{x}'(s) = \begin{cases} \underline{e}_2 & s \in (0,H) \\ \underline{e}_1 & s \in (H, H+L) \\ -\underline{e}_2 & s \in (H+L, 2H+L) \end{cases}$$

Notice that $\underline{x}$ is continuous, while $\underline{x}'$ is discont. Wherever defined, $|\underline{x}'(s)| \equiv 1$

$$\underline{t}(s) = \begin{cases} \underline{e}_2 \\ \underline{e}_1 \\ -\underline{e}_2 \end{cases} ; \quad \alpha(s) = \begin{cases} \pi/2 \\ 0 \\ -\pi/2 \end{cases} \qquad c(s) = \alpha'(s) \equiv 0 \text{ wherever defined}$$

Clearly every segment is straight (zero curvature $\alpha'(s)$). But $s \mapsto \alpha(s)$ is discontinuous

Can we encode the "non-straightness" of the portal thanks to the jumps of α ?

[If a function has jump discontinuities, we cannot reconstruct all its variations from the derivative. We need to add the contributions due to the jumps]

Using (*) from last page

$$\alpha(s-) = \alpha(0+) + \int_0^s c(\sigma)\,d\sigma + \sum_{0<s_i<s} [\![\alpha]\!](s_i) \qquad (*)$$

where $[\![\alpha]\!](s_i) = \alpha(s_i+) - \alpha(s_i-) =: \theta_{s_i}$

then

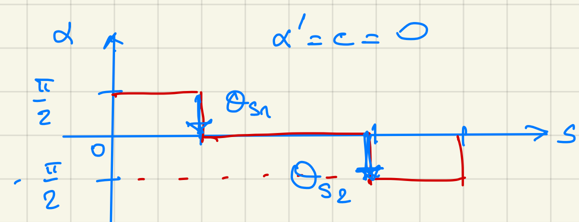
$s_1 = H, \quad \theta_{s_1} = 0 - \frac{\pi}{2} = -\frac{\pi}{2}$

$s_2 = H+L, \quad \theta_{s_2} = -\frac{\pi}{2} - 0 = -\frac{\pi}{2}$

$s \in (0,H) \qquad \alpha(s) = \alpha(0) + 0 + 0 = \frac{\pi}{2}$

$s \in (H, H+L) \qquad \alpha(s) = \frac{\pi}{2} + 0 + \left(-\frac{\pi}{2}\right) = 0$

$s \in (H+L, 2H+L) \qquad \alpha(s) = \frac{\pi}{2} + 0 + \left(-\frac{\pi}{2} - \frac{\pi}{2}\right) = -\frac{\pi}{2}$

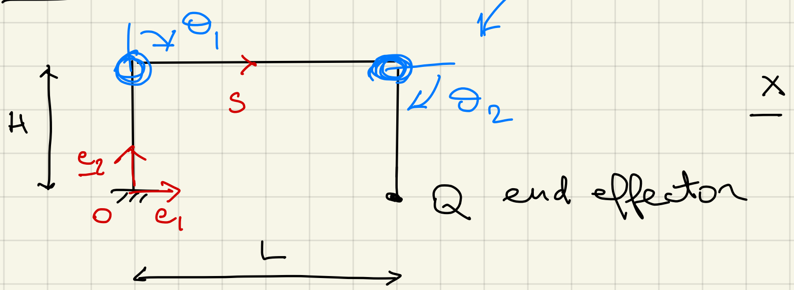


[Alternative: integrate in sub-intervals]

Remark If we differentiate (*) away from a discontinuity, we still get $\alpha'(s) = 0 + \underbrace{\frac{d}{ds}\int^s c(\sigma)\,d\sigma}_{=0 \text{ in this special example}} + 0 = c(s)$

Re: $F(s) = \int_0^s f(\sigma)\,d\sigma \Rightarrow F'(s) = f(s)$

Remark



$$\underline{x}(s;\overbrace{\theta_1,\theta_2}^{\text{shape parameters}})$$

$\underline{x}\left(s;-\frac{\pi}{2},-\frac{\pi}{2}\right)$ is the previous case (Example 4)

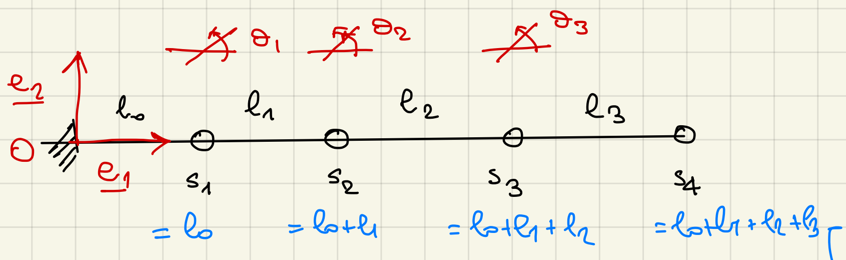
<u>Ex</u> Find the position of the end effector Q when $\theta_1=\frac{\pi}{2}$, $\theta_2=\frac{\pi}{2}$

$$\underline{x}\left(2H+L;\frac{\pi}{2},\frac{\pi}{2}\right)=\,?$$

by writing the arc-length parametrization of the "curve" with $\theta_1=\frac{\pi}{2}$, $\theta_2=\frac{\pi}{2}$

$+$ $\underline{x}\left(s;\frac{\pi}{2},\frac{\pi}{2}\right)$

Example 5 Parametrize by arc-length a "robotic arm"



4 rigid segments of lengths l_i, $i = 0, \dots, 3$
Pin-jointed at common ends
Relative rotations θ_i, $i = 1, \dots, 3$ at s_i, $i = 1, \dots, 3$

[θ_i are concentrated curvatures / shape parameters]

$$\alpha(s-) = \underbrace{\alpha(0+)}_{\alpha_0 = 0} + \underbrace{\int_0^s \underbrace{c(\sigma)}_{=0} d\sigma}_{=0} + \sum_{0 < s_i < s} \theta_{s_i} = \begin{cases} 0 & = \alpha_0 & s \in (0, s_1) \\ \theta_1 & = \alpha_1 & s \in (s_1, s_2) \\ \theta_1 + \theta_2 & = \alpha_2 & s \in (s_2, s_3) \\ \theta_1 + \theta_2 + \theta_3 & = \alpha_3 & s \in (s_3, s_4) \end{cases}$$

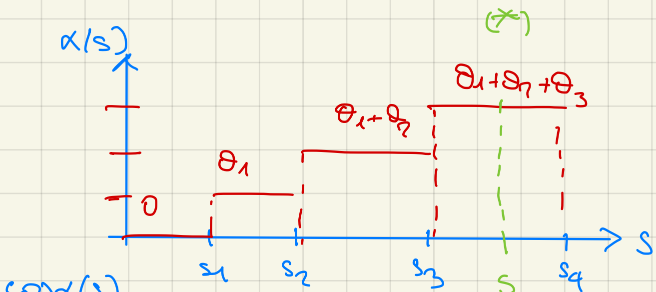
Pattern:
$$\alpha_3 = \sum_{i=1}^{3} \theta_i$$

$$\underline{t}(s) = \underline{x}'(s) = (\cos\alpha(s), \sin\alpha(s)) = \left(\cos\left(\sum_{0 < s_i < s} \theta_i\right), \sin\left(\sum_{0 < s_i < s} \theta_i\right)\right)$$

$x_1(s)$ and $x_2(s)$ are continuous (no jumps). Can recover them from derivatives, with no need to correct for jumps the formula from fundam. thm. of Calculus.

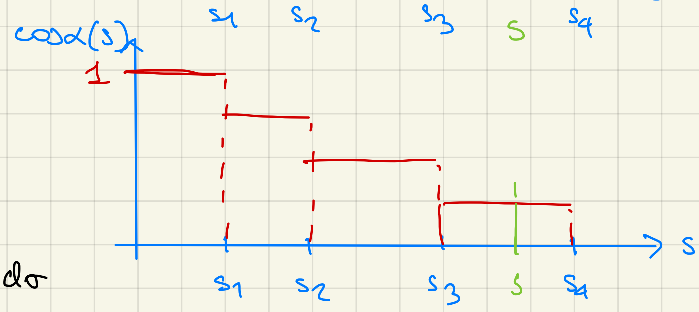
$s \in (0, s_1)$

$$x_1(s-) = \underbrace{x_1(0+)}_{=0} + \underbrace{\int_0^s \cos \underbrace{\alpha_0}_{=0} d\sigma}_{=1} = s$$



$s \in (s_1, s_2)$

$$x_1(s-) = x_1(0+) + \int_0^{s_1} \cos \alpha_0 d\sigma + \int_{s_1}^{s} \cos\theta_1 d\sigma$$
$$= 0 + s_1 \cos\alpha_0 + (s - s_1)\cos\theta_1$$



$s \in (s_2, s_3)$

$$x_1(s-) = \cancel{x_1(0+)} + \int_0^{s_1} \cos\alpha_0 d\sigma + \int_{s_1}^{s_2} \cos\theta_1 d\sigma + \int_{s_2}^{s} \cos(\theta_1 + \theta_2) d\sigma$$
$$= \cancel{0} + s_1 \cos\alpha_0 + (s_2 - s_1)\cos\theta_1 + (s - s_2)\cos(\theta_1 + \theta_2)$$

$s \in (s_3, s_4)$ (*)

$$x_1(s-) = \cancel{x_1(0+)} + \int_0^{s_1} \cos\alpha_0 d\sigma + \int_{s_1}^{s_2} \cos\theta_1 d\sigma + \int_{s_2}^{s_3} \cos(\theta_1+\theta_2) d\sigma + \int_{s_3}^{s} \cos(\theta_1+\theta_2+\theta_3) d\sigma$$
$$= 0 + \underbrace{s_1}_{l_0} \cos\alpha_0 + \underbrace{(s_2 - s_1)}_{l_1} \cos\underbrace{\theta_1}_{\alpha_1} + \underbrace{(s_3 - s_2)}_{l_2} \cos\underbrace{(\theta_1+\theta_2)}_{\alpha_2} + (s - s_3)\cos\underbrace{(\theta_1+\theta_2+\theta_3)}_{\alpha_3}$$

General formula for $s > s_k$

$$x_1(s-) = \underbrace{x_1(0+)}_{=0} + l_0 \underbrace{\cos\alpha_0}_{=1} + \sum_{j=1}^{k-1} l_j \cos\Big(\underbrace{\sum_{i=1}^{j} \theta_i}_{\alpha_j}\Big) + (s - s_k)\cos\Big(\underbrace{\sum_{i=1}^{k} \theta_i}_{\alpha_k}\Big)$$

Similarly for x_2

$$x_2(s-) = \underbrace{x_2(0+)}_{=0} + \underbrace{l_0 \sin\alpha_0}_{=0} + \sum_{j=1}^{k-1} l_j \sin\underbrace{\Big(\sum_{i=1}^{j} \theta_i\Big)}_{\alpha_j} + (s-s_k)\sin\underbrace{\Big(\sum_{i=1}^{k}\theta_i\Big)}_{\alpha_k}$$

Remark: We are seeing the possibility of shaping a "curve" through

continuous curvature (tape measure)

discrete curvature (robotic arm, foldable meter)

or a combination of both.

Remark: the sums of the angles θ_i composing the total rotation from 0 to α is strongly reminiscent of the exponential formalism

$$e^{\alpha_k \underline{\hat{e}}_3} = e^{\sum_{i=1}^{k} \theta_i \underline{\hat{e}}_3} = \prod_{i=1}^{k} e^{\theta_i \underline{\hat{e}}_3}$$

Suggestion for exercise

How do things change if there is a pin-joint at $s=0$, and one imposes there a rotation θ_0 ?