Mechanics

Bionics Eng.

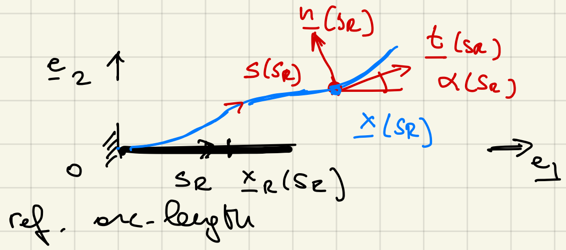
Lecture 5

15/10/2025

Lecture 5 Equilibrium equations in referential form 15/11/25

Recap

Kinematics



s arc-length in def. config.

Reference or initial config. $\underline{x}_0(s_R)$

$\alpha_0(s_R)$

$c_0(s_R) = \alpha_0'(s_R) \equiv 0$ (straight ref. config.)

Deformed config. $s(s_R)$, $\dfrac{ds}{ds_R} = |\underline{x}'(s_R)| = 1 + \varepsilon(s_R)$

$$k = \omega' = \alpha'(s_R) - \alpha_0'(s_R) = (1+\varepsilon)c - c_0$$

$$\Rightarrow (1+\varepsilon)c = c_0 + k$$

$$\underline{t}' = c\underline{n}, \quad \underline{n}' = -c\underline{t}, \quad c = \frac{c_0 + k}{1+\varepsilon}$$

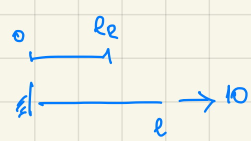
Equilibrium

$$\frac{d}{ds} M(s) + T(s) + m(s) = 0$$

$$\frac{d}{ds} \underline{R}(s) + \underline{f}(s) = 0$$

In this formulation, all quantities are expressed as functions of current arc-length s, which is in general an unknown function of reference arc-length s_R.(*) Change variables to express all quantities as functions of s_R rather than s (referential formulation, analog of Lagrangian vs. Eulerian in 3D)

(*) Think, e.g., of the BC $\underline{R}(\ell) = \underline{F}_\ell = F_{1\ell}\,\underline{e}_1$, $F_{1\ell} = 10\,N$



or of the action of the centrifugal force, that depends on the unknown $r = x_1(s_R)$

Equilibrium equations in referential form (i.e., in terms of functions of s_R rather than s, which is in principle unknown)

Change (independent) variable: $s = s(s_R)$

1. Functions and derivatives

$$s = s(s_R) \Rightarrow \quad f(s) \longrightarrow f(s(s_R)) = \tilde{f}(s_R) \qquad [f \text{ generic function of } s]$$

$$\frac{d}{ds_R}\tilde{f}(s_R) = \frac{d}{ds} f(s(s_R)) \frac{ds}{ds_R} = (1+\varepsilon(s_R)) \frac{d}{ds} f \Big|_{s=s(s_R)}$$

$$\boxed{\frac{d}{ds} f(s) \Big|_{s=s(s_R)} = \frac{1}{1+\varepsilon(s_R)} \frac{d}{ds_R} f(s(s_R)) = \frac{1}{1+\varepsilon(s_R)} \frac{d}{ds_R} \tilde{f}(s_R)}$$

2. Integrals

Set $s = s(s_R)$ and change variable in the integrals

$$\int_0^{\ell} m(s)\, ds \qquad ds = \frac{ds}{ds_R} ds_R = s'(s_R)\, ds_R = (1+\varepsilon)\, ds_R$$

$$= \int_0^{\ell_R} \underbrace{m(s(s_R))\,(1+\varepsilon(s_R))}_{=:\, m_R(s_R)} ds_R \qquad \text{torque per unit reference length}$$

$$\int_0^{\ell} \underline{f}(s)\, ds \underset{s=s(s_R)}{=} \int_0^{\ell_R} \underbrace{\underline{f}(s(s_R))\,(1+\varepsilon(s_R))}_{=:\, \underline{f}_R(s_R)} ds_R \qquad \text{force per unit reference length}$$

Example $\quad \underline{f}(s) = -\rho g\, \underline{e}_2 + \rho\, \Omega^2 x_1(s)\, \underline{e}_1$

$$\underline{f}_R(s_R) = -\underbrace{\rho(1+\varepsilon)}_{=:\, \rho_R(s_R)} g\, \underline{e}_2 + \underbrace{\rho(1+\varepsilon)}_{\rho_R} \Omega^2 x_1(s(s_R))\, \underline{e}_1 \qquad \text{mass p.u. ref. length}$$

$$\left\{ \begin{array}{l} \rho_R = \rho(1+\varepsilon) \\ \dfrac{dm}{ds_R} = \dfrac{dm}{ds}\dfrac{ds}{ds_R} \\ \text{Remark: if } \rho_R \text{ "data" then } \rho \text{ "unknown"} \end{array} \right.$$

$$\underline{f}_R(s_R) = f_{Rt}\, \underline{t}(s_R) + f_{Rn}\, \underline{n}(s_R), \quad \underline{t}(s_R) = (\cos\alpha(s_R), \sin\alpha(s_R)),\ \underline{n}(s_R) = (-\sin\alpha(s_R), \cos\alpha(s_R))$$

$$\underline{e}_1 = (1,0),\ \underline{e}_2 = (0,1) \Rightarrow \underline{e}_2 \cdot \underline{t} = \sin\alpha(s_R),\ \underline{e}_1 \cdot \underline{t} = \cos\alpha(s_R) \ \ldots \text{etc.}$$

$$f_{Rt}(s_R) = \underline{f}_R \cdot \underline{t} = -\rho_R\, g \sin\alpha(s_R) + \rho_R\, \Omega^2 x_1(s(s_R)) \cos\alpha(s_R)$$

$$f_{Rn}(s_R) = \underline{f}_R \cdot \underline{n} = -\rho_R\, g \cos\alpha(s_R) - \rho_R\, \Omega^2 x_1(s(s_R)) \sin\alpha(s_R)$$

Remark:

"data"

"unknowns"

Now work on the equations

- $$\frac{d}{ds} M(s) + T(s) + m(s) = 0$$

Set $s = s(s_R)$ and recall $\left.\frac{d}{ds} M(s)\right|_{s=s(s_R)} = \frac{1}{1+\varepsilon} \frac{d}{ds_R} M(s(s_R))$. Get

$$\frac{1}{1+\varepsilon} \frac{d}{ds_R} M(s(s_R)) + T(s(s_R)) + m(s(s_R)) = 0$$

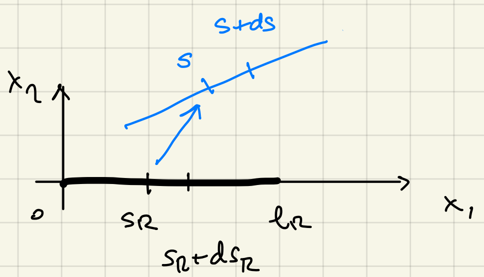
Multiply by $1+\varepsilon$

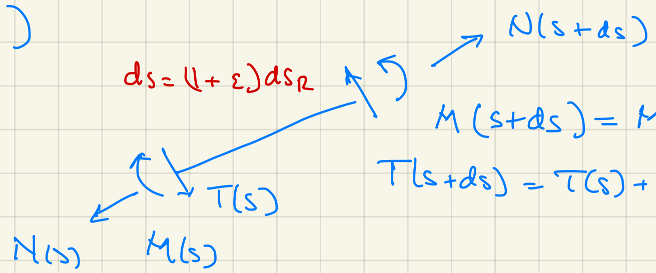
$$\frac{d}{ds_R} \underbrace{M(s(s_R))}_{=: M_R(s_R)} + (1+\varepsilon) \underbrace{T(s(s_R))}_{=: T_R(s(s_R))} + \underbrace{(1+\varepsilon) m(s(s_R))}_{= m_R(s(s_R))} = 0$$

or: $$\boxed{M_R' + (1+\varepsilon) T_R + m_R = 0}$$

moment balance in referential form

Remark (Assume $m = 0$; heuristics)





$$M(s+ds) = M(s) + M'(s) ds$$

$$T(s+ds) = T(s) + T'(s) ds$$

$$M'(s) ds + T(s) \underbrace{ds} \simeq 0$$

$$M_R'(s_R) ds_R + T_R(s_R) \overbrace{(1+\varepsilon) ds_R} \simeq 0$$

- $$\underbrace{\frac{d}{ds}\left(N\underline{t} + T\underline{n}\right)} + f_t\underline{t} + f_n\underline{n} = 0$$ where $f_t = \underline{f}\cdot\underline{t} = \underline{f}\cdot(\cos\alpha, \sin\alpha)$

$f_n = \underline{f}\cdot\underline{n} = \underline{f}\cdot(-\sin\alpha, \cos\alpha)$

$$N'\underline{t} + N\underline{t}' + T'\underline{n} + T\underline{n}'$$ $\underline{t}' = c\underline{n}$, $\underline{n}' = -c\underline{t}$

$$(N' - cT)\underline{t} + (T' + cN)\underline{n}$$

Hence we have, by taking components along $\underline{t}$ and $\underline{n}$

$$\frac{d}{ds}N(s) - c(s)T(s) + f_t(s) = 0$$

$$\frac{d}{ds}T(s) + c(s)N(s) + f_n(s) = 0$$

Set $s = s(s_R)$ to get $\left[\text{Re}: \left.\frac{d}{ds}N\right|_{s=s(s_R)} = \frac{1}{1+\varepsilon}\frac{d}{ds_R}N(s(s_R))\,;\ c = \frac{c_0+\kappa}{1+\varepsilon}\right]$

$$\frac{1}{\cancel{1+\varepsilon}}\frac{d}{ds_R}\underbrace{N(s(s_R))}_{=:N_R(s_R)} - \frac{c_0+\kappa}{\cancel{1+\varepsilon}}(s_R)\underbrace{T(s(s_R))}_{=T_R(s_R)} + \underbrace{(1+\varepsilon)f_t(s(s_R))}_{f_{Rt}(s_R)} = 0$$

similarly

$$\frac{d}{ds_R}N_R(s_R) - (c_0+\kappa)T_R(s_R) + f_{Rt}(s_R) = 0$$

$$\frac{d}{ds_R}T_R(s_R) + (c_0+\kappa)(s_R)N_R(s_R) + f_{Rn}(s_R) = 0$$

or:
$$\boxed{\begin{cases} N_R' - (c_0+\kappa)T_R + f_{Rt} = 0 \\ T_R' + (c_0+\kappa)N_R + f_{Rn} = 0 \end{cases}}$$

force balance in referential form

Remark : 3 ODEs, with many more unknown functions : $M_R, T_R, N_R, \varepsilon, \kappa, \ldots\ldots$

$\ldots\ldots x_1(s_R), x_2(s_R), \alpha(s_R)$

$$M_R' + (1+\varepsilon)\,T_R + m_R = 0$$

$$T_R' + \underbrace{(\underbrace{c_0}_{=0}+\kappa)\,N_R + f_{Rn} = 0}_{(*)}$$

$$N_R' - (\underbrace{c_0}_{=0}+\kappa)\,T_R + f_{Rt} = 0$$

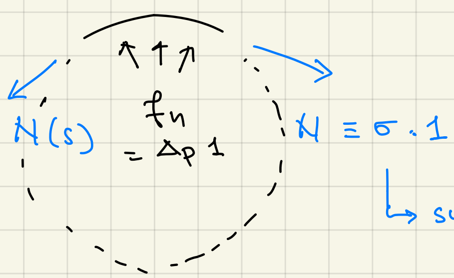
[$c_0 = 0$, straight ref. config.]

Remark The equilibrium equations above show the interplay of loads and geometry. In particular, (*) is reminiscent of Laplace's equation in capillary drops, showing how tangential forces coupled with curvature can balance a normal load coming from a pressure drop across the bounding surface of the drop.

Capillary drop in 2d :

only surface tension N, no shear and bending moment ($T=0$, $M=0$).

These are constitutive assumptions for a fluid film.



$$N_R' = 0 \implies N_R = \text{const}$$

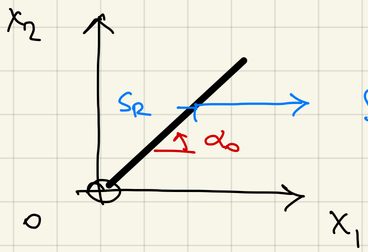
$$\kappa N_R + f_{Rn} = 0 \quad \sim \quad \frac{\sigma}{R} = \Delta p$$

(Young-Laplace law)

Case of centrifugal forces : configurational forces depend on the (unknown)

Forces typically depend on the configuration, which is unknown, so that they are not "data"

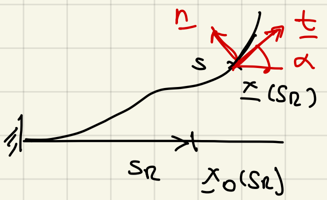
Think, e.g., of centrifugal force



$$\rho_R(s_R)\,\Omega^2\,\underbrace{x_1(s_R)}_{s_R\cos\alpha}\,ds_R\,\underline{e}_1$$

unknown. to be determined by equilibrium

General scheme of equilibrium problem (Counting of equations and unknowns)



	equations	unknowns
Kinematics		
$\begin{cases} \alpha'(s_R) = K(s_R) + \underbrace{c_R(s_R)}_{=0 \text{ in this case}} \\ \underline{x}'(s_R) = (1+\varepsilon(s_R))(\cos\alpha(s_R)\,\underline{e}_1 + \sin\alpha(s_R)\,\underline{e}_2) \end{cases}$	3 eq	x_1, x_2, α ε, K
+ kinematic BC's on $\alpha(0)$, $\underline{x}(0)$, ...		
$\Rightarrow$ recover α, $\underline{x}$ by integration from ε, K ↳ placement of the rod's axis		
Equilibrium		
$\begin{cases} M_R' + (1+\varepsilon)T_R + m_R = 0 \\ N_R' - (c_R + K)T_R + f_{Rt} = 0 \\ T_R' + (c_R + K)N_R + f_{Rn} = 0 \end{cases}$	3 eq	M_R, N_R, T_R
where		
$\underline{R}_R = N_R\,\underline{t} + T_R\,\underline{n}$		
$\underline{f}_R = f_{Rt}\,\underline{t} + f_{Rn}\,\underline{n}$		
+ static BC's $M_R(\ell)$, $T_R(\ell)$, $N_R(\ell)$, ...	6 equations	8 unknowns
	$\Rightarrow$ Need 2 extra equations	

Constitutive equations

Use for now linear dependence of N_R on ε (resp. M_R on K) as in linear theory. More later.

$$N_R = EA\,\varepsilon$$

$$M_R = EI\,K$$

In summary

$$8 = \overset{x' \quad \alpha'}{2 + 1} + \underset{\text{equil}}{3} + \overset{\text{const.}}{2}$$

ODEs / AEs
in the unknown functions

$$\underbrace{x_1, x_2, \alpha}_{3\ \text{geom}}, \underbrace{M_R, N_R, T_R}_{3\ \text{int. forces}}, \underbrace{\varepsilon, \kappa}_{2\ \text{strains}}$$

= 8 unknown functions

[Non-linear equations]

SKIP

Even in the simplest case $\underline{f}_R = -\rho_R g \underline{e}_2$, known, the components of $\underline{f}_R$ along $\underline{t}$ and $\underline{n}$ contain the unknown function $\alpha(s_R)$, see above. We have extra equation

$$\alpha'(s_R) = \kappa_0(s_R) + \kappa(s_R)$$

and, in addition,

$$\underline{x}'(s_R) = \left(1+\varepsilon(s_R)\right)\left[\cos\alpha(s_R)\,\underline{e}_1 + \sin\alpha(s_R)\,\underline{e}_2\right]$$

two more ODEs with two more unknowns $x_1(s_R)$, $x_2(s_R)$. Knowledge of $x_1(s_R)$ is necessary to specify centrifugal forces.

Still, it seems that we have too many unknowns. We need TWO additional equ's. We'll see how to "fix" this in the next lecture, thanks to the use of "constitutive equations", that specify the dependence of internal forces M_R, N_R, T_R on strain measures ε, κ, given the geometric characteristics of the cross section and the physical properties of the material of which the rod is made of ("constituted")

Our typical assumptions:

$$N = (EA)\ \varepsilon$$

$$M = (EI)\ \kappa$$