T2-1)

$$\begin{cases} X = \mathbb{R} \\ \rho(x,y) = (x-y)^2 \end{cases}$$

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$$\begin{cases} (i)\ \forall(x,y) \Rightarrow (x-y)^2 \geq 0 \quad \boxed{\checkmark} \\ (ii)\ x = y \Rightarrow \rho(x,y) = \rho(x,x) = (x-x)^2 = 0 \quad \boxed{\checkmark} \\ (iii)\ \rho(x,y) = (x-y)^2 = ((-1)(y-x))^2 = (y-x)^2 = \rho(y,x) \quad \boxed{\checkmark} \\ (iv) \begin{cases} \rho(x,z) = (x-z)^2 = x^2 + z^2 - 2xz \\ \rho(x,y) + \rho(y,z) = (x-y)^2 + (y-z)^2 = x^2 + y^2 - 2xy + y^2 + z^2 - 2yz \end{cases} \end{cases}$$

$$\Rightarrow \rho(x,z) - \rho(x,y) - \rho(y,z) = \cancel{x^2} + \cancel{z^2} - 2xz - \cancel{x^2} - y^2 + 2xy - y^2 - \cancel{z^2} + 2yz$$

$$= -2y^2 + 2y(x+z) - 2xz = -2\left(y^2 - y(x+z) + xz\right) = -2(y-x)(y-z) \lessgtr 0 \quad \boxed{\times}$$

$\Rightarrow$ The given pair is not a metric space.

$$\begin{cases} X = \mathbb{R}^n \\ \rho(\underline{x},\underline{y}) = \sqrt{\sum_{1}^{n} (x_k - y_k)^2} \end{cases} \qquad \begin{cases} (i)\ \forall(\underline{x},\underline{y}) \Rightarrow \rho(\underline{x},\underline{y}) = \sqrt{\sum (x_k - y_k)^2} \\ = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \dots + (x_n-y_n)^2} \end{cases}$$

Recalling that $\sqrt{x^2} = \pm|x|$, it is concluded that: $\forall(\underline{x},\underline{y}) \Rightarrow \rho(\underline{x},\underline{y}) \geq 0$ $\boxed{\times}$

The first condition is not satisfied. Hence the given pair is not a metric space. Also other conditions could be studied too, just like the previous example.

T2-2) Obtain Column Hermitian for:

$$d = \begin{bmatrix} s^2 & 0 \\ 0 & s^2 \\ 1 & s+1 \end{bmatrix} \xrightarrow{C_1 \leftrightarrow C_2} \begin{bmatrix} 0 & s^2 \\ s^2 & 0 \\ s+1 & 1 \end{bmatrix} \xrightarrow[s^2R_3 - R_1]{R_1 =} \begin{bmatrix} -s^2(s+1) & s^2 - s^2 \\ s^2 & 0 \\ s+1 & 1 \end{bmatrix} = \begin{bmatrix} -s^2(s+1) & 0 \\ s^2 & 0 \\ s+1 & 1 \end{bmatrix}$$

$$\xrightarrow[R_1 + (s+1)R_2]{R_1 =} \begin{bmatrix} -s^2(s+1) + s^2(s+1) & 0+0 \\ s^2 & 0 \\ s+1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ s^2 & 0 \\ s+1 & 1 \end{bmatrix};$$

T2-3) obtain gcrd for: $A = \begin{bmatrix} s^2 & -1 \\ -s & s^2 \end{bmatrix}$; $B = \begin{bmatrix} s & -s \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} s^2 & -1 \\ -s & s^2 \\ s & -s \\ 0 & 1 \end{bmatrix}$

$$\xrightarrow[R_3 + sR_4]{R_3 =} \begin{bmatrix} s^2 & -1 \\ -s & s^2 \\ s & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow[R_1 + sR_3]{R_1 =} \begin{bmatrix} 0 & -1 \\ -s & s^2 \\ s & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow[R_4 + R_1]{R_4 =} \begin{bmatrix} 0 & -1 \\ -s & s^2 \\ s & 0 \\ 0 & 0 \end{bmatrix}$$

$$\xrightarrow[R_2 + R_3]{R_2 =} \begin{bmatrix} 0 & -1 \\ 0 & s^2 \\ s & 0 \\ 0 & 0 \end{bmatrix} \xrightarrow[R_2 + s^2R_1]{R_2 =} \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ s & 0 \\ 0 & 0 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{R_2 \leftrightarrow R_3} \begin{bmatrix} s & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \text{gcrd} = \begin{bmatrix} s & 0 \\ 0 & -1 \end{bmatrix}$$

Note that gcrd matrix is not unique.

for gcld: $\Rightarrow [A \;\; B] = \begin{bmatrix} s^2 & -1 & s & -s \\ -s & s^2 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} s^2 & -1 & s & 0 \\ -s & s^2 & 0 & 1 \end{bmatrix}$

$\longrightarrow \begin{bmatrix} 0 & -1 & s & 0 \\ -s & s^2 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & -1 & s & 0 \\ -s & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\longrightarrow \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \Rightarrow \text{gcld} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$;

T2-4) The row operations we took in previous problem was as:

$$U(s) = \begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&s \\ 0&0&0&1 \end{bmatrix}\begin{bmatrix} 1&0&s&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix}\begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 1&0&0&1 \end{bmatrix}\begin{bmatrix} 1&0&0&0 \\ 0&1&1&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix}\begin{bmatrix} 1&0&0&0 \\ s^2&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix}\begin{bmatrix} 1&0&0&0 \\ 0&0&1&0 \\ 0&1&0&0 \\ 0&0&0&1 \end{bmatrix}\begin{bmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix}$$

$\Rightarrow U(s) = \begin{bmatrix} s&1&0&0 \\ 1&s^2&1&0 \\ 1&s&0&s \\ 0&1&0&1 \end{bmatrix}$; we know that $U(s)$ is unimodular. For right-coprimeness, we must satisfy: $XN + YD = I$; so we take:

$\bar{X} = \begin{bmatrix} s & 1 \\ 1 & s^2 \end{bmatrix}$, $\bar{Y} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \Rightarrow \bar{X}N + \bar{Y}D = \begin{bmatrix} s^3 - s & s^2 - s \\ -s^3+s^2+s & s^4-s-1 \end{bmatrix} = R(s)$ where

$\begin{cases} N = \begin{bmatrix} s^2 & -1 \\ -s & s^2 \end{bmatrix} \\ D = \begin{bmatrix} s & -s \\ 0 & 1 \end{bmatrix} \end{cases}$; Now, multiply R^{-1} in both sides: $R^{-1}\bar{X}N + R^{-1}\bar{Y}D = R^{-1}R = I$;

$\Rightarrow \begin{cases} X = R^{-1}\bar{X} \\ Y = R^{-1}\bar{Y} \end{cases} \Rightarrow XN + YD = I \Rightarrow P(s)$ & $Q(s)$ are right-coprime.

T2-5) check column reduction for:

$P(s) = \begin{bmatrix} s^3+s & s+2 \\ s^2+s+1 & 1 \end{bmatrix} \rightarrow \det(P(s)) = (1)(s^3+s) - (s+2)(s^2+s+1)$

$= s^3+s - s^3 - s^2 - s - 2s^2 - 2s - 2$

$= -3s^2 - 2s - 2 \Rightarrow \deg\{\det(P(s))\} = 2$;

$\sum_{i=1}^{2} \deg\{\text{col } i\} = 3 + 1 = 4 \Rightarrow 4 \neq 2$

$\Rightarrow$ The given matrix $P(s)$ is not column-reduced.

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T2-6) Transfer to column-reduced form:

$P(s) = \begin{bmatrix} (s+1)^2(s+2)^2 & -(s+1)^2(s+2) \\ 0 & s+2 \end{bmatrix} \xrightarrow[(s+1)^2R_2+R_1]{R_1 =} \begin{bmatrix} (s+1)^2(s+2)^2 & 0 \\ 0 & s+2 \end{bmatrix}$ ~~Now the maximum degree in the matrix is occurring~~

$\Rightarrow$ Now the maximum degree on the jth column is occurring on the jth element which means the matrix is in column-reduced form.

T2 - 11) $T_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & s(s+1) & 0 & s \\ 0 & 0 & s(s+2) & -s \\ 0 & 0 & -1 & 0 \end{bmatrix} \xrightarrow[R_3 + s(s+2)R_4]{R_3 =} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & s(s+1) & 0 & s \\ 0 & 0 & 0 & -s \\ 0 & 0 & -1 & 0 \end{bmatrix}$

$\xrightarrow[R_2 + R_3]{R_2 =} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & s(s+1) & 0 & 0 \\ 0 & 0 & 0 & -s \\ 0 & 0 & -1 & 0 \end{bmatrix} \xrightarrow{C_4 \leftrightarrow C_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & s(s+1) & 0 & 0 \\ 0 & 0 & -s & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = T_{r_1}(s)$

$T_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & s^2(s+1) & s(s+2) & -s \\ 0 & 0 & s+2 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \xrightarrow[R_2 - sR_3]{R_2 =} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & s^2(s+1) & 0 & 0 \\ 0 & 0 & s+2 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$

$\xrightarrow[R_3 + (s+2)R_4]{R_3 =} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & s^2(s+1) & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \xrightarrow{C_4 \leftrightarrow C_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & s^2(s+1) & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \xrightarrow[R_3 = s^{+1} R_3]{R_2 = s^{-1} R_2}$

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & s(s+1) & 0 & 0 \\ 0 & 0 & -s & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = T_{r_2}(s) \implies T_1 \, \alpha \, T_2$ Because T_1 & T_2 have the same Smith-form.

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