Q1-

Two matrices are said to be similar if there exists a nonsingular similarity matrix T that satisfies:

$$A_1 = TA_2T^{-1} \qquad A_2 = T^{-1}A_1T$$

The corresponding Jordan-Block form of the matrix A is always a similar matrix for A and the the similarity transition matrix providing this relation is spanned by the eigenvectors of A. Accordingly:

i-

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow |\lambda I - A| = 0 \rightarrow \begin{vmatrix} \lambda & 0 & 0 \\ -1 & \lambda & -2 \\ 0 & -1 & \lambda - 1 \end{vmatrix} = 0 \rightarrow \ \lambda_1 = 0\,.\lambda_2 = 2\,.\lambda_3 = -1$$

$$(\lambda I - A)v_i = 0 \rightarrow \begin{cases} \lambda = 0 \rightarrow v_1 = [-2 \quad -1 \quad 1]^{\mathrm{T}} \\ \lambda = -1 \rightarrow v_2 = [0 \quad -2 \quad 1]^{\mathrm{T}} \\ \lambda = 2 \rightarrow v_3 = [0 \quad 1 \quad 1]^{\mathrm{T}} \end{cases}$$

$$T = [v_1 \quad v_2 \quad v_3] = \begin{bmatrix} -2 & 0 & 0 \\ -1 & -2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow T^{-1} = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \rightarrow \Lambda = T^{-1}AT = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

ii-

$$B = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 4 & -13 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow |\lambda I - B| = 0 \rightarrow \begin{vmatrix} \lambda + 1 & -1 & -1 \\ 0 & \lambda - 4 & 13 \\ 0 & -1 & \lambda \end{vmatrix} = 0 \rightarrow \ \lambda_1 = -1\,.\lambda_{2.3} = 2 \pm 3j$$

$$\rightarrow \begin{cases} \lambda = 0 \rightarrow v_1 = [1 \quad 0 \quad 0]^{\mathrm{T}} \\ \lambda = 2 - 3j \rightarrow v_2 = [1 \quad 2 - 3j \quad 1]^{\mathrm{T}} \end{cases}$$

$$T = [v_1 \quad \mathrm{Re}\{v_2\} \quad \mathrm{Im}\{v_2\}] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow T^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$\rightarrow \Lambda = T^{-1}AT = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & -3 \\ 0 & 3 & 2 \end{bmatrix}$$

Q2-

Method One:

Using Cayley-Hamilton theorem,

$$P(A) = A^{100} \longrightarrow P(A) = R(A) = c_0 + c_1 A$$

$$\longrightarrow R(\lambda) = c_0 + c_1\lambda$$

$$\longrightarrow R(-1) = P(-1) = c_0 + c_1 = -1^{100} = 1$$

$$\longrightarrow \dot{R}(-1) = \dot{P}(-1) = c_1 = -100 \Longrightarrow c_0 = -99 \longrightarrow R(\lambda) = -100\lambda - 99$$

$$\Longrightarrow R(A) = -100A - 99 = \begin{bmatrix} -99 & -100 \\ 100 & 101 \end{bmatrix}$$

Method Two:

By taking benefit of Cayley-Hamilton theorem,

$$Q(\lambda) = |\lambda I - A| = \lambda(\lambda + 2) + 1 = \lambda^2 + 2\lambda + 1$$

Considering $P(A) = A^{100}$,

$$\frac{P(\lambda)}{Q(\lambda)} = F(\lambda) + \frac{R(\lambda)}{Q(\lambda)} \Longrightarrow \lambda^{100} = \lambda^{98}(\lambda^2 + 2\lambda + 1) + (-2\lambda^{99} - \lambda^{98})$$

$$\Longrightarrow \lambda^{100} = (\lambda^{98} - 2\lambda^{97})(\lambda^2 + 2\lambda + 1) + (3\lambda^{98} + 2\lambda^{97})$$

$$\vdots$$

$$\Longrightarrow \lambda^{100} = (\lambda^{98} - 2\lambda^{97} + \cdots + 99)(\lambda^2 + 2\lambda + 1) - (100\lambda + 99)$$

$$\rightarrow R(\lambda) = -100\lambda - 99$$

$$P(A) = R(A) \rightarrow A^{100} = -100A - 99I = \begin{bmatrix} -99 & -100 \\ 100 & 101 \end{bmatrix}$$

Q3-

i-

$$|\lambda I - A| = 0 \to \begin{vmatrix} \lambda - 1 & 0 & -2 \\ 0 & \lambda - 1 & 1 \\ 1 & -1 & \lambda - 1 \end{vmatrix} = 0 \to \lambda_1 = 1 \,.\, \lambda_{2.3} = 1 \pm j\sqrt{3}$$

$$(\lambda I - A)v_i = 0 \to\to \begin{cases} \lambda = 1 \to v_1 = [1 \quad 1 \quad 0]^{\mathrm{T}} \\ \lambda = 1 + j\sqrt{3} \to v_2 = \left[2 \quad -1 \quad j\sqrt{3}\right]^{\mathrm{T}} \\ \lambda = 1 - j\sqrt{3} \to v_3 = \left[2 \quad -1 \quad -j\sqrt{3}\right]^{\mathrm{T}} \end{cases}$$

ii-

$$|\lambda I - B| = 0 \to \begin{vmatrix} \lambda & -1 & 2 \\ -1 & \lambda & 0 \\ -1 & -1 & \lambda - 1 \end{vmatrix} = 0 \to \lambda_1 = -1 \,.\, \lambda_{2.3} = 1 \pm j\sqrt{2}$$

$$(\lambda I - B)v_i = 0 \to \begin{cases} \lambda = -1 \to v_1 = [1 \quad -1 \quad 0]^{\mathrm{T}} \\ \lambda = 1 - j\sqrt{2} \to v_2 = \left[1 + j\sqrt{2} \quad 1 \quad 1 - j\sqrt{2}\right]^{\mathrm{T}} \\ \lambda = 1 + j\sqrt{2} \to v_3 = \left[1 - j\sqrt{2} \quad 1 \quad 1 + j\sqrt{2}\right]^{\mathrm{T}} \end{cases}$$

Q4-

Diagonalize using $A = T^{-1}\Lambda T$ and we have:

$$\Delta(A) = (\lambda - 1)^2(\lambda - 2) = 0 \to \lambda = 1.1.2$$
$$\mathrm{rank}(I - A) = 2$$

So there exist an independent eigenvector and a dependent eigenvector for $\lambda = 1$.

$$(\lambda I - A)v_i = 0\,;\;\; (\lambda I - A)\varphi_i = v_i \to \begin{cases} \lambda = 1 \to v_1 = [2 \quad 0 \quad -1]^{\mathrm{T}} \\ \lambda = 1 \to \varphi_i = [2 \quad 1 \quad -1]^{\mathrm{T}}\,; \\ \lambda = 2 \to v_3 = [1 \quad 0 \quad -1]^{\mathrm{T}} \end{cases} \qquad \Lambda = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$T = [v_1 \quad v_2 \quad v_3] = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & -1 \end{bmatrix} \to T^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & -2 \end{bmatrix}; \qquad \Lambda = T^{-1}AT$$

$$e^{At} = Te^{\Lambda t}T^{-1} = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} e^{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & -2 \end{bmatrix} \qquad e^{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}t} = e^t \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 2e^t - e^{2t} & 2te^t & 2e^t - 2e^{2t} \\ 0 & e^t & 0 \\ e^{2t} - e^t & -te^t & 2e^{2t} - e^t \end{bmatrix}$$

Q5-

We already know:

$$e^{Jt} = e^{\lambda_1 t}\begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{p-1}}{(p-2)!} \\ 0 & 1 & t & \cdots & \frac{t^{p-1}}{(p-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & t \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where p is the dimension of the square matrix, J is the Jordan form and λ_1 is the eigenvalue with a more than one repetition. Accordingly, it could be said:

$$e^{At} = e^{\lambda_1 t}\begin{bmatrix} 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} \\ 0 & 1 & t & \frac{t^2}{2!} \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Q6-

In $A^{\mathrm{T}}A$, rows of A^{T} are multiplied by the columns of A. Considering that rows of A^{T} are the orthogonal unit vectors forming the column space of the matrix A. Therefore, in $A^{\mathrm{T}}A$, the unit vectors are being multiplied by themselves that equals to 1. Remember that due to the orthogonality property of the vectors respect to each other, the other elements are obtained 0. For example, we have:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow A^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}; A^{\mathrm{T}}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

or

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow A^{\mathrm{T}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}; A^{\mathrm{T}}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

Q7-

Note that Cholesky factorization could be used on symmetric positive definite matrices in which is presented:

$$A = \begin{bmatrix} a_{11} & A_{21}^{\mathrm{T}} \\ A_{21} & A_{22} \end{bmatrix} = LL^{\mathrm{T}}; \quad L = \begin{bmatrix} l_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix}$$

where

$$l_{11} = \sqrt{a_{11}} \quad ; \quad L_{21} = \frac{1}{l_{11}} A_{21} \quad ; \quad A_{22} - L_{21}L_{21}^{\mathrm{T}} = L_{22}L_{22}^{\mathrm{T}}$$

To provide what the problem is seeking us to, let

$$A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix} \quad ; \quad B = \begin{bmatrix} 4 & 1 & 1 & 1 \\ 1 & 3 & 0 & -1 \\ 1 & 0 & 2 & 1 \\ 1 & -1 & 1 & 4 \end{bmatrix}$$

Hence for A:

$$A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix} \rightarrow a_{11} = 4 \quad ; \quad A_{21} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad ; \quad A_{22} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$l_{11} = 2 \quad ; \quad L_{21} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Notation: $L_{22}(i)$ *means the element in the ith place*

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2.75 & 0.25 \\ 0.25 & 1.75 \end{bmatrix} = \begin{bmatrix} L_{22}(1) & 0 \\ L_{22}(2) & L_{22}(3) \end{bmatrix}\begin{bmatrix} L_{22}(1) & L_{22}(2) \\ 0 & L_{22}(3) \end{bmatrix} = L_{22}L_{22}^{\mathrm{T}}$$

$$\Rightarrow \begin{bmatrix} L_{22}^2(1) & L_{22}(1)L_{22}(2) \\ L_{22}(2)L_{22}(1) & L_{22}^2(2) + L_{22}^2(3) \end{bmatrix} = \begin{bmatrix} 2.75 & 0.25 \\ 0.25 & 1.75 \end{bmatrix}$$

$$\Rightarrow L_{22}(1) = \sqrt{2.75} \quad ; \quad L_{22}(2) = \frac{1}{2\sqrt{11}} \quad ; \quad L_{22}(3) = \frac{\sqrt{19}}{\sqrt{11}} \quad ;$$

$$\Rightarrow L \cong \begin{bmatrix} 2 & 0 & 0 \\ -0.5 & 1.66 & 0 \\ 0.5 & 0.15 & 1.34 \end{bmatrix}$$

And for B:

$$B = \begin{bmatrix} 4 & 1 & 1 & 1 \\ 1 & 3 & 0 & -1 \\ 1 & 0 & 2 & 1 \\ 1 & -1 & 1 & 4 \end{bmatrix} \rightarrow b_{11} = 4 \quad ; \quad B_{21} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad ; \quad B_{22} = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & 4 \end{bmatrix}$$

$$l_{11} = 2 \quad ; \quad L_{21} = \frac{1}{2}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$L_{22}L_{22}^{\mathrm{T}} = \begin{bmatrix}3 & 0 & -1\\0 & 2 & 1\\-1 & 1 & 4\end{bmatrix} - \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\end{bmatrix}\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{2}\end{bmatrix} = \begin{bmatrix}2.75 & -0.25 & -1.25\\-0.25 & 1.75 & 0.75\\-1.25 & 0.75 & 3.75\end{bmatrix}$$

$$\Rightarrow L_{22}L_{22}^{\mathrm{T}} = \begin{bmatrix}L_{22}(1) & 0 & 0\\L_{22}(2) & L_{22}(4) & 0\\L_{22}(3) & L_{22}(5) & L_{22}(6)\end{bmatrix}\begin{bmatrix}L_{22}(1) & L_{22}(2) & L_{22}(3)\\0 & L_{22}(4) & L_{22}(5)\\0 & 0 & L_{22}(6)\end{bmatrix} = \begin{bmatrix}2.75 & -0.25 & -1.25\\-0.25 & 1.75 & 0.75\\-1.25 & 0.75 & 3.75\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}L_{22}^2(1) & L_{22}(1)L_{22}(2) & L_{22}(3)L_{22}(1)\\L_{22}(2)L_{22}(1) & L_{22}^2(2)+L_{22}^2(4) & L_{22}(2)L_{22}(3)+L_{22}(4)L_{22}(5)\\L_{22}(3)L_{22}(1) & L_{22}(2)L_{22}(3)+L_{22}(4)L_{22}(5) & L_{22}^2(3)+L_{22}^2(5)+L_{22}^2(6)\end{bmatrix} = \begin{bmatrix}2.75 & -0.25 & -1.25\\-0.25 & 1.75 & 0.75\\-1.25 & 0.75 & 3.75\end{bmatrix}$$

$$\rightarrow L_{22}(1) = \sqrt{2.75} \quad ; \quad L_{22}(2) = -\frac{1}{2\sqrt{11}} \quad ; \quad L_{22}(3) = -\frac{5}{2\sqrt{11}} \quad ; \quad L_{22}(4) = \frac{\sqrt{31}}{10} \quad ;$$

$$L_{22}(5) = \frac{70}{11\sqrt{31}} \quad ; \quad L_{22}(6) = \sqrt{\frac{2444}{801}} \quad ;$$

$$\Rightarrow L \cong \begin{bmatrix}2 & 0 & 0 & 0\\0.5 & 1.66 & 0 & 0\\0.5 & -0.15 & 1.34 & 0\\0.5 & -0.75 & 0.48 & 1.72\end{bmatrix}$$

Q8-

You can find all the **.m* scripts attached to the assignment report within the zip file.

Q9-

The system is proposed as follows:

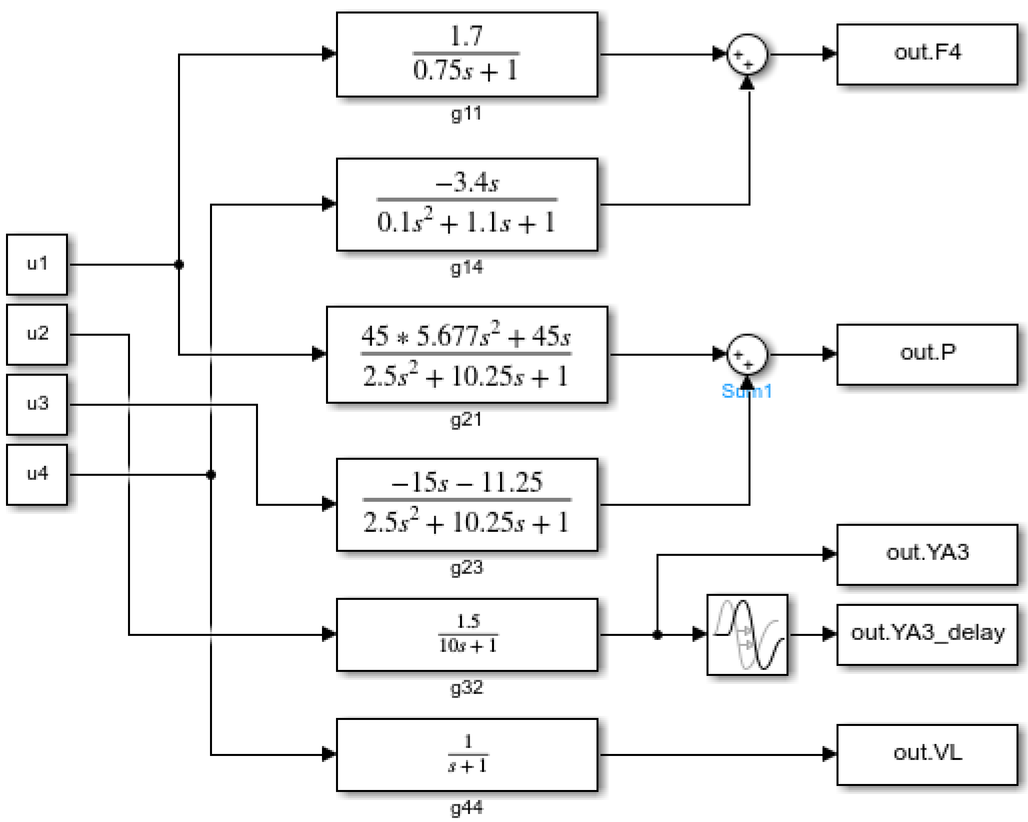
$$y_1 = g_{11}u_1 + g_{14}u_4$$

$$y_2 = g_{21}u_1 + g_{23}u_3$$

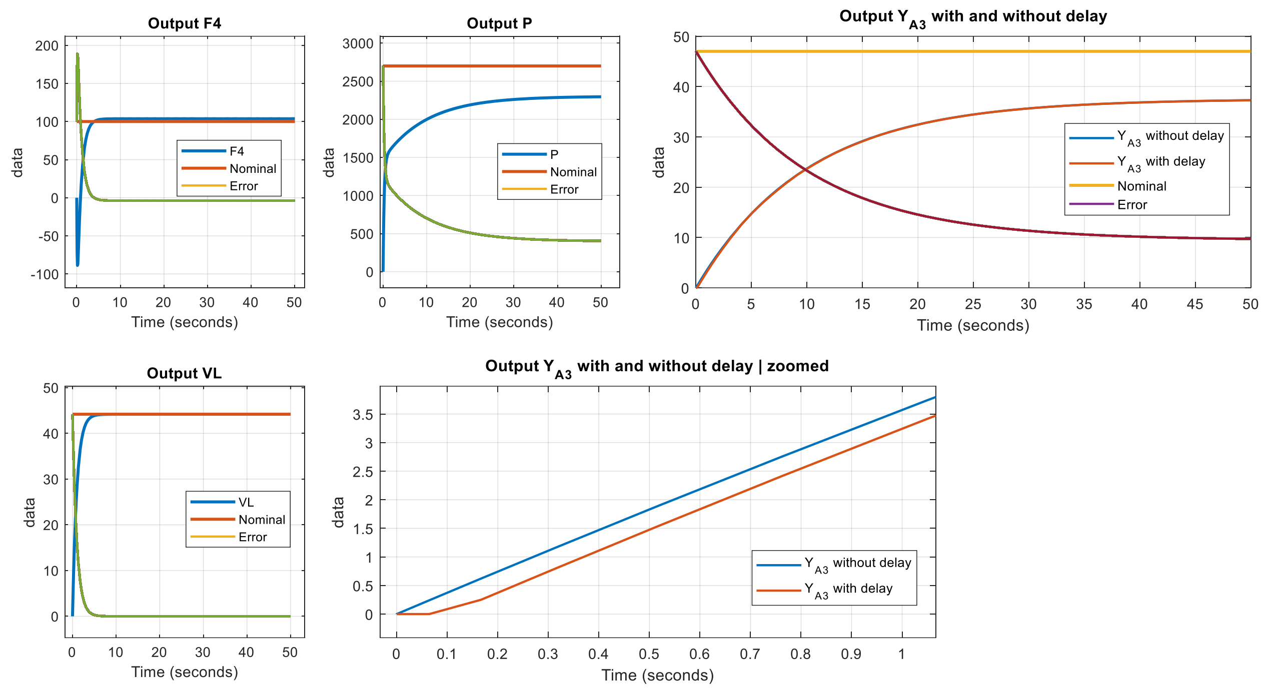
$$y_3 = g_{32}u_2$$

$$y_4 = g_{44}u_4$$

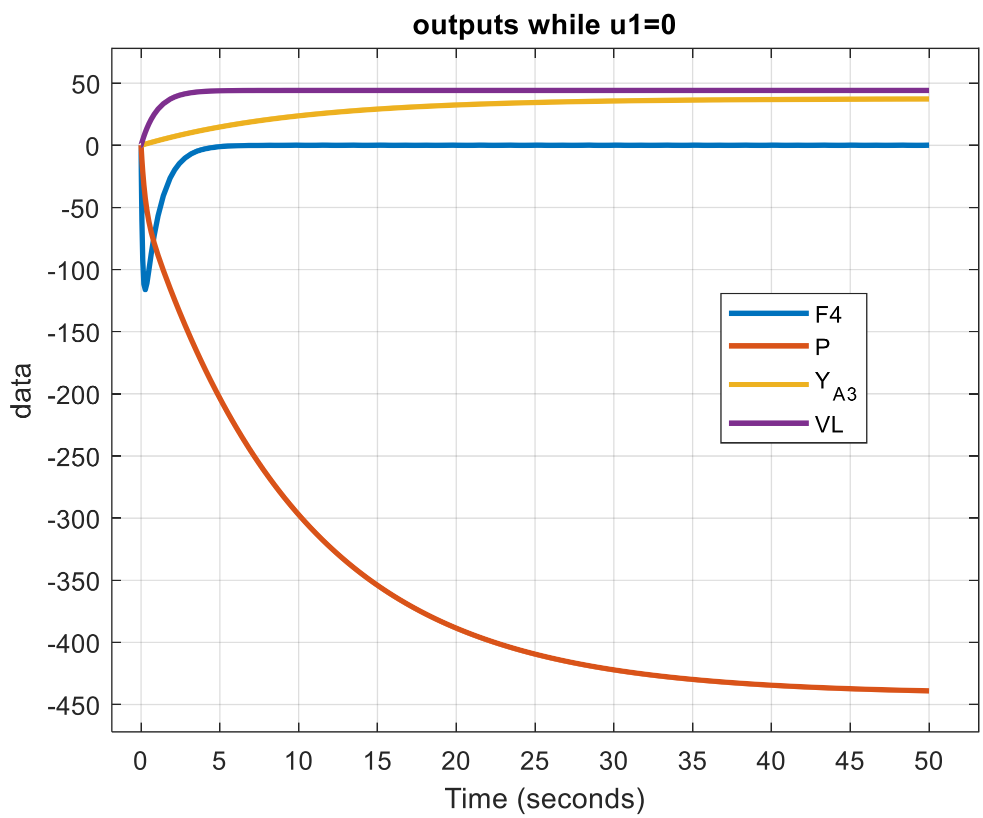
This system could be implemented as in the following figure. Try to open *main_09.slx* to explore more in the simulated system.



All of the outputs of the system are illustrated in the following figure. For a better grasp, the subsystem with the delay factor is plotted again with a little zoom in.

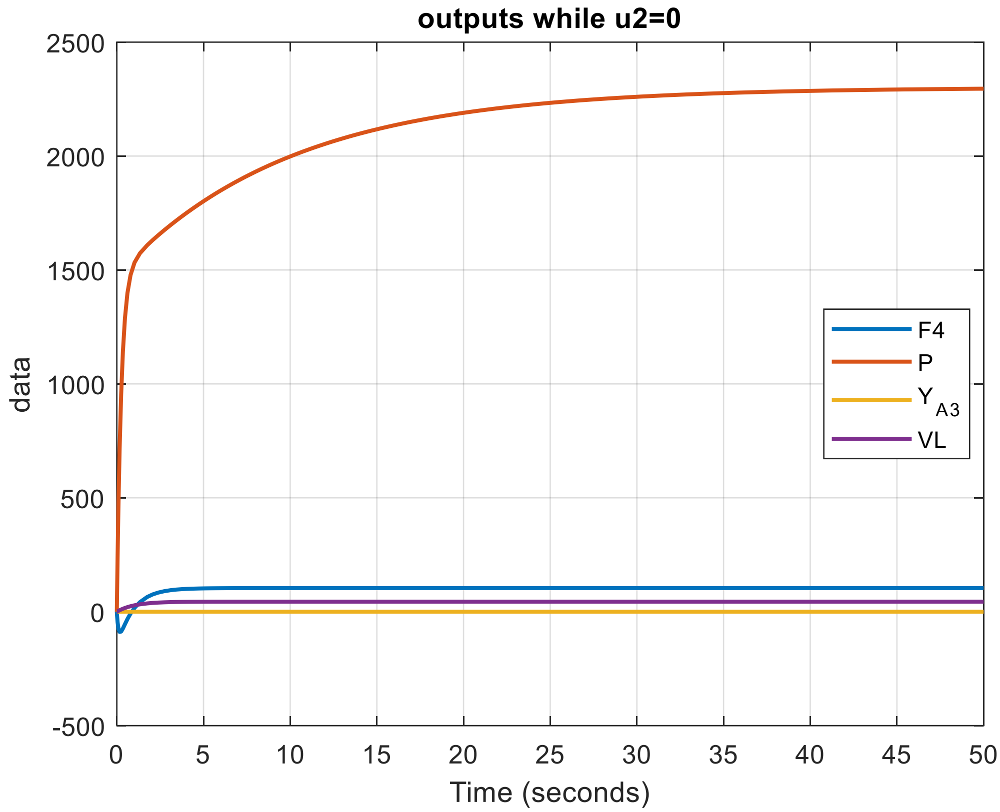


Let $u_1 = 0$:



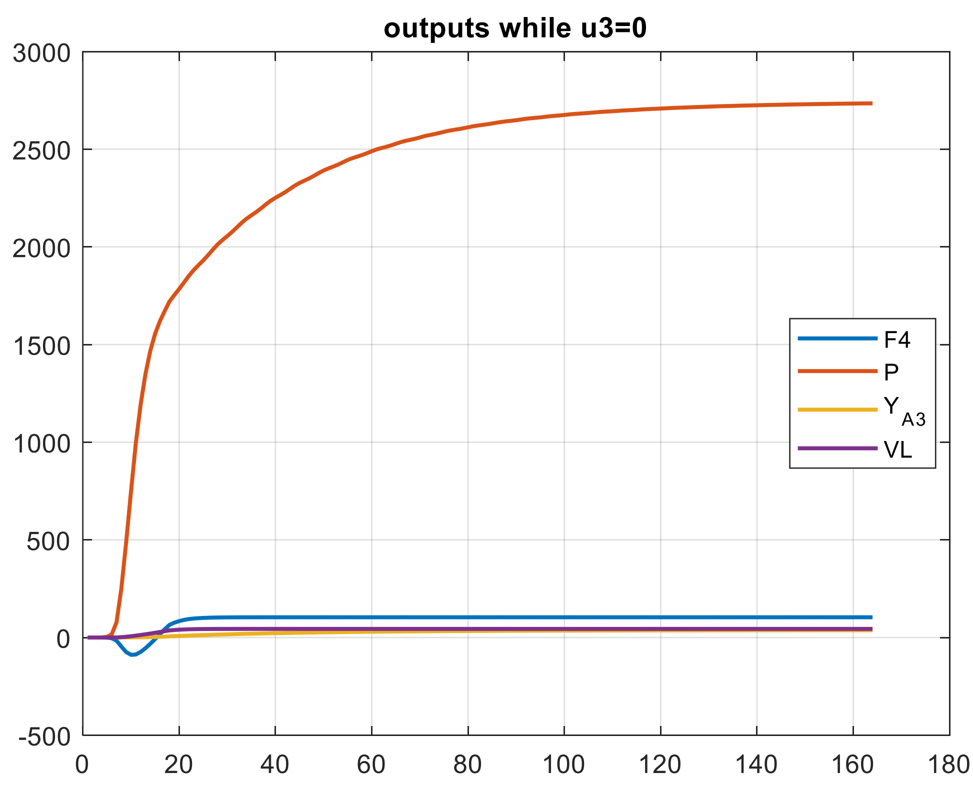
F4 and P are under effect of this input.

Let $u_2 = 0$:



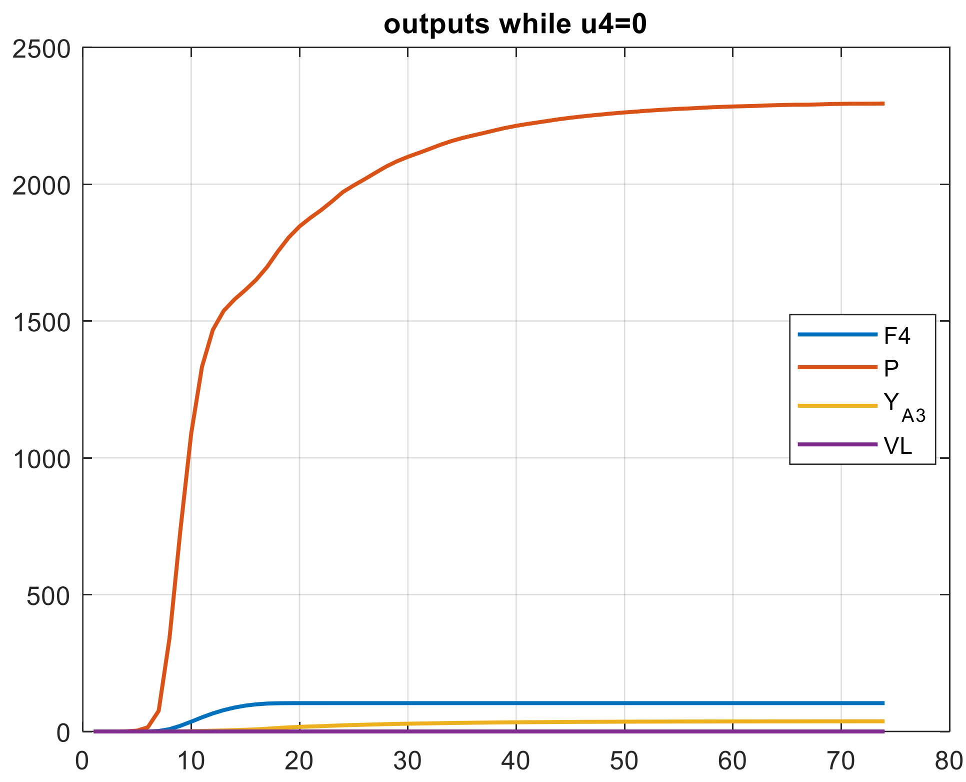
This input affects on Y_{A3}.

Let $u_3 = 0$:



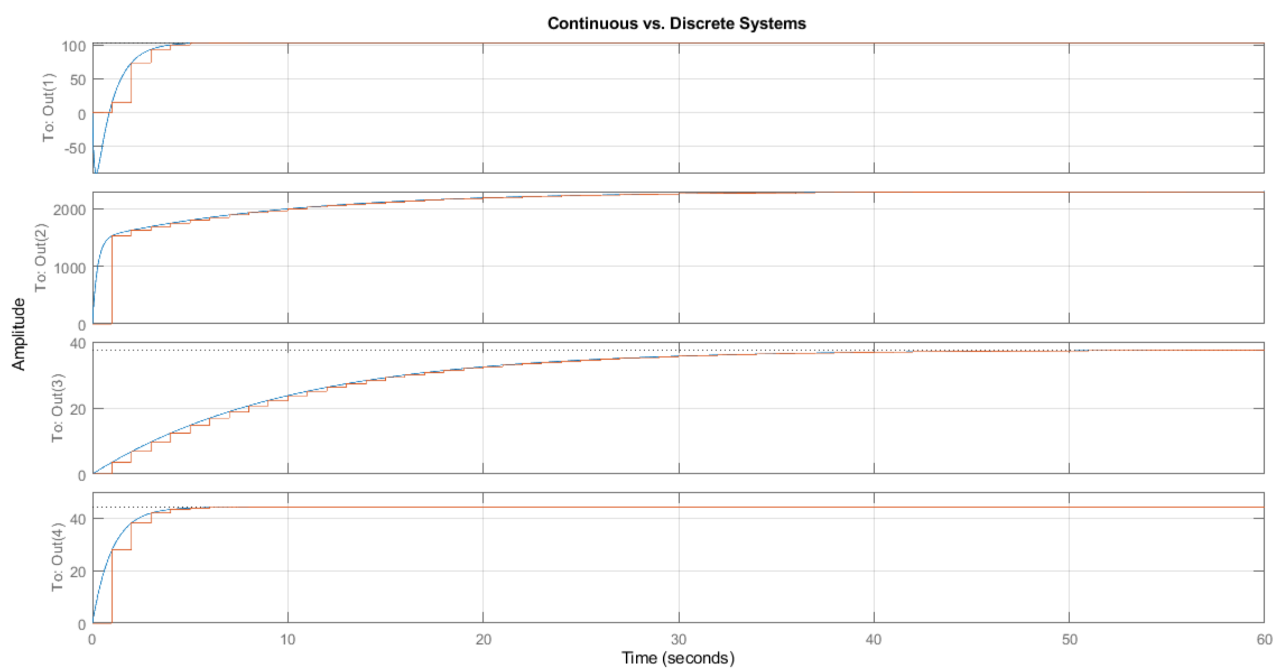
This input affects on P.

Let $u_4 = 0$:



This input affects on $F4$ and VL.

Use *c2d* command to convert the system from continuous time to discrete time. Having converted the system, the system outputs could be plotted along the discrete time form. Results could be found in the following:



It is obvious that the $T_s = 1s$ is not a quite proper sampling time value. Better results could be obtained using a higher sampling time value.