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Exercises - Chapter 04

Problem 01

$$\begin{cases} \underline{\dot{\boldsymbol{x}}} = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} \underline{\boldsymbol{x}} + \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix} \underline{\boldsymbol{u}} \\ \underline{\boldsymbol{y}} = \begin{bmatrix} 0.1 & 0 \\ 0 & 10 \end{bmatrix} \underline{\boldsymbol{x}} \end{cases}$$

To investigate controllability and observability, we form matrices below. If the rank of these matrices equals n, which is 2 here, then they are controllable and observable.

$$Controllability = [B \quad AB]$$

$$Observability = \begin{bmatrix} C \\ CA \end{bmatrix}$$

```
U =

    1.0000         0   -1.0000         0
         0    0.1000    1.0000   -0.2000
```

$$rank(U) = 2$$

The system is controllable.

```
V =

    0.1000         0
         0   10.0000
   -0.1000         0
   10.0000  -20.0000
```

The system is observable.

Hence, there is no uncontrollable nor observable mode.

Problem 02

$$\boldsymbol{A} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} , \quad \boldsymbol{B} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 2 & 2 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Given the matrices above that are in Jordan form, one can examine controllability and observability for each eigenvalue:

$$\lambda = -1 \quad \Rightarrow \quad \begin{cases} \hat{B}_1 = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \\ \hat{C}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{cases}$$

Considering the controllability matrix, this eigenvalue is uncontrollable because two rows are dependent on each other, so this mode is uncontrollable. However, regarding the observability matrix, since two columns are independent of each other, it is observable.

$$\lambda = -2 \quad \Rightarrow \quad \begin{cases} \hat{B}_2 = \begin{bmatrix} 1 & 0 \end{bmatrix} \\ \hat{C}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{cases}$$

Regarding the eigenvalue -2, since it corresponds to only one Jordan block and none of them are zero, it is both controllable and observable.

$$\lambda = 0 \quad \Rightarrow \quad \begin{cases} \hat{B}_3 = \begin{bmatrix} 0 & 1 \end{bmatrix} \\ \hat{C}_3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \end{cases}$$

For the eigenvalue 0, the situation is similar to the previous case, so it is also both controllable and observable.

Problem 03

To examine controllability, the greatest common left divisor (gcld) of the matrices (P) and (Q) must be unimodular, meaning its determinant is independent of (s). Therefore, we have:

$$det \begin{bmatrix} s+2 & s+1 \\ 0 & s \end{bmatrix} = s(s+2)$$

Given that this determinant is not independent of (s), it means that the system is not controllable, and the roots of this matrix correspond to uncontrollable modes, which in this case are 0 and -2, the uncontrollable modes.

For observability, similarly, the greatest common right divisor (gcrd) of the matrices (P) and (R) must be unimodular for the system to be observable. Therefore, we have:

$$\det \begin{bmatrix} s+1 & s \\ 0 & 1 \end{bmatrix} = s+1$$

Since this determinant is also not unimodular, it means the system is not observable, and the roots of this determinant correspond to unobservable modes, which in this case is -1, the unobservable mode.

Problem 04

$$G(s)=\frac{1}{(s+1)(s+2)}\begin{bmatrix} s-1 & s \\ -6 & s-2 \end{bmatrix}$$

First, we verify a realization that is not minimal: Initially, we calculate $\lim_{s\to\infty} G(s) = D$ and decompose the original transfer function as follows, where $\hat{G}(s)$ is the strictly proper part of (G(s)):

$$G(s)=\hat{G}(s)+D$$

$$G(s)=\begin{bmatrix} \frac{s-1}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \\ \frac{-6}{(s+1)(s+2)} & \frac{s-2}{(s+1)(s+2)} \end{bmatrix}$$

$$\lim_{s\to\infty} G(s) = D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow G(s) = \hat{G}(s)$$

Now, for each row of the matrix (G^\prime(s)), we find the greatest common divisor:

$$\hat{G}(s)=\begin{bmatrix} \frac{1}{s^2+3s+2}[s-1 \quad s] \\ \frac{1}{s^2+3s+2}[-6 \quad s-2] \end{bmatrix}$$

Now, we form the matrices (A), (B), and (C):

$$A=\begin{bmatrix} 0 & -2 & 0 & 0 \\ 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & -3 \end{bmatrix},\quad B=\begin{bmatrix} -1 & 0 \\ 1 & 1 \\ -6 & -2 \\ 0 & 1 \end{bmatrix},C=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},D=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$rank\{A\}=4$$

Now, using the Gilbert realization form and considering the function written in MATLAB, for a non-reducible Gilbert realization, one can easily reach the desired state space form of the system which is as follows:

```
A =

[-2,  0]
[ 0, -1]
```

```
B =

[ 0.83205029433784368302751260018 55,   0.55470019622522912201834173345 7]
[0.89442719099991587856366946749251, 0.44721359549995793928183473374626]
```

```
C =

[3.6055512754639892931192212674705, -2.2360679774997896964091736687313]
[ 7.2111025509279785862384425349410, -6.7082039324993690892275210061938]


D =

[0, 0]
[0, 0]
```

where

$rank(A) = 2 < 4$ because of minimality.

Problem 05

$$\boldsymbol{S(s)} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{(s+2)} & \mathbf{0} \\ \mathbf{-(s+2)} & \mathbf{(s+1)(s+2)} & \mathbf{1} & \mathbf{-1} & \mathbf{0} & \mathbf{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{(s+1)(s+2)} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{-(s+2)} & \mathbf{(s+2)} & \mathbf{0} & \mathbf{(s+1)} \\ \mathbf{0} & \mathbf{-(s+2)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{(s+1)} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

To examine controllability and determine uncontrollable modes for (P), (Q), we write a left coprime matrix fraction description (MFD) which will be structured as follows:

$$[P \quad Q] = \begin{bmatrix} 1 & 0 & -1 & 0 & s+2 & 0 \\ -(s+2) & (s+1)(s+2) & 1 & -1 & 0 & -1 \\ 0 & 0 & (s+1)(s+2) & 0 & 1 & 0 \\ 0 & 0 & -(s+2) & s+2 & 0 & s+1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 & s+2 & 0 \\ -(s+2) & (s+1)(s+2) & 0 & -1 & 0 & -1 \\ 0 & 0 & (s+1)(s+2) & 0 & 1 & 0 \\ 0 & 0 & 0 & s+2 & 0 & s+1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 & s+2 & 0 \\ -(s+2) & (s+1)(s+2) & 0 & -1 & 0 & 0 \\ 0 & 0 & (s+1)(s+2) & 0 & 1 & 0 \\ 0 & 0 & 0 & s+2 & 0 & -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 & s+2 & 0 \\ -(s+2) & (s+1)(s+2) & 0 & -1 & 0 & 0 \\ 0 & 0 & (s+1)(s+2) & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 & s+2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & (s+1)(s+2) & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & (s+1)(s+2) & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = L$$

$$det\{L\} = 1$$

Since the determinant obtained is unimodular, it implies that the system is controllable and does not have any uncontrollable modes.

For observability:

$$\begin{bmatrix} P \\ R \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ -(s+2) & (s+1)(s+2) & 1 & -1 \\ 0 & 0 & (s+1)(s+2) & 0 \\ 0 & 0 & -(s+2) & s+2 \\ 0 & s+2 & 0 & 0 \\ 0 & 0 & 0 & -(s+1) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ -(s+2) & 0 & 1 & -1 \\ 0 & 0 & (s+1)(s+2) & 0 \\ 0 & 0 & -(s+2) & s+2 \\ 0 & s+2 & 0 & 0 \\ 0 & 0 & 0 & -(s+1) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ -(s+2) & 0 & 1 & -1 \\ 0 & 0 & (s+1)(s+2) & 0 \\ 0 & 0 & -(s+2) & 1 \\ 0 & s+2 & 0 & 0 \\ 0 & 0 & 0 & -(s+1) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & -(s+1) & -1 \\ 0 & 0 & (s+1)(s+2) & 0 \\ 0 & 0 & -(s+2) & 1 \\ 0 & s+2 & 0 & 0 \\ 0 & 0 & 0 & -(s+1) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & -(s+1) & -1 \\ 0 & 0 & 0 & s+1 \\ 0 & 0 & -(s+2) & 1 \\ 0 & s+2 & 0 & 0 \\ 0 & 0 & 0 & -(s+1) \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & -(s+1) & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -(s+2) & 1 \\ 0 & s+2 & 0 & 0 \\ 0 & 0 & 0 & -(s+1) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & -(s+1) & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & s+2 & 0 & 0 \\ 0 & 0 & 0 & -(s+1) \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & -(s+1) & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & s+2 & 0 & 0 \\ 0 & 0 & -(s+1)/2 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & s+2 & 0 & 0 \\ 0 & 0 & -(s+1)/2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & s+2 & 0 & 0 \\ 0 & 0 & -(s+1)/2 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & s+2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & s+2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = R$$

$$\det\{R\} = s + 2$$

Given that the determinant of this matrix is not unimodular, it indicates that the system is not observable, and the unobservable modes are the roots of this determinant, which in this case is only -2.