4-3

Assuming controllability of the system output, y(t) could be transferred into $y(t_1) = Cx(t_1) = 0$ within a bounded time $0 \leq t \leq t_1$ from any $y(0)$.

The system response:

$$y(t_1) = Cx(t_1) = C^{At_1}[x(0) + \int_0^{t_1} e^{-A\tau} Bu(\tau)d\tau] = 0$$

$$Ce^{At_1 x(0)} = -Ce^{At_1}\int_0^{t_1} e^{-A\tau} Bu(\tau)d\tau = -C\int_0^{t_1} e^{A\tau} Bu(t_1 - \tau)d\tau$$

Define:

$$B = [b_1 \dots b_m]$$

and

$$e^{A\tau} = \sum_{i=0}^{n-1} \alpha_i(\tau)A^i$$

we have:

$$\int_0^{t_1} \sum_{i=0}^{n-1} \alpha_i(\tau)A^i Bu(t_1 - \tau)d\tau = \sum_{i=0}^{n-1}\sum_{j=1}^{m} \gamma_{ij}\, A^i b_j$$

where:

$$\gamma_{ij} = \int_0^{t_1} \alpha_i(\tau)u_j(t_1 - \tau)d\tau$$

Therefore:

$$Ce^{At_1 x(0)} = -\sum_{i=0}^{n-1}\sum_{j=1}^{m} \gamma_{ij} C\, A^i b_j$$

$Ce^{At_1 x(0)}$ is linear combination of $CA^i b_j$. Then if $rank\left\{\Phi_{op}\right\}$ = l, $Cx(0)$ spans the l-dimensional space and the system is output-controllable. Also, if this rank is less than l, then the assumption of being controllable is violated.

If the system is defined as:

$\dot{x} = Ax(t) + Bu(t)$

$y = Cx(t) + Du(t)$

then output-controllability matrix is:

$$\phi_{op} = [CB \quad CAB \quad \dots \quad CA^{n-1}B \quad D]$$