First Order Logic

Exercises

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0.1 Warm-up exercises

Exercise 0.1. *Non Logical symbols:*

constants a, b*; functions* f, g *(*f *is unary,* g *is binary); predicates* P, R, Q *(unary, binary and ternary, respectively).*

Say whether the following strings of symbols are well formed FOL formulas or terms:

1. $Q(a)$
2. $P(y)$
3. $P(g(b))$
4. $\neg R(x, a)$
5. $Q(x, P(a), b)$
6. $P(g(f(a), g(x, f(x))))$
7. $Q(f(a), f(f(x)), f(g(f(z), g(a, b))))$
8. $R(a, R(a, a))$

Solution. *Well formed formulas: 2., 4., 6., and 7. All other strings are NOT well formed FOL formulas nor terms.*

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Exercise 0.2. *Non Logical symbols:*

constants a, b*; functions* f^1, g^2*; predicates* P^1, R^2, Q^3 *(the exponent denotes the arity).*

Say whether the following strings of symbols are well formed FOL formulas or terms:

1. $R(a, g(a, a))$*;*

2. $g(a, g(a, a))$;
3. $\forall x.\neg P(x)$;
4. $\neg R(P(a), x)$;
5. $\exists a.R(a, a)$;
6. $\exists x.Q(x, f(x), b) \to \forall x.R(a, x)$;
7. $\exists x.P(R(a, x))$;
8. $\forall R(x, a)$;

Solution. *Well formed formulas: 1., 3., and 6. Well formed terms: 2. All other strings are NOT well formed FOL formulas nor terms.*

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Exercise 0.3. *Non Logical symbols:*

constants a, b; functions f^1, g^2; predicates P^1, R^2, Q^3 (the exponent denotes the arity).

Say whether the following strings of symbols are well formed FOL formulas or terms:

1. $a \to P(b)$;
2. $R(x, b) \to \exists y.Q(y, y, y)$;
3. $R(x, b) \vee \neg\exists y.g(y, b)$;
4. $\neg y \vee P(y)$;
5. $\neg\neg P(a)$;
6. $\neg\forall x.\neg P(x)$;
7. $\forall x\exists y.(R(x, y) \to R(y, x))$;

8. $\forall x \exists y.(R(x, y) \to (R(y, x) \vee (f(a) = g(a, x))))$;

Solution. *Well formed formulas: 2., 5., 6., 7., and 8. All other strings are NOT well formed FOL formulas nor terms.*

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Exercise 0.4. *Obtain a Skolem Normal Form for the formula:*

$$\exists x.\forall y.\forall z.\exists u.\forall v.\exists w.P(x, y, z, u, v, w).$$

Solution. *We have the following.*

1. x *is not preceded by any* $\forall$ *thus* x *becomes* c.
2. u *is preceded by* $\forall y.$ *and* $\forall z.$ *thus* u *becomes* $f(y, z)$.
3. w *is preceded by* $\forall y.$ *and* $\forall z.$ *and* $\forall v.$ *thus* w *becomes* $g(y, z, v)$.

The result is

$$\forall y.\forall z.\forall v.P(c, y, z, f(y, z), g(y, z, v)).$$

Exercise 0.5. *Obtain a Skolem Normal Form for the formula:*

$$\forall x.\exists y.\neg\forall z.\neg((\neg P(x, y) \wedge Q(x, z)) \vee R(x, y, z))$$

with the matrix in CNF.

Solution. *First convert the formula in prenex form with the matrix in CNF:*

$$\forall x.\exists y \exists z((\neg P(x, y) \vee R(x, y, z)) \wedge (Q(x, z) \vee R(x, y, z)).$$

Then,

1. y *is preceded by* $\forall x$ *thus* y *becomes* $f(x)$
2. z *is preceded by* $\forall x$ *thus* z *becomes* $g(x)$

We have

$$\forall x.((\neg P(x, f(x)) \vee R(x, f(x), g(x))) \wedge (Q(x, g(x)) \vee R(x, f(x), g(x)))$$

Exercise 0.6. *Obtain a Skolem Normal Form for the formula:*

$$\forall x.\exists y((\neg P(x,y) \equiv \forall z.Q(x,z)) \to R(x,y,z))$$

with the matrix in CNF.

Solution. *First eliminate implications and equivalences:*

$$\forall x \exists y(((\neg P(x,y) \vee \forall z.Q(x,z)) \wedge (P(x,y) \vee \exists w.\neg Q(x,w))) \vee R(x,y,z)).$$

Notice that by eliminating the equivalence, we get two quantifiers corresponding to the initial $\forall z$. *Then, write the formula in prenex form:*

$$\forall x.\exists y.\exists w.\forall z.(((\neg P(x,y) \vee Q(x,z)) \wedge (P(x,y) \vee \neg Q(x,w))) \vee R(x,y,z)).$$

Finally either you Skolemize and then write the matrix in CNF or you write the matrix in CNF and then Skolemize. The result is:

$$\forall x.\forall z.((\neg P(x,f(x)) \vee Q(x,z) \vee R(x,f(x),z)) \wedge (P(x,f(x)) \vee \neg Q(x,g(x)) \vee R(x,f(x),z)).$$

Exercise 0.7. *Consider a knowledge base containing just two sentences:* $P(a)$ *and* $P(b)$. *Does this knowledge base entail* $\forall x.\ P(x)$? *Explain your answer in terms of models.*

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0.2 FOL Formalization

Note: Some exercises require the use of equality which can be either formalized as a congruence relation or assumed as an additional language symbol with a fixed semantics.

Exercise 0.8. *What is the meaning of the following FOL formulas?*

1. $bought(Frank, dvd)$
2. $\exists x.bought(Frank, x)$

3. $\forall x.(bought(Frank, x) \to bought(Susan, x))$

4. $\forall x.bought(Frank, x) \to \forall x.bought(Susan, x)$

5. $\forall x \exists y.bought(x, y)$

6. $\exists x \forall y.bought(x, y)$

Solution.

1. *"Frank bought a dvd."*

2. *"Frank bought something."*

3. *"Susan bought everything that Frank bought."*

4. *"If Frank bought everything, so did Susan."*

5. *"Everyone bought something."*

6. *"Someone bought everything."*

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Exercise 0.9. *Which of the following formulas is a formalization of the sentence:*
"There is a computer which is not used by any student"

- $\exists x.(Computer(x) \wedge \forall y.(\neg Student(y) \wedge \neg Uses(y, x)))$
- $\exists x.(Computer(x) \to \forall y.(Student(y) \to \neg Uses(y, x)))$
- $\exists x.(Computer(x) \wedge \forall y.(Student(y) \to \neg Uses(y, x)))$

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Exercise 0.10. *Define an appropriate language and formalize the following sentences using FOL formulas.*

1. *All Students are smart.*
2. *There exists a student.*
3. *There exists a smart student.*
4. *Every student loves some student.*
5. *Every student loves some other student.*
6. *There is a student who is loved by every other student.*
7. *Bill is a student.*
8. *Bill takes either Analysis or Geometry (but not both).*
9. *Bill takes Analysis and Geometry.*
10. *Bill doesn't take Analysis.*
11. *No students love Bill.*

Solution.

1. $\forall x.(Student(x) \to Smart(x))$
2. $\exists x.Student(x)$
3. $\exists x.(Student(x) \wedge Smart(x))$
4. $\forall x.(Student(x) \to \exists y.(Student(y) \wedge Loves(x, y)))$
5. $\forall x.(Student(x) \to \exists y.(Student(y) \wedge \neg(x = y) \wedge Loves(x, y)))$
6. $\exists x.(Student(x) \wedge \forall y.(Student(y) \wedge \neg(x = y) \to Loves(y, x)))$
7. $Student(Bill)$
8. $Takes(Bill, Analysis) \leftrightarrow \neg Takes(Bill, Geometry)$
9. $Takes(Bill, Analysis) \wedge Takes(Bill, Geometry)$
10. $\neg Takes(Bill, Analysis)$
11. $\neg \exists x.(Student(x) \wedge Loves(x, Bill))$

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Exercise 0.11. *Define an appropriate language and formalize the following sentences using FOL formulas.*

1. *Bill has at least one sister.*
2. *Bill has no sister.*
3. *Bill has at most one sister.*
4. *Bill has (exactly) one sister.*
5. *Bill has at least two sisters.*
6. *Every student takes at least one course.*
7. *Only one student failed Geometry.*
8. *No student failed Geometry but at least one student failed Analysis.*
9. *Every student who takes Analysis also takes Geometry.*

Solution.

1. $\exists x.SisterOf(x, Bill)$
2. $\neg\exists x.SisterOf(x, Bill)$
3. $\forall x\forall y.(SisterOf(x, Bill) \wedge SisterOf(y, Bill) \rightarrow x = y)$
4. $\exists x.(SisterOf(x, Bill) \wedge \forall y.(SisterOf(y, Bill) \rightarrow x = y))$
5. $\exists x\exists y.(SisterOf(x, Bill) \wedge SisterOf(y, Bill) \wedge \neg(x = y))$
6. $\forall x.(Student(x) \rightarrow \exists y.(Course(y) \wedge Takes(x, y)))$
7. $\exists x.(Student(x) \wedge Failed(x, Geometry) \wedge \forall y.(Student(y) \wedge Failed(y, Geometry) \rightarrow x = y))$

8. $\neg\exists x.(Student(x) \wedge Failed(x, Geometry)) \wedge \exists x.(Student(x) \wedge Failed(x, Analysis))$

9. $\forall x.(Student(x) \wedge Takes(x, Analysis) \rightarrow Takes(x, Geometry))$

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Exercise 0.12. *Define an appropriate language and formalize the following sentences in FOL:*

- *someone likes Mary.*
- *nobody likes Mary.*
- *nobody loves Bob but Bob loves Mary.*
- *if David loves someone, then he loves Mary.*
- *if someone loves David, then he (someone) loves also Mary.*
- *everybody loves David or Mary.*

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Exercise 0.13. *Define an appropriate language and formalize the following sentences in FOL:*

1. *there is at least one person who loves Mary.*
2. *there is at most one person who loves Mary.*
3. *there is exactly one person who loves Mary.*
4. *there are exactly two persons who love Mary.*
5. *if Bob loves everyone that Mary loves, and Bob loves David, then Mary doesn't love David.*
6. *Only Mary loves Bob.*

Solution. *We assume to have a predicate "=" defined as a congruence relation on the elements of the domain.*

1. $\exists x.Person(x) \wedge Loves(x, Mary)$
2. $\exists x.(Person(x) \wedge Loves(x, Mary) \wedge \forall y.((Person(y) \wedge \neg(y = x)) \rightarrow \neg Loves(y, Mary))) \vee \forall x.(Person(x) \rightarrow \neg Loves(x, Mary))$
3. $\exists x.(Person(x) \wedge Loves(x, Mary) \wedge \forall y.((Person(y) \wedge \neg(y = x)) \rightarrow \neg Loves(y, Mary)))$
4. $\exists x.\exists y.(Person(x) \wedge Person(y) \wedge \neg(x = y) \wedge Loves(x, Mary) \wedge Loves(y, Mary) \wedge \forall z.((Person(z) \wedge \neg(z = x) \wedge \neg(z = y) \rightarrow \neg Loves(z, Mary)))$
5. $(\forall x.(Person(x) \wedge Loves(Mary, x) \rightarrow Loves(Bob, x)) \wedge Loves(Bob, David)) \rightarrow \neg Loves(Mary, David)$
6. $Loves(Mary, Bob) \wedge \forall x.(Person(x) \wedge \neg(x = Mary) \rightarrow \neg Loves(x, Bob))$

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0.3 Formalization and reasoninig

Exercise 0.14. *Consider the following sentences:*

1. *All actors and journalists invited to the party are late.*
2. *There is at least a person who is on time.*
3. *There is at least an invited person who is neither a journalist nor an actor.*

Formalize the sentences and prove that 3. is not a logical consequence of 1. and 2.

Solution. *Let A stand for "Actor", J stand for "Journalist", I for "Invited" and and L for "Late". Then we have the following:*

1. $\varphi_1 := \forall x.((A(x) \vee J(x)) \wedge I(x) \rightarrow L(x))$

2. $\varphi_2 := \exists x.\neg L(x)$

3. $\varphi_3 := \exists x.(I(x) \wedge \neg A(x) \wedge \neg J(x))$

It is sufficient to find an interpretation $\mathcal{I} = \langle D, g\rangle$ for which the logical consequence does not hold. Let Bob, Tom and Mary be three constants. The following table:

	$L(x)$	$A(x)$	$J(x)$	$I(x)$
Bob	*F*	*T*	*F*	*F*
Tom	*T*	*T*	*F*	*T*
Mary	*T*	*F*	*T*	*T*

describes the function g such that, e.g., $g(L) = \{Tom, Mary\}$ and so on. Clearly $\models_{\mathcal{I}} \varphi_1 \wedge \varphi_2$ but $\not\models_{\mathcal{I}} \varphi_3$.

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Exercise 0.15. *Consider the following sentences:*

1. *All actors and journalists invited to the party are late.*
2. *There is at least a person who is on time.*
3. *There is at least an invited person who is neither a journalist nor an actor.*

Formalize the sentences and prove that 3. is not a logical consequence of 1. and 2. using Herbrand theorem specifying an Herbrand domain.

Solution. *Let A stand for "Actor", J stand for "Journalist", I for "Invited" and and L for "Late". Then we have the following:*

1. $\forall x.((A(x) \vee J(x)) \wedge I(x) \rightarrow L(x))$

2. $\exists x.\neg L(x)$

3. $\exists x.(I(x) \wedge \neg A(x) \wedge \neg J(x))$

We show that the third is entailed by the first two by proving that the negation of the third and the first two

1. $\forall x.((A(x) \vee J(x)) \wedge I(x) \rightarrow L(x))$
2. $\exists x.\neg L(x)$
3. $\forall x.(\neg I(x) \vee A(x) \vee J(x))$

are inconsistent, which is true if and only if their skolemize form

1. $\forall x.((A(x) \vee J(x)) \wedge I(x) \rightarrow L(x))$
2. $\neg l(C)$
3. $\forall x.(\neg I(x) \vee A(x) \vee J(x))$

are incosistent, which is true, by Hebrand theorem with domain $D_H = \{c\}$, if and only if

1. $((A(c) \vee J(c)) \wedge I(c) \rightarrow L(c))$
2. $\neg L(c)$
3. $(\neg I(c) \vee A(c) \vee J(c))$

is unsatisfiable. The three formulas are satisfiable (a model is obtained by assigning $l(C)$ and $i(C)$ to $\perp$) and thus the third formula is not a logical consequence of the first 2.

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