Propositional Logic

Artficial Intelligence for Robotics I

Academic year 2021-2022

0.1 Warm-up exercises

Exercise 0.1. *Reduce to Negation Normal Form (NNF) the formula*

$$\neg(\neg p \vee q) \vee (r \to \neg s)$$

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Exercise 0.2. *Reduce to CNF the formula*

$$(\neg p \to q) \to (q \to \neg r)$$

✻

Exercise 0.3. *Reduce to Conjunctive Normal Form (CNF) the formula*

$$\neg(\neg p \vee q) \vee (r \to \neg s)$$

Solution.

1. $\neg(\neg p \vee q) \vee (\neg r \vee \neg s)$
2. $(\neg\neg p \wedge \neg q) \vee (\neg r \vee \neg s)$
3. $(p \wedge \neg q) \vee (\neg r \vee \neg s)$ NNF
4. $(p \vee \neg r \vee \neg s) \wedge (\neg q \vee \neg r \vee \neg s)$

✻

Exercise 0.4. *Reduce to Conjunctive Normal Form (CNF) the formula*

$$(\neg p \to q) \to (q \to \neg r)$$

Solution.

1. $\neg(\neg p \to q) \vee (q \to \neg r)$

2. $\neg(p \vee q) \vee (\neg q \vee \neg r)$

3. $(\neg p \wedge \neg q) \vee (\neg q \vee \neg r)$ **NNF**

4. $(\neg p \vee \neg q \vee \neg r) \wedge (\neg q \vee \neg r)$

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Exercise 0.5. *Reduce to CNF the following formulas (using Tseitin's conversion):*

- $p \to (q \wedge r)$
- $(p \vee q) \to r$
- $\neg(\neg p \vee q) \vee (r \to \neg s)$
- $\neg((p \to (q \to r))) \to ((p \to q) \to (p \to r))$
- $p \vee (\neg q \wedge (r \to \neg p))$
- $\neg((((a \to b)) \to a) \to a)$
- $\neg(a \vee (a \to b))$

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Exercise 0.6. *Use a semantic method (e.g., truth tables) to verify whether the following formulas are valid, satisfiable or unsatisfiable:*

- $(p \to q) \wedge \neg q \to \neg p$
- $(p \to q) \to (p \to \neg q)$

- $(p \vee q \to r) \vee p \vee q$
- $(p \vee q) \wedge (p \to r \wedge q) \wedge (q \to \neg r \wedge p)$
- $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$
- $(p \vee q) \wedge (\neg q \wedge \neg p)$
- $(\neg p \to q) \vee ((p \wedge \neg r) \leftrightarrow q)$
- $(p \to q) \wedge (p \to \neg q)$
- $(p \to (q \vee r)) \vee (r \to \neg p)$

✻

Exercise 0.7. *Prove that the formula $(p \to (q \to p))$ is a tautology using resolution/variable elimination or the DPLL procedure.*

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Exercise 0.8. *Prove that the negation of the formula*

$$((p \to (q \to r)) \to ((p \to q) \to (p \to r)))$$

is unsatisfiable, using resolution/variable elimination or the DPLL procedure..

Solution. *A set of clauses corresponding to the negation of the formula is*

$$\{\{\neg P, \neg Q, R\}, \{\neg P, Q\}, \{P\}, \{\neg R\}\}$$

Assuming unit clauses are detected and propagated according to variable elimination or DPLL:

1. *P is propagated because of $\{P\}$, producing the set of clauses $\{\{\neg Q, R\}, \{Q\}, \{\neg R\}\}$*

2. *Q is propagated because of $\{Q\}$, producing the set of clauses $\{\{R\}, \{\neg R\}\}$*
3. *$\neg R$ is propagated producing the empty clause and $False$ is returned.*

The given set of clauses is unsatisfiable.

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Exercise 0.9. *Use resolution / variable elimination or DPLL to verify whether the following logical consequences and equivalences are correct:*

- $(p \to q) \models \neg p \to \neg q$
- $(p \to q) \wedge \neg q \models \neg p$
- $p \to q \wedge r \models (p \to q) \to r$
- $p \vee (\neg q \wedge r) \models q \vee \neg r \to p$
- $\neg(p \wedge q)$ *is eqquivalent to* $\neg p \vee \neg q$
- $(p \vee q) \wedge (\neg p \to \neg q)$ *is equivalent to* q
- $(p \wedge q) \vee r$ *is equivalent to* $(p \to \neg q) \to r$
- $(p \vee q) \wedge (\neg p \to \neg q)$ *is equivalent to* p
- $((p \to q) \to q) \to q$ *is equivalent to* $p \to q$

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0.2 Formalization

Exercise 0.10. *Let's consider a propositional language where*

- p *means* "x is a prime number",

- *q means* "x is odd".

with $x > 2$.

Formalize the following sentences:

1. "x being prime is a sufficient condition for x being odd"

2. "x being odd is a necessary condition for x being prime"

Solution. *The solution to both 1. and 2. is simply $p \to q$*

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Exercise 0.11. *Let $A =$*"Aldo is Italian" *and $B =$*"Bob is English".
Formalize the following sentences:

1. "Aldo isn't Italian"

2. "Aldo is Italian while Bob is English"

3. "If Aldo is Italian then Bob is not English"

4. "Aldo is Italian or if Aldo isn't Italian then Bob is English"

5. "Either Aldo is Italian and Bob is English, or neither Aldo is Italian nor Bob is English"

Solution.

1. $\neg A$

2. $A \wedge B$

3. $A \to \neg B$

4. $A \vee (\neg A \to B)$ *logically equivalent to $A \vee B$*

5. $(A \wedge B) \vee (\neg A \wedge \neg B)$ *logically equivalent to* $A \leftrightarrow B$

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Exercise 0.12. *Angelo, Bruno and Carlo are three students that took the Logic exam. Let's consider a propositional language where*

- $A =$"Aldo passed the exam"*,*
- $B =$"Bruno passed the exam"*,*
- $C =$"Carlo passed the exam".

Formalize the following sentences:

1. "Carlo is the only one passing the exam"
2. "Aldo is the only one not passing the exam"
3. "Only one, among Aldo, Bruno and Carlo, passed the exam"
4. "At least one among Aldo, Bruno and Carlo passed"
5. "At least two among Aldo, Bruno and Carlo passed the exam"
6. "At most two among Aldo, Bruno and Carlo passed the exam"
7. "Exactly two, among Aldo, Bruno and Carlo passed the exam"

Solution.

1. $\neg A \wedge \neg B \wedge C$
2. $\neg A \wedge B \wedge C$
3. $\neg(A \wedge B) \wedge \neg(A \wedge C) \wedge \neg(B \wedge C) \wedge (A \vee B \vee C)$
4. $A \vee B \vee C$
5. $(A \vee B) \wedge (A \vee C) \wedge (B \vee C)$
6. $\neg A \vee \neg B \vee \neg C$

7. $(A \vee B) \wedge (A \vee C) \wedge (B \vee C) \wedge (\neg A \vee \neg B \vee \neg C)$

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Exercise 0.13. *Let's consider a propositional language where*

- $A =$"Angelo comes to the party",
- $B =$"Bruno comes to the party",
- $C =$"Carlo comes to the party",
- $D =$"Davide comes to the party".

Formalize the following sentences:

1. "If Davide comes to the party then Bruno and Carlo come too"
2. "Carlo comes to the party only if Angelo and Bruno do not come"
3. "Davide comes to the party if and only if Carlo comes and Angelo doesn't come"
4. "If Davide comes to the party, then, if Carlo doesn't come then Angelo comes"
5. "Carlo comes to the party provided that Davide doesn't come, but, if Davide comes, then Bruno doesn't come"
6. "A necessary condition for Angelo coming to the party, is that, if Bruno and Carlo aren't coming, Davide comes"

Solution.

1. $D \rightarrow B \wedge C$
2. $C \rightarrow \neg A \wedge \neg B$
3. $D \leftrightarrow (C \wedge \neg A)$
4. $D \rightarrow (\neg C \rightarrow A)$

5. $(\neg D \to C) \land (D \to \neg B)$
6. $A \to (\neg B \land \neg C \to D)$

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Exercise 0.14. *Anna and Barbara carpool to work. On any day, either Anna drives Barbara or Barbara drives Anna. In the former case, Anna is the driver and Barbara is the passenger; in the latter case Barbara is the driver and Anna is the passenger. Formalize the problem using the following propositions:*

1. Anna drives Barbara
2. Barbara drives Anna
3. Anna is the driver
4. Barbara is the driver
5. Anna is the passenger
6. Barbara is the passenger

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Exercise 0.15. *Define a propositional language which allows to describe the state of a traffic light on different instants.*

With the language defined above provide a (set of) formulas which expresses the following facts:

1. *the traffic light is either green, or red or orange;*
2. *the traffic light switches from green to orange, from orange to red, and from red to green;*
3. *it can keep the same color over at most 3 successive states.*

Solution.

Language

- g_k="traffic light is green at instant k"
- r_k="traffic light is red at instant k"
- o_k="traffic light is orange at instant k"

Axioms

1. "the traffic light is either green, or red or orange"
   $(g_k \leftrightarrow (\neg r_k \wedge \neg o_k)) \wedge (r_k \leftrightarrow (\neg g_k \wedge \neg o_k)) \wedge (o_k \leftrightarrow (\neg r_k \wedge \neg g_k))$
2. "the traffic light switches from green to orange, from orange to red, and from red to green"
   $(g_{k-1} \rightarrow (g_k \vee o_k)) \wedge (o_{k-1} \rightarrow (o_k \vee r_k)) \wedge (r_{k-1} \rightarrow (r_k \vee g_k))$
3. "it can keep the same color over at most 3 successive states"
   $(g_{k-3} \wedge g_{k-2} \wedge g_{k-1} \rightarrow \neg g_k) \wedge (r_{k-3} \wedge r_{k-2} \wedge r_{k-1} \rightarrow \neg r_k) \wedge (o_{k-3} \wedge o_{k-2} \wedge o_{k-1} \rightarrow \neg o_k)$

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0.3 Formalization and reasoning

In the following exercises (similar to those you will find in the final test), you are asked to come up with a formalization of the problem and provide a correct deduction using either resolution/variable elimination or DPLL.

Exercise 0.16. *Socrate says:*

> "If I'm guilty, I must be punished;
> I'm guilty. Thus I must be punished."

Is the argument logically correct?

Solution. *The argument is logically correct: if* p *means* "I'm guilty" *and* q *means* "I must be punished"*, then:*
$(p \to q) \wedge p \models q$ *(modus ponens)*

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Exercise 0.17. *Socrate says:*

> "If I'm guilty, I must be punished;
> I'm not guilty. Thus I must not be punished."

Is the argument logically correct?

Solution. *The argument is not logically correct:*
$(p \to q) \wedge \neg p \not\models \neg q$
☞ *consider for instance* $v(p) = \boldsymbol{F}$ *and* $v(q) = \boldsymbol{T}$

✻

Exercise 0.18. *Socrate says:*

> "If I'm guilty, I must be punished;
> I must not be punished. Thus I'm not guilty."

Is the argument logically correct?

✻

Exercise 0.19. *Socrate says:*

> "If I'm guilty, I must be punished;
> I must be punished. Thus I'm guilty."

Is the argument logically correct?

✻

Exercise 0.20. *Aladdin finds two trunks A and B in a cave. He knows that each of them either contains a treasure or a fatal trap.*

> *On trunk A is written:* "At least one of these two trunks contains a treasure."
>
> *On trunk B is written:* "In A there's a fatal trap."

Aladdin knows that either both the inscriptions are true, or they are both false.
Can Aladdin choose a trunk being sure that he will find a treasure?
If this is the case, which trunk should he open?

Solution. *Let's consider a propositional language where a =*"Trunk A contains the treasure" *and b =*"Trunk B contains the treasure".
☞ *Obviously $\neg a$ =*"Trunk a contains a trap" *(and similarly for $\neg b$), since each trunk either contains a treasure or a trap (exclusive or).*
Let's formalize what Aladdin knows:

- *Formalization of the inscriptions:*
  $a \vee b$ "At least one of these two trunks contains a treasure."
  $\neg a$ "A contains a trap"
- *Formalization of the problem:*
  1. "either both the inscriptions are true, or they are both false"
  $(a \vee b) \leftrightarrow \neg a$

What we can do is to verify whether there is any interpretation satisfying the formula in 1. :

- *The only interpretation satisfying 1. is:*
  $v(a) = $ ***F*** *and* $v(b) = $ ***T***

- *Thus Aladdin can open trunk B, being sure that it contains a treasure.*

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Exercise 0.21. *Suppose we know that:*

- "if Paolo is thin, then Carlo is not blonde or Roberta is not tall"
- "if Roberta is tall then Sandra is lovely"
- "if Sandra is lovely and Carlo is blonde then Paolo is thin"
- "Carlo is blonde"

Can we deduce that "Roberta is not tall" *?*

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Exercise 0.22. *Three boxes are presented to you. One contains gold, the other two are empty. Each box has imprinted on it a clue as to its contents; the clues are:*

Box 1 "The gold is not here"

Box 2 "The gold is not here"

Box 3 "The gold is in Box 2"

Only one message is true; the other two are false. Which box has the gold?

Solution. *Let B_i with $i \in \{1, 2, 3\}$ stand for "gold is in the i-th box". We can formalize the statements of the problem as follows:*

1. *One box contains gold, the other two are empty.*

$$(B_1 \land \neg B_2 \land \neg B_3) \lor (\neg B_1 \land B_2 \land \neg B_3) \lor (\neg B_1 \land \neg B_2 \land B_3) \tag{1}$$

2. *Only one message is true; the other two are false.*

$$(\neg B_1 \wedge \neg\neg B_2 \wedge \neg B_2) \vee (\neg\neg B_1 \wedge \neg B_2 \wedge \neg B_2) \vee (\neg\neg B_1 \wedge \neg\neg B_2 \wedge B_2) \quad (2)$$

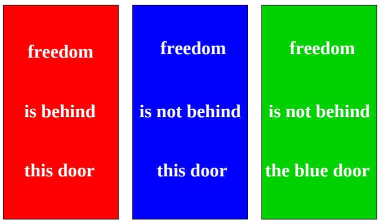
(2) is equivalent to:

$$(B_1 \wedge \neg B_2) \vee (B_1 \wedge B_2) \quad (3)$$

Using, e.g., the DPLL procedure it is easy to see that the only assignment I that verifies both (1) and (3) is the one with $I(B_1) = T$ and $I(B_2) = I(B_3) = F$, which implies that the gold is in the first box.

✻

Exercise 0.23. *Kyle, Neal, and Grant find themselves trapped in a dark and cold dungeon. After a quick search the boys find three doors. Behind one of the doors is a path to freedom. Behind the other two doors, however, is an evil fire-breathing dragon. Opening a door to the dragon means almost certain death. On each door there is an inscription:*



Given that at LEAST ONE of the three statements is true and at LEAST ONE of them is false, which door would lead the boys to safety?

Solution.

Language

- r: "freedom is behind the red door"
- b: "freedom is behind the blue door"
- g: "freedom is behind the green door"

Axioms

1. "behind one of the door is a path to freedom, behind the other two doors is an evil dragon"

$$(r \land \neg b \land \neg g) \lor (\neg r \land b \land \neg g) \lor (\neg r \land \neg b \land g) \tag{4}$$

2. "at least one of the three statements is true"

$$r \lor \neg b \tag{5}$$

3. "at least one of the three statements is false"

$$\neg r \lor b \tag{6}$$

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Exercise 0.24.
The Labyrinth Guardians. *You are walking in a labyrinth and all of a sudden you find yourself in front of three possible roads: the road on your left is paved with gold, the one in front of you is paved with marble, while the one on your right is made of small stones. Each street is protected by a guardian. You talk to the guardians and this is what they tell you:*

- *The guardian of the gold street:* "This road will bring you straight to the center. Moreover, if the stones take you to the center, then also the marble takes you to the center."
- *The guardian of the marble street:* "Neither the gold nor the stones will take you to the center."
- *The guardian of the stone street:* "Follow the gold and you'll reach the center, follow the marble and you will be lost."

Given that you know that all the guardians are liars, can you choose a road being sure that it will lead you to the center of the labyrinth? If this is the case, *which road you choose?*

Provide a propositional language and a set of axioms that formalize the problem and show whether you can choose a road being sure it will lead to the center.

Solution.

Language

- g: "the gold road brings to the center"
- m: "the marble road brings to the center"
- s: "the stone road brings to the center"

Axioms

1. "The guardian of the gold street is a liar"

$$\neg(g \wedge (s \rightarrow m)) \tag{7}$$

which can be simplified to obtain

$$\neg g \vee (s \wedge \neg m)$$

2. "The guardian of the marble street is a liar"

$$\neg(\neg g \wedge \neg s) \tag{8}$$

which can be simplified to obtain

$$g \vee s$$

3. "The guardian of the stone street is a liar"

$$\neg(g \wedge \neg m) \tag{9}$$

which can be simplified to obtain

$$\neg g \vee m$$

Using, e.g., DPLL, we can see that there are two possible interpretations that satisfy the axioms, and in both of them the stone street brings to the center.
Thus I can choose the **stone street** being sure that it leads to the center.

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