Main topic: Correlation, energy density spectra, Simple filter design:

problem 1:)

a) Derive the energy density spectrum $S_{xx}(f)$ of the Signal:

$$x(n) = \begin{cases} a^n, & n \geq 0, |a| < 1 \\ 0, & n < 0 \end{cases}$$

We know that!

$$r_{xx}[l] = x[l] * x[-l] \xleftrightarrow{\mathcal{F}} S_{xx}(\omega) = X(\omega) X^*(\omega) = |X(\omega)|^2$$

$$x(n) = u(n) a^n$$

$$X(\omega) = \mathcal{F}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(a e^{-j\omega}\right)^n$$

$$\Rightarrow \quad S_{xx}(\omega) = \frac{1}{1 - ae^{-j\omega}} \cdot \frac{1}{1 - ae^{j\omega}}$$

$$= \frac{1}{1 - ae^{j\omega} - ae^{-j\omega} + a^2}$$

eulers' formula:

$$-ae^{j\omega} = -(a\cos(\omega) + j\sin(\omega))$$

$$-ae^{-j\omega} = -(a\cos(\omega) - j\sin(\omega))$$

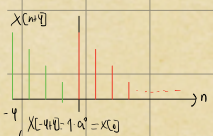
$$\Rightarrow \quad S_{xx}(\omega) = \frac{1}{1 - 2a\cos(\omega) + a^2}$$

We know that $\omega = 2\pi f$

$$\Rightarrow \quad S_{xx}(f) = \frac{1}{1 - 2a\cos(2\pi f) + a^2}$$

b) Derive the autocorrelation $r_{xx}(l)$:

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x[n+l]\,x[n],$$



$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} u[n]a^{n} \cdot u[n+l]a^{n+l}$$

When $u[n] \neq 0$ we know that $u[n+l] \neq 0$

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} u[n]a^{n}a^{n+l}$$

$$= \sum_{n=0}^{\infty} a^{2n+l} = \sum_{n=0}^{\infty} a^{2n}a^{l} = a^{l}\sum_{n=0}^{\infty} a^{2n}$$

$$= a^{l}\sum_{n=0}^{\infty} \left(a^{2}\right)^{n}$$

$$r_{xx}(l) = \frac{a^{|l|}}{1 - a^{2}}, \quad |a^{2}| < 1$$

Use $r_{xx}(l)$ to verify the expression for $S_{xx}(f)$:

$$\mathcal{F}(r_{xx}(l)) = S_{xx}(\omega)$$

We take the DTFT:

$$S_{xx}(\omega) = \sum_{l=-\infty}^{\infty} x(l) e^{-j\omega l}$$

$$= \sum_{l=-\infty}^{\infty} \frac{a^{|l|}}{1-a^2} e^{-j\omega l}$$

$$= \frac{1}{1-a^2} \sum_{l=-\infty}^{\infty} a^{|l|} e^{-j\omega l}$$

$$= \frac{1}{1-a^2} \left(\sum_{l=-\infty}^{-1} a^{|l|} e^{-j\omega l} + \sum_{l=0}^{\infty} a^{|l|} e^{-j\omega l} \right)$$

$$= \frac{1}{1-a^2} \left(\sum_{l=1}^{\infty} a^{l} e^{j\omega l} + \sum_{l=0}^{\infty} a^{l} e^{-j\omega l} \right)$$

$$= \frac{1}{1-a^2}\left(\sum_{l=1}^{\infty}\left(ae^{j\omega}\right)^l + \sum_{l=0}^{\infty}\left(ae^{-j\omega}\right)^l\right)$$

this diverges

$$= \frac{1}{1-a^2}\left(\sum_{l=0}^{\infty} a^{l+1} e^{j\omega(l+1)} + \sum_{l=0}^{\infty} a^l e^{-j\omega l}\right)$$

$$= \frac{1}{1-a^2}\left(\sum_{l=0}^{\infty} a^{l+1} e^{j\omega l} e^{j\omega} + \sum_{l=0}^{\infty} a^l e^{-j\omega l}\right)$$

$$= \frac{1}{1-a^2}\left(ae^{j\omega}\sum_{l=0}^{\infty}\left(ae^{j\omega}\right)^l + \sum_{l=0}^{\infty}\left(ae^{-j\omega}\right)^l\right)$$

$$= \frac{1}{1-a^2}\left(\frac{ae^{j\omega}}{1-ae^{j\omega}} + \frac{1}{1-ae^{-j\omega}}\right)$$

$$= \frac{1}{1-a^2}\left(\frac{ae^{j\omega}\left(1-ae^{-j\omega}\right)+1\left(1-ae^{j\omega}\right)}{\left(1-ae^{j\omega}\right)\left(1-ae^{-j\omega}\right)}\right)$$

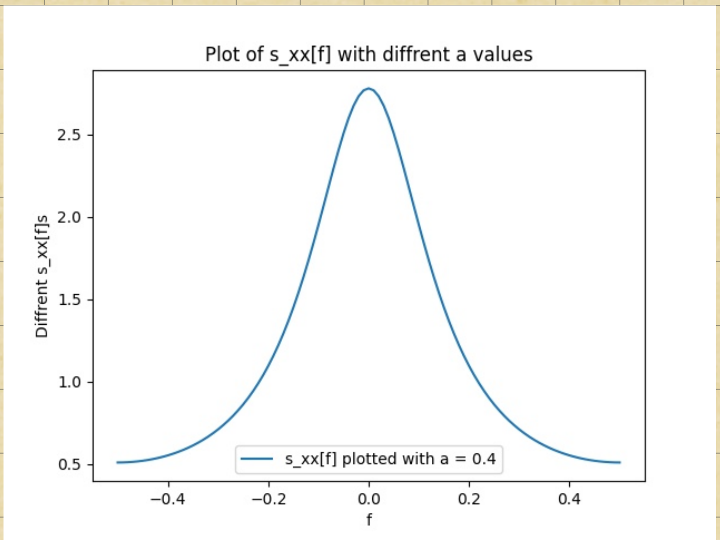
$$= \frac{1}{1-a^2}\left(\frac{1-a^2}{1-ae^{-j\omega}-ae^{j\omega}+a^2}\right)$$

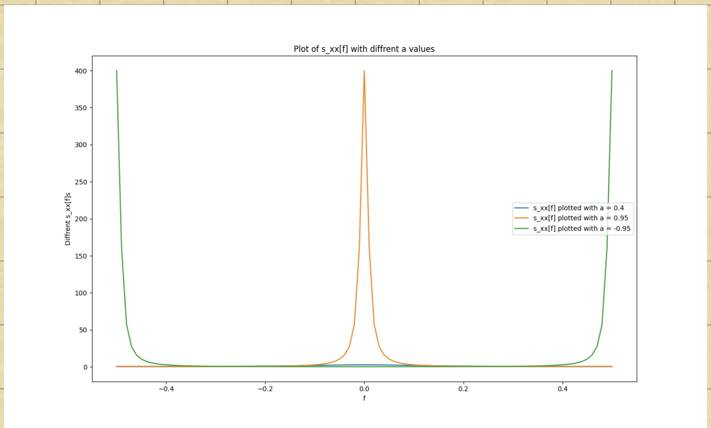
$$= \frac{1}{\cancel{1-a^2}}\left(\frac{\cancel{1-a^2}}{1+a^2-2a\cos(\omega)}\right)$$

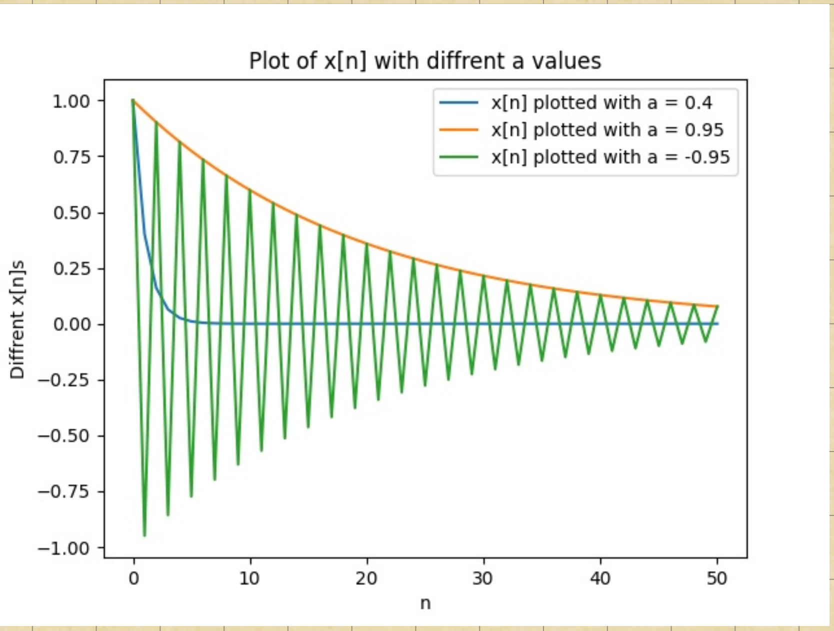
$$S_{xx}(f) \overset{\omega=2\pi f}{=} \frac{1}{1+a^2-2a\cos(2\pi f)}$$

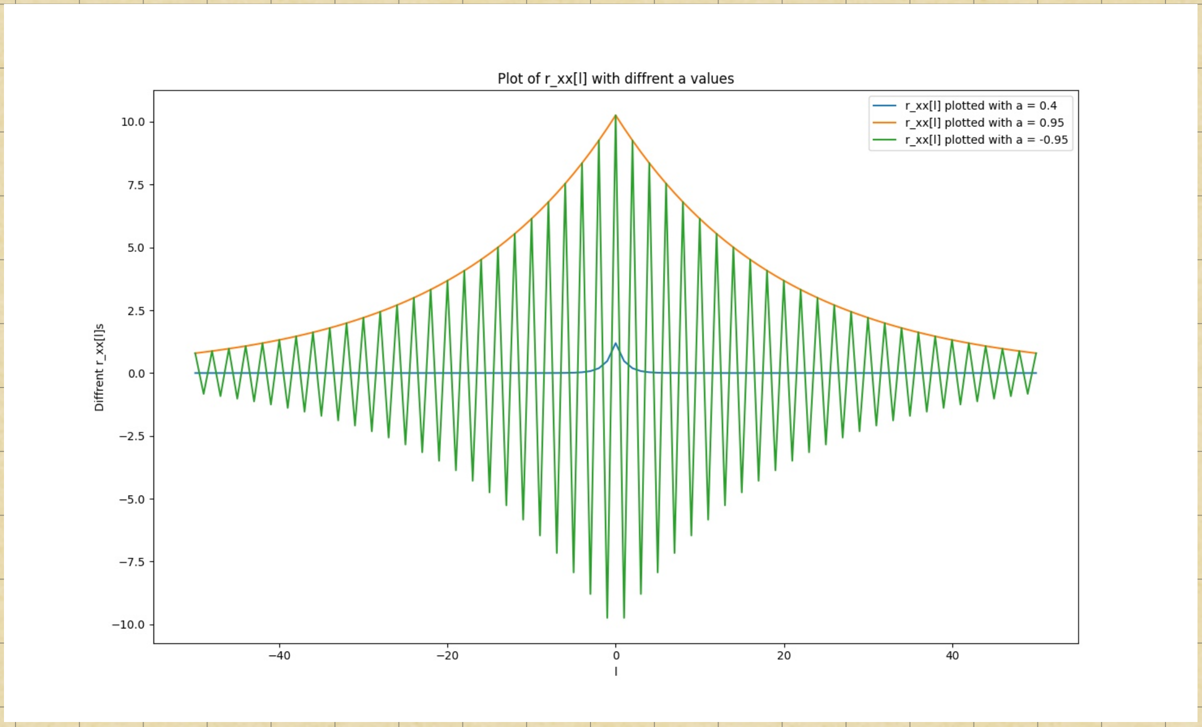
, same as in a

c) Plots!


Plot of s_xx[f] with diffrent a values
Diffrent s_xx[f]s
f
s_xx[f] plotted with a = 0.4


Plot of s_xx[f] with diffrent a values
Diffrent s_xx[f]s
f
s_xx[f] plotted with a = 0.4
s_xx[f] plotted with a = 0.95
s_xx[f] plotted with a = -0.95


Plot of x[n] with diffrent a values
Diffrent x[n]s
n
x[n] plotted with a = 0.4
x[n] plotted with a = 0.95
x[n] plotted with a = -0.95



- We can start by compring $a = 0.4$ with $a = 0.95$.
  As we can see from the $X[n]$ plots $X[n]$ varies much faster for $a = 0.4$. We can therfore see that the Correlation is much smaller with $a = 0.4$ in contrast to $a = 0.95$ wich has a way bigger Correlation.
  It is also clear that since $X[n]$ $a = 0.4$ drastically falls it Contains higher frequencys, we can see this with the energy spectral wich is way "Wider" with $a = 0.4$ then it is with $a = 0.95$

- By comparing $a = 0.95$ and $a = -0.95$, we see some symmetry. The only difference is that with $a = -0.95$ the $x[n]$ value and the autocorrelation "jumps" from positive to negative values. This large changes in sign means the signal contains high frequency components. Which can be verified by looking at the energy spectral density.

d)

$$E_x = r_{xx}[0] = \frac{a^{|0|}}{1 - a^2} = \underline{\underline{\frac{1}{1-a^2}}}$$

e)

$$x[n] \longrightarrow \boxed{h_1[n]} \xrightarrow{y_1[n]} \boxed{H_2(f)} \xrightarrow{y[n]}$$

$$h_1(n) = \delta(n) - a\delta(n-1)$$

$$H_2(f) = \begin{cases} \cos(2\pi f) & |f| \leq \frac{1}{4} \\ 0 & \frac{1}{4} < |f| \leq \frac{1}{2} \end{cases}$$

We know that

$$S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega)$$

Need to find We have this

$$|H(\omega)| = |H_1(\omega) H_2(\omega)|$$

$$H_1(\omega) = \sum_{n=-\infty}^{\infty} \left(\delta(n) - a\delta(n-1)\right) e^{-j\omega n}$$

$$= \frac{1 - ae^{-j\omega}}{}$$

$$|H_1(f)|^2 = \left(1 - ae^{-j2\pi f}\right)\left(1 - ae^{j2\pi f}\right) = 1 + a^2 - 2a\cos(2\pi f)$$

$$S_{y_1y_1}(f) = |H_1(f)|^2 S_{xx}(f) = \frac{1 + a^2 - 2a\cos(2\pi f)}{1 - 2a\cos(2\pi f) + a^2} = \frac{1}{}$$

$$S_{yy}(f) = |H_2(f)|^2 S_{y_1y_1}(f) = \begin{cases} \cos^2(2\pi f), & |f| \leq \frac{1}{4} \\ 0, & \text{else} \end{cases}$$

$$S_{xx}(f) = \frac{1}{1 - 2a\cos(2\pi f) + a^2},$$

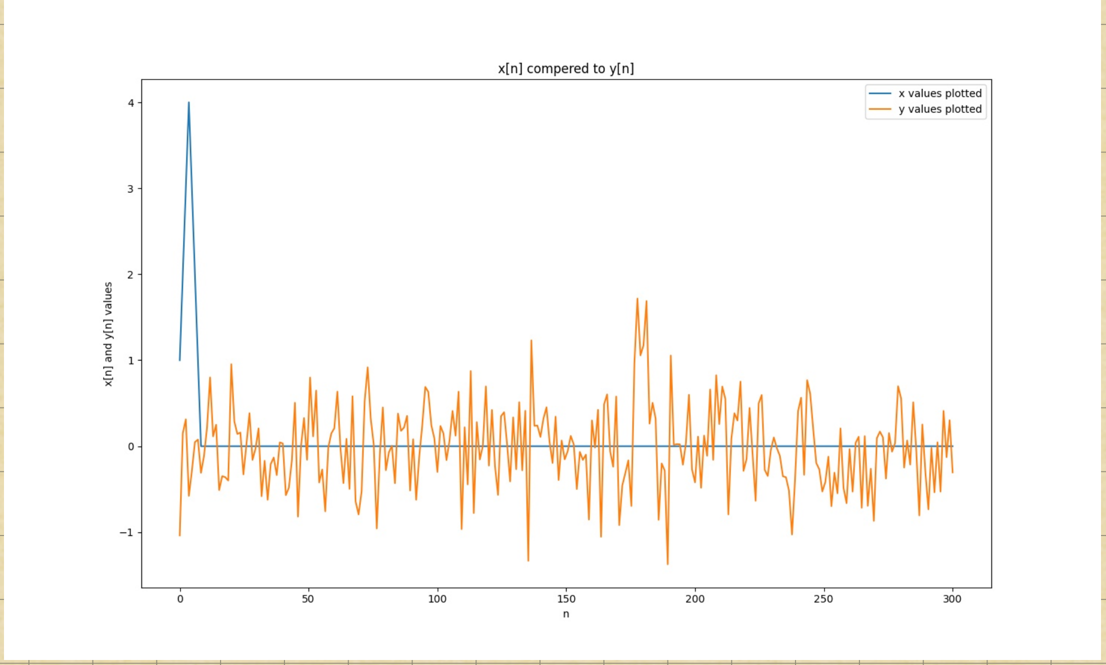
We can see that the energy is all contained in $|f| < 1/4$, this is because of the lowpass filter.

$$E_y = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_{yy}(f)\,df = \underline{\underline{\frac{1}{4}}}$$

<u>problem 2)</u>

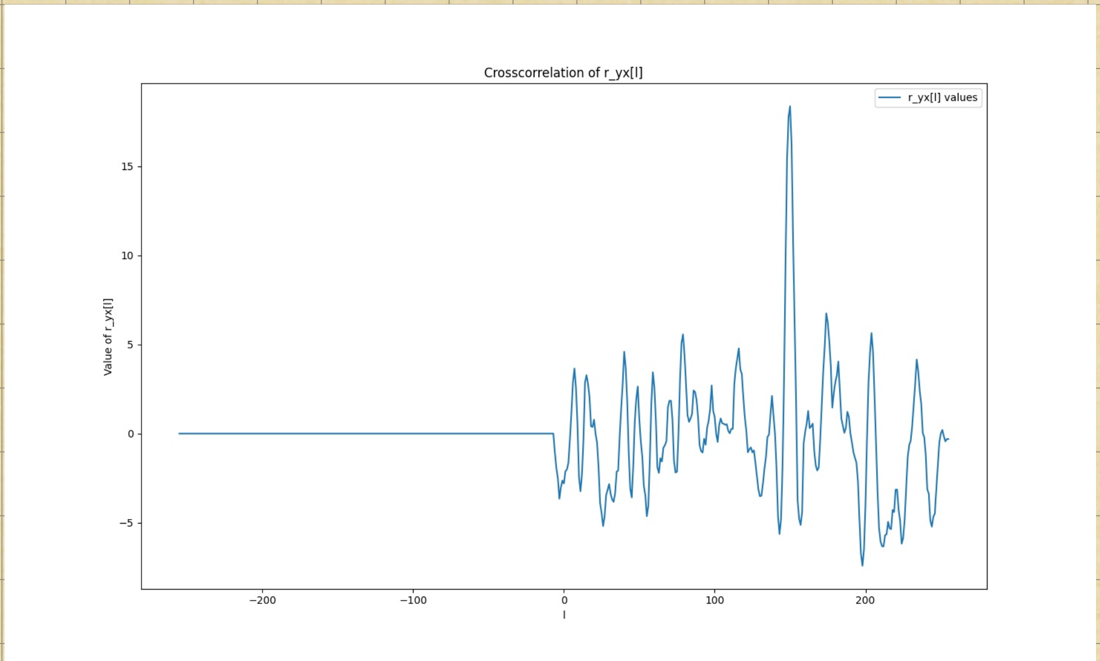
$$y[n] = \begin{cases} \alpha\, x[n-D] + w(n), & \text{if object hit} \\ w(n), & \text{if no object hit} \end{cases}$$

a)

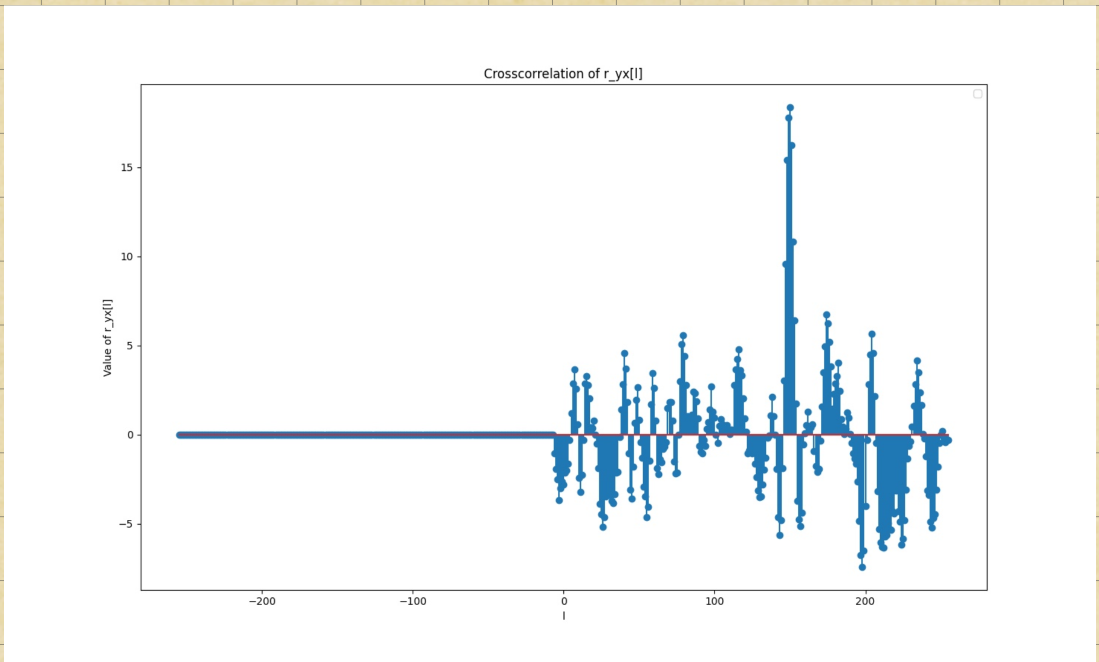


- It is not easy to see if a object has been hit, this is because of the extrem noice in y[n] wich makes it hard to see if y[n] is correlated to x[n].

b)



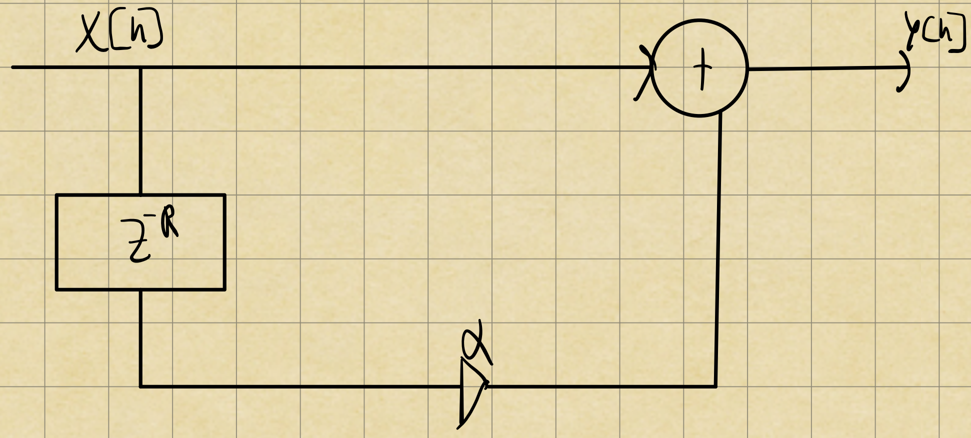
c)



d) By looking at the Crosscorrelation we can see that the function has a suspicious peak at $l = 150$. This suggests that the $y(n)$ signal contains a reflected component with $D = 150$

This method is way more credible then the direct comparison done in 2a)

Problem 3)



- Transfer function $H(z)$ for the filter:

$$H(z) = \frac{Y(z)}{X(z)}$$

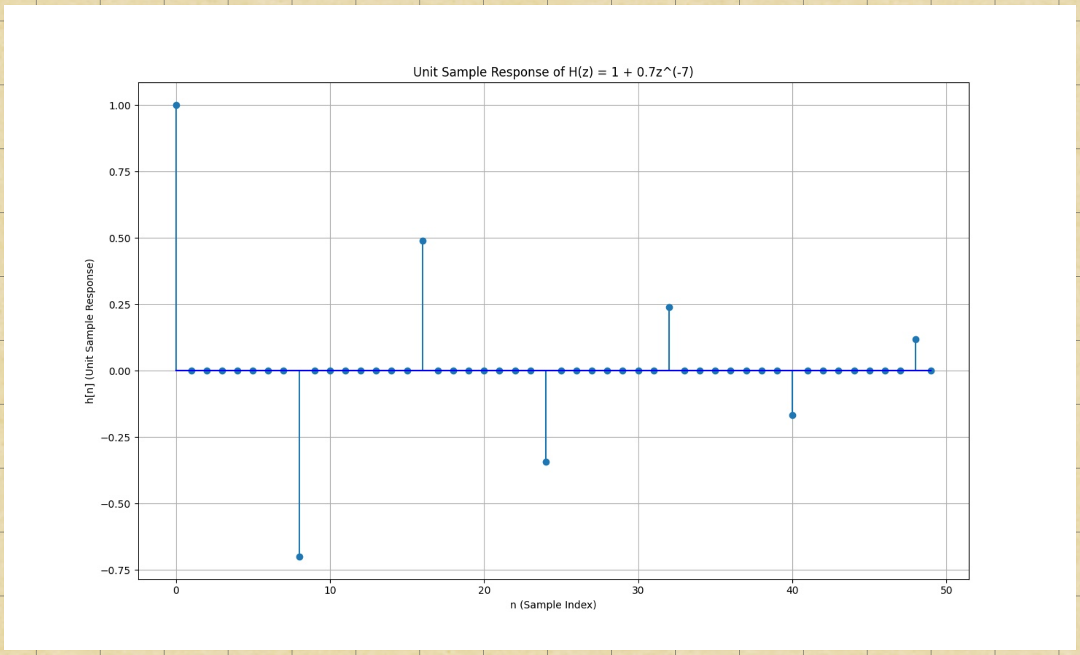
$$Y(z) = X(z) + z^{-R}\alpha X(z)$$

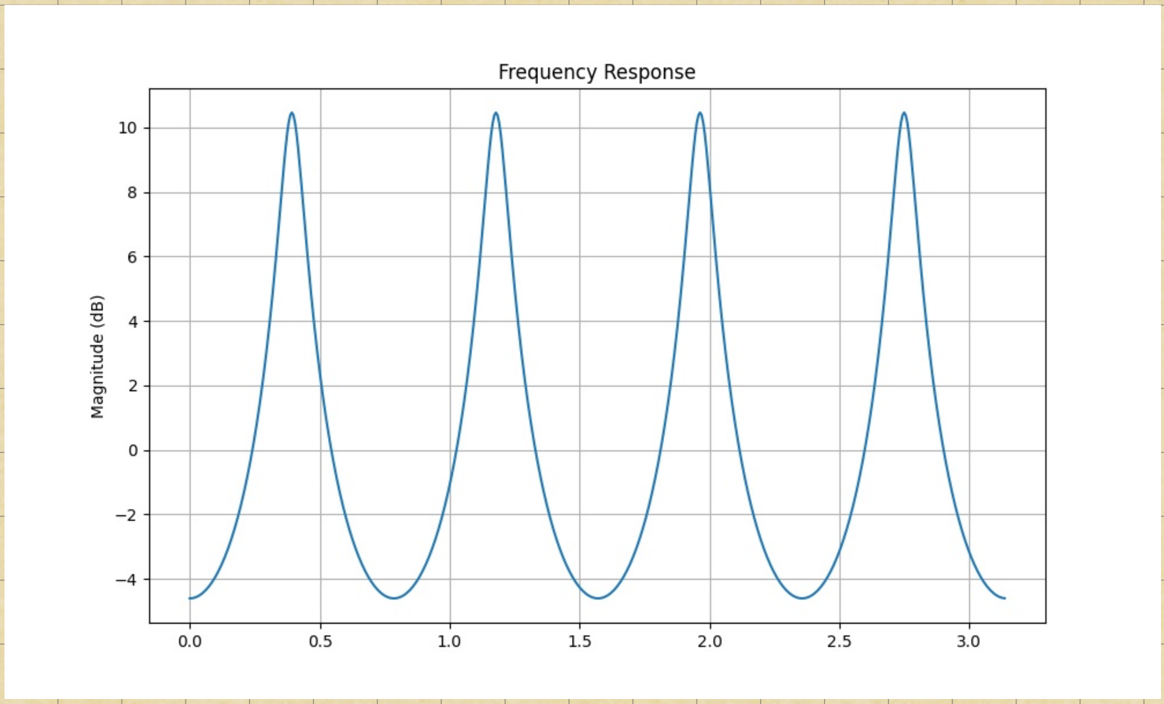
$$\Rightarrow \underline{\underline{H(z) = 1 + \alpha z^{-R}}}$$

- $F_s = 22050$ Hz ,

  - $D_s = \dfrac{R}{F_s}$

- plot the unit sample response and the frequency response.

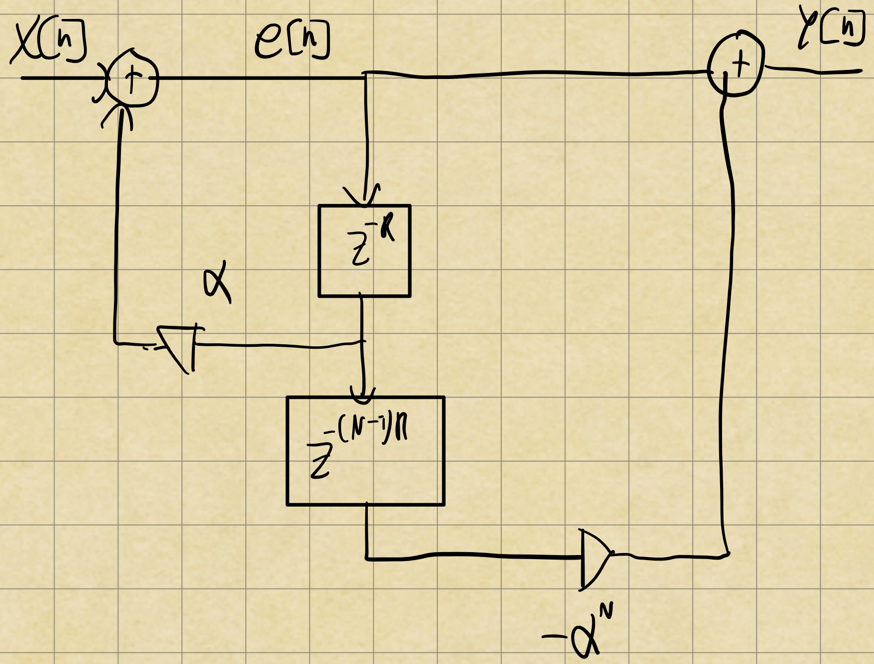




↑ Same α and R values

- By largening R and α we create a more pronounced echo effect.

New filter:



$$E(z) = X(z) + \alpha E(z) z^{-R}$$

$$E(z) - \alpha E(z) z^{-R} = X(z)$$

$$E(z)(1 - \alpha z^{-R}) = X(z)$$

$$E(z) = \frac{X(z)}{1 - \alpha z^{-R}}$$

$$z^{-R-NR+R}$$

$$Y(z) = E(z) + z^{-R} z^{-(N-1)R} \cdot (-\alpha^N) E(z)$$

$$= E(z)\left(1 + z^{-R} z^{-(N-1)R} (-\alpha)^N\right)$$

$$= \frac{X(z)}{1-\alpha z^{-R}} \left(1 - \alpha^N z^{-NR}\right)$$

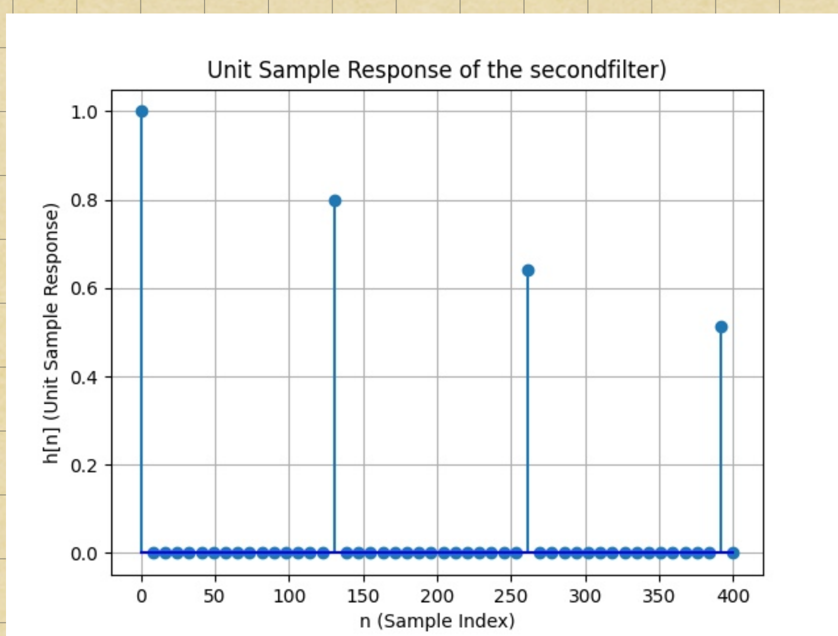
$$\Rightarrow H(z) = \frac{1-\alpha^N z^{-NR}}{1-\alpha z^{-R}}$$

Using:

$\alpha = 0.7$

$N = 6$

$R = 17$



- Something wrong with the code so it doesnt work. Couldnt figure out the problem.