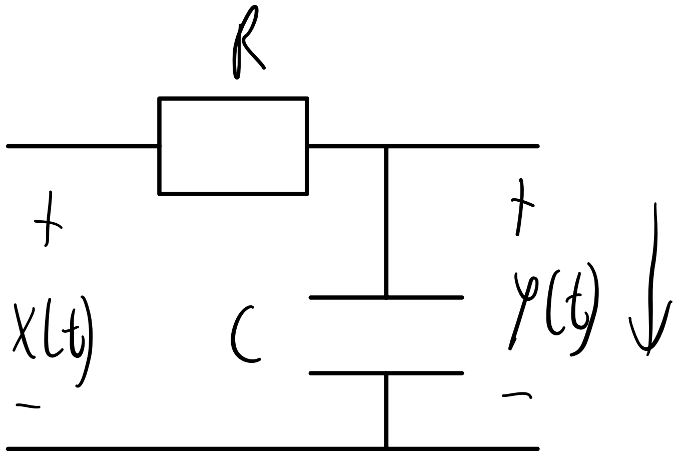
problem 1:)



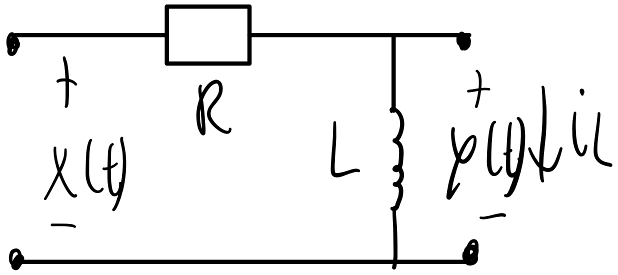
$i_C = C \cdot \frac{dV_C}{dt} = C \cdot \frac{dy(t)}{dt}$

a) Differential equations:

1) $X(t) = R \cdot i_C + y(t)$

$X(t) = R \cdot C \cdot \frac{dy(t)}{dt} + y(t)$

$X(t) = \tau \frac{dy(t)}{dt} + y(t)$, $\tau = RC$



, $V_L = y(t) = \frac{di_L}{dt} \cdot L$

2) $X(t) = R \cdot i_L + y(t)$

$$\frac{dX(t)}{dt} = R \frac{di_L}{dt} + \frac{dy(t)}{dt}$$

$$\frac{dX(t)}{dt} = \frac{R}{L} \cdot y(t) + \frac{dy(t)}{dt}$$

Transferfuction:

1) $X(t) = \tau \frac{dy(t)}{dt} + y(t)$

Laplace-transform:

$$X(s) = RCs\,Y(s) + Y(s)$$

$$X(s) = Y(s)(RCs + 1)$$

$$\frac{X(s)}{RCs+1} = Y(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \underline{\underline{\frac{1}{RCs+1}}}$$

2)

$$\frac{dX(t)}{dt} = \frac{R}{L} \cdot y(t) + \frac{dy(t)}{dt}$$

Laplace-tranform:

$$X(s) \cdot s = \frac{R}{L} \cdot Y(s) + Y(s) \cdot s$$

$$X(s) \cdot s = Y(s)\left(\frac{R}{L} + s\right)$$

$$\frac{X(s) \cdot s}{\frac{R}{L} + s} = Y(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\cancel{X(s)} \cdot s}{\frac{R}{L} + s} \cdot \frac{1}{\cancel{X(s)}}$$

$$= \underline{\underline{\frac{s}{\frac{R}{L} + s}}}$$

b) Determin frequency responses and filter types:

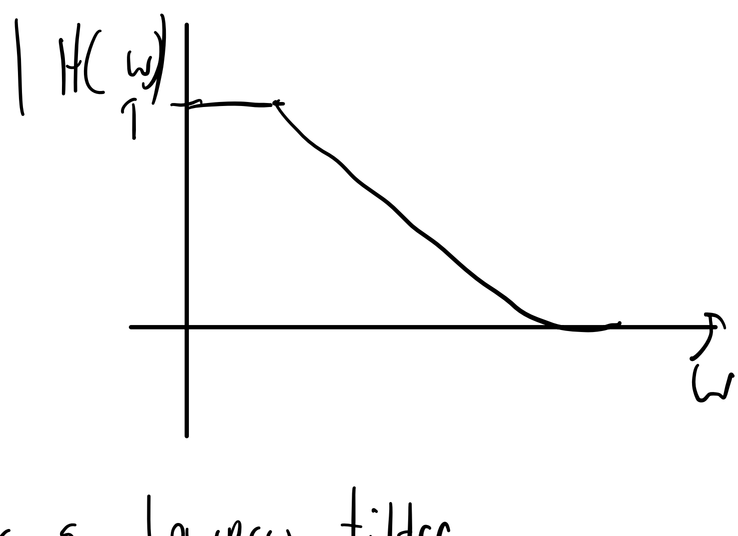
$$s = j\omega$$

$\Rightarrow$ 1)

$$H(j\omega) = \frac{1}{RCj\omega + 1}$$

$$|H(\omega)| = \frac{1}{\sqrt{(RC\omega)^2 + 1}}$$

We can easily see that when $w = 0$, $|H(\omega)| = 1$ and when $w \to \infty$, $|H(\omega)| \to 0$.



It is therfor a lowpass filter

2)

$$H(s) = \frac{s}{\frac{R}{L} + s}$$

$$H(j\omega) = \frac{j\omega}{\frac{R}{L} + j\omega}$$

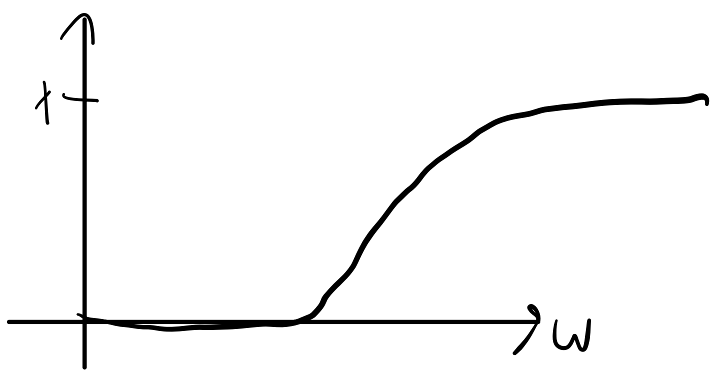
$$|H(\omega)| = \frac{\omega}{\sqrt{\frac{R^2}{L^2} + \omega^2}} = \frac{\omega}{\sqrt{\omega^2\left(\frac{R^2}{L^2\omega^2} + 1\right)}}$$

$$|H(\omega)| = \frac{1}{\sqrt{\frac{R^2}{L^2\omega^2} + 1}}$$

We see that when $\omega \to \infty$, $H(\omega) \to 1$.

and $\omega \to 0$, $H(\omega) \to 0$

this is the characteristic of a highpass filter



c) <u>Unit-pulse response for the filters:</u>

Impulsrespons 1) .

First we rewrite

$$H(s) = \frac{1}{RCs+1} = \frac{\frac{1}{RC}}{s+\frac{1}{RC}}$$

$$h(t) = \mathcal{L}^{-1}\left\{\frac{\frac{1}{RC}}{s+\frac{1}{RC}}\right\} = \frac{1}{RC}\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{RC}}\right\}$$

$$= \frac{1}{RC}e^{-\frac{1}{RC}t}u(t)$$

2)

$$H(s) = \frac{s}{\frac{R}{L}+s} = 1 - \frac{R/L}{R/L+s}$$

$$\frac{s}{\frac{R}{L}+s} = 1$$

$$s - \left(\frac{R}{L}+s\right) = -\frac{R}{L}$$

$$\frac{R}{L}+s$$

$$h(t) = \mathcal{L}^{-1}\{1\} - \frac{R}{L}\cdot\mathcal{L}^{-1}\left\{\frac{1}{R/L+s}\right\}$$

$$= \delta(t) - \frac{R}{L}e^{-\frac{R}{L}t}u(t)$$

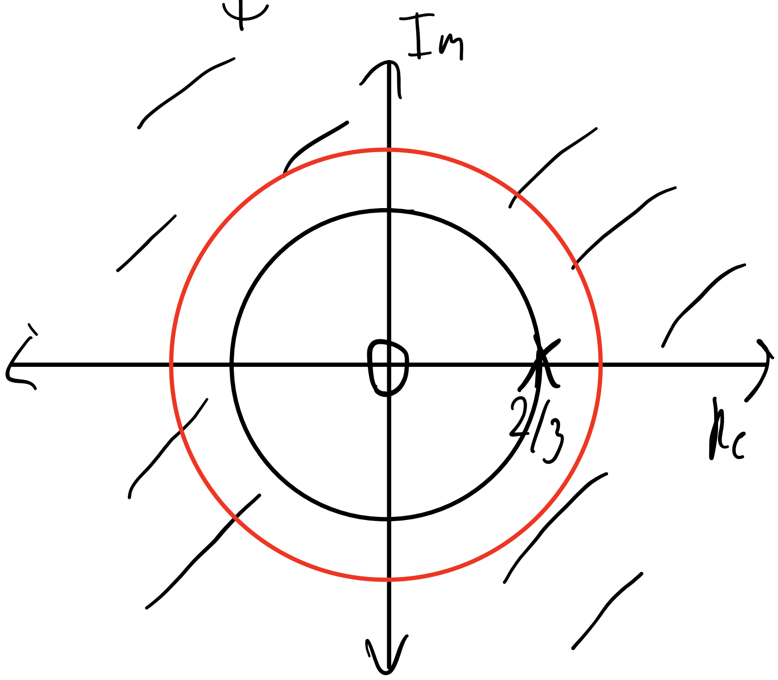
Problem 2:)

Find the region of Convergence and unit pulse response $h[n]$ for the following digital filters

a) Causal filter:

ROC

$$H(z) = \frac{1}{1 - \frac{2}{3}z^{-1}} = \frac{z}{z - \frac{2}{3}}$$



$$ROC: |z| > \frac{2}{3}$$

Unit pulse response $h[n]$:

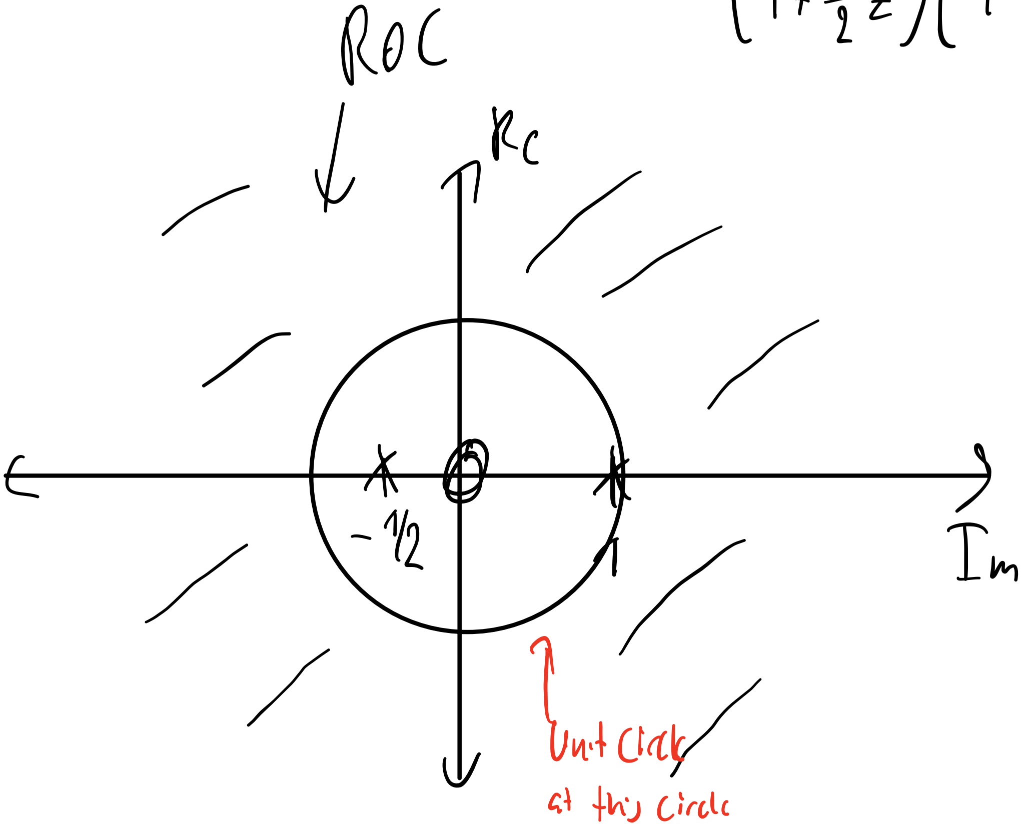
$$H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - p_k z^{-1}}$$

$$h[n] = \sum_{k=1}^{N} c_k p_k^n u[n]$$

$$= 1 \cdot \frac{2}{3}^n u[n] = \left(\frac{2}{3}\right)^n u[n]$$

b) A - Causal filter:

$$H(z) = \frac{1}{(1+\frac{1}{2}z^{-1})(1-z^{-1})}$$



ROC: $|z| > \overbrace{|p_{max}|}^{\text{Causal}}$

$|z| > 1$

Unit pulse response:

$$H(z) = \frac{1}{(1+\frac{1}{2}z^{-1})(1-z^{-1})} = \frac{A}{(1+\frac{1}{2}z^{-1})} + \frac{B}{(1-z^{-1})}$$

$$1 = A(1-z^{-1}) + B(1+\frac{1}{2}z^{-1})$$

$$H(z=1) = 1 = B\left(1+\frac{1}{2}\right)$$

$$1 = \frac{3}{2}B \Rightarrow \underline{B = \frac{2}{3}}$$

$$H\left(z=-\frac{1}{2}\right) = 1 = A\left(1-\frac{1}{-\frac{1}{2}}\right)$$

$$1 = A(1+2)$$

$$\underline{A = \frac{1}{3}}$$

$$H(z) = \frac{\frac{1}{3}}{1+\frac{1}{2}z^{-1}} + \frac{\frac{2}{3}}{1-z^{-1}}$$

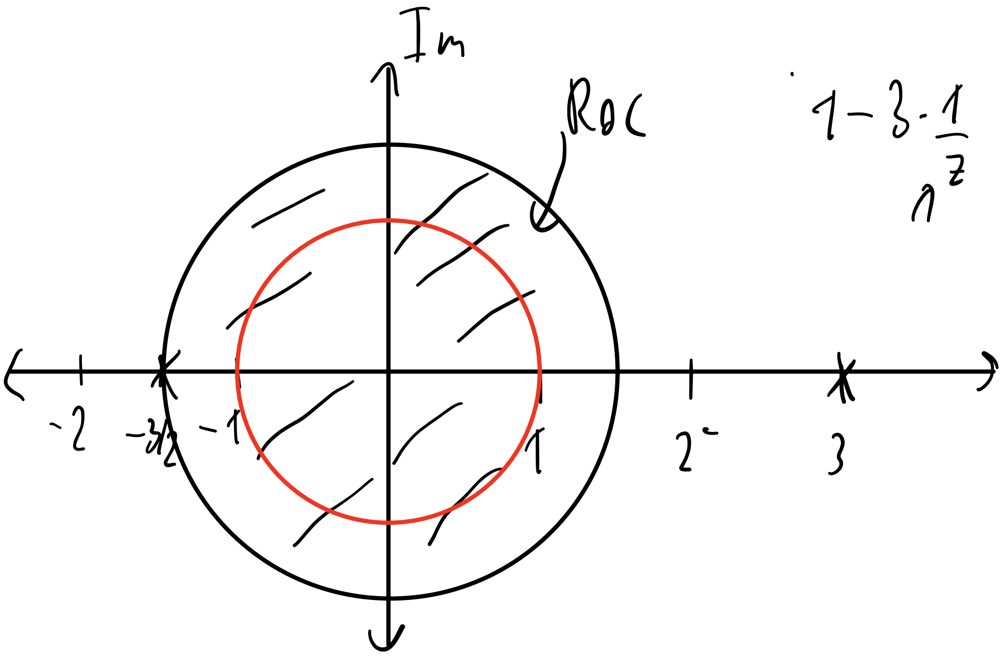
$$h[n] = \sum_{k=1}^{N} C_k p_k^n u[n]$$

n

$$h[n] = \frac{1}{3}(-1)^n\left(\frac{1}{2}\right)^n u[n] + \frac{2}{3}u[n]$$

c) An anti-causal filter!

$$H(z) = \frac{z^{-1}}{(1+3/2\,z^{-1})(1-3z^{-1})}$$



$1 - 3 \cdot \frac{1}{z}$

ROC: $|z| < |p_{min}|$

$|z| < \frac{3}{2}$

Unit pulse response:

$$H(z) = \frac{z^{-1}}{(1+3/2\,z^{-1})(1-3z^{-1})} = \frac{A}{1+3/2\,z^{-1}} + \frac{B}{1-3z^{-1}}$$

$$z^{-1} = A(1-3z^{-1}) + B(1+3/2\,z^{-1})$$

$$z=3 \,\Big|\, \frac{1}{3} = B\left(1 + \frac{3}{2}\cdot\frac{1}{3}\right) \Rightarrow \underline{\underline{B = \frac{2}{9}}}$$

$$z = -\frac{3}{2} \Big| \quad \frac{1}{-\frac{3}{2}} = A\left(1 - 3 \cdot \left(\frac{1}{-\frac{3}{2}}\right)\right)$$

$$-\frac{2}{3} = A\left(1 - \cancel{3} \cdot \left(-\frac{2}{\cancel{3}}\right)\right)$$

$$-\frac{2}{3} = 3A$$

$$\underline{A = -\frac{2}{9}}$$

$$H(z) = \frac{-2/9}{1 + 3/2\, z^{-1}} + \frac{2/9}{1 - 3z^{-1}}$$

Since it is anti-causal

$$\underline{\underline{h[n] = -\left(-2/9\left(-\frac{3}{2}\right)^n u[-n-1] + 2/9\left(3^n\right) u[-n-1]\right)}}$$

c)

As we can see from the drawings the only filter where the unit-circle is inside of ROC is in C and A. Therefor this is the only stable systems (A and C)

Problem 3:)

LTI

$$h[n] = \begin{cases} \frac{1}{2^n}, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$$x[n] = \begin{cases} 1, & n \geq 2 \\ 0, & \text{else} \end{cases}$$

a) Find z-transform, and ROCs.

$$h[n] = \left(\frac{1}{2}\right)^n, \quad n \geq 0$$

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n$$

This is a geometric-series and has the solution

$$H(z) = \frac{1}{1 - \frac{1}{2} z^{-1}}, \quad \frac{1}{2} < |z|$$

And for $x[n]$:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=2}^{\infty} (1 \cdot z^{-1})^n$$

Geometric-series!

$$= \sum_{n=2}^{\infty} \left(z^{-1}\right)^n$$

$$\Rightarrow \underline{\underline{X(z) = \frac{z^{-2}}{1 - z^{-1}}}} \quad , \quad |z^{-1}| < 1$$

$$\Rightarrow 1 < |z|$$

b) Derive an expression for the output signal $y[n]$ by performing the convolution in time domain.

$$, \quad h[n] = \frac{1}{2^n} u[n], \quad x[n] = u(n-2)$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{2^k} u(k) u(n-2-k)$$

$$y[n] = \sum_{k=0}^{\infty} \frac{1}{2^k} u(n-2-k)$$

We notice that the Series is zero when $n-2-k<0. \Rightarrow k > n-2$

$$\Rightarrow y[n] = \begin{cases} \sum_{k=0}^{n-2} \frac{1}{2^k}, & n-2 \geq 0 \\ 0, & n-2 < 0 \end{cases}$$

$$y[n] = \begin{cases} \frac{1-\left(\frac{1}{2}\right)^{n-1}}{1-\frac{1}{2}}, & n-2 \geq 0 \\ 0, & n-2 < 0 \end{cases}$$

$$y[n] = \begin{cases} 2 - 2\left(\frac{1}{2}\right)^{n-1}, & n-2 \geq 0 \\ 0, & n-2 < 0 \end{cases} \qquad 2 = \left(\frac{1}{2}\right)^{-1}$$

$$\underline{\underline{y[n] = 2u(n-2) - \left(\frac{1}{2}\right)^{n-2} u(n-2)}}$$

c) <u>Now using Z-transform</u>

$$Y(z) = H(z)\,X(z)$$

$$Y(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \cdot \frac{z^{-2}}{1-z^{-1}}$$

$$Y(z) = \frac{z^{-2}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)}$$

$$y[n] = \mathcal{Z}^{-1}\left\{z^{-2}\right\} \cdot \mathcal{Z}^{-1}\underbrace{\left\{\frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)}\right\}}_{Y_1(z)}$$

$$\frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)} = \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1-z^{-1}}$$

$$1 = A\left(1-z^{-1}\right) + B\left(1-\frac{1}{2}z^{-1}\right)$$

$$\Big|_{z=1} \quad 1 = B\left(1-\frac{1}{2}\right) \Rightarrow B = 2$$

$\left. \right| z = \frac{1}{2} \quad 1 = A\left(1 - \frac{1}{\frac{1}{2}}\right)$

$-1 = A$

$$Y_1(z) = \frac{-1}{1 - \frac{1}{2}z^{-1}} + \frac{2}{1 - z^{-1}}$$

$$y_1(t) = -1 \cdot \left(\frac{1}{2}\right)^n u[n] + 2u[n]$$

$$y[n] = Z^{-1}\left(z^{-2} \cdot Y_1[n]\right)$$

$$= y_1[n-2] = -1 \cdot \left(\frac{1}{2}\right)^{n-2} u[n-2] + 2u[n-2]$$

Same as in b

Problem 4)

$$y[n] = x[n] - x[n-2] - \frac{1}{4}y[n-2]$$

c) Transfer function:

$$Y(z) = X(z) - X(z)z^{-2} - \frac{1}{4}z^{-2}Y(z)$$

$$Y(z) + \frac{1}{4}z^{-2}Y(z) = X(z) - X(z)z^{-2}$$

$$Y(z) = \frac{X(z) - X(z)z^{-2}}{1 + \frac{1}{4}z^{-2}}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-2}}{1 + \frac{1}{4}z^{-2}}$$

$$1 - \frac{1}{z^2} = 0$$

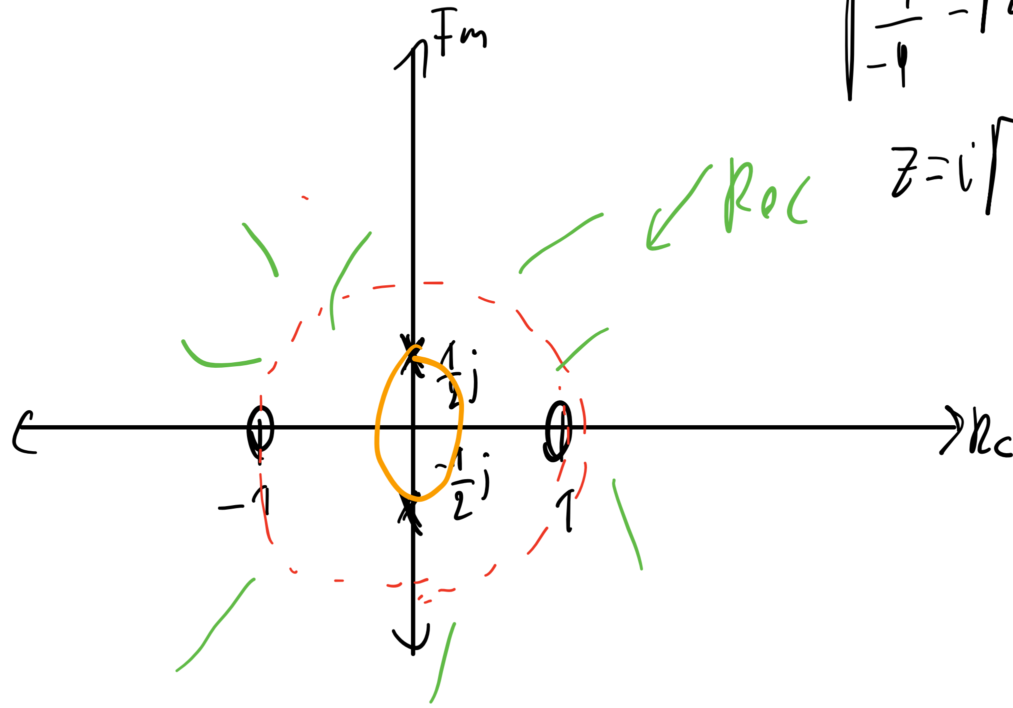
$$1 + \frac{1}{4} \cdot \frac{1}{z^2} = 0$$

$$\frac{1}{z^2} = -4$$

b)

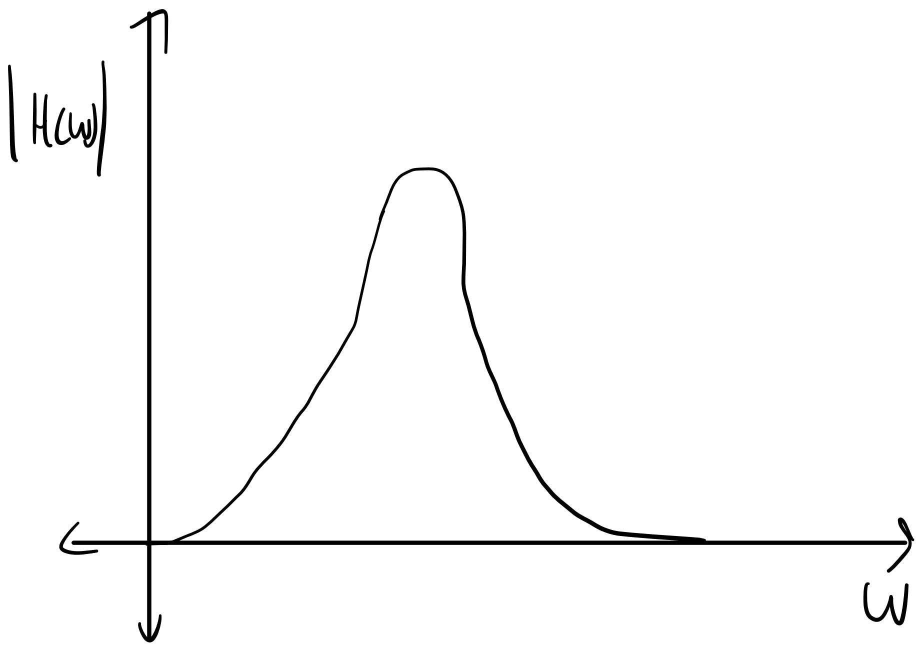
$$\sqrt{\frac{1}{-4}} = \sqrt{z^2}$$

$$z = i\sqrt{\frac{1}{4}} = j \pm \frac{1}{2}$$



c) Since the system is causal, and $|p_{max}| < 1$ (therefore the unit circle is defined in the ROC. The system is stable. See figure over.

d)



We see from the figure that it will look something like this. This is a bandpass-filter.