Problem 1:

$$x[n] = \begin{cases} 2, & n=0 \\ 1, & n=\pm 1 \\ 0, & \text{else} \end{cases} \quad , \quad y[n] = \begin{cases} 1, & -M \leq n \leq M \\ 0, & \text{else} \end{cases}$$

a)

$$\underline{X}[\omega] = \mathcal{F}\{x[n]\} = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$= x(0) e^{-j\omega 0} + x(-1) e^{-j\omega(-1)} + x(1) e^{-j\omega 1}$$

$$= 2 + e^{j\omega} + e^{-j\omega}$$

$$= 2 + \left(\cos(\omega) + j\sin(\omega)\right) + \left(\cos(-\omega) + j\sin(-\omega)\right)$$

We know that:

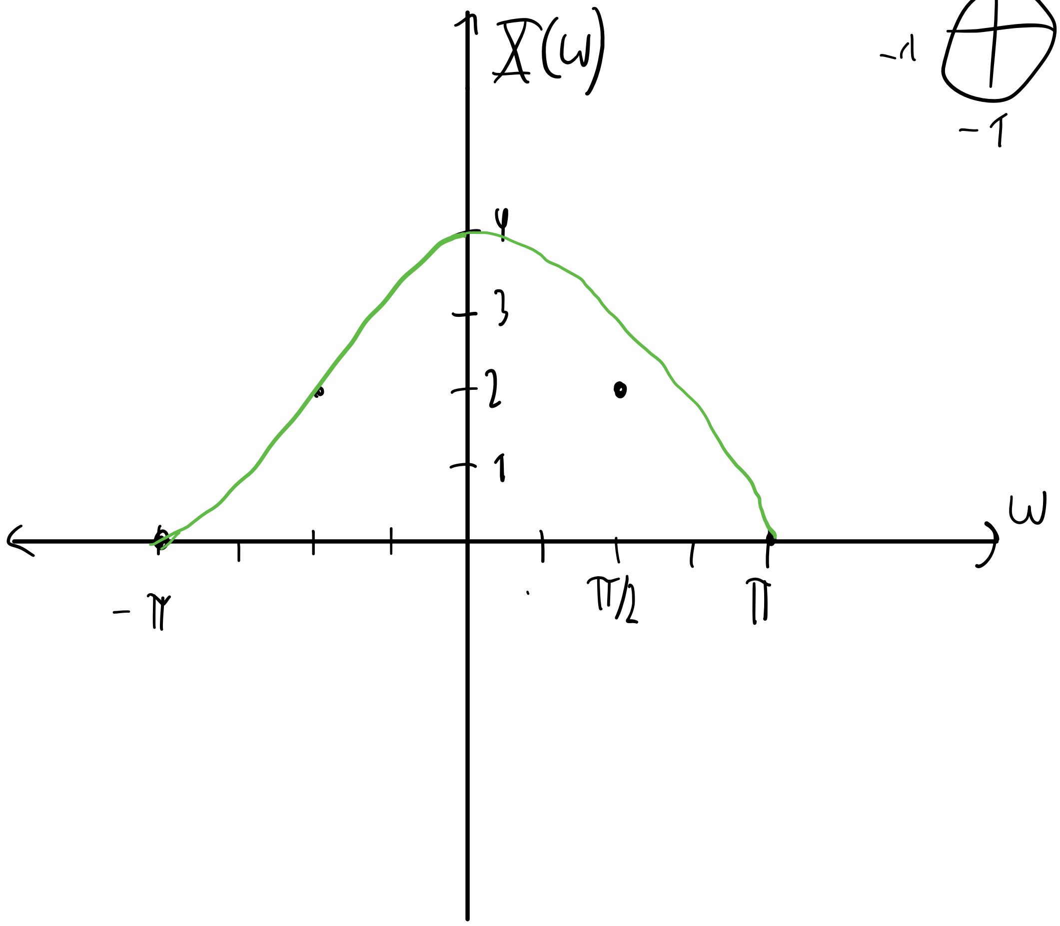
$$\cos(x) = \cos(-x)$$

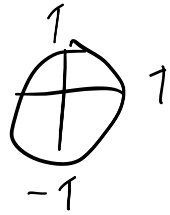
and

$$\sin(-x) = -\sin(x)$$

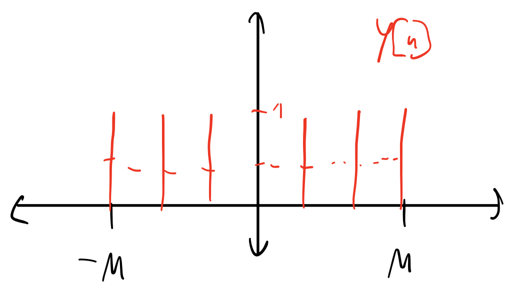
$$\Rightarrow \quad \underline{X}(\omega) = \underline{\underline{2 + 2\cos(\omega)}}$$

Sketch:





b)



$$Y(\omega) = \mathcal{F}\{y[n]\} = \sum_{n=-\infty}^{\infty} y(n)\, e^{-j\omega n}$$

$$= \sum_{n=-M}^{M} e^{-j\omega n}$$

$l = n + M$

$$\Rightarrow Y(\omega) = \sum_{l=0}^{2M} e^{-j\omega(l-M)}$$

$$= \sum_{l=0}^{2M} e^{j\omega M} e^{-j\omega l}$$

$$Y(w) = e^{jwM} \sum_{l=0}^{2M} e^{-jwl}$$

$$= e^{jwM} \left(\frac{1\left(1 - e^{-jw(2M+1)}\right)}{1 - e^{-jw}} \right)$$

$$= e^{jwM} \left(\frac{1 - e^{-jw(2M+1)}}{1 - e^{-jw}} \right)$$

$$= e^{jwM} \left(\frac{e^{\frac{jw}{2}} - e^{-jw((2M+1)+\frac{1}{2})}}{e^{\frac{jw}{2}} - e^{-\frac{jw}{2}}} \right)$$

$$= \frac{e^{jw(M+\frac{1}{2})} - e^{jw(-2M+M+\frac{1}{2}-1)}}{e^{\frac{jw}{2}} - e^{-\frac{jw}{2}}}$$

$$= \frac{e^{jw(M+\frac{1}{2})} - e^{-jw(M+\frac{1}{2})}}{e^{\frac{jw}{2}} - e^{-\frac{jw}{2}}}$$

$$Y(w) = \frac{\sin\left(w\left(M+\frac{1}{2}\right)\right)}{\sin\left(\frac{w}{2}\right)}$$

Sketch $M=10$

Python

c) Becase they are even real values

d)

$$Z[n] = \sum_{l=-\infty}^{\infty} X[n-lN]$$, $N=4$, $-8, -4, 0, 4, 8, 16$

$$z[0] = \sum_{-\infty}^{\infty} X[0-l\cdot 4] = 2$$

- Sketch $z[n]$

$$z[0]=\sum_{l=-\infty}^{\infty} X[-lN] = 2$$, Sidan $N>3$, $X[n]=0$, $N>3$

$$z[1] = \sum_{-\infty}^{\infty} X[1-lN] = 1$$

$z[-1] = 1$

$z[2] = X[2-lN] = 0$

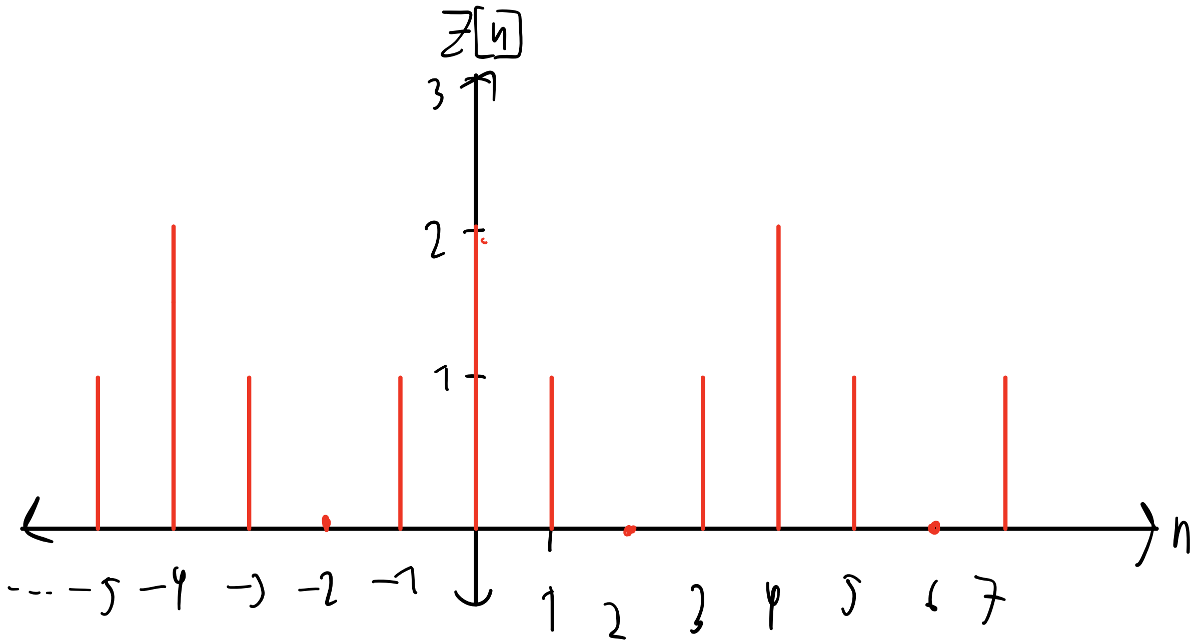
$z[3] = X[3-lN] = 1$

$z[4] = 2$

$z[5] = 1$

$z[6] = 0$

$z[7] = 1$



Find and sketch the Fourier Coefficients $\{C_k\}$,

$N=4$

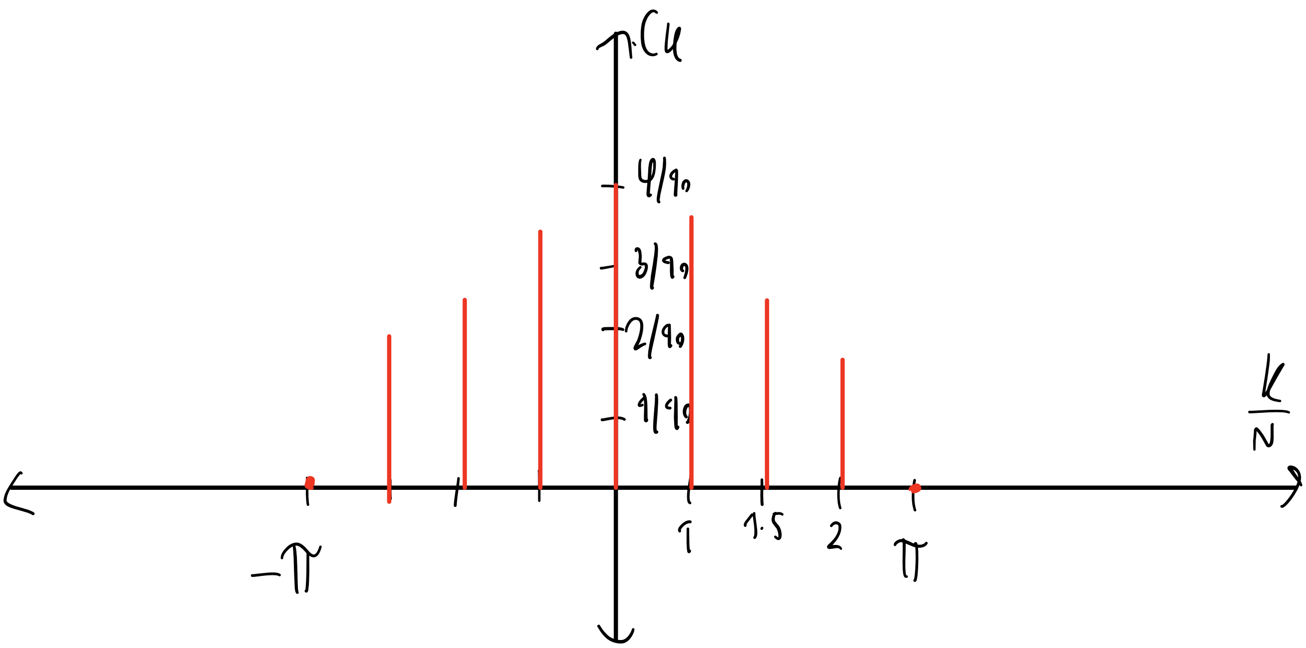
$$C_k = \frac{1}{N}\sum_{n=0}^{N-1} z[n]\, e^{-j2\pi kn/N}$$

$$C_k = \frac{1}{N}\left(2 + e^{-\frac{j2\pi k}{N}} + e^{-\frac{j2\pi k(N-1)}{N}}\right)$$

$$= \frac{1}{N}\left(2 + e^{-\frac{j2\pi k}{N}} + \underbrace{e^{-j2\pi k}}_{=1}\, e^{\frac{j2\pi k}{N}}\right)$$

$$= \underline{\underline{\frac{1}{N}\left(2+2\cos(2\pi k/N)\right)}}$$

- Sketch $\{C_k\}$ with $N=10$, $w=2\pi$, $\frac{k}{N} \in (-\pi, \pi)$



e) We know that $X(f) = 2 + 2\cos(2\pi f)$

and

$$C_k = \frac{1}{N}\left(2 + 2\cos(2\pi k/N)\right)$$

It is clear that C_k is just a scaled sample of $X(f)$.

problem 2:

$$X(\omega) = \mathcal{F}\{x[n]\}$$

a) $x_1[n] = x[n+3]$

So just $x[n]$ timeshifted!

$$\Rightarrow X_1(\omega) = \mathcal{F}\{x[n+3]\} = \underline{\underline{e^{j3\omega} X(\omega)}}$$

b)

$$x_2[n] = x[-n]$$

$$\Rightarrow \underline{\underline{X_2(\omega) = X(-\omega)}}$$

c)

$$x_3 = x[3-n] = x[-(3+n)] = x_2[3+n]$$

$$\Rightarrow \underline{\underline{X_3[\omega] = e^{-j3\omega} X(-\omega)}}$$

d)

$$X_y[n] = x[n] * x[n]$$

$$X_y[w] = X[w]X[w]$$

<u>problem 3:</u>

$$y[n] = x[n] + 2x[n-1] + x[n-2]$$

$$y[n] = -0.9y[n-1] + x[n]$$

a) $$H(w) = \sum_{k=-\infty}^{\infty} h[k] e^{-jwk}$$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = h[n] * x[n]$$

$$Y(w) = X(w)H(w)$$

$$\Rightarrow H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

We find $Y(\omega)$:

$$Y(\omega) = \mathcal{F}\{y[n]\} = \underline{X(\omega) + 2X(\omega)e^{-j\omega} + X(\omega)e^{-2j\omega}}$$

$$\Rightarrow H(\omega) = \frac{\cancel{X(\omega)} + 2\cancel{X(\omega)}e^{-j\omega} + \cancel{X(\omega)}e^{-2j\omega}}{\cancel{X(\omega)}}$$

$$H(\omega) = 1 + 2e^{-j\omega} + e^{-2j\omega}$$

$$= e^{-j\omega}\left(e^{j\omega} + 2 + e^{-j\omega}\right)$$

$$= \underline{\underline{e^{-j\omega}\left(2\cos(\omega) + 2\right)}}$$

(2):

$$H(w) = \frac{Y(w)}{X(w)}$$

$$Y(w) = -0.9Y(w)e^{-jw} + X(w)$$

$$Y(w) + 0.9Y(w)e^{-jw} = X(w)$$

$$\Rightarrow \quad \underline{Y(w) = \frac{X(w)}{1+0.9e^{-jw}}}$$

$$H(w) = \frac{Y(w)}{X(w)} = \frac{\cancel{X(w)}}{1+0.9e^{-jw}} \cdot \frac{1}{\cancel{X(w)}} = \underline{\underline{\frac{1}{1+0.9e^{-jw}}}}$$

b) Find magnitude and phase response!

Magnitude (1)!

$$|H(\omega)| = \sqrt{H(\omega)\,\overline{H(\omega)}} = \sqrt{\left(e^{-j\omega}(2\cos(\omega)+2)\right)\left(e^{j\omega}(2\cos(\omega)+2)\right)}$$

$$= \sqrt{(2\cos(\omega)+2)^2} = \underline{\underline{2\cos(\omega)+2}}$$

Magnitude (2)!

$$|H(\omega)| = \left(H(\omega)\,\overline{H(\omega)}\right)^{1/2} = \left(\left(\frac{1}{1+0.9e^{-j\omega}}\right)\left(\frac{1}{1+0.9e^{j\omega}}\right)\right)^{1/2}$$

$$= \frac{1}{\left((1+0.9e^{-j\omega})(1+0.9e^{j\omega})\right)^{1/2}}$$

$$= \frac{1}{(1+(0.9)^2+2\cdot 0.9\cos(\omega))^{1/2}}$$

$$\underline{\underline{|H(\omega)| = \frac{1}{(1+1.8\cos(\omega)+0.81)^{1/2}}}}$$

phaseresponse (1):

Since

$$2+2\cos(\omega)\geq 0 \quad \forall \omega$$

$$\Rightarrow \theta_1(\omega) = \angle H(\omega) = -\omega$$

phaseresponse (2):

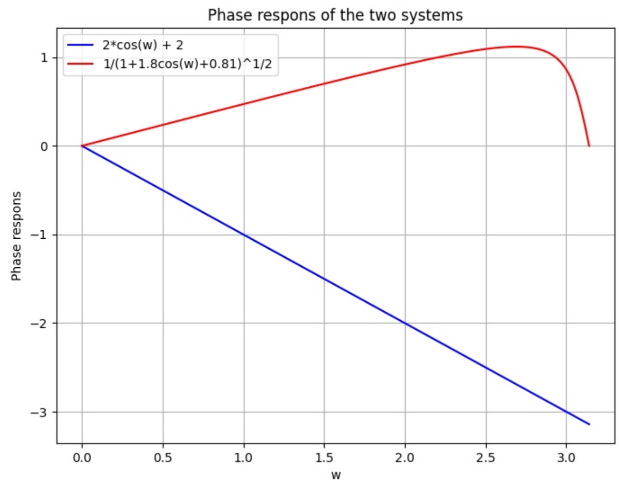
$$\theta_2(\omega) = \angle H(\omega) = \arctan\left(\frac{Im}{Re}\right)$$

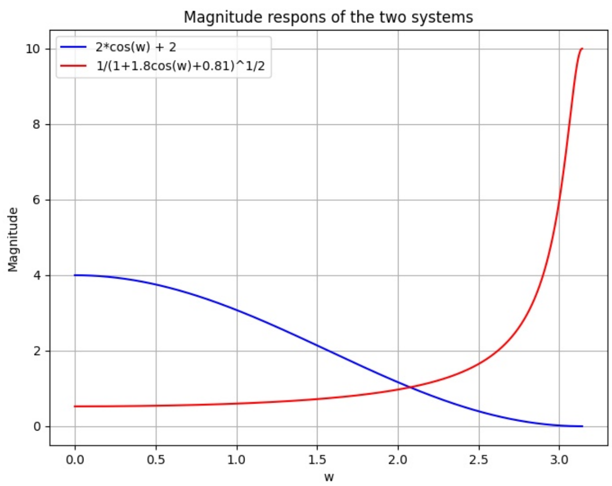
$$= -\arctan\left(\frac{-0.9\sin(\omega)}{1+0.9\cos(\omega)}\right)$$

$$= \arctan\left(\frac{0.9\sin(\omega)}{1+0.9\cos(\omega)}\right)$$

All phase function are odd and all magnitude even.

c)





d)

We can see that (1) is a lowpass since it blocks high frequencys.

And we can see that (2) is a highpass as it does the opposite

e) $X[n] = \frac{1}{2}\cos\left(\frac{\pi}{2}n + \frac{\pi}{4}\right)$ is psud through the two systems

Frequency:

(1) Respons of a LTI-System to a cos input Signal is (

$$y[n] = A|H(w)|\cos(w_0 n + \theta + \Theta(w_0))$$

Thrfor first system: Output

$$y_1[n] = A|H_1(\tfrac{\pi}{2})|\cos\left(\frac{\pi}{2}n + \frac{\pi}{4} + \Theta_1\left(\frac{\pi}{2}\right)\right)$$

$$= \frac{1}{2}\left|2+2\cos\left(\frac{\pi}{2}\right)\right|\cos\left(\frac{\pi}{2}n + \frac{\pi}{4} - \frac{\pi}{2}\right)$$

$$= \underline{\underline{\cos\left(\frac{\pi}{2}n - \frac{\pi}{4}\right)}}$$

We can easly see Amplitude, phase, and frequency from this

Output (2):

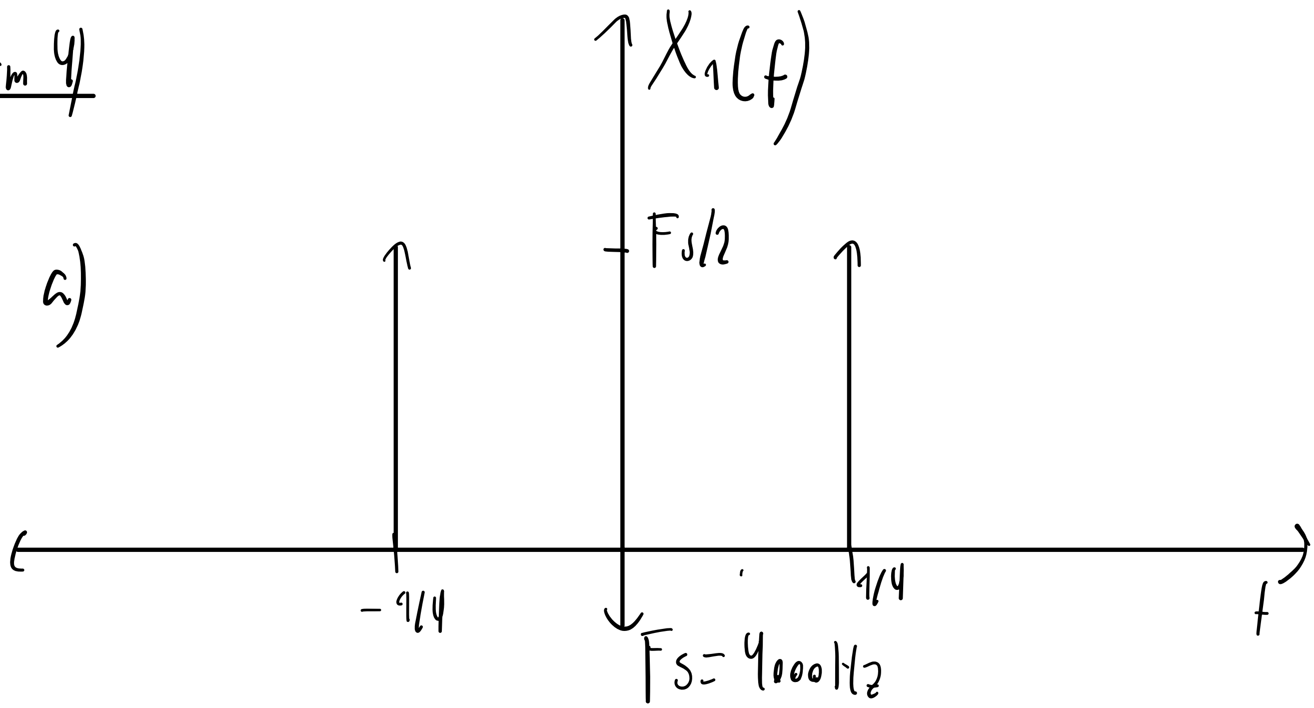
$$y_2[n] = A\left|H_2\left(\tfrac{\pi}{2}\right)\right| \cos\left(\frac{\pi}{2}n + \frac{\pi}{4} + \theta_2\left(\frac{\pi}{2}\right)\right)$$

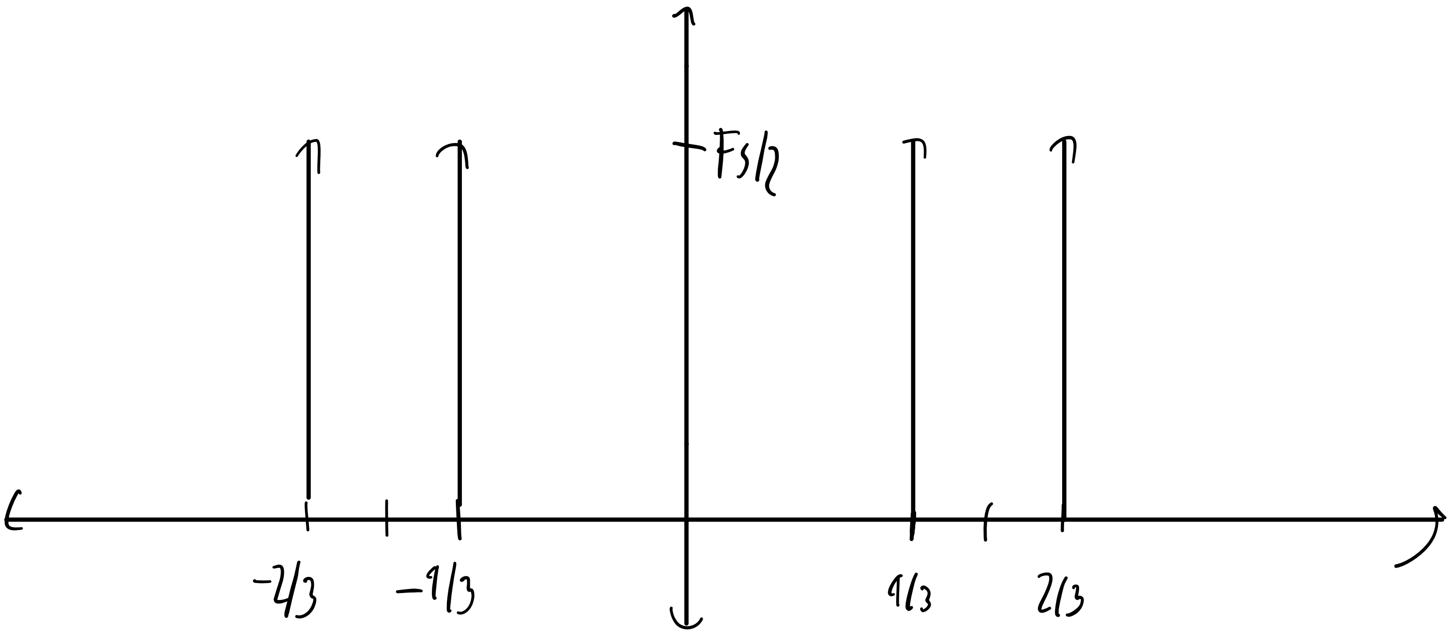
$$= \frac{1}{2}\left|\frac{1}{\left(1 + 1.8\cos\left(\frac{\pi}{2}\right) + 0.81\right)^{1/2}}\right| \cos\left(\frac{\pi}{2}n + \frac{\pi}{4} + \arctan\left(\frac{0.9}{1}\right)\right)$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{1.81}} \cdot \cos\left(\frac{\pi}{2}n + 1.52\right)$$

Problem 4

a)





$F_s = 1500\,\text{Hz}$

b)

They sound diffrent because the sample theorem requires that $F_s > 2F_{max}$. So we require $F_s > 2000\,\text{Hz}$. Since this is not true we will get aliasing problems