# Integral Calculus

Some standard results:..

① $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$

② $\int dx = x + c.$

③ $\int e^x \, dx = e^x + c.$

④ $\int \sin x \, dx = -\cos x + c.$

⑤ $\int \cos x \, dx = \sin x + c.$

⑥ $\int \tan x \, dx = \log(\sec x) + c = -\log(\cos x) + c$

⑦ $\int \cot x \, dx = \log(\sin x) + c$

⑧ $\int \sec x \, dx = \log((\sec x) + \tan x) + c$

⑨ $\int \operatorname{cosec} x \, dx = \log(\operatorname{cosec} x - \cot x) + c. = \log(\tan \frac{x}{2}) + c$

⑩ $\int \sec^2 x \, dx = \tan x + c.$

⑪ $\int \operatorname{cosec}^2 x \, dx = -\cot x + c.$

## * Integration by substitution:

→ Sometimes integration becomes easier by substitution of some fixed terms or functions

→ In general to evaluate the integrals

① $\int f'(x) e^{f(x)} dx$ , put $f(x) = y$

Note: put power of exponential

Then $f'(x) dx = y$.

② $\int f'(x) \{f(x)\} dx$

put $f(x) = y$

Eg $I = \int \sec x \, e^{\tan x} dx$

let $y = \tan x$ & $dy = \sec x \, dx$

$I = \int \sec x \, dx \cdot e^{\tan x}$

$= \int e^{y} . dy$

$= e^{y} = e^{\tan x} + c.$

③ $\int f'(x) \log\{f(x)\} dx$

put $f(x) = y$.

④ $\int f'(x) \sin\{f(x)\} dx$,

put $y = f(x)$.

# Trigonometric Substitution

↳ Sometimes integration can be made easier by converting the function from algebric to trigonom-etric form. For this, we adopt the following rule.

(1) In, $x^2+a^2$ form, put $x = a\tan\theta$

(2) In, $x^2-a^2$ form, put $x = a\sec\theta$

(3) In, $a^2-x^2$ form, put $x = a\sin\theta$.

⇒ Some standard results got using above ideas.

1. $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}+c$

2. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)+c \quad (x>a)$

3. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)+c \quad (a>x)$

4. $$\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left(x+\sqrt{x^2+a^2}\right)+c$$

$$= \sinh^{-1}\frac{x}{a}+c$$

5. $$\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left(x+\sqrt{x^2-a^2}\right)+c$$

$$= \cosh^{-1}\frac{x}{a}+c$$

6. $$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}+c$$

There are many others results & ideas that are adopted. Some common things to Recall:

$$\int \frac{f'(x)}{f(x)} = \log(f(x)) + c.$$

$$\int \frac{f'(x)}{\sqrt{f(x)}} = 2\sqrt{f(x)} + c.$$

Note:

To evaluate the integral of the form:

ⓐ $\displaystyle\int \frac{\text{linear eqn}}{\text{quadratic eqn}} \cdot dx$

OR

ⓑ $\displaystyle\int \frac{\text{linear eqn}}{\sqrt{\text{quadratic eqn}}}\, dx.$

Convert the numerator in the form of derivative of quadratic expression of denominator and then use the standard formula.

Eg:- $\displaystyle\int \frac{(2x+3)\,dx}{4x^2+4x+5}$

soln $I = \displaystyle\frac{1}{4}\int \frac{8x+4+8}{4x^2+4x+5}$

* Some trigonometric formulae that are common and need to be remember.

① $1 + \cos x = 2\cos^2 x/2$

② $1 - \cos x = 2\sin^2 x/2$

③ $\cos^2 x/2 + \sin^2 x/2 = 1$

or $\cos^2 x + \sin^2 x = 1$

④ $\cos x = \cos^2 x/2 - \sin^2 x/2$

⑤ $\sin x = 2\sin x/2 \cdot \cos x/2$.

* Standard Integrals:

↳ To evaluate the integrals of the form.

(a) $\int \frac{dx}{a \pm b\cos x}$ (b) $\int \frac{dx}{a \pm b\sin x}$

(c) $\int \frac{dx}{a^2\sin^2 x + b^2\cos^2 x}$ (d) $\int \frac{dx}{a\cos x + b\sin x + c}$

↳ Convert these integrals in the form of

$\int \frac{\sec^2 x \, dx}{f(\tan x)}$

& then put $\tan x = y$ & finally use any of the formula i.e going to be followed.

- To evaluate the integrals of the form.

(a) $\int \frac{a\cos x + b}{p\sin x + q\cos x + r}\,dx$

(b) $\int \frac{a\sin x + b}{p\sin x + q\cos x + r}\,dx$

(c) $\int \frac{a\sin x + b\cos x + c}{p\sin x + q\cos x + r}\,dx$

then use

Numerator $= A(\text{denominator}) + B\frac{d}{dx}[\text{denomi}$

And then find A & B.

* Some specific problems:-

→ (1) To evaluate the integrals of the form.

$\int \frac{dx}{\sqrt{(x-a)(x-b)}}$

put, $(x-a) = y^2$ & find the value of $x$ & substitute

OR

change the above integral in the form.

$\int \frac{dx}{\sqrt{px^2 + qx + r}}$

(2) To evaluate the integral of the form

$$\int \sqrt{\frac{an+b}{cn+d}}\, dn$$

put denominator $= y^2$ i.e. $cn+d = y^2$

Formulae below will be taken in action on above form.

(1) $\int \sqrt{n^2+a^2}\, dn = \left\{\frac{n}{2}\sqrt{n^2+a^2} + \frac{a^2}{2}\log\left(n+\sqrt{n^2+a^2}\right)\right\}+c$

(2) $\int \sqrt{n^2-a^2}\, dn = \left\{\frac{n}{2}\sqrt{n^2-a^2} + \frac{a^2}{2}\log\left(n+\sqrt{n^2+a^2}\right)\right\}+c.$

(3) $\int \sqrt{a^2-n^2}\, dn = \left\{\frac{n}{2}\sqrt{a^2-n^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{n}{a}\right)\right\}+c$

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* Integration by using partial fraction:
  - The idea of partial fraction is used if degree of denominator is greater than degree of numerator.

| Expression in denominator | Corresponding p. fraction |
| --- | --- |
| 1. a) $(x-a)(x-b)(x-c)$ | 1. a) $\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$ |
| b) $(x-a)^2(x-b)$ | b) $\frac{A}{(x-a)}+\frac{B}{(x-a)^2}+\frac{C}{(x-b)}$ |
| c) $(x-a)^3$ | c) $\frac{A}{(x-a)}+\frac{B}{(x-a)^2}+\frac{C}{(x-a)^3}$ |
| 2. a) $(ax^2+bx+c)(px^2+qx+r)$ | 2. a) $\frac{A}{(ax^2+bx+c)}+\frac{B}{(px^2+qx+r)}$ |
| b) $(x-a)(px^2+qx+r)$ | b) $\frac{A}{(x-a)}+\frac{B}{(px^2+qx+r)}$ |
| c) $(px^2+qx+r)^2$ | c) $\frac{A}{(px^2+qx+r)}+\frac{B}{(px^2+qx+r)^2}$ |

3. a) $$\frac{x^2}{(x-a)(x-b)} = 1+\frac{A}{(x-a)}+\frac{B}{(x-b)}$$

b) $$\frac{x^3}{(x-a)(x-b)(x-c)} = 1+\frac{A}{(x-a)}+\frac{B}{(x-b)}+\frac{C}{(x-c)}$$

* Some other problems:

Note ①:

6 If both numerator and denominator have even powers in 'x' then

Step I: Put $x^2 = y$ or even power $= y$

Step 2: Find partial fraction.

Step 3: Replace $y$ by $x^2$

Step 4: Integrate w.r.t $x$.

Eg.

Evaluate $\int \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx$

let $I = \int \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx$

~~p~~ put $x^2 = y$.

Now, $I = \int \frac{y}{(y+a^2)(y+b^2)} dx$

let, $\int \frac{y}{(y+a^2)(y+b^2)} dx = \int \frac{A}{(y+a^2)} dx + \int \frac{B}{(y+b)^2} dx$ --- ①

~~put y =~~

or $\cancel{\frac{y}{(y+a^2)(y+b^2)}} = \frac{A(y+b^2) + B(y+a^2)}{\cancel{(y+a^2)(y+b^2)}}$

or $y = A(y+b^2) + B(y+a^2)$

put $y = -b^2$

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$$-b^2 = B(a^2 - b^2$$

$$B = \frac{-b^2}{a^2-b^2}$$

& put $y = -b^2$

$$A = \frac{-a^2}{b^2-a^2} = \frac{a^2}{a^2-b^2}$$

Now, eqn ①.

$$\int \frac{x^2 \, dx}{(x^2+a^2)(x^2+b^2)} = \int \frac{(-b^2/(a^2-b^2))\, dx}{x^2+a^2} + \int \frac{(b^2/(a^2-b^2))\, dx}{x^2+b^2}$$

$$= \frac{-b^2}{a^2-b^2} \cdot \frac{1}{a} \tan^{-1}\frac{x}{a} + \frac{a^2}{x^2+b^2}\frac{1}{b}\tan^{-1}\frac{x}{b} + C$$

<u>Note 2:</u>

If numerator have odd powers only in $x$ & denominator have even power only.

① put $x^2 = y$ (i.e. $x$ with even power)

② put value of $y$ on eqn i.e. fund ~~dx~~ on dy. & do partial fraction.

③ Integrate it w.r.t. $y$.

④ Replace value of $y$ using $x^2$ or consideration.

* Integration by parts (- Integration of the product of 2 functions)

⇒

let, u & v be any two functions of $x$, then integration of their product $(u.v)$ w.r.t $x$ is given by

$$\int uv\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx$$

↳ Inorder to choose first & second function, we use ILATE.

# Short-cuts.

→ Let u and v be given 2 functions and

$$u' = \frac{du}{dx}, \quad u'' = \frac{du'}{dx} \text{ etc.,}$$

and let derivative of 1st function u becomes constant after some finite steps then,

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$$

until derivate of u becomes constant.

## Definite Integrals:-

let $\int f(x) = F(x)+c$, then the integral

$$\int_a^b f(x)\,dx = [F(x)+c]_a^b$$
$$= F(b) - F(a)$$

is called definite integral. Here a is lower limit and b is upper limit.

## properties of definite integral:

(i) $\int_a^a f(x)\,dx = 0$

(ii) $\int_a^b 0\,dx = 0.$

(iii) $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

(iv) if c is an interior point in (a, b) i.e. $a<c<b$ then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

(v) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$

(vi) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$

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