* Integral Function :-

A function which is regular in every finite region of the z-plane is called an integral function or entire function.

In other words, an integral function is an analytic function having no singularity except at ∞.

e.g.

e^{∞} d.n.e.
non-isolated essential singular

e^{2z} is an integral function.
[$\because$ e^{2z} has only singularity at $z = \infty$]

* The simplest functions are polynomials. We know that a polynomial can be uniquely expressed as the product of linear factors as

$$f(z) = f(0)\left(1-\frac{z}{z_1}\right)\left(1-\frac{z}{z_2}\right)\cdots\left(1-\frac{z}{z_n}\right).$$

This polynomial has zeros at the points $z = z_1, z_2, \ldots, z_n$.

An integral function which is not a polynomial may have an infinity of zeros z_n and the product $\prod\left(1-\frac{z}{z_n}\right)$ taken over these zeros may be divergent so an integral function can not be always factorized in the same way as in polynomials and thus we have to consider less simple factors than $\left(1-\frac{z}{z_n}\right)$ only.

Remark : 1. An integral function may have no zeros.
e.g. e^z has no zero.

2. An integral functions may have finite no. of zeros.
for example: A polynomial of finite degree.

3. An integral functions may have infinite no. of zeros.
e.g. $\sin z = z - \frac{z^3}{\lfloor 3} + \frac{z^5}{\lfloor 5} - - - - -$

$\sin z = 0$
$z = n\pi$
$n = 0, 1, 2, \ldots$

Thm. The most general integral functions with no zeros is of the form $e^{g(z)}$, where $g(z)$ is itself an integral function.

Pf. Let $f(z)$ be an integral function with no zero then $f'(z)$ is also an integral function so is $\frac{f'(z)}{f(z)}$.

Let $F(z) = \int_{z_0}^{z} \frac{f'(t)}{f(t)} dt$, where the integral is taken along any path from any fixed point z_0 to point z.

$$F(z) = \log f(t) \Big|_{z_0}^{z}$$

$$= \log f(z) - \log f(z_0)$$

$$\log f(z) = F(z) + \log f(z_0)$$

$$f(z) = \exp.[F(z) + \log f(z_0)]$$

$$= \exp.(g(z))$$

i.e. $f(z) = e^{g(z)}$

where $g(z) = F(z) + \log f(z_0)$ and $g(z)$ is an integral function.

* **Construction of an Integral function with given zeros**

If $f(z)$ is an integral function with only a finite no. of zeros say, $z_1, z_2, \ldots, z_n$.

Then the functions $\dfrac{f(z)}{(z-z_1)(z-z_2)\cdots(z-z_n)}$

is an integral function with no zeros.

Also, we know that most general form of an integral function having no zeros $e^{g(z)}$ where $g(z)$ is an integral function. [By Prev. Thm.]

Thus we put

$$e^{g(z)} = \frac{f(z)}{(z-z_1)(z-z_2)\cdots(z-z_n)}$$

$$f(z) = e^{g(z)}(z-z_1)(z-z_2)\cdots(z-z_n)$$

If, however, $f(z)$ is an integral function with an infinite no. of zeros then the only

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

limit point of the seq. of zeros $z_1, z_2, \ldots, z_n, \ldots$ is the point at ∞.

To determine an integral function $f(z)$ with an infinity of zeros we have an important thm. due to Weierstrass.

Weierstrass Primary Factors.

The expressions

$$E(z,0) = 1 - z$$

$$E(z,1) = (1-z)e^{z}$$

$$E(z,2) = (1-z)e^{z+\frac{z^2}{2}}$$

$$\cdots\cdots$$

$$E(z,p) = (1-z)e^{z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}}, \quad (p \geq 0)$$

are called Weierstrass primary factors. Each primary factor is an integral function has only a simple zero at $z=1$.

Case I : For $|z|<1$

$$E(z,p) = (1-z)e^{z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}}$$

$$= \exp.\left[\log(1-z) + \left(z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}\right)\right]$$

$$= \exp\left[-\frac{z}{1} - \frac{z^2}{2} - \cdots - \frac{z^p}{p} - \frac{z^{p+1}}{p+1} - \frac{z^{p+2}}{p+2} - \cdots\right.$$

$$\left. + \left(z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}\right)\right]$$

$$\exp.\left[-\frac{z^{p+1}}{p+1}-\frac{z^{p+2}}{p+2}-\cdots\right]$$

$$\log E(z,p) = -\frac{z^{p+1}}{p+1}-\frac{z^{p+2}}{p+2}-\cdots \quad \text{---} \circledast$$

Case (ii) : For $|z| \leq \frac{1}{2}$;

From ⊛, we have

$$|\log E(z,p)| = \left|-\frac{z^{p+1}}{p+1}-\frac{z^{p+2}}{p+2}-\cdots\right|$$

$$\leq \frac{|z|^{p+1}}{p+1}+\frac{|z|^{p+2}}{p+2}+\cdots$$

$$= |z|^{p+1}\left[\frac{1}{p+1}+\frac{|z|}{p+2}+\frac{|z|^2}{p+3}+\cdots\right]$$

$$< |z|^{p+1}\left[1+|z|+|z|^2+\cdots\right] \qquad \left\{\begin{array}{l}\because p+1<p+2\\ \quad <p+3<\cdots\\ p+1>1\\ \Rightarrow \frac{1}{p+1}<1\end{array}\right\}$$

$$\leq |z|^{p+1}\left[1+\frac{1}{2}+\frac{1}{4}+\cdots\right]_{|z|\leq\frac{1}{2}}$$

$$\leq |z|^{p+1}\frac{1}{1-\frac{1}{2}}$$

$$= 2|z|^{p+1}$$

imp. result

C+M

$$\boxed{|\log E(z,p)| \leq 2|z|^{p+1} \quad \text{whenever } |z|\leq \tfrac{1}{2}}$$

31 Imp Thm. Exams 9/4 (!!)

If $z_1, z_2, \cdots, z_n, \cdots$ be any seq. of numbers whose only limit point is a point at infinity then it is possible to construct an integral function which vanishes at each of point z_n and nowhere else.

$$z_1, z_2, \cdots z_n, \cdots \to \infty.$$

Pf. I Let the given zeros $z_1, z_2, \ldots, z_n, \ldots$ be arranged in order of non-decreasing modulus

i.e. $0 \le |z_1| \le |z_2| \le \ldots \le |z_n| \le \ldots$

Let $|z_n| = r_n$

Since r_n is increasing indefinitely with n, we can find a seq. of +ve integer $p_1, p_2, \ldots, p_n$ s.t. the series $\sum_{n=1}^{\infty} \left(\frac{r}{r_n}\right)^{p_n}$ ——— ①

is cgt for all +ve values of r.

It is always possible to find such a seq.; we can take $p_n = n$

Since $\sum \left(\frac{r}{r_n}\right)^n < \sum \frac{1}{2^n}$, for $\boxed{r_n > 2r}$ or $\frac{r_n}{r} > 2$ so $\left(\frac{r}{r_n}\right)^n < \left(\frac{1}{2}\right)^n$

and hence the series is cgt.

II Let $f(z) = \prod_{n=1}^{\infty} E\left(\frac{z}{z_n}, p_n - 1\right)$ ——— ②

This function has the req. property, for if $|z_n| > 2|z|$

Then, we have

$$\left|\log E\left(\frac{z}{z_n}, p_n - 1\right)\right| \le 2\left|\frac{z}{z_n}\right|^{p_n} = 2\left(\frac{r}{r_n}\right)^{p_n}$$

and hence by Weierstrass test, the series $\sum_{n=1}^{\infty} \log E\left(\frac{z}{z_n}, p_n - 1\right)$ is uniformly cgt

for $|z_n| > 2R$ and $|z| \le R$

and also by Weierstrass test for the uniform cgs of an infinite product to be cgt, so is the.

product, $\prod_{n=1}^{\infty} E\left(\frac{z}{z_n}, p_n - 1\right)$, $|z_n| > 2R$

[II] Hence $f(z)$ is regular for $|z| \le R$ and its only zeros in the region are those of

$$\prod_{n=1}^{\infty} E\left(\frac{z}{z_n}, p_n - 1\right), \quad |z_n| > 2R.$$

i.e. the point $z_1, z_2, \dots$

Since R can be taken as large as we please, therefore, we conclude that $f(z)$ is the integral function which vanishes only at $z_1, z_2, \dots$ and nowhere else.

Hence the result

Remark: 1. The function $f(z)$ is not uniquely defined, since we have a wide choice of seq. of integers $p_1, p_2, \dots$

Imp 13.

Weierstrass Factorization Theorem.

Let $f(z)$ be an entire function and let $\{z_n\}$ be the non-zero zeros of $f(z)$ repeated according to multiplicity, suppose $f(z)$ has a zero at $z=0$ of order $m \ge 0$ (a zero of order $m=0$ at $z=0$ means $f(0) \ne 0$). Then there is an entire function $g(z)$ and a seq. of integers $\{p_n\}$ s.t.

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E\left(\frac{z}{z_n}, p_n - 1\right)$$

Pf. According to preceding thm., $\{p_n\}$ can be chosen s.t.

$$h(z) = z^m \prod_{n=1}^{\infty} E\left(\frac{z}{z_n}, p_n - 1\right)$$

has the same zeros as $f(z)$ with the same multiplicity, it follows that $\frac{f(z)}{h(z)}$ has removable singularity at $z = 0, z_1, z_2, \ldots$

Thus $\frac{f}{h}$ is an entire function and furthermore has no zeros.

Since $\mathbb{C}$ is simply connected, there is an entire function $g(z)$ s.t.

$$\frac{f(z)}{h(z)} = e^{g(z)}$$

i.e. $\quad f(z) = e^{g(z)} \cdot h(z)$

$$\Rightarrow \quad f(z) = e^{g(z)} \cdot z^m \prod_{n=1}^{\infty} E\left(\frac{z}{z_n}, p_n - 1\right)$$

$$= z^m \cdot e^{g(z)} \prod_{n=1}^{\infty} E\left(\frac{z}{z_n}, p_n - 1\right)$$

Hence proved

Cor. Every function which is meromorphic in the entire z-plane is the quotient of the integral function.

Pf. Let $\phi(z)$ be a function, which is meromorphic in the entire z-plane

Then, we can find an entire function $g(z)$ with the poles of $\phi(z)$ as its zeros.

Thus the product $\phi(z)\cdot g(z)$ is analytic function say $f(z)$.

$$\therefore \phi(z) = \frac{f(z)}{g(z)}$$

where $f(z)$ & $g(z)$ both are integral function.

2/2

Thm. If $f(z)$ is an integral function and $f(0) \neq 0$, then $f(z) = f(0)\, P(z)\, e^{g(z)}$, where $P(z)$ is the product of primary factor and $g(z)$ is an integral function.

Pf. I We form $P(z)$ from the zeros of $f(z)$. Let

$$\boxed{\phi(z) = \frac{f'(z)}{f(z)} - \frac{P'(z)}{P(z)}} \quad \text{CTM}$$

Thus $\phi(z)$ is an integral function, since the poles of one term are cancelled by the poles of other or zeros of others.

II Let $$g(z) = \int_0^z \phi(t)\,dt = \int_0^z \left[\frac{f'(t)}{f(t)} - \frac{P'(t)}{P(t)}\right] dt$$

$$= \log f(z) - \log f(0) - \log P(z) + \log P(0)$$

$= \log f(z) - \log f(0) - \log P(z)$ [$\because \log P(0) = 0$]

[$P(0)=1$, since it is the product of primary factor]

$\Rightarrow \log f(z) = g(z) + \log f(0) + \log P(z)$

$\Rightarrow f(z) = f(0)\, P(z)\, e^{g(z)}$

Thm. Let $f(z)$ be an integral function of z and suppose it has simple zero at the point $z = a_1, a_2, \ldots$ then $f(z)$ can be expressed as

$$f(z) = f(0)\, e^{z f'(0)/f(0)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{z/a_n}$$

Pf. Since $a_1, a_2, a_3, \ldots$ are the simple zeros of $f(z)$

$\therefore$ In the nbhd. of $z = a_n$, we have

$f(z) = (z - a_n)\, g(z)$ — ①

where $g(z)$ is analytic and $g(a_n) \neq 0$

Taking log on both sides of ①

$\log f(z) = \log(z - a_n) + \log g(z)$

Diff. w.r.to z, we get

$$\frac{f'(z)}{f(z)} = \frac{1}{z - a_n} + \frac{g'(z)}{g(z)} \quad \text{— ②}$$

where $\frac{g'(z)}{g(z)}$ is analytic $\boxed{\text{at } z = a_n}$ [$\because g(a_n) \neq 0$]

Eq. ② shows that $\frac{f'(z)}{f(z)}$ has a simple pole at

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$z = a_n$

Also $\text{Res.}(z=a_n) = \lim_{z\to a_n} (z-a_n)\dfrac{f'(z)}{f(z)}$

$$= \lim_{z\to a_n} (z-a_n)\left[\frac{1}{z-a_n} + \frac{g'(z)}{g(z)}\right]$$

$$= 1 + 0 = 1$$

II Now, suppose that $\dfrac{f'(z)}{f(z)}$ satisfies the condition of M.L. Theorem.

M.L. Theorem :-

Let $f(z)$ be a meromorphic function whose only singularities except at ∞ are the poles $a_1, a_2, --, a_n$, where they are arranged in ascending order

$$0 \le |a_1| \le |a_2| \le --- \le |a_n|$$

and let $b_1, b_2, --, b_n$ respt. be residue at these poles. Let $\{C_n\}$ be a seq. of closed contours s.t. C_n includes $a_1, a_2, --, a_n$ and no other pole. Also suppose that

(i) The minimum distance R_n of points of C_n from the origin tends to ∞ as $n\to\infty$

(ii) $f(z)$ is bounded on C_n i.e. $|f(z)| \le M$ on C_n.

Then $f(z)$ can be expanded as

$$f(z) = f(0) + \sum_{n=1}^{\infty} b_n\left[\frac{1}{z-a_n} + \frac{1}{a_n}\right]$$

Then by M.L. Thm., we have

$$\frac{f'(z)}{f(z)} = \frac{f'(0)}{f(0)} + \sum_{n=1}^{\infty} 1 \cdot \left[\frac{1}{z-a_n} + \frac{1}{a_n}\right] \quad \text{--- (3)} \quad [b_n = 1]$$

Integrating eq. (3) w.r.to 'z' from 0 to z along a path not passing through any pole

$$[\log f(z)]_0^z = \left[\frac{f'(0)}{f(0)} \cdot z\right]_0^z + \sum_{n=1}^{\infty} \left[\log(z-a_n) + \frac{z}{a_n}\right]_0^z$$

$$\log \frac{f(z)}{f(0)} = \frac{f'(0)}{f(0)} \cdot z + \sum_{n=1}^{\infty} \log \frac{(z-a_n)}{-a_n} + \frac{z}{a_n}$$

$$= \frac{f'(0)}{f(0)} \cdot z + \sum_{n=1}^{\infty} \log \frac{(z-a_n)}{-a_n} \cdot e^{z/a_n}$$

$$\log \frac{f(z)}{f(0)} = \frac{f'(0)}{f(0)} \cdot z + \log \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{z/a_n}$$

$$= \log e^{\frac{f'(0)}{f(0)} \cdot z} + \log \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{z/a_n}$$

$$= \log \left[e^{\frac{f'(0)}{f(0)} \cdot z} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{z/a_n}\right]$$

$$f(z) = f(0)\, e^{\frac{f'(0)}{f(0)} \cdot z} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{z/a_n}$$

i.e. $$f(z) = f(0)\, e^{\frac{f'(0)}{f(0)} z} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{z/a_n}$$

Hence the proof.

Ex. Show that $\dfrac{\sin z}{z} = \prod_{n=1}^{\infty} \left(1 - \dfrac{z^2}{n^2\pi^2}\right)$.

Sol. let $f(z) = \dfrac{\sin z}{z}$

Poles of $f(z)$ are given by $z=0$
but $z=0$ is a removable singularity

$$f(0) = \lim_{z\to 0} \frac{\sin z}{z} = 1$$

Also $$f'(z) = \frac{z\cos z - \sin z}{z^2}$$

so $$\frac{f'(z)}{f(z)} = \frac{z\cos z - \sin z}{z^{\cancel{2}}} \cdot \frac{\cancel{z}}{\sin z} = \frac{z\cos z - \sin z}{z\sin z}$$

$$\lim_{z\to 0} \frac{f'(z)}{f(z)} = 0$$

$$\lim_{z\to 0} \frac{f'(z)}{f(z)} = \frac{z\cos z - \sin z}{z\sin z}$$

$$= \frac{0}{0}$$

Applying L-Hospital's rule

$$\lim_{z\to 0} \frac{f'(z)}{f(z)} = \lim_{z\to 0} \frac{\cancel{\cos z} - z\sin z \cancel{- \cos z}}{\sin z + z\cos z}$$

$$= \frac{0}{0}$$

Again — - - -

$$\lim_{z\to 0} \frac{f'(z)}{f(z)} = \lim_{z\to 0} \frac{-\sin z - z\cos z}{\cos z + \cos z - z\sin z}$$

$$= \frac{0}{2} = 0$$

Poles of $\frac{f'(z)}{f(z)}$ are given by :

$z\sin z = 0$

$\Rightarrow \boxed{z=0}$ and $\sin z = 0$

$\Rightarrow z = \boxed{n\pi}$ where $n = \pm 1, \pm 2, \dots$

which are simple poles

$$\Rightarrow \frac{f'(0)}{f(0)} = 0.$$

Now by M.L. Thm. under the condition that $\frac{f'(z)}{f(z)}$ satisfies the condition of this thm.

We have

$$f(z) = f(0)\, e^{\frac{f'(0)}{f(0)}\cdot z} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{z/a_n}$$

$$\therefore \frac{\sin z}{z} = 1 \cdot e^{0\cdot z} \prod_{n=1}^{\infty} \left(1 - \frac{z}{n\pi}\right) e^{z/n\pi} \cdot \prod_{n=-1}^{-\infty} \left(1 - \frac{z}{n\pi}\right) e^{z/n\pi}$$

$$= \prod_{n=1}^{\infty} \left(1 - \frac{z}{n\pi}\right) e^{z/n\pi} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n\pi}\right) e^{-z/n\pi}$$

$$= \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right)$$

Ex. Prove that $\cos z = \prod_{n=1}^{\infty}\left(1-\frac{z^2}{(n-\frac{1}{2})^2\pi^2}\right)$.

Sol. Let $f(z) = \cos z$

Ist of all we find $f(0)$

$f(0) = \cos 0 = 1$

Also $f'(z) = -\sin z$

so $\frac{f'(z)}{f(z)} = \frac{-\sin z}{\cos z}$

$$\lim_{z\to 0}\frac{f'(z)}{f(z)} = 0$$

Poles of $\frac{f'(z)}{f(z)}$ are given by $\cos z = 0$

$\Rightarrow z = (2n-1)\frac{\pi}{2}, \quad n = \pm 1, \pm 2, \ldots$

which are simple poles.

$\frac{f'(0)}{f(0)} = 0$

Now by M.L. Thm. under the condition that $\frac{f'(z)}{f(z)}$ satisfies the condition of this thm.

We have

$$f(z) = f(0)\, e^{\frac{f'(0)}{f(0)}\cdot z} \prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{z/a_n}$$

$$\therefore \cos z = 1\cdot e^{0.z}\prod_{n=1}^{\infty}\left(1-\frac{z}{(2n-1)\frac{\pi}{2}}\right)e^{z/(2n-1)\frac{\pi}{2}}$$

$$\prod_{n=-1}^{-\infty}\left(1+\frac{z}{(2n-1)\frac{\pi}{2}}\right)e^{+z/(2n-1)\frac{\pi}{2}}$$

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$$= \prod_{n=1}^{\infty}\left(1 - \frac{z}{(2n-1)\pi/2}\right) e^{z/(2n-1)\pi/2} \prod_{n=1}^{\infty}\left(1 + \frac{z}{(2n-1)\pi/2}\right) e^{-z/(2n-1)\pi/2}$$

$$= \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{(2n-1)^2 \frac{\pi^2}{2^2}}\right)$$

$$= \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{(n-1/2)^2 \cdot \pi^2}\right)$$

memorphic = only singularity are poles

16) (19) **Gamma Function**

Here, we shall construct a function called Gamma function or Euler's Gamma function which is meromorphic with poles at non +ve integers i.e. $z = 0, -1, -2, \ldots$.

There are two natural approaches to construct the Gamma function. One is via the Weierstrass product and the other is via a Mellin integral. Certain properties are clear from one definition but not from the other, although the two definitions give the same function.

We start with the former approach which involves more algebraic properties of the Gamma function for this we introduce functions which have only negative zeros.

Step I

The simplest function of the type is

$$\boxed{G(z) = \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}} \quad \text{(C.T.M.)} \quad (1)$$

Then $G(-z) = \prod_{n=1}^{\infty}\left(1-\frac{z}{n}\right)e^{z/n}$

$\therefore \; z\,G(z)\,G(-z) = z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right) = \frac{\sin \pi z}{\pi}$

$$= \frac{\sin \pi z}{\pi} \quad \text{—— (2)}$$

$\left[\because \frac{\sin \pi z}{\pi} = z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)\right]$

Now, $G(z-1) = \prod_{n=1}^{\infty}\left(1+\frac{(z-1)}{n}\right)e^{-(z-1)/n}$ —— (3)

The zeros of $G(z-1)$ are given by

$1+\frac{(z-1)}{n} = 0 \implies z = 1-n$

$\implies n = 1, 2, 3, \ldots$

so $z = 0, -1, -2, -3, \ldots$

Thus $G(z-1)$ has same zeros as $G(z)$ and in addition a simple zero at origin.

$\therefore$ By Weierstrass factorization thm., we can write

$G(z-1) = z\,e^{g(z)}\,G(z)$ —— (4)

where $g(z)$ is an integral function.

Step II

Using (1) and (3) in (4), we obtain

$$\prod_{n=1}^{\infty}\left(\frac{z-1+n}{n}\right)e^{-\frac{(z-1)}{n}} = z\,e^{g(z)}\prod_{n=1}^{\infty}\left(\frac{z+n}{n}\right)e^{-z/n} \quad \text{—— (5)}$$

For determining $g(z)$, we take logarithmic derivativ on both sides of (5).

$$\log\left[\prod_{n=1}^{\infty}\left(\frac{z-1+n}{n}\right)e^{-(z-1)/n}\right] = \log z + g(z) + \sum_{n=1}^{\infty}\log\left(\frac{z+n}{n}\right)e^{-z/n}$$

$$\Rightarrow \sum_{n=1}^{\infty}\left[\log\left(\frac{z-1+n}{n}\right) - \left(\frac{z-1}{n}\right)\right] = \log z + g(z) + \sum_{n=1}^{\infty}\left[\log\left(\frac{z+n}{n}\right) - \frac{z}{n}\right]$$

Diff. both sides w.r. to z, we get

$$\sum_{n=1}^{\infty}\left[\frac{\cancel{n}}{z-1+n}\cdot\frac{1}{\cancel{n}} - \frac{1}{n}\right] = \frac{1}{z} + g'(z) + \sum_{n=1}^{\infty}\left(\frac{\cancel{n}}{z+n}\cdot\frac{1}{\cancel{n}} - \frac{1}{n}\right)$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{z-1+n} - \frac{1}{n}\right) = \frac{1}{z} + g'(z) + \sum_{n=1}^{\infty}\left(\frac{1}{z+n} - \frac{1}{n}\right) \quad \text{— (6)}$$

3.- Replacing n by n+1, the series on L.H.S. of (6) can be written as

$$\sum_{n=1}^{\infty}\left(\frac{1}{z-1+n} - \frac{1}{n}\right) = \sum_{n+1=1}^{\infty}\left[\frac{1}{z-1+n+1} - \frac{1}{n+1}\right]$$

$$= \sum_{n=0}^{\infty}\left[\frac{1}{z+n} - \frac{1}{n+1}\right]$$

$$= \frac{1}{z} - 1 + \sum_{n=1}^{\infty}\left(\frac{1}{z+n} - \frac{1}{n+1} \cancel{+\frac{1}{n}} \cancel{-\frac{1}{n}}\right)$$

$$= \left(\frac{1}{z} - 1\right) + \sum_{n=1}^{\infty}\left(\frac{1}{z+n} - \frac{1}{n}\right) + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \frac{1}{z} - \cancel{1} + \sum_{n=1}^{\infty}\left(\frac{1}{z+n} - \frac{1}{n}\right) + \cancel{1} \qquad \left[\because \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = 1\right]$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z+n} - \frac{1}{n}\right)$$

Comparing it with (6), we conclude that

$g'(z) = 0$

$\therefore$ $g(z)$ is constant which we denote by γ.

Hence (4) can be written as

$$\boxed{G(z-1) = z e^{\gamma} G(z)} \quad \text{--- (7)} \checkmark$$

Step-(4) To determine γ, we put $z=1$ in (7) to get

$G(0) = e^{\gamma} G(1)$ $\qquad 1 = e^{\gamma} G(1)$ $\qquad \frac{1}{e^{\gamma}} = G(1)$

from (1), $G(0) = 1$ $\qquad \Rightarrow e^{-\gamma} = G(1)$

and $G(1) = \prod_{n=1}^{\infty} \left(1+\frac{1}{n}\right) e^{-1/n}$

$$\text{Hence} \quad \boxed{e^{-\gamma} = \prod_{n=1}^{\infty} \left(1+\frac{1}{n}\right) e^{-1/n}} \quad \text{--- (8)}$$

The nth partial product of this infinite product can be written as

$(n+1) e^{-(1+1/2+1/3+ \cdots + 1/n)}$

$$\left[(1+1)e^{-1}\left(1+\frac{1}{2}\right)e^{-1/2} \cdots \left(1+\frac{1}{n}\right)e^{-1/n} = \frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdots\frac{n+1}{n}\, e^{-\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)} \right]$$

and therefore

$$-\gamma = \lim_{n\to\infty} \left[\log(n+1) - \left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) \right] \qquad \left[\prod_{n=1}^{\infty} = \lim_{k\to\infty} \prod_{n=1}^{k} \right]$$

$$= \lim_{n\to\infty} \left[\log n - \left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) \right]$$

$$\boxed{\gamma = \lim_{n\to\infty} \left[1+\frac{1}{2}+\cdots+\frac{1}{n} - \log n \right]}$$

The constant γ is Euler's constant. Its approximate value is $.57716$

* Gamma function (Γ) :-

The Γ (Gamma) function is defined as

$$\boxed{\Gamma(z) = \frac{1}{z e^{\gamma z} G(z)}} \quad \text{--- (9)}$$

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$$= \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right)^{-1} e^{z/n} \quad \text{--- (10)}$$

We note that $\Gamma(z)$ is well-defined in the whole complex plane except for $z = 0, -1, -2, \ldots$ which are simple poles of the function. Hence $\Gamma(z)$ is meromorphic with these poles but no zeros.

The following are the direct consequences i.e. simple properties of the $\Gamma(z)$ function.

(a) $\Gamma(z+1) = z\,\Gamma(z)$ $\qquad G(z-1) = z e^{\gamma} G(z)$

Using (7) in (9), we get

Solve for $\Gamma(z)$

$$\Gamma(z-1) = \frac{e^{-(z-1)\gamma}}{(z-1)G(z-1)} = \frac{e^{-(z-1)\gamma}}{(z-1)\,z e^{\gamma}\cdot G(z)}$$

$$= \frac{1}{z-1}\left(\frac{e^{-\gamma z}}{z G(z)}\right) = \frac{\Gamma(z)}{z-1}$$

i.e. $\Gamma(z) = (z-1)\,\Gamma(z-1)$

$$\therefore \quad \Gamma(z+1) = z\,\Gamma(z) \quad \text{--- (11)}$$

(b) $\Gamma(1-z)\,\Gamma(z) = \dfrac{\pi}{\sin \pi z}$

$\Gamma(1-z)$ Using (2) and (9) and (11), we obtain

$$\Gamma(1-z)\,\Gamma(z) = -z\,\Gamma(-z)\,\Gamma(z)$$

$\Gamma(z-1) = z\Gamma$

$z\,G(-z)\,G(z) = \dfrac{\pi}{\sin \pi z}$

$$= +\cancel{z}\,\frac{1}{+\cancel{z}\,e^{-\gamma z} G(-z)} \times \frac{1}{z e^{\gamma z} G(z)}$$

$$= \frac{1}{z\,G(-z)\,G(z)}$$

$$= \frac{\pi}{\sin \pi z} \quad \text{--- (12)}$$

(c) ~~$\Gamma(1/2) = \sqrt{\pi}$~~

Putting $z = 1/2$ in (12), we get

$$\Gamma(1/2) = \sqrt{\pi}$$

(d) (i) $\Gamma(1) = 1$

for $z = 1$, (8) and (10) result, we get

$$\Gamma(1) = e^{-\gamma} \prod_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{-1} e^{1/n}$$

$$= e^{-\gamma} \cdot e^{\gamma} = 1$$

(ii) $\Gamma(n+1) = \lfloor n$

$\Gamma(1) = 1$

Further if n is a +ve integer, then using eq. (11) repeatedly, we get

$\Gamma(1) = 1, \quad \Gamma(2) = 1$

$\Gamma(3) = 2\,\Gamma(2) = 2 \cdot 2 = \lfloor 2$

$\Gamma(4) = 3\,\Gamma(3) = 3 \cdot 2 \cdot 1 = \lfloor 3$ and so on.

Thus finally we get $\Gamma(n+1) = \lfloor n$

Thus Γ function is considered as a generalization of the factorial function.

(e) (12) $\boxed{\text{Duplication Formula or Legendre's Formula}}$

$$\Gamma(z)\,\Gamma(z+1/2) = \frac{\sqrt{\pi}}{2^{2z-1}}\,\Gamma(2z)$$

Pf.

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right)^{-1} e^{z/n} = \frac{1}{e^{\gamma z} G(z)}$$

Taking log both sides

$$\log \Gamma(z) = -\gamma z + \sum_{n=1}^{\infty} \log\left[\left(1+\frac{z}{n}\right)^{-1} e^{z/n}\right] - \log z$$

$$= -\gamma z + \sum_{n=1}^{\infty}\left[-\log\left(\frac{z+n}{n}\right) + \frac{z}{n}\right] - \log z$$

Diff. w.r.to z

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma + \sum_{n=1}^{\infty}\left[\frac{-n}{(z+n)}\cdot\frac{1}{n} + \frac{1}{n}\right] - \frac{1}{z}$$

Again Diff. w.r.to z, we get

$$\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) = -0 + \sum_{n=1}^{\infty}\frac{1}{(z+n)^2} + \frac{1}{z^2}$$

$$\Rightarrow \frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) = \sum_{n=0}^{\infty}\frac{1}{(z+n)^2} \quad \text{--- (10)}$$

lly replacing z by 2z in (10), we get

$$\frac{d}{dz}\left(\frac{\Gamma'(2z)}{\Gamma(2z)}\right) = 2\sum_{n=0}^{\infty}\frac{1}{(2z+n)^2}$$

~~Suppose~~ Suppose Thus, we have

$$\frac{d}{dz}\left[\frac{\Gamma'(z)}{\Gamma(z)}\right] + \frac{d}{dz}\left[\frac{\Gamma'(z+\frac{1}{2})}{\Gamma(z+\frac{1}{2})}\right] = \frac{4}{4}\left[\sum_{n=0}^{\infty}\frac{1}{(z+n)^2} + \sum_{n=0}^{\infty}\frac{1}{(z+n+\frac{1}{2})^2}\right]$$

$$= 4\left[\sum_{n=0}^{\infty}\frac{1}{(2z+2n)^2} + \sum_{n=0}^{\infty}\frac{1}{(2z+2n+1)^2}\right]$$

$$= 4\left[\sum_{m=0}^{\infty}\frac{1}{(2z+m)^2}\right] \qquad \left[\begin{matrix}\text{even} = 2n \\ \text{odd} = 2n+1\end{matrix}\right]$$

$$= 2\frac{d}{dz}\left[\frac{\Gamma'(2z)}{\Gamma(2z)}\right]$$

Thus, we get

$$\frac{d}{dz}\left[\frac{\Gamma'(z)}{\Gamma(z)}\right] + \frac{d}{dz}\left[\frac{\Gamma'(z+\frac{1}{2})}{\Gamma(z+\frac{1}{2})}\right] = 2\frac{d}{dz}\left[\frac{\Gamma'(2z)}{\Gamma(2z)}\right]$$

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~~which on integration gives~~

$$\frac{\Gamma'(z)}{\Gamma(z)} + \frac{\Gamma'(z+\frac{1}{2})}{\Gamma(z+\frac{1}{2})} = 2\frac{\Gamma'(2z)}{\Gamma(2z)} + a, \quad \text{where } a \text{ is}$$

the coeff. of integration

Integrating again, we get

$$\log\Gamma(z) + \log\Gamma(z+\tfrac{1}{2}) = \log\Gamma(2z) + az + b$$

$$\log\Gamma(z)\Gamma(z+\tfrac{1}{2}) = \log\Gamma(2z) + \log e^{az+b}$$

$$\boxed{\Gamma(z)\Gamma(z+\tfrac{1}{2}) = [\Gamma(2z)]\, e^{az+b}} \quad \text{—— (13)}$$

Put $z = \frac{1}{2}$ in eq. (13) & $z = 1$ in eq. (13)

$$\Gamma(\tfrac{1}{2})\Gamma(1) = \Gamma(1)\, e^{\frac{a}{2}+b}$$

$$\Rightarrow \sqrt{\pi} = e^{a/2+b}$$

$$\Rightarrow \frac{a}{2} + b = \frac{1}{2}\log\pi \quad \text{—— (A)}$$

$$\Gamma(1)\Gamma(\tfrac{3}{2}) = \Gamma(2)\, e^{a+b}$$

$$\Gamma(\tfrac{3}{2}) = e^{a+b}$$

$$\frac{1}{2}\Gamma(\tfrac{1}{2}) = e^{a+b}$$

$$\frac{\sqrt{\pi}}{2} = e^{a+b}$$

$$\log\frac{\sqrt{\pi}}{2} = a+b$$

$$\Rightarrow \frac{1}{2}\log\pi - \log 2 = a+b \quad \text{—— (B)}$$

subtracting (A) from (B), we get

$$\frac{a}{2} = -\log 2 \quad \Rightarrow \quad \boxed{a = -2\log 2} \quad \text{—— (C)}$$

$$\& \quad \frac{1}{2}\log\pi - \log 2 = -2\log 2 + b.$$

$$\Rightarrow \frac{1}{2}\log\pi = -\log 2 + b$$

$$\Rightarrow \boxed{\frac{1}{2}\log\pi + \log 2 = b} \quad \text{—— (D)}$$

Using the value of a & b in (13), we get

$$\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+\tfrac{1}{2}) \quad \text{—— (14)}$$

$$\Gamma(z)\,\Gamma(z+\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2^{2z-1}}\,\Gamma(2z)$$

This is reqd. Duplication formula.

Emani

(f) Show that residue of gamma function at $z=-n$ is $\frac{(-1)^n}{\lfloor n}$.

Pf. We know that $\Gamma(z)$ has simple poles at $z=0,-1,-2,\ldots$

So, residue at $z=-n$ is

$$\mathrm{Res}(z=-n) = \lim_{z\to -n} (z+n)\,\Gamma(z) \quad \text{--- (1)}$$

Again, we have

$$\Gamma(z+1) = z\,\Gamma(z)$$

$$\Rightarrow \Gamma(z) = \frac{\Gamma(z+1)}{z}$$

Also, $\Gamma(z+2) = (z+1)\,\Gamma(z+1)$

$$\Rightarrow \Gamma(z+1) = \frac{\Gamma(z+2)}{z+1}$$

$$\therefore \Gamma(z) = \frac{\Gamma(z+2)}{z\cdot(z+1)} = \frac{\Gamma(z+3)}{z(z+1)(z+2)} \cdots = \frac{\Gamma(z+n+1)}{z(z+1)\cdots(z+n)}$$

$$\text{i.e.}\quad \Gamma(z) = \frac{\Gamma(z+n+1)}{z(z+1)\cdots(z+n)} \quad \text{--- (2)}$$

$$\mathrm{Res}(z=-n) = \lim_{z\to -n} \frac{\cancel{(z+n)}\,\Gamma(z+n+1)}{z(z+1)\cdots\cancel{(z+n)}}$$

$$= \lim_{z\to -n} \frac{\Gamma(z+n+1)}{z(z+1)\cdots(z+n-1)}$$

$$= \frac{\Gamma(1)}{(-n)(-n+1)\cdots(-1)}$$

$$= \frac{\Gamma(1)}{(-1)^n n(n-1)\cdots 3\cdot 2\cdot 1}$$

$$= \frac{1}{(-1)^n \lfloor n}$$

$$= \frac{(-1)^n}{\lfloor n}$$

$$\therefore \quad \text{Res.}(z=-n) = \frac{(-1)^n}{\lfloor n}$$

* **Another approach to Gamma function**

Here, we define gamma function, is an integral representation. This representation was given by Mellin and thus integral given in def. below is sometimes called Mellin's integral.

Def. We define for $\boxed{Re(z) > 0}$

$$\boxed{\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, dt}$$

Ex. Using the integral representation $f(z)$, prove that $\Gamma(z+1) = z\Gamma(z)$.

Sol. We have $\Gamma(z+1) = \int_0^\infty e^{-t}\, t^{z}\, dt$

$$= \left[t^{z}\cdot \frac{(e^{-t})}{-1}\right]_0^\infty - \int_0^\infty z\, t^{z-1}\cdot \frac{e^{-t}}{-1}\, dt$$

$$= z\int_0^\infty e^{-t}\, t^{z-1}\, dt$$