A Short Historical Review of Kepler Conjecture

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1. Introduction

This report gives a short historical review of Kepler conjecture, which states that the density of a packing of congruent spheres in three dimensions is never greater than $\pi/\sqrt{18} \approx 0.74048\ldots$ [1]. In other words it actually means that FCC (face-centered cubic) and HCP (hexagonal closed-pack) are the densest ways of packing spheres in three dimensions. All physicists, Materials scientists and even a student studying todays science knows that it is true but it took 400 years to be accepted as an actual theorem from a famous conjecture. In 1900, Hilbert made a famous list consisting of twenty three unsolved problems in mathematics for the coming century. This Kepler conjecture was listed there as 18th and considered as one of the oldest problems in discrete geometry.

A packing of spheres is an arrangement of nonoverlapping spheres in Euclidean space. The density of packing is defined as that fraction of the total volume of the unit cell occupied by the nuclei and inner electron which compromise the spheres.

2. Early History

The arrangement probabilities of spheres in three dimensions have been starting to fascinate the mathematicians from a long time ago. An ancient Sanskrit work of finding the formula for the number of balls in a triangular piles which was composed around 499 CE is an example. These historical thirsts for knowledge pave the way of studying mathematical properties of density packing in FCC. The work was composed by the great mathematician Aryabhatiya and the commentary on it written in 629 CE by Bhaskaraiya. What he said is this:

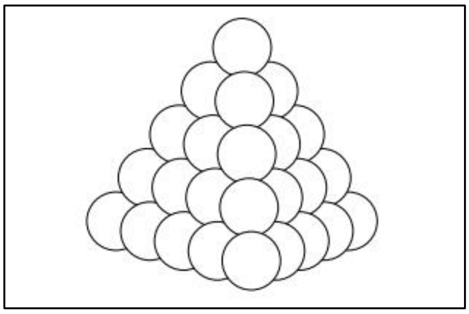

Figure 1: Spheres in triangular piles

Aryabhatiya, Ganitapada 21:

"*For a series with a common difference and first term of 1, product of three increased by 1 from the total, or else the cube of plus 1 diminished by root, divided by 6, is the total of the pile*." [2]

The rule is actually a series of arithmetic progression where the sum of total can be calculated by summing up the spheres in layer by layer. Where sum of spheres in i_{th} layer is $S_i = 1 + 2 + 3 + \ldots + i$. So the total number of spheres in n layers is:

$$\sum_{i=1}^{n} S_i = \frac{n\,(n+1)(n+2)}{6}$$

3. Origin of Kepler Conjecture

The Szpiro's book "Kepler's Conjecture" which is famous for biographical materials of mathematics says a story behind the origin of this conjecture.

"Somewhere toward the end of the 1590s, stocking his ships for yet another expedition, [Sir Walter] Raleigh asked his sidekick and mathematical assistant Thomas Harriot to develop a formula [for the number] of cannonballs in a given stack…" [3]

In 1591, Harriot made a chart of triangular numbers for Raleigh. Harriot connected sphere packings to Pascal's triangle long before Pascal introduced the triangle! Harriot was the first to distinguish between the face-centered cubic and hexagonal closed-packings.

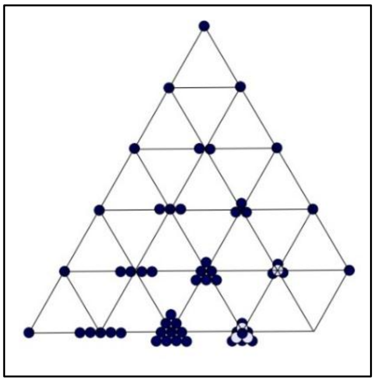

Figure 2: Harriot's chart of triangular numbers

The event which brought Kepler to atomic theory is an interesting part in the history of condensed matter physics. Harriot's theory of matter appears to have been virtually that of Democritus, Hero of Alexandria. Harriot believed that the universe is composed of atoms with void space interposed. The atoms themselves are eternal and continuous. This theory could manage a large influence in the field of optics. In a letter to Kepler on 2 December 1606 Harriot asked when a light ray falls upon the surface of a transparent medium, is it partially reflected or partially refracted? Since the principle of uniformity, a single point cannot both reflect and transmit light, the answer must lie in the supposition that the ray is resisted by some points and not others.

In this letter Harriot advised Kepler to abstract himself mathematically into an atom in order to enter "Nature's House". But in his reply of 2 August 1607, Kepler declined to follow Harriot saying that he believed the reflection-refraction problem in terms of the union of two opposing qualities- Transparence and Opacity. In 1611 Kepler published an essay that explores the consequences of a theory of matter composed of small spherical particles. This essay was the first recorded step towards a mathematical theory of the genesis of inorganic and organic form.

Kepler's essay describes the density packing of FCC and asserts that "the packing will be the tightest possible, so that in no other arrangement could more pellets be stuffed into the same container". This assertion has come to be known as Kepler conjecture.

4. Journey to the Final Proof

Kepler could not prove his own conjecture but his conjecture fascinated the most beautiful minds on earth. After Kepler the debate appeared between Isaac Newton and David Gregory. The debate started with the question of how many spheres of equal radius can be arranged to touch a given sphere. This is the famous **Kissing number problem in n-dimensions [4]**. Newton claimed that the maximum was 12 spheres but Gregory said that 13 might be possible. It is proved later in 19^{th} century that Newton was correct. This Newton-Gregory debate about Kissing number problem gave mathematicians a new strategy to solve Kepler Conjecture.

The next step was taken by world's greatest mathematician Carl Friedreich Gauss. In 1831 Gauss claimed that Kepler conjecture is true if the spheres have to be arranged in a regular lattice. That means any packing arrangement that disproved the Kepler conjecture would have to be an irregular lattice. But eliminating all possible irregular lattice is too much difficult and this what made the Kepler conjecture so hard to prove. The most important part of this statement is irregular arrangement of spheres would be denser but any attempt to extend this structure to fill a larger volume always reduces its density.

After Gauss no further progress happened till 19^{th} century. In 1953 L Fejes Toth proved a correlation between Kepler conjecture and Newton-Gregory Kissing problem in n-dimensions. In 1993 International Journal of Mathematics published a proof by Wu-Yi Hsiang. This incredible proof considered only closed neighbors and used mainly trigonometry. But Gabor

Fejes Toth (son of L Fejes Toth) and Thomas Hales claimed individually that many statements in Hsiang's final proof have no acceptable proofs. Thomas Hales later followed the G Fejes Toth strategy and in 1994 published his own proof of Kepler conjecture. Then Wu-Yi Hsiang rejoined him and in 2000 Thomas Hales wrote in one of his another expository article "Cannonballs and Honeycombs":

"A jury of twelve referees has been deliberating on the proof since September 1998."

It was a tough job and after the controversies of Hsiang's proof they had to be careful. The work consisted of six papers including the thesis of Hales's Ph.D. student Samuel Ferguson. At that stage it was consisted of 250 pages of notes and 3 gigabytes of computer programs, data and results. The main problem was computation complexity as many cases in that proof would have required more computation time than the age of the universe. After so many hard work referees gave up and at the Joint Mathematics Meeting in Baltimore in January 2003, Hales received the Chauvenet Prize of the Mathematical Association of America for his unproved extra-ordinary proof. The jury board of Mathematics Association declared in the meeting that they were unable to judge his proof.

Thomas Hales could not accept this five years delay of the jury board and rather he started to prove his own proof by himself. He launched a worldwide cooperative project called "**Flyspeck**" (based on the letters FPK, for "Formal Proof of Kepler") to establish a verification of his own proof by computer analysis. He estimated it would take around 20 years of personal-work to complete the proof. In 2005 Hales published a 100 page paper describing the non-computer part of his proof. The project announced completed on August 10, 2014 **[5]**. In January 2015, Thomas Hales and 21 collaborators published "A formal proof of Kepler conjecture" **[6]**. Thus one of the oldest math problems in mathematics is finally proved by the help of the greatest minds in history.

5. Conclusions

It was a scandalous situation as every mathematician believed Kepler conjecture was true since the time of Gauss. All that is missing a final complete proof. The "**Flyspeck**" project not only proved this oldest discrete geometry problem, it has opened a window of computational science to the world. It has made the scientists more fascinated and thirsty for knowledge which is beyond remarkable.

References

[1] http://mathworld.wolfram.com/KeplerConjecture.html [accessed 27th February, 2015]

[2] Thomas C. Hales (2002) An Overview of the Kepler Conjecture, arXiv: math/9811071v2

[3] Frank Morgan (2005) Kepler's Conjecture and Hales's Proof, A book review on AMS

[4] http://en.wikipedia.org/wiki/Kissing_number_problem [accessed 27th February, 2015]

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[6] Thomas C. Hales and 20 collaborators (2015), A formal Proof of the Kepler Conjecture, arXiv: 1501.02155v1