Discrete Probability Problems

Problem 1: Tossing a Coin Twice

A fair coin is tossed twice. Let X be the number of heads observed.

(a) List the sample space and define the probability mass function (PMF) of X.

(b) Find $P(X = 1)$.

(c) Find $P(X \leq 1)$.

Problem 1: Tossing a Coin Twice – Tabular Representation

A fair coin is tossed twice. Let X be the number of heads observed.

Sample Space: $S = \{HH, HT, TH, TT\}$

Definition of Random Variable:

Outcome	**Value of X (Number of Heads)**
HH	2
HT	1
TH	1
TT	0

Probability Mass Function (PMF) Table:

$X = x$	$P(X = x)$
0	$\frac{1}{4}$
1	$\frac{2}{4} = \frac{1}{2}$
2	$\frac{1}{4}$

(b) Compute $P(X = 1)$:

$$P(X = 1) = \frac{1}{2}$$

(c) Compute $P(X \leq 1)$:

$$P(X \leq 1) = P(X = 0) + P(X = 1) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$$

Problem 2: Rolling a Die

A fair six-sided die is rolled once. Let X be the outcome.

(a) Define the probability mass function $f(x)$ of X.

(b) Find $P(X > 4)$.

(c) Find $P(X \text{ is even})$.

Solution to Problem 2: Rolling a Die

A fair six-sided die is rolled once. Let X be the number that appears on the die.

(a) **Probability Mass Function:**
The sample space is:

$$S = \{1, 2, 3, 4, 5, 6\}$$

Since the die is fair, each outcome has an equal probability of $\frac{1}{6}$. The PMF of X is:

$X = x$	$P(X = x)$
1	$\frac{1}{6}$
2	$\frac{1}{6}$
3	$\frac{1}{6}$
4	$\frac{1}{6}$
5	$\frac{1}{6}$
6	$\frac{1}{6}$

(b) **Compute $P(X > 4)$:**
The values greater than 4 are $X = 5$ and $X = 6$, so:

$$P(X > 4) = P(X = 5) + P(X = 6) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$$

(c) **Compute $P(X$ is even$)$:**
The even values are $X = 2, 4, 6$, so:

$$P(X \text{ is even}) = P(X = 2) + P(X = 4) + P(X = 6) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$$

Final Answers:

(a) PMF: $P(X = x) = \frac{1}{6}$ for $x = 1, 2, 3, 4, 5, 6$

(b) $P(X > 4) = \frac{1}{3}$

(c) $P(X \text{ is even}) = \frac{1}{2}$

Problem 3: Tossing a Coin Three Times

A fair coin is tossed three times. Let X be the number of heads observed.

(a) Write the probability mass function for X.

(b) Find $P(X = 2)$.

(c) Find $P(X \geq 1)$.

Solution to Problem 3: Tossing a Coin Three Times

A fair coin is tossed three times. Let X be the number of heads observed.

(a) **Probability Mass Function (PMF):**
The sample space consists of $2^3 = 8$ outcomes:

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

The random variable X (number of heads) takes values $0, 1, 2, 3$.

$X = x$	$P(X = x)$
0	$\frac{1}{8}$ (TTT)
1	$\frac{3}{8}$ (HTT, THT, TTH)
2	$\frac{3}{8}$ (HHT, HTH, THH)
3	$\frac{1}{8}$ (HHH)

(b) **Compute $P(X = 2)$:**

$$P(X = 2) = \frac{3}{8}$$

(c) **Compute $P(X \geq 1)$:**

$$P(X \geq 1) = P(X = 1) + P(X = 2) + P(X = 3) = \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = \frac{7}{8}$$

Final Answers:

(a) PMF:

$$P(X = 0) = \frac{1}{8}, \quad P(X = 1) = \frac{3}{8}, \quad P(X = 2) = \frac{3}{8}, \quad P(X = 3) = \frac{1}{8}$$

(b) $P(X = 2) = \frac{3}{8}$

(c) $P(X \geq 1) = \frac{7}{8}$

Problem 4: Two Dice are Rolled

Two fair dice are rolled simultaneously. Let X be the sum of the two faces.

(a) Write the possible values of X.

(b) Find the probability mass function for X.

(c) Find $P(X = 7)$ and $P(X \leq 5)$.

Solution to Problem 4: Two Dice are Rolled

Two fair six-sided dice are rolled simultaneously. Let X be the sum of the two faces.

(a) **Possible Values of X:**

The smallest possible sum is $1 + 1 = 2$, and the largest is $6 + 6 = 12$. So, the possible values of X are:

$$X \in \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

(b) **Probability Mass Function (PMF):**

There are $6 \times 6 = 36$ equally likely outcomes when two dice are rolled.

The number of ways each sum can occur:

Sum X	**Number of Outcomes**
2	1 (1+1)
3	2 (1+2, 2+1)
4	3 (1+3, 2+2, 3+1)
5	4 (1+4, 2+3, 3+2, 4+1)
6	5 (1+5, 2+4, 3+3, 4+2, 5+1)
7	6 (1+6, 2+5, 3+4, 4+3, 5+2, 6+1)
8	5 (2+6, 3+5, 4+4, 5+3, 6+2)
9	4 (3+6, 4+5, 5+4, 6+3)
10	3 (4+6, 5+5, 6+4)
11	2 (5+6, 6+5)
12	1 (6+6)

Therefore, the PMF is:

$X = x$	$P(X = x)$
2	$\frac{1}{36}$
3	$\frac{2}{36}$
4	$\frac{3}{36}$
5	$\frac{4}{36}$
6	$\frac{5}{36}$
7	$\frac{6}{36}$
8	$\frac{5}{36}$
9	$\frac{4}{36}$
10	$\frac{3}{36}$
11	$\frac{2}{36}$
12	$\frac{1}{36}$

(c) **Compute $P(X = 7)$ and $P(X \leq 5)$:**

$$P(X = 7) = \frac{6}{36} = \frac{1}{6}$$

$$P(X \leq 5) = P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) = \frac{1 + 2 + 3 + 4}{36} = \frac{10}{36} = \frac{5}{18}$$

Final Answers:

(a) Possible values of X: $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

(b) PMF as given in the table above.

(c) $P(X = 7) = \frac{1}{6}$, $\quad P(X \leq 5) = \frac{5}{18}$

Problem: Verify a Probability Mass Function

Let a discrete random variable X take the values $x = 1, 2, 3, 4$ with the probability function defined as:

$$f(x) = \frac{x}{10}$$

(a) Show that $f(x) \geq 0$ for all $x \in \{1, 2, 3, 4\}$.

(b) Show that $\sum_{x=1}^{4} f(x) = 1$.

(c) Conclude whether $f(x)$ is a valid probability mass function.

Solution: Verify a Probability Mass Function

Let the function be:

$$f(x) = \frac{x}{10}, \quad \text{for } x \in \{1, 2, 3, 4\}$$

We verify whether this function is a valid probability mass function (PMF).

(a) **Non-negativity:**

We check that $f(x) \geq 0$ for all values in the domain:

$$f(1) = \frac{1}{10}, \quad f(2) = \frac{2}{10}, \quad f(3) = \frac{3}{10}, \quad f(4) = \frac{4}{10}$$

Since all values are non-negative, this condition is satisfied.

(b) **Sum of Probabilities:**

We compute the total sum:

$$\sum_{x=1}^{4} f(x) = \frac{1}{10} + \frac{2}{10} + \frac{3}{10} + \frac{4}{10} = \frac{1+2+3+4}{10} = \frac{10}{10} = 1$$

Hence, the total probability is 1.

(c) **Conclusion:**

Since:

- $f(x) \geq 0$ for all $x \in \{1, 2, 3, 4\}$, and
- $\sum f(x) = 1$,

we conclude that $f(x) = \frac{x}{10}$ is a valid probability mass function (PMF) for the random variable X.

Similar Problems to Example 2.5

Problem 1. Let the function

$$f(x) = \begin{cases} c(4 - x^2), & -2 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the value of c such that $f(x)$ is a probability density function.

(b) Compute $P(-1 < X < 1)$.

Problem 2. Let

$$f(x) = \begin{cases} cxe^{-x}, & 0 < x < 5 \\ 0, & \text{otherwise} \end{cases}$$

(a) Determine the constant c so that $f(x)$ is a valid PDF.

(b) Calculate the probability $P(1 \leq X \leq 3)$.

Problem 3. Let

$$f(x) = \begin{cases} cx^3, & 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the value of c that makes $f(x)$ a probability density function.

(b) Find $P(0.5 < X < 1.5)$.

Solution to Problem 1

Let the function

$$f(x) = \begin{cases} c(4 - x^2), & -2 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

(a) To find the constant c such that $f(x)$ is a probability density function, we must ensure that:

$$\int_{-\infty}^{\infty} f(x)\, dx = 1.$$

Since $f(x) = 0$ outside $(-2, 2)$, we compute:

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-2}^{2} c(4 - x^2)\, dx = c \int_{-2}^{2} (4 - x^2)\, dx.$$

We compute the integral:

$$\int_{-2}^{2} (4 - x^2)\, dx = \int_{-2}^{2} 4\, dx - \int_{-2}^{2} x^2\, dx.$$

$$\int_{-2}^{2} 4\, dx = 4 \cdot (2 - (-2)) = 4 \cdot 4 = 16.$$

$$\int_{-2}^{2} x^2\, dx = 2 \int_{0}^{2} x^2\, dx = 2 \cdot \left[\frac{x^3}{3}\right]_0^2 = 2 \cdot \frac{8}{3} = \frac{16}{3}.$$

So,

$$\int_{-2}^{2} (4 - x^2)\, dx = 16 - \frac{16}{3} = \frac{48}{3} - \frac{16}{3} = \frac{32}{3}.$$

Therefore,

$$\int_{-\infty}^{\infty} f(x)\,dx = c \cdot \frac{32}{3} = 1 \Rightarrow c = \frac{3}{32}.$$

(b) Now compute the probability $P(-1 < X < 1)$. We use:

$$P(-1 < X < 1) = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} \frac{3}{32}(4 - x^2)\,dx.$$

Factor out the constant:

$$= \frac{3}{32}\int_{-1}^{1}(4 - x^2)\,dx = \frac{3}{32}\left(\int_{-1}^{1} 4\,dx - \int_{-1}^{1} x^2\,dx\right).$$

Compute each integral:

$$\int_{-1}^{1} 4\,dx = 4 \cdot (1 - (-1)) = 4 \cdot 2 = 8,$$

$$\int_{-1}^{1} x^2\,dx = 2\int_{0}^{1} x^2\,dx = 2 \cdot \left[\frac{x^3}{3}\right]_0^1 = 2 \cdot \frac{1}{3} = \frac{2}{3}.$$

So,

$$P(-1 < X < 1) = \frac{3}{32}\left(8 - \frac{2}{3}\right) = \frac{3}{32} \cdot \left(\frac{24 - 2}{3}\right) = \frac{3}{32} \cdot \frac{22}{3} = \frac{22}{32} = \frac{11}{16}.$$

Final Answers:

(a) $c = \frac{3}{32}$

(b) $P(-1 < X < 1) = \frac{11}{16}$

Solution to Problem 2

Let

$$f(x) = \begin{cases} cxe^{-x}, & 0 < x < 5 \\ 0, & \text{otherwise} \end{cases}$$

(a) To determine the constant c so that $f(x)$ is a probability density function, we require:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

Since $f(x) = 0$ outside $(0, 5)$, we compute:

$$\int_{0}^{5} cxe^{-x}\,dx = c\int_{0}^{5} xe^{-x}\,dx.$$

We compute $\int_0^5 xe^{-x}\,dx$ using integration by parts:
Let $u = x \Rightarrow du = dx, \quad dv = e^{-x}dx \Rightarrow v = -e^{-x}$.
Then,

$$\int xe^{-x}dx = -xe^{-x} + \int e^{-x}dx = -xe^{-x} - e^{-x} + C = -(x+1)e^{-x} + C.$$

Now evaluate:

$$\int_0^5 xe^{-x}\,dx = \left[-(x+1)e^{-x}\right]_0^5 = -(6e^{-5}) + (1e^0) = 1 - 6e^{-5}.$$

Therefore,

$$\int_0^5 cxe^{-x}dx = c(1 - 6e^{-5}) = 1 \Rightarrow c = \frac{1}{1 - 6e^{-5}}.$$

(b) To calculate the probability $P(1 \le X \le 3)$, we evaluate:

$$P(1 \le X \le 3) = \int_1^3 f(x)dx = c\int_1^3 xe^{-x}dx.$$

Again using the known integral:

$$\int xe^{-x}dx = -(x+1)e^{-x}.$$

So,

$$\int_1^3 xe^{-x}dx = \left[-(x+1)e^{-x}\right]_1^3 = -(4e^{-3}) + (2e^{-1}) = 2e^{-1} - 4e^{-3}.$$

Hence,

$$P(1 \le X \le 3) = c\left(2e^{-1} - 4e^{-3}\right).$$

Substitute $c = \dfrac{1}{1 - 6e^{-5}}$ to get the final answer.

Final Answers:

(a) $c = \dfrac{1}{1 - 6e^{-5}}$

(b) $P(1 \le X \le 3) = \dfrac{2e^{-1} - 4e^{-3}}{1 - 6e^{-5}}$

Solution to Problem 3

Let

$$f(x) = \begin{cases} cx^3, & 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

(a) To determine the constant c so that $f(x)$ is a probability density function, we require:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

Since $f(x) = 0$ outside $(0, 2)$, we compute:

$$\int_0^2 cx^3\,dx = c\int_0^2 x^3\,dx.$$

Compute the integral:

$$\int_0^2 x^3\,dx = \left[\frac{x^4}{4}\right]_0^2 = \frac{2^4}{4} = \frac{16}{4} = 4.$$

So:

$$c \cdot 4 = 1 \Rightarrow c = \frac{1}{4}.$$

(b) To find $P(0.5 < X < 1.5)$, we compute:

$$P(0.5 < X < 1.5) = \int_{0.5}^{1.5} f(x)\,dx = \int_{0.5}^{1.5} \frac{1}{4}x^3\,dx = \frac{1}{4}\int_{0.5}^{1.5} x^3\,dx.$$

Compute the integral:

$$\int_{0.5}^{1.5} x^3\,dx = \left[\frac{x^4}{4}\right]_{0.5}^{1.5} = \frac{1.5^4}{4} - \frac{0.5^4}{4} = \frac{(5.0625 - 0.0625)}{4} = \frac{5.0}{4} = 1.25.$$

So:

$$P(0.5 < X < 1.5) = \frac{1}{4} \cdot 1.25 = 0.3125.$$

Final Answers:

(a) $c = \frac{1}{4}$

(b) $P(0.5 < X < 1.5) = \frac{5}{16} = 0.3125$

Problem and Solution related to probability density function similar to Example 2.6

Let

$$f(x) = \begin{cases} k(4-x), & 0 \le x \le 4 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the value of k and the distribution function $F(x)$

We first determine k such that the total probability is 1:

$$\int_0^4 k(4-x)\,dx = 1 \Rightarrow k\int_0^4 (4-x)\,dx = 1$$

$$\int_0^4 (4-x)\,dx = \left[4x - \frac{x^2}{2}\right]_0^4 = 16 - 8 = 8 \Rightarrow k \cdot 8 = 1 \Rightarrow k = \frac{1}{8}$$

Now compute the cumulative distribution function $F(x)$:

$$F(x) = \int_0^x \frac{1}{8}(4-t)dt = \frac{1}{8}\left[4t - \frac{t^2}{2}\right]_0^x = \frac{1}{8}\left(4x - \frac{x^2}{2}\right)$$

So:

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{8}\left(4x - \frac{x^2}{2}\right), & 0 \le x \le 4 \\ 1, & x > 4 \end{cases}$$

(b) Compute $P(1 \leq X \leq 3)$ using $F(x)$

$$P(1 \leq X \leq 3) = F(3) - F(1) = \frac{1}{8}\left(4 \cdot 3 - \frac{9}{2}\right) - \frac{1}{8}\left(4 \cdot 1 - \frac{1}{2}\right) = \frac{1}{8}\left(12 - 4.5 - (4 - 0.5)\right) = \frac{1}{8}(7.5 - 3.5) = \frac{4}{8} = \frac{1}{2}$$

Answer: $P(1 \leq X \leq 3) = \frac{1}{2}$

Problem and Solution related to probability density function similar to Example 2.6

Let

$$f(x) = \begin{cases} kx^2, & 0 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the value of k and the distribution function $F(x)$

We compute:

$$\int_0^2 kx^2\,dx = 1 \Rightarrow k\int_0^2 x^2\,dx = 1 \Rightarrow k \cdot \left[\frac{x^3}{3}\right]_0^2 = k \cdot \frac{8}{3} = 1 \Rightarrow k = \frac{3}{8}$$

Now find $F(x) = \int_0^x \frac{3}{8}t^2 dt = \frac{3}{8} \cdot \left[\frac{t^3}{3}\right]_0^x = \frac{x^3}{8}$

So:

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{x^3}{8}, & 0 \leq x \leq 2 \\ 1, & x > 2 \end{cases}$$

(b) Compute $P(0.5 \leq X \leq 1.5)$

$$P(0.5 \leq X \leq 1.5) = F(1.5) - F(0.5) = \frac{(1.5)^3}{8} - \frac{(0.5)^3}{8} = \frac{3.375 - 0.125}{8} = \frac{3.25}{8} = \frac{13}{32}$$

Answer: $P(0.5 \leq X \leq 1.5) = \frac{13}{32}$