TEST 6

úloha 1

$A = 28, \quad B = 6$

$$M = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 3 \end{pmatrix}$$

a) $V = \{\bar{v} \in \mathbb{R}^3 : \bar{v} \perp \bar{x}\}, \quad \bar{x} = (28, -6, 28)$

$$\bar{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \qquad \langle \bar{v} | \bar{x} \rangle = v^T M \cdot \bar{x}$$

$$\langle \bar{v}, \bar{x} \rangle = (v_1\ v_2\ v_3) \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 3 \end{pmatrix} \begin{pmatrix} 28 \\ -6 \\ 28 \end{pmatrix} =$$

$$= (v_1 - v_3,\ 2v_2 - v_3,\ -v_1 - v_2 + 3v_3) \begin{pmatrix} 28 \\ -6 \\ 28 \end{pmatrix} =$$

$$= 28(v_1 - v_3) - 6(2v_2 - v_3) + 28(-v_1 - v_2 + 3v_3) =$$

$$= \cancel{28v_1} - 28v_3 - 12v_2 + 6v_3 \cancel{- 28v_1} - 28v_2 + 84v_3 =$$

$$= -40v_2 + 62v_3$$

$\bar{v} \perp \bar{x} \Rightarrow \langle \bar{v}, \bar{x} \rangle = 0.$

$$-40v_2 + 62v_3 = 0$$

$$62v_3 = 40v_2$$

$$v_3 = \frac{40}{62} v_2 \qquad v = \begin{pmatrix} v_1 \\ v_2 \\ \frac{20}{31} v_2 \end{pmatrix} \qquad v_1, v_2 \in \mathbb{R}$$

b) $\|\bar{v}\| = 1$

$$\langle \bar{v}, v \rangle = 1 \Rightarrow \left\langle \begin{pmatrix} v_1 \\ v_2 \\ \frac{20}{31} v_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ \frac{20}{31} v_2 \end{pmatrix} \right\rangle =$$

$$= \left(v_1, v_2, \frac{20}{31} v_2\right) \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \frac{20 v_2}{31} \end{pmatrix} =$$

$$= \left(v_1 + \frac{20}{31} v_2,\ 2v_2 - \frac{20}{31} v_2,\ -v_1 - v_2 + \frac{60}{31} v_2\right) \begin{pmatrix} v_1 \\ v_2 \\ \frac{20 v_2}{31} \end{pmatrix} =$$

$$= v_1 \left(v_1 + \frac{20}{31} v_2\right) + \frac{42}{31} v_2^2 + \left(-v_1 - v_2 + \frac{60}{31} v_2\right) \cdot \frac{20 v_2}{31} =$$

$$\Rightarrow v_1\left(v_1-\frac{20v_2}{31}\right)+\frac{42v_2^2}{31}+\frac{20}{31}v_2\left(-v_1+\frac{29v_2}{31}\right)\Rightarrow$$

$$\Rightarrow v_1^2-\frac{20}{31}v_1v_2+\frac{42v_2^2}{31}-\frac{20}{31}v_1v_2+\frac{20\cdot 29v_2^2}{31\cdot 31}\Rightarrow$$

$$\Rightarrow v_1^2-\frac{40}{31}v_1v_2+\frac{1882v_2^2}{31\cdot 31};\quad \langle\bar v,\bar v\rangle=1\Rightarrow$$

$$v_1^2-\frac{40}{31}v_1v_2+\frac{1882v_2^2}{961}-1=0$$

$$v_1=\frac{+\frac{40}{31}v_2\pm\sqrt{\left(\frac{40}{31}\right)^2v_2^2-4\left(\frac{1882}{961}v_2^2-1\right)}}{2}=$$

$$=\frac{\frac{40}{31}v_2\pm\sqrt{\left(\frac{40}{31}\right)^2v_2-\frac{4\cdot 1882v_2^2+961\cdot 4}{961}}}{2}=$$

$$=\frac{\frac{40}{31}v_2\pm\frac{2}{31}\sqrt{400v_2^2-4882v_2^2+961}}{2}=\frac{20}{31}v_2\pm\frac{1}{31}\sqrt{961-1482v_2}$$

$$v_1=\frac{1}{31}\left(20v_2\pm\sqrt{961-1482v_2}\right)\qquad 961-1482v_2\geq 0,\quad v_2\leq\left|\frac{961}{1482}\right|$$

$$\left\{\begin{pmatrix}\frac{1}{31}\left(20v_2\pm\sqrt{961-1482v_2}\right)\\ v_2\\ \frac{20}{31}v_2\end{pmatrix},\quad v_2\leq\left|\frac{961}{1482}\right|,\ \|v\|=1\right\}$$

$$\vec v=\begin{pmatrix}v_1\\ v_2\\ \frac{20}{31}v_2\end{pmatrix}$$

Necht

a) $\left.\begin{matrix}v_1=1\\ v_2=31\end{matrix}\right\}\Rightarrow \bar v_1=\begin{pmatrix}1\\31\\20\end{pmatrix}$

b) $\left.\begin{matrix}v_1=2\\ v_2=62\end{matrix}\right\}\Rightarrow \bar v_1=\begin{pmatrix}2\\62\\40\end{pmatrix}$

c) $\left.\begin{matrix}v_1=v_1\\ v_2=0.\end{matrix}\right\}\Rightarrow \begin{pmatrix}v_1\\0\\0\end{pmatrix}$

$\langle \bar v_1,\bar v_1\rangle = \langle 1,31,20$

d) $\begin{matrix}v_1=0\\ v_2=31\end{matrix}$

(2

a) $\langle \bar{v}_1, \bar{v}_1 \rangle = (1, 31, 20) \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 31 \\ 20 \end{pmatrix} =$

$= (-19, 42, 28) \begin{pmatrix} 1 \\ 31 \\ 20 \end{pmatrix} = -19 + 42 \cdot 31 + 28 \cdot 20 = 1843$

$\| \bar{v}_1 \| = \sqrt{1843}$

$\bar{v}_1 = \begin{pmatrix} \frac{1}{\sqrt{1843}} \\ \frac{31}{\sqrt{1843}} \\ \frac{20}{\sqrt{1843}} \end{pmatrix}$

b) $\langle \bar{v}_2, \bar{v}_2 \rangle = (2, 62, 40) \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 62 \\ 40 \end{pmatrix} = (-38, 84, 56) \begin{pmatrix} 2 \\ 62 \\ 40 \end{pmatrix} = 7372.$

$\bar{v}_2 = \frac{1}{\sqrt{7372}} \begin{pmatrix} 2 \\ 62 \\ 40 \end{pmatrix}$, v_1 je LZ $\bar{v}_2$!

c) $\langle \bar{v}_3, \bar{v}_3 \rangle = (5, 0, 0) \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 3 \end{pmatrix} \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} = 25$

$v_3 = \frac{1}{5} \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$

d) $\langle \bar{u}_4, \bar{u}_4 \rangle = (0, 31, 20) \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 31 \\ 20 \end{pmatrix} = 1882$

$\bar{v}_4 = \frac{1}{\sqrt{1882}} \begin{pmatrix} 0 \\ 31 \\ 20 \end{pmatrix}$

úloha 1

úloha 2

$\| \cdot \|$ norma

$\langle \cdot | \cdot \rangle$ - sk. súčin

V – vek. prost. nad $\mathbb{R}$

$\bar{x}, \bar{y} \in V$

$\bar{u} = \bar{x}$

$\bar{v} = \bar{y}$

$\| \bar{x} \| = 28, \quad \| \bar{x} + \bar{y} \| = 6, \quad \| \bar{x} - \bar{y} \| = \sqrt{4 \cdot 28^2 + 6^2} = \sqrt{3172}$

$\| \bar{y} \|$ - ?

$\|\bar{x}\| = \sqrt{\langle \bar{x}, \bar{x} \rangle} = 28 \Rightarrow \langle \bar{x}, \bar{x} \rangle = 28^2 \quad / \ \langle \bar{x}, \bar{x} \rangle > 0$ (

$\|\bar{x}+\bar{y}\| = \sqrt{\langle \bar{x}+\bar{y}, \bar{x}+\bar{y} \rangle} = 6 \Rightarrow \langle \bar{x}+\bar{y}, \bar{x}+\bar{y} \rangle = 36$ $\qquad \langle \bar{x}, \bar{x} \rangle > 0$ (pozitívna definitnosť)

$\|\bar{x}-\bar{y}\| = \sqrt{\langle \bar{x}-\bar{y}, \bar{x}-\bar{y} \rangle} = \sqrt{3172} \Rightarrow \langle \bar{x}-\bar{y}, \bar{x}-\bar{y} \rangle = 3172$

a) $\langle \bar{x}+\bar{y}, \bar{x}+\bar{y} \rangle = 36$ (linearita v prvom arg) 2b.

$\langle \bar{x}, \bar{x}+\bar{y} \rangle + \langle \bar{y}, \bar{x}+\bar{y} \rangle = 36$ (1.)

$\langle \bar{x}+\bar{y}, \bar{x} \rangle + \langle \bar{x}+\bar{y}, \bar{y} \rangle = 36$ (2b)

$\langle x, x \rangle + \langle \bar{y}, \bar{x} \rangle + \langle \bar{x}, \bar{y} \rangle + \langle \bar{y}, \bar{y} \rangle = 36$ (1)

$\langle \bar{x}, \bar{x} \rangle + 2\langle \bar{x}, \bar{y} \rangle + \langle \bar{y}, \bar{y} \rangle = 36$ (*)

b) $\langle \bar{x}-\bar{y}, \bar{x}-\bar{y} \rangle = 3172$ (2b)

$\langle \bar{x}, \bar{x}-\bar{y} \rangle + \langle -\bar{y}, \bar{x}-\bar{y} \rangle = 3172$ (2a) $\wedge$ (1)

$\langle \bar{x}-\bar{y}, \bar{x} \rangle - \langle \bar{x}-\bar{y}, \bar{y} \rangle = 3172$ (2b)

$\langle \bar{x}, \bar{x} \rangle + \langle -\bar{y}, \bar{x} \rangle - \langle \bar{x}, \bar{y} \rangle - \langle -\bar{y}, \bar{y} \rangle = 3172$ (2a)

$\langle \bar{x}, \bar{x} \rangle - \langle \bar{y}, \bar{x} \rangle - \langle \bar{x}, \bar{y} \rangle + \langle \bar{y}, \bar{y} \rangle = 3172$

$\langle \bar{x}, \bar{x} \rangle - 2\langle \bar{y}, \bar{x} \rangle + \langle \bar{y}, \bar{y} \rangle = 3172$ (**)

$(*) + (**) \Rightarrow 2\langle \bar{x}, \bar{x} \rangle + 2\langle \bar{y}, \bar{y} \rangle = 3172 + 36$

$\langle \bar{x}, \bar{x} \rangle + \langle \bar{y}, \bar{y} \rangle = 1604$

$\langle \bar{y}, \bar{y} \rangle = 1604 - \langle \bar{x}, \bar{x} \rangle$

$\langle \bar{y}, \bar{y} \rangle = 1604 - 784$

$\langle \bar{y}, \bar{y} \rangle = 820$

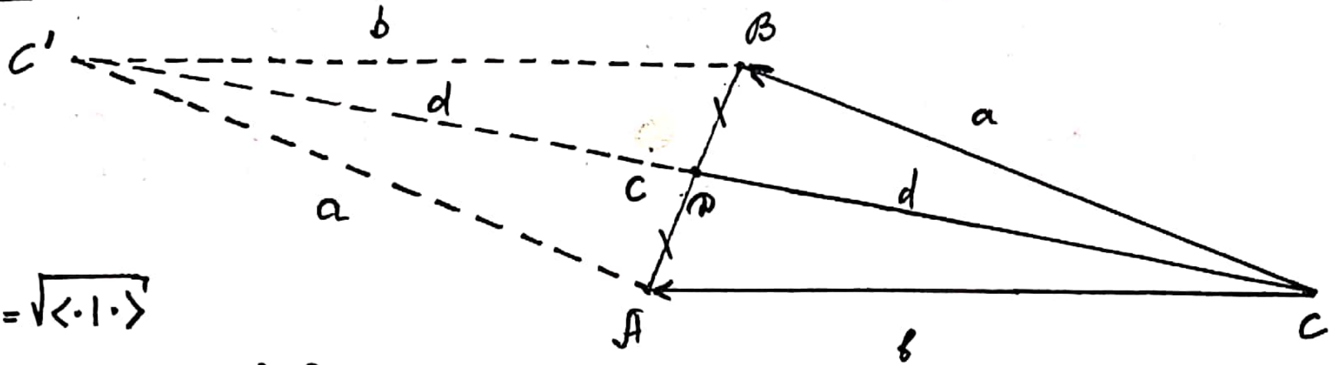
$\|\bar{y}\| = \sqrt{\langle \bar{y}, \bar{y} \rangle} = \sqrt{820} = 2\sqrt{205}$

$\|\bar{y}\| = 2\sqrt{205}$

úloha 2.

Bonusové úlohy

B.10.



$\|\cdot\| = \sqrt{\langle \cdot | \cdot \rangle}$

$a^2+b^2 = 2d^2+\frac{c^2}{2}$?

$AC \parallel C'B \Rightarrow \|AC\| = \|C'B\| = b$

$CB \parallel C'A \Rightarrow \|CB\| = \|C'B\| = a$

$\|CD\| = \|CC'\| = d$.

1. $\bar{CB} + \bar{BC'} = CC' \Rightarrow \bar{CC'} = 2\bar{CD} \Rightarrow 2\bar{CD} = \bar{CB} + BC'$

Necht $\bar{CD} = \bar{d}$
$CB = \bar{a}$
$BC' \parallel AC \wedge \|BC'\| = \|AC\|$
$BC' = \bar{b}'$

$2\bar{d} = \bar{a} + \bar{b}'$

$\langle 2\bar{d}, 2\bar{d} \rangle = \langle a+b', a+b' \rangle$

$4d^2 = \langle a, a+b' \rangle + \langle b', a+b' \rangle$

$4d^2 = \langle a+b', a \rangle + \langle a+b', b' \rangle$

$4d^2 = \langle a,a \rangle + \langle b',a \rangle + \langle a, b' \rangle + \langle b', b' \rangle$

$4d^2 = a^2 + \langle a, b' \rangle + \langle a, b' \rangle + b'^2$

$4d^2 = a^2 + 2\langle a, b' \rangle + b'^2$ $\quad$ / $\|\bar{b}'\| = \|\bar{b}\|$

$4d^2 = a^2 + 2\langle a, b \rangle + b^2$ $(*)$

$\vec{c} = \bar{a} - \bar{b} \Rightarrow \langle c, c \rangle = \langle \bar{a}-b, a-\bar{b} \rangle \Rightarrow c^2 = \langle \bar{a}, \bar{a}-\bar{b} \rangle + \langle -\bar{b}, \bar{a}-\bar{b} \rangle \Rightarrow$

$\Rightarrow c^2 = \langle \bar{a}-\bar{b}, \bar{a} \rangle - \langle \bar{b}, \bar{a}-\bar{b} \rangle = \ldots \Rightarrow c^2 = \langle \bar{a}, \bar{a} \rangle - 2\langle a, b \rangle + \langle b, b \rangle =$

$\Rightarrow c^2 = a^2 - 2\langle a, b \rangle + b^2 \Rightarrow 2\langle a, b \rangle = a^2+b^2-c^2$ $(**) \rightarrow (*)$

$4d^2 = a^2 + (a^2+b^2-c^2) + b^2$

$4d^2 = 2a^2 + 2b^2 - c^2 \;/ :2 \Rightarrow \boxed{2d^2 + \frac{c^2}{2} = a^2 + b^2}$

B10.

B.11. $P_0(x), P_1(x), P_2(x) \dots$ – Legendrove polynómy.

$$(n+1)P_{n+1}(x) - (2n+1)x P_n(x) + n P_{n-1}(x) = 0 \quad !$$

$$I_n = \int_{-1}^{1} P_n^2(x)\,dx \quad -? \qquad \|\cdot\|^2 = \langle \cdot, \cdot \rangle$$

$$P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x)$$

$$n+1 \to n$$

$$\boxed{P_n(x) = \frac{2n-1}{n} x P_{n-1}(x) - \frac{n-1}{n} P_{n-2}(x)} \quad (*)$$

$$\int_{-1}^{1} P_n^2(x)\,dx = \|P_n(x)\|^2 = \overset{*}{=}$$

$$= \int_{-1}^{1} P_n \cdot \left(\frac{2n-1}{n} x P_{n-1}(x) - \frac{n-1}{n} P_{n-2}(x)\right) dx$$

$$= \frac{1}{n} \int_{-1}^{1} P_n(x)(2n-1) x P_{n-1}(x) - (n-1) \underbrace{P_n(x) P_{n-2}(x)\,dx}_{0} =$$

0 ($P_n(x)$ a $P_{n-2}(x)$ ortogonalní podle zadaní)

$$= \frac{1}{n} \int_{-1}^{1} P_n(x) \cdot (2n-1) x P_{n-1}(x)\,dx = \frac{2n-1}{n} \int_{-1}^{1} \underbrace{P_n(x) \cdot x}_{(**)} \cdot P_{n-1}(x)\,dx =$$

Podle ! $$\boxed{x \cdot P_n(x) = \frac{n+1}{2n+1} P_{n+1}(x) + \frac{n}{2n+1} P_{n-1}(x)} \quad (**)$$

$$= \frac{2n-1}{n} \int_{-1}^{1} \left(\frac{n+1}{2n+1} P_{n+1}(x) + \frac{n}{2n+1} P_{n-1}(x)\right) P_{n-1}(x)\,dx =$$

$$= \frac{2n-1}{n(2n+1)} \int_{-1}^{1} \Big((n+1) \underbrace{P_{n+1}(x) \cdot P_{(n-1)}(x)}_{=0 \text{ (ortogonalní)}} + n P_{n-1}(x) P_{n-1}(x)\Big)\,dx =$$

$$\frac{\cancel{n}(2n-1)}{\cancel{n}(2n+1)} \int_{-1}^{1} P_{n-1}^2(x)\,dx = \frac{2n-1}{2n+1} \|P_{n-1}(x)\|^2$$

$$\|P_n(x)\|^2 = \frac{2n-1}{2n+1}\,\|P_{n-1}(x)\|^2$$

$2(n-k)-1 =$

$\cancel{2n - (2k+1) = 2n - 2k + 1 = 2(n-k)+1}$

$$\|P_n(x)\|^2 = \frac{2n-1}{2n+1}\cdot\frac{2n-3}{2n-1}\,\|P_{n-2}(x)\|^2$$

$$= \frac{2n-1}{2n+1}\cdot\frac{2n-3}{2n-1}\cdot\frac{2n-5}{2n-3}\,\|P_{n-3}(x)\|^2$$

$2n - (2k+1)$

$2(n-k)+1$

$$= \frac{\cancel{2n-1}}{2n+1}\cdot\frac{\cancel{2n-3}}{\cancel{2n-1}}\cdot\frac{\cancel{2n-5}}{\cancel{2n-3}}\cdots\frac{2(n-k)+1}{2(n-k)+3}\cdots\frac{\cancel{5}\cdot\cancel{3}\cdot 1}{\cancel{5}\cdot\cancel{3}}\,\|P_0(x)\|^2$$

$$\boxed{\|P_n(x)\|^2 = \int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}}$$

$\|P_0(x)\|^2 = 2$

$\int_{-1}^{1} 1\cdot 1\,dx = [x]_{-1}^{1} = 1-(-1) = 2$

to byla výzva!