Fomenko Dmitriy

1

I.

a) Nechť P, Q jsou vektorové prostory nad T, a nechť $\mathcal{A} \in \mathcal{L}(P, Q)$, a nechť $\dim P$ je konečná. Pak

$$\dim P = \dim(\ker \mathcal{A}) + \dim(\operatorname{Im}(A))$$

б) $B: \mathbb{R}^3 \to \mathbb{R}^3 : B(x_1, x_2, x_3) = (x_1 + x_2 + x_3, x_1 - x_2, -x_1 + x_2)$

$\ker(B) = \{\overline{x} \in \mathbb{R}^3 : f(\overline{x}) = \vec{0}\} = \{ \left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right\} \}$

$$+\left(\begin{array}{ccc} 1 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$\operatorname{Im} f = \{ f(\overline{x}) : \overline{x} = \mathbb{R}^3 \}$

$? \exists \overline{x} \in \mathbb{R}^3$ zobr. $(a, b, c) \in \mathbb{R}^3$

$$\begin{aligned} x_1 + x_2 + x_3 &= a \\ x_1 - x_2 + 0x_3 &= b \\ -x_1 + x_2 + 0x_3 &= c \end{aligned} \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 1 & -1 & 0 & b \\ -1 & 1 & 0 & c \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 1 & -1 & 0 & b \\ 0 & 0 & 0 & b+c \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & -2 & -1 & b-a \\ 0 & 0 & 0 & b+c \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & b \\ 0 & -2 & -1 & b-a \\ 0 & 0 & 0 & b+c \end{array}\right)$$

$\longrightarrow$

$$Im\, f = \{ f(\bar{x}) : \bar{x} \in \mathbb{R}^3 \}$$

$$? \exists \bar{x} \in \mathbb{R}^3 \quad \text{тоб} \quad (a, b, c) \in \mathbb{R}^3$$

$$x_1 + x_2 + x_3 = \alpha$$
$$x_1 - x_2 + 0x_3 = \beta$$
$$-x + x_2 + 0x_3 = \gamma$$

$$\begin{array}{ccc} ① & ② & ③ \end{array}$$
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & \alpha \\ 1 & -1 & 0 & \beta \\ -1 & 1 & 0 & \gamma \end{array}\right) \sim$$

$$\begin{array}{ccc} ③ & ① & ② \end{array}$$
$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & \alpha \\ 0 & 1 & -1 & \beta \\ 0 & -1 & 1 & \gamma \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & \alpha \\ 0 & 1 & -1 & \beta \\ 0 & 0 & 0 & \gamma+\beta \end{array}\right)$$

$$\left(\begin{pmatrix} \alpha+\beta \\ 0 \\ -\beta \end{pmatrix} \begin{pmatrix} \alpha-\beta \\ \beta \\ 0 \end{pmatrix} \right) \qquad \alpha = 1 \quad \beta = 1$$

$$\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1\cdot 0 + (-1)\cdot 0 + 2\cdot 0 = 0$$

$$\left(\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) \quad \left\langle \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \middle| \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle = 2\cdot 0 + 0\cdot 1 + 0(-1) = 0$$

je Ortogon. baze

$$\sqrt{2^2 + (-1)^2} = \sqrt{5}$$

$$\chi = \left(\frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) \text{ – ortonormalna baze}$$

$$\chi = \left(\frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) \text{ – ortonorm. baze}$$

Об. Ортог. база $\ker(B)$.. $\ker(B) = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

$$\sqrt{1^2+1^2+(-2)^2} = \sqrt{2+4} = \sqrt{6}$$

Об. база $= \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$.

Fomenko D. 2

II $\forall \alpha \in \mathbb{R} \quad A_\alpha : \mathbb{R}^3 \to \mathbb{R}^3$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & \alpha & 0 \\ 1 & 2 & 1 \end{pmatrix} \qquad ?\alpha \text{ je diag.}$$

$$\det \begin{pmatrix} 1-\lambda & 2 & 1 \\ 0 & \alpha-\lambda & 0 \\ 1 & 2 & 1-\lambda \end{pmatrix} = (\alpha-\lambda) \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} =$$

$$= (\alpha-\lambda)\left((1-\lambda)^2 - 1\right) = \cancel{(\alpha-\lambda)(1-2\lambda}$$

$$= (\alpha-\lambda)\left(((1-\lambda)-1)(1-\lambda+1)\right) = (\alpha-\lambda)(-\lambda)(2-\lambda)$$

$$\sigma(A) = \{0, 2, \alpha\}$$

(I) $\alpha = 0 \Rightarrow \nu_a(0) = 2$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \nu_g(0) = 2.$$

$$\nu_a(0) = \nu_g(0) = 2. \quad \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$\alpha = 2 \Rightarrow \nu_a(2) = 2$

$$\begin{pmatrix} -1 & 2 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} -1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 4 & 0 \end{pmatrix} \Rightarrow \nu_g(2) = 1.$$

$\nu_g(2) \neq \nu_a(2)$. $\quad \alpha \neq 2$.

$\alpha = \alpha$

$$\begin{pmatrix} 1-\alpha & 2 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & 1-\alpha \end{pmatrix} \sim \begin{pmatrix} 1-\alpha & 2 & 1 \\ 0 & 0 & 0 \\ \alpha & 0 & -\alpha \end{pmatrix} \Rightarrow \nu_g(\alpha) = \nu_a(\alpha) \begin{pmatrix} 1 \\ -1+\frac{\alpha}{2} \\ 1 \end{pmatrix}$$

Fomenko D.

matic. $\mathcal{A}$ je diagonalizovatelná pro $\forall \alpha \in \mathbb{R} \setminus \{2\}$

$$[\mathcal{A}_\alpha]_{XX} = \begin{pmatrix} -2 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = X$$

$$\alpha = 4 \Rightarrow \begin{pmatrix} 1 \\ -1+\frac{\alpha}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

T je lin. transf.

$T^2 = 2T$

λ je vlast. hodnota $T \Rightarrow \lambda \in \{0, 2\}$

$$T \cdot \bar{x} = \lambda \cdot \bar{x}$$

$$T^2 = 2T$$

$$T^2 - 2T \cancel{} = 0$$

$$\cancel{(\lambda E - 2T)\bar{x} = \lambda x}$$

$$\cancel{(T^2 - 2T)\bar{x}} =$$

$$(T^2 - 2T)\bar{x} = 0$$

$$T^2 \cdot \bar{x} - 2T\bar{x} = 0$$

$$\cancel{T} \; T \cdot \underbrace{T \cdot \bar{x}} - 2 \underbrace{T \cdot \bar{x}} = 0 \quad / \; T \cdot \bar{x} = \lambda \cdot \bar{x} \; /$$

$$\underbrace{T \cdot \lambda\bar{x}} - 2\lambda\bar{x} = 0 \Rightarrow$$

Linearita $\Rightarrow T \cdot \lambda\bar{x} = \lambda T \bar{x}$ / λ je vlast. číslo /

$$\Rightarrow \lambda \underbrace{T\bar{x}}_{\lambda\bar{x}} - 2\lambda\bar{x} = 0$$

$\lambda^2 \bar{x} - 2\lambda\bar{x} = 0$ / aksiom distributivita $\odot$ vzhledem ke sčítání čísel /

$\bar{x} \cdot (\lambda^2 - 2\lambda) = 0$ / $\bar{x}$ je vl. vektor $\neq \vec{0}$ /

$\Rightarrow \lambda^2 - 2\lambda = 0 \Rightarrow \lambda(\lambda - 2) = 0$

$\lambda = 0 \quad \lambda = 2$

$$\boxed{\lambda \in \{0, 2\}}$$

Famenko D. 4

4). Nech V je priestor so $\langle . | . \rangle$.

Nech $S: V \to V$ injektívne ($\forall x,y \Rightarrow$ ~~f(x)~~ $x \neq y \Rightarrow f(x) \neq f(y)$)

$$\langle x, y \rangle_s := \langle Sx, Sy \rangle$$

def. na V iný, nový sk. súčin.

a) dokážeme aks. sk. súčinu

1) $\forall \bar{x}, \bar{y} \in V$. $\langle \bar{x} | \bar{y} \rangle = \langle \bar{y} | \bar{x} \rangle$.

2) $\langle \alpha \bar{x} + \bar{y} | \bar{z} \rangle = \alpha \langle \bar{x} | \bar{z} \rangle + \langle \bar{y} | \bar{z} \rangle$, $\alpha \in T$, $\bar{x}, \bar{y}, \bar{z} \in V$

3) $\langle \bar{x} | \bar{x} \rangle \geq 0$ $\forall x \in V$, a $\langle \bar{x} | \bar{x} \rangle = 0 \Leftrightarrow \bar{x} = 0$.

① $\langle \bar{x} | \bar{y} \rangle_s = \langle S\bar{x} | S\bar{y} \rangle = \langle S\bar{y} | S\bar{z} \rangle = \langle y | z \rangle_s$

$\langle Sx | Sy \rangle \overset{def}{=} \langle Sy | Sx \rangle$
$\langle y | x \rangle_s \overset{def}{=} \langle Sy | Sx \rangle$ $\Big\} \Rightarrow$ sym. platí

② $\langle \alpha \bar{x} + \bar{y} | z \rangle_s = $ ~~α~~ rozdel. na 2 prípady

2.1 $\boxed{\langle \bar{x} + \bar{y} | \bar{z} \rangle_s = \langle S(\bar{x} + \bar{y}) | \bar{z} \rangle = \langle S(\bar{x}) + S(\bar{y}) | S\bar{z} \rangle = \langle S\bar{x} | S\bar{z} \rangle + \langle S\bar{y} | S\bar{z} \rangle = \langle x + z \rangle_s + \langle x | y \rangle_s}$

$\langle S(\bar{x} + \bar{y}) | S\bar{z} \rangle = $ ~~S~~ $\langle x, z \rangle_s + \langle \bar{y} | \bar{z} \rangle_s$

$\langle S(\bar{x}) + S(\bar{y}) | S\bar{z} \rangle = \langle S\bar{x} | S\bar{z} \rangle + \langle S\bar{y} | S\bar{z} \rangle =$

$= \langle \bar{x} | \bar{z} \rangle_s + \langle \bar{y} | \bar{z} \rangle_s$

2.2 $\langle \alpha x, | y \rangle_s$

$$\underline{\langle S(\alpha\bar{x})|Sy\rangle} = \langle \alpha S\bar{x} \mid Sy\rangle = \alpha\langle S\bar{x} \mid S\bar{y}\rangle =$$

$$= \underline{\alpha\langle \bar{x}|\bar{y}\rangle_S}$$

③ nezápornost.

$\langle \bar{x}|\bar{x}\rangle_S = 0$ vždy pro $\bar{x} = \vec{o}$.

$$\langle \bar{x}|\bar{x}\rangle_S = \underbrace{\langle S\bar{x} \mid S\bar{x}\rangle}_{} = 0$$

$\longmapsto$ S je injektivní $\Rightarrow$

z původního skal. součinu $\bar{x} = \vec{o}$ $\qquad S\vec{x} = 0 \Leftrightarrow \vec{x} = \vec{o}$

Fomonko R.

a) Nechť V je vektorový prostor nad T, a $\mathcal{A} \in \mathcal{L}(V)$ Číslo $\underline{\lambda \in T}$ je <u>vlastnou</u> ~~číslom~~ ~~dime~~ <u>hodnotou</u> lin. transformácie, pokud $\exists \bar{x} \in V, \bar{x} \neq \bar{0}$, tak že $\mathcal{A}\bar{x} = \lambda \bar{x}$, vektor $\bar{x}$ je <u>vlastní vektor</u> lineární transformáci ~~príslu~~ příslušným λ.
Množinu vlastnych hodnot nazveme spektrem $\mathcal{A}$.

b) $\mathbb{R}^\infty$ vek. prostor nad $\mathbb{R}$, $(a_i)_{i=1}^\infty$ posl. $\mathbb{R}$.

$\mathcal{L} : \mathbb{R}^\infty \to \mathbb{R}^\infty$

$$\mathcal{L}(a_1, a_2, a_3 \ldots) = (a_2, a_3, \ldots) \quad ?$$

$\mathcal{L}$ je lin. transf pokud

$\mathcal{L}(\bar{x} + \bar{y}) = \mathcal{L}(\bar{x}) + \mathcal{L}(\bar{y}), \ \forall \bar{x}, \bar{y} \in V$

$\mathcal{L}(c \cdot \bar{x}) = c\mathcal{L}(\bar{x})., \ c \in T, \bar{x} \in V.$

• $(a_i)_{i=1}^\infty \quad (b_i)_{i=1}^\infty$

a) $\mathcal{L}((a_1, a_2 \ldots) + (b_1, b_2, \ldots)) = \mathcal{L}((a_1 + b_1), (a_2 + b_2), \ldots)$

$= ((a_2 + b_2), (a_3 + b_3) + \ldots +)$

Fomenko D

$$L(a_1, a_2\, a_3 \ldots) + L(b_1, b_2\, b_3 \ldots) =$$

$$= (a_2 a_3 \ldots) + (b_2\, b_3 \ldots) =$$

$$= \underline{((a_2 + b_2), (a_3 + b_3) \ldots)} \text{ platí}$$

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b) $L(c(a_1, a_2, a_3 \ldots)) = L(ca_1, ca_2\, ca_3 \ldots) =$

$$= \underline{(ca_2, ca_3, ca_4 \ldots)}.$$

$$cL(a_1, a_2\, a_3 \ldots) = c(a_2\, a_3 \ldots) =$$

$$= \underline{(ca_2, ca_3 \ldots)}$$

platí

tudíž L je lin. zobraz.

- a vl. hod. a vl. vektory?

$$L\bar{x} = \lambda \bar{x}$$

$$\underline{L \cdot (a_i)_{i=1}^{\infty}} = \lambda \cdot (a_i)_{i=1}^{\infty}$$

$$\downarrow$$

$$(a_2\, a_3 \ldots \cancel{a_n}) = (\lambda a_1 + \lambda a_2, \lambda a_3 \ldots)$$

R^{∞} -! pokud prostor ~~byl~~ Bych byl konečnorozměrný vl. hod. a vl. vektory neexist.

mněním dimenze vynecháním prvku. ale máme R^{∞}? tak stusím.

Vynechám prvého prvka nemením postupnosť nekonečných ~~prvkov~~.

~~$a_2 \to a_1, a_3 \to a_2$~~. $2 \to 1,\ 3 \to 2 \dots$

$$(\bar{a}_1, \bar{a}_2 \dots) = (\lambda a_1, \lambda a_2, \lambda a_3)$$

$$\left.\begin{array}{l} \bar{a}_1 = \lambda a_1 \\ \bar{a}_2 = \lambda a_2 \end{array}\right. \Rightarrow \left.\begin{array}{l} \lambda = \dfrac{\bar{a}_1}{a_1} = \dfrac{a_2}{a_1} \\ \lambda = \dfrac{\bar{a}_2}{a_2} = \dfrac{a_3}{a_2} \end{array}\right\} \Rightarrow \dfrac{a_{n+1}}{a_n} = \lambda.$$

vlastný vektora ~~je~~ sú množina geometr. post.

vlast. hodnoty ~~je~~ sú koef. geom. post.

napríklad. $\lambda = 3.\ \{1, 3, 9, 27, 81 \dots\}$

$$L(1, 3, 9 \dots) = \underline{(3, 9, 27, 81) = 3(1, 3, 9, 27 \dots)}$$

- U je podpriestor $\mathbb{R}^\infty$

$$2a_n + a_{n+1} = a_{n+2} \quad \forall n$$

χ báze? $\dim(U) = 2$.

$(1, 2, 4, 8,$ $\qquad (a_n, a_n, 2a_n + a_{n+1}, 2a_{n-1} + 2a_n + a_{n+1},$

$(1, 3, 5, 11, 21 \dots$ $\qquad 2(2a_n + a_{n+1}) + 2a_{n-1} + 2a_n + a_{n+1}$

$$\{a_1, a_2, \underbrace{2a_1 + a_2}_{a_3}, \underbrace{2a_2 + a_3}_{a_4}, \underbrace{2a_3 + a_4}_{a_5} \dots\}$$

- Invar?

  NH: V je končnorazs. v.p. U je podp. a $A: V \to V$ je l. trans. U je invar, če $\underline{\forall \vec{v} \in U}$ $A\vec{v} \in U$.

  $(U)_{n=1}^{n}$ posl. če $2a_n + a_{n+1} = \underline{a_{n+2}}$.

  $$L(a_1 \ldots, 2a_n + a_{n+1} = a_{n+2}) = \underbrace{(a_2, 2a_1 + a_2, 2a_2 + 2a_1 + a_2 \ldots)}_{\text{nekončno mnogo prvih}}$$

  a pro k-tý pravék plac.

  $$2a_k + a_{k+1} = a_{k+2}$$