Fomenko Dmitriy

(1)

I.

a) Nechť P, Q jsou vektorové prostory nad T, a nechť $\mathcal{A} \in \mathcal{L}(P,Q)$, a nechť $\dim P$ je konečná. Pak

$$\dim P = \dim(\ker \mathcal{A}) + \dim(\operatorname{Im}(A))$$

б) $B: \mathbb{R}^3 \to \mathbb{R}^3$: $B(x_1, x_2, x_3) = (x_1+x_2+x_3, x_1-x_2, -x_1+x_2)$

$$\ker(B) = \{\bar{x} \in \mathbb{R}^3 : f(\bar{x}) = \vec{0}\} = \{\begin{pmatrix}1\\1\\-2\end{pmatrix}\}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix}1\\1\\-2\end{pmatrix}$$

$$\operatorname{Im} f = \{f(\bar{x}) : \bar{x} = \mathbb{R}^3\}$$

$? \exists \bar{x} \in \mathbb{R}^3$ zobr. $(a,b,c) \in \mathbb{R}^3$

$$\begin{aligned} x_1 + x_2 + x_3 &= a \\ x_1 - x_2 + 0x_3 &= b \\ -x_1 + x_2 + 0x_3 &= c \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 1 & -1 & 0 & b \\ -1 & 1 & 0 & c \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 1 & -1 & 0 & b \\ 0 & 0 & 0 & b+c \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & -2 & -1 & b-a \\ 0 & 0 & 0 & b+c \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & b \\ 0 & -2 & -1 & b-a \\ 0 & 0 & 0 & b+c \end{array}\right)$$

$\longrightarrow$

$$\mathrm{Im}\, f = \{ f(\bar{x}) : \bar{x} = \mathbb{R}^3 \}$$

$$? \ \exists\, \bar{x} \in \mathbb{R}^3 \ \cancel{\text{zod}} \ (a, b, c) \in \mathbb{R}^3$$

$$x_1 + x_2 + x_3 = \alpha$$
$$x_1 - x_2 + 0x_3 = \beta$$
$$-y + x_2 + 0x_3 = \gamma$$

$$\begin{array}{c} \begin{array}{ccc} ① & ② & ③ \end{array} \\ \left(\begin{array}{ccc|c} 1 & 1 & 1 & \alpha \\ 1 & -1 & 0 & \beta \\ -1 & 1 & 0 & \gamma \end{array}\right) \end{array} \sim$$

$$\sim \begin{array}{c} \begin{array}{ccc} ③ & ① & ② \end{array} \\ \left(\begin{array}{ccc|c} 1 & 1 & 1 & \alpha \\ 0 & 1 & -1 & \beta \\ 0 & -1 & 1 & \gamma \end{array}\right) \end{array} \sim \begin{array}{c} \begin{array}{ccc} ③ & ① & ② \end{array} \\ \left(\begin{array}{ccc|c} 1 & 1 & 1 & \alpha \\ 0 & 1 & -1 & \beta \\ 0 & 0 & 0 & \gamma+\beta \end{array}\right) \end{array}$$

$$\begin{pmatrix} \alpha+\beta \\ 0 \\ -\beta \end{pmatrix} \begin{pmatrix} \alpha-\beta \\ \beta \\ 0 \end{pmatrix} \qquad \alpha = 1 \quad \beta = 1$$

$$\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \left\langle \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \middle| \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle = 2\cdot 0 + 0\cdot 1 + 0(-1) = 0$$

je Ortogon. baze

$$\sqrt{2^2 + (-1)^2} = \sqrt{5}$$

$$X = \left(\frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) \text{ - ortonormalna baze}$$

Об. Ортог. база $\ker(B)$.. $\ker(B) = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

$$\sqrt{1^2+1^2+(-2)^2} = \sqrt{2+4} = \sqrt{6}$$

$$\text{Об. база} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$$

Fomenko D. (2)

II $\forall \alpha \in \mathbb{R}$ $\mathcal{A}_\alpha : \mathbb{R}^3 \to \mathbb{R}^3$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & \alpha & 0 \\ 1 & 2 & 1 \end{pmatrix} \quad ?\alpha \text{ je diag.}$$

$$\det \begin{pmatrix} 1-\lambda & 2 & 1 \\ 0 & \alpha-\lambda & 0 \\ 1 & 2 & 1-\lambda \end{pmatrix} = (\alpha-\lambda) \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} =$$

$$= (\alpha-\lambda)\left((1-\lambda)^2 - 1\right) = \cancel{(\alpha-\lambda)(1-2\lambda}$$

$$= (\alpha-\lambda)\left(((1-\lambda)-1)(1-\lambda+1)\right) = (\alpha-\lambda)(-\lambda)(2-\lambda)$$

$$\sigma(\mathcal{A}) = \{0, 2, \alpha\}$$

(I) $\alpha = 0 \Rightarrow \nu_a(0) = 2$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \nu_g(0) = 2.$$

$$\nu_a(0) = \nu_g(0) = 2. \quad \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$\alpha = 2 \Rightarrow \nu_a(2) = 2$

$$\begin{pmatrix} -1 & 2 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} -1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 4 & 0 \end{pmatrix} \Rightarrow \nu_g(2) = 1.$$

$\nu_g(2) \neq \nu_a(2)$. $\quad \alpha \neq 2$.

$\alpha = \alpha$

$$\begin{pmatrix} 1-\alpha & 2 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & 1-\alpha \end{pmatrix} \sim \begin{pmatrix} 1-\alpha & 2 & 1 \\ 0 & 0 & 0 \\ \alpha & 0 & -\alpha \end{pmatrix} \Rightarrow \nu_g(\alpha) = \nu_a(\alpha) \begin{pmatrix} 1 \\ -1+\frac{\alpha}{2} \\ 1 \end{pmatrix}$$

Fomenko D.

matic. $\mathcal{A}$ je diagonalizovatelná pro $\forall \alpha \in \mathbb{R} \setminus \{2\}$

$$[\mathcal{A}]_{\mathcal{X}\mathcal{X}} = \begin{pmatrix} -2 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = X$$

$$\alpha = 4 \Rightarrow \begin{pmatrix} 1 \\ -1+\frac{\alpha}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

T je lin. transf.

$T^2 = 2T$

λ je vlast. hodnota $T \Rightarrow \lambda \in \{0, 2\}$

$$T \cdot \bar{x} = \lambda \cdot \bar{x}$$

$T^2 = 2T$

$T^2 - 2T = 0$

$(T^2 - 2T)\bar{x} = 0$

$T^2 \cdot \bar{x} - 2T\bar{x} = 0$

$T \cdot \underbrace{T \cdot \bar{x}} - 2\underbrace{T \cdot \bar{x}} = 0$ / $T \cdot \bar{x} = \lambda \cdot \bar{x}$ /

$\underbrace{T \cdot \lambda\bar{x}} - 2\lambda\bar{x} = 0 \Rightarrow$

Linearita $\Rightarrow T \cdot \lambda\bar{x} = \lambda T\bar{x}$ / λ je vlast. číslo /

$\Rightarrow \lambda \underbrace{T\bar{x}}_{\lambda\bar{x}} - 2\lambda\bar{x} = 0$

$\lambda^2\bar{x} - 2\lambda\bar{x} = 0$ / axiom distributivita $\odot$ vzhledem ke sčítání čísel /

$\bar{x} \cdot (\lambda^2 - 2\lambda) = 0$ / $\bar{x}$ je vl. vektor $\neq \vec{0}$ /

$\Rightarrow \lambda^2 - 2\lambda = 0 \Rightarrow \lambda(\lambda - 2) = 0$ $\boxed{\lambda \in \{0, 2\}}$

$\lambda = 0 \quad \lambda = 2$

Famenko D. 4

4). Nech V je priestor so $\langle . | . \rangle$.

Nech $S: V \to V$ injektívne ($\forall x, y \Rightarrow$ ~~f(x)≠~~ $x \neq y \Rightarrow f(x) \neq f(y)$)

$$\langle x, y \rangle_s := \langle Sx, Sy \rangle$$

def. na V iný, nový. sk. súčin.

a) dokážeme aks. sk. súčinu

1) $\forall \bar{x}, \bar{y} \in V.\ \langle \bar{x} | \bar{y} \rangle = \langle \bar{y} | \bar{x} \rangle$.

2) $\langle \alpha \bar{x} + \bar{y} | \bar{z} \rangle = \alpha \langle \bar{x} | \bar{z} \rangle + \langle \bar{y} | \bar{z} \rangle$, $\alpha \in T$, $\bar{x}, \bar{y}, \bar{z} \in V$

3) $\langle \bar{x} | \bar{x} \rangle \geq 0\ \forall x \in V$, a $\langle \bar{x} | \bar{x} \rangle = 0 \Leftrightarrow \bar{x} = 0$.

①. $\underline{\langle \bar{x} | \bar{y} \rangle_s} = \langle S\bar{x} | S\bar{y} \rangle = \langle S\bar{y} | S\bar{z} \rangle = \underline{\langle y | z \rangle_s}$

$$\left.\begin{array}{l} \langle Sx | Sy \rangle \overset{def}{=} \langle Sy | Sx \rangle \\ \langle y | x \rangle_s \overset{def}{=} \langle Sy | Sx \rangle \end{array}\right\} \Rightarrow \text{sym. platí}$$

② $\langle \alpha \bar{x} + \bar{y} | \bar{z} \rangle_s =$ ~~α~~ rozdel. na 2 prípady

2.1 $\langle \bar{x} + \bar{y} | \bar{z} \rangle_s = \langle S(\bar{x} + \bar{y}) | \bar{z} \rangle = \langle S(\bar{x}) + S(\bar{y}) | S\bar{z} \rangle = \langle S\bar{x} | S\bar{z} \rangle + \langle S\bar{y} | S\bar{z} \rangle = \langle x + z \rangle_s + \langle x | y \rangle_s$

$\langle S(\bar{x} + \bar{y}) | S\bar{z} \rangle =$ ~~S~~ $\langle x, z \rangle_s + \langle \bar{y} | \bar{z} \rangle_s$

$\langle S(\bar{x}) + S(\bar{y}) | S\bar{z} \rangle = \langle S\bar{x} | S\bar{z} \rangle + \langle S\bar{y} | S\bar{z} \rangle =$

$= \langle \bar{x} | \bar{z} \rangle_s + \langle \bar{y} | \bar{z} \rangle_s$

2.2 $\langle \alpha x, | y \rangle_s$

~~R~~ $\underline{\langle S(\alpha\bar{x}) | S y\rangle} = \langle \alpha S\bar{x} | S y\rangle = \alpha\langle S\bar{x} | S\bar{y}\rangle =$

$= \underline{\alpha\langle \bar{x} | \bar{y}\rangle_s}$

3 nezápornost.

$\langle \bar{x} | \bar{x}\rangle_s = 0$ vždy pro $\bar{x} = \vec{o}$.

$\langle \bar{x} | \bar{x}\rangle_s = \underbrace{\langle S\bar{x} | S\bar{x}\rangle}_{\longmapsto \text{ S je injektivní} \Rightarrow} = 0$

z povodniho skal. součinu $\bar{x} = \vec{o}$ $\quad S\vec{x} = 0 \Leftrightarrow \vec{x} = \vec{o}$

Fomonko D.

a) Nechť V je vektorový prostor nad T, a $\mathcal{A} \in \mathcal{L}(V)$ Číslo $\lambda \in T$ je vlastnou ~~číslem~~ ~~dime~~ hodnotou lin. transformácie, pokud $\exists \bar{x} \in V, \bar{x} \neq \bar{0}$, tak že $\mathcal{A}\bar{x} = \lambda \bar{x}$, vektor $\bar{x}$ je vlastní vektor lineární transformáci ~~príslu~~ příslušným λ.

Množinu vlastných hodnot nazveme spektrem $\mathcal{A}$.

b) $\mathbb{R}^\infty$ vek. prostor nad $\mathbb{R}$, $(a_i)_{i=1}^{\infty}$ posl. $\mathbb{R}$.

$\mathcal{L}: \mathbb{R}^\infty \to \mathbb{R}^\infty$

$\mathcal{L}(a_1, a_2, a_3 \dots) = (a_2, a_3, \dots)$?

$\mathcal{L}$ je lin. transf. pokud

$\mathcal{L}(\bar{x} + \bar{y}) = \mathcal{L}(\bar{x}) + \mathcal{L}(\bar{y}), \ \forall \bar{x}, \bar{y} \in V$

$\mathcal{L}(c \cdot \bar{x}) = c\mathcal{L}(\bar{x})$, $c \in T$, $\bar{x} \in V$.

• $(a_i)_{i=1}^{\infty}$ $(b_i)_{i=1}^{\infty}$

a) $\mathcal{L}((a_1, a_2 \dots) + (b_1, b_2, \dots)) = \mathcal{L}((a_1 + b_1), (a_2 + b_2), \dots)$

$= ((a_2 + b_2), (a_3 + b_3) + \dots +)$

Fomenko ①

$$\mathcal{L}(a_1, a_2\, a_3 \dots) + \mathcal{L}(b_1, b_2\, b_3 \dots) =$$

$$= (a_2 a_3 \dots) + (b_2\; b_3 \dots) =$$

$$= \underline{((a_2+b_2), (a_3+b_3) \dots\dots)}$$ ~~to~~ platí

b) $\mathcal{L}(c(a_1, a_2, a_3 \dots)) = \mathcal{L}(ca_1, ca_2\; ca_3 \dots) =$

$$= \underline{(ca_2, ca_3, ca_4 \dots)}.$$

$$c\mathcal{L}(a_1, a_2\; a_3 \dots) = c(a_2\; a_3 \dots) =$$

$$= \underline{(ca_2, ca_3 \dots)}$$

platí

~~tro~~ $\mathcal{L}$ je lin. zobraz.

- a vl. hod. a vl. vektory?

$$\mathcal{L}\bar{x} = \lambda\bar{x}$$

$$\underline{\mathcal{L}\cdot(a_i)_{i=1}^{\infty}} = \lambda\cdot(a_i)_{i=1}^{\infty}$$

$\swarrow$

$$(a_2\; a_3 \dots) = (\lambda a_1 + \lambda a_2, \lambda a_3 \dots)$$

$\mathbb{R}^\infty$ - ! pokud prostor ~~byl~~ bych byl konecnorozmerny vl. hod. a vl. vektoru neexist mnemim divenze vynechanim prvka. ale mame $\mathbb{R}^\infty$? tak stusim.

vynechám prvého prvka nemením postupnosť nekonečných ~~prvkov~~

~~a2 -> a1 a3 -> a2~~ 2 -> 1, 3 -> 2...

$(\bar{a}_1, \bar{a}_2 \ldots) = (\lambda a_1 + \lambda a_2 \; \lambda a_3)$

$$\left.\begin{array}{l} \bar{a}_1 = \lambda a_1 \\ \bar{a}_2 = \lambda a_2 \end{array} \Rightarrow \begin{array}{l} \lambda = \frac{\bar{a}_1}{a_1} = \frac{a_2}{a_1} \\ \lambda = \frac{\bar{a}_2}{a_2} = \frac{a_3}{a_2} \end{array} \right\} \Rightarrow \frac{a_{n+1}}{a_n} = \lambda.$$

vlastný vektora sú množina geometr. post.

vlast. hodnoty je sú koef. geom. post.

napríklad. $\lambda = 3$. $\{1, 3, 9, 27, 81 \ldots\}$

$L(1, 3, 9 \ldots) = \underline{(3, 9, 27, 81) = 3(1, 3, 9, 27 \ldots)}$

- U je podpriestor $\mathbb{R}^\infty$

$2a_n + a_{n+1} = a_{n+2}$ ~~ne~~ $\forall n$

x báze? $\dim(U) = 2$.

$(1, 2, 4, 8,$ $\quad (a_n, a_n, 2a_n + a_{n+1}, 2a_{n-1} + 2a_n + a_{n+1},$

$(1, 3, 5, 11, 21 \ldots$ $\quad 2(2a_n + a_{n+1}) + 2a_{n-1} + 2a_n + a_{n+1}$

$\{a_1, a_2, \underbrace{2a_1 + a_2}_{a_3}, \underbrace{2a_2 + a_3}_{a_4}, \underbrace{2a_3 + a_4}_{a_5} \ldots\}$

- Invar?

NH! V je konečnoroz. v.p. U je podp. a $A: V \to V$ je l. trans. U je invar, al $\underline{\forall \vec{u} \in U}$ $A\vec{u} \in U$.

$(\vec{u})_{n=1}^{n}$ posl. če $2a_n + a_{n+1} = \underline{a_{n+2}}$.

$$L(a_1 \ldots, 2a_n + a_{n+1} = a_{n+2}) = \underbrace{(a_2, 2a_1 + a_2, 2a_2 + 2a_1 + a_2 \ldots)}_{\text{nekonečno mnogo prvkov}}$$

a pro k-ty porvěk plače

$2a_k + a_{k+1} = a_{k+2}$