Fomenko

$10+10+10+10+7=47$

Fomenko Dmitriy (1)

I.

a) Nechť P, Q jsou vektorové prostory nad T, a nechť $\mathcal{A} \in \mathcal{L}(P,Q)$, a nechť $\dim P$ je konečná. Pak

$$\dim P = \dim(\ker \mathcal{A}) + \dim(\operatorname{Im}(A)) \qquad 1$$

b) $B: \mathbb{R}^3 \to \mathbb{R}^3 \ : \ B(x_1, x_2, x_3) = (x_1+x_2+x_3,\ x_1-x_2,\ -x_1+x_2)$

$$\ker(B) = \{\bar{x} \in \mathbb{R}^3 : f(\bar{x}) = \vec{0}\} = \left\{ \left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right\} \right\}$$

$$+C_1 \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$\operatorname{Im} f = \{ f(\bar{x}) : \bar{x} = \mathbb{R}^3 \}$$

$? \ \exists \bar{x} \in \mathbb{R}^3$ zobr. $(a, b, c) \in \mathbb{R}^3$

$$\begin{aligned} x_1 + x_2 + x_3 &= a \\ x_1 - x_2 + 0x_3 &= b \\ -x_1 + x_2 + 0x_3 &= c \end{aligned} \quad +C_1 \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 1 & -1 & 0 & b \\ -1 & 1 & 0 & c \end{array}\right) \sim -C_1 \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 1 & -1 & 0 & b \\ 0 & 0 & 0 & b+c \end{array}\right) \sim$$

$$\sim {}^{+C_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & -2 & -1 & b-a \\ 0 & 0 & 0 & b+c \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & b \\ 0 & -2 & -1 & b-a \\ 0 & 0 & 0 & b+c \end{array}\right)$$

$\longrightarrow$

Fomenko D (1)

$$\mathrm{Im}\, f = \{ f(\bar{x}) : \bar{x} \in \mathbb{R}^3 \}$$

$$? \exists \bar{x} \in \mathbb{R}^3 \text{ тоб } (\alpha, \beta, \gamma) \in \mathbb{R}^3$$

$$x_1 + x_2 + x_3 = \alpha$$
$$x_1 - x_2 + 0x_3 = \beta$$
$$-x + x_2 + 0x_3 = \gamma$$

$$\begin{array}{ccc} ① & ② & ③ \\ \end{array}$$
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & \alpha \\ 1 & -1 & 0 & \beta \\ -1 & 1 & 0 & \gamma \end{array}\right) \sim$$

$$\begin{array}{ccc} ③ & ① & ② \\ \end{array}$$
$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & \alpha \\ 0 & 1 & -1 & \beta \\ 0 & -1 & 1 & \gamma \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & \alpha \\ 0 & 1 & -1 & \beta \\ 0 & 0 & 0 & \gamma + \beta \end{array}\right)$$

$$\left(\begin{pmatrix} \alpha+\beta \\ 0 \\ -\beta \end{pmatrix} \begin{pmatrix} \alpha-\beta \\ \beta \\ 0 \end{pmatrix} \right) \quad \alpha = 1 \quad \beta = 1$$

$$\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1 \cdot 0 + (-1) \cdot 0 + 2 \cdot 0 = 0$$

$$\left(\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) \quad \left\langle \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \middle| \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle = 2 \cdot 0 + 0 \cdot 1 + 0(-1) = 0$$

je Ortogon. baze ✓

$$\sqrt{2^2 + (-1)^2} = \sqrt{5}$$

$$X = \left(\frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) \text{ - ortonormalna baze}$$

$$X = \left(\frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) \text{ - ortonorm. baze}$$

Ob. Ortog. baza ker(B).. $\ker(B) = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

$$\sqrt{1^2+1^2+(-2)^2} = \sqrt{2+4} = \sqrt{6}$$

Ob. baze $= \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$.

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Fomenko D. ②

II $\forall \alpha \in \mathbb{R}$ $\mathcal{A}_\alpha : \mathbb{R}^3 \to \mathbb{R}^3$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & \alpha & 0 \\ 1 & 2 & 1 \end{pmatrix} \qquad ?\alpha \text{ je diag.}$$

$$\det \begin{pmatrix} 1-\lambda & 2 & 1 \\ \boxed{0 \quad \alpha-\lambda \quad 0} \\ 1 & 2 & 1-\lambda \end{pmatrix} = (\alpha-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} =$$

$$= (\alpha-\lambda)\left((1-\lambda)^2 - 1\right) = \cancel{(\alpha-\lambda)(1-2\lambda}$$

$$= (\alpha-\lambda)\left(((1-\lambda)-1)(1-\lambda+1)\right) = (\alpha-\lambda)(-\lambda)(2-\lambda)$$

$$\sigma(A) = \{0, 2, \alpha\}$$

① $\alpha = 0 \Rightarrow \nu_a(0) = 2$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \nu_g(0) = 2.$$

$$\nu_a(0) = \nu_g(0) = 2. \quad \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$\alpha = 2 \Rightarrow \nu_a(2) = 2$

$$+\begin{pmatrix} -1 & 2 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} -1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 4 & 0 \end{pmatrix} \Rightarrow \nu_g(2) = 1.$$

$\nu_g(2) \neq \nu_a(2)$. $\quad \alpha \neq 2$.

$\alpha = \alpha$

$$-\begin{pmatrix} 1-\alpha & 2 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & 1-\alpha \end{pmatrix} \sim \begin{pmatrix} 1-\alpha & 2 & 1 \\ 0 & 0 & 0 \\ \alpha & 0 & -\alpha \end{pmatrix} \Rightarrow \nu_g(\alpha) = \nu_a(\alpha) \begin{pmatrix} 1 \\ -1+\frac{\alpha}{2} \\ 1 \end{pmatrix}$$

Fomenko D.

matic. $\mathcal{A}$ je diagonalizovatelná pro $\forall \alpha \in \mathbb{R} \setminus \{2\}$

$$[\mathcal{A}]_{\mathcal{X}\mathcal{X}} = \begin{pmatrix} -2 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = X$$

$$\alpha = 4 \Rightarrow \begin{pmatrix} 1 \\ -1+\frac{\alpha}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

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Fomenko D 3

T je lin. transf.

$T^2 = 2T$

λ je vlast. hodnota $T \Rightarrow \lambda \in \{0, 2\}$

$T \cdot \bar{x} = \lambda \cdot \bar{x}$

$T^2 = 2T$

$T^2 - 2T = 0$

$(T^2 - 2T)\bar{x} = 0$

$T^2 \cdot \bar{x} - 2T\bar{x} = 0$

$T \cdot \underbrace{T \cdot \bar{x}} - 2\underbrace{T \cdot \bar{x}} = 0$ / $T \cdot \bar{x} = \lambda \cdot \bar{x}$ /

$\underbrace{T \cdot \lambda\bar{x}} - 2\lambda\bar{x} = 0 \Rightarrow$

Linearita $\Rightarrow T \cdot \lambda\bar{x} = \lambda T\bar{x}$ / λ je vlast. číslo /

$\Rightarrow \lambda \underbrace{T\bar{x}}_{\lambda\bar{x}} - 2\lambda\bar{x} = 0$

$\lambda^2\bar{x} - 2\lambda\bar{x} = 0$ / aksiom distributivita $\odot$ vzhledem ke sčítání čísel /

$\bar{x} \cdot (\lambda^2 - 2\lambda) = 0$ / $\bar{x}$ je vl. vektor $\neq \vec{0}$ /

$\Rightarrow \lambda^2 - 2\lambda = 0 \Rightarrow \lambda(\lambda - 2) = 0$

$\lambda = 0 \quad \lambda = 2$

$\boxed{\lambda \in \{0, 2\}}$

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4). Nech V je priestor so $\langle . | . \rangle$.

Nech $S: V \to V$ injektívne ($\forall x,y \Rightarrow$ ~~f(x)~~ $x \neq y \Rightarrow f(x) \neq f(y)$)

$$\langle x, y \rangle_s := \langle Sx, Sy \rangle$$

def. na V iný, nový. sk. súčin.

a) dokážeme aks. sk. súčinu

1) $\forall \bar{x}, \bar{y} \in V$. $\langle \bar{x} | \bar{y} \rangle = \langle \bar{y} | \bar{x} \rangle$.

2) $\langle \alpha \bar{x} + \bar{y} | \bar{z} \rangle = \alpha \langle \bar{x} | \bar{z} \rangle + \langle \bar{y} | \bar{z} \rangle$, $\alpha \in T$, $\bar{x}, \bar{y}, \bar{z} \in V$

3) $\langle \bar{x} | \bar{x} \rangle \geq 0 \ \forall x \in V$, a $\langle \bar{x} | \bar{x} \rangle = 0 \Leftrightarrow \bar{x} = 0$.

① $\underline{\langle \bar{x} | \bar{y} \rangle_s} = \langle S\bar{x} | S\bar{y} \rangle = \langle S\bar{y} | S\bar{z} \rangle = \underline{\langle y | z \rangle_s}$

$\left.\begin{array}{l} \langle Sx | Sy \rangle \overset{def}{=} \langle Sy | Sx \rangle \\ \langle y | x \rangle_s \overset{def}{=} \langle Sy | Sx \rangle \end{array}\right\} \Rightarrow$ sym. platí

② $\langle \alpha \bar{x} + \bar{y} | \bar{z} \rangle_s = \cancel{\alpha}$ rozdel. na 2 prípady

2.1 $\langle \bar{x} + \bar{y} | \bar{z} \rangle_s = \langle S(\bar{x} + \bar{y}) | \bar{z} \rangle = \langle S(\bar{x}) + S(\bar{y}) | S\bar{z} \rangle =$
$= \langle S\bar{x} | S\bar{z} \rangle + \langle S\bar{y} | S\bar{z} \rangle =$
$= \langle x | z \rangle_s + \langle x | y \rangle_s$

$\langle S(\bar{x} + \bar{y}) | S\bar{z} \rangle = \cancel{S} \langle x, z \rangle_s + \langle \bar{y} | \bar{z} \rangle_s$

$\langle S(\bar{x}) + S(\bar{y}) | S\bar{z} \rangle = \langle S\bar{x} | S\bar{z} \rangle + \langle S\bar{y} | S\bar{z} \rangle =$

$= \langle \bar{x} | \bar{z} \rangle_s + \langle \bar{y} | \bar{z} \rangle_s$

2.2 $\langle \alpha x, | y \rangle_s$

~~$\cdot\langle S$~~ $\underline{\langle S(\alpha \bar{x}) | S y \rangle} = \langle \alpha S \bar{x} \mid S y \rangle = \alpha \langle S\bar{x} \mid S\bar{y} \rangle =$

$= \underline{\alpha \langle \bar{x} | \bar{y} \rangle_s}$

③ nezápornost.

$\langle \bar{x} | \bar{x} \rangle_s = 0$ vždy pro ~~$\forall$~~ $\bar{x} = \vec{0}$.

$\langle \bar{x} | \bar{x} \rangle_s = \underbrace{\langle S\bar{x} \mid S\bar{x} \rangle}_{\longmapsto \text{ S je injektivní} \Rightarrow} = 0$

z povodního skal. súčinu $\bar{x} = \vec{0}$ $\qquad$ $S\vec{x} = 0 \Leftrightarrow \vec{x} = \vec{0}$

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Fomonko D. 10

a) Nechť V je vektorový prostor nad T, a $A \in L(V)$ Číslo $\underline{\lambda \in T}$ je $\underline{\text{vlastnou}}$ ~~číslem~~ ~~dime~~ $\underline{\text{hodnotou}}$ lin. transformácie, pokud $\exists \bar{x} \in V, \bar{x} \neq \bar{0}$, tak že $A\bar{x} = \lambda \bar{x}$, vektor $\bar{x}$ je $\underline{\text{vlastní vektor}}$ linearní transformáci ~~pozad~~ příslušným λ.
Množinu vlastnych hodnot nazveme spektrem A.

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b) $\mathbb{R}^\infty$ vek. prostor nad $\mathbb{R}$, $(a_i)_{i=1}^\infty$ posl. $\mathbb{R}$.

$$L: \mathbb{R}^\infty \to \mathbb{R}^\infty$$

$$\underline{L(a_1, a_2, a_3 \ldots) = (a_2, a_3, \ldots)} \quad ?$$

L je lin. transf pokud

$$L(\bar{x} + \bar{y}) = L(\bar{x}) + L(\bar{y}), \ \forall \bar{x}, \bar{y} \in V$$

$$L(c \cdot \bar{x}) = c L(\bar{x}), \ c \in T, \bar{x} \in V.$$

✓ 2

• $(a_i)_{i=1}^\infty \quad (b_i)_{i=1}^\infty$

a) $L((a_1, a_2 \ldots) + (b_1, b_2, \ldots)) = L((a_1 + b_1), (a_2 + b_2), \ldots)$
$= ((a_2 + b_2), (a_3 + b_3) + \ldots +)$

Fomenko ②

$$\mathcal{L}(a_1, a_2\, a_3 \dots) + \mathcal{L}(b_1\, b_2\, b_3 \dots) =$$

$$= (a_2 a_3 \dots) + (b_2\, b_3 \dots) =$$

$$= \underbrace{((a_2+b_2), (a_3+b_3) \dots)}$$ ~~to~~ platí

b) $\mathcal{L}(c(a_1, a_2, a_3 \dots)) = \mathcal{L}(ca_1, ca_2\, ca_3 \dots) =$

$$= \underbrace{(ca_2, ca_3, ca_4 \dots)}_{1}.$$

$$c\mathcal{L}(a_1, a_2\, a_3 \dots) = c(a_2\, a_3 \dots) =$$

$$= \underbrace{(ca_2, ca_3 \dots)}_{\text{plat}}$$

✓

proto $\mathcal{L}$ je lin. zobraz.

- a vl. hod. a vl. vektory?

$$\mathcal{L}\bar{x} = \lambda \bar{x}$$

$$\underbrace{\mathcal{L} \cdot (a_i)_{i=1}^{\infty}} = \lambda \cdot (a_i)_{i=1}^{\infty}$$

$$\downarrow$$

$(a_2\, a_3 \dots \cancel{a_n}) = (\lambda a_1 + \lambda a_2, \lambda a_3 \dots)$

R^∞ - ! pokud prostor ~~byl~~ by(ch) byl konečnorozměrný vl. hod. a vl. vektoru neexist. měním dimenze vynecháním prvku. ale máme R^∞? tak nevím.

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Fomonko
D

vynechám prvého prvka nemením postupnosť nekonečných ~~pr…~~

~~2→1, 1→2~~ $2 \to 1,\ 3 \to 2 \ldots$

$$(\bar{a}_1, \bar{a}_2 \ldots) = (\lambda a_1, \lambda a_2, \lambda a_3)$$

$$\left.\begin{array}{l}\bar{a}_1 = \lambda a_1 \\ \bar{a}_2 = \lambda a_2\end{array}\right. \Rightarrow \left.\begin{array}{l}\lambda = \dfrac{\bar{a}_1}{a_1} = \dfrac{a_2}{a_1} \\ \lambda = \dfrac{\bar{a}_2}{a_2} = \dfrac{a_3}{a_2}\end{array}\right\} \Rightarrow \frac{a_{n+1}}{a_n} = \lambda.$$

vlastný vektora ~~je~~ jsou množina geometr. post. ✓

vlast. hodnoty ~~je~~ jsou koef. geom. post. ✓

napríklad. $\lambda = 3.\ \{1, 3, 9, 27, 81 \ldots\}$ ✓

$$L(1, 3, 9 \ldots) = \underline{(3, 9, 27, 81) = 3(1, 3, 9, 27 \ldots)}$$

✓ 3

- U je podpriestor $\mathbb{R}^\infty$

$$2a_n + a_{n+1} = a_{n+2} \quad \forall n$$

x báze? $\dim(U) = 2$

$(1, 2, 4, 8,$ $\quad (a_n, a_n, 2a_n + a_{n+1}, 2a_{n-1} + 2a_n + a_{n+1},$

$(1, 3, 5, 11, 21 \ldots$ $\quad 2(2a_n + a_{n+1}) = 2a_{n-1} + 2a_n + a_{n+1}$

✓ ale je toto X, ako viete, že je lineárne nezávislá?

$$\{a_1, a_2, \underbrace{2a_1 + a_2}_{a_3}, \underbrace{2a_2 + a_3}_{a_4}, \underbrace{2a_3 + a_4}_{a_5} \ldots\}$$

toto mi nie je jasné

• Invar?

MH! V je končnorazs. V.p. U je podp. a

$A: V \to V$ je l. trans. U je invar,

ak $\underline{\forall \vec{v} \in U}$ $A\vec{v} \in U$.

$(u)_{i=1}^{n}$ posl. če $2a_n + a_{n+1} = a_{n+2}$.

$L(a_1 \dots, 2a_n + a_{n+1} = a_{n+2}) = \underbrace{(a_2, 2a_1 + a_2, 2a_2 + 2a_1 + a_2 \dots)}_{\text{nekončno mnogo prvkov}}$

a pro. k-ty prvek plat.

$2a_k + a_{k+1} = a_{k+2}$ / 2

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