

# Linear Algebra

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# 前言

这是数学系线性代数的笔记，写给自己。如有错误请见谅，这些只是作为分享。

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# 第一章 Linear Equations

## 1.1 环和域

### 1.1.1 加群和环的定义

#### Definition 1.1.1 (加群)

假如一个 *Abel* 群的代数运算为加法，并且用符号 ‘+’ 表示，则该群叫做加群。

**Remark:** 加群的单位元  $e$  是唯一的，且  $e=0$ 。称作零元，我们有以下的计算规则：

$$0 + a = a + 0 = a$$

#### Definition 1.1.2 (环)

一个集合  $R$  称之为环满足：

1.  $R$  是一个加群
2.  $R$  对一个乘法来说是一个半群 (半群是一个群胚 + 结合律)
3. 在集合  $R$  上，乘法对加法满足分配率  $a(b+c)=(ab+ac)$

### 1.1.2 交换元、单位元、零因子、整环

#### Definition 1.1.3 (交换环)

一个环叫做一个交换环，假如：

$$ab = ba$$

在环中乘法运算下的单位元，叫做环的单位元。

**Definition 1.1.4 (单位元)**

环中的单位元  $e$ , 假如对于  $R$  的任意元素  $a$  来说, 有:

$$\boxed{e} a = a \boxed{e} = a$$

↑      ↑  
单位元

**Definition 1.1.5 (零因子)**

一个环中的两个元素  $a, b$  之间如果有一个是  $0$ , 那么  $ab=0$ . 但反之不成立.

$$ab = 0 \xrightarrow{\text{不成立}} a = 0 \text{ or } b = 0$$

**Example:** 例如模  $n$  的剩余类环: 假设  $n=ab$

若  $n$  不是素数, 假设:

$$[a] \neq [0], [b] \neq [0], [a][b] = [ab] = [n] = [0]$$

则我们可以得知  $ab = 0 \xrightarrow{\text{不成立}} a = 0 \text{ or } b = 0$

**Remark:** 若是在一个环里,

$$a \neq 0, b \neq 0, ab = 0$$

则  $a$  被称为左零因子,  $b$  被称为右零因子

**Definition 1.1.6 (整环)**

一个环叫做整环, 满足:

1. 乘法交换律:

$$ab = ba$$

2.  $R$  有单位元  $1$ :

$$1a = a1 = a$$

3.  $R$  没有零因子:

$$ab = 0 \implies a = 0 \text{ or } b = 0$$

**Remark:**  $a, b$  可以是任意  $R$  中的元.

## 1.1.3 除环、域

**Definition 1.1.7 (除环)**

一个环  $R$  叫做一个除环，满足：

1.  $R$  至少包含一个不为  $0$  的元
2.  $R$  有一个单位元
3.  $R$  的每个不等于  $0$  的元有一个逆元

**Definition 1.1.8 (除环)**

一个集合  $F$  被称为域，如果满足以下条件：

1. 加法封闭性： $\forall a, b \in F$ ，有  $a + b \in F$ 。
2. 加法可交换性： $\forall a, b \in F$ ，有  $a + b = b + a$ 。
3. 加法单位元素：存在加法单位元素  $0$ ，使得  $\forall a \in F$ ，有  $a + 0 = a$ 。
4. 加法逆元素： $\forall a \in F$ ，存在加法逆元素  $-a$ ，使得  $a + (-a) = 0$ 。
5. 乘法封闭性： $\forall a, b \in F$ ，有  $a \cdot b \in F$ 。
6. 乘法可交换性： $\forall a, b \in F$ ，有  $a \cdot b = b \cdot a$ 。
7. 乘法单位元素：存在乘法单位元素  $1$ ，使得  $\forall a \in F$ ，有  $a \cdot 1 = a$ 。
8. 乘法逆元素： $\forall a \in F$ ，对于非零元素，存在乘法逆元素  $a^{-1}$ ，使得  $a \cdot a^{-1} = 1$ 。
9. 分配律： $\forall a, b, c \in F$ ，满足  $(a + b) \cdot c = a \cdot c + b \cdot c$ 。

**Definition 1.1.9 (Subfield)**

设  $F$  是一个域。如果  $K \subseteq F$  满足以下条件，则称  $K$  是  $F$  的子域：

1.  $K$  非空，并且包含域  $F$  中的加法单位元素  $0$  和乘法单位元素  $1$ 。
2. 对于任意的  $a$  和  $b$  属于  $K$ ， $a + b$  和  $a \cdot b$  也都属于  $K$ （其中  $+$  和  $\cdot$  分别表示域  $F$  中的加法和乘法运算）。
3. 对于任意的  $a$  属于  $K$ ，它的相反元素  $-a$  也属于  $K$ 。
4. 对于任意的非零元素  $a$  属于  $K$ ，它的乘法逆元素  $a^{-1}$  也属于  $K$ 。

**Definition 1.1.10 (Characteristic)**

In abstract algebra, "characteristic" is an important concept for a ring or a field. The characteristic is used to describe the smallest positive integer  $n$  for which  $n$  times the multiplicative identity 1 equals the additive identity (usually denoted as 0) in the algebraic structure.

For a ring (a set with addition and multiplication operations, satisfying certain algebraic rules), the characteristic refers to the smallest positive integer  $n$  such that  $n$  times 1 equals 0 (or defined as 0 if there is no such  $n$ ).

For a field (a special type of ring where every non-zero element has a multiplicative inverse), the characteristic is also a positive integer  $n$  or zero, representing  $n$  times 1 equals 0 or having characteristic zero if there is no such  $n$ .

The significance of the characteristic lies in its impact on the properties and structure of the ring or field. Particularly, in the case of a field, the characteristic is either a prime number or zero. This distinction is useful as it allows us to differentiate between fields of different characteristics and has important applications in properties of algebraic equations and polynomials.

**1.2 System of linear Equations**

Suppose  $F$  is a field, We consider the problem of finding  $n$  scalars (element of  $F$ )  $x_1, \dots, x_n$  which satisfy the conditions

$$\begin{aligned} A_{11} + A_{12}x_1 + \dots + A_{1n}x_n &= 0 \\ A_{21} + A_{22}x_1 + \dots + A_{2n}x_n &= 0 \\ &\vdots \\ A_{m1} + A_{m2}x_1 + \dots + A_{mn}x_n &= 0 \end{aligned} \tag{1.1}$$

where  $y_1, y_2, \dots, y_n$  and  $A_{ij}, 1 \leq i, j \leq n$  are given elements of  $F$ . We call 1.1 this a **system of  $m$  linear equations** in  $n$  unknowns. Any  $n$ -tuple  $(x_1, x_2, \dots, x_n)$  of elements of  $F$  which satisfies each of the equation in 1.1 is called a solution of the system. If  $y_1 = y_2 = \dots = y_m = 0$ , we say that the system is **homogeneous**, or that each of equations is homogenous,

**Definition 1.2.1 (linear combination)**

For the 1.1, suppose we select  $m$  scalars  $c_1, \dots, c_m$ , multiply the  $j$ th equation by  $c_j$  and then add.

$$(c_1A_{11} + \dots + c_mA_{m1})x_1 + \dots + (c_1A_{1n} + \dots + c_mA_{mn})x_n = c_1y_1 + \dots + c_my_m$$

**Note:** Evidently, any solution of the entire system of equations 1.1 will also be a solution of this new equation

**Definition 1.2.2 (*Linear equivalent*)**

Let us say that two systems of linear equations are **linearly equivalent** if each equation of one is a linear combination of the equations of the other.

$$\begin{aligned}
 B_{11} + B_{12}x_1 + \cdots + B_{1n}x_n &= z_1 \\
 B_{21} + B_{22}x_1 + \cdots + B_{2n}x_n &= z_2 \\
 &\vdots \\
 B_{m1} + B_{m2}x_1 + \cdots + B_{mn}x_n &= z_m
 \end{aligned} \tag{1.2}$$

**Theorem 1.2.1**

Equivalent system of linear equations have exactly the same solutions.

**1.3 Matrix and Elementary Row Operations**

there is no need to continue writing the 'unknowns'  $x_1, x_2, \dots, x_n$  in the system of linear equations 1.1, since one actually compute only with the coefficient  $A_{ij}$  and the scalars  $y_i$

We shall now abbreviate the system 1.1 by writing:

$$AX = Y$$

where:

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} \tag{1.3}$$

**A is the matrix of coefficient of the system**

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$



**Remark:**

1. The entries of the matrix  $A$  are the scalars  $A(i,j) = A_{ij}$
2. The matrix  $A$  is an  $m \times n$  matrix
3. The matrix  $X$  is an  $n \times 1$  matrix
4. The matrix  $Y$  is an  $m \times 1$  matrix
5. The  $AX = Y$  is nothing more than a compact way of writing the system of linear equations 1.3

The elementary row operations on an  $m \times n$  matrix  $A$  over the field  $F$ :

1. multiply of one row of  $A$  by a none-zero scalar  $c$ ;
2. interchange of two rows of  $A$ ;
3. replacement of the  $r$ th row of  $A$  by row  $r$  plus  $c$  time row  $s$ ,  $c$  any scalar and  $r \neq s$ ,

An elementary row operation is thus a special type of function with domain the set of all  $m \times n$  matrices over  $F$  and range the same set. One can describe  $e$  in the three cases as follows:

1.  $e(A)_{ij} = A_{ij}$  if  $i \neq r, e(A)_{rj} = cA_{rj}$
2.  $e(A)_{ij} = A_{ij}$  if  $i \neq r, s, e(A)_{rj} = A_{sj}, e(A)_{sj} = A_{rj}$
3.  $e(A)_{ij} = A_{ij}$  if  $i \neq r, e(A)_{rj} = A_{rj} + cA_{sj}$

**Theorem 1.3.1**

To each elementary row operation  $e$  there corresponds an elementary row operation  $e_1$ , of the same type as  $e$ , such that  $e_1(e(A)) = e(e_1(A)) = A$ . In other words, the inverse operation of an elementary row operation is also an elementary row operation of the same type.

**Definition 1.3.1**

If  $A$  and  $B$  are  $m \times n$  matrices over  $F$ , we say that  $A$  is row equivalent to  $B$  if there is a finite sequence of elementary row operations which transforms  $A$  into  $B$ .

**Remark:** Using Theorem 1.3.1, we can find a easy way to verify the following. Each matrix is row-equivalent to itself; if  $B$  is row-equivalent to  $A$ , then  $A$  is row-equivalent to  $B$ ; if  $B$  is row-equivalent to  $A$  and  $C$  is row-equivalent to  $B$ , then  $C$  is row-equivalent to  $A$ .

**Theorem 1.3.2**

If  $A$  and  $B$  are row-equivalent  $m \times n$  matrices over the field  $F$ , we say that  $B$  is row-equivalent to  $A$  if  $B$  can be obtained from  $A$  by a finite sequence of elementary row operations.

**Theorem 1.3.3**

If  $A$  and  $B$  are row-equivalent  $m \times n$  matrices over the field  $F$ , then the system of linear equations  $AX = 0$  is equivalent to the system of linear equations  $BX = 0$ .

**证明:** suppose we pass from  $A$  to  $B$  by a finite sequence of elementary row operations:

$$A = A_0 \rightarrow A_1 \rightarrow \cdots \rightarrow A_n = B \quad (1.4) \quad \blacksquare$$

**Definition 1.3.2**

An  $m \times n$  matrix  $R$  is called row-reduced if:

1. the first non-zero entry in each row of  $R$  is 1;
2. each column of  $R$  which contains the leading non-zero entry of some row has all its other entries 0.

**Remark:** The item 2 implies the num of row is more than the num of column. because if the num of row is less than the num of column, there must be a column which has no leading non-zero entry of some row, then the item 2 can't be satisfied.

There is a example:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

This matrix is not row-reduced matrix because the num of row is less than the num of column.

**Theorem 1.3.4**

Every  $m \times n$  matrix over the field  $F$  is row-equivalent to a row-reduced matrix.

**Definition 1.3.3**

An  $m \times n$  matrix  $R$  is called a **row-reduced echelon matrix** if:

1.  $R$  is a row-reduced matrix;
2. every row of  $R$  which has all its entries 0 occurs below every row which has a non-zero entry;
3. if rows  $1, \dots, r$  are the non-zero rows of  $R$ , and if the leading non-zero entry of row  $i$  occurs in column  $k_i$ ,  $i = 1, \dots, r$ , then  $k_1 \leq k_2 < \cdots < k_r$

**Theorem 1.3.5**

Every  $m \times n$  matrix  $A$  is row-equivalent to a row-reduced echelon matrix.

**Theorem 1.3.6**

If  $A$  is an  $m \times n$  matrix and  $m \leq n$ , the homogeneous system of linear equations  $AX = 0$  has a non-trivial solution.

**Theorem 1.3.7**

if  $A$  is an  $n \times n$  matrix, then  $A$  is row-equivalent to the identity matrix if and only if the system of linear equations  $AX = 0$  has only the trivial solution. ■

证明：