Linear Algebra

——HOFFMAN AND RAY KUNZE

ManJack

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前言

这是数学系线性代数的笔记，写给自己。如有错误请见谅，这些只是作为分享。

MaJack

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第一章 Linear Equations

1.1 环和域

1.1.1 加群和环的定义

Definition 1.1.1 (加群)

*假如一个 **Abel** 群的代数运算为加法，并且用符号'+'表示，则该群叫做**加群**。*

Remark: *加群的单位元 e 是唯一的，且 e=0。称作零元，我们有以下的计算规则:*

$$0 + a = a + 0 = a$$

Definition 1.1.2 (环)

一个集合 R 称之为环满足:

1. *R 是一个加群*
2. *R 对一个乘法来说是一个半群 (半群是一个群胚 + 结合律)*
3. *在集合 R 上，乘法对加法满足分配率 a(b+c)=(ab+ac)*

1.1.2 交换元、单位元、零因子、整环

Definition 1.1.3 (交换环)

一个环叫做一个交换环，假如:

$$ab = ba$$

在环中乘法运算下的单位元，叫做环的单位元。

Definition 1.1.4 (单位元)

环中的单位元 e, 假如对于 R 的任意元素 a 来说，有:

$$\underbrace{e}_{\text{单位元}} a = a \underbrace{e}_{\text{单位元}} = a$$

Definition 1.1.5 (零因子)

一个环中的两个元素 a,b 之间如果有一个是 0, 那么 ab=0. 但反之不成立.

$$ab = 0 \xRightarrow{\text{不成立}} a = 0 \ or \ b = 0$$

Example: 例如模 n 的**剩余类环**: 假设 n=ab

若 n 不是素数, 假设:

$$[\alpha] \neq [0], [b] \neq [0], [\alpha][b] = [\alpha b] = [n] = [0]$$

则我们可以得知 $ab = 0 \xRightarrow{\text{不成立}} a = 0 \ or \ b = 0$

Remark: 若是在一个环里,

$$a \leq 0, b \neq 0, ab = 0$$

则 a 被称为**左零因子**,b 被称为**右零因子**

Definition 1.1.6 (整环)

一个环叫做**整环**, 满足:

1. 乘法交换律:

$$ab = ba$$

2. R 有单位元 1:

$$1a = a1 = a$$

3. R 没有零因子:

$$ab = 0 \Longrightarrow a = 0 \ or \ b = 0$$

Remark: a,b 可以是任意 R 中的元.

1.1.3 除环、域

Definition 1.1.7 (除环)

一个环 R 叫做一个除环，满足:

1. R 至少包含一个不为 0 的元
2. R 有一个单位元
3. R 的每个不等于 0 的元有一个逆元

Definition 1.1.8 (除环)

一个集合 F 被称为域，如果满足以下条件:

1. 加法封闭性: $\forall a, b \in F$，有 $a + b \in F$。
2. 加法可交换性: $\forall a, b \in F$，有 $a + b = b + a$。
3. 加法单位元素: 存在加法单位元素 0，使得 $\forall a \in F$，有 $a + 0 = a$。
4. 加法逆元素: $\forall a \in F$，存在加法逆元素 $-a$，使得 $a + (-a) = 0$。
5. 乘法封闭性: $\forall a, b \in F$，有 $a \cdot b \in F$。
6. 乘法可交换性: $\forall a, b \in F$，有 $a \cdot b = b \cdot a$。
7. 乘法单位元素: 存在乘法单位元素 1，使得 $\forall a \in F$，有 $a \cdot 1 = a$。
8. 乘法逆元素: $\forall a \in F$，对于非零元素，存在乘法逆元素 a^{-1}，使得 $a \cdot a^{-1} = 1$。
9. 分配律: $\forall a, b, c \in F$，满足 $(a + b) \cdot c = a \cdot c + b \cdot c$。

Definition 1.1.9 (*Subfiled*)

设 F 是一个域。如果 $K \subseteq F$ 满足以下条件，则称 K 是 F 的子域:

1. K 非空，并且包含域 F 中的加法单位元素 0 和乘法单位元素 1。
2. 对于任意的 a 和 b 属于 K，$a + b$ 和 $a \cdot b$ 也都属于 K（其中 $+$ 和 $\cdot$ 分别表示域 F 中的加法和乘法运算）。
3. 对于任意的 a 属于 K，它的相反元素 $-a$ 也属于 K。
4. 对于任意的非零元素 a 属于 K，它的乘法逆元素 a^{-1} 也属于 K。

Definition 1.1.10 (*Characteristic*)

In abstract algebra, "characteristic" is an important concept for a ring or a field. The characteristic is used to describe the smallest positive integer n for which n times the multiplicative identity 1 equals the additive identity (usually denoted as 0) in the algebraic structure.

For a ring (a set with addition and multiplication operations, satisfying certain algebraic rules), the characteristic refers to the smallest positive integer n such that n times 1 equals 0 (or defined as 0 if there is no such n).

For a field (a special type of ring where every non-zero element has a multiplicative inverse), the characteristic is also a positive integer n or zero, representing n times 1 equals 0 or having characteristic zero if there is no such n.

The significance of the characteristic lies in its impact on the properties and structure of the ring or field. Particularly, in the case of a field, the characteristic is either a prime number or zero. This distinction is useful as it allows us to differentiate between fields of different characteristics and has important applications in properties of algebraic equations and polynomials.

1.2 System of linear Equations

Suppose F is a field,We consider the problem of finding n scalars(element of F)$x_1, \cdot, x_n$ which satisfy the conditions

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n &= 0 \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n &= 0 \\ &\vdots \\ A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n &= 0 \end{aligned} \tag{1.1}$$

where $y_1, y_2, \cdots, y_n$ and $A_{ij}, 1 \le i, j \le n$ are given elements of F. We call 1.1 this a **system of m linear equations** in n unknowns. Any n-tuple $(x_1, x_2, \cdots, x_n)$ of elements of F which satisfies each of the equation in 1.1 is called a solution of the system.If $y_1 = y_2 = \cdots = y_m = 0$, we say that the system is **homogeneous**,or that each of equations is hommogenous,

Definition 1.2.1 (*linear combination*)

For the 1.1,suppose we select m scalars $c_1, \cdots, c_m$,multiply the jth equation by c_j and then add.

$$(c_1A_{11} + \cdots + c_mA_{m1})x_1 + \cdots + (c_1A_{1n} + \cdots + c_mA_{mn})x_n = c_1y_1 + \cdots + c_my_m$$

Note: *Evidently,any solution of the entire system of equations 1.1 will also be a solution of this new equaiton*

Definition 1.2.2 (*Linear equivalent*)

*Let us say that two systems of linear equations are **linearly equivalent** if each equation of one is a linear combination of the equations of the other.*

$$\begin{aligned} B_{11} + B_{12}x_1 + \cdots + B_{1n}x_n &= z_1 \\ B_{21} + B_{22}x_1 + \cdots + B_{2n}x_n &= z_2 \\ &\vdots \\ B_{m1} + B_{m2}x_1 + \cdots + B_{mn}x_n &= z_m \end{aligned} \tag{1.2}$$

Theorem 1.2.1

Equivalent system of linear equations have exactly the same solutions.

1.3 Matrix and Elementary Row Operations

there is no need to continue writing the 'unkonwns' $x_1, x_2, \cdots, x_n$ in the system of linear equations 1.1, since one actually compute only with the coefficient A_{ij} and the scalars y_i

We shall now abbreviate the system 1.1 by writing:

$$AX = Y$$

where:

$$\boxed{A} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} \tag{1.3}$$

A is the matrix of coefficient of the system

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

Remark:

1. *The entries of the matrix A are the scalars A(i,j) =* A_{ij}
2. *The matrix A is an* $m \times n$ *matrix*
3. *The matrix X is an* $n \times 1$ *matrix*
4. *The matrix Y is an* $m \times 1$ *matrix*
5. *The* $AX = Y$ *is nothing more than a compact way of writing the system of linear equations 1.3*

Proposition 1.3.1

The elementary row operations on an $m \times n$ *matrix A over the field F:*

1. *multiply of one row of A by a none-zero scalar c;*
2. *interchange of two rows of A;*
3. *replacement of the rth row of A by row r plus c time row s,c any scalar and r* $\neq$ *s,*

An elementary row operation is thus a special type of function with domain the set of all $m \times n$ matrices over F and range the same set. One can describe e in the three cases as follows:

1. $e(A)_{ij} = A_{ij} \quad if \quad i \neq r, e(A)_{rj} = cA_{rj}$
2. $e(A)_{ij} = A_{ij} \quad if \quad i \neq r, s, e(A)_{rj} = A_{sj}, e(A)_{sj} = A_{rj}$
3. $e(A)_{ij} = A_{ij} \quad if \quad i \neq r, e(A)_{rj} = A_{rj} + cA_{sj}$

Theorem 1.3.2

To each elementaty row opeatrion e there corrspoonds an elementaty row opeation e_1 *, of the same type as e,such that* $e_1(e(A)) = e(e_1(A)) = A$*. In other words, the inverse opeation of an elementaty row operation is also an elementaty row operation of the same type.*

Definition 1.3.1

If A and B are $m \times n$ *matrices over F,we say that A is row equivalent to B if there is a finite sequence of elementary row operations which transforms A into B.*

Remark: *Using Theorem 1.3.2,we can find a easy way to verify the following.Each matrix is rwo-equivalent to itself;if B is row-equivalent to A.then A is row-equivalent to B;if B is row-equivalent to A and C is row-equivalent to B,then C is row-equivalent to A.*

Definition 1.3.2

If A and B are row-equivalent $m \times n$ *matrices over the field F,we say that B* ***is row-equivalent to A*** *if B can be obtained from A by a finite sequence of elementaty row operations.*

Theorem 1.3.3

If A and B are row-equivalent $m \times n$ matrices over the field F,then the system of linear equations $AX = 0$ is equivalent to the system of linear equations $BX = 0$.

证明： suppose we pass from A to B by a finite sequence of elementary row operations:

$$A = A_0 \to A_1 \to \cdots \to A_n = B \tag{1.4}$$

■

Definition 1.3.3

An $m \times n$ matrix R is called row-reduced if:

1. *the first non-zero entry in each row of R is 1;*
2. *each column of R which contains the leading non-zero entry of some row has all its other entries 0.*

Remark: *The item 2 implies the num of row is more than the num of column.because if the num of row is less than the num of column,there must be a column which has no leading non-zero entry of some row,then the item 2 can't be satisfied.*

There is a example:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

This matrix is not row-reduced matrix because the num of row is less than the num of column.

Theorem 1.3.4

Every $m \times n$ matrix over the field F is row-equivalent to a row-reduced matrix.

证明： Let A be an $m \times n$ matrix over F.If every entry in the first row of A is 0,then condition(a) is satified in so far as row is concerned.If row 1 has non-zero entry,let k be the smallest positive integer j for which $A_{1j} \neq 0$.Multiply row 1 by A_{1k}^{-1},and then condition (a) is satisfied with regard to rwo 1.Now for each $i \geq 2$,add $(-A_{ik})$ times row 1 to row i,Now the leading non-zero entry of row 1 occurs in column k,that entry is 1,and every other entry in column. ■

Definition 1.3.4

An $m \times n$ matrix R is called a ***row-reduced echelon matrix*** *if:*

1. *R is a row-reduced matrix;*
2. *every row of R which has all its entries o occurs below every row which has a non-zero entry;*
3. *if rows $1, \cdots, r$ are the non-zero rows of R,and if the leading non-zero entry of row i occurs in column $k_i, i = 1, \ldots, r$, then $k_1 \leq k_2 < \cdots < k_r$*

Theorem 1.3.5

Every $m \times n$ matrix A is row-equivalent to a row-reduced echelon matrix.

Theorem 1.3.6

If A is an $m \times n$ matrix and $m \leq n$,the the homogeneous system of linear equations AX = 0 has a non-trivial solution.

Theorem 1.3.7

if A is an $n \times n$ matrix, then A is row-equivalent to the identity matrix if and only if the system of linear equations AX = 0 has only the trivial solution.

证明： If A is row-equivalent to I, then AX = 0 and IX = 0 has the same solution.Conversely,suppose AX = 0 has only the trivial solution X = 0.Let R be an $n \times n$ row-reduced echelon matrix which is row-equivalent to A,and let r be the number of non-zero rows of R.Then RX = 0 has no non-trivial solution.Thus $r \geq n$,but $r \leq n$.So r = n,We know R is **RREF**,it has r non-zero rows.So R is the identity matrix. ■

Theorem 1.3.8

If A is an $n \times n$ matrix,then A is row-equivalent to the identity matrix if and only if and oly if the system of equations AX = 0 hase only he trivial solution.

证明： If a is row-equivalent to I. then AX = 0 and IX=0 have the same solutions. We assume that AX = 0 has only the trivial solution X = 0. Let R be an $n \times n$ row-reduced echelon matrix which is equivalent to A,and let r be the number of non-zero rows of R.Then RX = 0 has no non-trivial solution. Thus $r \geq r$.But R only has n rows.So certainly $r \leq n$, and we hav r = n. Since this means that R acturally has a leading non-zero entry of 1 in each of it n rows, and since these 1's occur each in a different one of the n columns,R must be the $n \times n$ identity matrix ■

1.3.1 Matrix Multiplication

Definition 1.3.5

Let A be an $m \times n$ matrix ove the field F and let B be an $n \times p$ matrix over F.The ***Product*** *AB is the $m \times p$ matrix C whose i,j entry is*

$$C_{ij} = \sum_{r=1}^{n} A_{ir} B_{rj}.$$

Remark: *Acrossing this definition,we can draw two conclusions of calculating the product of two matrix AB: The first:*
We calculate the product of two matrix by the following row-operations: In randon j entry of i row in A and j row in B,we multiply corrspoondently them and add them up. And we can get the i row of C. we can proof that why can get the conclusion.

证明： *we know the i row of C:*

$$\begin{aligned}\sum_j C_{ij} &= \sum_j \sum_r A_{ir}B_{rj} \\ &= \sum_j (A_{i1}B_{1j} + \cdots + A_{ir}B_{rj} + \cdots + A_{in}B_{nj}) \\ &= \begin{bmatrix} A_{i1}B_{11} + \ldots + & A_{ir}B_{r1} + & \cdots + & A_{in}B_{n1} \\ \vdots & \vdots & \vdots & \vdots \\ A_{i1}B_{1m} + \ldots + & A_{ir}B_{rm} + & \ldots + & A_{in}B_{nm} \end{bmatrix} \\ &= A_{i1}\sum_j B_{1j} + \cdots + A_{ir}\sum_j B_{rj} + \cdots + A_{in}\sum_j B_{nj} \\ &= \sum_r \sum_j A_{ir}B_{rj}\end{aligned}$$

This means that j entry of i row in A and j row in B.

■

Example： Here are some products of matrices twith rational entries

$$\begin{bmatrix} 5 & -1 & 2 \\ 0 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 5 & -1 & 2 \\ 15 & 4 & 8 \end{bmatrix}$$

Here:

$$\gamma_1 = (9\ 12\ -8) = -2(0\ 6\ 1) + 3(3\ 8\ -2)$$

$$\gamma_2 = (12\ 62\ -3) = 5(0\ 6\ 1\ ) + 4(3\ 8\ -2)$$

Theorem 1.3.9

If A,B,C are matrices over the filed F such that the products BC and A(BC) are defined, then so are the products AB,(AB)C and

$$A(BC) = (AB)C$$

证明： To show that A(BC) = (AB)C means to show that

$$[A(BC)]_{ij} = [(AB)C]_{ij}$$

for each i,j By definition

$$\begin{aligned}[A(BC)]_{ij} &= \sum_r A_{ir}(BC)_{rj} \\ &= \sum_r A_{ir}\sum_s B_{rs}C_{sj} \\ &= \sum_r \sum_s A_{ir}B_{rs}C_{sj} \\ &= \sum_s \sum_r A_{ir}B_{rs}C_{sj} \\ &= \sum_s (\sum_r A_{ir}B_{rs})C_{sj} \\ &= \sum_s (AB)_{is}C_{sj} \qquad = [(AB)C]_{ij}\end{aligned}$$

■

Definition 1.3.6

An $m \times n$ matrix is asaid to be an elementaty matrix if it can be obtained frome the $m \times m$ identity matrix by means of as single elementary row operation.

Theorem 1.3.10

Let e be an elementary row operation and let E be the $m \times m$ elementary matrix $E = e(I)$. Then,for every $m \times n$ matrix A,

$$e(A) = EA$$

Corollary 1.3.11

Let A and B be $m \times n$ matrices over the filed F,Then B is row-equaivalent to A if and only if $B = PA$,where P is a product of $m \times m$ elementary matrices

1.4 Invertible Matrices

Definition 1.4.1

Let A be an $n \times n$ (square) matrix over the field F. An $n \times n$ matrix B such that $BA = I$ is called let inverse of A;an $n \times n$ matrix B suc h that $AB = I$ is called a right invers of A.If $AB = BA = I$,then B is called a two-side inverse of A and A is said to be invertible

Lemma 1.4.1

If A has a left inverse B and a right inverse C, then $B = C$.

证明： *Suppose $BA = I$ and $AC = I$.Then:*

$$B = BI = B(AC) = (BA)C = IC = C$$

■

Theorem 1.4.2

Let A and B be $n \times n$ matrices over F.

1. *If A is invertible, so is A^{-1} and $(A^{-1})^{-1}$*
2. *If both A and are invertible,so is AB,and $(AB)^{-1} = B^{-1}A^{-1}$*

证明： *The first statement is evident from the symmetry of the definition. The second follows upon verification of the relations.*

$$(AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = I$$

■

Corollary 1.4.3

A product of invertible matrices is ivertible.

Theorem 1.4.4

An elementary matrix is invertible.

■

证明：

Theorem 1.4.5

If A is an $n \times n$ matrix, the flowing are equivalent.

1. *A is invertible.*
2. *A is row-equivalent to the $n \times n$ identity matrix.*
3. *A is a product of elementary matrices.*

■

证明：

Corollary 1.4.6

If A is an invertible $n \times n$ matrix and if a sequence of elementary row operations reduces A to be identity, then thta same sequence of operations when applied to I yields A^{-1}

Corollary 1.4.7

Let A and B be $m \times n$ matrices.Then B is row-equivalent to A if and only if $B = PA$ where P is an invertible $m \times m$ matrix

Theorem 1.4.8

1. *A is invertible*
2. *The homogeneous system $AX = 0$ has only the trivial solution $X = 0$*
3. *The system of equations $AX = Y$ has a solution X for each $n \times 1$ matrix Y.*

■

证明：

Corollary 1.4.9

A square matrix with either a left or right inverse is invertible.

证明： Let A be an matrix.Suppose A has a left inverse,i.e,a matrix B such that BA = I.We know that A is row-equivalent to I,so A is invertible.Because X = IX = (BA)X = 0,so B is a right inverse of A.On the orther hand,Suppose A has a right inverse,i.e,a matrix C such that AC = I.We can know C is invertible,and $A = C^{-1}$.So A is invertible with inverse C ■

Corollary 1.4.10

Let $A = A_1A_2 \cdots A_k$,where $A_1, \ldots, A_k$ are $n \times n$ (square) matrices.Then A is invertible if and *ovly if each* A_j *is invertible.* ■

证明：

第二章　Vector Spaces

2.1 Vector Spaces

Definition 2.1.1

A vector space *(or linear space) consists of the following:*

1. *a field F of scalars;*
2. *a set V of objects,called vecots;*
3. *a rule (or opearion),called vector addition, which associates with each pair of vectors α and β in V a vector $\alpha+\beta$ in V a vector $\alpha+\beta$ in V, called the sun of α and β insuch a way that the following conditions hold:*
   (a) *addition is coommutative:*
   $$\alpha+\beta=\beta+\alpha$$
   (b) *addition is associative:*
   $$(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma)$$
   (c) *there is a unique vector 0 in V,called the zero vector,such that:*
   $$\alpha+0=\alpha$$
   (d) *for each vector α in V there is a unique vector $-\alpha$ in V such that:*
   $$\alpha+(-\alpha)=0$$
4. *a rule (or operation),called scalar multiplication,which associates with each scalar c in F and vetor α in V a vector $c\alpha$,called the product of c and α, in such a way that the following conditions hold:*
   (a) $1\alpha=\alpha$
   (b) $(c_1c_2)\alpha=c_1(c_2\alpha)$
   (c) $c(\alpha+\beta)=c\alpha+c\beta$
   (d) $(c_1+c_2)\alpha=c_1\alpha+c_2\alpha$

It is importatnt to observe, the vector space is a composite object consisting of a field.Teh same set of vetors may be part of a number of distinct vector sapcesWe mat simply refer to the vetors as **V**,or then it is desirable to specify the field,we shall say **V is a vector psace over the field F**.

We can know the following poprties:

The n-tuple space,Let F be ant field,and let V be the set of all n-tuples $\alpha = (x_1, x_2, \ldots, x_n)$ of scalars x_i in F,If $\beta = (y_1, y_2, \ldots, y_n)$ with y_i in F,the sum of α and β is defined by

$$\alpha + \beta = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$