GACD

——Teacher Wu

ManJack

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前言

This is a GACD note-book for xupt. If there has much error in note-book,forgive me. It's just writes for me.

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目录

第一章　Bezier Curves

1.1 Introduction

Bezier curves are widely used in computer graphics and related fields. They are used to model smooth curves and surfaces, and are used in computer animation, CAD, and other fields. The curves, which are related to Bernstein polynomials, are named after Pierre Bézier, who used it in the 1960s for designing curves for the bodywork of Renault cars. Generalizations of Bézier curves to higher dimensions are called Bézier surfaces, of which the Bézier triangle is a special case.

1.2 Bezier Curves

Definition 1.2.1 (*Bezier Curve*)

$$F(t) = P_0 + \sum_{i=1}^{n} a_i f_i^n(t)$$

$$f_i^t(t) = \frac{-t^i}{(i-1)!}\frac{d^{i-1}}{dt^{i-1}}\left[\frac{(1-n)^n - 1}{t}\right] = \sum_{j=1}^{n}(-1)^{i+j}\binom{n}{j}\binom{j-1}{i-1}t^j$$

Remark: *This polynomial funcitons is given by Bezier*

Now we can write the Bezier curve as

1.2.1 Bezier Curves definition

Definition 1.2.2

n power Bezier curve is given by:

$$P(t) = \sum_{i=0}^{n} b_i B_i^n(t), t \in [0,1]$$

Remark: *The $b_i \in R^3$ is the control point of the curve. The $B_i^n = \binom{n}{j}(1-t)^{n-i}t^i$ is the Bernstein polynomial.*

Property 1.2.1

1. *unit decompostion*

$$1 = [t + (1-t)]^n = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} = \sum_{i=0}^{n} B_i^n(t)$$

2. *non-negative*

$$B_i^n(t) \geq 0$$

3. *The endpoint* $B_i^n(0) = \begin{cases} 1, & i = 0 \\ 0, & i \neq 0 \end{cases}$, $B_i^n(1) = \begin{cases} 1, & i = n \\ 0, & i \neq n \end{cases}$, $\frac{dB_i^n(t)}{dt}\Big|_{t=0} = \begin{cases} -n, & i = 0 \\ n, & i = n \\ 0 = i \neq 0, n \end{cases}$,

$\frac{dB_i^n(t)}{dt}\Big|_t = 1 = \begin{cases} -n, & i = 0 \\ n, & i = n \\ 0 = i \neq 0, n \end{cases}$

4. *Symmetry*

$$B_i^n(t) = C_{n-i}^n (1-t)^{n-i} t^i = B_{n-i}^n(1-t)$$

5. *Derivative*

$$\frac{dB_i^n(t)}{dt} = n\left[B_{i-1}^{n-1}(t) - B_i^{n-1}(t)\right]$$

6. *Recursion*

$$B_i^n(t) = (1-t)B_i^{n-1}(t) + tB_{i-1}^{n-1}(t)$$

7. *The Maxium : The max of*

$$B_i^n(t) is B_{\lfloor n/2 \rfloor}^n(t)$$

8. *elevation*

$$B_i^n(t) = (1 - \frac{i}{n+1})B_i^{n+1}(t) + \frac{i+1}{n+1}B_{i-1}^{n+1}(t)$$

9. *partion formula:*

$$B_i^n(ct) = \sum_{n}^{j} B_i^j B_j^n(t)$$

10. *integral formula:*

$$\int_0^1 B_i^n(t)dt = \frac{1}{n+1}$$

11. *conversion formula with power basis:*

$$t^j = \sum_{i=j}^{n} \frac{C_{n-j}^{i-j}}{C_n^i} B_i^n(t)$$

12. *Recursion formula:*

$$P_n(t) = P_n(b_0, b_1, ..., b_n; t) = (1-t)P_{n-1}(b_0, b_1, ..., b_{n-1}; t) + tP_{n-1}(b_1, b_2, ..., b_n; t)$$

13. *end point char:*

$$P_n(0) = b_0, P_n(1) = b_n$$

14. *the Bezier curve will not extend outside the boundary of the convex polygon formed by its control points.*

15. *Geometric Invaiance: The Bezier curve is invariant under affine transformation.*

Definition 1.2.3 (*de Casteljau's algorithm*)

$$\begin{cases} P_t^0(t_0) = b_i, i = 0, 1, ..., n \\ P_i^j(t) = (1-t)P_i^{j-1}(t) + tP_{i+1}^{j-1}(t) \end{cases}$$

Remark: *In the end,we can get* $P_0^n(t)$ *and that* $P_0^n(t)$ *is* $P_n(t_0)$. *And The original curves is divided into two curves. The two curves are the same as the original curve.*$P_n(P_0^0(t), P_0^1(t), \cdots, P_0^n(t_0); t)$ *and* $P_n(P_0^n(t_0), P_1^{n-1}(t_0), \cdots, P_n^0(t_0); t)$

1.2.2 Bezier Curves other forms

Representing Bezier curve using edge vectors:

Definition 1.2.4

We know The

$$P_n(t) = \sum_{i=0}^{n} B_i^n(t)b_i, 0 \le t \le 1$$

let

$$a_0 = b_0, a_i = b_i - b_{i-1}, i = 1, 2, ..., n$$

we have

$$P_n = \sum_{i=0}^{n} f_{i,n}(t)a_i, 0 \le t \le 1.$$

where

$$\begin{cases} f_{0,n}(t) = 1 \\ f_{i,n}(t) = 1 - \sum_{j=0}^{i-1} B_j^n(t) \text{ or } f_{i,n}(t) = \sum_{j=i}^{n} B_j^n(t) \end{cases}$$

The same as we can conclude that Bezier curves's Derivative.
Assume that

$$P_n(t) = \sum_{i=0}^{n} B_i^n(t) b_i$$

we can detive

1. $\frac{dP_n(t)}{dt} = nP_{n-1}(\Delta b_0, \Delta b_1, ..., \Delta b_{n-1}; t)$
2. $\frac{d^2P_n(t)}{dt^2} = n(n-1)P_{n-2}(\Delta^2 b_0, \Delta^2 b_1, ..., \Delta^2 b_{n-2}; t)$
3. $\frac{d^kP_n(t)}{dt^k} = \frac{n!}{(n-k)!}P_{n-k}(\Delta^k b_0, \Delta^k b_1, \ldots, \Delta^k b_{n-2}; t)$

Remark:

1. $\dot{P}_n(0) = n\Delta b_0, \dot{P}_n(1) = n\Delta b_{n-1}$

证明： *Proof of* $\dot{P}_n(0) = n\Delta b_0$*:*

$$\begin{aligned} \dot{P}_n(0) &= n\sum_{i=0}^{n-1} B_i^{n-1}(0)\Delta b_i = \sum_{i=0}^{n-1} \binom{n-1}{i} t^i (1-t)^{n-i-1}\Big|_{t=0} \Delta b_i \\ &= n\Delta b_0 \end{aligned}$$

Proof of $\dot{P}_n(1) = n\Delta b_{n-1}$*:*

$$\begin{aligned} \dot{P}_n(1) &= n\sum_{i=0}^{n-1} B_i^{n-1}(1)\Delta b_i = \sum_{i=0}^{n-1} \binom{n-1}{i} t^i (1-t)^{n-i-1}\Big|_{t=1} \Delta b_i \\ &= n\Delta b_{n-1} \end{aligned}$$

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2. In which that :$\Delta b_i = b_{i+1} - b_i$, $\Delta^k b_i = \Delta^{k-1} b_{i+1} - \Delta^{k-1} b_i$