Approximating C_{free} space topology by constructing Vietoris-Rips complex

Aakriti Upadhyay, Chinwe Ekenna and Weifu Wang

***Abstract*— In this work, we present a memory efficient representation of the roadmap that approximates the underlying topology of the C_{free} space. First, we perform sampling in the configuration space while meeting pre-conditions in the workspace and connect samples such that the resulting graph becomes a *skeleton* graph that can be used to construct a Vietoris-Rips (VR) complex. Second, we show that a series of topological collapses can be performed to remove vertices from graph so that the resulting graph reconstructs homotopy equivalent shape to the η-offset of the C_{free} space. In our experiments, we use a range of sample points in four different test environments, construct VR-complex from densely connected graphs and perform topology collapses on it. Our result shows that on satisfying certain pre-conditions, the VR-complex gives homotopy equivalent shape to the η-offset of the C_{free} space.**

I. Introduction

Configuration spaces ($\mathcal{C}_{space}$) determine possible placements of robots. This space can be segmented into regions with obstacles and regions that are free of obstacles and a potential position to place feasible configurations. Configuration space construction can be categorized into two different methods: geometry-based and topology-based. Geometry-based methods compute the exact geometric representation of the configuration space, while topology-based methods capture the connectivity of the configuration space. Recently a more advanced topology based methods called topology data analysis have been developed that creates a map of the configurations space using point cloud data [33] which helps identify connectable regions where trajectories can be built.

Topology data analysis methods has been studied extensively in the field of geometry and mathematics with applications in point cloud data [6], [33]. One well-used concept - the delaunay triangulation [5], [16], [29], [31], [35] has been used to triangulate simplexes projection onto the configuration space. This method however, has shown a deficit in higher and more complicated dimensions (existence of holes and voids) [14].

The study of analyzing the topological properties of configuration space present researchers with the challenge of utilizing these properties for the beneficial guidance of robots during motion planning in complex environments. In this work, our algorithm extracts the information of the environment in the form of a simplicial complex and performs simplicial collapse to generate a memory efficient yet topologically equivalent representation of the C_{free} space.

Our work builds upon the fact that a VR complex preserves the topology of the sampling space. In addition, the work by Attali *et al.* [4] shows that the constructed VR-complex can even be homotopy equivalent to the sampling space if the samples satisfy a collection of constraints. We introduce the constraints in detail in Section III-C.

Once we have the VR-complex that is homotopy equivalent to the underlying sampling space (C_{free} space), we then perform a series of topological collapses to remove samples from the VR-complex while maintaining the topological properties. We therefore, end up with a memory efficient representation (roadmap) of the configuration space. We implemented our approach and conducted experiments in several benchmark environments, and our approach can remove from 40% to 90% of the samples generated and still approximate the C_{free} space topology.

Even though our approach is not currently incremental, the capability of preserving topological information of the sampling space while being memory efficient can be beneficial for planning in complex spaces.

II. Related Work

Topological features are defined as the basic representation of mathematical or geometrical space and refer to a feature that supports continuity, connectivity and convergence that is established and maintained based on geometric coincidence. These topological features can be extracted using various mathematical concepts such as sheaf theory, persistent homology, Vietoris-Rips (VR) complexes and landmarking approach. Past results have shown the beneficial use of topological feature for improved behaviors or actions of machines in areas like signal processing, cohomology, topological motion planning, etc.

To understand the application of homotopy classes for 2D and 3D space objects, Bhatacharya et al. in [7], [9], [10], proposed the use of homology classes for 2D objects, as the application of homotopy classes cannot be practically applied to robots path planning problems. They proposed an application of complex analysis and electromagnetism for the path planning through the 3D objects with K genus (holes in the obstacles) by using the concept of homotopy classes. Later in his work, a more practical application approach was researched, Bhatacharya et al. in [8], used the concept of persistent homology to find the homology class of trajectories that is most persistent for a given probability map. The work proposed the use of persistent homology concept to solve the fundamental problem of goal-directed path planning in an uncertain environment represented by probability map.

Research by Pokorny et. al. in [32], studies homotopy classes of trajectories in general configuration space using Delaunay-Cech Complexes filtration method and abstract the global information of trajectories using persistent homology. This work focuses on trajectory classification and present a sampling-based approach which can handle noise, and can be applied to general configuration spaces which relies solely on the knowledge of samples in the free configuration space. Pokorny et. al. further showed in [33], the application of a sampling-based approach to topological motion planning that is fully data-driven in nature and used persistent cohomology with coefficients in a finite field to solve the path planning problem. The work used the Delaunay-Cech filtration method to filter the data from point-cloud dataset and improvises Dijkstras algorithm to generate distance vector trajectories for a source vertex.

The above-cited research have shown improvements that are more inclined towards extracting topological information of the space and then performing approximate sampling with performance guarantees. These methods however, do not provide a measure of the approximation that has been performed.

A. *Sampling Based Motion Planning*

Sampling-based methods [15] are a state-of-the-art approach to solving motion planning problems. These methods are known to be probabilistic complete because the probability of finding a solution if it exists tends towards 1 as the number of samples generated also increases. Sampling-based methods are broadly classified into two main classes: graph-based methods such as the Probabilistic Roadmap Method (PRM) [27] and tree-based methods such as Expansive-Space tree planner (ESTs) [25] and Rapidly-exploring Random Tree (RRT) [28]. PRM variants consider different topology which include uniformly generating samples in the environment [27], sampling near obstacles [1], [3], [12], [24], [38], sampling with constraints placed on the robots [30] and planning with uncertainty in the environment [26]. Other methods exist that investigate the heterogeneous nature of the planning environment using reinforcement learning [17]–[20], [36].

III. Preliminaries

A. *VR-complex and Čech-complex*

There have been various research done to better understand the VR and $\check{C}$ech complexes and how they provide topologically correct approximations. More formally, the VR and $\check{C}$ech complex can be defined as follows:

Given a set X of points in Euclidean space E, the VR complex R(X) is the abstract simplicial complex whose k-simplices are determined by subsets of k + 1 points in X with a diameter that is at most ε, whereas the $\check{C}$ech-complex C(X) is the abstract simplicial complex where a subset of k + 1 points in X determines a k-simplex if and only if they lie in a ball of radius ε/2.

In work by Attali et. al., [4] they presented mathematical proofs to show VR complexes can provide topologically correct approximations of shapes with the notion of distances between points in the metric space. This previous research provides conditions under which the VR complex of the point set at some scale reflects the homotopy type of the shape for a finite point set that samples a shape. Research in [34], proposed an alternative filtration called the sparse VR filtration for approximating the persistent homology property.

To perform better topological analysis, Erin et. al. in work [13] presents a new concept called shadow complexes, it is determined as projection map from the VR complex to Euclidean n-space that has as its image a more accurate n-dimensional approximation to the homotopy type of sampled space. This projection map was 1D-connected for the planar case $n = 2$, so that the VR complex accurately captures connectivity and fundamental group data which implied that the fundamental group of a VR complex for a planar point set is a free group or free space. However, the projection map did not preserve higher-order topological data for planar sets, nor did it preserve fundamental group data for point sets in dimension larger than three.

Previous work has shown that VR Complex have been used to extract the topological feature of space using various methods. In a recent work [39], Zomorodian proposes three different algorithms to faster compute VR Complex from a generated neighborhood graph of topological space. The neighborhood graphs were computed using k-nearest neighbor and landmarking (witness graph) approach. This graph is augmented with higher-dimensional simplices to compute VR complex by applying inductive, incremental and maximal clique algorithm.

In this work, we will use the concept of VR Complex to get the topological approximations of configuration space for better path planning in difficult and complex environments by providing a measure of the approximation to better guide sampling in difficult regions.

B. *Simplicial collapses*

A simplicial complex K, i.e., a collection of sets closed under the subset operation, it is a generalization of a graph and is useful in representing higher-than-pairwise connectivity relationships. The elements of any set are called vertices and the set itself is called a simplex. Topological thinning (simplicial collapse) [11] is an important preprocessing operation that aims to shrink simplicial complexes to a smaller, simpler simplex which retains a lot of the significant information of the space. for example, A thinned simplicial complex is a subcomplex of K with the condition that all the faces of the maximal simplices are shared. Thus, it can no longer be possible to collapse them.

To establish the conceptual understanding of topology collapse, the work in [22] gives a theoretical proof of simplicial collapse of simplicial complexes with the implementation shown on folded manifolds. Presenting the application of simplicial collapses, research in [23], proposed a method for computing the cohomology ring of three-dimensional (3D) digital binary-valued pictures via a simplicial complex and algebraic thinning using topological representations. In algebraic thinning, they compute a chain contraction of a chain complex to its homology to perform primary and secondary cohomology operations. In another work [37], presents a novel algorithm for simplifying homology and hole location computations on a complex by reducing it to it's core using a strong collapse (a concept of simplicial collapse). The proposed algorithm for strong collapsing reduces the complexity of finding holes in the number of simplices and preserves homology and hole locations in the network with applications in sensor and social networks. A k-dimensional hole, in the geometric realization, is the empty space bounded by a collection of $(k-1)$-simplices in the network.

In this work, the concept of simplicial collapse or topological thinning will be investigated. In particular, the simplicial collapse will be potentially used to reduce the complexity of maximal simplices through vertex deletion down to a core simplex on maintaining the topological structure of the configuration space.

C. From VR complex to sampled-space topology

Generally, a VR-complex does not preserve the topology of the underlying sampled space. However, in [4], the authors showed that a VR complex can be retracted to a Cech complex to approximate the topology of the underlying sampled space. Let us define the *flag complex* of a graph G, denoted Flag G as the maximal simplicial complex whose 1-skeleton is G. More precisely, this is the largest simplicial complex sharing with the $\check{C}$ech complex the same 1-skeleton. In addition, let us denote the VR-complex $R(P,t)$ the abstract simplicial complex whose k-simplicies correspond to subsets of $k+1$ points in P with a diameter that is at most $2t$. The Cech complex $C(P,t)$ as the abstract simplicial complex whose k-simplicies correspond to subsets of $k+1$ points that can be enclosed in a ball of radius t. Define α as an *inert value* of P if $\mathrm{Rad}(\delta) \neq \alpha$ for all non-empty subsets $\delta \subset P$.

Then, given any point set $P \in \mathbb{R}^n$ and any real numbers $\alpha, \beta \geq 0$ with $\alpha \leq \beta$, define the flag complex of any graph G satisfying $R(P,\alpha) \subset \mathrm{Flag}\ G \subset R(P,\beta)$ an (α,β)-almost Rips complex of P. Also, let $v_n = \sqrt{\frac{2n}{n+1}}$. We can then have the following theorems, which is Theorem 7 from [4].

Theorem 1. *Let $P \subset \mathbb{R}^n$ be a finite set of points. For any real numbers $\beta \geq \alpha \geq 0$ such that α is an inert value of P and $c_P(v_n\beta) < 2\alpha - v_n\beta$, there exists a sequence of collapses from any (α,β)-almost Rips complex of P to the Cech complex $C(P,\alpha)$.*

Further, the graph can be shown to be equivalent to η-offset of the sampling space X, from Theorem 10 in [4].

Theorem 2. *Let ϵ, α and β be three non-negative real numbers such that $\alpha \leq \beta$ and $\eta = 2\alpha - v_n\beta - 2\epsilon > 0$. Let P be a finite set of points whose Hausdorff distance to a compact subset X is ϵ or less. Then, any (α,β)-almost Rips complex of P is homotopy equivalent to the η-offset of X whenever α is an inert value of P and $h_X(v_n\beta+\epsilon) < 2\alpha - v_n\beta - 2\epsilon$.*

where $\mathrm{Hull}(X)$ denotes the convex hull of X, and

$$h_X(t) = d_H(\mathrm{Hull}(X,t)|X) \tag{1}$$

$$Hull(X,t) = \bigcup_{\substack{\emptyset \neq \delta \subset X \\ \mathrm{Rad}(\delta) < t}} \mathrm{Hull}(\delta) \tag{2}$$

From the theorem, we can derive that in order to use a graph-like structure to approximate the underlying topology of the sampling space, we need to first have sufficiently dense samples, so that P is no more than ϵ away from the set X based on Hausdorff distance. Here, X is the set we would like to approximate using samples in P. Recall, Hausdorff distance $d_H(X,Y)$ is

$$\begin{aligned} d_H(X,Y) &= \max\{\sup_{x\in X}\inf_{y\in Y} d(x,y), \sup_{y\in Y}\inf_{x\in X} d(x,y)\} \\ d(y,X) &= \inf_{x\in X} d(y,x) \\ d_H(Y|X) &= \sup_{y\in Y} d(y,X) \end{aligned}$$

Therefore, if the samples P satisfy the above properties, we can construct a graph based on P and use the relations to approximate the underlying topology of X. On the other hand, given a set of samples P, we can compute the relevant parameters so that the samples can approximate a compact subset of the sampled space X.

IV. METHODOLOGY

In this section, we show how to first find samples P satisfying conditions stated in Theorems 2, then construct the VR complex from the samples. Then, we will show how to remove vertices on the graph while maintaining topological equivalence by performing topological collapses.

A. Sampling the configuration space

We implement theorems and lemmas from [4] that checks the sampling conditions for sample points in the $\mathcal{C}_{space}$. The workspace representation of the $\mathcal{C}_{space}$ is considered to densely sample points, set P, in the environment. These samples are used to compute Hausdorff distance (ϵ) with the boundary points of the workspace, set X. The Hausdorff distance between two compact sets X and P is analyzed on generating different range of sample points at a time in the workspace. The convex hull points of set P is calculated to compute the two-sided Hausdorff distance with set X, i.e. the supremum distance between sets X and P. In [4], the authors state that if the Hausdorff distance is smaller than some notion of topological feature size of the shape, then the output is topologically correct. Taking this into consideration from theorem 2 we validate the expression $2\epsilon < 2\alpha - v_n\beta$, where $\beta = \alpha$ in our experiments. On verifying the sampling condition in the workspace, the output is a densely sampled $\mathcal{C}_{space}$ graph G.

B. Collapsing a VR-complex

The convex hull of any nonempty subset of the $n+1$ points that define an n-simplex is called a face of the simplex (complex). A maximal face (facet) is any simplex in a complex that is *not* a face of any larger simplex. Given $\tau, \delta \in K$, if $\tau \subset \delta$, in particular $\dim \tau < \dim \delta$, and δ is a maximal face of K and no other maximal face of K contains τ, then τ is called a *free face*. A *simplicial collapse* of K is the removal of all simplices γ such that $\tau \subseteq \gamma \subset \delta$.

Given a simplicial complex K of dimension $n \geq d$, a d-skeleton of K is the subcomplex of K consisting of all the faces of K that have dimension at most d. Then, a graph can be used to represent the 1-skeleton of K, and let us refer to the graph as the *underlying graph* and denote the graph as G_K. For simplicity, in this work we will refer to the 0-skeleton of K as vertices of G_K, and faces of K that have dimension 1 as edges of G_K. Then, we can derive the following results.

Lemma 1. *Given a complex K and its underlying graph G_K, let δ be a maximal face of K, if a vertex v of G_K is a subset of δ ($v \subset \delta$) and no other maximal face of K contains v, then there exist a sequence of simplicial collapses on K that can remove vertex v.*

Proof: Let there exist a sequence of free faces $s_0, s_1, s_2, \ldots, s_m$, so that $s_0 \subset s_1 \subset s_2 \subset \ldots \subset s_m \subset \delta$ and $s_0 = v$. Let s_1 be one of the edges on G_K with v being one endpoint of the edge, let s_2 be the tetrahedron containing s_1, etc. Because each s_i is a free face, a simplicial collapse can remove it. Then, let the sequence of collapse start from s_m, and move towards s_0. Each collapse of s_i will not change the fact that s_i still is a free face of δ. Therefore, v can be removed. □

Then, we can extend the results to get the following theorem.

Theorem 3. *Given a complex K and its underlying graph G_K, let δ be a maximal face of K, and let V_s be the set of all the vertices where v is a subset of δ and no other maximal face of K contains v. Then, after removing all vertices in V_s, there does not exist other simplicial collapse on K involving subsets of δ.*

Proof: Let us assume that after removing all vertices in V_s, there still exist at least one free face $\tau \subset \delta$. If τ is of dimension 0, then it is a vertex that only belongs to δ, so it must have been part of V_s, so τ can only be of dimension 1 or above. If τ is of dimension 1, i.e. an edge on G_K, then at least one vertex of the edge will belong only to δ otherwise the edge cannot be a free face. Therefore, removing all vertices of V_s will remove this edge. Inductively, we can extend this to higher dimensions. Therefore, there cannot be any free face left after removing all vertices in V_s. □

The results lead to the following corollary,

Corollary 1. *Given a complex K and its underlying graph G_K, let δ be a maximal face of K, and let V_s be the set of all the vertices where v is a subset of δ and no other maximal face of K contains v, then removing V_s removes all free faces of δ.*

C. From roadmaps to collapsed VR-complex

Algorithm 1 constructs vietoris-rips complex using maximal clique technique from [21] on a sampled graph. The maximal clique technique uses a hybrid algorithm to compute quick cliques. These cliques are binary represented to perform $\oplus$ operation on it to simplicially collapse by pruning vertices from the graph.

The algorithm returns a sampled graph with vertices of non-colliding regions of $\mathcal{C}_{space}$ after completing topological collapse on the graph. This densely sampled graph gives an approximate topological shape representation of the objects and available free region in the $\mathcal{C}_{space}$.

Algorithm 1 Graph-Collapse(G)

Input: Let G be the sampled graph, M be maximal clique, B be set of binary representation for each clique and T be set of vertices on topological collapse.

for all nodes in graph G **do**
  compute maximal clique M.
while M is not empty **do**
  for each clique in M **do**
    if node in clique **then**
      Set binary value '1' for node in B
    else
      Set binary value '0' for node in B
if B is not empty **then**
  T = B $\oplus$ B
  for each node in T **do**
    project node in graph G_{new}.
return G_{new}

Each node in the graph was represented in binary form based on the clique in which it belongs. For the graph with n nodes, the binary representation of a clique (or a sub-graph) is the binary string of length n in which the i^{th} character is "1" if the clique (sub-graph) contains the i^{th} node and "0" if the clique (sub-graph) does not contain the i^{th} node. During $\oplus$ operation, the algorithm checks for i^{th} node with '0' to perform the pruning process. The algorithm collapses all the edges between the two nodes and removes nodes that are not part of the cliques from the graph. The resulting topological reconstruction of the environment gives a sampled facet graph G_{new}.

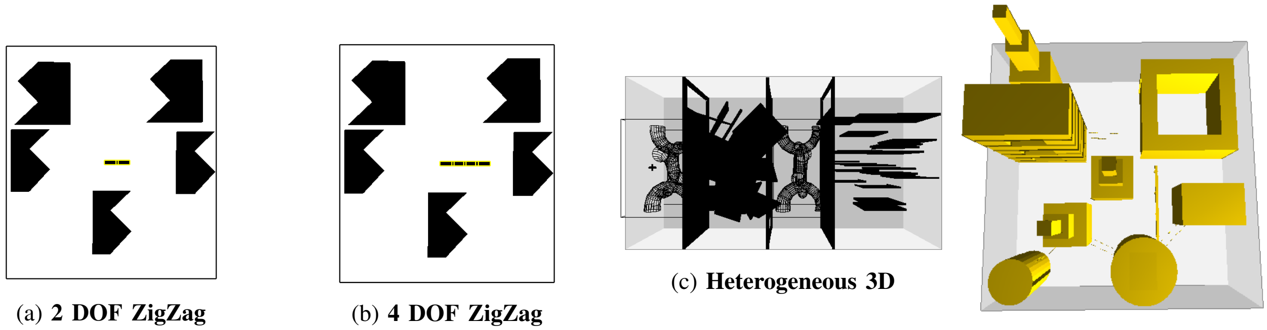
V. Experiments and Results

A. Experimental Setup

All experiments were executed on a Dell Optiplex 7040 desktop machine running OpenSUSE operating system and were implemented in C++.

We perform experiments in four different environments as shown in Figure 1 and generate samples ranging from 100 to 10,000. The environments are taken from the Parasol Lab benchmarks at Texas A & M University [2].

- **ZigZag environment**: This environment is 2D space with zig-zag kind of structured obstacles randomly placed in it as shown in Figure 1a and 1b. The robot has



(a) **2 DOF ZigZag** (b) **4 DOF ZigZag** (c) **Heterogeneous 3D** (d) **Helico**

Fig. 1: Environments with Obstacles

snake-like shape with joints increased based on degree of freedom (dof) required for robot. The robot here have 2 $dofs$ and 4 $dofs$.

- **Heterogeneous 3D**: This environment is a 3D maze kind of structure with walls and narrow passages between the walls. The robot has to pass through the maze-like tunnels to reach the end as shown in Figure 1c. The robot here is toroidal.
- **Helico**: This environment has city-like structure with tall buildings and cable wires between buildings as shown in Figure 1d. The robot here is a helicopter. The robot might have to change its vertical position to reach some goals.

B. Results

We perform two sets of experiments on our four testbeds, as described in Section V-A, in free space (no obstacles in the environment) and obstacle space. We first perform experiments for the sampling conditions of P based on preconditions i.e., $2\epsilon < 2\alpha - v_n\beta$, where $\beta = \alpha$ previously discussed in Section IV-A and another condition as defined in [4], states that as the sampled space becomes denser, the Hausdorff distance (ϵ) distance reduces or approaches a constant value. The second experiments utilize this dense sampled space graph G to construct VR-complex and perform topology collapse. We have results of the produced space graph after the topological collapse which show that with the vertex deletion we still produce a graph with coverage of the $\mathcal{C}_{space}$.

1) Sampling conditions: In Figures 2a, 2b, 3a and 3b, the Hausdorff distance (ϵ) decreases in an empty environment as well as in environment with obstacles. The trend in Figures clearly satisfy from [4] that the value of ϵ become constant above $Rad(X)$ where X is the set of border points on $\mathcal{C}_{space}$ and $Rad()$ is the radius of the circle covering the $\mathcal{C}_{space}$. Figures 2a to 3b also show that the precondition $2\epsilon < 2\alpha - v_n\beta$ is satisfied where the purple and blue bars are representing 2ϵ (E) and green and yellow bars represents $2\alpha - v_n\beta$ (A) and in all cases this condition is satisfied. Histograms show values of Hausdorff distance computed w.r.t. a different number of samples generated in the environment (in horizontal axis).

In Figure 3b, it can be seen that the ϵ value (purple bar) for Helico environment in $\mathcal{C}_{free}$ (FreeSpace) shows a low value at first and increases its value for denser graph and slowly decreasing to become constant later. The position of a robot in this environment is at the corner of the $\mathcal{C}_{space}$, so when samples are generated at first for the lower range they are generated only near the boundary of the $\mathcal{C}_{space}$ and hence ϵ value is low. But when the number of samples increases in the environment they are generated all over the space making the $\mathcal{C}_{space}$ densely sampled and thus ϵ value is computed for points all over the space leading to the initial increase before decreasing and then becoming constant.

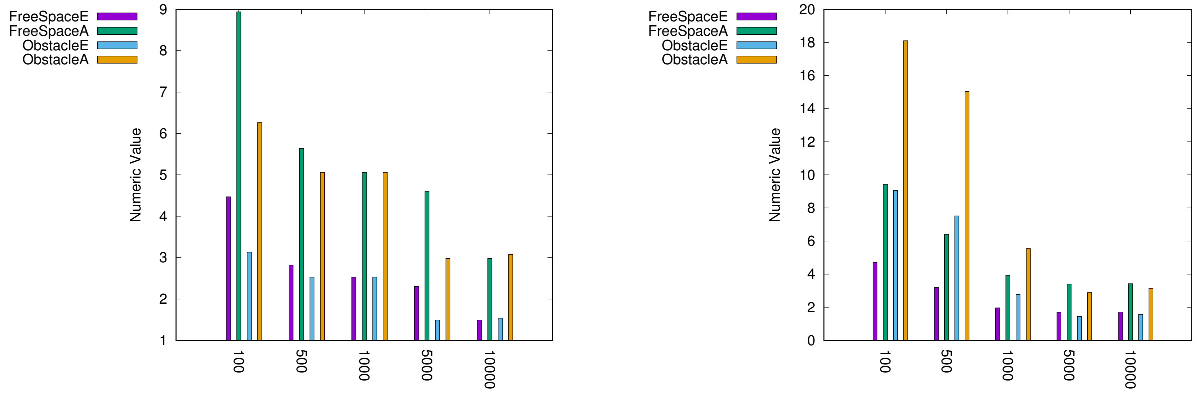
2) Topology Collapse: Table I contains results for topology collapse experiments that utilize theorems and algorithms presented in Section IV-B and IV-C. Results substantiate the ability to delete vertices thus substantiating Lemma 1. We show a 40 to 90% reduction across all the environments which is promising towards deleting vertices while still maintaining the topological information of the space with the produced maps.

Figure 4 gives a pictorial representation for two environments produced after the graph topology collapse.

In addition, because the samples we generated are used to construct VR-complex, which are locally complete subgraphs (cliques), we can actually skip the storage of all the edges and store only the information of which cliques a vertex belongs to. Therefore, the storage needed to store the entire resulting roadmap is only a linear scale of the number of samples left after the collapse, which is comparable to k-nearest neighbor PRM but provides much richer topological information.

VI. Discussion and Future Work

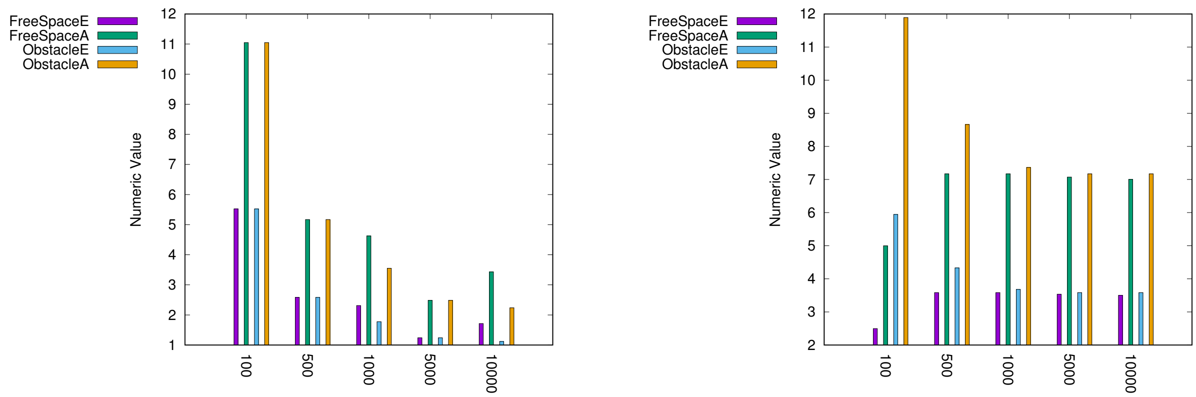
We presented an approach derived from topological data analysis formal theorems and algorithms and show that a correct approximation of topology of $\mathcal{C}_{free}$ space that satisfies pre conditions in $\mathcal{C}_{space}$ exists. We constructed Vietoris-Rips complexes and performed topology collapses on a densely sampled connected graph. We show in our results that these methods give homotopy equivalent shape on reconstruction with a VR complex of sample points set P. We have also shown the trend Hausdorff distance follows w.r.t. the number of sample points in $\mathcal{C}_{space}$ for the two sampling conditions



(a) **2 DOF ZigZag**

(b) **4 DOF ZigZag**

Fig. 2



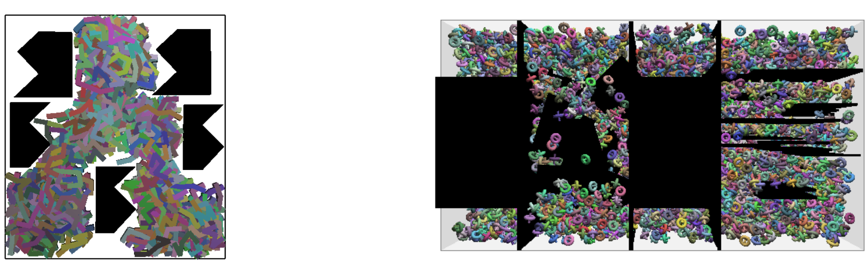
(a) **Heterogeneous 3D**

(b) **Helico**

Fig. 3

Environment	Nodes Before	Nodes After- Free	% Reduction	Nodes After- Obstacle	% Reduction
2DOF Zig Zag	10,000	5081	49.2	4826	51.7
4DOf Zig Zag	10,000	637	93.6	896	91.1
Heterogeneous 3D	10,000	4968	50.3	5061	49.3
Helico	10,000	5041	49.6	5023	49.8

TABLE I: **Results after the Topology Collapse in the Free and Obstacle Environment**



(a) **4 DOF ZigZag**

(b) **Heterogeneous 3D**

Fig. 4: Environments after the Topological collapse

and reaches to a constant value as the sample size increases leading the graph to become denser.

For future work, we will further enhance the approach to identify critical points in $\mathcal{C}_{space}$, i.e. sample points closest to the $\mathcal{C}_{space}$ curvature, and using the properties of Vietoris-Rips, perform path planning on smaller sized graphs.

References

[1] Amato, N.M., Wu, Y.: A randomized roadmap method for path and manipulation planning. In: Proc. IEEE Int. Conf. Robot. Autom. (ICRA). pp. 113–120 (1996)

[2] Amato, N.M.: Motion planning benchmarks, http://parasol.tamu.edu/groups/amatogroup/benchmarks/

[3] Amato, N.M., Bayazit, O.B., Dale, L.K., Jones, C., Vallejo, D.: OBPRM: an obstacle-based PRM for 3d workspaces. In: Proceedings of the third Workshop on the Algorithmic Foundations of Robotics. pp. 155–168. A. K. Peters, Ltd., Natick, MA, USA (1998), (WAFR '98)

[4] Attali, D., Lieutier, A., Salinas, D.: Vietoris–rips complexes also provide topologically correct reconstructions of sampled shapes. Computational Geometry 46(4), 448–465 (2013)

[5] Aurenhammer, F.: Voronoi diagramsa survey of a fundamental geometric data structure. ACM Computing Surveys (CSUR) 23(3), 345–405 (1991)

[6] Beksi, W.J., Papanikolopoulos, N.: 3d point cloud segmentation using topological persistence. In: Robotics and Automation (ICRA), 2016 IEEE International Conference on. pp. 5046–5051. IEEE (2016)

[7] Bhattacharya, S.: Identification and representation of homotopy classes of trajectories for search-based path planning in 3d (2011)

[8] Bhattacharya, S., Ghrist, R., Kumar, V.: Persistent homology for path planning in uncertain environments. IEEE Transactions on Robotics 31(3), 578–590 (2015)

[9] Bhattacharya, S., Likhachev, M., Kumar, V.: Topological constraints in search-based robot path planning. Autonomous Robots 33(3), 273–290 (2012)

[10] Bhattacharya, S., Lipsky, D., Ghrist, R., Kumar, V.: Invariants for homology classes with application to optimal search and planning problem in robotics. Annals of Mathematics and Artificial Intelligence 67(3-4), 251–281 (2013)

[11] Björner, A.: Topological methods. Handbook of combinatorics 2, 1819–1872 (1995)

[12] Boor, V., Overmars, M.H., van der Stappen, A.F.: The Gaussian sampling strategy for probabilistic roadmap planners. In: Proc. IEEE Int. Conf. Robot. Autom. (ICRA). vol. 2, pp. 1018–1023 (May 1999)

[13] Chambers, E.W., De Silva, V., Erickson, J., Ghrist, R.: Vietoris–rips complexes of planar point sets. Discrete & Computational Geometry 44(1), 75–90 (2010)

[14] Chen, L., Xu, J.c.: Optimal delaunay triangulations. Journal of Computational Mathematics pp. 299–308 (2004)

[15] Choset, H., Lynch, K.M., Hutchinson, S., Kantor, G.A., Burgard, W., Kavraki, L.E., Thrun, S.: Principles of Robot Motion: Theory, Algorithms, and Implementations. MIT Press, Cambridge, MA (June 2005)

[16] Cignoni, P., Montani, C., Perego, R., Scopigno, R.: Parallel 3d delaunay triangulation. In: Computer Graphics Forum. vol. 12, pp. 129–142. Wiley Online Library (1993)

[17] Ekenna, C.: Improved Sampling Based Motion Planning Through Local Learning. Ph.D. thesis, Texas A&M University, College Station, Texas (2016)

[18] Ekenna, C., Thomas, S., Jacobs, S.A., Amato, N.M.: Adaptive neighbor connection for PRMs, a natural fit for heterogeneous environments and parallelism. Tech. Rep. TR13-006, Texas A&M (May 2013)

[19] Ekenna, C., Uwacu, D., Thomas, S., Amato, N.M.: Improved roadmap connection via local learning for sampling based planners. In: Proc. IEEE Int. Conf. Intel. Rob. Syst. (IROS). pp. 3227–3234. Hamburg, Germany (October 2015)

[20] Ekenna, C., Uwacu, D., Thomas, S., Amato, N.M.: Studying learning techniques in different phases of prm construction. In: Machine Learning in Planning and Control of Robot Motion Workshop (IROS-MLPC). Hamburg, Germany (October 2015)

[21] Eppstein, D., Strash, D.: Listing all maximal cliques in large sparse real-world graphs. In: International Symposium on Experimental Algorithms. pp. 364–375. Springer (2011)

[22] Forman, R.: Combinatorial differential topology and geometry. New Perspectives in Algebraic Combinatorics 38, 177 (1999)

[23] González-Díaz, R., Real, P.: On the cohomology of 3d digital images. Discrete Applied Mathematics 147(2-3), 245–263 (2005)

[24] Hsu, D., Jiang, T., Reif, J., Sun, Z.: Bridge test for sampling narrow passages with probabilistic roadmap planners. In: Proc. IEEE Int. Conf. Robot. Autom. (ICRA). pp. 4420–4426. IEEE (2003)

[25] Hsu, D., Latombe, J.C., Motwani, R.: Path planning in expansive configuration spaces. Int. J. Comput. Geom. & Appl. pp. 495–517 (1999)

[26] Jaillet, L., Hoffman, J., van den Berg, J., Abbeel, P., Porta, J.M., Goldberg, K.: EG-RRT: Environment-guided random trees for kinodynamic motion planning with uncertainty and obstacles. In: Proc. IEEE Int. Conf. Intel. Rob. Syst. (IROS) (2011)

[27] Kavraki, L.E., Švestka, P., Latombe, J.C., Overmars, M.H.: Probabilistic roadmaps for path planning in high-dimensional configuration spaces. IEEE Trans. Robot. Automat. 12(4), 566–580 (August 1996)

[28] LaValle, S.M., Kuffner, J.J.: Randomized kinodynamic planning. In: Proc. IEEE Int. Conf. Robot. Autom. (ICRA). pp. 473–479 (1999)

[29] Liu, Y., Su, X., Liang, H., Xu, H., Wang, Y., Chen, X.: An improved three-dimension reconstruction method based on guided filter and delaunay. In: 2017 International Conference on Optical Instruments and Technology: Optoelectronic Imaging/Spectroscopy and Signal Processing Technology. vol. 10620, p. 106201D. International Society for Optics and Photonics (2018)

[30] McMahon, T., Thomas, S., Amato, N.M.: Motion planning with reachable volumes. Tech. Rep. 13-001, Parasol Lab, Department of Computer Science, Texas A&M University (Jan 2013)

[31] Plaku, E., Plaku, E., Simari, P.: Clearance-driven motion planning for mobile robots with differential constraints. Robotica pp. 1–23 (2018)

[32] Pokorny, F.T., Hawasly, M., Ramamoorthy, S.: Multiscale topological trajectory classification with persistent homology. In: Robotics: science and systems (2014)

[33] Pokorny, F.T., Kragic, D.: Data-driven topological motion planning with persistent cohomology. In: Robotics: Science and Systems (2015)

[34] Sheehy, D.R.: Linear-size approximations to the vietoris–rips filtration. Discrete & Computational Geometry 49(4), 778–796 (2013)

[35] Tarutoko, Y., Kobayashi, K., Watanabe, K.: Topological map generation based on delaunay triangulation for mobile robot. In: SICE-ICASE, 2006. International Joint Conference. pp. 492–496. IEEE (2006)

[36] Upadhyay, A., Ekenna, C.: Investigating heterogeneous planning spaces. In: Simulation, Modeling, and Programming for Autonomous Robots (SIMPAR), 2018 IEEE International Conference on. pp. 108–115. IEEE (2018)

[37] Wilkerson, A.C., Moore, T.J., Swami, A., Krim, H.: Simplifying the homology of networks via strong collapses. In: Acoustics, Speech and Signal Processing (ICASSP), 2013 IEEE International Conference on. pp. 5258–5262. IEEE (2013)

[38] Yeh, H.Y.C., Denny, J., Lindsey, A., Thomas, S., Amato, N.M.: UMAPRM: Uniformly sampling the medial axis. In: Proc. IEEE Int. Conf. Robot. Autom. (ICRA). pp. 5798–5803. Hong Kong, P. R. China (June 2014)

[39] Zomorodian, A.: Fast construction of the vietoris-rips complex. Computers & Graphics 34(3), 263–271 (2010)