# Martingale Theory and Risk Neutral Pricing

**Manan Rawat**

**Prof. Shrikrishna G. Dani, Centre for Excellence In Basic Sciences, Mumbai**

**Semester Project**

**School of Mathematics**

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# Abstract

This semester project aims to explore various properties and applications of Martingale theory and its applications in Financial Mathematics and derivative pricing.

We begin with formal definitions related to probability theory and sigma and sub-sigma algebras. By exploring the topics related to conditional expectations, filtration and adapted processes, the project reaches formal definition of Martingales, submartingales and supermartingales. Further the project details several properties of martingales. The project looks at convergence theorems related to martingales. The project explores Radon-Nikodym derivatives and the Radon-Nikodym theorem, which is used later for purpose of pricing derivatives in the latter part of the project. The material of the first section is drawn from various books, namely Schilling 2005, Rosenthal 2006 and Williams 1991.

In the second part of the project, terms and concepts of mathematical finance are introduced. Concepts such as arbitrage are discussed in detail. The project then looks at Risk-Neutral Measures and how it relates to our study of martingale measure. We look at several market models, discrete and continuous and explore the concepts of complete and arbitrage free markets and conditions pertaining to it. We discuss the ways to price contingent claims. We conclude the second part of the project by obtaining the Black-Scholes Option pricing formulation using Risk-Neutral pricing methods. The second part of the project is largely inspired from the book, Bingham and Kiesel 2004.

# Chapter 1

# Martingale Theory

## 1.1 Basic Definitions

Before we study Martingales, we must be familiar with basic definitions and topics on the subject.

**Definition 1.1** ($\sigma-$Algebra)**.** *Consider a set $\Omega$. A $\sigma-$algebra $\mathcal{G}$ of subsets of $\Omega$ is a collection $\mathcal{G}$ of subsets of $\Omega$ satisfying the following*

1. $\phi \in \mathcal{G}$
2. *if* $S \in \mathcal{G}$ *then* $S^c \in \mathcal{G}$
3. *if* $S_1, S_2, ... \in \mathcal{G}$ *then* $\bigcup_{n=1}^{\infty} S_n \in \mathcal{G}$.

*A function $f : \Omega \to \mathbb{R}$ is said to be measurable if for every $a \in \mathbb{R}$, the set $\{\omega \in \Omega | f(\omega) < a\}$ is measurable.*

**Definition 1.2** (Sub $\sigma-$algebras)**.** *Let $\Omega$ be a set. Let $\mathcal{F}, \mathcal{G}$ be $\sigma-$algebras on $\Omega$, then $\mathcal{G}$ is said to be sub$-\sigma-$algebra of $\mathcal{F}$ if and only if $\mathcal{G} \subseteq \mathcal{F}$.*

**Definition 1.3** (Tail $\sigma-$algebras)**.** *Let $X_1, X_2, ...$ be random variables. Define*

$$\mathcal{T}_n = \sigma(X_{n+1}, X_{n+2}, ....), \quad \mathcal{T} = \bigcap_n \mathcal{T}_n.$$

*The $\sigma-$algebra $\mathcal{T}$ is called the tail $\sigma-$algebra of the sequence $(X_n : n \in \mathbb{N})$.*

**Definition 1.4** (Measurable Space)**.** *A measurable space is a pair $(\Omega, \mathcal{G})$ consisting of a set $\Omega$*

*and its $\sigma-$algebra $\mathcal{G}$ of subsets of $\Omega$.*

**Definition 1.5** (Random Variable)**.** *A random variable $X$ is a measurable function $X : \Omega \to E$ from a measurable space referred to as a sample space $\Omega$ consisting of a a set of possible outcomes to a measurable space $E$.*

**Definition 1.6** (Random Process)**.** *A random process is a collection of random variables $\{X_t\}$ indexed by time.*

**Definition 1.7** (Probability Measure)**.** *A **Probability measure** $\mathbb{P}$ on the sample space $(S,\mathcal{S}')$ is a real valued function defined on the collection of events $\mathcal{S}'$ that satisfies the following*

1. *$\mathbb{P}(A) \geq 0$ for every event $A$.*
2. *$\mathbb{P}(S)=1$.*
3. *If $\{A_i : i \in I\}$ is a countable, pairwise disjoint collection of events then*

$$\mathbb{P}(\bigcup_{i\in I} A_i) = \sum_{i\in I} \mathbb{P}(A_i)$$

*A **Probability space** is a triple $(\Omega, \mathcal{G}, \mathbb{P})$, where $\Omega$ is a sample space, $\mathcal{G}$ is a $\sigma-$algebra of events and $\mathbb{P}$ is a probability measure on $\mathcal{G}$.*

**Definition 1.8** (Independent $\sigma-$algebras)**.** *Sub $\sigma-$algebras $\mathcal{G}_1, \mathcal{G}_2, .....$ of $\mathcal{F}$ are called independent if, whenever $G_i \in \mathcal{G}_i (i \in \mathbb{N})$ and $i_1, i_2, ...., i_n$ are distinct, then*

$$\mathbb{P}(G_{i_1} \bigcap ... \bigcap G_{i_n}) = \prod_{k=1}^{n} \mathbb{P}(G_{i_k}).$$

**Definition 1.9** (Almost surely)**.** *A sequence of random variables $X_1, X_2, X_3, ..$ is said to converge almost surely to a random variable $X$, shown by $X_n \xrightarrow{a.s.} X$, if*

$$\mathbb{P}(\{s \in S : \lim_{n\to\infty} X_n(s) = X(s)\}) = 1.$$

**Definition 1.10** (Conditional Probability)**.** *If $A$ and $B$ are events, and $\mathbb{P}(B) > 0$, then we define the conditional probability of $A$ given $B$, denoted by $\mathbb{P}(A|B)$, as*

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \bigcap B)}{\mathbb{P}(B)}.$$

**Definition 1.11** (Probability Distributions). *A probability distribution is a function that gives the probabilities of the occurrence of different possible outcomes of an experiment.*

**Definition 1.12** (Joint Probability Distributions). *Given a random process, where the random variables are defined on the same probability space, the joint probability distribution is the corresponding probability distributions on all possible outcomes.*

**Definition 1.13** (IID). *A sequence of random variables $X_1, X_2, \ldots$ is said to be independent and identically distributed, if the random variables are*

1. *are pairwise independent*
2. *have the same distribution function, i.e. $\mathbb{P}(X_i \leq x) = \mathbb{P}(X_j \leq x)$ for all $i, j$ and $x$.*

## 1.2 Introduction to Martingale

### 1.2.1 Conditional Expectation

**Definition 1.14** (Expectation). *For a random variable $X \in L^1 = L^1(\Omega, \mathcal{G}, \mathbb{P})$, we define the expectation $\mathbb{E}(X)$ of $X$ by*

$$\mathbb{E}(X) := \int_\Omega X dP = \int_\Omega X(\omega)\mathbb{P}(d\omega).$$

**Theorem 1.1.** *Suppose that $(X_n)$ is a sequence of RVs, that $X$ is a RV, and that $X_n \to X$ almost surely*

$$\mathbb{P}(X_n \to X) = 1.$$

We define

$$E(X;G) = \int_G X(\omega)\mathbb{P}(d\omega) = \mathbb{E}(X\mathbb{1}_F),$$

where, as ever,

$$\mathbb{1}_G(\omega) = 1 \ \ if \ \ \omega \in G,$$

$$\mathbb{1}_G(\omega) = 0 \ \ if \ \ \omega \notin G.$$

Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability triple and $X$ and $Z$ are random variables,

$X$ taking the distinct values $x_1, x_2, ...., x_m$,

$Z$ taking the distinct values $z_1, z_2, ...., z_n$.

We know the elementary conditional probability and conditional expectation are as follows

$$\mathbb{P}(X = x_i | Z = z_j) = \mathbb{P}(X = x_i, Z = z_j)/\mathbb{P}(Z = z_j)$$

and elementary conditional expectation

$$\mathbb{E}(X|Z = z_j) = \sum x_i \mathbb{P}(X = x_i | Z = z_j).$$

the random variable $Y = \mathbb{E}(X|Z = z_j)$,the conditional expectation of $X$ given $Z$, is defined as follows

$$if \;\; Z(\omega) = z_j, \;\; then \;\; Y(\omega) = \mathbb{E}(X|Z = z_j) = y_j.$$

**Theorem 1.2** (Fundamental Theorem)**.** *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a triple, and* $X$ *a random variable* $\mathbb{E}(|X|) < \infty$*. Let* $\mathcal{G}$ *be sub*$-\sigma-$*algebra of* $\mathcal{F}$*. Then there exist a random variable* $Y$ *such that*

1. $Y$ *is* $\mathcal{G}$ *measurable,*
2. $\mathbb{E}(|Y|) < \infty$,
3. *for every set* $G$ *in* $\mathcal{G}$ *(equivalently, for every set* $G$ *in some* $\pi-$*system which contains* $\Omega$ *and generates* $\mathcal{G}$*), we have*

$$\int_G Y d\mathbb{P} = \int_G X d\mathbb{P}, \;\; G \in \mathcal{G}.$$

*If* $\tilde{Y}$ *is another random variable with these properties then* $\tilde{Y} = Y$*, almost surely, that is,* $\mathbb{P}[\tilde{Y} = Y] = 1$*. A random variable* $Y$ *with the above properties is called a version of the conditional expectation* $\mathbb{E}(X|\mathcal{G})$ *of* $X$ *given* $\mathcal{G}$*, and we write* $Y = \mathbb{E}(X|\mathcal{G})$*, almost surely.*

### 1.2.2 Filtration and Adapted Processes

**Definition 1.15** (Filtration)**.** *On a measurable space,* $(\Omega, \mathcal{G}, \mathbb{P})$ *,a filtration is an increasing sequence*

$$\mathcal{G}_0 \subset \mathcal{G}_1 \subset ... \subset \mathcal{G}_j \subset .... \subset \mathcal{G}$$

*of sub*$-\sigma-$*algebras of* $\mathcal{G}$*.*

**Definition 1.16** ($\sigma-$finite measurable space). *A space $(\Omega, \mathcal{G}_0, \mathbb{P})$ is $\sigma-$finite if $(G_j)_{j\in\mathbb{N}} \subset \mathcal{G}_0$ with $F_j \uparrow \Omega$ and $\mathbb{P}(G_j) < \infty$.*

**Definition 1.17** ($\sigma-$finite filtered measurable space). *If $(\Omega, \mathcal{G}_0, \mathbb{P})$ is $\sigma-$finite, then $(\Omega, \mathcal{G}, \mathcal{G}_j, \mathbb{P})$ is a $\sigma-$finite filtered measure space.*

**Definition 1.18** (Adapted Process). *A process $X = (X_n : n \geq 0)$ is called adapted ( to the filtration $\{\mathcal{G}_n\}$) if for each $n$, is $\mathcal{G}_n-$measurable.*

**Definition 1.19** (Martingale). *Let $(\Omega, \mathcal{G}, \mathcal{G}_j, \mathbb{P})$ be a $\sigma-$finite filtered measure space. A sequence of $\mathcal{G}-$measurable functions $(u_j)_{j\in\mathbb{N}}$ is called a martingale (with respect to the filtration $(\mathcal{G}_j)_{j\in\mathbb{N}}$), if $u_j \in \mathcal{L}^1(\mathcal{G}_j)$ for each $j \in \mathbb{N}$ and if*

$$\int_G u_{j+1} d\mathbb{P} = \int_G u_j d\mathbb{P} \ \ \forall\, G \in \mathcal{G}_j.$$

*or*

*A process $X$ is called a martingale (relative to $(\{\mathcal{G}_n\}, \mathbb{P})$) if*

1. *$X$ is adapatd,*
2. *$\mathbb{E}(|X_n|) < \infty, \ \ \forall n,$*
3. *$\mathbb{E}[X_n|\mathcal{G}_{n-1}] = X_{n-1}, \ \ almost\ surely \ \ (n \geq 1).$*

**Definition 1.20** (Submartingale). *$(u_j)_{j\in\mathbb{N}}$ is a submartingale (with respect to $(\mathcal{G}_j)_{j\in\mathbb{N}}$) if $u_j \in \mathcal{L}^1(\mathcal{G}_j)$ and*

$$\int_G u_{j+1} d\mathbb{P} \geq \int_G u_j d\mathbb{P} \ \ \forall\, G \in \mathcal{G}_j.$$

*or*

*A process $X$ is called a submartingale (relative to $(\{\mathcal{G}_n\}, \mathbb{P})$) if*

1. *$X$ is adapatd,*
2. *$\mathbb{E}(|X_n|) < \infty, \ \ \forall n,$*
3. *$\mathbb{E}[X_n|\mathcal{G}_{n-1}] \geq X_{n-1}, \ \ almost\ surely \ \ (n \geq 1).$*

**Definition 1.21** (Supermartingale). *$(u_j)_{j\in\mathbb{N}}$ is a supermartingale (with respect to $(\mathcal{G}_j)_{j\in\mathbb{N}}$) if*

*$u_j \in \mathcal{L}^1(\mathcal{G}_j)$ and*

$$\int_G u_{j+1} d\mathbb{P} \leq \int_G u_j d\mathbb{P} \;\; \forall\, G \in \mathcal{G}_j.$$

*or*

*A process $X$ is called a supermartingale (relative to $(\{\mathcal{G}_n\}, \mathbb{P})$) if*

1. *$X$ is adapatd,*
2. *$\mathbb{E}(|X_n|) < \infty, \;\; \forall n,$*
3. *$\mathbb{E}[X_n|\mathcal{G}_{n-1}] \leq X_{n-1},$ almost surely $(n \geq 1)$.*

### 1.2.3 Properties of Martingales

If $(X, \mathcal{G}.\mathcal{G}_j, \mathbb{P})$ is a $\sigma-$finite filtered measure space then

1. $(u_j)_{j\in\mathbb{N}}$ is a martingale if, and only if, it is both a sub$-$ and supermartingale.
2. $(u_j)_{j\in\mathbb{N}}$ is a supermartingale if, and only if, $(-u_j)_{j\in\mathbb{N}}$ is a submartingale.
3. Let $(u_j)_{j\in\mathbb{N}}$ and $(w_j)_{j\in\mathbb{N}}$ be [sub-]martingales and $\alpha, \beta$ be [positive] real numbers. Then $(\alpha u_j + \beta w_j)_{j\in\mathbb{N}}$ is a [sub-]martingale.
4. Let $(u_j)_{j\in\mathbb{N}}$ be a submartingale. Then $(u_j^+)_{j\in\mathbb{N}}$ is a submartingale. Take $G \in \mathcal{G}_j$ and observe that $\{u_j \geq 0\} \in \mathcal{G}_j$. Then

   $$\int_G u_{j+1}^+ d\mathbb{P} \geq \int_{G\bigcap\{u_j\geq 0\}} u_{j+1}^+ d\mathbb{P} \geq \int_{G\bigcap\{u_j\geq 0\}} u_{j+1} d\mathbb{P} \geq \int_{G\bigcap\{u_j\geq 0\}} u_j d\mathbb{P} \geq \int_G u_j^+ d\mathbb{P}.$$

   where $u^+ = max\{u(x), 0\}$.
5. Let $(u_j)_{j\in\mathbb{N}}$ be a martingale. Then $(|u_j|)_{j\in\mathbb{N}}$ is a submartingale as $|u_j| = 2u_j^+ - u_j$.
6. Let $(u_j)_{j\in\mathbb{N}}$ be a martingale. If $u_j \in \mathcal{L}^p(\mathcal{G}_j)$ for some $p \in [1, \infty)$, then $(|u_j|^p)_{j\in\mathbb{N}}$ a submartingale.
7. Let $u_j \in \mathcal{L}^1(\mathcal{G}_j),\; j \in \mathbb{N}$, and $u_1 \leq u_2 \leq u_3 \leq .....$ Then $(u_j)_{j\in\mathbb{N}}$ is a submartingale.
8. Let $(\Omega, \mathcal{G}, \mathbb{P}) = ([0,1), \mathcal{B}, \lambda = \lambda^1_{[0,1)})$ and consider the finite $(\sigma-)$algebras generated by all dynamic intervals of $[0,1)$ of length $2^{-j}, j \in \mathbb{N}_0$ :

   $$\mathcal{G}_j^{\Delta} = \sigma([0, 2^{-j}), ...., k2^{-j}, (k+1)2^{-j}), ....., [(2^j - 1)2^{-j}, 1)).$$

As, $\mathcal{G}_0^{\Delta} \subset \mathcal{G}_1^{\Delta} \subset .... \subset \mathcal{B}[0,1)$ and $([0,1), \mathcal{B}[0,1), \mathcal{G}_j^{\Delta}, \lambda)$ is a $(\sigma-)$ finite filtered measure space. Then $(u_j)_{j\in\mathbb{N}_0}$, $u_j = 2^j \mathbb{1}_{[0,2^{-j})}$, is martingale.

As the sets $[k2^{-j}, (k+1)2^{-j}), k = 0, 1, 2...., 2^j - 1$ are a disjoint partition of $[0,1)$, every $G \in \mathcal{G}$ consists of a (finite) disjoint union of such sets. If $[0, 2^{-j}) \subset G$, we have

$$\int_G u_{j+1} d\lambda = \int 2^{j+1} \mathbb{1}_{G\bigcap[0,2^{-j})} d\lambda = 2^{j+1} 2^{-(j+1)} = 2^j 2^{-j} = 2^j \mathbb{1}_{G\bigcap[0,2^{-j})} d\lambda = \int_G u_j d\lambda.$$

and, otherwise,

$$\int_G u_{j+1} d\lambda = \int_G 2^{j+1} \mathbb{1}_{[0,2^{-(j+1)})} d\lambda = 0 = \int_G u_j d\lambda.$$

9.

**Definition 1.22** (Independent functions). *Assume that $(\Omega, \mathcal{G}, \mathbb{P})$ is a probability space, that is, a measure space where $\mathbb{P}(\Omega) = 1$. A family of real functions $(u_j)_{j\in\mathbb{N}} \subset \mathcal{L}^1(\mathcal{G})$ is called independent, if*

$$\mathbb{P}(\bigcap_{j=1}^{M} u_j^{-1}(B_j)) = \prod_{j=1}^{M} \mathbb{P}(\mathbb{P}^{-1}(B_j))$$

*holds for all $M$ in $\mathbb{N}$ and any choice of $B_1, B_2, ...., B_M \in \mathcal{B}(\mathbb{R})$.*

If $\mathcal{G}_k = \sigma(u_1, u_2, ...., u_k)$ is the $\sigma-$algebra generated by $u_1, u_2, ..., u_k$, then the sequence of partial sums

$$s_k = u_1 + u_2 + .... + u_k, \quad k \in \mathbb{N},$$

is an $(\mathcal{G}_k)_{k\in\mathbb{N}}-$ submartingale if, and only if, $\int u_j d\mathbb{P} \geq 0 \ \forall \ j$.

10. Let $(u_j)_{j\in\mathbb{N}} \subset \mathcal{L}_+^1(\mathcal{G}) \bigcap \mathcal{L}_+^{\infty}(\mathcal{G})$ be independent functions. Then $p_k = u_0 \cdot u_1 \cdot ..... \cdot u_k, k \in \mathbb{N}$, is a submartingale with respect to the filtration $\mathcal{G} = \sigma(u_0, u_1, ...., u_k)$ if, and only if, $\int u_j d\mathbb{P} \geq 1$ for all $j$.

$$\int_G p_{k+1} d\mathbb{P} = \int \mathbb{1}_G p_k u_{k+1} d\mathbb{P} = \int \mathbb{1}_G \cdot \int u_{k+1} d\mathbb{P} = \int_G p_k d\mathbb{N} \cdot \int u_{k+1} d\mathbb{P} \ \ \forall \ G \in \mathbb{G}_k.$$

11. Let $u_1, u_2, ...., u_{k+1}$ be independent integrable functions on the probability space $(\Omega, \mathcal{G}, \mathbb{P})$. Then

$$\int_G u_{k+1} d\mathbb{P} = \mathbb{P}(G) \int u_{k+1} d\mathbb{P} \ \ G \in \sigma(u_1, u_2, ...., u_k)$$

and

$$\int \phi u_{k+1} d\mathbb{P} = \int \phi d\mathbb{P} \cdot \int u_{k+1} d\mathbb{P} \quad \phi \in \mathcal{L}^1(\sigma(u_1, ...., u_k)).$$

### 1.2.4 Stopping Time and Doob's Optional Stopping Theorem

**Definition 1.23** (Stopping Time). *Let $(\Omega, \mathcal{G}, \mathcal{G}_j, \mathbb{P})$ be a $\sigma-$finite filtered measure space. A stopping time is a map $\tau : \Omega \to \mathbb{N} \bigcup \{\infty\}$ which satisfies $\{\tau \leq j\} \in \mathcal{G}_j \;\; \forall j \in \mathbb{N}$. The associated $\sigma-$algebra is given by*

$$\mathcal{G}_\tau = \{G \in \mathcal{G} \; : \; A \bigcap \{\tau \leq j\} \in \mathcal{G}_j \;\; \forall j \in \mathbb{N}\}.$$

**Lemma 1.3.** *Let $\sigma, \tau$ be stopping times on a $\sigma-$ finite filtered measure space $(\Omega, \mathcal{G}, \mathcal{G}_j, \mathbb{P})$.*

1. *$\sigma \wedge \tau, \sigma \vee \tau, \sigma + k, k \in \mathbb{N}_0$ are stopping times.*
2. *$\{\sigma < \tau\} \in \mathcal{G}_\sigma \cap \mathcal{G}_\tau$ and $\mathcal{G} + \sigma \subset \mathcal{G}_\tau$ if $\sigma \leq \tau$.*
3. *If $u_j$ is a sequence of real functions such that $u_j \in \mathcal{M}(\mathcal{G}_j)$, then $u_\sigma$ is $\mathcal{G}_\sigma/\mathcal{B}(\mathbb{R})-$measurable.*

**Theorem 1.4.** *For a $\sigma-$finite filtered measure space $(\Omega, \mathcal{G}.\mathcal{G}_j, \mathbb{P})$, for a sequence $(u_j)_{j\in\mathbb{N}}, u_j \in \mathcal{L}^1(\mathcal{G}_j)$, the following assertion are equivalent:*

1. *$(u_j)_{j\in\mathbb{N}}$ is a submartingale;*
2. *$\int u_\sigma d\mathbb{P} \leq \int u_\tau d\mathbb{P}$ for all bounded stopping times $\sigma \leq \tau$;*
3. *$\int_G u_\sigma d\mathbb{P} \leq \int_G u_\tau d\mathbb{P}$ for all bounded stopping times $\sigma \leq \tau$ and $G \in \mathcal{G}_\sigma$.*

**Definition 1.24** (Martingale Transform). *Let $C$ and $X$ be stochastic processes. Let process $C \circ X$ is martingale transform, where*

$$(C \circ X)_n = \sum_{k=1}^{n} C_k (X_k - X_{k-1}) = \sum_{k=1}^{n} C_k \Delta X_k,$$

*when $n > 1$ and $(C \circ X)_0 = X_0$.*

**Theorem 1.5** (Doob's Optional-Stopping Theorem). *1. Let $\tau$ be a stopping time. Let $X$ be a supermartingale. Then $X_\tau$ is integrable and*

$$\mathbb{E}(X_\tau) \leq \mathbb{E}(X_0).$$

*in each of the following situation:*

- $\tau$ *is bounded (for some* $N$ *in* $\mathbb{N}$, $\tau(\omega) \leq N$, $\forall\, \omega$.
- $X$ *is bounded (for some* $K$ *in* $\mathbb{R}^+, |X_n(\omega)| \leq K$ *for every* $n$ *and every* $\omega$*) and* $\tau$ *is almost surely finite;*
- $\mathbb{E}(\tau) < \infty$, *and, for some* $K$ *in* $\mathbb{R}^+$,

$$|X_n(\omega) - X_{n-1}(\omega)| \leq K \;\; \forall (n, \omega).$$

2. *If any of the above conditions holds and* $X$ *is a martingale, then*

$$\mathbb{E}(X_\tau) = \mathbb{E}(X_0).$$

**Corollary 1.5.1.** *1. Suppose that* $M$ *is martingale, the increments* $M_n - M_{n-1}$ *of which are bounded by some constant* $K_1$*. Suppose that* $C$ *is previsble process bounded by some constant* $K_2$*, and* $\tau$ *is a stopping time such that* $\mathbb{E}(\tau) < \infty$*. Then*

$$\mathbb{E}(C \circ M)_\tau = 0.$$

2. *If* $X$ *is a non−negative supermartingale, and* $\tau$ *is a stopping time which is almost surely finite, then*

$$\mathbb{E}(X_\tau) \leq \mathbb{E}(X_0).$$

**Theorem 1.6.** *1. Let* $C$ *be a bounded non−negative previsible process so that, for some* $K$ *in* $[0, \infty), |C_n(\omega)| \leq K$ *for every* $n$ *and every* $\omega$*. Let* $X$ *be a supermartingale. Then* $C \circ X$ *is a supermartingale null at* $0$.

2. *Let* $C$ *be a bounded non−negative previsible process so that, for some* $K$ *in* $[0, \infty), |C_n(\omega)| \leq K$ *for every* $n$ *and every* $\omega$*. Let* $X$ *be a martingale. Then* $C \circ X$ *is a martingale null at* $0$.
3. *If* $C$ *is a bounded previsible process and* $X$ *is a martingale, then* $(C \circ X)$ *is a martingale null at* $0$.
4. *In (i) and (ii), the boundedness condition on* $C$ *may be replaced by the condition* $C_n \in$

*$\mathcal{L}^2 \; \forall \; n$, provided $X_n \in \mathcal{L}^2, \forall \; n$.*

**Lemma 1.7.** *Suppose that $\tau$ is a stopping time such that for some $N$ in $\mathbb{N}$ and some $\epsilon > 0$, we have. for every $n$ in $\mathbb{N}$ :*

$$\mathbb{P}(\tau \leq n + \mathcal{N} | \mathcal{G}_n) > \epsilon, \;\; almost \;\; surely.$$

## 1.3 Convergence Theorem

**Definition 1.25** (Upcrossing). *The number $U_N[a,b](\omega)$ of upcrossing $[a,b]$ made by $n \to X_n(\omega)$ by time $N$ is defined to be the largest $k$ in $\mathbb{Z}^+$ such that we can find*

$$0 \leq s_1 < t_1 < s_2 < t_2 < .... < s_k < t_k \leq N$$

*with*

$$X_{s_i}(\omega) < a, \;\; X_{t_i}(\omega) > b \;\; (1 \leq i \leq k).$$

**Lemma 1.8** (Doob's Upcrossing Lemma). *Let $X$ be a supermartingale. Let $U_N[a,b]$ be the number of upcrossings of $[a.b]$ by time $N$. Then*

$$(b-a)\mathbb{E}U_N[a,b] \leq \mathbb{E}[(X_N - a)^-)].$$

**Corollary 1.8.1.** *Let $X$ be a supermartingale bounded in $\mathcal{L}^1$ in that*

$$\sup_n \mathbb{E}(|X_n|) < \infty.$$

*Let $a, b \in \mathbb{R}$ with $a < b$. Then, with $U_\infty[a,b] =\uparrow \lim_N U_N[a,b]$,*

$$(b-a)\mathbb{E}U_\infty[a,b] \leq |a| + \sup_n \mathbb{E}(|X_n|) < \infty$$

*so that*

$$\mathbb{P}(U_\infty[a,b] = \infty) = 0.$$

**Definition 1.26.** *Define $X_\infty(\omega) = \limsup X_n(\omega), \;\; \forall \; \omega$, so that $X_\infty$ is $\mathcal{G}_\infty$ measurable and*

$X_\infty = \lim X_n$, *almost surely.*

**Theorem 1.9** (Doob's 'Forward' Convergence Theorem). *Let $X$ be a supermartingale bounded in $\mathcal{L}^1 : \sup_n \mathbb{E}(|X_n|) < \infty$. Then, almost surely, $X_\infty = \lim X_n$ exists and is finite.*

**Corollary 1.9.1.** *If $X$ is a non−negative supermartingale, then $X_\infty = \lim X_n$ exists almost surely.*

**Corollary 1.9.2.** *Under any of the following conditions the pointwise limit $\lim_{j\to\infty} u_j$ exists a.e. in $\mathbb{R}$*

- $(u_j)_{j\in\mathbb{N}}$ *is a supermartingale and* $\sup_{j\in\mathbb{N}} \int u_j^- d\mu < \infty$.
- $(u_j)_{j\in\mathbb{N}}$ *is a positive supermartingale.*
- $(u_j)_{j\in\mathbb{N}}$ *is a martingale and* $\sup_{j\in\mathbb{N}} |u_j^-| d\mu < \infty$.

**Definition 1.27** (Reversed or Backward Submartingale). *Let $(w_j, \mathcal{A}_l)_{l\in-\mathbb{N}}$ be a backwards submartingale. If on $\mathcal{A}_{-\infty} = \bigcap_{l\in\mathbb{N}} \mathcal{A}_l$ the measure $\mu_{|\mathcal{A}_{-\infty}}$ is $\sigma-$finite, then $\lim_{j\to+\infty} w_{-j}(x) \in [-\infty, +\infty)$ exists for almost all $x$ and define an $\mathcal{A}_{-\infty}/\mathcal{B}[-\infty,+\infty)$ exists for almost all $x$ and defines $\mathcal{A}_{-\infty}/\mathcal{B}[-\infty,\infty)-$ measurable function.*

**Theorem 1.10.** *Let $(u_j)_{j\in\mathbb{N}}$ be a submartingale on the $\sigma-$ finite filtered measurable space $(X, \mathcal{A}, \mathcal{A}_j, \mu)$. Then the following assertions are equivalent:*

- $u_\infty(x) = \lim_{j\to\infty} u_j(x)$ *exists a.e.,* $u_\infty \in \mathcal{L}^1(\mathcal{A}_\infty, \mu)$. $\lim_{j\to\infty} \int u_j d\mu = \int u_\infty d\mu$, *and* $(u_j)_{j\in\mathbb{N}\bigcup\{\infty\}}$ *is a submartingale.*
- $(u_j)_{j\in\mathbb{N}}$ *is uniformly integrable.*
- $(u_j)_{j\in\mathbb{N}}$ *convergences in* $\mathcal{L}^1(A_\infty)$.

**Theorem 1.11.** *Let $(w_l, \mathcal{A}_l)_{l\in-\mathbb{N}}$ be a backwards submartingale and assume $\mu|_{\mathcal{A}_{-\infty}}$ is a $\sigma-$finite. Then*

- $\lim_{j\to+\infty} w_{-j} = w_{-\infty} \in [-\infty, \infty)$*exists a.e.*
- $\mathcal{L}^1-\lim_{j\to+\infty} w_{-j} = w_{-\infty}$ *if, and only if,* $\inf_{j\in\mathbb{N}} \int w_{-j} d\mu > -\infty$. *In this case,* $(w_l, \mathcal{A}_l)_{l\in\mathbb{N}\bigcup\{-\infty\}}$ *is a.e. real valued.*

*For a backward martingale, the condition in $(ii)$ is automatically satisfied.*

**Theorem 1.12** (Borel-Cantelli)**.** *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $(A_j)_{j\in\mathbb{N}} \subset \mathcal{A}$. Then*

$$\sum_{j=1}^{\infty} \mathbb{P}(A_j) < \infty \Longrightarrow \mathbb{P}(\limsup_{j\to\infty} A_j) = 0;$$

*if the sets $A_j$ are pairwise independent, then*

$$\sum_{j=1}^{\infty} \mathbb{P}(A_j) = \infty \Longrightarrow \mathbb{P}(\limsup_{j\to\infty} A_j) = 1.$$

## 1.4 The Radon-Nikodym Theorem and Martingale Inequalities

### 1.4.1 Radon-Nikodym Theorem

**Lemma 1.13.** *Let $(X, \mathcal{A}, \mu)$ be a measure space. For any $f \in \mathcal{L}^1_+(\mathcal{A})-$ or indeed for $f \in \mathcal{M}^+(\mathcal{A})-$ the set$-$function $\nu := f\mu$ given by $\nu(A) = \int_A f(x)\mu(dx)$ is again a measure. Then*

$$N \in \mathcal{A},\ \mu(N) = 0 \Longleftarrow \nu(N) = 0.$$

**Definition 1.28** (Absolutely continuous)**.** *Let $\mu, \nu$ be two measures on the measurable space $(X, \mathcal{A})$. Then for the set function $\nu$ as defined above, we call nu absolutely continuous w.r.t $\mu$ and write $\nu << \mu$.*

**Theorem 1.14** (Radon-Nikodym)**.** *Let $\mu, \nu$ be two measures on the measurable space $(X, \mathcal{A})$. If $\mu$ is $\sigma-$finite, then the following assertion are equivalent*

- *$\nu(A) = \int_A f(x)\mu(dx)$ for some a.e. unique $f \in \mathcal{M}^+(\mathcal{A})$;*
- *$\nu << \mu$.*

*The unique function $f$ is called the Radon$-$Nikodym derivative and denoted by $f = \frac{d\nu}{d\mu}$.*

**Definition 1.29** (Upward Filtering)**.** *$I$ is called an upward filtering or upwards directed if*

$$\alpha, \beta \in I \Longleftarrow \gamma \in I : \alpha \leq \gamma, \beta \leq \gamma.$$

*Martingales allows us to prove maximal inequalities which are quite useful. Let us introduce the* *notations*

$$u_N^*(x) = \max_{1\le j\le N} |u_j(x)|$$

$$u_\infty^*(x) = \sup_{j\in\mathbb{N}} |u_j(x)|.$$

### 1.4.2 Martingale Inequalities

**Lemma 1.15.** *Let $(X, \mathcal{A}, \mathcal{A}_|, \mu)$ be a $\sigma-$finite filtered measure space and let $(u_j)_{j\in\mathbb{N}}$ be a sub-martingale. Then we have $\forall s > 0$*

$$\mu(\{\max_{1\le j\le N} u_j \ge s\}) \le \frac{1}{s}\int_{\{\max_{1\le j\le N} u_j \ge s\}} u_N d\mu \le \frac{1}{s} u_N^+ d\mu.$$

*If $u_j \in \mathcal{L}_+^p(\mu)$ or if $(u_j)_{j\mathbb{N}} \subset \mathcal{L}^p(\mu), p \in [1,\infty)$, is a martingale then*

$$\mu(\{u_N^* \ge s\}) \le \frac{1}{s^p}\int_{\{u_N^*\ge s\}} |u_N|^p d\mu \le \frac{1}{s^p}\int |u_N|^p d\mu.$$

**Theorem 1.16** (Doob's Maximal Inequality)**.** *Let $(X, \mathcal{A}, \mathcal{A}_j, \mu)$ be a $\sigma-$finite filtered measure space, $1 < p < \infty$ and let $(u_j)_{j\in\mathbb{N}}$ be a martingale or $(|u_j|^p)_{j\in\mathbb{N}}$ be a martingale. Then we have*

$$||u_N^*|| \le \frac{p}{p-1}||u_N||_p \le \frac{p}{p-1}\max_{1\le j\le N}||u_j||_p.$$

**Corollary 1.16.1.** *Let $(u_j)_{j\in\mathbb{N}}$ be a martingale on the $\sigma-$finite filtered measure space $(X, \mathcal{A}, \mathcal{A}_j, \mu)$. Then*

$$\mu(\{u_\infty^* \ge s\}) \le \frac{1}{s}\sup_{j\in\mathbb{N}}||u_j||_1;$$

$$||u_\infty^*|| \le \frac{p}{p-1}\sup_{j\in\mathbb{N}}||u_j||, \quad p\in(1,\infty).$$

*If $(u_j)_{j\in\mathbb{N}\bigcup\{\infty\}}$ is a martingale, we may replace $\sup_{j\in\mathbb{N}}||u_j||_p, p \in [1,\infty)$ in the above expressions by $||u_\infty||_p$.*

**Definition 1.30** (Hardy-Littlewood Maximal Function)**.** *The Hardy–Littlewood maximal function $u \in \mathcal{L}^p(\lambda^n), 1 \le p < \infty$ is defined by*

$$u^*(x) := \sup_{B:B\ni x}\frac{1}{\lambda^n(B)}\int_B |u| d\lambda^n,$$

*where $B \subset \mathbb{R}^n$ denotes ball of any radius.*

**Lemma 1.17.** *Let $u \in \mathcal{L}^p(\lambda^n), 1 \leq p < \infty$. The Hardy–Littlewood maximal function satisfies*

$$u^*(x) = \sup\{\frac{1}{\lambda^n(B_r(c))}\int_{B_r(c)} |u| d\lambda^n : r \in \mathbb{Q}_+, c \in \mathbb{Q}^n, x \in B_r(c)\}.$$

*In particular, $u^*$ is Borel measure.*

**Theorem 1.18** (Hardy, Littlewood)**.** *Let $u \in \mathcal{L}^p(\lambda^n), 1 \leq p < \infty$, and write $u^*$ for the maximal function. Then*

$$\lambda^n(\{u^* \geq s\}) \leq \frac{c_n}{s}||u||_1,$$

$$||u^*||_p \leq \frac{pc_n}{p-1}||u||_p,$$

*with the universal constant $c_n = (\frac{16}{\sqrt{\pi}})^n \Gamma(\frac{n}{2}+1)$.*

**Definition 1.31** (Maximal Function)**.** *Let $\mu$ be a local finite measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. The maximal function is given by*

$$\mu^*(x) := \sup_{B: B \ni x} \frac{\mu(B)}{\lambda^n(B)}.$$

**Corollary 1.18.1.** *Let $\mu$ be a finite measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ with total mass $||\mu||$ and the maximal function $\mu^*$. Then*

$$\lambda^n(\{\mu^* \geq s\}) \leq \frac{c_n}{s}||\mu||, \quad s > 0,$$

*with the universal constant $c_n = (\frac{16}{\sqrt{\pi}})^n \Gamma(\frac{n}{2}+1)$.*

# Chapter 2

# Risk Neutral Pricing

## 2.1 Introduction To Mathematical Finance

**Definition 2.1** (Derivative Security)**.** *A derivative security, or contingent claim, is a financial contract whose value at expiration date $T$ is determined by the price (or prices within a pre-specified time-interval) of the underlying financial assets (or instruments) at time $T$ (within the time interval* $[0, T]$.

**Definition 2.2** (Forward)**.** *A forward contract is an agreement to buy or sell a given underlying asset $S$ at a future date $T$ at a predetermined price $F$.*

**Definition 2.3** (Forward)**.** *A forward contract is an agreement to buy or sell an asset $S$ at a certain future date $T$ for a certain price $K$.*

**Definition 2.4** (Options)**.** *An option is a financial instruments giving one the right but not the obligation to make a specified transaction at (or buy) a specified date at a specified price.*

1. *Call options gives one the right to buy.*
2. *Put options gives the right to sell.*

*A* ***European option*** *gives one the right to buy/sell on the specified date, the expiry date, on which the option has to be expires or matures.*
*An* ***American option*** *gives one the right to buy/sell at any time prior to or at expiry.*

**Definition 2.5** (Underlying)**.** *The asset to which an option refers to is called the underlying asset or the underlying.*

**Definition 2.6** (Strike Price). *The price at which the parties agree to buy/sell the underlying, on/by the expiry date (if exercised), is called the strike price denoted by $K$.*

*For a European Call option with strike price $K$; write $S(t)$ for the value (or price) of the underlying at time $t$.*

1. *If $S(t) > K$, the option is in the money.*

2. *If $S(t) = K$, the option is said to be at the money.*

3. *If $S(t) < K$, the option is out of money.*

*The payoff from the option is*

$$S(t) - K \quad if \quad S(t) > K$$

$$0 \; otherwise$$

*written as $[S(T) - K]^+$.*

**Definition 2.7** (Short and Long Position). *The following positions are assumed when a derivative contract is written.*

1. *An agent who agrees to buy the underlying asset is said to have a long position.*

2. *The other agent who has the obligation to sell the underlying assumes the short position.*

**Definition 2.8** (Delivery Date, Price and Forward price). *The following is the terminology we must be aware of associated with a derivative contract.*

1. *The settlement date of the future/forward/option is called the delivery date.*

2. *The specified price is referred to as delivery price.*

3. *The forward price $f(t, T)$ is the delivery price that would make the contract have zero value at time $t$.*

**Definition 2.9** (Standard Brownian Motion). *A stochastic process $X = (X(t))_{t \geq 0}$ is a standard (one−dimensional) Brownian motion, on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, if*

1. $X(0) = 0 \;\; a.s.$

2. *$X$ has independent increments : $X(t + u) - X(t)$ depends only on $u$*

3. *$X$ has Gaussian increments : $X(t+u) - X(t)$ is normally distributed with mean $0$ and variance $u, X(t+u) - X(t) \sim N(0, u)$,*

4. *$X$ has continuous paths: $X(t)$ is a continuous functions of $t$, i.e. $t \to X(t, \omega)$ is continuous for all $\omega \in \Omega$.*

**Definition 2.10** (Closed Martingale)**.** *Some Martingale of the form $X_t = \mathbb{E}[X|\mathcal{F}_t]$ $(t \geq 0)$ for some integrable random variable $X$. Then $X$ is said to close $(X_t)$ which is called a close martingale/ For a close martingalee*

$$X_t \to X_\infty \ \ (t \to \infty) \ \ a.s.$$

*and then also*

$$X_t = \mathbb{E}[X_\infty|\mathcal{F}_t] \ \ a.s.$$

*This property is equivalent also to uniform integrability*

$$\sup_t \int_{\{|X_t|>x\}} |X_t| d\mathbb{P} \to 0 \ \ (x \to \infty).$$

**Definition 2.11** (Predictable)**.** *A Process $C = (C_n)_{n=1}^{\infty}$ predictable if $C_n$ if $\mathcal{F}_{n-1}-$measurable for all $n \geq 1$.*

**Definition 2.12** (Complete Market)**.** *A market $\mathcal{M}$ is complete if every contingent claim is attainable, i.e. for every $\mathcal{F}_T-$measurable random variable $X \in L^0$ there exists a replicating self–financing strategy $\phi \in \Phi$ such that $V_\phi(T) = X$.*

## 2.2 Arbitrage

**Definition 2.13** (Arbitrage)**.** *Arbitrage is the simultaneous purchase and sale of an asset in different market to exploit price difference. It is an investment strategy where an investor buys and sells assets in different market conditions to generate a profit.*

Generally speaking, the greater the risk, the greater the return required to make investments an attractive enough prospects to attract funds. In essence of arbitrage, it is not possible to guarantee a profit without exposure to risk, were is possible to do so, the arbitrageurs would do so, in unlimited quantity, using the market as a money pump to extract arbitrarily large quantities of riskless profits, this would make impossible for the market to be in equilibrium, which we assume in this text. Thus a market with arbitrage opportunity would be a disorderly market, i.e. too disordered to model. The minimum requirements of absence of arbitrage opportunity is enough to allow one to build a model of a financial market, is idealized, yet realistic enough to provide real insights and handle the mathematics necessary to price standard, contingent claims. One is able to determine prices, using the **arbitrage pricing techniques.** An arbitrage opportunity arises in a market, where prices quoted are not arbitrage prices. One developes financial models, to spot such pricing errors.

### 2.2.1 Arbitrage Relationships

Consider the following determinants of the value of the option

1. Current Stock Price - $S(t)$
2. Strike Price - $K$
3. Stock Volatility - $\sigma$
4. Time to expiry $T - t$
5. Interest Rates $r$

At expiry, the only variables that matter are the stock prices $S(t)$ and strike price $K$, the payoff $C$ is given by $(S(T) - K)^+, P = (S(T) - K)^-(:= \max\{K - S(T), 0\})$.
We see that an increase in the stock price will increase the value of a call option, provided all the other factors are unchanged over the period of time. The opposite happens when the strike price is increased: the price of a call option goes down.
When one buys an option, they bet on a favourable outcome. The outcome is uncertain, this can be represented by a probability density; favourable outcomes are governed by the tail of the density. An increase in volatility flattens out the density and hence thickens the tails and increase the value of both call and put options. This holds true for all cases where one does not suffer from severe unfavourable outcomes.
In a simple model, one might argue that longer the period of expiry the more can happen to the price of the stock. So a longer period increases the possibility of movements of the stock price and hence the value of a call should be higher the more time remains before the expiry. An increase in volatility increase the likelihood of favourable outcomes at expiry, whereas the

stock price movements before expiry, whereas the stock price movements before the expiry may cancel themselves out. A longer time until expiry might also increase the possibility of the adverse effects from which the stock price has to recover before the expiry.

Interest rates also changes the price of an option. An increase in the interest rate tends to increase the expected growth rate in an economy and hence the stock price tends to increase.On the other hand, the present value of any future cash flows decrease. This, decreases the value of a put option and increase the value of a call option.

### 2.2.2 Arbitrage Bounds

For European Options (puts and calls) with identical underlying (say a stock $S$), strike $K$ and expiry date $T$.

**Theorem 2.1** (Fundamental Relationship)**.** *We have the following put-call parity between the prices of the underlying asset $S$ and European call and put options on stocks that pay no dividends:*

$$S + P - C = Ke^{-r(T-t)}.$$

**Theorem 2.2.** *The following bounds hold for European call options:*

$$\max\{S(t) - e^{-r(T-t)}K, 0\} = (S(t) - e^{-r(T-t)}K)^{+} \leq C(t) \leq S(t).$$

**Theorem 2.3.** *For a stock not paying dividends we have*

$$C_A(t) = C_E(t).$$

Qualitatively, there are two reasons why an American call should not be exercised early:

1. Insurance - An investor who holds a call option instead of the underlying stock is 'insured; against a fall in stock price below $K$, and if he exercised early, he loses this insurance.

2. Interested on the strike price. When the holder exercise the option, he buys the stock and pays the strike price, $K$. Early exercise at $t < T$ deprives the holder of the interest on K between times $t$ and $T$ : the later he pays out $K$, the better.

We consider a one$-$period model. i.e. we allow trading only at $t = 0$ and $t = T = 1$. Our aim is to value at $t = 0$ a European derivative on a stock $S$ with maturity $T$.

Model $S_T$ as a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The derivative is given by $H = f(S_T)$, i.e. it is a random variable for a suitable function $f(.)$). We could then price the derivatives using some discount factor $\beta$ by using the expected value of the discounted future payoff:

$$H_0 = \mathbb{E}(\beta H).$$

The real question that arises here is how do one pick the probability $\mathbb{P}$. Investors have different opinions about the distribution of the price $S_T$.

## 2.3 Risk-Neutral Measure

We can use the no-arbitrage principle and construct a hedging portfolio using only known (and already priced) securities to duplicate the payoff $H$. We assume

1. Investors are nonsatiable, i.e. they always prefer more to less.
2. Markets do not allow arbitrage, i.e. the possibility of risk$-$free profits.

From the no$-$arbitrage principle we see: *If it is possible to duplicate a payoff $H$ of a derivative using a portfolio $V$ of underlying (basic) securities, i.e. $H(\omega) = V(\omega), \forall\omega$, the price of the portfolio at $t = 0$ must equal the price of the derivative at $t = 0$.* The no$-$arbitrage price is independent of the individual preferences of the investor (given by certain probability assumptions about the future, i.e. a probability measure $\mathbb{P}$). We can thus identify a special, so$-$called **risk-neutral probablity measure** $\mathbb{P}^*$, such that

$$H_0 = \mathbb{E}^*(\beta H) = (p^* \cdot \beta(S_1 - K) + (1 - p^*) \cdot 0) = 1.$$

**Definition 2.14** (First Fundamental Theorem of Asset Pricing)**.** *The no$-$arbitrage condition is equivalent to the existence of an equivalent martingale measure.*

*We can price assets using the expectation operator is equivalent to the uniqueness of the equivalent martingale measure.*

### 2.3.1 A Single Period Model

Consider a single period model, i.e. we have two time$-$indexes, say $t = 0$, which is the current date (time) and $t = T$, which is the terminal date for all economic activities. The financial market contains $d + 1$ traded financial assets, whose price at time $t = 0$ are denoted by the vector $S(0) \in \mathbb{R}^{d+1}$,

$$S(0) = (S_0(0), S_0(1), ..., S_d(0))^{'}$$

At the time $T$, the owner of financial asset number $i$ receives a random payment depending on the state of the world. We can model this randomness, by introducing a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with a finite $|\Omega| = N$ of points (each corresponding to a certain state of the world) $\omega_1, \omega_2, ..., \omega_j, ..., \omega_N$, each of the positive probability: set of subsets of $\Omega$ on which $\mathbb{P}(.)$ is set of defined, here $\mathcal{F} = \mathcal{P}(\Omega)$ the set of all subsets of $\Omega$. We write the random payment arising from financial asset $i$ as

$$S_i(T) = (S_i(T, \omega_1), ..., S_i(T, \omega_j), ...., S_i(T, \omega_N))^{'}.$$

At the time $t = 0$, the agents can buy and sell financial assets. The portfolio position of an individual agent by trading strategy $\phi$, which is an $\mathbb{R}^{d+1}$ vector,

$$\phi = (\phi_0, \phi_1, ...\phi_d)^{'}.$$

Here $\phi_i$ denote the quantity of the $i$th asset bought at time $t = 0$, which may be negative as well as positive.

At time $t = 0$ we invest the amount $S(0)^{'}\phi = \sum_{i=0}^{d} \phi_i S_i(0)$ and at time $t = T$ we receive the random payment $S(T, \omega)^{'}\phi = \sum_{i=0}^{d} \phi_i S_i(T, \omega)$ depending on the realized state $\omega$. Using the $(d + 1) \times N-$matrix $S$,whose columns are vectors $S(T, \omega)$. An arbitrage strategy is a vector $\phi \in \mathbb{R}^{d+1}$ such that $S(0)^{'}\phi = 0$, our net investment at time $t = 0$ is zero, and

$$S(T, \omega)^{'}\phi \geq 0, \forall \omega \in \Omega$$

and there exists a $\omega \in \Omega$ such that $S(T, \omega)^{'}\phi > 0$.

We equivalently, formulate this as: $S(0)^{'}\phi < 0$, we borrow money consumption at time $t = 0$, and

$$S(T, \omega)^{'}\phi \geq 0, \forall \omega \in \Omega$$

i.e. we don't have to repay anything at $t = T$.

### 2.3.2 Complete and Arbitrage Free Market

**Theorem 2.4.** *There is no arbitrage if and only if there exists a vector*

$$\phi \in \mathbb{R}^N, \phi_i > 0, \; \forall\, 1 \leq i \leq N$$

*such that*

$$S\phi = S(0).$$

The vector $\phi$ is called a state-price vector. We can thus reformulate the above statement as *There is no arbitrage if and only there exists a state-price vector.*
We could choose an any asset $i$ as long as its price vector $(S_i(0), S_i(T, \omega_1), ..., S_i(T, \omega_N))'$ only contains positive entries and express all prices in units of this asset.
Let us assume that asset 0 is a riskless bond paying 1 unit in all states $\omega \in \Omega$ at time $T$. This means that $S_0(T, \omega) = 1$ in all states of the world $\omega \in \Omega$. Thus

$$\frac{S_0(0)}{\psi_0} = \sum_{j=1}^{N} q_j S_0(T, \omega_j) = \sum_{j=1}^{N} q_j 1 = 1,$$

and $\psi_0$ is the discount on riskless borrowing. Introducing an interest rate $r$, we must have

$$S_0(0) = \sum_{j=1}^{N} q_j \frac{S_i(T, \omega_j)}{(1+r)^T} = \mathbb{E}_Q(\frac{S_i(T)}{(1+r)^T}).$$

We can write this as

$$\frac{S_i(T)}{(1+r)^0} = \mathbb{E}_Q(\frac{S_i(T)}{(1+r)^T}).$$

We show that processes $S_i(t)/(1+r)^t, t = 0$, are $\mathbb{Q}-$martingales. We can now see

**Theorem 2.5.** *The (arbitrage-free) market is complete if and only if there exists a unique martingale measure.*

Risk-Neutral pricing corresponds to using the expectation operator with respect to an equivalent martingale measure. Once one knows how the given prices of our $(d+1)$ financial assets should be related in order to exclude arbitrage opportunities, one should be able to price the newly introduced financial instruments by its random payments $\delta(T) = (\delta(T, \omega_1), ...., \delta(T, \omega_j), ...\delta(T, \omega_N))'$

at time $t = T$ and ask for its pricing $\delta(0)$ at time $t = 0$. The idea is to use an equivalent probability measure $Q$ and set

$$\delta(0) = \mathbb{E}_Q(\delta(T)/(1+r)^T)$$

Unfortunately as we do not have a unique maratingales measure in general, we cannot guarantee the uniqueness martingale measure in general, we cannot guarantee the uniqueness of the $t = 0$ price. However, we can see that there is only one equivalent martingale measure at our disposal. Given a set of financial assets on a market, the underlying question is whether we are able to price any new financial asset which might be introduced in the market, or equivalently whether we can replicate the cash-flow of the new asset by means of a portfolio of our original assets. If we can replicate every new asset, the market is called complete.

**Theorem 2.6.** *Suppose there is arbitrage opportunities. Then the model is complete if and only if the matrix equation*

$$S'\phi = \delta$$

*has a solution $\phi \in \mathbb{R}^{d+1}$ for any vector $\delta \in \mathbb{R}^N$.*

## 2.4 Market Models

**Definition 2.15** (Numeraire)**.** *A numeraire is a price process $(X(t))_{t=0}^T$ (a sequence of random variable), which is strictly positive for all $t \in \{0, 1, 2, ..., T\}$.*

For the standard approach the risk−free bank account process is used as numeraire. We take $S_0(0) = 1$, (i.e. units of the initial value off our numeraire), and define $\beta(t) := \frac{1}{S_0(t)}$ as a discount factor.

**Definition 2.16** (Trading Strategy)**.** *A trading strategy (or dynamic portfolio) $\phi$ is a $\mathbb{R}^{d+1}$ vector stochastic process $\phi = (\phi(t))_{t=1}^T = ((\phi_0(t,\omega), \phi_1(t,\omega), ..., phi_d(t,\omega))')_{t=1}^T$ which is predictable: each $\phi_i(t)$ is $\mathcal{F}_{t-1}$ measurable for $t \geq 1$. Here $\phi_i(t)$ denote the number of shares of asset $i$ held in the portfolio at time $t-$ to be determined on the basis of information available before time $t$; i.e. the investor selects his time $t$ portfolio after observing the $S(t-1)$.*

However, the portfolio $\phi(t)$ must be established before, and held until after, announcement of the prices $S(t)$. The components $\phi_i(t)$ may assume negative as well as positive values. Reflecting the fact that one allow short sales and assume the assets are perfectly divisible.

**Definition 2.17** (Wealth or Value Process)**.** *The value of the portfolio at time $t$ is the scalar product*

$$V_\phi(t) = \phi(t) \cdot S(t) = \sum_{i=0}^{d} \phi_i(t) S_i(t), \ (t = 1, 2, ...., T) \text{ and } V_p hi(0) = \phi(1) \cdot S(0).$$

*The process $V_\phi(t, \omega)$ is called wealth or value process of the trading strategy $\phi$.*

The initial wealth $V_\phi(t, \omega)$ is called the initial investment or the endowment of the investor. Now $\phi(t) \cdot S(t-1)$ reflects the market value of the portfolio just after it has been established at time $t-1$, whereas $\phi(t) \cdot S(t)$ is the value just after time $t$ prices are observed, but before changes are made in the portfolio. Hence,

$$\phi(t) \cdot (S(t) - S(t-1)) = \phi(t) \Delta S(t)$$

is the change in the market value due to changes in security prices which occur between time $t-1$ and $t$. This motivates

**Definition 2.18** (Gains process)**.** *The gains process $G_\phi$ of trading strategy $\phi$ is given by*

$$G_\phi(t) := \sum_{\tau=1}^{t} \phi(\tau) \cdot (S(\tau) - S(\tau - 1))$$

$$= \sum_{\tau=1}^{t} \phi(\tau) \cdot \Delta S(\tau), \ \ (t = 1, 2, ...., T).$$

One can define $\tilde{S}(t) = (1, \beta(t) S_1(t), ..., \beta(t) S_d(t))'$, the vector of discounted prices and consider the discounted value process

$$\tilde{V}_\phi(t) = \beta(t)(\phi(t) \cdot S(t)) = \phi(t) \cdot \tilde{S}(t), \ \ (t = 1, 2, ...., T).$$

and the discounted gains process

$$\tilde{G}_\phi(t) := \sum_{\tau=1}^{t} \phi(\tau) \cdot (\tilde{S}(\tau) - \tilde{S}(\tau - 1))$$

$$= \sum_{\tau=1}^{t} \phi(\tau) \cdot \Delta\tilde{S}(\tau), \quad (t = 1, 2, ...., T).$$

### 2.4.1 Trading Strategies

Observe the discounted gains process reflects the gains from trading with assets 1 to $d$ only, which in case of standard model are risky assets.

**Definition 2.19** (Self Financing Strategy). *The strategy $\phi$ is self−financing, $\phi \in \Phi$, if*

$$\phi(t) \cdot S(t) = \phi(t+1) \cdot S(t)(t = 1, 2, ..., T-1).$$

This means when new prices $S(t)$ are quoted at time $t$, the investors adjusts his portfolio $\phi(t)$ to $\phi(t+1)$ without bringing in any wealth.

**Proposition 1**

Let $X(t)$ be a numeraire. A trading strategy $\phi$ is self financing with respect to $S(t)$ if and only if $\phi$ is self−financing with respect to $X(t)^{-1}S(t)$.

**Corollary 2.6.1.** *A trading strategy $\phi$ is self−financing with respect to $S(t)$ if and only if $\phi$ is self−financing with respect to $\tilde{S}(t)$.*

**Proposition 2**

A trading strategy $\phi$ belongs to $\Phi$ if and only if

$$\tilde{V}_\phi(t) = V_\phi(0) + \tilde{G}_\phi(t), \quad (t = 0, 1, 2, ..., T).$$

**Proposition 3**

If $(\phi_1(t), ..., \phi_d(t))'$ is predictable and $V_0$ is $\mathcal{F}_0$−measurable, there is a unique predictable process $(\phi_0(t))_{t=1}^{T}$ such that $\phi = (\phi_0, \phi_1, ..., \phi_d)'$ is a self−financing with initial value of the corresponding portfolio $V_\phi(0) = V_0$.

**Definition 2.20** (Contingent Claim). *A contingent claim $X$ with maturity date $T$ is an arbitrary $\mathcal{F}_T = \mathcal{F}$−measurable random variables (which is by finiteness of the probability space bounded). W denote the class of all contingent claims by*

$$L^0 = L^0(\Omega, \mathcal{F}, \mathbb{P}).$$

### 2.4.2 Equivalent Martingale Measure

**Definition 2.21** (Arbitrage Strategy)**.** *Let $\tilde{\Phi} \subset \Phi$ be a set of self–financing strategies. A strategies $\phi \in \tilde{\Phi}$ is called an arbitrage opportunity or arbitrage strategy with respect to $\tilde{\Phi}$ if* $\mathbb{P}\{V_\phi(T) = 0\} = 1$, *and the terminal wealth of $\phi$ satisfies*

$$\mathbb{P}\{V_\phi(T) \geq 0\} = 1 \textit{ and } \mathbb{P}\{V_\phi(T) > 0\} > 0.$$

**Definition 2.22** (Arbitrage Free Security Market)**.** *We say that a security market $\mathcal{M}$ is arbitrage–free if there are no arbitrage opportunity in the class $\Phi$ of trading strategies.*

One can use the above to analyze the sample path of the price process. One observes a realization $S(t, \omega)$ of the price process $S(t)$. One can switch to unique sequences of partitions $\{\mathcal{P}_t\}$ corresponding to the filtration $\{\mathcal{F}_t\}$. So at time $t$ one knows the set $A_t \in \mathcal{P}_t$ with $\omega \in A_t$. A set $A \in \mathcal{P}_t$ is the disjoint union of sets $A_1, A_2, ..., A_K \in \mathcal{P}_{t+1}$. Since $S(u)$ is $\mathcal{F}_u-$ measurable $S(t)$ is constant on $A$ and $S(t+1)$ is constant on the $A_k$, $k = 1, 2..., K$. So one can think of $A$ as the time 0 state in a single−period model and each $A_k$ corresponds to a state at time 1 in the single−period model.

**Lemma 2.7.** *If the market model contains no arbitrage opportunities, then for all* $t \in \{0, 1, 2...., T-1\}$, *for all self–financing trading strategies $\phi \in \Phi$ and any $A \in \mathcal{P}_t$, we have*

1. $\mathbb{P}(\tilde{V}_\phi(t+1) - \tilde{V}_\phi(t) \geq 0|A) = 1 \Longrightarrow \mathbb{P}(\tilde{V}_\phi(t+1) - \tilde{V}_\phi(t) = 0|A) = 1,$
2. $\mathbb{P}(\tilde{V}_\phi(t+1) - \tilde{V}_\phi(t) \leq 0|A) = 1 \Longrightarrow \mathbb{P}(\tilde{V}_\phi(t+1) - \tilde{V}_\phi(t) = 0|A) = 1.$

**Proposition 4**

If an equivalent martingale measure−exists − that is, if $\mathcal{P}(\tilde{S}) \neq \emptyset-$ then the market $\mathcal{M}$ is arbitrage−free.

**Definition 2.23** (Martingale Measure)**.** *A probability measure $\mathbb{P}^*$ on $(\Omega, \mathcal{F}_T\}$ equivalent to $\mathbb{P}$ is called a martingale measure for $\tilde{S}$ if the process $\tilde{S}$ follows a $\mathbb{P}^*-$martingale with respect to filtration $\mathcal{F}$. We denote by $\mathcal{P}(\tilde{S})$ the class of equivalent martingale measure.*

**Proposition 5**

Let $\mathbb{P}^*$ be an equivalent martingale measure ($\mathbb{P}^* \in \mathcal{P}(\tilde{S})$) and $\phi \in \Phi$ any self financing strategy. Then the wealth process $\tilde{V}_\phi(t)$ is a $\mathbb{P}^*-$martingale with respect to the filtration $\mathcal{F}$.

**Proposition 6**

If the market $\mathcal{M}$ is arbitrage−free, then the class $\mathcal{P}(\tilde{S})$ of equivalent martingale measures is non−empty.

Let $L^0 = L^0(\Omega, \mathcal{F}, \mathbb{P})$ the set of random variable on $(\Omega, \mathcal{F})$ and

$$L^0_{++}(\Omega, \mathcal{F}, \mathbb{P}) := \{X \in L^0 : X(\omega) \geq 0 \; \forall \omega \in \Omega \; and \; \exists \, \omega \in \Omega \; s.t. \; X(\omega) > 0\}.$$

Here $L^0_{++}$ is a cone−closed under vector addition and multiplication by positive scalars, using $L^0_{++}$ we can write the arbitrage condition more compactly as

$$V_\phi(0) = \tilde{V}(0) = 0 \Rightarrow \tilde{V}_\phi(T) \neq L^0_{++}(\Omega, \mathcal{F}, \mathbb{P})$$

for any self−financing strategy $\phi$.

**Lemma 2.8.** *In an arbitrage−free market any predictable vector process $\phi' = (\phi_1, \phi_2, ..., \phi_d)$ satisfies*

$$\tilde{G}_{\phi'}(T) \notin L^0_{++}(\Omega, \mathcal{F}, \mathbb{P}).$$

**Definition 2.24** (Attainable Contingent Claims)**.** *We call the subspace $K$ of $L^0(\Omega, \mathcal{F}, \mathbb{P})$ defined by*

$$K = \{X \in L^0(\Omega, \mathcal{F}, \mathbb{P}) : X = \tilde{G}(T), \phi \;\; predictable\}$$

*the set of contingent claims attainable at price* 0*. Thus, we can say a market is arbitrage−free if and only if*

$$K \bigcap L^0_{++}(\Omega, \mathcal{F}, \mathbb{P}) = \emptyset.$$

**Theorem 2.9** (No−arbitrage Theorem)**.** *The market $\mathcal{M}$ is arbitrage−free if and only if there exists a probability measure $\mathbb{P}^*$ measure equivalent to $\mathbb{P}$ under which the discounted $d-$dimensional asset price process $\tilde{S}$ is a $\mathbb{P}^*-$martingale.*

## 2.5 Contingent Claims

**Definition 2.25** (Attainable Contingent Claim)**.** *A contingent claim is attainable if there exists a replicating strategy $\phi \in \Phi$ such that*

$$V_\phi(T) = X$$

So the replicating strategy generates the same time $T$ cash−flow as does $X$. Working with discounted values, we find

$$\beta(T)X = \tilde{V}_\phi(T) = V(0) + \tilde{G}_\phi(T).$$

So the discounted value of a contingent claim is given by the initial cost of setting up a replicating strategy and the gains from trading. In a highly efficient security market we expect that the law of one price holds true, that is for a specified cash−flow there exists only one price at any time instant. Arbitrageurs would use the opportunity to cash in a risk−less profit. So the no arbitrage condition implies that for an attainable contingent claim its time $t$ price must be given by the value of any replicating strategy. This is the basic idea of arbitrage pricing theory.
The idea is to replicate a given cash−flow at a given point in time. Using a self−financing trading strategy the investor's wealth may go negative at time $t < T$, but he must be able to cover his debt at the final date. To avoid negative wealth the concept of admissible strategies is introduced. A self−financing strategy $\phi \in \Phi$ is called admissible if $V_\phi(t) \geq 0$ for each $t = 0, 1, 2, ...., T$. Assume $\Phi_a$ for the class of admissible trading strategies.

### 2.5.1 Contingent Claim pricing

**How can we determine the value of a contingent claim $X$ i.e. a cash−flows at time $T$, at time $t < T$?**

**Proposition 7**

Suppose the market $\mathcal{M}$ is arbitrage−free. Then any attainable contingent claim $X$ is uniquely replicated in $\mathcal{M}$.

**Definition 2.26** (Arbitrage Price Process)**.** *Suppose the market is arbitrage−free. Let $X$ be attainable contingent claim with time $T$ maturity. Then the arbitrage price process $\pi_X(t)$, $0 \leq$*

$t \leq T$ *or simply arbitrage price of* $X$ *is given by the value process of any replicating strategy* $\phi$ *for* $X$.

**Proposition 8**

The arbitrage price process of any attainable contingent claim $X$ is given by the **risk-neutral valuation problem**

$$\pi_X(t) = \beta(t)^{-1}\mathbb{E}^*(X\beta(T)|\mathcal{F}_t) \ \ \forall t = 0, 1, 2, ..., T,$$

where $\mathbb{E}^*$ is expectation operator with respect to an equivalent martingale measure $\mathbb{P}^*$.

In case of an arbitrage$-$free market $\mathcal{M}$ one can even insist on replicating non negative contingent claims by an admissible strategy $\phi \in \Phi_a$. Indeed, if $\phi$ is self$-$financing and $\mathbb{P}^*$ is an equivalent martingale measure under which discounted prices $S$ are $\mathbb{P}^*-$martingale, $\tilde{V}_\phi(t)$ is also a $\mathbb{P}^*-$martingale, being the martingale transform of the martingale $\tilde{S}$ by $\phi$. So

$$\tilde{V}_\phi(t) = \mathbb{E}(\tilde{V}_\phi(T)|\mathcal{F}_t) \ (t = 0, 1, 2, ...T).$$

If $\phi$ replicates $X, V_\phi(T) = X \geq 0$, so discounting, $\tilde{V}_\phi(T) \geq 0$, so the above equation gives $\tilde{V}_\phi(t) \geq 0$ for each $t$. Thus all the values at each time $t$ are non$-$negative, not just the final value at time $T-$so $\phi$ is admissible.

**Theorem 2.10** (Completeness Theorem)**.** *An arbitrage$-$free market* $\mathcal{M}$ *is complete and only if there exists a unique probability measure* $\mathbb{P}^*$ *equivalent to* $\mathbb{P}$ *under which discounted asset prices are martingales.*

## 2.6 Continuous Time Financial Model

To say every contingent claim can be replicated means that every $\mathbb{P}^*-$martingale can be represented as a martingale transform by a replicating trading strategy $\phi$.

**Definition 2.27** ($\mathbb{P}^*-$admissible )**.** *A self$-$financing trading strategy* $\phi$ *is called* $(\mathbb{P}^*-)$ *admissible if the relative gains process*

$$\tilde{G}_\phi(t) = \int_0^t \phi(u) d\tilde{S}(u)$$

*is a* $(\mathbb{P}^*-)$ *martingale. The class of all* $(\mathbb{P}^*-)$ *admissible trading strategies is denoted by* $\Phi(\mathbb{P}^*)$.

**Theorem 2.11.** *The financial market model $\mathcal{M}$ contains no arbitrage opportunities in $\Phi(\mathbb{P}^*)$.*

**Theorem 2.12** (Risk-Neutral Valuation Formula)**.** *The arbitrage process of any attainable claim is given by the risk−neutral valuation formula*

$$\Pi_X(t) = S_0(t)\mathbb{E}_{\mathbb{P}^*}[\frac{X}{S_0(T)}|\mathcal{F}_t].$$

**Corollary 2.12.1.** *For any two replicating portfolios $\phi, \psi \in \Phi(\mathbb{P}^*)$, we have*

$$V_\psi(t) = V_\phi(t).$$

**Lemma 2.13.** *Assume that the discounted contingent claim $X/S_0(t)$ is $\mathbb{P}^*$ integral. If the $(\mathbb{P}^*)-$martingale $M$ defined by*

$$M(t) = \mathbb{E}_{\mathbb{P}^*}[\frac{X}{S_0(T)}|\mathcal{F}_t]$$

*admits an integral representations of the form*

$$M(t) = x + \sum_{i=1}^{d} \int_0^t \phi_i(u) d\tilde{S}_i(u).$$

**Theorem 2.14.** *If the strong martingale measure $\mathbb{P}^*$ is the unique martingale measure for the financial market model $\mathcal{M}$, then $\mathcal{M}$ is complete, in the restricted sense that every contingent claim $X$ satisfying*

$$\frac{X}{S_0(T)} \in L^1(\mathcal{F}, \mathbb{P})$$

*is attainable.*

One can develop the technique of change of numeraire which is going to be useful in valuation of contingent claims. Assume that we use $S_0(t)$ as numeraire, and that there exists a (strong) equivalent martingale measure $\mathbb{P}^*$, i.e. the basic security prices discounted with respected to $S_0$ are $\mathbb{P}-$martingales.

**Theorem 2.15.** *Let $X(t)$ be a non−dividend−paying numeraire such that $\frac{X(t)}{S_0(t)}$ is a $\mathbb{P}^*-$martingale. Then there exists a probability measure $Q_X$, defined by its Radon−Nikodym derivative $\eta(T)$ with respect to $\mathbb{P}^*$,*

$$\eta(t) = \frac{dQ_X}{d\mathbb{P}^*}|_{\mathcal{F}_\sqcup} = \frac{X(t)}{X(0)S_0(t)},$$

*such that*

1. *The basic security prices discounted with respect to* $X$ *are* $Q_X-$*martingales;*

2. *If a contingent claim* $Y$ *is attainable under* $(S_0(t), \mathbb{P}^*)$*, then it is attainable under* $(X(t), Q_X)$*, and the replicating portfolio is the same. So the arbitrage price processes, given by risk−neutral valuation formula coincide.*

**Corollary 2.15.1.** *Let* $X(t), Y(t)$ *be a numeraires satisfying the assumption of the above proposition and* $Z$ *a contingent claim. Then we have the change−of−numeraire formula*

$$X(t)\mathbb{E}_{Q_X}[\frac{Z}{X(T)}|\mathcal{F}_t] = Y(t)\mathbb{E}_{Q_Y}[\frac{Z}{Y(t)}|\mathcal{F}_t].$$

**Proposition 9**

Denote by $T$ and $K$ respectively the maturity and exercise of a European call option on an asset $Z$. The time $t = 0$ price $C(0)$ of the call can be written as

$$\frac{C(0)}{p(0,T)} = \mathbb{E}_{Q_T}[(Z(T) - K)^+],$$

or

$$C(0) = Z(0)Q_Z(A) - K_p(0,T)Q_T(A).$$

where $A = \{\omega \ : \ Z(T,\omega) > K\}$.

## 2.7 The Generalized Black-Scholes Model

### 2.7.1 The Model

We consider a financial market $\mathcal{M}^{BS}$ is which $d+1$ assets are traded continuously. The first of these is an asset without systematic risk, called account, which price process defined. Its dynamics is given by

$$dB(t) = B(t)r(t)dt, \ \ B(0) = 1.$$

The remaining $d$ assets (usually referred to as stocks) are subject to systematic risk. The price process $S_i(t), \ 0 \leq t \leq T$ of the stock $1 \leq i \leq d$ is modelled by linear Stochastic differential equation

$$dS_i(t) = S_(t)(b_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t)dW_j(t)) \ , \quad S_i(0) = p_i \in (0, \infty);$$

here $W(t) = (W_1(t), W_2(t), ..., W_n(t)), 0 \leq t \leq T$ is a standard $n-$dimensional Brownian motion defined on filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ . We assume the underlying filtration $\mathbb{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$ is Brownian filtration.

We interpret the process $\{r(t), 0 \leq t \leq T\}$ as the interest-rate process - the short rate or the spot rate $-$i.e. $r(t)$ is the (instantaneously riskless) instantaneous interest rate. The vector process $\{b(t)' = (b_1(t), b_2(t), ...., b_d(t)), 0 \leq t \leq T)$ is the vector of appreciation rates for the stocks, measuring the instantaneous rate of change of $S$ at time $t$. Finally the matrix process of volatilities $\{\sigma(t) = (\sigma_{ij}(t)), 1 \leq i \leq d, 1 \leq j \leq n, 0 \leq t \leq T\}$ models the instantaneous intensity with which the $j^{th}$ source of uncertainty influences the price of the $i^{th}$ stock at time t. The processes $r(t), b(t), \sigma(t)$ are referred to as the coefficients of the model $\mathcal{M}^{BS}$.

**Theorem 2.16.** *1. If $\mathcal{M}^{BS}$ is arbitrage-free (with respect to tame strategies), then there exists a progressively measurable process $\gamma : [0, T] \times \Omega \to \mathbb{R}^d$, called the market price of risk, such that*

$$(b(t) - r(t))\mathbb{1}_d = \sigma(t)\gamma(t), \quad 0 \leq t \leq T \ \ a.s.$$

*2. Conversely, if such a process $\gamma(.)$ exists and satisfies, in addition to the above requirements,*

*3.*

$$\int_0^T ||\gamma(t)||^2 dt < \infty, \quad a.s.$$

*and*

$$\mathbb{E}[\exp\{-\int_0^T \gamma(t)' dW(t) - \frac{1}{2}\int_0^T ||\gamma(t)||^2 dt\}] = 1$$

*then $\mathcal{M}^{BS}$ is arbitrage$-$free.*

### 2.7.2 Multi-Dimensional Price Process

Consider a simple model with two securities $S_1, S_2$. Let the price-process dynamics be given by

$$dS_1(t) = S_1(t)(b_1(t)dt + \sigma_1(t)dW(t)),$$

$$dS_2(t) = S_2(t)(b_2(t)dt + \sigma_2(t)dW(t)),$$

Assume the process $\gamma(.)$ as

$$\frac{b_1(t) - r(t)}{\sigma_1(t)} = \frac{b_2(t) - r(t)}{\sigma_2(t)} = \gamma(t).$$

Observe that in the numerators we have the excess rate of return of the risky assets over the risk-free rate and in the denominators the volatility of the assets. So these quotients can be interpreted as the risk premium per unit of volatility. As in the theorem above this ratio is often called the market price of risk. We can thus rewrite the equations as

$$dS_1(t) = S_1(t)(r(t) + \gamma(t)\sigma_1(t))dt + \sigma_1(t)dW(t)),$$

$$dS_2(t) = S_2(t)(r(t) + \gamma(t)\sigma_2(t))dt + \sigma_2(t)dW(t)),$$

If, for example $\gamma = 0$, then

$$S_1(t) = S_1(t)(r(t)dt + \sigma_1(t)dW(t)),$$

$$S_2(t) = S_2(t)(r(t)dt + \sigma_2(t)dW(t)),$$

and we see that $\tilde{S} = S_i/B,\ i = 1, 2$ are (local) martingales under $\mathbb{P}$ r IP, so we are already in our usual risk-neutral setting. If we set $\gamma = \sigma_2$, we get

$$d[\frac{S_1(t)}{S_2(t)}] = (\sigma_1(t) - \sigma_2(t))\frac{S_1(t)}{S_2(t)}dW(t)$$

So $S_1/S_2$ is a (local) martingale in this setting. Since the attitude towards risk is described by $\sigma_2(= \gamma)$(the risk in holding the asset $S_2$), $\mathbb{P}$ is called a risk-neutral measure with respect to $S_2$. Now assume that $d = n$, that the coefficients of the model satisfy the integrability conditions, that $\sigma$ is non$-$singular and that $\gamma(.)$ exists. Then the financial market model admits a unique equivalent martingale measure $\mathbb{P}^*$ with Radon-Nikodym derivative given by the Girsanov transformation

$$L(t) = \exp\{-\int_0^t \gamma(u)'dW(u) - \frac{1}{2}\int_0^t ||\gamma(u)||^2 du\},\ \ 0 \le t \le T$$

and $\gamma(t) = \sigma^{-1}(t)(b(t) - r(t)\mathbb{1}_d)$. This is an arbitrage free model. This is what we call a standard model.

### 2.7.3 Black Scholes Formula Valuation

**Theorem 2.17** (Risk-Neutral Valuation Formula)**.** *Let $\mathcal{M}^{BS}$ be a standard multi−dimensional Black-Scholes model and $X$ a contingent claim. The arbitrage price process of $X$ is given by the risk-neutral valuation formula*

$$\Pi_X(t) = B(t)\mathbb{E}^*[\frac{X}{B(T)}|\mathcal{F}_\sqcup] = \mathbb{E}^*[X \exp\{-\int_t^T r(u)du\}|\mathcal{F}_T].$$

By the risk-neutral valuation principle $\Pi_X(t) = e^{rt}M(t)$, and so $F(t,s) = e^{rt}G(t,s)$, and computing the partial derivatives of $F$ we obtain the representation. Specialising further and considering a European call with strike $K$ and maturity $T$ on the stock $S$(so $\Phi(T) = (S(T)-K)^+$), we can evaluate the above expected value and obtain The Black-Scholes price process of a European call is given by

$$C(t) = S(T)N(d_1(S(t), T-t)) - Ke^{-r(T-t)}N(d_2(S(t), T-t))$$

The functions $d_1(s,t)$ and $d_2(s,t)$ are given by

$$d_1(s,t) = \frac{\log(s/K) + (r + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}},$$

$$d_2(s,t) = d_1(s,t) = \frac{\log(s/K) + (r - \frac{\sigma^2}{2}t}{\sigma\sqrt{t}}$$

To obtain a replicating portfolio we use Ito's lemma to find the dynamics of the $\mathbb{P}^*-$martingale $M(t) = G(t, S(t))$ :

$$dM(t) = \sigma S(t)G_s(t, S(t))d\tilde{W}(t).$$

U sing this representation, we get in terms of the notation in the Black-Scholes model

$$h(t) = \sigma S(t)G_s(t, S(t))$$

which gives for the stock component of the replicating portfolio using the discounted assets

$$\phi_1(t) = G_s(t, S(t))B(t)$$

and using the self-financing condition the cash component is

$$\phi_0(t) = G(t, S(t)) - G_s(t, S(t))S(t).$$

To transfer this portfolio to undiscounted values we multiply it by the discount factor i.e. $F(t, S(t)) = B(t)G(t, S(t))$ and get:

**Proposition 10**

The replicating strategy in the classical Black−Scholes model is given by

$$\phi_0 = \frac{F(t, S(t)) - F_s(t, S(t))S(t)}{B(t)},$$

$$\phi_1 = F_s(t, S(t))$$

# References


Bingham, N.H. and Kiesel, Rudiger (2004). *Risk-Neutral Valuation*. Springer.

Rosenthal, Jeffrey S. (2006). *A first look at Rigorous Probability*. World Scientific Publishing Co. Pte. Ltd.

Schilling, Rene L. (2005). *Measures, Integrals and Martingales*. Cambridge University Press.

Williams, David (1991). *Probability with Martingales*. Cambridge University Press.