Method ② to find P. Soln of NHDE

→ Method of variation of parameters (MOVP)

Consider NHDE → $y'' + P(x)y' + Q(x)y = r(x)$ — ①

→ In MOUC 1) P and Q should be constants

2) $R(x)$ should be in some simple particular form.

→ Method of variation of parameters has no above limitations and works regardless of nature of P, Q & R provided that the G. Soln of HDE $y'' + P(x)y' + Q(x)y = 0$ — ② is already known.

For a NHDE → $y'' + P(x)y' + Q(x)y = r(x)$ — ①

If Gen. soln of $y'' + P(x)y' + Q(x)y = 0$ (HDE) — ②

is $y_g = c_1 y_1 + c_2 y_2$ — ③

Then P. soln by MOVP is given by

$$\boxed{y_p = v_1(x)\, y_1 + v_2(x)\, y_2} \quad (4)$$

where $v_1(x) = -\int \frac{y_2\, r(x)}{W(y_1, y_2)}\, dx$ & $v_2(x) = \int \frac{y_1\, r(x)}{W(y_1, y_2)}\, dx$

eg 1 Solve the D.E → $y'' + 4y = \sec 2x$ — (1)
using MOVP.

Soln →
Step 1 HDE → $y'' + 4y = 0$ — (2)
A.E → $m^2 + 4 = 0$, Roots → $m = \pm 2i$

So $y_g = c_1 \cos 2x + c_2 \sin 2x$ — (3)
$= c_1 y_1 + c_2 y_2$

where $y_1 = \cos 2x$ & $y_2 = \sin 2x$. (We can choose any fn as y_1 & y_2 out of two)

Step 2 So by MOVP the P. soln will be →

$$y_p = v_1(x) y_1 + v_2(x) y_2 \quad \text{— (4)}$$

where $v_1(x) = -\int \frac{y_2 \, r(x)^{*}}{W(y_1, y_2)} dx$ | $v_2(x) = \int \frac{y_1 \, r(x)^{*}}{W} dx$

(* Plz note here $r(x)$ is taken from given NHDE after reducing it to standard form $y'' + P(x)y' + Q(x)y = r(x)$ → If it is not in standard form 1st make it in std form & then take $r(x)$)

Now $W(y_1, y_2) = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$

So $v_1(x) = -\int \frac{(\sin 2x)(\sec 2x)}{2} dx = \frac{1}{4} \ln(\cos 2x)$

& $v_2(x) = \int \frac{(\cos 2x)(\sec 2x)}{2} dx = \frac{1}{2} x$.

So $y_p = v_1 y_1 + v_2 y_2 = \left(\frac{1}{4} \ln(\cos 2x)\right)(\cos 2x) + \left(\frac{1}{2} x\right)(\sin 2x)$ — (4)

So G. Soln of NHDE (1) by MOVP is

$= y_g + y_p$

where y_g is given by eqn (3)

& y_p is given by eqn (4).

eg ② $y'' + 3y' + 2y = 2e^{x}$ — ①

Solve the given D.E by MOVOP.

Soln → HDE → $y'' + 3y' + 2y = 0$

A.E → $m^2 + 3m + 2 = 0$, Roots ⇒ $m = -1, -2$.

So $y_g = c_1 e^{-x} + c_2 e^{-2x}$ — ②

$= c_1 y_1 + c_2 y_2$

$y_1 = e^{-x}$, $y_2 = e^{-2x}$ — ③

So now by MOVP →

$y_p = v_1(x) y_1 + v_2(x) y_2$ — ④

where $v_1(x) = -\int \frac{y_2\, r(x)}{W} dx$ | $v_2(x) = \int \frac{y_1\, r(x)}{W} dx$

$W(y_1, y_2) = W = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -e^{-3x}$.

So $v_1 = -\int \frac{e^{-2x}\,(2e^{x})}{-e^{-3x}} dx = e^{2x}$.

$v_2 = \int \frac{e^{-x}\,(2e^{x})}{-e^{-3x}} = -\frac{2}{3} e^{3x}$.

So $y_p = v_1 y_1 + v_2 y_2 = (e^{2x})(e^{-x}) + (+e^{-2x})\left(-\frac{2}{3} e^{3x}\right)$

$= \frac{1}{3} e^{x}$.

So Gen. Soln of given NHDE is → $= y_g + y_p$

$= (c_1 e^{-x} + c_2 e^{-2x}) + \frac{1}{3} e^{x}$

eg ③ $xy'' - 4xy = e^{2x}$ — ①

Solve the given D.E by MOVP.

Soln → As the L.H.S of ① is not in Euler-Cauchy form. So 1st of all we will reduce it to standard form

$$y'' - 4y = e^{2x}/x \quad \text{— ②} \qquad \text{So that } r(x) = e^{2x}/x, \quad P(x) = 0, \quad Q(x) = -4$$

Now → HDE → $y'' - 4y = 0$

A.E → $m^2 - 4 = 0$, $m = \pm 2$.

So $y_g = c_1 e^{2x} + c_2 e^{-2x}$ — ③

Again → Let $y_g = c_1 y_1 + c_2 y_2$ where $y_1 = e^{2x}$, $y_2 = e^{-2x}$

So by MOVP →

$$y_p = v_1(x) y_1 + v_2(x) y_2 \quad \text{— ④}$$

where $v_1(x) = -\int \frac{y_2\, r(x)}{W(y_1, y_2)} dx$ — ⑤ | $v_2(x) = \int \frac{y_1\, r(x)}{W(y_1, y_2)} dx$ — ⑥

Now $W(y_1, y_2) = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -2 - 2 = \underline{\underline{-4}}$

So $v_1(x) = \int \frac{e^{-2x} \cdot e^{2x}}{4 \cdot x} dx = \frac{1}{4} \int \frac{1}{x} dx = \frac{1}{4} \ln x$ — ⑦

& $v_2(x) = \int \frac{e^{2x} \cdot e^{2x}}{-4x} dx = -\frac{1}{4} \boxed{\int \frac{e^{4x}}{x} dx}$ → Not solvable

So $y_p = v_1 y_1 + v_2 y_2 = \frac{1}{4} \ln x \,(e^{2x}) - \left(\frac{1}{4} \int \frac{e^{4x}}{x} dx\right) e^{-2x}$ — ⑧

So that the G.S of NHDE ① is →

$$= y_g + y_p$$

$$= (c_1 e^{2x} + c_2 e^{-2x}) + \frac{1}{4} e^{2x} \ln x - \frac{1}{4} e^{-2x} \int \frac{e^{4x}}{x} dx.$$

32A

Clearly here the term $\left(\int \frac{e^{4x}}{x}dx\right)$ is not Solvable. We have to keep it as such. So it is one of the shortcoming of the MOVP when either integral becomes very difficult to solve or they are not solvable by usual known methods.

Qs for Practice (Try yourself)

Method of variation of parameters

32B

Q1 $y'' + y = \operatorname{cosec} x$

Q2 $y'' - y = e^{-x}\sin(e^{-x}) + \cos(e^{-x})$

Q3 $2y'' + 3y' + y = e^{-3x}$

Q4 $y'' - 2y' - 3y = 64xe^{-x}$

Find only $\boxed{y_p}$ by MOVP

Soln

1) $c_1\cos x + c_2\sin x - x\cos x + \sin x(\ln\sin x)$

2) $c_1e^{x} + c_2e^{-x} - e^{x}\sin(e^{-x})$

3) $y_p = \frac{1}{10}e^{-3x}$

4) $y_p = -e^{-x}\left[8x^2 + 4x + 1\right]$

Some Qs related to ~~NH~~ Legendre' D.E (NHDE)
Or NHDE form of general Euler–Cauchy ~~form~~ D.E

$$a_0 (ax+b)^2 \frac{d^2y}{dx^2} + a_1(ax+b)\frac{dy}{dx} + a_2 y = r(x) \quad \text{(1)}$$

eg(1). Find the G.S of the D.E

$$x^2y'' - 2xy' + 2y = x^3 \cos x \quad \text{(1)}$$

by MOVP.

Soln

Method (1). Clearly here R.H.S or D.E (1) is of ~~Euler~~ Cauchy form.

(*) So before proceeding to solve HDE first (as we were doing usually), here we will first change whole D.E (1) by using substitution →

$$x = e^z \quad \text{(2)} \quad \Rightarrow z = \ln x$$

So $x\frac{dy}{dx} = \frac{dy}{dz}$ | $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$ (3)

Put (2) & (3) in D.E (1) we get →

$$\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) - 2\frac{dy}{dz} + 2y = e^{3z}\cdot\cos(e^z)$$

$$\frac{d^2y}{dz^2} - 3\frac{dy}{dz} + 2y = e^{3z}\cdot\cos(e^z) \quad \text{(4)}$$

So this is linear NHDE with constant Coefficients

→ After changing (1) to this form (4), now we can proceed for (y_g) & (y_p) (as we were doing earlier)

So $\rightarrow$ H.D.E $\rightarrow$ $\dfrac{d^2y}{dz^2} - 3\dfrac{dy}{dz} + 2y = 0$ — (5)

A.E $\rightarrow$ $m^2 - 3m + 2 = 0$

Roots $\rightarrow$ $m = 1, 2$

So $y_g = c_1 e^{z} + c_2 e^{2z}$ — (6)

$= \cancel{c_1 x + c_2 x^2}$

~~[scribbled]~~

$\rightarrow$ Now for P.Soln by MOVP $\rightarrow$

$\therefore$ $y_g = c_1 e^{z} + c_2 e^{2z}$

$= c_1 y_1 + c_2 y_2$ | $y_1 = e^{z}$, $y_2 = e^{2z}$

So $y_p = v_1(z) y_1 + v_2(z) y_2$ — (7) (Here we will work in variable (z) as D.E (4) is in (z))

where

$v_1(z) = -\displaystyle\int \frac{y_2(z) r(z)}{W} dz$ — (8) | $v_2(z) = \displaystyle\int \frac{y_1(z) r(z)}{W} dz$ — (9)

$r(z) = e^{3z}\cos(e^{z})$ [Always check D.E is in standard form or not! ~~By using this method the D.E so obtained (4) will always in stan~~ while choosing $r(z)$]

$W = \begin{vmatrix} e^{z} & e^{2z} \\ e^{z} & 2e^{2z} \end{vmatrix}$

$= e^{3z}$

So $\rightarrow$ $v_1(z) = -\displaystyle\int \frac{e^{2z} e^{3z}\cos(e^{z})}{e^{3z}} dz = -\int e^{2z}\cos(e^{z}) dz$

$= -e^{z}\sin(e^{z}) - \cos(e^{z})$ — (10) [Put $e^{z} = t$]

$\rightarrow$ $v_2(z) = \displaystyle\int \frac{e^{z}\, e^{3z}\cos(e^{z})}{e^{3z}} dz = \int e^{z}\cos(e^{z}) dz = \sin(e^{z})$ — (11)

So that →

$$\rightarrow \quad y_p = v_1(z)\, y_1 + v_2(z)\, y_2$$

$$= \left[-e^z \sin(e^z) - \cos(e^z)\right](e^z) + (\sin(e^z))\cdot e^{2z}$$

$$= -\cos(e^z)\, e^z$$

So $y_p = -e^z \cos(e^z)$ — (12)

Hence G.S of NHDE (1) is

$$= y_g + y_g$$

$$= (c_1 e^z + c_2 e^{2z}) - e^z \cos(e^z)$$

$$= c_1 x + c_2 x^2 - x\cos(x)$$

Method (2)

$$x^2 y'' - 2xy' + 2y = x^3 \cos x \quad \text{— (1)}$$

So here we proceed as usual

HDE → $x^2 y'' - 2xy' + 2y = 0$ — (2)

Clearly it is Cauchy D.E

So Put $x = e^z$ — (3), $z = \ln x$

Hence $x\dfrac{dy}{dx} = \dfrac{dy}{dz}$ — (4) $\quad \Big| \quad x^2\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{dz^2} - \dfrac{dy}{dz}$ — (5)

Put (4) + (5) in (2) we get →

$$\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) - 2\left(\frac{dy}{dz}\right) + 2y = 0$$

$$\Rightarrow \frac{d^2y}{dz^2} - 3\frac{dy}{dz} + 2y = 0 \quad \text{— (6)}$$

A·E → $m^2-3m+2=0$

Roots → $m=1,2$

So $y_g = c_1 e^z + c_2 e^{2z}$
$\quad = c_1 x + c_2 x^2$ — (7)

→ Now to find P. Soln by MOVP →

∵ $y_g = c_1 x + c_2 x^2$
$\quad = c_1 y_1 + c_2 y_2$ $\left| \begin{array}{l} y_1 = x \\ y_2 = x^2 \end{array} \right.$

So $y_P = v_1(x) y_1 + v_2(x) y_2$ — (8)

where → $v_1(x) = -\int \frac{y_2\, r(x)}{W} dx$ (9) $\Big|$ $v_2(x) = \int \frac{y_1\, r(x)}{W} dx$ (10)

[Now here everything is fine but while finding v_1 & v_2 the $r(x)$ should not be taken $x^3 \cos x$. We must change D·E (1) to standard form, that is

$$y'' - \frac{2}{x} y' + \frac{2}{x^2} y = x \cos x$$

So that $\boxed{r(x) = x\cos x}$]

→ Take $r(x) = x\cos x$ & also $\underline{\underline{W = x^2}}$

we get →

$v_1(x) = -x\sin x - \cos x$
$v_2(x) = \sin x$

So that →
$y_P = v_1 y_1 + v_2 y_2$
$\quad = \underline{\underline{-x\cos x}}$ — (11)

So G.S of NHDE $= y_g + y_P$

y_g → given by (7)
y_P → given by (11)

eg (2) Solve the D.E

$$(2x+1)^2 y'' + 2(2x+1)y' - 4y = 8x + 4\ln(2x+1) \quad \text{--- (1)}$$

by MOUC.

Soln →

→ Clearly R.H.S ($r(x)$, particularly $\ln(2x+1)$) is not in form where we can apply MOUC. So it is better to change all D.E in variable (z) by using substitution $(2x+1) = e^z$ as given D.E is Legendre D.E.

→ So take $(2x+1) = e^z$ --- (2), $z = \ln(2x+1)$

Hence $(2x+1)\dfrac{dy}{dx} = 2\dfrac{dy}{dz}$ --- (3) (important) | $(2x+1)^2 \dfrac{d^2y}{dx^2} = 4\left(\dfrac{d^2y}{dz^2} - \dfrac{dy}{dz}\right)$ --- (4) (Don't forget)

Put (2) – (4) in (1) we get

$$4\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) + 4\frac{dy}{dz} - 4y = 4(e^z - 1) + 4z$$

$$\Rightarrow \boxed{\frac{d^2y}{dz^2} - y = e^z + z - 1} \quad \text{--- (5)}$$

which is linear NHDE in (z) & now can be solved by MOUC.

→ HDE → $\dfrac{d^2y}{dz^2} - y = 0$ | A.E → $m^2 - 1 = 0$

Roots → $m = \pm 1$

So $y_g = c_1 e^z + c_2 e^{-z}$ --- (6)

Now to find $y_p \rightarrow$ $\therefore r(z) = e^z + z - 1$

So $y_p = Aze^z + Bz + C$ — (7) $\left[\because e^z \text{ in } r(z) \text{ has } a=1, \text{ which is single root of A.E}\right]$ (e^{az})

$y_p' = A(ze^z + e^z) + B$ — (8)

$y_p'' = Aze^z + 2Ae^z$ — (9)

Put (7)–(9) in (5) we get $\rightarrow$ (after solving L.H.S)

$2Ae^z - Bz - C = e^z + z - 1$

Compare both sides we get

$A = \frac{1}{2},\ B = -1,\ C = 1$

So $\boxed{y_p = \frac{1}{2}ze^z - z + 1}$ — (7)

Hence the G.S of NHDE (1) is

$= y_g + y_p$

$= (c_1 e^z + c_2 e^{-z}) + \frac{1}{2}ze^z - z + 1$

$= \left(c_1(2x+1) + \frac{c_2}{(2x+1)}\right) + \frac{1}{2}\left(\ln(2x+1)\right)(2x+1) - \ln(2x+1) + 1.$

(*) If D.E is like $\rightarrow$ $x^2y'' + xy' - y = x^3$, say. And we have been asked to solve it by MOUC. Can we solve it by $\boxed{\text{method (2)}}$ as discussed earlier or not.

[Hint: Not, Method of undetermined Coeffecients is, Check $P(x)$ + $Q(x)$ of D.E in standard form]

Qs for Practice → Solution of NHDE & Euler-Cauchy, Legendre D.E type

38 A

Q1 $(x+2)^2 y'' + (x+2)y' - y = x$.
(Solve by both MOVP and MOVC)

Q2 $(2x+1)^2 y'' - 2(2x+1)y' - 12y = 6x$.
(Solve by both MOVP & MOVC)

Soln →
1) $\left(c_1(x+2) + \frac{c_2}{x+2}\right) + \left[\frac{1}{2}(x+2)\ln(x+2) - \frac{1}{4}(x+2) + 2\right]$

2) $\frac{c_1}{2x+1} + c_2(2x+1)^3 - \frac{3}{8}x + \frac{1}{16}$]