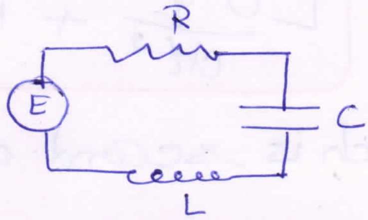
Note: Application of D.E to LRC – circuit

→ Here we will consider the application of ~~[illegible]~~ D.E to series circuit containing 1) electromotive force 2) resistor 3) Inductor + 4) Capacitor

→ Electromotive force (eg battery or generator) produces a flow of current in a closed circuit + this current produces so called voltage drop across each resistor, inductor + Capacitor.



→ Fundamental laws in the theory of electric circuits

Kirchhoff's voltage law → The algebraic sum of instantaneous voltage drop ~~across resistor, inductor +~~ around a close circuit in a specific direction is zero.

OR

The sum of the voltage drop across inductor, resistor + Capacitor is equal to total electromotive force (emf) in a closed circuit

→ Mathematically

$$\boxed{\underbrace{L\frac{di}{dt}}_{\substack{\text{Voltage drop across}\\ \text{Inductor}}} + \underbrace{Ri}_{\substack{\text{voltage drop}\\ \text{across resistor}}} + \underbrace{\frac{1}{C}q}_{\substack{\text{Voltage drop}\\ \text{across capacitor}}} = \underbrace{E}_{\text{emf}}} \quad \text{—(1)}$$

L → inductance
R → Resistance
C → Capacitance

So we have

$$L\frac{di}{dt} + Ri + \frac{1}{C}q = E \quad \text{—(1)}$$

$\therefore i = \frac{dq}{dt}$ —(2) So Put it in (1)

$$\boxed{L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E} \quad \text{—(3)}$$

which is second order non homogeneous linear D.E in (q)

Again, diff (1) w.r.t (t)

$$\boxed{L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = \frac{dE}{dt}} \quad \text{—(4)}$$

which is 2nd order NHDE (linear) in (i).

(*) As per Q asked we can use either (3) or (4).

Note → The 2nd order linear NHDE reduces to 1st order in two cases →

1) If circuit has no capacitor →

From eqn (1)
$$\boxed{L\frac{di}{dt} + Ri = E}$$

2) If circuit has no inductor →

From (1)
$$Ri + \frac{1}{C}q = E$$

$$\Rightarrow \boxed{R\frac{dq}{dt} + \frac{1}{C}q = E}$$

eg ① A circuit has in series an emf given by $E = \sin t$ volt, resistor of $2\,\Omega$, inductor of $1\,H$. If initial current is zero. Find current at any time $t > 0$.

Soln → ∵ the D.E governing the current in LRC-circuit is given by →

$$L\frac{di}{dt} + Ri + \frac{1}{C}q = E \quad \text{— ①}$$

∵ there is no capacitor so ① reduces to

$$L\frac{di}{dt} + Ri = E \quad \text{— ②}$$

Ω = ohm
H = henry

$$\frac{di}{dt} + 2i = \sin t \quad \text{— ③}$$

So $$I.F = e^{\int 2\,dt} = e^{2t} \quad \text{— ④}$$

So G.S of ③ is →

$$i(e^{2t}) = \int (\sin t)\, e^{2t}\, dt + C$$

$$i(e^{2t}) = \frac{e^{2t}}{5}(2\sin t - \cos t) + C \quad \text{← (Solving)} \quad \text{— ⑤}$$

∵ given that at $t = 0$, $i = 0$

So $0 = \frac{1}{5}(-1) + C \Rightarrow \boxed{C = \frac{1}{5}}$

So ⑤ becomes →

$$i(e^{2t}) = \frac{1}{5}e^{2t}(2\sin t - \cos t) + \frac{1}{5}$$

$$\boxed{i = \frac{1}{5}(2\sin t - \cos t) + \frac{1}{5}e^{-2t}}$$

↳ Desired soln.

eg ②

A circuit has in series an emf given by $\cos t$ V, resistor of $3\,\Omega$, inductor of 1 H and capacitor of 0.5 farad. If initial current ~~+~~ charge on capacitor are zero. Find charge on capacitor at any time $t > 0$.

Soln → ∵ current at any time $(t>0)$ in LRC circuit is governed by D.E →

$$L\frac{di}{dt} + Ri + \frac{1}{C}q = E \quad \text{--- (1)}$$

So as per given Q, ① changes to

$$\frac{di}{dt} + 3i + 2q = \cos t \quad \text{--- (2)}$$

∵ we have to find charge at any $t>0$ on capacitor so using $i = \frac{dq}{dt}$ in ② we get

$$\frac{d^2q}{dt^2} + 3\frac{dq}{dt} + 2q = \cos t \quad \text{--- (3)}$$

which is linear NHDE in (q)

$$\text{HDE} \rightarrow \frac{d^2q}{dt^2} + 3\frac{dq}{dt} + 2q = 0 \quad \text{--- (4)}$$

$$\text{A.E} \rightarrow m^2 + 3m + 2 = 0$$

$$\text{Roots} \rightarrow m = -1, -2.$$

$$\text{G.S of HDE (4) is} \rightarrow \left[q_g = c_1 e^{-t} + c_2 e^{-2t}\right] \quad \text{--- (5)}$$

Now for finding ~~___~~ particular Soln of ③ we can either use MOUC or MOVP.

let us use MOVC $\rightarrow$

$$\text{So } q_p = A\cos t + B\sin t \quad \text{— (6)} \qquad \left(\because r(t) = \cos t \text{ in (3)} \right)$$

$$\text{So } q_p' = -A\sin t + B\cos t \quad \text{— (7)}$$

$$q_p'' = -A\cos t - B\sin t \quad \text{— (8)}$$

So put q_p, q_p' & q_p'' in (3) we get $\rightarrow$

$$(-A\cos t - B\sin t) + 3(-A\sin t + B\cos t) + 2(A\cos t + B\sin t) = \cos t$$

$$\cos t(A+3B) + \sin t(B-3A) = \cos t + 0\cdot\sin t$$

Comparing coefficients of $\cancel{\cos t}$ & $\sin t$ we get

$$\left.\begin{array}{l} A+3B = 1 \\ -3A+B = 0 \end{array}\right\} \rightarrow \text{Solving we get} \rightarrow \left[\begin{array}{l} A = 1/10 \\ B = 3/10 \end{array}\right]$$

$$\text{So } q_p = \tfrac{1}{10}\cos t + \tfrac{3}{10}\sin t \quad \text{— (9)}$$

Hence G.S of D.E (3) is

$$= q_g + q_p$$

$$= (c_1 e^{-t} + c_2 e^{-2t}) + \tfrac{1}{10}\cos t + \tfrac{3}{10}\sin t$$

RLC – Circuits (Application of D.E)

Q1 An emf $E = 200e^{-5t}$ is applied to a series circuit consisting of 20 ohm resistor + 0.01 F capacitor. Find charge + current at any time assuming that there is no initial charge on capacitor.

Q2 Find current in a series circuit with $R = 10\,\Omega$, $L = 2\,H$, $E(t) = 20\cos 5t$. Determine current at any time $t > 0$

Q3 A circuit consist of an inductance of 2 Henry, resistance of 4 ohm + capacitance of 0.05 farad. If the charge + current is initially zero then find charge at any time (t) if there is constant emf of 100 volts.

Soln (1) $q = 10te^{-5t}$

$i = 10e^{-5t} - 50te^{-5t}$

(2) $i(t) = \cos 5t + \sin 5t - e^{-5t}$

(3) $q(t) = 5 - \frac{5}{3}e^{-t}(3\cos 3t + \sin 3t)$