Part ① To find (y_g)

Note Use of known solution to find an another Solution (Reduction of order)

(Method ① to find G.S of HDE)

1st method to find G. Soln of HDE.

Consider a HDE → $y'' + P(x)y' + Q(x)y = 0$ — ①

We have discussed earlier that if y_1 & y_2 are any two particular solutions of HDE ① & if they are L.I then $c_1y_1 + c_2y_2$ is the general soln of HDE ①. But from where to get y_1 & y_2. Unfortunately there is no method to find y_1 & y_2. However if we have one soln (which can be found by inspection or some other techniques) then other L.I soln can be found easily

So if → $y'' + P(x)y' + Q(x)y = 0$ — ① is a HDE and y_1 (or $y_1(x)$) is its one (nonzero) solution then the other solution of this D.E is given by →

$$y_2 = v(x)\cdot y_1 \quad ② \qquad \text{where} \quad v(x) = \int\left(\frac{1}{y_1^2}e^{-\int P(x)dx}\right)dx \quad ③$$

The ~~two~~ second solution (y_2) so obtained in this manner is always L.I to (y_1).

eg ① If $y = x^2$ ① is one soln of $x^2y'' + xy' - 4y = 0$ ② then find the other solution. Hence G.S.

Soln ∵ $y_1 = x^2$ — ①

So $y_2 = v(x)y_1$ ② where $v(x) = \int\left(\frac{1}{y_1^2}e^{-\int P(x)dx}\right)dx$ ③

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From (2) $x^2y'' + xy' - 4y = 0$

$\Rightarrow y'' + \frac{1}{x}y' - \frac{4}{x^2}y = 0$ — (4)

Imp → [Changing given D.E to standard form $y'' + P(x)y' + Q(x)y = 0$]

$\Rightarrow P(x) = 1/x$.

Hence $v(x) = \int \left(\frac{1}{x^4} e^{-\int 1/x\, dx}\right) dx = \int \left(\frac{1}{x^4} e^{-\ln x}\right) dx$

$= \int \frac{1}{x^5} dx = \frac{x^{-5+1}}{-5+1}$ (∵ we are looking for Particular Soln so c can be taken zero)

$\Rightarrow v(x) = -\frac{1}{4}x^{-4} = \frac{-1}{4x^4}$ — (5)

Hence → $y_2 = y_1 v(x) = (x^2)\left(\frac{-1}{4x^4}\right)$

$\left[y_2 = \frac{-1}{4x^2} \right]$ — (6) ✱ (See next Page bottom)

Also the G.S of D.E (1) is $= c_1y_1 + c_2y_2$

$= c_1(x^2) + c_2\left(\frac{-1}{4x^2}\right)$

$= c_1x^2 + c_3 \cdot \frac{1}{x^2}$].

(∵ y_1 & y_2 are L.I so $c_1y_1 + c_2y_2$ is G.S)

We can see here ×→

$W(y_1, y_2) = \begin{vmatrix} x^2 & -1/4x^2 \\ 2x & \frac{1}{2}x^{-3} \end{vmatrix} = \frac{1}{2x} + \frac{1}{2x} = 1/x$

$\Rightarrow W = \frac{1}{x} \neq 0 \quad \forall\, x - \{0\}$ [At $x=0$ W is not defined]

So y_1 & y_2 are L.I $\forall\, x - \{0\}$.

So $c_1y_1 + c_2y_2$ is G.S of (1) D.E $\forall\, x - \{0\}$.

Eg ② If $y = e^x$ ① is one solution of. $xy'' - (2x+1)y' + (x+1)y = 0$ ② then find the other & hence G.S. ⑨

Soln

$\therefore y_1 = e^x$. Also from ② $y'' - (2+\frac{1}{x})y' + (1+\frac{1}{x})y = 0$ ③ (Changing to standard form)

Now →

$y_2 = v(x)y_1$ — ⑤ $\quad P(x) = -(2+\frac{1}{x})$ ④

where

$$v(x) = \int \left(\frac{1}{y_1^2} e^{-\int P(x)dx} \right) dx$$

$$= \int \left(\frac{1}{e^{2x}} e^{\int (2+\frac{1}{x})dx} \right) dx = \int \frac{1}{e^{2x}} \cdot \left(e^{2x + \ln x}\right) dx$$

$$= \int x\,dx = \frac{x^2}{2} \quad ⑤$$

So $y_2 = v(x)y_1 = (\frac{x^2}{2})e^x$ ⑥

So. G.S is → $c_1y_1 + c_2y_2$

$$= c_1(e^x) + c_2(\frac{x^2}{2})e^x$$

$$= c_1e^x + c_3x^2e^x = (c_1 + c_3x^2)e^x$$

(*) Can always check or verify whether it is correct or not

→ $y_2 = \frac{-1}{4x^2}$, $y_2' = \frac{1}{2x^3}$, $y_2'' = \frac{-3}{2x^4}$.

Put y_2, y_2', y_2'' in ~~the~~ HDE →

$x^2y'' + xy' - 4y = 0$ we get

$$L.H.S = x^2\left(\frac{-3}{2x^4}\right) + x\left(\frac{1}{2x^3}\right) - 4\left(\frac{-1}{4x^2}\right)$$

$$= \frac{-3}{2x^2} + \frac{1}{2x^2} + \frac{1}{x^2} = \frac{-3+1+2}{2x^2}$$

$$= 0 = R.H.S$$

~~Proved~~ True

Method second (To find y_g (G.S) of HDE)

→ Homogeneous D.E with Constant Coeffecients

Now here we are considering the more particular case of HDE where coeffecients $P(x), Q(x)$ are constants.
That is, we have → $y'' + Py' + Qy = 0$ — ①
where P & Q are constants.

Theory → To find solution (general) of such D.E we assume $y = e^{mx}$ — ② be possible solution of ① for suitable choices of m. [∵ derivative of exp fn are constant multiple of original exp fn]

So $y' = me^{mx}$, $y'' = m^2 e^{mx}$. Put y, y' & y'' in ① we get →

$$(m^2 + mP + Q)e^{mx} = 0$$

∵ $e^{mx} \neq 0 \Rightarrow m^2 + mP + Q = 0$ — ③

[③ is Auxiliary eqn or characteristic eqn of HDE ①]

(Roots of auxiliary eqn) ⇒ $m = \dfrac{-P \pm \sqrt{P^2 - 4Q}}{2}$ | $m_1 = \dfrac{-P + \sqrt{P^2 - 4Q}}{2}$, $m_2 = \dfrac{-P - \sqrt{P^2 - 4Q}}{2}$

So these are the values of m for which $y = e^{mx}$ is soln of ①. (That is, $y_1 = e^{m_1 x}$ & $y_2 = e^{m_2 x}$)

Case ① If roots of Auxiliary Eqn (AE) are real & distinct ($P^2 - 4Q > 0$)

Say, m_1, m_2

Then the two solutions of HDE ① are

$$y_1 = e^{m_1 x} \;\&\; y_2 = e^{m_2 x}$$

Now $W(y_1, y_2) = \begin{vmatrix} e^{m_1 x} & e^{m_2 x} \\ m_1 e^{m_1 x} & m_2 e^{m_2 x} \end{vmatrix}$

$= (m_2 - m_1) e^{(m_1+m_2)x} \neq 0 \ \forall x. \ (\because m_1 \neq m_2)$

Since $W \neq 0 \ \forall x$. So y_1 & y_2 are L.I

Hence the G.S of the HDE — if the roots of A.E are real & distinct (m_1, m_2) — is given by

$$\boxed{y_g = \boxed{c_1 e^{m_1 x} + c_2 e^{m_2 x}} = c_1 y_1 + c_2 y_2} \quad \text{—(A)}$$

Case (2) If roots of A.E are Complex. (Say $m = a \pm ib$)

then the G.S of HDE (1) is given as →

$$y_g = e^{ax}\left[c_1 \cos bx + c_2 \sin bx\right] \quad \text{—(B)}$$

Case (3) If roots of A.E are real & equal (Say $m_1 = m_2 = m$)

then the G.S of HDE (1) is given by →

$$\left.\begin{aligned} y_g &= c_1 e^{mx} + c_2 (x e^{mx}) \\ &= (c_1 + c_2 x) e^{mx} \end{aligned}\right\} \text{—(C)}$$

$y_1 = e^{mx}$ Using use of known soln to find other $y_2 = x e^{mx}$

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eg (1) Find the G.S of $2y'' - 4y' + 8y = 0$ —(1)

let $y = e^{mx}$ be possible soln of (1)

So $y' = m e^{mx}$, $y'' = m^2 e^{mx}$. Put y, y', y'' in (1)

we get $(2m^2 - 4m + 8) e^{mx} = 0$

So A.E is → $2m^2 - 4m + 8$

$\Rightarrow m = 1 \pm i\sqrt{3}$ (Complex Roots)

$\therefore$ $m = +1 \pm i\sqrt{3} = a \pm ib$.

So G.S $\rightarrow$ $y_g = e^{ax}[c_1 \cos bx + c_2 \sin bx]$

$= e^{x}[c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x)]$

eg(2) Given D.E $4y'' - 12y' + 9y = 0$ — (1). Find its G.S.

Soln

Let $y = e^{mx}$ be trial soln of (1).

Put value of y, y' & y'' in (1) we get A.E as $\rightarrow$

$4m^2 - 12m + 9 = 0$

Solving we get $\rightarrow$ $m = 3/2, 3/2$ (Equal roots)

So $y_g = (c_1 + c_2 x)e^{mx}$

$= (c_1 + c_2 x)e^{\frac{3}{2}x}$

eg(3) Obtain the G.S of D.E $\rightarrow$

$y'' + 4y' - 5y = 0$ — (1)

Soln

Here A.E $\longrightarrow$ $m^2 + 4m - 5 = 0$

Roots $\longrightarrow$ $1, -5$ (Real & distinct)

So $y_g = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$= c_1 e^{x} + c_2 e^{-5x}$

Method 3 (To find G.S of HDE)

Note Cauchy–Euler Differential equation → (or simply Cauchy D.E)

The general form of nth order Cauchy-Euler D.E is given by →

$$a_0 x^n y^n + a_1 x^{n-1} y^{n-1} + \cdots + a_{n-1} x y' + a_n y = r(x) \quad \text{—(1)}, \; x \neq 0, \; a_0 \neq 0$$

where $a_0, a_1, \ldots a_n$ are constants.

→ For second order → $a_0 x^2 y'' + a_1 x y' + a_2 y = r(x)$

→ Cauchy-Euler (HDE) → $a_0 x^2 y'' + a_1 x y' + a_2 y = 0$ —(2) $\quad a_0 \neq 0, \; x \neq 0$

Method to solve Cauchy-Euler 2nd order HDE

Put $x = e^z$ —(3) $\Rightarrow z = \ln x$ —(4) so that $\dfrac{dz}{dx} = 1/x$ —(5)

(This substitution reduces (2) to HDE with constant coeffecients)

Now $\dfrac{dy}{dx} = \dfrac{dy}{dz} \cdot \dfrac{dz}{dx}$ (Using chain rule)

$$= \frac{1}{x}\frac{dy}{dz} \Rightarrow \boxed{x\frac{dy}{dx} = \frac{dy}{dz}} \text{—(6)}$$

Again

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$$

$$= \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x}\left(\left(\frac{d}{dz}\left(\frac{dy}{dz}\right)\right)\cdot\frac{dz}{dx}\right)$$

$\dfrac{dy}{dz} \to \dfrac{d}{dz}\left(\dfrac{dy}{dz}\right)$, $z \to \dfrac{dz}{dx}$, x

$$= \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

$$\Rightarrow \boxed{x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}} \text{—(7)}$$

(*) [We will use (6) + (7) directly in Q5.]

Put (6) + (7) in (2) we get

$$\boxed{a_0 \frac{d^2y}{dz^2} + (a_1 - a_0)\frac{dy}{dz} + a_2 y = 0}$$

which is 2nd order HDE with constant coeffecients + can be solved.

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eg ① Find the G.S of → $2x^2y'' + xy' - 6y = 0$ — ①

Clearly it is a Cauchy-Euler D.E.

So let $x = e^z$ ② $\Rightarrow z = \ln x$ ③ so that $\frac{dz}{dx} = \frac{1}{x}$ ④

Now $x\frac{dy}{dx} = \frac{dy}{dz}$ ⑤ | $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$ ⑥

Put ⑤ + ⑥ in ① we get

$$2\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) + \frac{dy}{dz} - 6y = 0$$

$$\Rightarrow 2\frac{d^2y}{dz^2} - \frac{dy}{dz} - 6y = 0 \quad \text{— ⑦}$$

(which is 2nd order HDE with constant coeffecients)

Here A.E → $2m^2 - m - 6 = 0$

Roots → $m = -\frac{3}{2}, 2$

So $y_g = c_1 e^{-\frac{3}{2}z} + c_2 e^{2z}$ (Plz note G.S in variable (z) and not in (x))

$= c_1 x^{-3/2} + c_2 x^2$ $[\because x = e^z]$

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eg ② Obtain G.S of $x^2y'' - 5xy' + 13y = 0$ — ①

let $x = e^z$ ② $\Rightarrow z = \ln x$ ③

So that $x\frac{dy}{dx} = \frac{dy}{dz}$ ④ | $x^2\frac{d^2y}{dx^2} = \left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$ — ⑤

Put (4) + (5) in (1) we get

$$\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right)+5\left(\frac{dy}{dz}\right)+13y=0$$

$$\Rightarrow \frac{d^2y}{dz^2}-6\frac{dy}{dz}+13y=0 \quad -(6)$$

A.E $\rightarrow$ $m^2-6m+13=0$

Roots $\rightarrow$ $m=3+2i$

So $y_g = e^{3z}[C_1\cos 2z + C_2\sin 2z]$ (In variable z as (6) is in (2) D.E)

$= x^3[C_1\cos(2\ln x)+C_2\sin(2\ln x)]$ ($\because x=e^z$)

eg (2). Determine G.S of $x^2y''-3xy'+4y=0$ — (1)

Let $x=e^z$ — (2) $\Rightarrow z=\ln x$. — (3)

So that $x\frac{dy}{dx}=\frac{dy}{dz}$ — (4) | $x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dz^2}-\frac{dy}{dz}$ — (5)

Put (4) + (5) in (1) we get

$$\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right)-3\left(\frac{dy}{dz}\right)+4y=0$$

$$\Rightarrow \frac{d^2y}{dz^2}-4\frac{dy}{dz}+4y=0$$

$\Rightarrow$ A.E $\rightarrow$ $m^2-4m+4=0$

Roots $\rightarrow$ $m=2,2$

$y_g = (C_1+C_2z)e^{2z}$ (in variable z)

$= (C_1+C_2\ln x)x^2$

~~Method (2) to find the~~

Note → If the HDE is given in form → (Legendre's D.E)

$$a_0(ax+b)^2\frac{d^2y}{dx^2}+a_1(ax+b)\frac{dy}{dx}+a_2y=0 \quad (1)$$

Then it can be reduced to HDE with Constant coeffecients by taking the substitution →

$$(ax+b)=e^z \quad (2) \Rightarrow z=\ln(ax+b) \quad (3) \Rightarrow \frac{dz}{dx}=\frac{a}{ax+b} \quad (4)$$

And here using same methodology as we did earlier

$$(ax+b)\frac{dy}{dx}=a\frac{dy}{dz} \quad (5) \qquad \Big| \qquad (ax+b)^2\frac{d^2y}{dx^2}=a^2\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right) \quad (6)$$

Putting (5) + (6) in (1) will again reduces the HDE (1) into HDE with constant coeffecients.

eg(1) Solve the D.E →

$$(2x+1)^2y''+(2x+1)y'+y=0 \quad (1)$$

Soln → Put $(2x+1)=e^z$ (2), $\Rightarrow z=\ln(2x+1)$ (3) $\Rightarrow \frac{dz}{dx}=\frac{2}{2x+1}$ (4)

So that $$(2x+1)\frac{dy}{dx}=2\frac{dy}{dz} \quad (5) \qquad \Big| \qquad (2x+1)^2\frac{d^2y}{dx^2}=(2)^2\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right) \quad (6)$$

Put (5) + (6) in (1) we get

$$4\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right)+2\frac{dy}{dz}+y=0$$

$$\Rightarrow 4\frac{d^2y}{dz^2}-2\frac{dy}{dz}+y=0 \quad (6)$$

A.E → $4m^2 - 2m + 1 = 0$

Roots → $m = \frac{1}{4} \pm i\frac{\sqrt{3}}{4}$

So $y_g = e^{\frac{1}{4}z}\left[C_1 \cos\left(\frac{\sqrt{3}}{4}z\right) + C_2 \sin\left(\frac{\sqrt{3}}{4}z\right)\right]$

$= (2x+1)^{1/4}\left[C_1 \cos\left(\frac{\sqrt{3}}{4}\ln(2x+1)\right) + C_2 \sin\left(\frac{\sqrt{3}}{4}\ln(2x+1)\right)\right]$

eg (2) Solve the D.E →

$(3x+1)y'' + 6y' = 0$ — (1)

Soln → (1) can be written as →

Multiply both sides by $(3x+1)$

$(3x+1)^2 y'' + 6(3x+1)y' = 0$ — (2) → (Clearly it is general form of E-C HDE or Legendre D.E)

So Put $3x+1 = e^z$

So $z = \ln(3x+1)$

Also $(3x+1)\frac{dy}{dx} = 3\frac{dy}{dz}$ — (3)

$(3x+1)^2\frac{d^2y}{dx^2} = 9\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$ — (4)

Put (3) & (4) in (2) we get

$9\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) + 6\left(3\frac{dy}{dz}\right) = 0$

$\Rightarrow 9\frac{d^2y}{dz^2} + 9\frac{dy}{dz} = 0 \Rightarrow \frac{d^2y}{dz^2} + \frac{dy}{dz} = 0$ — (5)

Which is 2nd order HDE with constant coeffecients

→ A.E → $m^2 + m = 0$

$m(m+1) = 0$

$m = 0, -1$

So $y_g = C_1 e^{0z} + C_2 e^{-z}$

$= C_1 + \frac{C_2}{3x+1}$

* {You can always verify the correctness of soln by substituting it & its derivatives in D.E.}

~~[illegible]~~ Try yourself (17A)

Q1 $4x^2\frac{d^2y}{dx^2}+16x\frac{dy}{dx}+9y=0$

$\left[y=(c_1+c_2\ln x)x^{-3/2}\right]$

Q2 $x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+2y=0$ $\left[y=x\left(c_1\cos(\ln x)+c_2\sin(\ln x)\right)\right]$

Q3 $(x+1)^2\frac{d^2y}{dx^2}+(x+1)\frac{dy}{dx}=0$

$\left[y=c_1+c_2\ln(x+1)\right]$

Q4 $(2+3x)^2\frac{d^2y}{dx^2}+3(2+3x)\frac{dy}{dx}-36y=0$

$\left[y=c_1(2+3x)^2+\frac{c_2}{(2+3x)^2}\right]$