Lemma 1 If y_1 & y_2 are any two solutions of

$$y'' + P(x)y' + Q(x)y = 0 \quad \text{--- (1)}$$

on $[a,b]$ then their Wronskian $(W = W(y_1, y_2))$ is either identically zero or never zero on $[a,b]$.

Proof →

Given → y_1 & y_2 are two solutions of $y'' + P(x)y' + Q(x)y = 0$ --- (1)

To show → Wronskian of y_1 & y_2 is either identically zero or never zero on $[a,b]$

Soln → $\because$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2 \quad \text{--- (2)}$$

So

$$\frac{dW}{dx} = y_1 y_2'' + y_1' y_2' - y_1' y_2' - y_1'' y_2$$

$$= y_1 y_2'' - y_1'' y_2 \quad \text{--- (3)}$$

$\because$ y_1 & y_2 are solns of (1) then →

$$y_1'' + P(x)y_1' + Q(x)y_1 = 0 \quad \text{--- (4)}$$

$$\& \quad y_2'' + P(x)y_2' + Q(x)y_2 = 0 \quad \text{--- (5)}$$

Multiply (4) by y_2 & (5) by (y_1) and subtracting (5) by (4) we get

$$(y_1 y_2'' - y_2 y_1'') + P(x)(y_1 y_2' - y_1' y_2) = 0$$

$$\Rightarrow \frac{dW}{dx} + P(x) \cdot W = 0 \quad \text{--- (6)} \quad (\text{Use (2) \& (3)})$$

Which is linear 1st order D.E in W

$$I.F = e^{\int P(x)dx}$$

So G.S. of (6) is →

$$W(e^{\int P(x)dx}) = \int (0 \cdot e^{\int P(x)dx}) + C$$

$$\Rightarrow W = C e^{-\int P(x)dx}$$

So we have

$$W(y_1, y_2) = W$$
$$= c\, e^{-\int P(x)dx}, \quad c \text{ is constant of integration} \quad \text{---}(7)$$

Clearly $\boxed{\therefore e^{-\int P(x)dx} \neq 0}$ So

1) If $c = 0$ then $W = 0$ or W is identically zero $\forall x \in [a, b]$

(Identically zero means zero $\forall x \in [a, b]$. It is not like zero for some x & nonzero for other in $[a, b]$)

2) If $c \neq 0$, $W \neq 0$ $\forall x \in [a, b]$

So Proved.

Lemma (2) If y_1 & y_2 are two solutions of $y'' + P(x)y' + Q(x)y = 0$ —(1) on $[a, b]$ then they are L.D on this interval iff ~~their~~ their Wronskian $(W(y_1, y_2)) = y_1 y_2' - y_2 y_1'$ is identically zero on $[a, b]$.

Proof →

Case A Given → y_1 & y_2 are two L.D soln of (1)

To show → $W(y_1, y_2) = 0$

Proof → $\therefore W = W(y_1, y_2) = y_1 y_2' - y_1' y_2$ —(2)

Case (i)

If y_1 (or y_2) is identically zero on $[a, b]$

Say, $y_1 = 0$.

Put $y_1 = 0$ so that $y_1' = 0$ in (2) we get $W(y_1, y_2) = 0$ $\forall x \in [a, b]$

So proved

Case ② If both y_1 & y_2 are nonzero on $[a, b]$ or do not vanish on $[a, b]$

∴ y_1 & y_2 are L.D. So let $y_2 = c y_1$ — ③, c is constant

$\Rightarrow y_2' = c y_1'$ — ④

Now $W(y_1, y_2) = y_1 y_2' - y_1' y_2 = y_1 (c y_1') - y_1' (c y_1)$

$\Rightarrow W = 0$ So proved

Case B (Conversely)

Given → $W(y_1, y_2) = 0 \quad \forall x \in [a, b]$

To show → y_1 & y_2 are L.D on $[a, b]$

Proof →

Case ① If y_1 or y_2 is identically zero on $[a, b]$

→ Say, $y_1 = 0 \quad \forall x \in [a, b]$

So $y_1 = 0 \cdot y_2 \Rightarrow y_1$ & y_2 are L.D

So proved

∴ $f(x) = 0$ is L.D with any other fn nonzero

Case ② If y_1 & y_2 are non zero on $[a, b]$

∴ $W(y_1, y_2) = y_1 y_2' - y_1' y_2 = 0$

$\Rightarrow \dfrac{y_1 y_2' - y_1' y_2}{y_1^2} = 0 \Rightarrow d\left(y_2/y_1 \right) = 0$

$\Rightarrow y_2/y_1 = k \Rightarrow \boxed{y_2 = k y_1} \quad \forall x \in [a, b]$, k is constant

$\Rightarrow y_1$ & y_2 are L.D

So proved.