Integrating Factor → (Using integrating factors to solve the D.E ~~s~~ or to make D.E exact).

Def → I.F is a function or expression in general which when multiplied to a differential equation makes it integrable (or atleast facilitate the integration)

→ Consider an eg → $y\,dx + (x^2 y - x)\,dy = 0$ —— (1)

Here $M = y \mid N = x^2 y - x$.

$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = 2xy - 1 \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \rightarrow$ $\boxed{\text{(1) is Not exact D.E}}$

Now → let us ~~diff~~ multiply both sides of D.E (1) by $\frac{1}{x^2}$ so that (1) becomes

$\frac{y}{x^2}dx + (y - \frac{1}{x})\,dy = 0$ —— (2)

Here $M = \frac{y}{x^2} \mid N = y - \frac{1}{x}$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{1}{x^2}$ and $\frac{\partial N}{\partial x} = \frac{1}{x^2}$ So now $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

So D.E (2) becomes ~~is~~ exact. ~~now~~

→ So here term $\boxed{\frac{1}{x^2}}$ is I.F that makes D.E (1) exact.

Note We have two particular cases for finding the integrating factor when a D.E is not exact. ($M\,dx + N\,dy = 0$)

Case (1) If $\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)/N = f(x)$ (fn of x alone). —— (1)

Then $I.F = e^{\int f(x)\,dx}$ (Proof at Page 24A) (22–25)

Case (2) If $\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)/M = g(y)$ (fn of y alone) —— (2)

Then $I.F = e^{-\int g(y)\,dy}$ (Proof at Page 25) (22–25) eg ① Check whether given D.E →

$$(5x^3+12x^2+6y^2)\,dx + 6xy\,dy = 0 \quad — ①$$

is exact or not. If it is not exact, use I.F to make it exact + hence find the G. Soln.

Soln → Here $M = 5x^3+12x^2+6y^2$ | $N = 6xy$

So $\frac{\partial M}{\partial y} = 12y$ + $\frac{\partial N}{\partial x} = 6y$ So $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ step①

⇒ D.E ① is not exact in present form.

Now $\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = 12y - 6y = 6y$.

So $\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) / N = \frac{6y}{6xy} = \frac{1}{x}$ [fn of x alone] Case① step②

So $I.F = e^{\int \frac{1}{x}dx} = e^{\ln x} = \underline{\underline{x}}$

Multiply both sides of D.E ① by I.F we get

$$(\underbrace{5x^4+12x^3+6xy^2}_{M})\,dx + \underbrace{6x^2y}_{N}\,dy = 0 \quad — ②$$ step③

So now $\frac{\partial M}{\partial y} = 12xy$ | $\frac{\partial N}{\partial x} = 12xy$

Hence $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ ⇒ D.E ② is exact.

So the G.S of ① is →

$$\int M\,dx + \int N\,dy = C.$$ → step④

$$\int (5x^4+12x^3+6xy^2)\,dx + \int (0)\,dy = C$$

$$\boxed{x^5+3x^4+3x^2y^2 = C}$$ → Soln

[$f(x,y) = C.$]

where $f(x,y) = x^5+3x^4+3x^2y^2$

eg(2) $(3x^2y^3e^y + y^3 + y^2)dx + (x^3y^3e^y - xy)dy = 0$ —— (1) (19)

Statement of Q same as for eg(1)

Soln → Here $M = 3x^2y^3e^y + y^3 + y^2$

$N = x^3y^3e^y - xy$

So $\frac{\partial M}{\partial y} = 9x^2y^2e^y + 3x^2y^3e^y + 3y^2 + 2y$

$\frac{\partial N}{\partial x} = 3x^2y^3e^y - y$ Clearly $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

So (1) is not exact → step (1)

Consider $\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = 9x^2y^2e^y + 3y^2 + 3y$

So $\frac{\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}{M} = \frac{9x^2y^2e^y + 3y^2 + 3y}{3x^2y^3e^y + y^3 + y^2} = \frac{3(3x^2y^2e^y + y^2 + y)}{y(3x^2y^2e^y + y^2 + y)}$

$= 3/y$ (fn of y alone) — step (2)

So $I.F = e^{-\int 3/y \, dy} = e^{-3\ln y} = 1/y^3$

→ Multiply both sides of (1) by $I.F = 1/y^3$ we get

$\underbrace{(3x^2e^y + 1 + 1/y)}_{M}dx + \underbrace{(x^3e^y - x/y^2)}_{N}dy = 0$ —— (2) → step (3)

Now $\frac{\partial M}{\partial y} = 3x^2e^y - 1/y^2$ and $\frac{\partial N}{\partial x} = 3x^2e^y - 1/y^2$

Clearly $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. So (2) is exact now.

G.S is then given by

$\int (3x^2e^y + 1 + 1/y)dx + \int (0)dy = c$ → step (4)

$f(x, y) = c$

where $f(x,y) = x^3e^y + x + x/y$ ⇒ $\boxed{x^3e^y + x + x/y = c}$ → G.S

(*) Again we can always verify that $f(x,y) = c$, the solution we obtained is correct or not by finding → $\frac{\partial f}{\partial x} = M$ | $\frac{\partial f}{\partial y} = N$.

Try yourself →

Check whether given D.E is exact or not. If not make it exact by using I.F. Hence find the G.S of D.E

1) $xe^{x}(dx-dy)+e^{x}dx+ye^{y}dy=0$

$$\left[e^{(x-y)}x+y^2/2=c\right]$$

2) $(x^2+y^2+1)dx-2xy\,dy=0$

$$\left[x^2-y^2-1=cx\right]$$

3) $x\sin x\frac{dy}{dx}+y(x\cos x-\sin x)=2$

$$\left[\frac{2}{x}+\frac{y\sin x}{x}=c\right]$$

Note → <u>Finding the integrating factor by Inspection</u>

→ <u>Some formulas that are often useful during Inspection</u>

1) $d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$

2) $d(xy) = x\,dy + y\,dx$

3) $d(x^2+y^2) = 2(x\,dx + y\,dy)$

4) $d\left(\tan^{-1}\left(\frac{x}{y}\right)\right) = \frac{y\,dx - x\,dy}{x^2+y^2}$

5) $d\left(\ln\left(\frac{x}{y}\right)\right) = \frac{y\,dx - x\,dy}{xy}$

6) $\frac{1}{2}\,d\left(\ln(x^2+y^2)\right) = \frac{x\,dx + y\,dy}{x^2+y^2}$

<u>eg ①</u> Sometimes finding the I.F by particular case ① & ② (as done at Page ⑰) becomes little bit lengthy whereas using the I.F by inspection can make calculations very easy

$$y\,dx - x\,dy = xy^3\,dy \quad \text{--- ①}$$

<u>Case ①</u> <u>By Inspection</u> (keep above ⑥ formulas in mind)

So from ①, multiply both sides by $\frac{1}{xy}$

[I.F should be such that both sides becomes perfectly integrable]

So here we get

$$\frac{y\,dx - x\,dy}{xy} = \frac{xy^3}{xy}\,dy$$

$$d\left(\ln\left(\frac{x}{y}\right)\right) = y^2\,dy$$

<u>Integrate</u>

$$\int d\left(\ln\left(\frac{x}{y}\right)\right) = \int y^2\,dy$$

$$\Rightarrow \boxed{\ln\left(\frac{x}{y}\right) = \frac{y^3}{3} + C}$$

<u>Case ②</u> Try yourself by using Particular form ① or ② & see the difference $\left(\text{I.F} = \frac{e^{-y^3/3}}{y^2}\right)$

eg 2. (Where both particular cases 1 + 2 does not work)

$$x\,dy = (x^5 + x^3y^2 + y)\,dx \quad -(1)$$

Soln. (1) can be written as

$$y\,dx - x\,dy = -x^3(x^2+y^2) \quad -(2)$$

Multiply both sides by $\frac{1}{x^2+y^2}$ we get

$$\frac{y\,dx - x\,dy}{x^2+y^2} = -x^3 \; . \text{ Integrate both sides we get}$$

$$\int d\left(\tan^{-1}(x/y)\right) = \int -x^3\,dx.$$

$$\left[\tan^{-1}(x/y) = -x^4/4 + c\right]$$

(*) [Check yourself → Particular case 1 + 2 of I.F does not hold here.]

eg 3. Solve D.E → $x\,dy - y\,dx = (2x^2-3)\,dx$ —(1)

by using integrating factor by inspection

Soln → ∵ $x\,dy - y\,dx = (2x^2-3)\,dx.$

Now ~~So~~, multiply both sides by $(1/x^2)$ we get

$$\frac{x\,dy - y\,dx}{x^2} = \frac{(2x^2-3)}{x^2}\,dx$$

$$\Rightarrow \quad d(y/x) = \left(2 - 3/x^2\right)dx \; . \text{ Integrating both sides we get}$$

$$\boxed{y/x = 2x + 3/x + c} \rightarrow \text{G. Soln}$$

(*) → Can you Solve it using Particular case 1 or case 2

→ There is another way of finding I.F (to solve such Qs). See Page 26

21A

Try yourself (by Inspection only)

1) $x\,dx + y\,dy + 2(x^2+y^2)\,dx = 0$

2) $(xy - 2y^2)\,dx - (x^2 - 3xy)\,dy = 0$

3) $y^2\,dx + x(x-y)\,dy = 0$

Soln

1) $y^2 = e^{2(c-2x)} - x^2$

2) $y^3 = x^2 e^{\frac{cy-x}{y}}$

3) $y = x\ln y + cx$

Q Given $\dfrac{\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}{N - M} = g(z)$, — ① where $z = x + y$ — ②

Then show that

$u = e^{\int g(z)dz}$ — ③ is I.F of D.E → $M(x,y)dx + N(x,y)dy = 0$ — ④

$\boxed{\text{Q of Tutorial Sheet ②}}$

Soln →

Here u is given to be the I.F of D.E ④. So if we multiply both sides of ④ by I.F ⓤ then ④ will become exact D.E, that is, $u[(Mdx + Ndy) = 0]$.

u = u(z) (fn of x+y) from ③

$\Rightarrow (uM)dx + (uN)dy = 0$ — ⑤ is exact D.E.

So if ⑤ is exact then →

$$\frac{\partial}{\partial y}(uM) = \frac{\partial}{\partial x}(uN).$$

$$\left[u\frac{\partial M}{\partial y} + M\frac{\partial u}{\partial y} = u\frac{\partial N}{\partial x} + N\frac{\partial u}{\partial x}\right]$$

$$\Rightarrow u\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = N\frac{\partial u}{\partial x} - M\frac{\partial u}{\partial y}.$$

$$\Rightarrow \left[\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \Big/ \left(N\frac{\partial u}{\partial x} - M\frac{\partial u}{\partial y}\right)\right] = \frac{1}{u} \quad \text{— ⑥}$$

From here particular Case ① & ② of I.F will arise. (see page 24-25)

Here Given that u is a fn of z (from ③). Now since z is also fn of x + y so ⓤ is also fn of x + y

⟶ u (I.F) is a fn of both x + y. (or z)

So let $u = f(z)$ — ⑦

Can also take $u = u(z)$

Also in this Q it has been given in addition that

$z = x + y$ — ②

Now we have to find $\frac{\partial u}{\partial x}$ & $\frac{\partial u}{\partial y}$ from (7) & (2).

So $\frac{\partial u}{\partial x} = \frac{du}{dz} \cdot \frac{\partial z}{\partial x}$.

$$
\begin{array}{ccc} u & & u \\ | & & | \\ z & & z \\ | & & | \\ y & & x \end{array}
$$

$\boxed{u} \to$ dep

$\boxed{z} \to$ inter

$\boxed{x \quad y} \to$ indep

$$= \frac{du}{dz}\frac{\partial}{\partial x}(x+y) = \frac{du}{dz}\cdot 1$$

$$= \frac{du}{dz} = \frac{d}{dz}(f(z)) = f'(z)$$

$$\Rightarrow \frac{\partial u}{\partial x} = f'(z) \quad \text{—(8)} \quad (\text{or } u'(z))$$

Similarily we can find $\frac{\partial u}{\partial y} = \frac{du}{dz}\cdot\frac{\partial z}{\partial y} = \frac{du}{dz}(1)$

$$= \frac{du}{dz} = \frac{d}{dz}(f(z)) = f'(z) \quad \text{—(9)} \quad (\text{or } u'(z))$$

Put (8) & (9) in (6) we get

$$\frac{1}{u} = \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \Big/ \left[f'(z)(N-M)\right]$$

$$\Rightarrow \frac{f'(z)}{f(z)} = \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \Big/ (N-M) \qquad [\because u = f(z)]$$

$\left(\frac{u'(z)}{u(z)}\right)$

$$= g(z) \quad \left\{\begin{array}{l}\because \text{L.H.S is a fn of } z \text{ so} \\ \text{R.H.S will also be fn of } z\end{array}\right\}$$

$$\Rightarrow \frac{f'(z)}{f(z)} = g(z) \quad \text{—(10)} \qquad \text{where } g(z) = \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \Big/ (N-M) \quad \text{—(11)}$$

(Same as in eqn (1))

Integrate both sides of (10)

$$\int \frac{f'(z)}{f(z)}dz = \int g(z)\,dz \Rightarrow \ln(f(z)) = \int g(z)$$

$$\Rightarrow f(z) = e^{\int g(z)dz}$$

$$\Rightarrow \boxed{u = e^{\int g(z)dz}} \longrightarrow \text{Proved}$$

—(12) (Same as eqn (3))

So proved that $u = e^{\int g(z)dz}$ is I.F of $M\,dx + N\,dy = 0$

where $g(z)$ is given by (11) & $z = x+y$.

Note → In previous Q ~~case~~ (*), we have considered I.F $u(x, y)$ as a fn of both x + y. No doubt we were working on Q where $z = x+y$ was given in addition + we arrived at some conclusion as asked in Q.

However, if z has not been given or we are simply given that I.F is a fn of x + y, that is, $u(x, y)$. Then we would have reached atmost upto eqn (6).

Now if $u(x, y)$ (we assume, in general, any fn of x + y) is fn of $x+y$ then we are stuck at eqn (6). So we use a golden rule that "If you cannot solve problem, try to solve a simpler one — the result may be useful" [Credit: Advanced Engineering Math by Kreyszig].

→ So we consider simpler problem + assume that $u(x, y)$ is a fn of x only or fn of y only. Let us see what happens.

Case ① If $\underset{\times}{u}^{(I.F)}$ is a fn of x only. $u = u(x)$

So $\frac{\partial u}{\partial y} = 0$. Put it in eqn ⑥ at page ㉒

we get $\rightarrow$

$$1/u = \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) / N \frac{\partial u}{\partial x}$$

$$\Rightarrow \frac{\partial u / \partial x}{u} = \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) / N$$

$= \text{fn of } x \text{ alone}$

$= f(x)$ (say)

[$\because$ L.H.S is strictly fn of x only so R.H.S would also be]

$$\Rightarrow \frac{\partial u/\partial x}{u} = f(x) \Rightarrow \int \frac{\partial u/\partial x}{u} dx = \int f(x) dx$$

$$\Rightarrow \ln(u(x)) = \int f(x) dx$$

$$u(x) = e^{\int f(x) dx}$$

[$\because$ we need particular value so we consider constant of integration zero]

So $I.F = u(x) = e^{\int f(x) dx}$

where $f(x) = \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) / N$.

So from here we get a result that

If $\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) / N$ is a fn of x only

then $I.F = e^{\int f(x) dx}$ where $f(x) = \dfrac{\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}{N}$.

Case ② If I.F, u, is a fn of y only

$u = u(y)$.

So $\frac{\partial u}{\partial x} = 0$. Put it in eqn ⑥ we get

$$1/u = \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \Big/ \left(-M \frac{\partial u}{\partial y}\right)$$

$$\Rightarrow \frac{\partial u/\partial y}{u} = -\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \Big/ M.$$

$= -g(y)$ (Say) $\quad$ [∵ L.H.S is fn of y alone So R.H.S will also be fn of y alone]

= fn of y alone

$$\Rightarrow \frac{\partial u/\partial y}{u} = -g(y) \Rightarrow \int \frac{\partial u/\partial y}{u}\, dy = -\int g(y)\, dy$$

$$\Rightarrow \ln(u) = -\int g(y)\, dy$$

$$\Rightarrow u = u(y) = e^{-\int g(y)\, dy}$$

So $I.F = u(y) = e^{-\int g(y)\, dy}$

where $g(y) = \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \Big/ M.$

So from here we got a result that →

whenever → $\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \Big/ M$ = fn of y alone

$= g(y)$ (Say)

Then

$$I.F = e^{-\int g(y)\, dy}$$

Note (Leibnitz's D. Eqn)

Method to solve 1st order O.D.E of form

$$\frac{dy}{dx} + P(x)y = Q(x) \quad -(1)$$ by using I.F

→ For such D.E we take I.F →

$$I.F = e^{\int P(x)dy}$$

And G.S is

$$\rightarrow \boxed{y(I.F) = \int Q(x)\cdot(I.F)dx + c}$$

For students

Optional

Try to find ~~the~~ or derive the I.F + G.Soln of D.E (1) using methodology of integrating factors as we did in previous section. (page 22–25)

Optional

Hint → Convert (1) into Mdx + Ndy = 0 form and then use same strategy ~~...~~ of I.F as we did at page (22–25) to find I.F. Hence find G.S (See page 32–33 for proof)

eg (1) Solve D.E → $(x\ln x)\frac{dy}{dx} + y = \frac{2}{x}\ln x \quad -(1)$

using I.F.

Here (1) can be written as →

Soln →
$$\frac{dy}{dx} + \left(\frac{1}{x\ln x}\right)y = \frac{2}{x^2} \quad -(2)$$

Here $P(x) = \frac{1}{x\ln x}$ | $Q(x) = 2/x^2$

$$\text{So } I.F = e^{\int P(x)dx} = e^{\int \frac{1/x}{\ln x}} = e^{\ln(\ln x)} = (\ln x)$$

So G.S of D.E (2) is →

$$y(\ln x) = \int\left(\frac{2}{x^2}\ln x\right)dx + c$$

Solving we get → $y(\ln x) = -2\left[\frac{1}{x} + \frac{\ln x}{x}\right] + c$ → Soln.

eg(2) Solve D.E

$$(2x - 10y^3)\frac{dy}{dx} + y = 0 \quad \text{--- (1)}$$

using I.F

Soln → So (1) can be written as

$$\left.\frac{dy}{dx} + \frac{y}{2x - 10y^3} = 0\right] \times$$ Not in leibnitz form

However → If we write (1) as →

$$\frac{dx}{dy} = \frac{10y^3 - 2x}{y} = 10y^2 - \frac{2}{y}x$$

$$\Rightarrow \frac{dx}{dy} + \left(\frac{2}{y}\right)x = 10y^2 \quad \text{--- (2)}$$

$$\text{So I.F} = e^{\int \frac{2}{y} dy} = e^{2\ln y} = y^2$$

So G.S is →

$$x(y^2) = \int (10y^2)(y^2)\,dy + c$$

$$\boxed{xy^2 = 2y^5 + c} \quad \text{--- (3)}$$

[You can also use Particular case (2) of I.F. Try yourself]

Try yourself →

1) $(1 - x^2)y' + 2xy = x\sqrt{1 - x^2}$ $\left[y = \sqrt{1 - x^2} + c(1 - x^2)\right]$

2) $(1 + x + xy^2)dy + (y + y^3)dx = 0$ $\left[xy = c - \tan^{-1} y\right]$

3) If $\frac{dy}{dx} + y\tan x = \sin 2x$, $y(0) = 0$

Show that max value of y is $\frac{1}{2}$

$$\left[y = -2\cos^2 x + 2\cos x\right]$$

(* You can also use any other method to solve these Qs)

Note Special 1st order DE

Bernoulli D.Eqn → 1st order nonlinear D.E of the form →
$$\frac{dy}{dx} + P(x) \cdot y = Q(x) \cdot y^n \quad \text{—} (1)$$
is called Bernoulli D.E.

→ Here $P(x)$, $Q(x)$ are ~~functions~~ functions of x or constants and n is any real NO.

→ If $n=0$ or $n=1$, (1) becomes linear D.E of order 1 and can be solved by method of I.F or variable separable method, ~~Separa~~ respectively

→ For all other values of n, D.E (1) is nonlinear.

Method of solve Bernoulli D.E (by reducing it to linear 1st order D.E)

To solve (1) we put $v(x) = y^{1-n}$ — (2)

Optional

$$\Rightarrow \frac{dv}{dx} = (1-n) y^{-n} \frac{dy}{dx}$$
$$= (1-n) y^{-n} \left[Q y^n - P \cdot y \right] \quad (\text{Use eqn } (1))$$
$$\frac{dv}{dx} = (1-n) Q(x) - (1-n) P(x) \cdot y^{1-n} \quad \begin{bmatrix} P = P(x) \\ Q = Q(x) \end{bmatrix}$$
$$= (1-n) Q(x) - (1-n) P(x) \cdot v(x) \quad \left[\because v = y^{1-n} \right]$$
$$\Rightarrow \frac{dv}{dx} + \left[(1-n) P(x) \right] v(x) = (1-n) Q(x)$$
$$\Rightarrow \boxed{\frac{dv}{dx} + \left[(1-n) P \right] v = (1-n) Q} \quad (2)$$
→ which is 1st order linear D.E in v that can be solved easily

(*) It is better to remeber the methodology to solve the Bernoulli DE instead of remembering the formula (2).

eg① Solve the D.E →

$$xy' + y = 3x^3y^3 \quad — ①$$

Soln ① can be written as →

$$y' + \frac{1}{x}y = 3x^2y^3 \quad — ②$$

Clearly ② is Bernoulli D.E.

with $P(x) = 1/x$, $Q(x) = 3x^2$, $n = 3$

So let $v(x) = y^{1-n} = y^{1-3} = y^{-2}$ } ③

$$\frac{dv}{dx} = -2y^{-3}\frac{dy}{dx} \Rightarrow \left(\frac{dy}{dx} = -\frac{1}{2}y^3\frac{dv}{dx}\right) \quad ④$$

Put ③ + ④ in ②

we get →

$$-\frac{1}{2}y^3\frac{dv}{dx} + \frac{1}{x}y = 3x^2y^3$$

$$\Rightarrow \frac{dv}{dx} - \frac{2}{x}\cdot\frac{1}{y^2} = (-2)\,3x^2$$

$$\Rightarrow \frac{dv}{dx} + (-2/x)\cdot v(x) = -6x^2 \quad ⑤ \quad \left(\because \underbrace{v}_{v(x)} = 1/y^2\right)$$

which is linear D.E in $v \to v(x)$

So I.F $= e^{-\int 2/x\,dx} = e^{-2\ln x} = 1/x^2$

So G.S is →

$$v(1/x^2) = \int(-6x^2)(1/x^2) + c$$

$$\Rightarrow v/x^2 = -6x + c$$

$$\Rightarrow \frac{1}{y^2x^2} = -6x + c \Rightarrow \boxed{y = \pm\left[-6x^3 + cx^2\right]^{-1/2}} \to \text{G.S}$$

Q9 (2) Show that G.S of the D.E →

$$y' + 4xy + xy^2 = 0 \quad \text{---(1)} \quad \text{is}$$

$$y = [-1/4 + ce^{2x^2}]$$

Soln →

$$\therefore \ y' + (4x)y = -xy^2 \quad \text{---(1)}$$

is clearly a Bernoulli D.E.

With $\underline{P(x) = 4x}$, $\underline{Q(x) = -x}$, $\underline{n = 2}$

let $v(x) = y^{1-n} = y^{1-2} = y^{-1}$ ---(2)

So $\dfrac{dv}{dx} = -y^{-2}\dfrac{dy}{dx} \Rightarrow \dfrac{dy}{dx} = -y^2\dfrac{dv}{dx}$ ---(3)

Put (2) + (3) in (1)

$$-y^2\frac{dv}{dx} + (4x)y = -xy^2$$

$$\Rightarrow \frac{dv}{dx} - (4x)\cdot \frac{1}{y} = +x$$

$$\Rightarrow \boxed{\frac{dv}{dx} + (-4x)v = x} \quad \text{---(4)}$$

Linear D.E in v.

$$I.F = e^{-\int 4x\,dx} = e^{-2x^2} \quad \text{---(5)}$$

So the Gen. Soln is →

$$v(e^{-2x^2}) = \int (+x)e^{-2x^2}dx + c$$

[Put $x^2 = t \Rightarrow 2x\,dx = dt$

$\Rightarrow \int xe^{-2x^2}dx = \int e^{-2t}dt/2 = \frac{1}{2}e^{-2t}/-2$

$= -\frac{1}{4}e^{-2t} = -\frac{1}{4}e^{-2x^2}$]

So $v(e^{-2x^2}) = -\frac{1}{4}e^{-2x^2} + c$

$$\frac{1}{y} = -\frac{1}{4} + ce^{2x^2}$$

$$\Rightarrow \left[y = \left(-\frac{1}{4} + ce^{2x^2}\right)\right]$$

Try yourself →

Q1 $\dfrac{dy}{dx} = \dfrac{y^2x^3 - 2y}{x}$ $\left[\dfrac{1}{y} = -x^3 + cx^2 \right]$

Q2 $xy(1+xy^2)\dfrac{dy}{dx} = 1$ $\left[\dfrac{-1}{x} = y^2 - 2 + ce^{-y^2/2} \right]$

Q3 $y^2 dx + x(x-y)dy = 0$ $\left[y/x = \tan y + c \right]$

* [If you ~~can~~ want to solve it by some other method, you can do so.]

Proof $\frac{dy}{dx} + P(x)\,y = Q(x)$ —— 1 (Optional) 32

(Optional) Proof of I.F + G.S of Leibnitz EQN

$\rightarrow \frac{dy}{dx} = Q - Py \Rightarrow (Q - Py)\,dx - dy = 0$ —— 1

$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[Q(x) - P(x)\,y\right] = -P(x)$

$M = Q(x) - P(x)y$

$N = -1$

$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(-1) = 0$

Clearly 1 is not exact as $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Now $\because \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -P(x)$

So $\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) / N = P(x)$

= fn of x alone

$$\boxed{I.F = e^{\int P(x)\,dx}}$$

(Here is I.F of Leibnitz D.E)

So 1 becomes (on multiplying both sides by I.F.)

$$e^{\int P(x)dx}\left[(Q(x) - P(x)y)\,dx - dy\right] = 0 \quad \text{—— 2}$$

$$e^{\int P(x)dx}\left[Q(x) - P(x)y\right]dx - e^{\int P(x)dx}\,dy = 0$$

$\frac{\partial M}{\partial y} = -P(x)\,e^{\int P(x)dx}$ | $\frac{\partial N}{\partial x} = -P(x)\,e^{\int P(x)dx}$

So $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. So 2 is exact Now

G.S (To find the G. Soln now) 33

G.S is given by $\int M\,dx + \int N\,dy = C$

$$\int \left(e^{\int P(x)dx}\left(Q(x) - P(x)y\right)\right)dx + \int (0)\,dy = C$$

$$\int \left[(I.F)\left(Q(x) - P(x)y\right)\right]dx = C$$

$$\int (I.F)\,Q(x)\,dx - \int (I.F)\,P(x)\,y\,dx = C$$

$$\Rightarrow \int (I.F)\,P(x)\,y\,dx = \int (I.F)\,Q(x)\,dx + C_1$$

$$\Rightarrow y\int \left(e^{\int P(x)dx}\cdot P(x)\right)dx = \int (I.F)\,Q(x)\,dx + C_1$$

$$y\int \left[\frac{d}{dx}\left(e^{\int P(x)dx}\right)\right]dx = \int (I.F)\,Q(x)\,dx + C_1$$

$$\Rightarrow y\left[e^{\int P(x)dx}\right] = \int e^{\int P(x)dx}\cdot Q(x)\,dx + C_1$$

$$\boxed{y\,(I.F) = \int (I.F)\,Q(x)\,dx + C_1}$$

→ G.S

Note Proof of Short cut to find G.S of Exact D.E ($\boxed{\text{All Optional}}$)

$\because$ $M(x,y)\,dx + N(x,y)\,dy = 0$ —— (1)

is exact: So $\exists$ $f(x,y)$ s.t

(Optional all)

$$\underbrace{\frac{\partial f}{\partial x} = M}_{(2)} \ \& \ \frac{\partial f}{\partial y} = N \quad \text{—— (3)}$$

From (2) Integrate both sides w.r.t (x)

$$\underbrace{f(x,y) = \int M(x,y)\,dx + g(y)}_{(4)}$$

$g(y)$ → Constant of Integration (Fn of y or Constant)

Now to find $g(y)$ we use (3)

$$\frac{\partial}{\partial y}\left[\int M(x,y)\,dx + g(y)\right] = N$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\int M\,dx\right) + g'(y) = N$$

$$\Rightarrow g'(y) = N - \frac{\partial}{\partial y}\left(\int M\,dx\right)$$

$$\Rightarrow g(y) = \int\left[\left[N - \frac{\partial}{\partial y}\left(\int M\,dx\right)\right]dy\right] \quad \text{—— (5)}$$

$$= \underbrace{\int N(x,y)\,dy}_{(A)} - \underbrace{\int\left(\frac{\partial}{\partial y}\left[\int M(x,y)\,dx\right]\right)dy}_{(B)}$$

$\Rightarrow$ Now R.H.S ~~each term (A) & (B)~~ of eqn (5) should strictly be fn of (y) alone or Constant as L.H.S $= g(y)$

→ For that to happen, we can assume that term (A) → $\int N(x,y)dy$ and

term (B) → $-\int\left(\frac{\partial}{\partial y}\left[\int M(x,y)dx\right)dy$

Should strictly ~~be~~ contain fns of y only. Because if there is any term of x ~~[illegible]~~ in each of term (A) or (B) then after solving they will get cancelled at last so as to make R.H.S fn of y alone

→ So taking credit of this if we look at term $\int\left(\frac{\partial}{\partial y}\left[\int M(x,y)dx\right)dy$, for this to be fn of y alone we can ~~assume~~ take $M(x,y)$ fn of y alone (as any term of x after solving get cancelled with terms of term (A)). If this happen then this whole term (B) becomes zero

→ Now left is $\int N(x,y)dy$. So again we can assume here that it is strictly fn of y (no terms of x [using same logic as above]

So we get

$$g(y) = \int N(x,y)\,dy - \int\left(\frac{\partial}{\partial y}\left[\int M(x,y)\,dx\right]\right)dy$$

$$= \int N(x,y)\,dy - 0$$

$$= \int N(x,y)\,dy \quad \left[\begin{array}{l}\text{where } N(x,y) \text{ has} \\ \text{Only terms containing } y \text{ or no} \\ \text{term containing } x\end{array}\right] \quad (6)$$

Put (6) in (4) we get

$$f(x,y) = \int M(x,y)\,dx + \int N(x,y)\,dy$$

$\therefore$ G.S is $\rightarrow$

$$f(x,y) = c$$

$$\Rightarrow \underbrace{\int M(x,y)\,dx}_{\substack{\text{Taking } y \text{ Constant} \\ \text{(as usual)}}} + \underbrace{\int N(x,y)\,dy}_{\substack{\text{Taking no term} \\ \text{Containing } x \\ \text{(as explained)}}} = c$$

Verified