Que 1: Kinetic energy of an electron and Photon is 4.55×10^{-25} J. Calculate the velocity, momentum & wavelength of electron & Photon.

Solution: If m_0 is the rest mass and v is the velocity of electron then its kinetic energy (E_K) is given by

$$E_K = \frac{1}{2} m_0 v^2$$

given that $E_K = 4.55 \times 10^{-25}$ J

$$\Rightarrow 4.55 \times 10^{-25} = \frac{1}{2} \times 9.1 \times 10^{-31} \times v^2$$

$$\Rightarrow v^2 = \frac{4.55 \times 10^{-25}}{9.1 \times 10^{-31}} \times 2 = 1.00 \times 10^6$$

$$\Rightarrow v = 10^3 \ ms^{-1}$$

Momentum of electron is given as $P = m_0 v$

$$= 9.1 \times 10^{-31} \times 10^3$$
$$= 9.1 \times 10^{-28} \ kg\,ms^{-1}$$

wavelength of electron is $\lambda = h/p = \frac{6.62 \times 10^{-34}}{9.1 \times 10^{-28}} = 7.27 \times 10^{-7}$ m

---

IInd Part

for Photon $E = hc/\lambda = 4.55 \times 10^{-25} = \frac{6.62 \times 10^{-34} \times 3 \times 10^8}{\lambda}$

$$\Rightarrow \lambda = 4.365 \times 10^{-1} \ m$$

$V = 3 \times 10^8 \ ms^{-1}$

$$P = h/\lambda = \frac{6.62 \times 10^{-34}}{4.365 \times 10^{-1}}$$

$$= 1.517 \times 10^{-33} \ kg\,ms^{-1}$$

Que 2 Write down the conditions for acceptable wave function and Prove that $\psi = Ae^{-x^2}$ $(-\infty \le x \le \infty)$ is an acceptable wave function

Solution
(i) The wave function must be finite everywhere
(ii) Single Valued
(iii) It must be Continuous
(iv) Derivative of a given should also be Continuous

Proof given function is $\psi = Ae^{-x^2}$

Finite] (i) $\lim_{x \to \pm\infty} \psi(x) = Ae^{-x^2}$

$= 0 \Rightarrow$ this function is finite everywhere $\longrightarrow$ (1)

Single Valued] (ii) Check for some values $x = 1, 2, 3, \ldots$

$\psi(1) = Ae^{-1}$; $\psi(-1) = Ae^{-1}$
$\psi(2) = Ae^{-4}$; $\psi(-2) = Ae^{-4}$] It is Single valued $\longrightarrow$ (2)

Continuous] (iii) $\lim_{x \to \pm\infty} \frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x}(Ae^{-x^2}) = -2Ae^{-x^2} \cdot x$

$= -2Ax \cdot e^{-x^2} = -2A\frac{x}{e^{x^2}}$

Apply L-Hospital rule

$$\lim_{x \to \pm\infty} \frac{\partial \psi}{\partial x} = \frac{-2A \cdot 1}{2x \cdot e^{x^2}} = -\frac{A}{x \cdot e^{x^2}} = -\frac{A}{\infty} = 0$$ $\longrightarrow$ (3)

(iv) $$\lim_{x \to \pm\infty} \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) = \frac{\partial}{\partial x}\left(-2Ax \cdot e^{-x^2}\right)$$

$$= -2A\left[xe^{-x^2} \cdot (-2x) + e^{-x^2}\right]$$

$$= -2A\left[-2x^2e^{-x^2} + e^{-x^2}\right]$$

$$= -2A\left[-\frac{2x^2}{e^{x^2}} + e^{-x^2}\right] = -2A\left[\frac{-2x^2 + 1}{e^{x^2}}\right]$$

Apply L'Hospital rule

$$\Rightarrow \lim_{x \to \pm\infty} \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) = -2A\left[\frac{4x}{e^{x^2} \cdot 2x}\right] = \frac{4A}{2x^2e^{x^2} + e^{x^2}} = \frac{4A}{\infty} = 0$$ $\longrightarrow$ (4)

from (1), (2), (3) & (4) the above function is allowed

(3) The wave function of a free particle in normalized state is represented by $\psi = Ne^{-(x^2/2a^2)+ikx}$

Calculate the normalization factor N and the maximum probability of finding the particle.

Solution

The normalization condition is

$$\int_{-\infty}^{\infty} \psi^* \psi \, dx = 1$$

Putting the values of ψ & ψ^* in the above equation, we get

$$\Rightarrow \int_{-\infty}^{\infty} Ne^{-(x^2/2a^2)+ikx} \cdot Ne^{-(x^2/2a^2)+ikx} \, dx = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} N^2 e^{-2x^2/2a^2} \, dx = 1$$

$$\Rightarrow N^2 \int_{-\infty}^{\infty} e^{-x^2/a^2} \, dx = 1$$

$$\because \int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

$$\Rightarrow N^2 \cdot a\sqrt{\pi} = 1$$

$$\Rightarrow \boxed{N^2 = \frac{1}{a\sqrt{\pi}} \Rightarrow N = \frac{1}{a^{1/2}\pi^{1/4}}}$$

→ The maximum probability P(x) can be given as

$$P(x) = |\psi^*(x)\psi(x)|$$

$$= N^2 e^{-x^2/a^2}$$

$$= \frac{1}{a\sqrt{\pi}} e^{-x^2/a^2}$$

(4) Which of the following are eigenfunctions of the operator $\frac{\partial^2}{\partial x^2}$?
Find out the appropriate eigenvalue for them
(i) $\sin x$
(ii) $\sin^2 x$

Solution given that $f(x) = \sin x$
(i) operating $\frac{\partial^2}{\partial x^2}$ on $f(x)$, we get

$$\frac{\partial^2}{\partial x^2}(\sin x) = -\sin x = -f(x)$$

Hence, $\sin x$ is an eigenfunction having eigenvalue -1

(ii) $f(x) = \sin^2 x$

operating $\frac{\partial^2}{\partial x^2}$ on $f(x)$, we get

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}(\cancel{2}\sin^2 x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}(1 - \cancel{2}\cos 2x)$$

$\cos 2x = \frac{1 - 2\sin^2 x}{2}$

$\downarrow$

$1 - 2\cos 2x = 2\sin^2 x$

$$= \frac{1}{2}\frac{\partial}{\partial x}(0 + 2\sin 2x)$$

$$= \frac{1}{2} \cdot 4\cos 2x$$

$$= 2\cos 2x$$

$\rightarrow \left| \begin{array}{l} 2(1 - 2\sin^2 x) \\ = 2 - 4\sin^2 x \end{array}\right.$

Hence it is not an eigenfunction ~~of~~ for $f(x) = \sin^2 x$

---

(5) Solution

given $\psi = ax \quad 0 \le x \le 1$
$\psi = 0 \quad$ elsewhere

To find :- Probability of particle b/w 0.45 and 0.55

$$\text{Probability} = \int_{x_1}^{x_2} \psi^* \psi \, dx = \int_{0.45}^{0.55} a^2 x^2 \, dx = a^2 \int_{0.45}^{0.55} x^2 \, dx = \frac{a^2}{3} \left| x^3 \right|_{0.45}^{0.55}$$

$$= 0.0251 a^2$$

(b) The expectation $\langle x \rangle$ value of particle's position is

$$\langle x \rangle = \int_0^1 \psi^* \hat{x} \psi \, dx$$

$$= \int_0^1 \psi^* x \psi \, dx$$

$$= \int_0^1 ax \cdot x \cdot ax \, dx = \int_0^1 a^2 x^3 \, dx = a^2 \int_0^1 x^3 \, dx$$

$$= \frac{a^2}{4} \left| x^4 \right|_0^1$$

$$= \frac{a^2}{4} \text{ Ans}$$

---

6 solution Steady State Potential $\Rightarrow$ Steady State S.W.E

given $\psi = Ae^{-x^2/L^2}$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{2m}{\hbar^2}(E - U)\psi = 0 \longrightarrow (1)$$

$E = 0$; $\psi = Ae^{-x^2/L^2}$

$$\Rightarrow \frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x}\left(Ae^{-x^2/L^2}\right) = Ae^{-x^2/L^2} \cdot \left(-\frac{2x}{L^2}\right)$$

$$= -\frac{2A}{L^2} x \cdot e^{-x^2/L^2}$$

$$\Rightarrow \frac{\partial^2 \psi}{\partial x^2} = \frac{\partial}{\partial x}\left(-\frac{2}{L^2} A x \cdot e^{-x^2/L^2}\right)$$

$$= -\frac{2A}{L^2}\left[x \cdot e^{-x^2/L^2} \cdot \left(-\frac{2x}{L^2}\right) + e^{-x^2/L^2}\right]$$

$$= -\frac{2A}{L^2}\left[-\frac{2x^2}{L^2} e^{-x^2/L^2} + e^{-x^2/L^2}\right]$$

$$= -\frac{2}{L^2}\left[-\frac{2x^2}{L^2} + 1\right]\psi = \left[+\frac{4x^2}{L^4} - \frac{2}{L^2}\right]\psi$$

$$= \left[\frac{4x^2}{L^4} - \frac{2}{L^2}\right]\psi \longrightarrow (2)$$

Put this in eq. 1

$$\left[\frac{4x^2}{L^4} - \frac{2}{L^2}\right]\psi + \frac{2m}{\hbar^2}\left[E-U\right]\psi = 0$$

$$\frac{4x^2}{L^4} - \frac{2}{L^2} + \frac{2m}{\hbar^2}(-U) = 0$$

$$\Rightarrow \quad \frac{4x^2}{L^4} - \frac{2}{L^2} = \frac{2m}{\hbar^2}U$$

$$\Rightarrow \quad \boxed{U = \frac{\hbar^2}{2m}\left(\frac{4x^2}{L^4} - \frac{2}{L^2}\right)}$$

---

(7) Solution

The energy of Particle in an infinite square well is given by

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

The energy E and wavelength λ of a Photon emitted as the Particle makes a transition from the $n=2$ State to the $n=1$ State are

$$E = E_2 - E_1 = \frac{2^2\pi^2\hbar^2}{2mL^2} - \frac{1^2\pi^2\hbar^2}{2mL^2}$$

$$E = (2^2 - 1^2)\frac{\pi^2\hbar^2}{2mL^2} = \frac{3\pi^2\hbar^2}{2mL^2}$$

$$= 6.15 \text{ MeV}$$

and λ of Photon $= \frac{hc}{E} = 2.02 \times 10^{\cancel{-7}\,-6} m \times \frac{10^{\cancel{5}}}{\cancel{10^{-3}}}$

$= \cancel{2020 A^0}$ 202 fm

8 Solution

Transmission Probability is given by

$$T = e^{-2kL}$$

where k is wave number inside barrier and is given by $k = \frac{\sqrt{2m(U-E)}}{\hbar}$

for e^- with 1.0 eV energy

$$k_1 = \frac{\sqrt{2m(10-1)\times 1.6\times10^{-19}}}{1.054\times10^{-34}} = \frac{\sqrt{2\times(9.1\times10^{-31})\,\text{kg}\times 9\,\text{eV}\times 1.6\times10^{-19}\,\text{J/eV}}}{1.054\times10^{-34}}$$

kg m² s⁻²

$$k_1 = 1.542\times10^{10}\ \text{m}^{-1}$$

$$\Rightarrow T_1 = e^{-2k_1L}$$

$$= e^{-2\times1.542\times10^{10}\times0.50\times10^{-9}}$$

$$\boxed{T_1 = e^{-15.42} = 2.05\times10^{-7}}$$

II For 2.0 eV

$$k_2 = \frac{\sqrt{2\times9.1\times10^{-31}\times8\times1.6\times10^{-19}}}{1.054\times10^{-34}} = 1.448\times10^{10}\ \text{m}^{-1}$$

$$T_2 = e^{-2k_2L} = e^{-2\times1.448\times10^{10}\times0.50\times10^{-9}}$$

$$= e^{-14.48} = 5.14\times10^{-7}$$

2nd Part: if barrier (L) is doubled in width to 1.0 nm find T_1 & T_2

$$\cancel{T_1' = e^{-2k_1L'} = e^{-2\times1.542\times10^{10}\times10^{-9}}}$$

$$\cancel{k_1' = \sqrt{2\times9.1\times10^{-31}\times9\times1.6\times10^{-19}}}$$

$$T_1 = e^{-30.84} = 4.039\times10^{-14}$$

$$T_2 = e^{-28.96} = 2.647\times10^{-13}$$