Note Homogeneous D.E →

Homogeneous fn → A function $f(x, y)$ is said to be homogeneous fn of degree (n) if $\exists\ \lambda > 0$ s.t

$$f(\lambda x, \lambda y) = \lambda^n f(x, y) \quad [n \text{ is any real No}]$$

eg ①. $f(x, y) = x^2 + y^2$

Clearly $f(\lambda x, \lambda y) = (\lambda x)^2 + (\lambda y)^2 = \lambda^2 (x^2 + y^2)$

$= \lambda^2 f(x, y)$.

So it is H. fn of deg ②.

eg ② $f(x, y) = xy$

So here → $f(\lambda x, \lambda y) = (\lambda x)(\lambda y)$

$= \lambda^2 (xy)$

$= \lambda^2 f(x, y)$

So it is H. fn of deg ②

eg ③ Try yourself

Find whether it is H. Fn or not

$$\left[f(x, y) = \frac{x^{3/2} + y^{3/2}}{x + 2} \right]$$

Note Homogeneous 1st order D.E $\rightarrow$ A D.E of the form $y' = f(x,y)$ is said to be Homogeneous D.E if $f(x,y)$ is a homogeneous fn of degree zero.

That is, $f(\lambda x, \lambda y) = \lambda^0 f(x,y)$

eg ① $y' = \dfrac{x^2+y^2}{xy}$

Clearly here $f(x,y) = \dfrac{x^2+y^2}{xy}$

So $f(\lambda x, \lambda y) = \dfrac{(\lambda x)^2+(\lambda y)^2}{(\lambda x)(\lambda y)} = \dfrac{\lambda^2(x^2+y^2)}{\lambda^2(xy)}$

$= \lambda^0 \dfrac{(x^2+y^2)}{xy} \Rightarrow \boxed{f(\lambda x, \lambda y) = \lambda^0 f(x,y)}$

So clearly $f(x,y)$ is homogeneous fn of deg zero.

So $y' = f(x,y) = \dfrac{x^2+y^2}{xy}$ is a Homogeneous D.E.

eg ② $y' = \dfrac{x}{y}\tan\left(\dfrac{x-y}{x+y}\right)$

Check ??, whether it is HDE or not.

Method to Solve H.D.E $\rightarrow$

eg ① Solve the DE $\rightarrow$ $(x+y)dy + (x-y)dx = 0$ — ①

Soln $\dfrac{dy}{dx} = \dfrac{y-x}{y+x}$ — ① Clearly $f(x,y) = \dfrac{y-x}{y+x}$ is H. fn of deg 0. So it is HDE.

So Put $y = vx$ — ② $\Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ — ③

Put ② & ③ in ①

$$v + x\frac{dv}{dx} = \frac{vx - x}{vx + x} = \frac{v-1}{v+1} \Rightarrow \left[x\frac{dv}{dx} = \frac{-(v^2+1)}{(v+1)}\right] \text{ — ④}$$

Separating variables we get

$$\frac{v+1}{v^2+1}dv = \frac{-1}{x}dx \Rightarrow \int \frac{v+1}{v^2+1}dv = -\int \frac{1}{x}dx$$

$$\boxed{\frac{1}{2}\ln(v^2+1) + \tan^{-1}(v) = -\ln x + c} \rightarrow \text{General Soln}$$

where $v = y/x$.

Note D.E of the form → $\frac{dy}{dx} = \frac{ax+by+c}{lx+my+n}$ — (1)

Case (1) $\frac{a}{l} = \frac{b}{m} = k$

Case (2) $\frac{a}{l} \neq \frac{b}{m}$

Case (1) Solve the D.E → $(x-2y+1)dy - (3x-6y+2)dx = 0$ — (1)

Soln

$\Rightarrow \frac{dy}{dx} = \frac{3x-6y+2}{x-2y+1}$, Clearly $\frac{3}{1} = \frac{-6}{-2} = \underline{\underline{3}}$

$= \left.\frac{3(x-2y)+2}{(x-2y)+1}\right\}$ (2) So let $x-2y = v$ — (3)

$1 - 2\frac{dy}{dx} = \frac{dv}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(1-\frac{dv}{dx}\right)$ — (4)

Put (3) + (4) in (2) we get

$\frac{1}{2}\left(1-\frac{dv}{dx}\right) = \frac{3v+2}{v+1}$

Solving → $\frac{dv}{dx} = \frac{-5v-3}{v+1}$

Separating variables → $\frac{v+1}{5v+3}\,dv = -dx$

Integrating → $\int \frac{v+1}{5v+3}\,dv = \int -dx$

$\Rightarrow \int \left((1/5) + \frac{(2/5)}{5v+3} \right) dv = \int -dx$

$\Rightarrow \left[\frac{1}{5}v + \frac{2}{25}\ln(5v+3) = -x + c \right.$ → General Soln

where $v = x-2y$

Case (2) Obtain the G.S of the D.Eqn →

$$(y-x+1)\,dy - (y+x+2)\,dx = 0 \quad \text{--- (1)}$$

Soln →

Here $\dfrac{dy}{dx} = \dfrac{x+y+2}{-x+y+1}$ — (2) Clearly $\dfrac{1}{-1} \neq \dfrac{1}{1}$

So Put $\left.\begin{array}{l} x = X+h \\ y = Y+k \end{array}\right\}$ (3) $\Rightarrow \dfrac{dy}{dx} = \dfrac{dY}{dX}$ (4) $\left[\because \dfrac{dy}{dx} = \dfrac{dY}{dx}\right.$ as $dx = dX$ $\left(\because x = X+h,\ 1 = \dfrac{dX}{dx} \Rightarrow dx = dX\right)\Big]$

(h + k constants)

Put (3) + (4) in (2)

$$\frac{dY}{dX} = \frac{(X+Y)+(h+k+2)}{(-X+Y)+(-h+k+1)} \quad \text{--- (4)}$$

Now choose these values of h + k s.t

$\left.\begin{array}{l} h+k+2 = 0 \\ -h+k+1 = 0 \end{array}\right\} \longrightarrow$ Solving we get $\left[h = -\tfrac{1}{2},\ k = -\tfrac{3}{2}\right]$ (5)

Put (5) in (4) we get →

$$\frac{dY}{dX} = \frac{X+Y}{-X+Y} \quad \text{(6)}$$ [Clearly a HDE in X + Y]

So put $Y = VX$ — (7) $\qquad \dfrac{dY}{dX} = V + X\dfrac{dV}{dX}$ — (8)

Put (7) + (8) in (6)

$$V + X\frac{dV}{dX} = \frac{X+VX}{-X+VX} = \frac{1+V}{-1+V}$$

$$\Rightarrow X\frac{dV}{dX} = \frac{-V^2+2V+1}{V-1} \Rightarrow \frac{V-1}{V^2-2V-1}\,dV = \frac{-1}{X}\,dX$$

$$\Rightarrow \int \frac{V-1}{V^2-2V-1}\,dV = -\int \frac{1}{X}\,dX$$

$$\Rightarrow \boxed{\tfrac{1}{2}\ln(V^2-2V-1) = -\ln X + C}$$

General Soln

where $V = Y/X$, $Y = y-k$, $k = -\tfrac{3}{2}$, $X = x-h$, $h = -\tfrac{1}{2}$

Try yourself → Find the G.S of D.E →

1) $\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x}$ $\left[y+\sqrt{x^2+y^2} = cx^2 \right]$

2) $2ye^{x/y}\,dx + \left(y - 2xe^{x/y}\right)dy = 0$

$\left[\ln y + 2e^{x/y} + c = 0 \right]$

3) $(x+y-1)\,dx + (2x+2y-3)\,dy = 0$

$\left[x + 2y + \ln(x+y-2) = c \right]$

4) $(x+2y)\,dx + (y-1)\,dy = 0$

$\left[\ln(x+y+1) + \left(\frac{x+2}{x+y+1}\right) = c \right]$

5) $(-x+y+2)\,dy = (y-x)\,dx$ $\left[(y-x)^2 + 4y = c \right]$

6) $\frac{dy}{dx} = \frac{y+x-2}{y-x-4}$ $\left[(x+1)^2 - (y-3)^2 + 2(x+1)(y-3) = c \right]$

Note Total differential → let $f(x, y)$ be function having continous 1st order partial derivatives ~~then~~ in some domain D. Then total differential of f (denoted by df or $df(x, y)$) is given as →

$$\boxed{df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.}$$

(*) It represents change in the value of $z = f(x, y)$ w.r.t change in the value of x & y both.

Note Exact differential → An expression (or differential)

(A) of the form → $M(x, y)dx + N(x, y)dy$ ——(1)

is said to be exact differential in domain D if ∃ function $f(x, y)$ s.t the total differential of f $(= df)$ is equal to ~~equation~~ expression (1), That is,

→ (1) is exact differential in D if ∃ $f(x, y)$ s.t

$$\boxed{\frac{\partial f}{\partial x} = M \quad \& \quad \frac{\partial f}{\partial y} = N} \quad \text{——(2)}$$

→ So if $Mdx + Ndy$ is an exact differential then the D.E → $M(x, y)dx + N(x, y)dy = 0$ ——(3)

is called exact D.E

Note → Test for exactness →

(B) Consider 1st order D E → $M(x,y)dx + N(x,y)dy = 0$ — (1)

where M + N has continous 1st order partial derivatives in some domain D. Then D.E (1) will be exact $\boxed{\text{iff}}$ (if and only if)

$$\left[\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\right] \quad \text{— (2)}$$

(*)

Condition (2) is necessary + sufficent for the D.E (1) to be exact

Note → Solution of exact D E

(C) If the D.E → $M(x,y)dx + N(x,y)dy = 0$ — (1)

is exact then $\exists\, f(x,y)$ s.t

$$\frac{\partial f}{\partial x} = M \quad + \quad \frac{\partial f}{\partial y} = N \quad \text{— (2)}$$

So Put (2) in (1) we get

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

$$\Rightarrow d(f(x,y)) = 0$$

$$\Rightarrow \boxed{f(x,y) = c}, \quad c \text{ is constant}$$

↳ G.S of exact D.E

Note Exact differential Eqns → [Combining (A), (B) + (C) in previous pages we can say] → A differential equation of the form → $M(x,y)\,dx + N(x,y)\,dy = 0$ — (1)

is said to be exact D.E if ∃ function $f(x,y)$ s.t

$$\frac{\partial f}{\partial x} = M \quad \text{and} \quad \frac{\partial f}{\partial y} = N \qquad \text{— (2)}$$

And furthermore, $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$ — (3)

→ The Soln of such D.E is given by

$$\boxed{f(x,y) = c}, \quad c \text{ is constant}$$

---

1) Test for exactness → $\left(M(x,y)\,dx + N(x,y)\,dy = 0\right)$ (1)

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$ → Necessary + sufficent condition for (1) to be exact

2) G.S of (1) is → $\boxed{f(x,y) = c}$, c is constant

s.t $\frac{\partial f}{\partial x} = M \;\Big|\; \frac{\partial f}{\partial y} = N$ (will always hold)

(*) Exact D.E also comes under category $\boxed{\frac{dy}{dx} = f(x,y)}$.

Just here for our convenience we write it in form →
$M(x,y)\,dx + N(x,y)\,dy = 0$.

Note ~~Exact differential eqn~~ (Explanation in brief) (Optional)

→ If the D.E $M(x,y)\,dx + N(x,y)\,dy = 0$ — ①

is a exact D.E, then it means that differential on the L.H.S of eqn ① is exact differential. That is, L.H.S of ① is equal to total differential of some fn, say, $f(x,y)$

So that $M(x,y)\,dx + N(x,y)\,dy = d(f(x,y))$ → Total differential

$\Rightarrow M(x,y)\,dx + N(x,y)\,dy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ → By def of total differential

Compare both sides we get →

$$\boxed{\underbrace{\frac{\partial f}{\partial x} = M}_{②} \quad \& \quad \underbrace{\frac{\partial f}{\partial y} = N}_{③}}$$

Again differentiate ② w.r.t y

$$\underbrace{\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial M}{\partial y}}_{④}$$

And differentiate ③ w.r.t x

$$\underbrace{\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial N}{\partial x}}_{⑤}$$

So from ④ & ⑤ $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$ $\left(\because \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial y}\right)$

Finally → $\because d(f(x,y)) = M(x,y)\,dx + N(x,y)\,dy = 0$

$\Rightarrow d(f(x,y)) = 0$

$\Rightarrow \boxed{f(x,y) = c}$

eg 1 Check whether given D.E

$$(3x^2+2e^y)dx + (2xe^y+3y^2)dy = 0 \quad \text{--- (1)}$$

is exact or not. If exact find its solution.

Soln Compare D.E (1) with the

$$M(x,y)dx + N(x,y)dy = 0 \quad \text{--- (2)}$$

we get

$$M(x,y) = M = \underline{3x^2+2e^y} \;\Big|\; N(x,y) = N = \underline{2xe^y+3y^2}$$

Now $\dfrac{\partial M}{\partial y} = 2e^y$ & $\dfrac{\partial N}{\partial x} = 2e^y$

Clearly $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$. So D.E (1) is exact D.E

General soln → Method (1) [Shortcut] → (Proof at Page 34-36)

If a D.E ~~given~~ is exact then its G.S is formed using →

$$\int M\,dx + \int N\,dy = C \rightarrow \text{arbitrary constant}$$

(Take y as Constant in this integral) (Take all those term zero that contain x or any fn of x)

$$\Rightarrow \int (3x^2+2e^y)dx + \int (3y^2)dy = C$$

[∵ term $2xe^y$ of N contains x]

$$\Rightarrow x^3 + 2xe^y + y^3 + C_1 = C$$

$$\Rightarrow x^3 + 2xe^y + y^3 = C_2$$

where $C_2 = C - C_1$

[Without loss of generality we can assume $C_1 = 0$]

So the required G.S of given D.E is

$$f(x,y) = c_2$$

where $f(x,y) = x^3 + 2xe^y + y^3$

Verification → Once we found G.S, we can verify that it is correct or not.

∴ G.S is → $f(x,y) = c_2$

where $f(x,y) = x^3 + 2xe^y + y^3$

Now $\frac{\partial f}{\partial x} = 3x^2 + 2e^y = M$ (True)

Also $\frac{\partial f}{\partial y} = 2xe^y + 3y^2 = N$ (True)

So $\frac{\partial f}{\partial x} = M$ & $\frac{\partial f}{\partial y} = N$.

So our G.S is correct. Hence verified.

Method ② (General method) (Optional)

$\because$ D.E ① is exact (as we found earlier that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$)

So $\exists\, f(x,y)$ s.t

$$\underbrace{\frac{\partial f}{\partial x} = M}_{③} \quad \Big| \quad \underbrace{\frac{\partial f}{\partial y} = N}_{④}$$

$\because \frac{\partial f}{\partial x} = M \Rightarrow \left(\frac{\partial f}{\partial x} = 3x^2 + 2e^y\right)$

Integrate both sides w.r.t x, we get

$$\underbrace{f(x,y) = x^3 + 2xe^y}_{⑤} + g(y)$$ [where $g(y)$ is a fn of y only or constant]

Now to find $g(y)$ we will use eqn ④.

$\because \frac{\partial f}{\partial y} = N \Rightarrow \frac{\partial}{\partial y}\left[x^3 + 2xe^y + g(y)\right] = 2xe^y + 3y^2$

From ⑤ $\qquad$ From ①

$\Rightarrow 2xe^y + g'(y) = 2xe^y + 3y^2$

$\Rightarrow g'(y) = 3y^2$ Integrate both sides w.r.t ⓨ

we get $\rightarrow \quad g(y) = y^3 + c_1 \Big]$ — ⑥

Without loss of generality we can assume $c_1 = 0$ here

Put ⑥ in ⑤ we get $\rightarrow$

$$f(x,y) = x^3 + 2xe^y + y^3 + c_1$$

So G.S of this exact D.E ① is $\rightarrow$

$$f(x,y) = c$$

$$\Rightarrow x^3 + 2xe^y + y^3 + c_1 = c$$

$$\Rightarrow \boxed{x^3 + 2xe^y + y^3 = c_2}$$

$c_2 = c - c_1$.

eg (2) Solve the IVP →

$$(2x\cos y + 3x^2y)dx + (x^3 - x^2\sin y - y)dy = 0 \quad \text{--- (1)}$$

$$y(0) = 2$$

Soln From D.E (1) →

$$M(x,y) = 2x\cos y + 3x^2y$$

$$N(x,y) = x^3 - x^2\sin y - y$$

Clearly $\frac{\partial M}{\partial y} = -2x\sin y + 3x^2$

& $\frac{\partial N}{\partial x} = 3x^2 - 2x\sin y$.

So $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Hence D.E (1) is a exact D.E

Now since D.E (1) is exact so its G.S is →

$$\int M\,dx + \int N\,dy = c$$

$$\int (2x\cos y + 3x^2y)dx + \int (-y)\,dy = c$$

$$x^2\cos y + x^3y - y^2/2 = c \quad \text{--- (2)}$$

So G.S is → $f(x,y) = c$

where $f(x,y) = x^2\cos y + x^3y - y^2/2$

Also given → $y(0) = 2$ [That is at $x = 0$, $y = 2$]

Put it in (2) we get →

$$-4/2 = c$$

$$\boxed{c = -2}$$

→ Hence Soln of given IVP or Particular Soln of given D.E is

$$x^2\cos y + x^3y - y^2/2 = -2$$

Try yourself

Q1 Determine for what values of ⓐ + ⓑ, D.E

$(y + x^3)dx + (ax + by^3)dy = 0$. is exact

Hence find the G.S of exact D.E.

$\longrightarrow$ $a = 1$, $b \rightarrow$ any value , G.S $\rightarrow$ $xy + x^4/4 + by^4/4 = c$

Q2 Solve the D.E $\rightarrow$

$$(2xy\cos(x^2) - 2xy + 1)dx + (\sin x^2 - x^2)dy = 0$$

$\longrightarrow$ $y\sin x^2 - x^2y + x = c$

Q3 Solve IVP $\rightarrow$

$(\cos x + y\sin x)dx = (\cos x)\,dy$, $y(\pi) = 0$

$\longrightarrow$ $\sin x - y\cos x = c$ / $y = \tan x$.