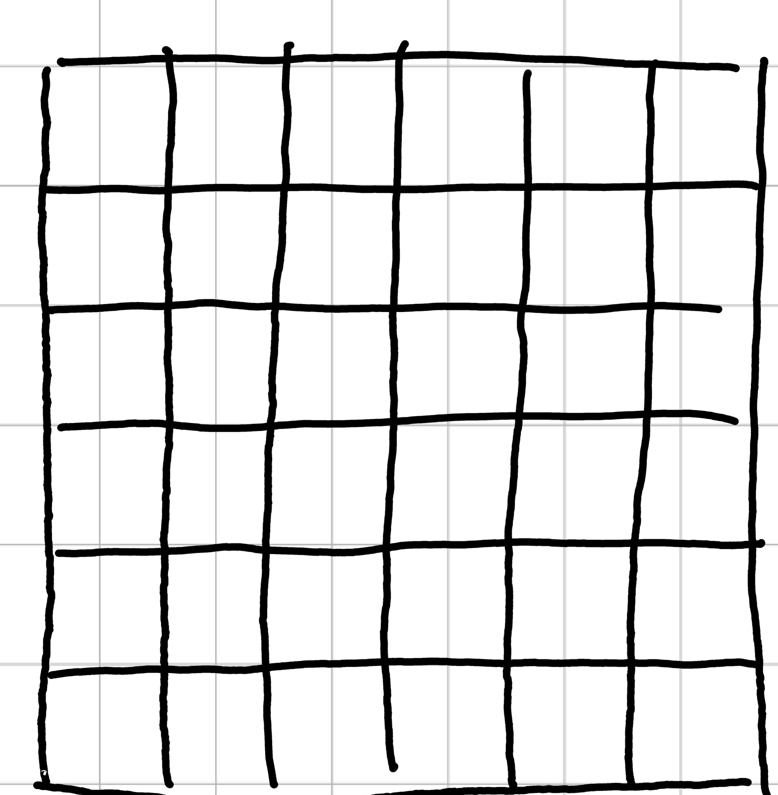
Big Foot Hunter

Part 1: Test



$N = 100$

$p_0 = 0.1$ - prob of random step

p_1 - prob of Bigfoot's step

$\hat{Y} = \{Y_{ij}\}$ - observation

$Y_{ij} = 1$ if step at (i,j)
$\quad\quad\; 0$, otherwise

$H_1 \to$ Bigfoot was there

$H_0 \to$ no Bigfoot

def $S = \sum_{i,j} Y_{i,j}$ - number of observed steps

- The prob to obtain a specific $\hat{Y}$ under H_0:

$$P(\hat{Y} \mid H_0) = p_0^S (1-p_0)^{N^2-S} = (1-p_0)^{N^2} \left(\frac{p_0}{1-p_0}\right)^S$$

- Let's find $P(\hat{Y} \mid H_1)$:

def $\hat{X}$ - true (hidden) Bigfoot's path:

- step(0) $X_{i_0 j_0} = 1 \to Y_{i_0 j_0} = 1$ with prob p_1
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad = 0$ with prob $1-p_1$

- step (1) chose (i_1, j_1) from $\{(i_0+1, j_0+1), (i_0-1, j_0+1), (i_0, j_0+1)\} \to X_{i_1, j_1} = 1 \to Y_{i_1, j_1} = 1$ with prob p_1

- step (2) as step(1) and so on.

Note that $\forall j \ \exists!\, i : \ X_{\underset{\text{row}}{i}\underset{\text{column}}{j}} = 1$, Let's call it as $i = f(j)$.

So transition matrix is $T\big((i_t, j_t) \to (i_{t+1}, j_{t+1})\big) =$

$= T\big((i_t, j_t) \to (i_{t+1}, j_t + 1)\big) \equiv T(i_t \to i_{t+1}) =$

$= T\big(f(j_t) \to f(j_t + 1)\big)$ and doesn't depend out.

$$T = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & & 0 \\ 0 & \frac{1}{3} & & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \vdots & & 0 \\ \frac{1}{3} & 0 & & \frac{1}{3} \\ & 0 & & \frac{1}{3} \end{pmatrix} \quad (\text{pereodic b.c.})$$

so $p(X|H_1) \equiv p(X_{H_1}) = \underset{\overset{\shortparallel}{\frac{1}{N}}}{p(j_0)}\, T\big(f(j_0) \to f(j_1)\big) \cdot \ldots \cdot$

$\cdot\, T\big(f(j_{N-2}) \to f(j_{N-1})\big)$

and $p(Y_{H_1} | X_{H_1}) = \prod_{i,j} \big[\underbrace{p_0^{Y_{i,j}} (1-p_0)^{1-Y_{i,j}}}_{\text{prob to fill } Y_{i,j} \text{ without BigFoot}}\big]^{1-X_{i,j}} \times$

$\times \big(\underbrace{(p_1 + p_0)^{Y_{i,j}} (1 - p_1 - p_0)^{1-Y_{i,j}}}_{\text{prob to fill } Y_{i,j} \text{ with BigFoot}}\big)^{X_{i,j}}$

so $p(Y|H_1) = \sum_{H_1} p(X|H_1)\, p(Y_{H_1}|X_{H_1})$

our Test is $T \sim \log \dfrac{p(Y|H_1)}{p(Y|H_0)} =$

$= \log\left(\sum_{X_{H_1}} \frac{1}{N} \left(\frac{1}{3}\right)^{N-1} \prod_{i,j} \left[\left(\frac{p_1}{p_0} + 1\right)^{Y_{i,j}} \left(1 - \frac{p_1}{1-p_0}\right)^{1-Y_{i,j}} \right]^{X_{i,j}} \right)$

But this is <u>exponentially</u> hard! :C

• Let us use dynamical approach:

define $y_i = \hat{Y}_{\omega i}$, $x_i = \hat{X}_{\omega i}$ ← columns

so $$P(Y_{H_1}) = P(\{y_i\}_{i=1}^{N}) = \sum_{x_N} p(y_N | x_N) \times$$
$$\times \sum_{\{x_i\}_{i=1}^{N-1}} \Big(p(x_1) \prod_{i=1}^{N-1} \underbrace{T(x_i \to x_{i+1})}_{\equiv\, T_{i,i+1}} p(y_i | x_i) \Big)$$

we can compute it by forward algorithm:

$$P(Y_{H_1}, x_N) = \sum_{x_{N-1}} P(\{y_i\}_{i=1}^{N-1}, x_{N-1}) T(x_{N-1} \to x_N) \times$$
$$\times P(y_N | x_N)$$

so $$P(Y_{H_1}) = \sum_{x_N} P(Y_{H_1}, x_N)$$

• also, we can write now: $P(Y_{H_0}) = P(\{y_i\}_{i=1}^{N} | H_0) =$
$$= \prod_{i=1}^{N} \Big(\prod_{j=1}^{N} p_0^{y_{i,j}} (1-p_0)^{1-y_{i,j}} \Big)$$

$$P(y_i | x_i) = \prod_j \Big[p_0^{y_{i,j}} (1-p_0)^{1-y_{i,j}} \Big]^{1-x_{i,j}} \Big((p_1+p_0)^{y_{i,j}} (1-p_1-p_0)^{1-y_{i,j}} \Big)^{x_{i,j}}$$

• So define $$\tilde{P}(Y_{H_1}, x_N) \equiv \sum_{x_{N-1}} \tilde{P}(\{y_i\}_{i=1}^{N-1}, x_{N-1}) \times$$
$$\times T(x_{N-1} \to x_N) \frac{P(y_N | x_N)}{P(y_N | H_0)} = \sum_{x_{N-1}} \tilde{P}(\{y_i\}_{i=1}^{N-1} x_{N-1}) \times$$
$$\times T(x_{N-1} \to x_N) \Big(\frac{p_1}{p_0} + 1 \Big)^{y_N \cdot x_N^T} \Big(1 - \frac{p_1}{1-p_0} \Big)^{\underbrace{sum(x_N)}_{1} - y_N x_N^T}$$

• So we have N recursion layers with $\sim N$ operations inside $\Rightarrow$ complexity is $O(N^2)$

so $$T \sim \log \Big(\sum_{x_N} \tilde{P}(Y_{H_1}, x_N) \Big)$$

- but actually, $x_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{pmatrix}$ is a vector with single "1" inside. So we can iterate over its position $\equiv j_i$

so $T \sim \log\Big(\sum_{j_N} \tilde{p}(Y_{H_1}, j_N) \Big)$, and

$$\tilde{p}(Y_{H_1}, j_N) = \sum_{\substack{j_{N-1} \in \\ \{j_N+1,\, j_N-1,\, j_N\}}} \tilde{p}(Y_{H_1}, j_{N-1}) \underbrace{T(j_{N-1} \to j_N)\left(1 - \frac{p_1}{1-p_0}\right)}_{T'(j_{N-1} \to j_N)} \times$$

$$\times \Bigg(\underbrace{\frac{(p_1+p_0)(1-\beta)}{p_0\,(1-p_0-p_1)}}_{\rho} \Bigg)^{Y_{j_N, N}} = \sum_{\substack{j_{N-1} \in \\ \{j_N+1,\, j_N-1,\, j_N\}}} \tilde{p}(Y_{H_1}, j_{N-1})\, T'(j_{N-1} \to j_N) \times \rho^{Y_{j_N, N}}$$