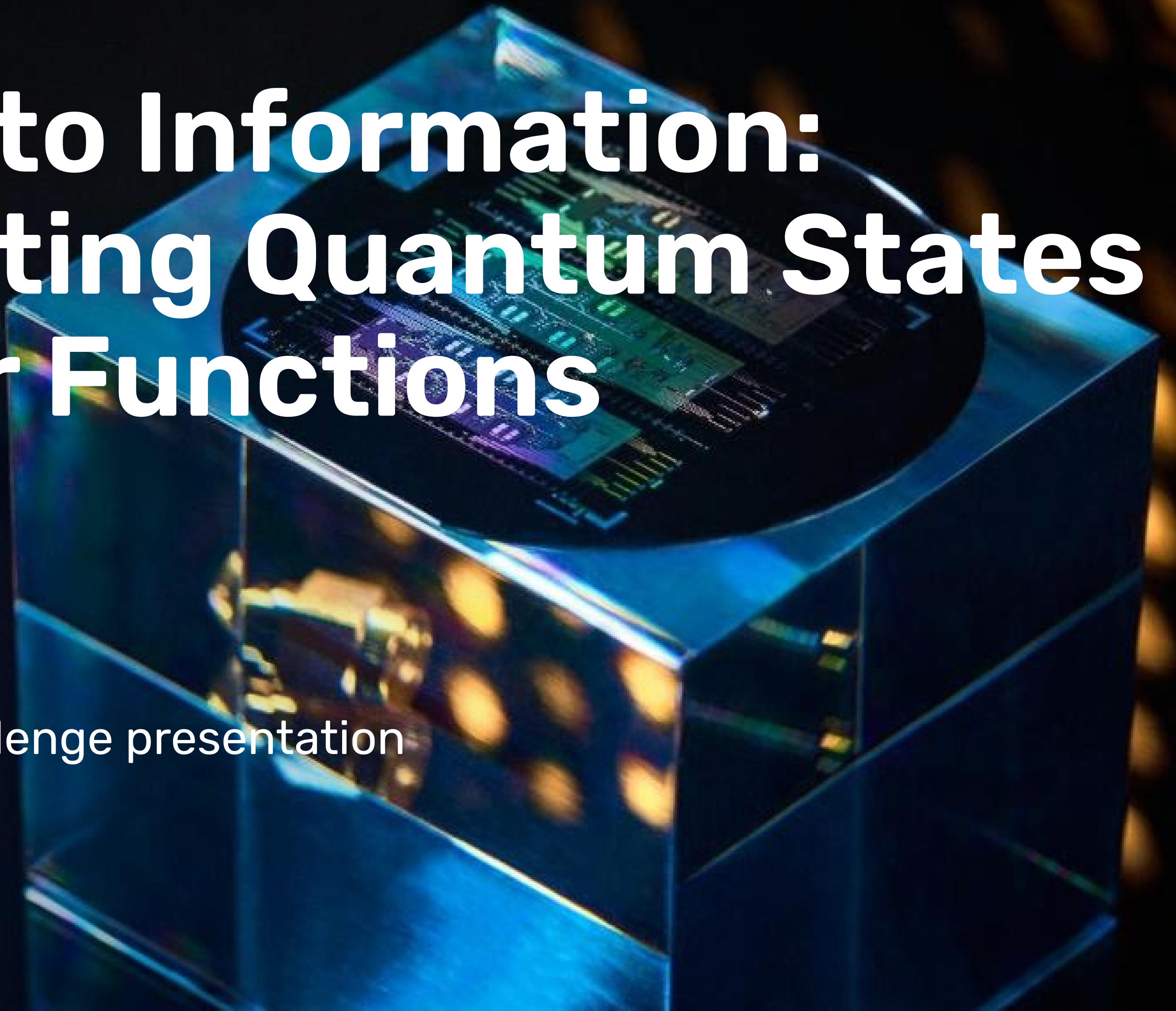
ALICE & BOB

From Noise to Information: Reconstructing Quantum States from Wigner Functions



ETH Quantum Hackathon challenge presentation

April 2025

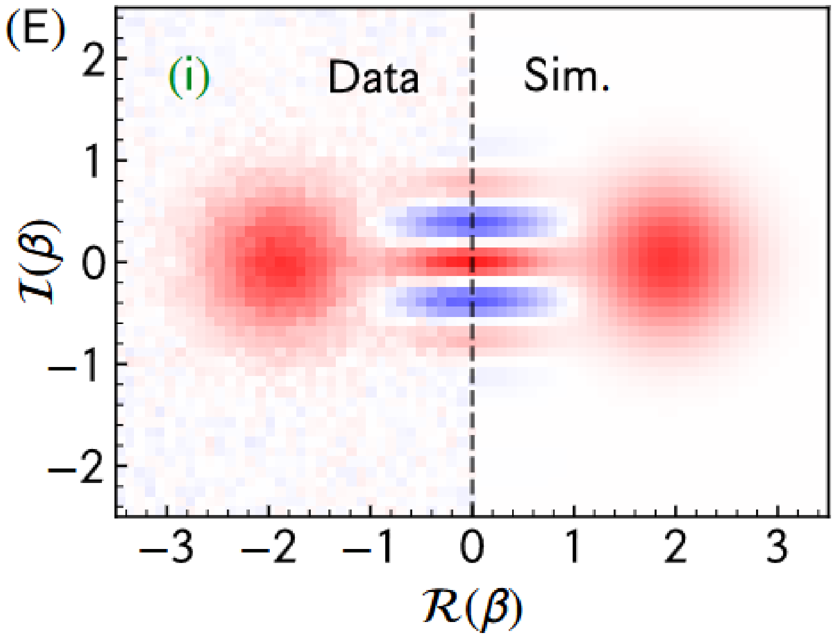
Reconstructing Quantum States from Wigner Functions

Real-World Problem:

How do we efficiently know our quantum state?

→ **Quantum Tomography**

For Alice & Bob:
Measuring Wigner Function



R. Rousseau et al. – arXiv (2025)

Density Matrix

Describe statistical mixtures of quantum states

$$\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i|$$

$|\psi_i\rangle$ Orthogonal quantum states

p_i Probability of the system being in state $|\psi_i\rangle$

→ very useful for describing decoherence, incomplete knowledge

Key Properties of the Density Matrix

- Hermitian: $\rho^\dagger = \rho$
- Positive semi-definite: $\lambda_i \geq 0$
- Normalization: $\mathrm{Tr}(\rho) = 1$

Pure vs. Mixed States

Pure state: $\rho^2 = \rho$ e.g.: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$$\rho = |\psi\rangle\langle\psi|$$

Mixed state: $\rho = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$

The Wigner Function

Quasi-probability distribution in phase-space:

$$W_\rho(x, p) = \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} \langle x + y|\, \rho \,|x - y\rangle\, e^{2ipy/\hbar} dy$$

- x and p: position and momentum of the system
- $\langle x + y|\, \rho \,|x - y\rangle$: density matrix in position-basis → coherence of the system
- Integral over $\int e^{-2ipy/\hbar} dy$: Fourier transform

  → how different position states interfere with each other in the momentum domain.
- $\frac{1}{\pi\hbar}$: normalization factor

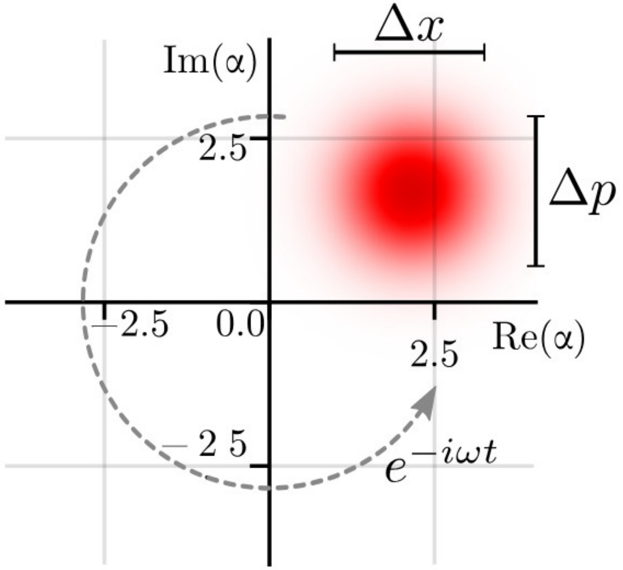
The Wigner Function

Quasi-probability distribution in phase-space:

$$W_\rho(x,p) = \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} \langle x+y|\,\rho\,|x-y\rangle\, e^{2ipy/\hbar} dy$$

In classical mechanics: State of the system described by a **single point** in phase space

In quantum mechanics:
→ Quasi-probability distribution
→ Heisenberg uncertainty



- positive regions of W(x,p): where the quantum state is more likely to be found.
- **negative regions:** indicate quantum interference effects.
- **zero regions**: the state has no contribution.

The Wigner Function

Quasi-probability distribution in phase-space:

$$W_\rho(x,p) = \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} \langle x+y|\,\rho\,|x-y\rangle\, e^{2ipy/\hbar} dy$$

Properties:

$$\int W_\rho(x,p)dxdp = 1$$ (Normalization)

$W_\rho(x,p)$ Real-valued function

$$W_\rho(x\to\pm\infty, p\to\pm\infty)\to 0$$

The Wigner Function II

Displacement and Parity Operators

Displacement operator:

$$\hat{D}(\alpha) = \exp\left(\alpha \hat{a}^\dagger - \alpha \hat{a}\right)$$

$\alpha \in \mathbb{C}$ $\quad$ $\hat{a}$: annihilation operator

$$D(\alpha)\left|0\right\rangle = \left|\alpha\right\rangle$$

Parity operator:

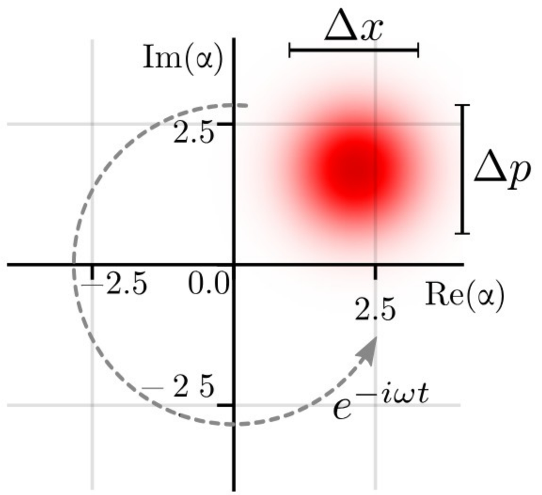
$$\hat{P} = e^{i\pi \hat{a}^\dagger \hat{a}}$$

In Fock basis:

$$\hat{P}\left|n\right\rangle = (-1)^n \left|n\right\rangle$$

Identical way of expressing the Wigner function:

$$W(\alpha) = \frac{2}{\pi}\mathrm{Tr}\left[D(\alpha) P D^\dagger(\alpha)\rho\right]$$



→ Measure Parity of a displaced state

Challenge: Density matrix reconstruction

One-to-one mapping: (Isomorphism)

$$\hat{\rho} \longleftrightarrow W(\alpha)$$

Numerically:

$$\tilde{\rho} = \mathrm{fit}(W(x, p))$$

How to implement this numerically?

How to obtain the density matrix from an existing Wigner function?

$$W(\alpha) \longrightarrow \hat{\rho}$$

Challenge: Density matrix reconstruction

Wigner Tomography: A Convex-Optimization Approach

Wigner function does not correspond to a measurement
→ **Parity** is a physical measurement

Measurement Operator:

$$E_\alpha = \frac{1}{2}(I + D(\alpha)PD^\dagger(\alpha))$$

Probability:

$$p_\alpha = \mathrm{Tr}(E_\alpha \rho) \qquad p_\alpha \in [0, 1]$$

$$W(\alpha) = \frac{2}{\pi}(2p_\alpha - 1) \Longleftrightarrow p_\alpha = \frac{1}{2}\left(1 + \frac{\pi}{2}W(\alpha)\right)$$

→ if we know $W(\alpha)$, we can compute the corresponding measurement probability

Switch of perspective:

$W(\alpha)$ as a given function → treat the **measured parity probabilities** p_α as **data points**

Each point α corresponds to a measurement operator E_α, find ρ s.t. $p_\alpha = \mathrm{Tr}(E_\alpha \rho)$

Challenge: Convex Optimization

Goal: From a given set of measurement results p_α, find a density matrix ρ that best explains the data.

Assume finite number of measurement results:

$$\{p_{\alpha_k}\}_{k=1}^{n}$$

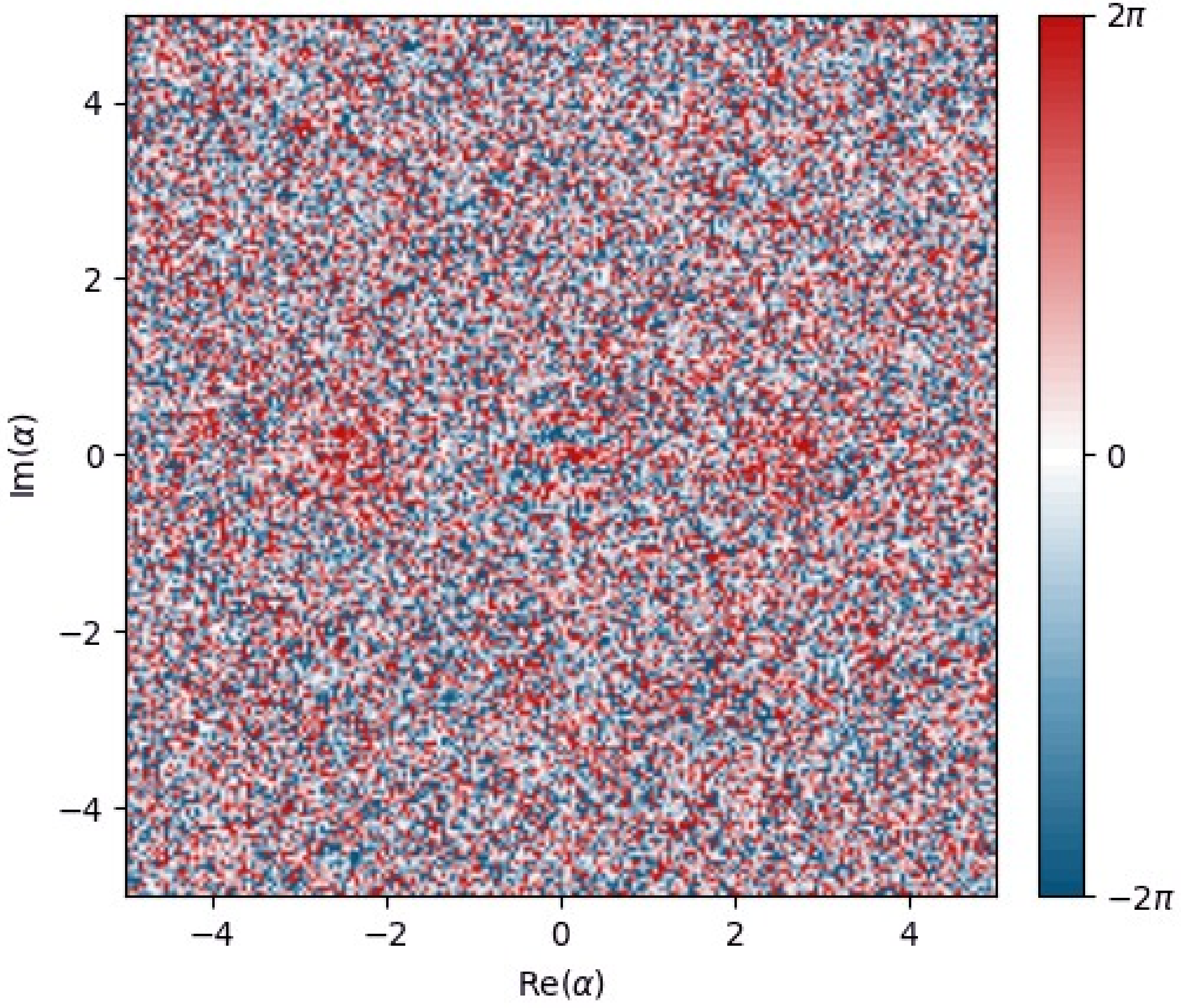
Cost function:

$$\mathcal{L}(\rho) = \sum_{k=1}^{n} |\mathrm{Tr}(E_{\alpha_k}\rho) - p_{\alpha_k}|^2$$

$$\rho \succeq 0 (\text{positive semidefinite}), \mathrm{Tr}(\rho) = 1$$

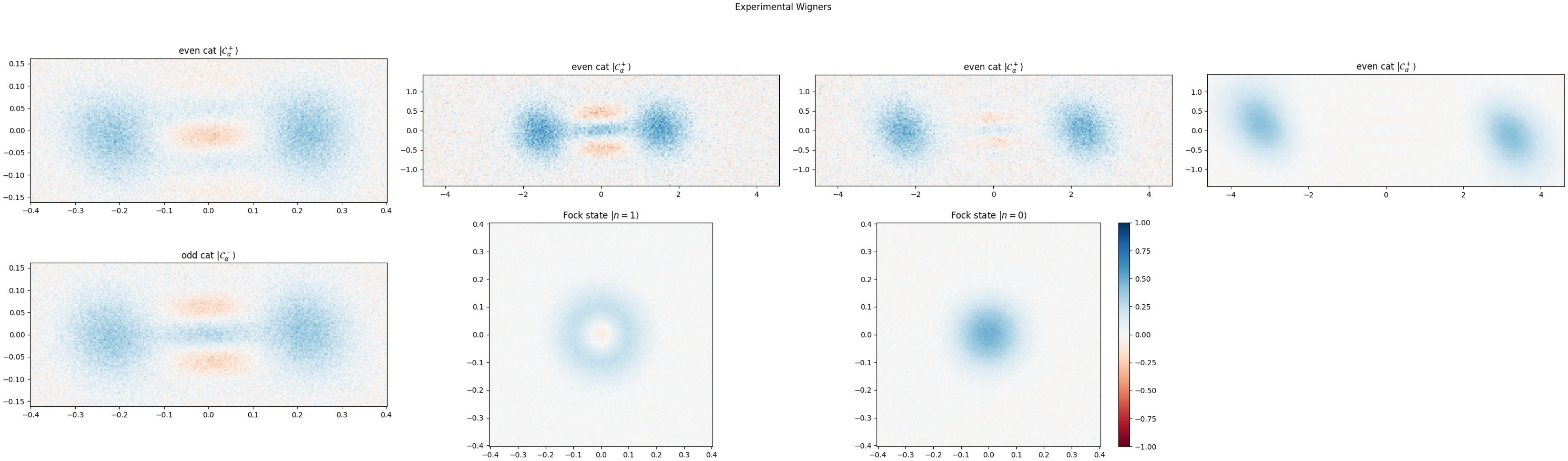
→ **Convex optimization problem**

Challenge: Robustness against noise



How does the reconstruction scale when (gradually) introducing noise to the system?

Challenge: Robustness against noise



Reglade et al., Nature 629, 778–783 (2024)

Challenge: Dealing with real-world data

In practice: Callibration of measurement outcomes necessary

How to correct the affine distortion numerically?

Affine distortion:

$$W_{measured}(\alpha) = aW_\rho(\alpha) + b + \text{noise}$$

How to filter or denoise the Wigner function?

Remember:

$$\int W(\alpha)d\alpha = 1$$

$$W_\rho(\alpha \to \pm\infty) \to 0$$

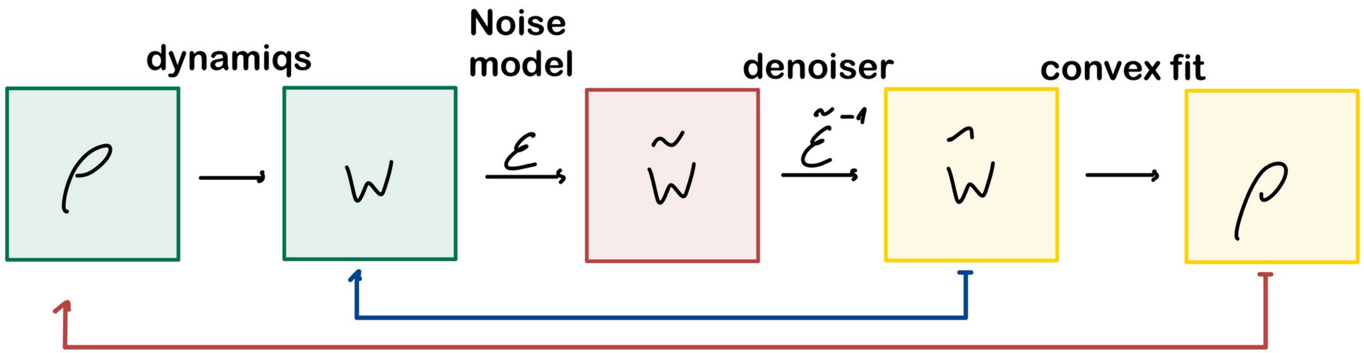
→ Apply 2D Gaussian filter

Challenge: Optional Task

Supervised Learning for Wigner Denoising

1. Design a realistic noise model

Create **multiple noise models** to simulate imperfections seen in experiments



2. Build a Supervised Dataset

- **Generate Clean Wigners**: Simulate a variety of pure quantum states
- **Corrupt the Data**: Apply your noise models to generate paired samples

$$\tilde{W}_i = \mathrm{NoiseModel}(W_i)$$

3. Train a Denoising Model

Accelerate Reconstruction Method

How to speed up the reconstruction?

Useful Tools

Software:

- Dynamiqs, JAX
- cvxpy or other convex optimization tools
- Scikit learn
- Scipy
- Linear algebra (numpy, etc.)
- Machine Learning libraries
  - Pytorch
  - Tensorflow
  - Keras
  - ...

Getting Help:

- Ask us!
- Explore suggested references
- Explore scientific articles