Engineering Economy_

[1-1]
Introduction
Basic Concepts and Definitions

1

Scratch your head.

- The presidents of two small businesses play racquetball each week. After several conversation, they have decided that, due to their frequent travel that they should evaluate the purchase of a plane jointly owned by the two companies.
- What are some of the typical economic-based questions they should answer as they evaluate the alternatives: 1. co-own…..2. continue as is?

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Some questions

- How much will it cost each year? (estimation)
- How will they pay for it? (financing plan)
- Are there tax advantages? (Tax law)
- Which alternative is more cost-effective?
- What is the expected rate of return? (equation)
- What if we use different amount each year? (sensitivity analysis)

So……..what

- As you see from previous example how we incorporated engineering with economy.
- Economy…………….(cost, tax, and financing)
- Engineering…………(mathematical model, and decision making)

What is Engineering Economy?

- Engineering economy is a subset of economy for application to engineering projects
- Engineers seek solutions to problems, and the economic viability of each potential alternative or design is normally considered along with the *technical aspects*
- Engineering economy involves the evaluation of the costs and benefits of proposed projects

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Why Engineering Economy is Important?

- There are lots of factors that are considered in making decisions
- These factors are combinations of economic and non-economic ones
- Engineers play a major role in investment by making decisions based on economic analysis and design considerations
- Thus, decisions often reflect the engineer's choice of *how to best invest funds* by choosing the proper alternative out of a set of alternatives

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Role of Engineering Economy in Decision Making

- Engineers make decisions but tools and computers do not
- Tools assist engineers in making decisions
- Decisions affect what will happen in the future and thus the time frame of engineering economy is the future
- So, engineering economy analysis presents the best estimates of what is expected to occur

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Role of Engineering Economy in Decision Making

- Understand the Problem
- Collect all relevant data/information
- Define the feasible alternatives
- Evaluate each alternative
- Select the "best" alternative
- Implement and monitor

This is the major role of engineering economy

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Role of Engineering Economy in Decision Making

- The economic evaluation of alternatives is based on the so called "Measure of Worth" such as:
  - Present worth: amount of money at the current time
  - Future worth: amount of money at some future time
  - Payback period: Number of years to recover the initial investment and a stated rate of return
  - Rate of return: Compound interest rate on unpaid or unrecovered balances
  - Benefit/cost ratio

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Role of Engineering Economy in Decision Making

- There are other factors that affect the decision making such as social, environmental, legal, political, personal, …

- This may place less reliance on the economic-based factors yet this also shows the importance of knowing all the involved factors including the economic ones

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Time Value of Money

- Time Value of Money (TVM) is an important concept
- TVM is based on the concept that money that you hold today is worth more because you can invest it and earn interest
- For instance, you can invest your dollar for one year at a 6% annual interest rate and accumulate $1.06 at the end of the year
- You can say that the future value of the dollar is $1.06 given a 6% interest rate and a one-year period
- It follows that the present value of the $1.06 you expect to receive in one year is only $1

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What is Interest?

- Interest is what you *earn* when you let people borrow your money
- Some call it the *price of renting your money*
- Interest can be thought of as the price a lender charges a borrower for the use of his money
- Interest is the difference between an ending amount of money and the beginning amount

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Interest Types

- There are two types of interest:
  - Interest paid: when a person borrows money and repays a larger amount
  - Interest revenue: when a person *saved*, *invested*, or *lent* money and obtains a *return* of a larger amount
- Numerical values are the same for both yet they are *different* in interpretation

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Interest Paid

- Interest paid = amount owed *now* – *original* amount

  where the interest is paid over a specific time unit

- If the interest is expressed as a percentage of the original amount then it is called the *interest rate* and expressed as in the following:

$$Interest\ rate\ (\%) = \frac{Interest\ accrued\ \text{per time unit}}{\text{original amount}} \times 100\% rued$$

- The time unit of the rate is called the interest period

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Interest Earned

- Interest earned = total amount *now* – *original* amount

- Interest earned over a specific period of time is expressed as a percentage of the original amount and is called rate of return (ROR) and is computed from the following:

$$\text{Rate of Return (\%)} = \frac{\text{interest accrued per time unit}}{\text{original amount}} \times 100\%$$

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Interest – Example [1]

- An employee borrows $10,000 on May 1 and must repay a total of $10,700 exactly 1 year later

- Determine the interest amount and interest rate paid

- Interest amount = $10,700 - $10,000 = $700

- Interest rate = $700/$10,000 = 7% per year

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Interest – Example [2]

- A company plans to borrow $20,000 from a bank for one year at 9% interest for a new recording equipment
- Compute the *interest* and the *total amount* due after 1 year

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- The total interest accrued:
  Interest = $20,000 × 0.09 = $1,800
- The total amount due is the sum of principal and interest:
  Total due = $20,000 + $1,800 = $21,800

Interest – Example [3]

- Calculate the amount deposited one year ago to have $1,000 now at an interest rate of 5% per year

The total amount accrued ($1,000) is the sum of the original deposit and the earned interest. If "**y**" is the original deposit then,

Total amount accrued = original + original × interest rate

$1,000 = y + y(0.05) which gives a value of y = $952.38

- Calculate the amount of interest earned during this time period
  Interest = $1,000 – 952.38 = $47.62

Interest – Example [4]

- Calculate the amount deposited one year ago to have \$1,000 as a net benefit now at an interest rate of 5% per year

The total amount accrued after one year of deposition equals the sum of the original deposit and the earned interest. If "**y**" is the original deposit then,

Total amount accrued = original + original × interest rate

$y + 1{,}000 = y + y(0.05)$ which gives a value of y = \$20,000

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Simple and Compound Interest

- In the previous examples, the interest period was 1 year and the interest amount was calculated at the end of one period
- When more than one interest period is involved (e.g. after *3 years*), it is then necessary to state whether the interest is accrued on a *simple* or *compound* basis from one period to the next

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Simple and Compound Interest

- Simple interest is named as such because the interest calculated is not compounded

- The total *simple* interest over several periods is computed as:

Interest = (principal) × (number of periods) × (interest rate)

Simple Interest – Example

- A company loaned money to an engineering staff member for a radio-controlled model airplane. The loan is for \$1,000 for 3 years at 5% per year simple interest

- How much money will the engineer repay at the end of 3 years?

- The interest for each of the 3 years is:
  Interest per year = \$1,000 × 0.05 = \$50
  Total interest for 3 years is \$1,000 × 0.05 × 3 = \$150
  The amount due after 3 years is \$1,000 + \$150 = \$1,150

Simple Interest – Example

- The $50 interest accrued in the first year and the $50 accrued in the second year do not earn interest
- The interest due each year is calculated **only** on the $1,000 principal

End of Year	Amount Borrowed	Interest	Amount Owed	Amount Paid
0	$1,000	0	0	0
1	-	$50	$1,050	0
2	-	$50	$1,100	0
3	-	$50	$1,150	$1,150

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Compound Interest

- Compound interest: it is the interest that accrued for each interest period and is calculated on the principal plus the total amount of interest accumulated in all previous periods
- Thus compound interest means *interest on top of interest*
- Compound interest reflects the effect of the time value of money on the interest
- Compound interest for ***one*** period = (principal + all accrued interest) × (interest rate)

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Compound Interest – Example

- If an engineer borrows \$1,000 at 5% per year compound interest, compute the total amount due after 3 years

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- The interest and total amount due each year are computed:

- Year 1 interest: \$1,000 × 0.05 = \$50.00
  Total amount due after year 1 = \$1,000 + \$50 = \$1,050

  Year 2 interest: \$1,050 × 0.05 = \$52.50
  Total amount due after year 2 = \$1,050 + \$52.5 = \$1,102.50

  Year 3 interest: \$1,102.5 × 0.05 = \$55.13
  Total amount due after year 3 = \$1,102.5 + \$55.13 = \$1,157.63

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Compound Interest – Example

- Another and shorter way to calculate the total amount due after 3 years is to combine calculations rather than perform them on a year-by-year basis. The total due each year is as follows:

Year 1: \$1,000 × $(1.05)^1$ = \$1,050.00
Year 2: \$1,000 × $(1.05)^2$ = \$1,102.50
Year 3: \$1,000 × $(1.05)^3$ = \$1,157.63

- In a general formula:

  Total due = principal × $(1 + \text{interest rate})^{\text{number of years}}$

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Inflation

- Inflation represents a decrease in the value of a given currency
- Also, inflation indicates a *loss in the purchasing power of money over time*
- For instance, $1 now will not purchase the same number of apples as $1 did 20 years ago

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Economic Equivalence

- To compare alternatives that provide the same service over extended periods of time when interest is involved, we must reduce them to an equivalent basis
- Equivalence factors are needed in engineering economy to make cash flows (CF) at different points in time comparable. For example, a cash payment that has to be made today cannot be compared directly to a cash payment that must be made by the end of 5 years

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Equivalence

- So, economic equivalence means that different sums of money at different times would be equal in economic value

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Economic Equivalence

- For example, if the interest rate is 6% per year, $100 today (*present time*) is equivalent to $106 *one year from today*
- So, if someone offered you a gift of $100 today or $106 one year from today, it would make no difference from an economic perspective
- The two sums of money are *equivalent* to each other only when the interest rate is 6% per year
- That is, at a *higher* or *lower* interest rate, $100 today is not equivalent to $106 one year from today
- The same concept applies a year ago, that is a total of $100 today is economically equivalent to $100/1.06 = $94.34

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Engineering Economy

[1-2]
Introduction
Examples and Additional Concepts

Example [1]
Time Value of Money

- You have $1,000 and you want to buy a $1,000 machine

- Suppose that you can invest money at 6% interest, but the price of the machine increases only at an annual rate of 4% due to inflation. After a year, you can still buy the machine and you will have $20 left over (*earning power exceeds inflation*)

- If the price of the machine increases at an annual rate of 8% instead, you will not have enough money to buy the machine a year from today. In this case, it is better to buy it today (*inflation exceeds earning power*)

Example [2]

The Concept of Equivalence

- Demonstrate the concept of equivalence using the different loan repayment plans described below. Each plan repays a $5,000 loan in 5 years at 8% interest per year
- *Plan 1*: *Simple interest, pay all at end*. No interest or principal is paid until the end of year 5. interest accumulates each year on the principal only
- *Plan 2*: *Compound interest, pay all at end*. No interest or principal is paid until the end of year 5. interest accumulates each year on the total of principle and all accrued interest
- *Plan 3*: *Simple interest paid annually, principal repaid at end*. The accrued interest is paid each year, and the entire principal is repaid at the end of year 5
- *Plan 4*: *Compound interest and portion of principal repaid annually*. The accrued interest and one-fifth of the principal is repaid each year

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Example [2]

The Concept of Equivalence – Plan 1

Microsoft Excel - Engineering Economy

	A	B	C	D	E
1	1	2	3	4	
2	**End of year**	**Interest owed for year**	**Total owed at end of year**	**End-of-year payment**	
3	*Plan 1: Simple interest, pay all at end*				
4	0				
5	1	$400.00	$5,400.00	-	
6	2	$400.00	$5,800.00	-	
7	3	$400.00	$6,200.00	-	
8	4	$400.00	$6,600.00	-	
9	5	$400.00	$7,000.00	$7,000.00	
10	Total			$7,000.00	
11					

Sheet1

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Example [2]

The Concept of Equivalence – Plan 2

Microsoft Excel - Engineering Economy

	A	B	C	D	E
13	**End of year**	**Interest owed for year**	**Total owed at end of year**	**End-of-year payment**	
14	*Plan 2: Compound interest, pay all at end*				
15	0				
16	1	$400.00	$5,400.00	-	
17	2	$432.00	$5,832.00	-	
18	3	$466.56	$6,298.56	-	
19	4	$503.88	$6,802.44	-	
20	5	$544.20	$7,346.64	$7,346.64	
21	Total			$7,346.64	
22					
23					

Sheet1

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Example [2]

The Concept of Equivalence – Plan 3

Microsoft Excel - Engineering Economy

	A	B	C	D	E
23					
24	**End of year**	**Interest owed for year**	**Total owed at end of year**	**End-of-year payment**	
25	*Plan 3: Simple interest paid annually, principal repaid at end*				
26	0				
27	1	$400.00	$5,400.00	$400.00	
28	2	$400.00	$5,400.00	$400.00	
29	3	$400.00	$5,400.00	$400.00	
30	4	$400.00	$5,400.00	$400.00	
31	5	$400.00	$5,400.00	$5,400.00	
32	Total			$7,000.00	

Sheet1

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Example [2]

The Concept of Equivalence – Plan 4

Microsoft Excel - Engineering Economy

	A	B	C	D	E
35	**End of year**	**Interest owed for year**	**Total owed at end of year**	**End-of-year payment**	
36	*Plan 4: Compound interest and portion of principal repaid annually. The* ***accrued interest*** *and* ***one-fifth*** *of the principal is repaid each year*				
37	0				
38	1	$400.00	$5,400.00	$1,400.00	
39	2	$320.00	$4,320.00	$1,320.00	
40	3	$240.00	$3,240.00	$1,240.00	
41	4	$160.00	$2,160.00	$1,160.00	
42	5	$80.00	$1,080.00	$1,080.00	
43	Total			$6,200.00	

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Example [2]

The Concept of Equivalence – *Comments*

$5,000 at time 0 is equivalent to each of the following:

- Plan 1: $7,000 at the end of year 5 at 8% simple interest
- Plan 2: $7,346.64 at the end of year 5 at 8% compound interest
- Plan 3: $400 per year for 4 years and $5,400 at the end of year 5 at 8% simple interest
- Plan 4: Decreasing payments of interest and partial principal in years 1 ($1,400) through 5 ($1,080) at 8% compound interest

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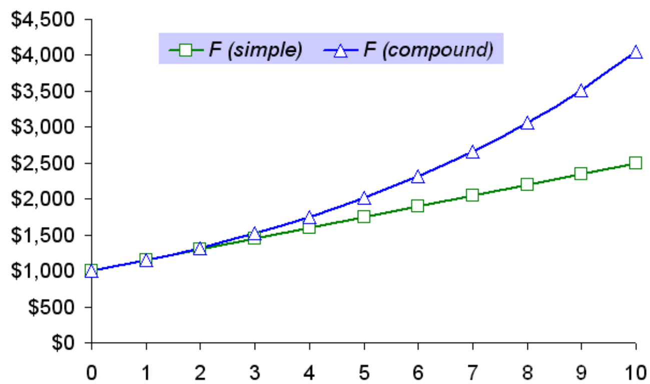
Just Keep in Mind

- Simple interest: $F = P(1+ni)$

- Compound interest: $F = P(1+i)^n$

where
F: future worth
P: present worth
i: interest rate
n: number of interest periods



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Simple and Compound Interest Comparison

- The total amount with *simple* interest grows *linearly*

- The total amount with *compound* interest grows *exponentially*

- This exponential growth is referred to as the power of compounding

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Terminology and Symbols

- The equations and procedures of *engineering economy* utilize the following terms and symbols:

- P: [dollars] value or amount of money at the present time (time 0). P is referred to as present worth (PW), present value (PV), net present value (NPV), discounted cash flow (DCF), and capitalized cost (CC)

- F: [dollars] value or amount of money at some future time. Also F is called future worth (FW) and future value (FV)

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Terminology and Symbols

- A: [dollars per year, dollars per months] series of consecutive, equal, end-of-period amounts of money. Also A is called the annual worth (AW) and equivalent uniform annual worth (EUAW)

- n: [years, months, days] number of interest periods

- i: [percent per year, percent per month, percent per day] interest rate or rate of return per time period. *Assume compound interest if not specified*

- All engineering economy problems involve the element of interest period n and interest rate i

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Example [3]

The Concept of Equivalence

- You are given the alternative of receiving either $3,000 at the end of five years or P dollars today
- What value of P would make you *indifferent* to $3,000 at the end of five years if the interest rate is 8%?

Symbols are as follows: F = $3,000, n = 5 years, and i = 8%. Find P

We know that $F = P(1+i)^n$

Substituting yields P = $2,042

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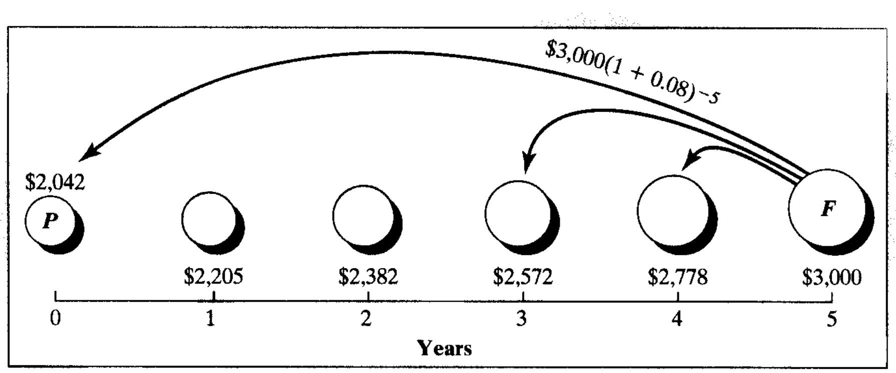
Example [3]

The Concept of Equivalence



Various dollar amounts that will be economically equivalent to $3,000 in five years given an interest rate of 8%

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Example [4]
Terminology and Symbols

- Someone plans to borrow $10,000 now to help in buying a car
- He arranged to repay the entire principal plus 8% per year interest after 5 years
- *Identify the engineering economy symbols* involved and their values for the total owed after 5 years

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Example [4]
Terminology and Symbols

- We have P = $10,000, i = 8%, and n = 5 years
- You must find the value of F which represents the amount to be repaid after 5 years

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Example [5]

Terminology and Symbols

- Assume you borrow $2,000 now at 7% per year for 10 years and you must repay the loan in equal yearly payments

- Determine the symbols involved and their values

Example [5]

Terminology and Symbols

- We have P = $2,000, i = 7% per year, and n = 10 years

- We need to find the value of A which represents the yearly equal (uniform) payments

Example [6]
Terminology and Symbols

- Last year Jane's grandmother offered to put enough money into a savings account to generate $1,000 to help pay Jane's expenses at college
- Identify the symbols
- Calculate the amount that had to be deposited exactly 1 year ago to earn $1,000 ***in interest*** now if the rate is 6% per year

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Example [6]
Terminology and Symbols

- We have i = 6% and n = 1
- We know that the amount of interest is $1,000
- Since $F = P(1+i)^n$ and F = P + interest amount
- Then P = $16,666.67 which represents the amount to be deposited at the beginning

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Example [7]

- What is the present worth of a lump sum of one million dollars to be received 50 years from today if the interest rate is 10%?

- $F = P(1+i)^n$ and or $P = F(1+i)^{-n}$ →
  $P = \$1{,}000{,}000 \times (1+10\%)^{-50} \approx \$8{,}519$

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Example [8]

- Show the yearly payments a borrower has to pay to the bank for the following three repayment plans. The loan is \$10,000 and must be repaid to the bank in 5 years at 10% interest per year
- *Plan 1*: *Compound interest*. However, there is no interest in the third year. Pay all at end. No interest or principal is paid until the end of year 5
- *Plan 2*: *Compound interest*. However, you pay the *interest* plus *one-quarter* of the principal by the end of each year
- *Plan 3*: Same as Plan 2 except in the third year you pay only the interest

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Example [8]

End of year	Interest owed for year	Total owed at end of year	End-of-year payment
Plan 1: Compound interest, pay all at end			
0			
1	$1,000.00	$11,000.00	-
2	$1,100.00	$12,100.00	-
3	$0.00	$12,100.00	-
4	$1,210.00	$13,310.00	-
5	$1,331.00	$14,641.00	$14,641.00
Total			*$14,641.00*

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Example [8]

End of year	Interest owed for year	Total owed at end of year	End-of-year payment
Plan 2: Compound interest and portion of principal repaid annually. The ***accrued interest*** *and* ***one-fourth*** *of the principal is repaid each year*			
0			
1	$1,000.00	$11,000.00	$3,500.00
2	$750.00	$8,250.00	$3,250.00
3	$500.00	$5,500.00	$3,000.00
4	$250.00	$2,750.00	$2,750.00
5	$0.00	$0.00	$0.00
Total			*$12,500.00*

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Example [8]

End of year	Interest owed for year	Total owed at end of year	End-of-year payment
*Plan 3: Compound interest and portion of principal repaid annually. The **accrued interest** and **one-fourth** of the principal is repaid each year except in the third year where you pay only the interest*			
0			
1	$1,000.00	$11,000.00	$3,500.00
2	$750.00	$8,250.00	$3,250.00
3	$500.00	$5,500.00	$500.00
4	$500.00	$5,500.00	$3,000.00
5	$250.00	$2,750.00	$2,750.00
Total			*$13,000.00*

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Solution by Excel

- Using the symbols P, F, A, i, and n exactly as defined earlier, the Excel functions that most used in engineering economy are the following:
  - P: PV(i%,n,A,F)
  - F: FV(i%,n,A,P)
  - A: PMT(i%,n,P,F) Just equal values
  - n: NPER(i%,A,P,F)
  - i: RATE(n,A,P,F)

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ENGINEERING ECONOMY_

INTRODUCTION
CASH FLOW DIAGRAMS

1

CASH FLOWS

- To financially analyze engineering projects, we need to model the projects in terms of cash flows
- Cash flows represent the flow or movement of money at some specific time over some period of time
- Outflows represent cash that is leaving an account such as a withdrawal (expenses or disbursements or losses or costs)
- Inflows represent cash that is entering an account such as a deposit (revenues or receipts or benefits or incomes)

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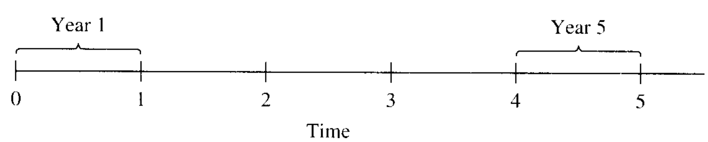
CASH FLOWS AND ENGINEERING PROJECTS

- An engineering project can be viewed as an account with *outflows* and *inflows*
- Cash flow movements can be visually displayed through the use of a cash flow diagram



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CASH FLOW DIAGRAM

- A cash flow diagram is a picture of a financial problem that shows all cash inflows and outflows plotted along a horizontal time line
- The cash flows over time are represented by arrows at relevant periods: upward arrows denote positive flows and downward arrows denote negative flows
- Arrows represent net cash flows since two or more values at the same time are summed and shown as a single arrow
- *Net cash flows = receipts – disbursement*
  = cash inflows – cash outflows

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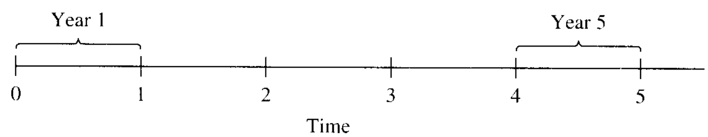
CASH FLOW DIAGRAM

- Generally, the start of the diagram represents the *beginning* of the interest period
- When t = 0, this is the present
- When t = 1, this is the the *end of the first year* (or *beginning of the second year*)



A typical cash flow time scale for 5 years

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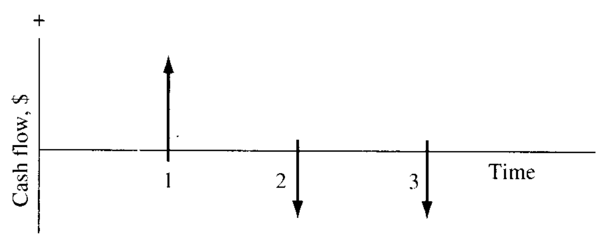
CASH FLOW DIAGRAM

The figure illustrates a receipt (cash *inflow*) at the end of year 1 and equal disbursements (cash *outflows*) at the end of years 2 and 3



Example of positive and negative cash flows

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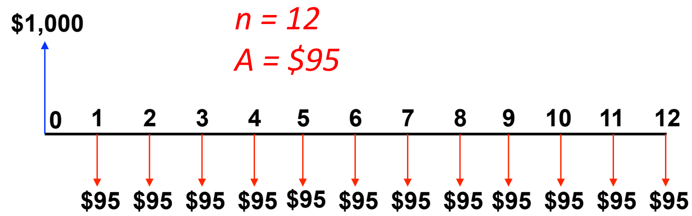
CASH FLOW DIAGRAM – EXAMPLE [1]

- You borrowed $1,000 from a bank to purchase a laptop. The bank requires you to make 12 equal monthly payments of $95 to pay off the loan

- Draw the cash flow diagram for this scenario

P = $1,000
n = 12
A = $95



CASH FLOW DIAGRAM – EXAMPLE [2]

- A company spent $2,500 on a new compressor 7 years ago

- The annual *income* from the compressor has been $750

- Additionally, the $100 spent on *maintenance* during the first year has increased each year by $25

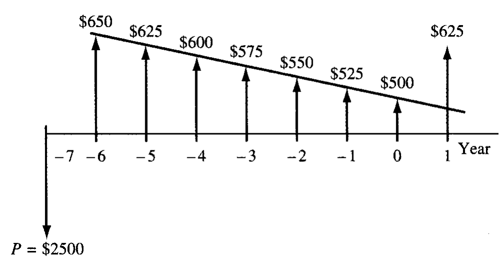
- The company plans to sell the compressor at the end of next year for $150

- Construct the cash flow diagram from the *company's perspective*

CASH FLOW DIAGRAM – EXAMPLE [2]

- Use now as time t = 0
- The incomes and costs for years -7 through 1 (next year) are tabulated

End of year	Income	Cost	Net Cash Flow
−7	$ 0	$2500	$−2500
−6	750	100	650
−5	750	125	625
−4	750	150	600
−3	750	175	575
−2	750	200	550
−1	750	225	525
0	750	250	500
1	750 + 150	275	625



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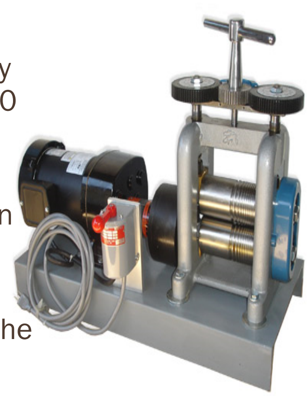
CASH FLOW DIAGRAM – EXAMPLE [3]

- A company expanded its operations with the *purchase* of a $10 million rolling mill in 2004
- Assume that the new mill runs at peak capacity (4.375 million pounds of output per year) for 10 years
- Assume that a pound of output generates $9 in revenues while costing $3.90 to produce
- Maintenance of the equipment is $10 million the first year and grows by $1 million per year
- Finally, the mill is to be scrapped at the end of 10 years for $500,000



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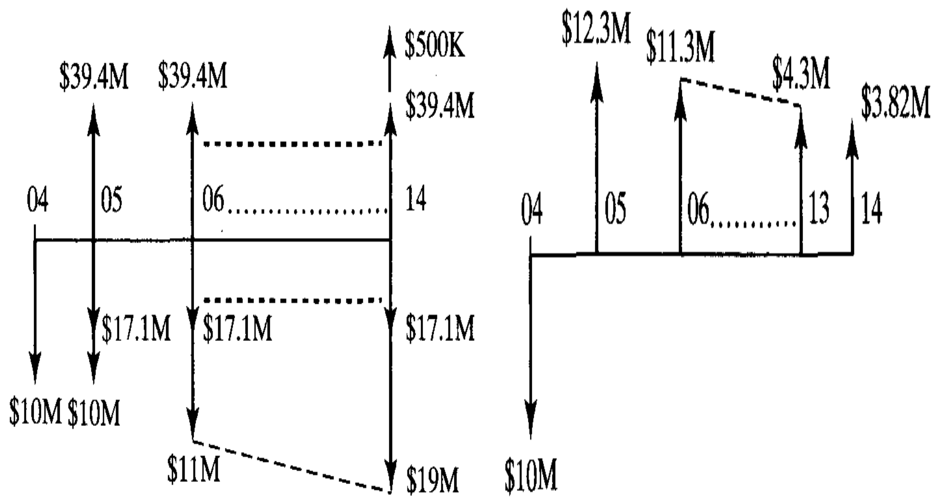
CASH FLOW DIAGRAM – EXAMPLE [3]



CASH FLOW DIAGRAM – EXAMPLE [4]

- You *deposited* a $1,000 in your account in a bank that gives a ***daily*** interest of 0.003% where interest is paid monthly. Assume simple interest

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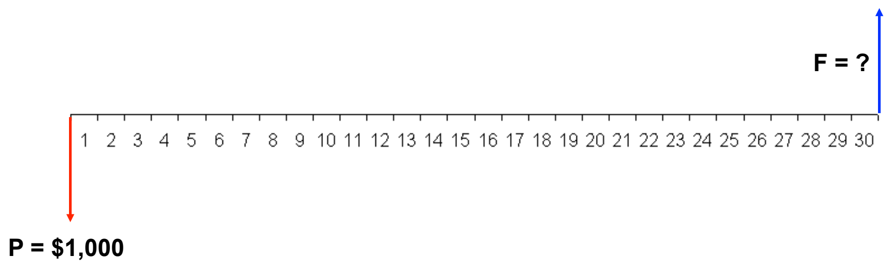
- [1] For this scenario, what is your balance after 30 days?
- [2] If you deposit another $2,000 on the 11th day and withdraw $500 on the 26th day, what is your balance at the end of the 30th day?
- In both cases, draw the cash flow diagram

CASH FLOW DIAGRAM – EXAMPLE [4]

- [1] Since we have simple interest, then $F = P(1+ni)$
  → $F = \$1,000\times(1+30\times0.003\%) = \$1,000.9$



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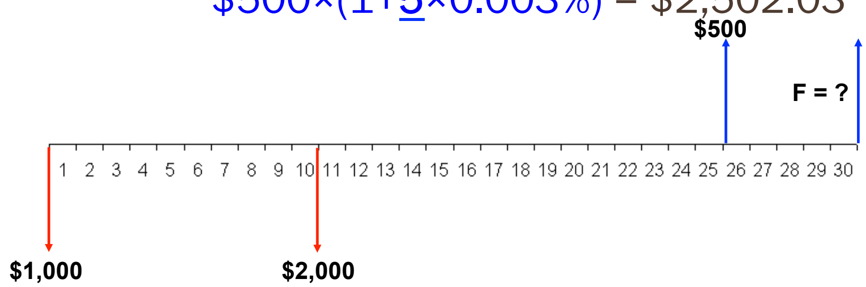
CASH FLOW DIAGRAM – EXAMPLE [4]

- [2] $F = \$1,000\times(1+\underline{30}\times0.003\%) + \$2,000\times(1+\underline{20}\times0.003\%) - \$500\times(1+\underline{5}\times0.003\%) = \$2,502.03$



Just keep in mind that the day is represented by its beginning

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CASH FLOW DIAGRAM – EXAMPLE [5]

- You have deposited $1,000 with an interest rate of 3% every 6 months where the interest is computed every 6 months
- How much you will have after 5 years?
- Two years later after the initial deposit of the money, you deposited additional $1,000 with an interest rate of 2% every 6 months (*applies only to this deposit*). How much will you have after 5 years?

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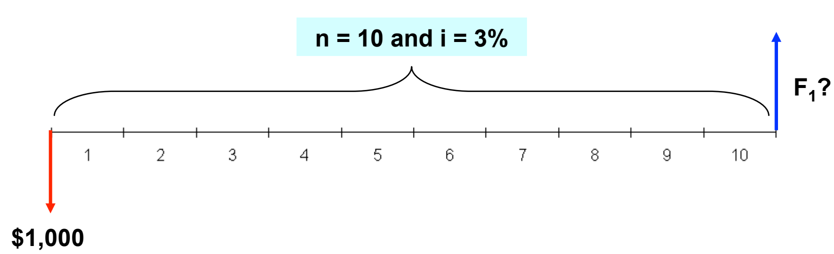
CASH FLOW DIAGRAM – EXAMPLE [5]

- We have a total of 10 periods each period of 6 months for the $1,000
- $F1 = P(1+i)^n = \$1{,}000 \times (1+3\%)^{10} = \$1{,}343.92$



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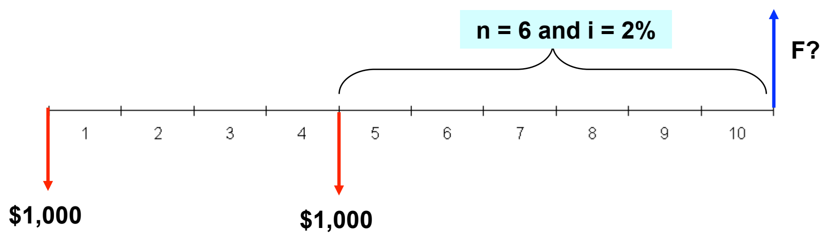
CASH FLOW DIAGRAM – EXAMPLE [5]

- We have a total of 10 periods each period of 6 months for the first $1,000 [F1]
- We have a total of 6 periods each period of 6 months for the second $1,000 [F2]
- F = F1+F2 = $\$1,000\times(1+3\%)^{10}$ + $\$1,000\times(1+2\%)^{6}$ = $2,470.08



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EXAMPLE [6]

- What would be the future worth after two years of a deposit of $1,000 now if the interest rate for the first year is 10% and for the second year is 5%?
- By the end of the *first* year, the total amount becomes:
  $1,000(1+10\%)^1 = \$1,100$
- By the end of the *second* year, the total amount becomes:
  $1,100(1+5\%)^1 = \$1,155$

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Engineering Economy

[2-1]

Time Value of Money

Single Cash Flow

The Five Types of Cash Flows

Cash flow transactions can be generally classified into five general categories:

(1) Single cash flow

(2) Uniform series

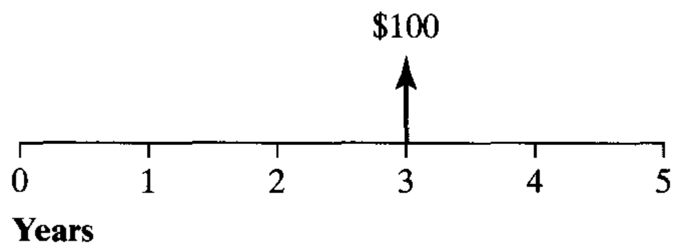
(3) Linear gradient series

(4) Geometric gradient series, and

(5) Irregular series

Single Cash Flow

- The simplest case involves the equivalence of a single present amount and its future worth
- Thus, the single-cash-flow formulas deal with only two amounts: a single present amount P and its equivalent future worth F



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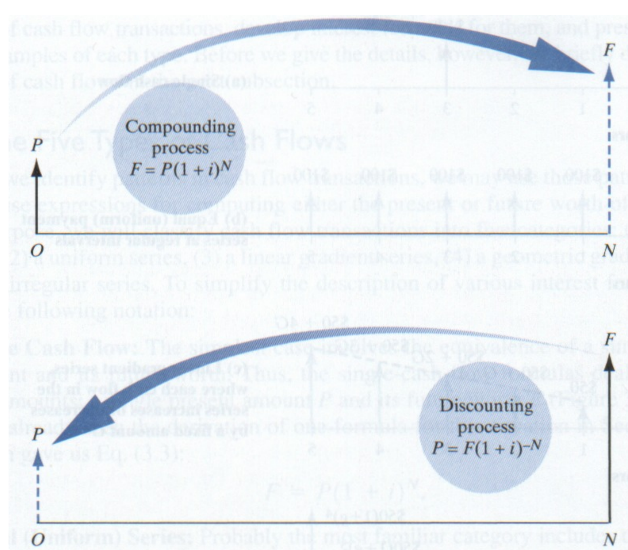
Single Cash Flow



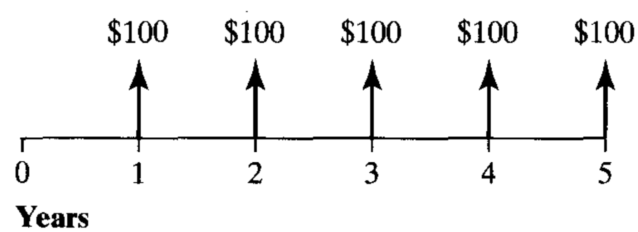
You have P find F

You have F find P

4

Equal (Uniform) Series

- Includes transactions arranged as a series of equal cash flows at regular intervals, known as an equal payment series (or uniform series)
- The equal-cash-flow formulas deal with the equivalence relations P, F, and A (where A is the constant amount of the cash flows in the series)



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Linear Gradient Series

- A common pattern of variation occurs when each cash flow in a series increases (or decreases) by a fixed amount
- A five-year loan repayment plan might specify, for example, a series of annual payments that increase by $500 each year
- We call this type a *linear gradient series* because its cash flow diagram produces an ascending (or descending) straight line
- In addition to using P, F, and A, the formulas employed in such problems involve a constant amount G of the change in each cash flow

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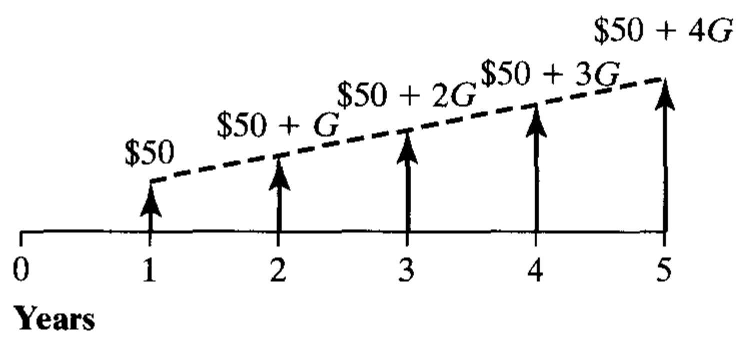
Linear Gradient Series



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Geometric Gradient Series

- This type is formed when the series in a cash flow is determined not by some fixed amount like $500, but by some fixed rate, expressed as a percentage
- The curving gradient in the diagram is named the *geometric gradient series*
- In the formulas dealing with such series, the rate of change is represented by a lowercase g

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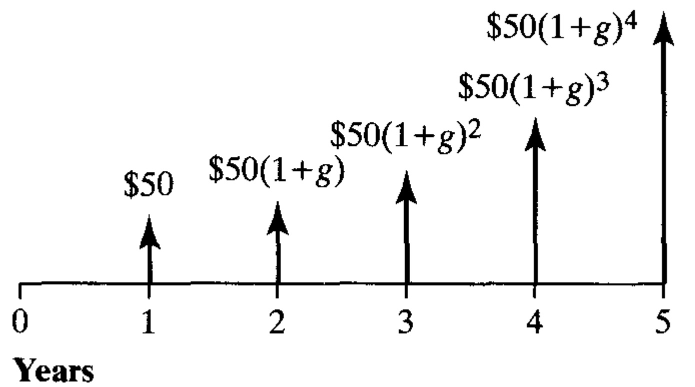
Geometric Gradient Series



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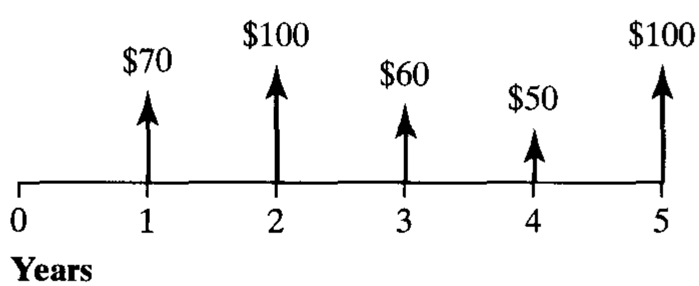
Irregular (Mixed) Series

A series of cash flows may be irregular, in that it does not exhibit a regular overall pattern



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Single-Payment Factors

- We know that the amount of money F accumulated after n years from a present worth P with interest compounded one time per year is given by the following equation → $F = P(1+i)^n$

- The factor $(1+i)^n$ is called the single-payment compound amount factor (SPCAF) and is usually referred to as the F/P factor

- The factor P/F is known as the single-payment present worth factor (SPPWF)

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Single-Payment Factors

- Note that single payment means that only one payment or receipt is involved

- A standard notation has been adopted for all the economic factors and is always in the general form (X/Y,i,n)

- The letter X represents what is sought, while the letter Y represents what is given

- For example, F/P means find F when P is given

- Thus, (F/P,6%,20) represents the ***factor*** that is used to calculate the future amount F accumulated in 20 periods if the interest rate is 6% per period. P is given

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Single-Payment Factors

Factor Notation	Factor Name	Find/Given	Standard Notation Equation	Equation with Factor Formula	Excel Functions
$(F/P,i,n)$	Single-payment compound amount	F/P	$F = P(F/P,i,n)$	$F = P(1+i)^n$	FV(i%,n,,P)
$(P/F,i,n)$	Single-payment present worth	P/F	$P = F(P/F,i,n)$	$P = F[1/(1+i)^n]$	PV(i%,n,,F)

The value of (P/F,5%,10) → $P = F[1/(1+i)^n] = 0.6139$

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6% TABLE 11 Discrete Cash Flow: Compound Interest Factors 6%

	Single Payments		Uniform Series Payments				Arithmetic Gradients	
n	Compound Amount F/P	Present Worth P/F	Sinking Fund A/F	Compound Amount F/A	Capital Recovery A/P	Present Worth P/A	Gradient Present Worth P/G	Gradient Uniform Series A/G
1	1.0600	0.9434	1.00000	1.0000	1.06000	0.9434		
2	1.1236	0.8900	0.48544	2.0600	0.54544	1.8334	0.8900	0.4854
3	1.1910	0.8396	0.31411	3.1836	0.37411	2.6730	2.5692	0.9612
4	1.2625	0.7921	0.22859	4.3746	0.28859	3.4651	4.9455	1.4272
5	1.3382	0.7473	0.17740	5.6371	0.23740	4.2124	7.9345	1.8836
6	1.4185	0.7050	0.14336	6.9753	0.20336	4.9173	11.4594	2.3304
7	1.5036	0.6651	0.11914	8.3938	0.17914	5.5824	15.4497	2.7676
8	1.5938	0.6274	0.10104	9.8975	0.16104	6.2098	19.8416	3.1952
9	1.6895	0.5919	0.08702	11.4913	0.14702	6.8017	24.5768	3.6133
10	1.7908	0.5584	0.07587	13.1808	0.13587	7.3601	29.6023	4.0220
11	1.8983	0.5268	0.06679	14.9716	0.12679	7.8869	34.8702	4.4213
12	2.0122	0.4970	0.05928	16.8699	0.11928	8.3838	40.3369	4.8113
13	2.1329	0.4688	0.05296	18.8821	0.11296	8.8527	45.9629	5.1920
14	2.2609	0.4423	0.04758	21.0151	0.10758	9.2950	51.7128	5.5635
15	2.3966	0.4173	0.04296	23.2760	0.10296	9.7122	57.5546	5.9260

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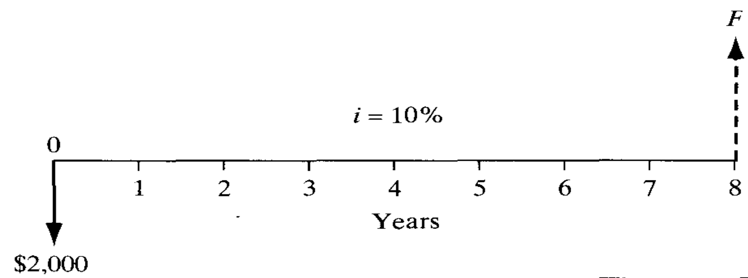
Single-Payment Factors
Example

If you had \$2,000 now and invested it at 10%, how much would it be worth in eight years?

$F = P(1+i)^n = \$2{,}000 \times (1+0.1)^8 = \$4{,}287.18$

<u>Or</u>

$F = P(F/P,i,n) = 2{,}000(F/P,10\%,8)$

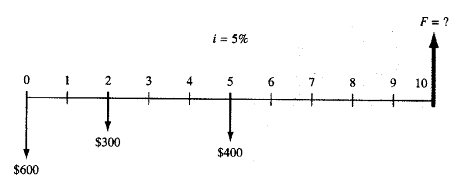


Single-Payment Factors
Example

- The office supplies for an engineering firm for different years were as follows:
  Year 0: \$600; Year 2: \$300; and Year 5: \$400
- What is the equivalent value in year 10 if the interest rate is 5% per year?

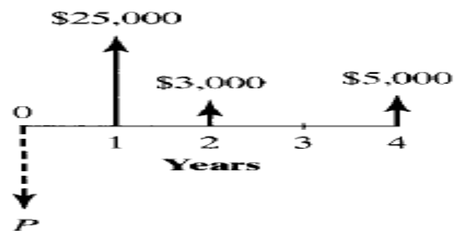
- Draw the cash flow diagram for the values \$600, \$300, and \$400
- Use *F/P* factors to find F in year 10
- $F = 600(F/P,5\%,10) + 300(F/P,5\%,8) + 400(F/P,5\%,5) =$

  $600\times(1.6289) + 300\times(1.4775) + 400\times(1.2763) = \$1931.11$



Example
Decomposition and Superposition

- A company wishes to set aside money now to invest over the next four years. The company can earn 10% on a lump sum deposited now, and it wishes to *withdraw* the money in the following increments:
  - Year 1: $25,000
  - Year 2: $3,000
  - Year 3: No expenses
  - Year 4: $5,000



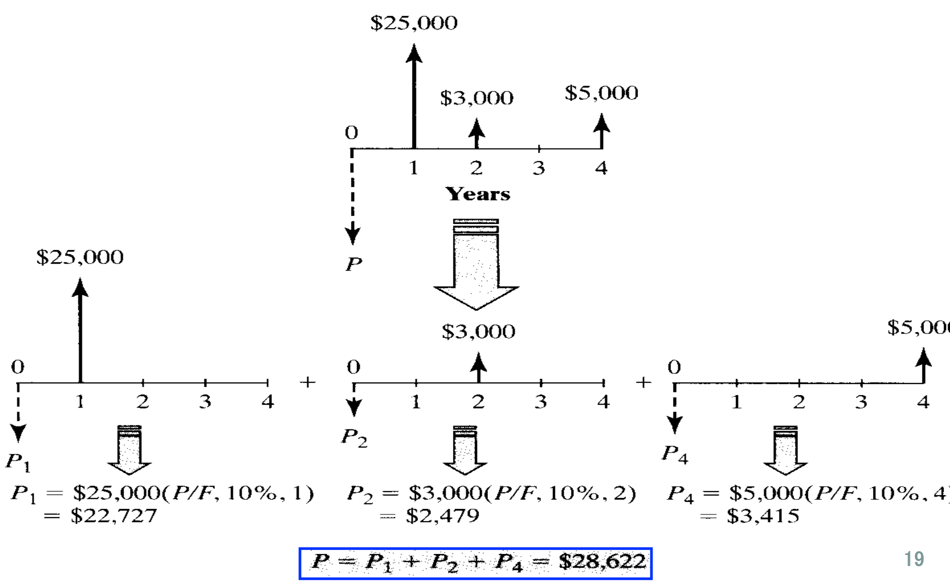
- How much money must be deposited now to cover the anticipated payments over the next 4 years?

Example
Decomposition and Superposition

- Apparently, one way to deal with an uneven series of cash flows is to calculate the equivalent present value of each single cash flow and to sum the present values to find P

- That is, the cash flow is broken into three components (decomposition) and later all the three present values are summed up (superposition)

Example
Decomposition and Superposition



Example
Decomposition and Superposition

- To see if the needed \$28,622 is sufficient, let's calculate the balance at the end of each year
- If you deposit \$28,622 now, it will grow to (1.10)(\$28,622), or \$31,484, at the end of year 1. From this balance, you pay out \$25,000
- The remaining balance, \$6,484, will again grow to (1.10)(\$6,484), or \$7,132, at the end of year 2. Now you make the second payment (\$3,000) out of this balance, which will leave you with only \$4,132 at the end of year 2
- Since no payment occurs in year 3, the balance will grow to $\$(1.10)^2(\$4,132)$, or \$5,000, at the end of year 4
- The final withdrawal in the amount of \$5,000 will deplete the balance completely

Engineering Economy_

[2-2]

Time Value of Money

Uniform Series

Present-Worth Factor
Equal Payment Uniform Series

- What would you have to *invest* now P in order to withdraw A dollars at the end of each of the next n periods?

- In this case, it is the P/A factor used to calculate the equivalent P value in year 0 for a uniform series of A values beginning at the end of period 1 and extending for n periods

$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right]$$

- The term in brackets is the conversion factor known as the uniform-series present worth factor (USPWF)

2

Capital Recovery Factor
Equal Payment Uniform Series

- To reverse the situation, the present worth P is known and the equivalent uniform-series amount A is sought
- The first A value occurs at the end of period 1, that is, one period after P occurs

$$A = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right]$$

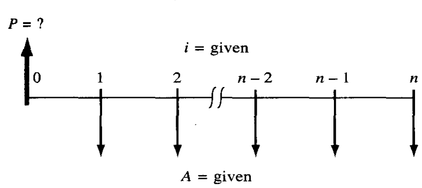
- The term in brackets is called the capital recovery factor (CRF), or A/P factor. It is like investing P now and getting the equivalent through annual uniform n number of equal payments (A)

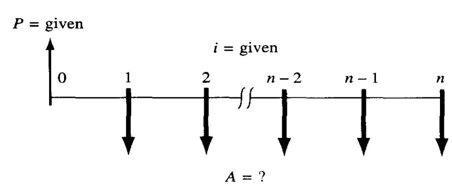
3

Present-Worth and Capital Recovery Factor
Equal Payment Uniform Series

Factor Notation	Factor Name	Find/Given	Factor Formula	Standard Notation Equation	Excel Function
$(P/A,i,n)$	Uniform-series present worth	P/A	$\frac{(1+i)^n - 1}{i(1+i)^n}$	$P = A(P/A,i,n)$	PV(i%,n,A)
$(A/P,i,n)$	Capital recovery	A/P	$\frac{i(1+i)^n}{(1+i)^n - 1}$	$A = P(A/P,i,n)$	PMT(i%,n,P)

If i = 15% and n = 25 years, the P/A factor value is (P/A, 15%,25) = 6.4641





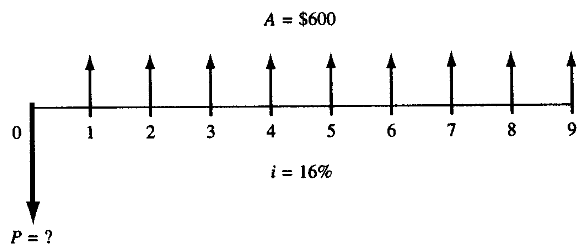
Very important to note that there is no A payment at t = 0 but only P payment

4

Present-Worth Factor
Equal Payment Uniform Series

- How much money should you be willing to pay now for a guaranteed \$600 per year for 9 years starting next year, at a rate of return of 16% per year?

- The present worth is:
  P = 600(P/A,16%,9) = 600×(4.6065) = \$2,763.90



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Present-Worth Factor
Equal Payment Uniform Series

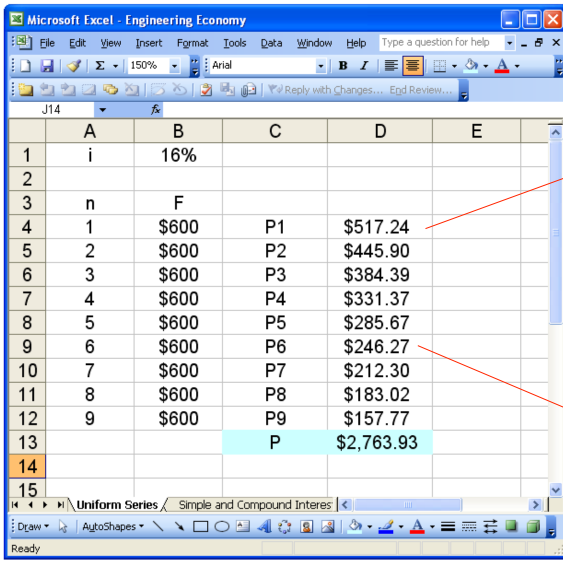
- We can solve the previous example in the following way using the superposition theory

- Simply assume each \$600 dollar due by the end of each year is the future value of a present value (at time = 0)

- Thereafter, sum up all these present values to arrive at the total present value that yield the equal payments of \$600 at the end of each year

6

Present-Worth Factor
Equal Payment Uniform Series



	A	B	C	D	E
1	i	16%			
2					
3	n	F			
4	1	$600	P1	$517.24	
5	2	$600	P2	$445.90	
6	3	$600	P3	$384.39	
7	4	$600	P4	$331.37	
8	5	$600	P5	$285.67	
9	6	$600	P6	$246.27	
10	7	$600	P7	$212.30	
11	8	$600	P8	$183.02	
12	9	$600	P9	$157.77	
13			P	$2,763.93	
14					
15					

PV(16%,1,,600)

PV(16%,6,,600)

7

Sinking Fund Factor and
Uniform Series Compound Amount

- Suppose we are interested in the future amount F of a fund to which we contribute A dollars each period and on which we earn interest at a rate of i per period

$$A = F\left[\frac{i}{(1+i)^n - 1}\right] \rightarrow \textbf{(A/F,i,n)}$$

- The expression in brackets is the A/F or sinking fund factor

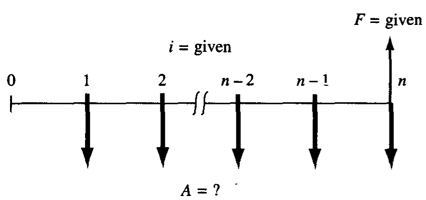
- The above equation can be rearranged to find F for a stated A series in periods 1 through n

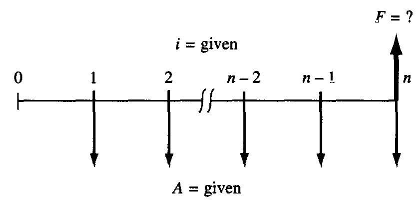
$$F = A\left[\frac{(1+i)^n - 1}{i}\right] \rightarrow$$

Uniform-series compound amount factor (F/A,i,n)

8

Sinking Fund Factor and
Uniform Series Compound Amount

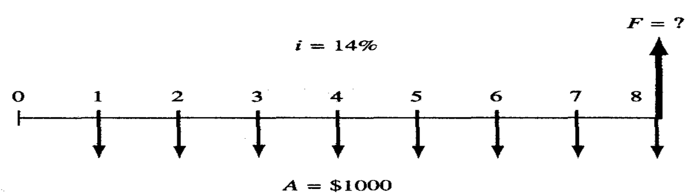




Note that A payments start at the beginning of the second year

Sinking Fund Factor and
Uniform Series Compound Amount

- What is the equivalent future worth of one thousand dollar of investment each year for 8 years starting 1 year from now with an interest rate of 14%?
- You need to find out the value of F
- The cash flow diagram shows the annual payments
- F = 1,000×(F/A,14%,8) = $13,232.8



Sinking Fund Factor and
Uniform Series Compound Amount

	A	B	C	D	E
15					
16					
17	i	14%			
18	n	A			
19	0	$1,000	F0	$1,000.00	
20	1	$1,000	F1	$1,140.00	
21	2	$1,000	F2	$1,299.60	
22	3	$1,000	F3	$1,481.54	
23	4	$1,000	F4	$1,688.96	
24	5	$1,000	F5	$1,925.41	
25	6	$1,000	F6	$2,194.97	
26	7	$1,000	F7	$2,502.27	
27			F	$13,232.76	
28					
29					

FV(14%,0,,1000)

FV(14%,5,,1000)

Engineering Economy

[2-3]

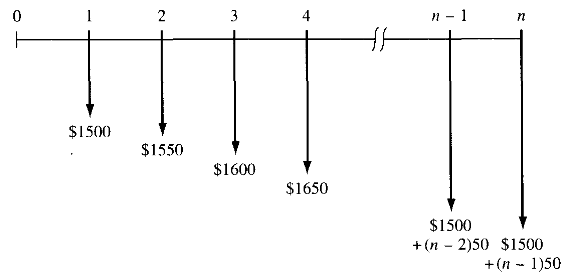
Time Value of Money

Arithmetic Gradient Series

Arithmetic (Linear) Gradient Series

- An *arithmetic* gradient is a cash flow series that either increases or decreases by a constant amount
- The cash flow, whether income or disbursement, changes by the same arithmetic amount each period
- The amount of the increase or decrease is the gradient (G)
- For example, if an engineer predicts that the cost of maintaining a machine will increase by $500 per year until the machine is retired, a gradient series is involved and the amount of the gradient is $500

Arithmetic (Linear) Gradient Series

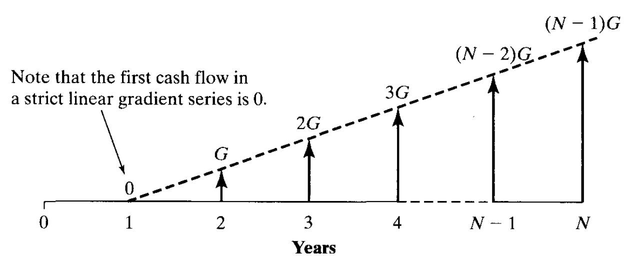


- The diagram is of an arithmetic gradient series with a *base amount* of $1,500 and a gradient of $50
- The *origin* of the series is at the end of the first period
- G is the constant arithmetic change in the magnitude of *receipts* or *disbursements* from one time period to the next

3

Strict Linear Gradient Series

- The *strict* linear gradient series has the ***origin** at the end of the first period* with a zero value
- The *gradient* G can be either positive or negative. If G > 0, the series is referred to as an increasing gradient series. If G < 0, it is a decreasing gradient series



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Arithmetic (Linear) Gradient Series
Example

- A company expects a revenue of \$80,000 in fees next year. Fees are expected to increase uniformly to a level of \$200,000 in nine years
- Determine the arithmetic gradient and construct the cash flow diagram

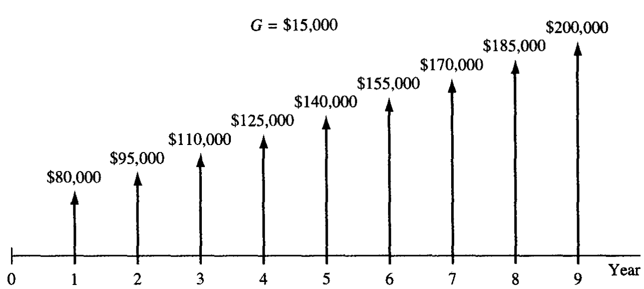
5

Arithmetic (Linear) Gradient Series
Example

- The cash flow in year n (CFn) may be calculated as:

  CFn = base amount + (n-1)G

- The base amount (generally A1) is \$80,000 and the total revenue increase in 9 years = 200,000 – 80,000 = 120,000
- G = increase/(n-1) = 120,000/(9-1) = \$15,000



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Arithmetic (Linear) Gradient Series
Analysis

Three factors will be considered for arithmetic gradient ***strict*** series:

- P/G factor for present worth: G(P/G,i,n)
  Convert an arithmetic gradient G (*without the base amount*) for n years into a present worth at year 0

- A/G factor for annual series: G(A/G,i,n)
  Convert an arithmetic gradient G (*without the base amount*) for n years into an equivalent uniform series of A value

- F/G factor for future worth: G(F/G,i,n)
  Convert an arithmetic gradient G (*without the base amount*) for n years into an equivalent future value at year n

7

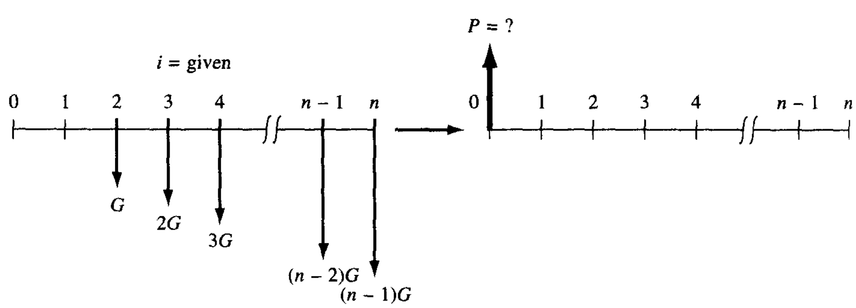
Arithmetic (Linear) Gradient Series
Present Worth Factor – P/G Factor

The Present worth factor (P/G) can be expressed in the following form:

P = G(P/G,i,n)→

gradient series present-worth factor

$$P = G\left[\frac{(1+i)^n - in - 1}{i^2(1+i)^n}\right]$$

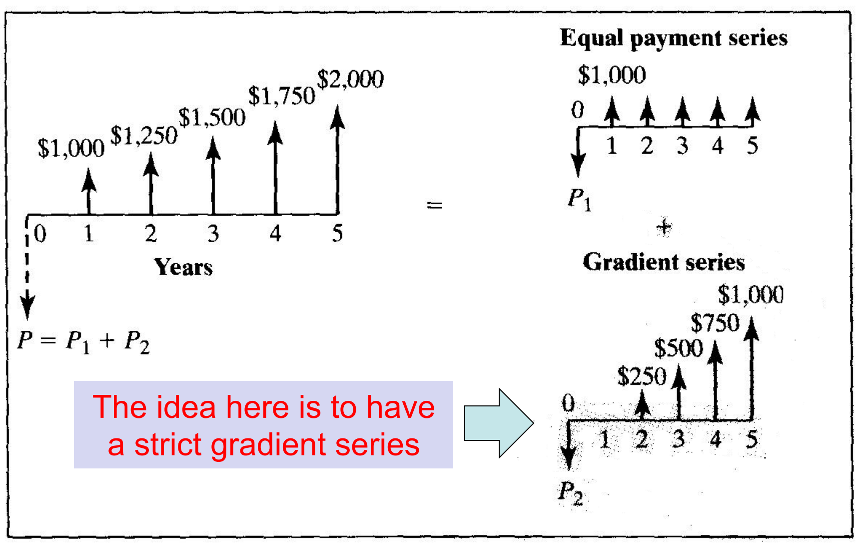


8

Arithmetic (Linear) Gradient Series
Present Worth Factor – Example

- A textile mill has just purchased a lift truck that has a useful life of five years. The engineer estimates that maintenance costs for the truck during the first year will be $1,000
- As the truck ages, maintenance costs are expected to increase at a rate of $250 per year over the remaining life
- Assume that the maintenance costs occur at the end of each year. The firm wants to set up a maintenance account that earns 12% annual interest. All future maintenance expenses will be paid out of this account. How much does the firm have to deposit in the account now?

Arithmetic (Linear) Gradient Series
Present Worth Factor – Example



Arithmetic (Linear) Gradient Series
Present Worth Factor – Example

- We have: A1=$1,000; G=$250; i=12%; and n=5 years. Find P
- The cash flow can be broken into two components where the first is an equal uniform payment series (A1) and the second is a strict linear gradient series (G)
- P = P1 + P2

  P = A1(P/A,12%,5) + G(P/G,12%,5) =

  $1,000(3.6048) + $250(6.397) = $5,204

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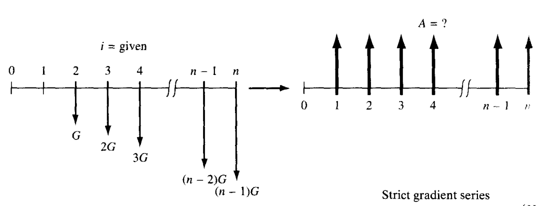
Arithmetic (Linear) Gradient Series
Annul Series Factor – A/G Factor

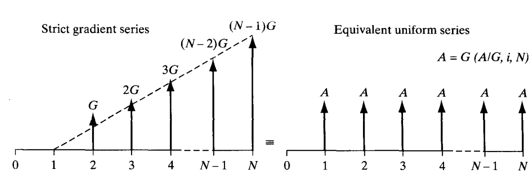
The equivalent uniform annual series (A value) for an arithmetic gradient G is found by the following formula:

A = G(A/G,i,n) →

$$A = G\left[\frac{1}{i} - \frac{n}{(1+i)^n - 1}\right]$$

Arithmetic-gradient uniform-series factor





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Arithmetic (Linear) Gradient Series
Annul Series Factor – Example

- You want to deposit \$1,000 in your saving account at the end of the first year and increase this amount by \$300 for each of the next five years
- Then what should be the size of an annual uniform deposit that yields an equal balance with the above by the end of six years if the interest rate is 10%?

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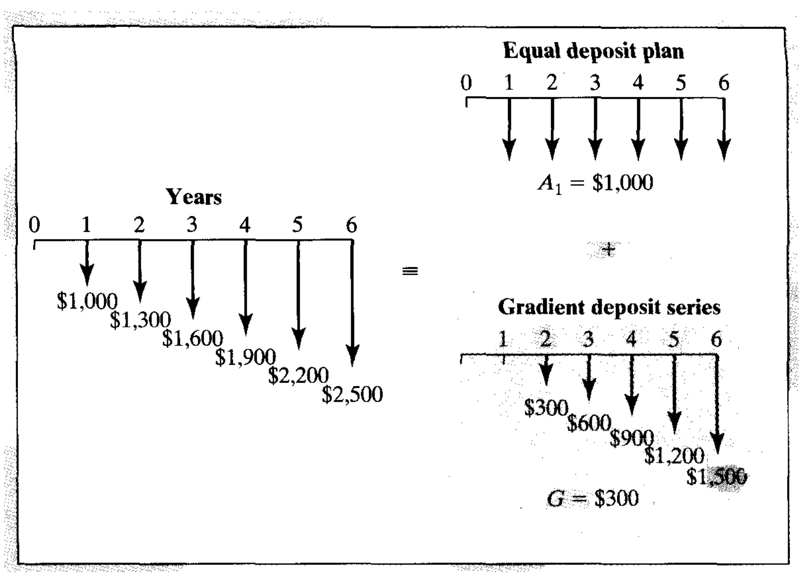
Arithmetic (Linear) Gradient Series
Annul Series Factor – Example



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Arithmetic (Linear) Gradient Series

Annul Series Factor – Example

- We have: A1=$1,000; G=$300; i=10%, and n=6. Find A
- We have to separate the constant portion of $1,000 from the series leaving the gradient series of 0; 0; 300; 600; ….; 1,500
- To find the equal payment series beginning at the end of year 1 and ending at year 6 we consider:

  A = $1,000 + $300(A/G,10%,6) =

  $1,000 + $300(2.2236) = $1,667.08

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Arithmetic (Linear) Gradient Series

Annul Series Factor – Example

Year	F	P
0	0	$0.00
1	$1,000.00	$909.09
2	$1,300.00	$1,074.38
3	$1,600.00	$1,202.10
4	$1,900.00	$1,297.73
5	$2,200.00	$1,366.03
6	$2,500.00	$1,411.18
Total	-	$7,260.51

A/P	0.2296074
A	$1,667.07

An alternative way to solve this question is by finding the present worth of all the payments and then to convert P to a uniform series of A

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Arithmetic (Linear) Gradient Series
Future Worth Factor – F/G Factor

The future worth factor (F/G) can be expressed in the following form:

F = G(F/G,i,n)→

Arithmetic-gradient future worth factor

$$F = G\left[\left(\frac{1}{i}\right)\left(\frac{(1+i)^n - 1}{i} - n\right)\right]$$

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Arithmetic (Linear) Gradient Series
Future Worth Factor – Example

- Suppose that you make a series of annual deposits into a bank account that pays 10% interest. The initial deposit at the end of the first year is $1,200

- The deposit amounts decline by $200 in each of the next four years

- How much would you have immediately after the fifth deposit?

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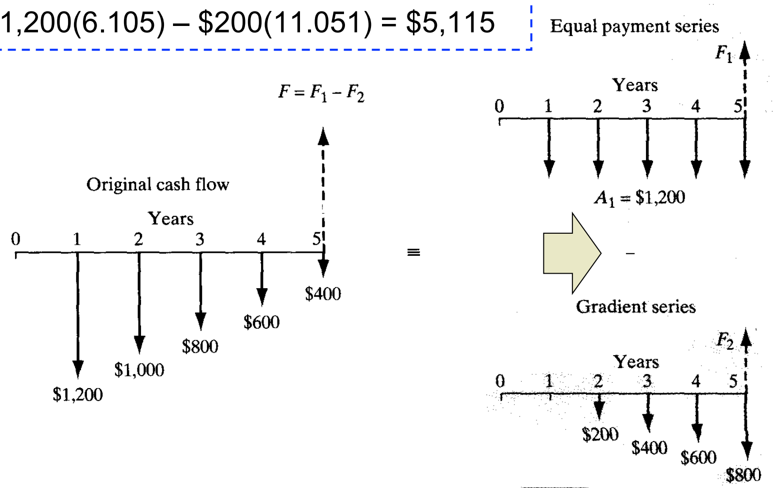
Arithmetic (Linear) Gradient Series

Future Worth Factor – Example

F = F1 – F2

F = A1(F/A,10%,5) – $200(F/G,10%,5) =

$1,200(6.105) – $200(11.051) = $5,115



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Engineering Economy_

[2-4]

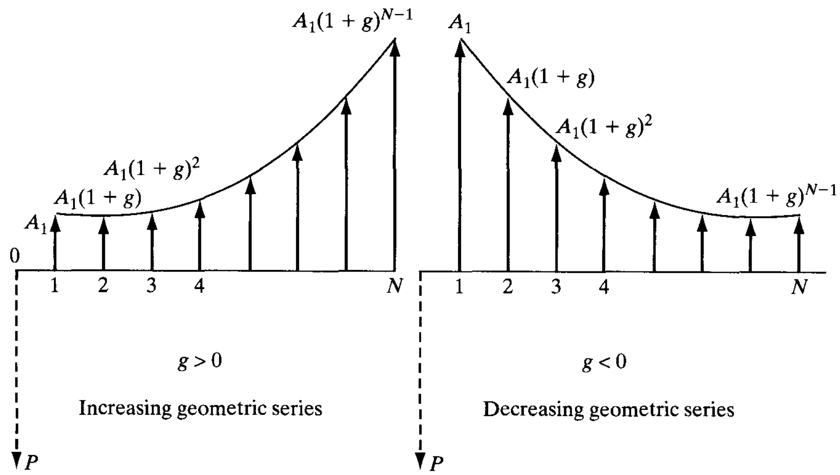
Time Value of Money

Geometric Gradient Series

Geometric Gradient Series

- In geometric gradient series, cash flow increases or decreases from period to period by a constant percentage
- This uniform rate of change defines a *geometric gradient* series of cash flows
- We will use the term g which is the constant rate of change by which amounts increase or decrease from one period to the next

Geometric Gradient Series



Geometric Gradient Series

- We need to find the value of the present worth at time = 0 based on geometric gradient series cash flows starting by the end of period 1 by an amount A1 and ***increasing*** by a constant rate of g each period

- P = A1(P/A,g,i,n)

$$(P/A,g,i,n) = \begin{cases} \dfrac{1-\left(\dfrac{1+g}{1+i}\right)^n}{i-g} & \text{when } g \neq i \\ \dfrac{n}{1+i} \quad \text{when } g = i \end{cases}$$

Geometric Gradient Series

Example

- Engineers at a specific company need to make some modifications to an existing machine
- The *modification* costs only $8,000 and is expected to last 6 years with a $1,300 *salvage* value
- The *maintenance* cost is expected to be high at $1,700 the first year, increasing by 11% per year thereafter
- Determine the equivalent present worth of the modification and maintenance cost. The interest rate is 8% per year

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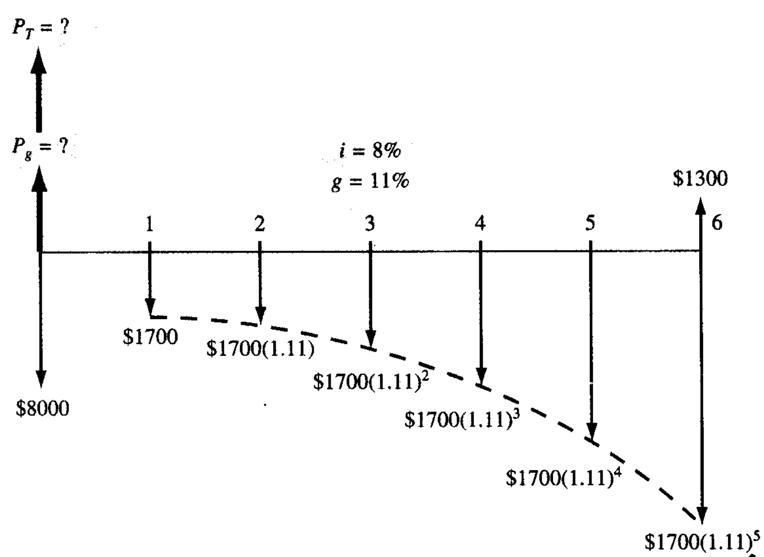
Geometric Gradient Series

Example



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Geometric Gradient Series

Example

- The present worth value is comprised of three components:
  - ✓The present modification cost = \$8,000
  - ✓The present value of the future salvage value
  - ✓The present value of all the maintenance values throughout the 6 years and these are represented by the geometric gradient series
- $P_T = -8{,}000 + 1{,}300(P/F,8\%,6) - P_g$
- $P_g = A1(P/A,g,i,n) \rightarrow (P/A,11\%,8\%,6) = \dfrac{1-\left(\dfrac{1+0.11}{1+0.08}\right)^6}{0.08-0.11}$
- $P_T = -8{,}000 + 819.26 - 1{,}700 \times 5.9559 =$ \$ –17,305.85

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Engineering Economy_

[3]

Combining Factors

Examples

General

- Most estimated cash flow series do not fit exactly the series for which the *factors* and equations were developed earlier
- Therefore, it is necessary to combine the equations
- However, there are several ways to address a particular sequence of cash flows in order to determine the present, future, or annual worth
- Different ways to address this will be explained herein through a set of solved examples

Combining Factors
The Different Cases

1. Shifted series: Determine P, F, or A of a uniform series starting at a time other than period 1

2. Calculate P, F, or A of randomly placed single amounts and uniform series

3. Make equivalence calculations for cash flows involving shifted arithmetic or geometric gradients

4. Make equivalence calculations for cash flows involving decreasing arithmetic gradients (refer to section [2-3])

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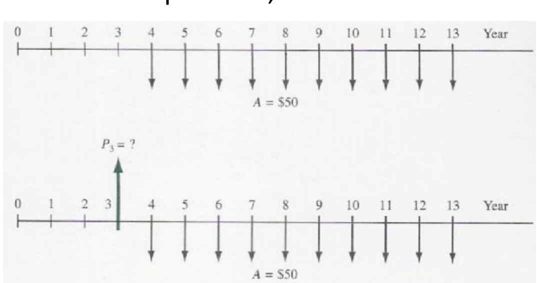
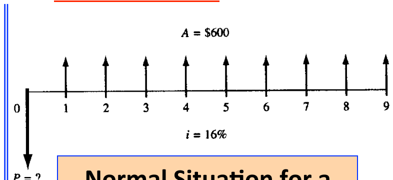
Shifted Uniform Series
Example – 1

- When dealing with uniform series, the normal situation is to have the series begins at the *end of period 1*

- Or, the present worth is always located one period prior to the first uniform series amount when using the P/A factor

- When we have a situation that the payment does not start at the end of period 1, then the series is called "shifted series"





Normal Situation for a nine-period uniform annual payments

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Shifted Uniform Series
Example – 1

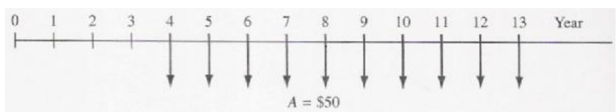
- When we have shifted uniform series, then **P** can be determined by any of the following methods:

  [1] Use the P/F factor to find the present worth of each disbursement at year 0 and add them up

  [2] Use the F/P factor to find the future worth of each disbursement in year 13, add them, and then find the present worth of the total P=F(P/F,i,13)

  [3] Use the F/A factor to find the future amount F=A(F/A,i,10) and then compute the present worth using P=F(P/F,i,13)

  [4] Use the P/A factor to compute the present worth (located in year 3 not 0) and then find the present worth in year 0 by using the (P/F,i,3) factor



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Shifted Uniform Series
Example – 1

- Assume that the interest rate is 8%

Year	F	P
0	0	$0.00
1	0	$0.00
2	0	$0.00
3	0	$0.00
4	50	$36.75
5	50	$34.03
6	50	$31.51
7	50	$29.17
8	50	$27.01
9	50	$25.01
10	50	$23.16
11	50	$21.44
12	50	$19.86
13	50	$18.38
Total	-	$266.33

$P = F[1/(1+i)^n]$

$P = F(P/F,i,n)$

Find the present worth corresponding to the future values

$P = F(P/F,8\%,7)$

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Shifted Uniform Series
Example – 1

Year	P	F
0	0	$0.00
1	0	$0.00
2	0	$0.00
3	0	$0.00
4	50	$99.95
5	50	$92.55
6	50	$85.69
7	50	$79.34
8	50	$73.47
9	50	$68.02
10	50	$62.99
11	50	$58.32
12	50	$54.00
13	50	$50.00
Total	-	$724.33
	P	$266.33

$F = P(1+i)^n$

Find the future worth corresponding to the present values, sum up these future values and convert back to find the present value

F = P(F/P,8%,4)

Shifted Uniform Series
Example – 1

$$\left[\frac{(1+i)^n - 1}{i}\right]$$

F/A	(F/A,i,10)	14.49
	F13	$724.33
	P0	$266.33

$$F = A\left[\frac{(1+i)^n - 1}{i}\right]$$

P = F(P/F,8%,13)

Shifted Uniform Series
Example – 1

$$\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right]$$

P/A	(P/A,i,10)	6.71
	P3	$335.50
	P0	$266.33

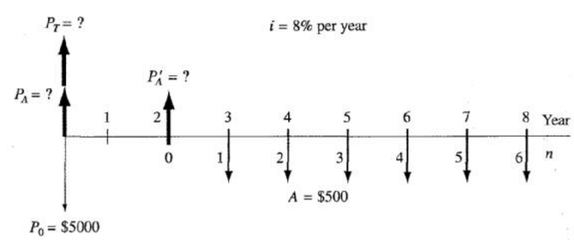
$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right]$$

P = F(P/F,8%,3)

Shifted Uniform Series
Example – 2

- An engineering technology group just *purchased* a software for $5,000 now and annual payments of $500 per year for 6 years starting 3 years from now for *annual upgrades*

- What is the *present worth* of the payments if the interest rate is 8% per year?



Shifted Uniform Series

Example – 2

- Find the value of P'_A for the shifted series
  $P'_A = \$500(P/A,8\%,6)$
- Since P'_A is located in year 2, now find P_A in year 0
  $P_A = P'_A(P/F,8\%,2)$
- The total present worth is determined by adding P_A and the initial payment P_0 in year 0
  $P_T = P_0 + P_A = 5{,}000 + 500(P/A,8\%,6)(P/F,8\%,2)$
  $= 5{,}000 + 500(4.6229)(0.8573) = \$6{,}981.6$

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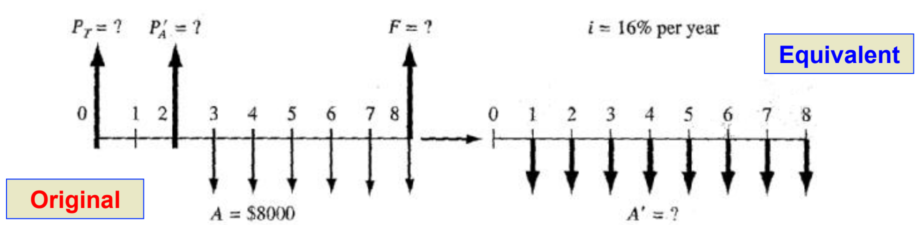
Shifted Uniform Series

Example – 3

- Recalibration of sensitive measuring devices costs $8,000 per year
- If the machine will be recalibrated for each of 6 years starting 3 years after purchase, calculate the 8-year equivalent uniform series at 16% per year



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Shifted Uniform Series
Example – 3

- To solve this question, first calculate P'_A from the uniform series A, and then find the P_T value. After that, compute A' based on the value of P_T

- P'_A = 8,000(P/A,16%,6)

- P_T= P'_A(P/F,16%,2) = 8,000(P/A,16%,6)(P/F,16%,2)
  = 8,000(3.6847)(0.7432) = $21,907.75

- The equivalent series A' for 8 years can now be determined via the A/P factor:
  A' = P_T(A/P,16%,8) = $5,043.60

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Shifted Uniform Series
Example – 3

- An alternative way to solve this question is by finding the future worth F in year 8 based on the uniform series A. Then use this future worth value to find the uniform series A'

- F = 8,000(F/A,16%,6) = $71,820

- The A/F factor is used to obtain A' over all 8 years
  A' = F(A/F,16%,8) = $5,043.2

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Uniform Series and Randomly Placed Amounts

- When cash flows include both a uniform series and randomly placed single amounts, the procedure to find the present worth value would be as follows:
  - Find the present worth for the uniform series using the P/A factor
  - Find the present worth for the single amounts using the P/F factor

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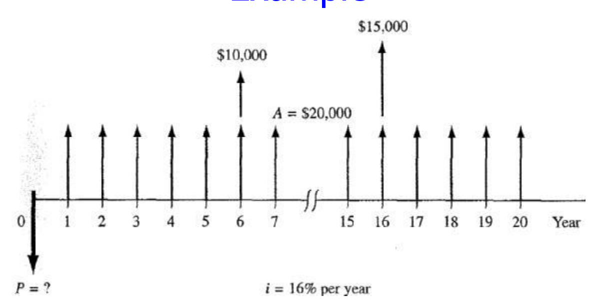
Uniform Series and Randomly Placed Amounts
Example



- To find the present worth, do the following:

  [1] Find P for the 20-year uniform series = 20,000(P/A,16%, 20)

  [2] Find P for the $10,000 amount = 10,000(P/F,16%,6)

  [3] Find P for the $15,000 amount = 15,000(P/F,16%,16)
- Sum up the three values and this equals $124,075

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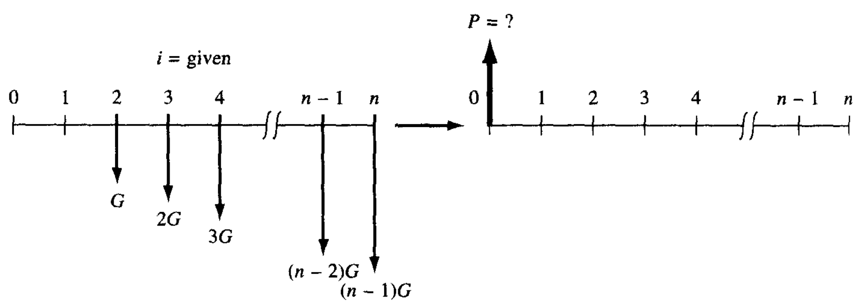
Shifted Gradients

- To find the present worth of an arithmetic gradient series, we use the relation P = G(P/G,i,n)
- Just keep in mind that the *present worth* of an arithmetic gradient will always be located two periods before the gradient starts



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Shifted Gradients

- A gradient that starts at any time that is not the end of second year is called a shifted gradient
- When having shifted gradients, then we can resort to renumbering the time scale
- The period in which the gradient first appears is labeled period **2** and the value n is obtained accordingly

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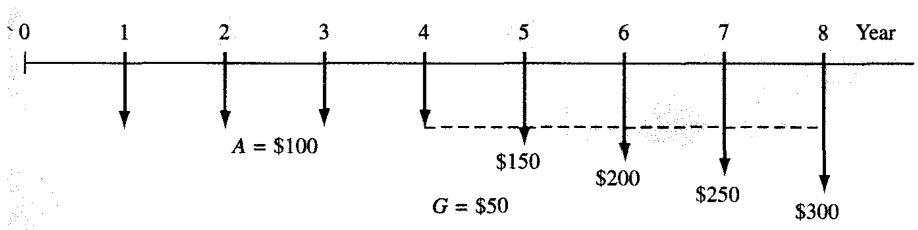
Shifted Gradients
Example – 1

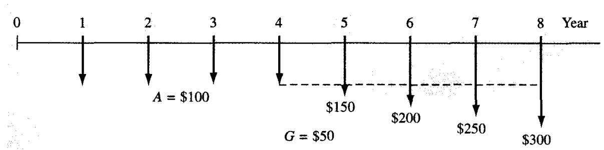
- An engineer has tracked the average inspection cost for 8 years. The cost average was steady at $100 for the first four years but have increased consistently by $50 for each of the last 4 years

- What is the total present worth (in year 0)?



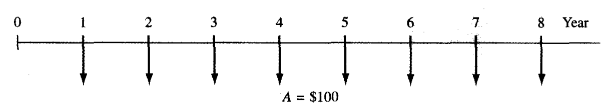
Shifted Gradients
Example – 1

To solve it, you need to decompose it as follows:



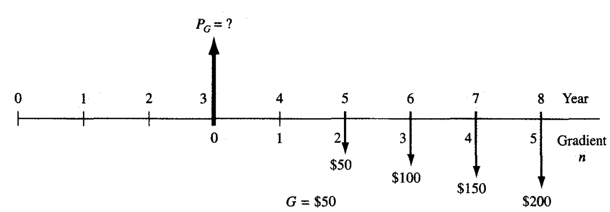
Original cash flow diagram

$P = P_A + P_G$



Uniform series cash flow diagram

P_A = 100(P/A,i,8)



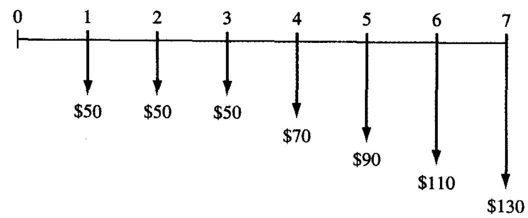
Arithmetic gradient cash flow diagram

P'_G = 50(P/g,I,5)

P_G = [50(P/G,i,5)](P/F,i,3)

Shifted Gradients
Example – 2

- Compute the *equivalent uniform* annual series in years 1 through 7



- Find the present worth for the arithmetic gradient series at year 2: $P_G = 20(P/G,i,5)$
- $P_0 = P_G(P/F,i,2)$
- Annualize the P_0: $A_G = P_0(A/P,i,7)$
- Add the base amount = $50 + A_G$

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