INFORMATION THEORY AND CODING

V C.S.

INTRODUCTION TO INFORMATION THEORY 1

PREVIOUS YEARS QUESTIONS

PART-A

Q.1 *Consider a source x that produces five symbols with $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}$ probabilities. Determine source entropy H(X).* *[R.T.U. 2017]*

Ans. $H(s) = \sum_{i=1}^{5} p_i \log_2 \frac{1}{p_i}$

$$= \frac{1}{2}\log_2(2) + \frac{1}{4}\log_2(4) + \frac{1}{8}\log_2(8) + \frac{1}{16}\log_2(16) + \frac{1}{16}\log_2(16)$$

$= 0.5 + 0.5 + 0.375 + 0.25 + 0.25 = 1.875$ bits/symbol

Information rate, R

$R = r_s H(s)$ bits/sec $= 1000 \times 1.875$ bits/sec

Q.2 ***A continuous signal is band limited to SkHz. The signal is quantized in 8 levels of a PCM system with the probabilities 0.25, 0.2, 0.2, 0.1, 0.1, 0.05, 0.05 and 0.05. Calculate the entropy and rate of information.***

[Note : Read SkHz = 5 kHz]. *[R.T.U. 2016]*

Ans. The signal should be sampled at a frequency $5 \times 2 = 10$ kHz (Sampling theorem). Each sample is then quantized to one of the eight levels. Looking at each quantized level as a message.

We get,

$$H = -(0.25 \log 0.25 + 0.2 \log 0.2 + 0.2 \log 0.2 + 0.1 \log 0.1 + 0.1 \log 0.1 + 0.05 \log 0.05 + 0.05 \log 0.05 + 0.05 \log 0.05)$$

$= 2.74$ bits/message

As the sampling frequency is 10 kHz, the message rate = 10,000 messages/sec. Hence, the rate of information is

$R = rH = 10{,}000 \times 2.74 = 27{,}400$ bits/sec.

Q.3 *Define Joint Entropy.* *[R.T.U. 2016]*

Ans. Joint Entropy : The joint entropy of two discrete random variables X and Y is merely the entropy of their pairing: (X,Y). This implies that if X and Y are independent, then their joint entropy is the sum of their individual entropies. For examples, if (X,Y) represents the position of a chess piece – X the row and Y the column, then the joint entropy of the row of the piece and the column of the piece will be the entropy of the position of the piece.

$$H(X,Y) = E_{X,Y}[-\log p(x,y)]$$

$$= -\sum_{x,y} p(x,y) \log p(x,y)$$

Q.4 *Define Information Rate.* *[R.T.U. 2016]*

Ans. Information Rate : The information rate is represented by R and it is given as,

Information Rate R = rH

Here R is the information rate.

H is the entropy or average information.

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And r is the rate at which messages are generated.

Information rate R is represented in average number of bits of information per second. It is calculated as follows :

$$R = \left(r \text{ in } \frac{\text{messages}}{\text{second}}\right) \times \left(H \text{ in } \frac{\text{bits}}{\text{messages}}\right)$$

= bits/ second

Q.5 ***A high resolution black and white TV picture consists of about 2×10^6 picture elements and 16 different brightness levels. Pictures are repeated at rate of 32 per sec. All picture elements are assumed to be independent and all levels have equal likelihood of occurance, calculate the average information conveyed by this TV picture service?***

Ans. Given

picture element $= 2 \times 10^6$

symbols = 16

repeatation rate = 32/sec

then $H = \log_2 M$

$= \log_2^{16}$

$= \log_2^{2^4} = 4$ bit/symbols

$r = 2 \times 10^6 \times 32$

$= 64 \times 10^6$ symbols/sec

then Info rate

$R = H$

$= 64 \times 10^6 \times 4$ bit/sec

$= 2.56 \times 10^8$ bit/sec

PART-B

Q.6 ***Discuss and categorize channels for information communication.*** *[R.T.U. 2018]*

Ans. Communication Channels : In communication, a channel is a mean of passing information from a sender to a recipient. Determining the most appropriate channel, or medium, is critical to the effectiveness of communication. Channels include oral means such as telephone calls and presentations, and written modes such as reports, memos, and emails.

Communication channels differ along a scale from rich to lean. Think about how you would select a steak—some have more fat than others; they are rich and full of flavor and body. If, however, you are on a diet and just want the meat, you will select a lean steak. Communication channels are the similar: rich channels are more interactive, provide opportunities for two-way communication, and allow both the sender and receiver to read the nonverbal messages. The leanest channels, on the other hand, trim the "fat" and present information without allowing for immediate interaction, and they often convey "just the facts". The main channels of communication are grouped below from richest to leanest:

- Richest channels: face-to-face meeting; in-person oral presentation
- Rich channels: online meeting; video conference
- Lean channels: teleconference; phone call; voice message; video (e.g., Face time)
- Leanest channels: blog; report; brochure; newsletter; flier; email; phone text; social media posts (e.g., Twitter, Facebook).

Oral communications tend to be richer channels because information can be conveyed through speech as well as nonverbally through tone of voice and body language. Oral forms of communication can range from a casual conversation with a colleague to a formal presentation in front of many employees. Richer channels are well suited to complex (or potentially unsettling) information, since they can provide opportunities to clarify meaning, reiterate information, and display emotions.

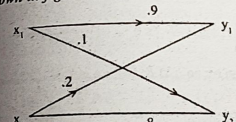
While written communication does not have the advantage of immediacy and interaction, it can be the most effective means of conveying large amount of information. Written communication is an effective channel when context, supporting data, and detailed explanations are necessary to inform or persuade others. One drawback of written communications is that they can be misunderstood or misinterpreted by an audience that doesn't have subsequent opportunities to ask clarifying questions or otherwise respond.

The following are some examples of different types of communication channels and their advantages:

- **Web-based communication**, such as video conferencing, allows people in different locations to hold interactive meetings. Other Web-based communication, such as information presented on a company website, is suited for sharing transaction details (such as order confirmation) or soliciting contact information (such as customer phone number and address)
- **Emails** provide instantaneous written communication; effective for formal notices and updates, as well as informal exchanges.

- **Letters** are a more formal method of written communication usually reserved for important messages such as proposals, inquiries, agreements, and recommendations.
- **Presentations** are usually oral and usually include an audiovisual component, like copies of reports, or material prepared in Microsoft PowerPoint or Adobe Flash.
- **Telephone meetings/conference calls** allow for long-distance interaction.
- **Message boards** and **Forums** allow people to instantly post information to a centralized location.
- **Face-to-face meetings** are personal, interactive exchanges that provide the richest communication and are still the preferred method of communication in business.

Q.7 ***Consider a discrete memory less binary channel shown in fig.***



(i) Find channel matrix of the channel.
(ii) Find $P(y_1)$, $P(y_2)$ when $P(x_1) = P(x_2) = .5$
(iii) Find $P(x_1, y_2)$ and $P(x_2, y_2)$ when $P(x_1) = P(x_2) = .5$ *[R.T.U. 2017]*

Ans.(i) Channel matrix

$$P(y|x) = \begin{bmatrix} P(y_1|x_1) & P(y_2|x_1) \\ P(y_1|x_2) & P(y_2|x_2) \end{bmatrix}$$

$$P(y|x) = \begin{bmatrix} .9 & .1 \\ .2 & .8 \end{bmatrix}$$

(ii) $P(y_1)$ and $P(y_2)$ when $P(x_1) = P(x_2) = .5$

$$[P(y)] = [P(x)]\,[P(y/x)] = [.5 \quad .5] \begin{bmatrix} .9 & .1 \\ .2 & .8 \end{bmatrix}$$

$[P(y)] = [.55 \quad .45]$

$[P(y_1)] = .55$ and $[P(y_2)] = .45$

(iii) $[P(x_1, y_2)] = [P(x)]\,[P(y/x)]$

$$= \begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix} \begin{bmatrix} .9 & .1 \\ .2 & .8 \end{bmatrix}$$

$$= \begin{bmatrix} .45 & .05 \\ .10 & .40 \end{bmatrix}$$

$P(x_1, y_2) = .05$

$P(x_2, y_1) = .10.$

Q.8 ***Define following terms:***
(i) Information
(ii) Mutual Information *[R.T.U. 2016]*

Ans.(i) Information : The principle of improbability (which is one of the basic principles of the media world)- "if dog bites a man, it's no news, but if a man bites a dog, it's a news" –help us in this regard. The probability of a dog biting a man is quite high, so this is not a news, i.e. very little amount of information is communicated by the message "a dog bites a man". On the other hand, the probability of a man biting a dog is extremely small, so this becomes a news, i.e. quite an amount of information is communicated by the message "a man bites a dog". Thus, we see that there should be some sort of inverse relationship between the probability of an event and the amount of information associated with it. The more the probability of an event, the less is the amount of information associated with it, and vice versa. Thus,

$$I(x_j) = f\left[\frac{1}{p(x_j)}\right]$$

Where x_j, is an event with a probability $p(x_j)$, and the amount of information associated with it is $I(x_j)$.

(ii) Mutual Information : Mutual information is a quantity that measures a relationship between two random variables that are sampled simultaneously. In particular, it measures how much information is communicated, on average, in one random variable about another. Intuitively, one might ask, how much does one random variable tell me about another.

For example, suppose X represents the roll of a fair 6-sided die, and Y represents whether the roll is even (0 if even, 1 if odd). Clearly, the value of Y tells us something about the value of X and vice versa. That is, these variables share mutual information.

Mutual information measures the amount of information that can be obtained about one random variable by observing another. It is important in communication where it can be used to maximize the amount of information shared between sent and received signals. The mutual information of X relative to Y is given by:

$$I(X;Y) = E_{X,Y}[SI(x,y)]$$

$$= \sum_{x,y} p(x,y) \log \frac{p(x,y)}{p(x)p(y)}$$

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Q.9 ***A black and white TV picture consists of 525 lines of picture information. Assume that each line consists of 525 picture elements and that each element can have 256 brightness levels. Pictures have repeated at the rate of 30 frames/sec. Calculate the average rate of information conveyed a set of TV set to a viewer.*** *[R.T.U. Dec. 2013]*

Ans. Given:

525 lines of picture information.
525 picture elements/line.
256 brightness levels/element.
Repeatation rate of pictures = 30 frames/sec.
Average rate of information = ?
We know that entropy is given by :

$$H(X) = -\sum_{i=1}^{m} P(x\,i)\log_2 P(x\,i) \text{ bits/symbol} \quad \text{... (i)}$$

Here, given that

$$m = 256 \quad P(xi) = \frac{1}{256}$$

substituting all values in equation (i) we get,

$$H(X) = -\sum_{i=1}^{256} \frac{1}{256}\log_2\left(\frac{1}{256}\right) = 8 \text{ bits/element.}$$

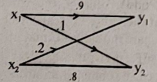
The rate 'r' at which symbols are generated is given by :

$r = (525)\,.\,(525)\,.\,(30) = 8.268 \times 10^6$ elements/sec.

Hence, average rate of information converged is given by :

$R = rH(X) = 8.268 \times 10^6 \times 8$
$= 66.144 \times 10^6$ bit/s $= \approx 66$ Mb/s

Q.10 ***(i) Consider a binary channel shown below.***



(a) Find the channel matrix of the channel
(b) Find P (y_1) and P(y_2) where $P(x_1) = P(x_2) = 5$
(c) Find the joint probabilities P (x_1, y_2) and P (x_2, y_1) when $P(x_1) = P(x_2) = 5$.
(ii) Show that I (x, y) ≥ 0. Under what condition does the equality holds ? *[R.T.U. 2012; Raj. Univ. 2000]*

Ans.(i) *Refer to Q.7.*

Ans. (ii) Mutual Info is given by

$$I(x;y) = \sum_{i=1}^{n}\sum_{j=1}^{m} P(x_i, y_j)\log_2 \frac{P(x_i/y_j)}{P(x_i)} \quad \text{... (1)}$$

but $$P\left(\frac{x_i}{y_j}\right) = \frac{P(x_i/y_j)}{P(y_j)} \quad \text{... (2)}$$

then from eq. (2) to (1)

$$I(x;y) = \sum_{i=1}^{n}\sum_{j=1}^{m} P(x_i, y_j)\log_2 \frac{P(x_i/y_j)}{P(x_i)P(x_j)}$$

$$= -\sum_{i=1}^{n}\sum_{j=1}^{m} P(x_i, y_j)\log_2 \frac{P(x_j)P(y_j)}{P(x_i, y_j)}$$

$$-I(x:y) = \sum_{i=1}^{n}\sum_{j=1}^{m} P(x_i, y_j)\log_2 \frac{P(x_i)P(y_j)}{P(x_i, y_j)} \quad \text{... (3)}$$

but we know that

$$\sum_{k=1}^{m} P_k \log_2\left(\frac{q_k}{p_k}\right) \le 0 \quad \text{... (4)}$$

then by eq. (3) & (4) we get

$-I(x; y) \le 0$

$\boxed{I(x;y) \ge 0}$

Q.11 ***An analog signal is bandlimited to 28 Hz and sampled at Nyquist rate. The samples are quantized into 4 levels. Each level represents on message. The probabilities of occurrence of these 4 levels (messages) are $P_1 = P_2 = \frac{1}{8}$ and $P_3 = P_4 = \frac{3}{8}$.***
Calculate
(i) Entropy (H)
(ii) Information rate (R) *[R.T.U. 2012, 2010]*

Ans. Given

$$P_1 = \frac{1}{8}, \qquad P_3 = \frac{3}{8}$$

$$P_2 = \frac{1}{8}, \qquad P_4 = \frac{3}{8}$$

(i) We know

Entropy :

$$H = \sum_{k=1}^{N} P_k \log_2\left(\frac{1}{P_k}\right)$$

$$H = \sum_{k=1}^{4} P_k \log_2\left(\frac{1}{P_k}\right)$$

$$H = P_1\log_2\left(\frac{1}{P_1}\right) + P_2\log_2\left(\frac{1}{P_2}\right) + P_3\log_2\left(\frac{1}{P_3}\right) + P_4\log_2\left(\frac{1}{P_4}\right)$$

$$H = \frac{1}{8}\log_2(8) + \frac{1}{8}\log_2(8) + \frac{3}{8}\log_2\left(\frac{8}{3}\right) + \frac{3}{8}\log_2\left(\frac{8}{3}\right)$$

$\boxed{H = 1.8 \text{ bits/message}}$ **Ans.**

(ii) Information Rate (R) : We know that $R = rH$

But signal is sampled at Nyquist rate. So the Nyquist rate = 2B sample/sec. and every sample generate are message signal. So message per second

$r = 2$ (2B) message/sec
$= 4$B message/sec.

So $R = rH$

$$= (4B).(1.8)\frac{\text{bits}}{\text{message}} \times \frac{\text{message}}{\text{sec.}}$$

$\boxed{R = 7.2 \text{ bits/sec.}}$ **Ans.**

PART-C

Q.12 ***(a) Show that for a discrete binding channel:***
(i) H(X, Y) = H(X/Y) + H(Y) *[R.T.U. 2018, Dec. 2013, 2013]*
(ii) H(X, Y) = H(X) + H(Y) *[R.T.U. 2018]*
(b) Prove the following properties of mutual information :
(i) I(X;Y) = H(X) − H(X/Y) *[R.T.U. 2018, 2013, 2012, 2010]*
(ii) I(X;Y) = H(X) + H(Y) − H(X, Y) *[R.T.U. 2018]*
(iii) I(X;Y) = H(X) = H(Y) (for noise free channel) *[R.T.U. 2018, 2013]*

Ans.(a)(i) H(X, Y) = H(X/Y) + H(Y) : H(X,Y) = H(X/Y) + H(Y) = H(Y/X) + H(X)

$$H(X,Y) = -\sum_{i=1}^{m}\sum_{j=1}^{m} p(X = i, Y = j)\log_2 p(X = i, Y = j)$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{m} p(X = i, Y = j)$$

$$[\log_2 p(X = i)(Y = j/X = i)]$$

$$\left[\because p(X = i, Y = j) = p(X = i)p(Y = j/X = i)\right]$$

$$= -\sum_{i=1}^{m}\sum_{j=1}^{m} p(X = i, Y = j)\log_2 p(X = i) - \sum_{i=1}^{m}\sum_{j=1}^{m} p(X = i, Y = j)\log_2 p(Y = j/X = i)$$

$$= \sum_{i=1}^{m}\log_2\frac{(X = i)}{(Y = j)}\sum_{j=1}^{m} p(Y = j/X = i) - \sum_{i=1}^{m}\sum_{j=1}^{m} p(X = i, Y = j)\log_2 p(Y = j/X = i)$$

$$\left[\because \sum_{j=1}^{m} p(Y = i/X = i) = 1\right]$$

$$\left[-\sum_{i=1}^{m} p(X = i)\log_2 p(X = i)\right]$$

$$+\left[-\sum_{i=1}^{m}\sum_{j=1}^{m} p(X = i, Y = j)\log_2 p(Y = j/X = i)\right]$$

$$= H(X) + H(Y/X)$$

Now

$$H(X, Y) = H(Y) + H(X/Y)$$

$$= -\sum_{i=1}^{m}\sum_{j=1}^{m} p(X = i, Y = j)\log_2 p(X = i, Y = j)$$

$$= -\sum_{i=1}^{m}\sum_{j=1}^{m} p(X = i, Y = j)\,[\log_2 p(Y = j) \quad p(X = i/Y = j)]$$

$$\left[\because p(X = i, Y = j) = p(Y = i)p(X = i/Y = j)\right]$$

$$= -\sum_{i=1}^{m}\sum_{j=1}^{m} p(X = i, Y = j)\log_2 p(Y = j) - \sum_{i=1}^{m}\sum_{j=1}^{m} p(X = i, Y = j)\log_2 p(X = i/Y = j)$$

$$= -\sum_{i=1}^{m}\sum_{j=1}^{m} p(Y = j)\,p(X = i/Y = j) + H(X/Y)\log_2 p(Y = j)$$

$$= \sum_{j=1}^{m} p(Y = j)\log_2 p(Y = j) + \sum_{i=1}^{m} p(X = i/Y = j) + H(X/Y)$$

$$\left[\because \sum_{i=1}^{m} p(X = i/Y = j) = 1\right]$$

$$= \sum_{j=1}^{m} p(Y = j)\log_2 p(Y = j) + H(X/Y)$$

$$H(X, Y) = H(Y) + H(X/Y)$$

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(ii) For a very noisy channel (independent), no relation can be established between transmission and receiver, these being independent:

$$\begin{cases} p_{i/j} = p_i \\ q_{j/i} = q_i \end{cases} \quad \text{... (1)}$$

It follows that:

$$\begin{cases} H(X/Y) = H(X) \\ H(Y/X) = H(Y) \end{cases} \quad \text{... (2)}$$

Also, we know that

$$H(XY) = H(Y) + H(X/Y) \quad \text{... (3)}$$
$$I(X;Y) = H(X) - H(X/Y) \quad \text{... (4)}$$

$$I(X;Y) = H(Y) - H(Y/X) \quad \text{... (5)}$$

From (2), (3), (4), (5) we get

$$H(X, Y) = H(X) + H(Y)$$
$$I(X;Y) = 0$$

Ans.(b)(i) Here $H(X/Y)$ is the conditional entropy and it is given as,

$$H(X/Y) = \sum_{i=1}^{n}\sum_{j=1}^{m} P(x_i, y_j) \log_2 \frac{1}{P(x_i/y_j)} \quad \text{...(1)}$$

$H(X/Y)$ is the information or uncertainty in X after Y is received. In other words $H(X/Y)$ is the information lost in the noisy channel. It is the average conditional self information.

Consider the equation

$$I(X;Y) = \sum_{i=1}^{n}\sum_{j=1}^{m} P(x_i, y_j) \log_2 \frac{P(x_i/y_j)}{P(x_i)}$$

Let us write the above equation as,

$$I(X;Y) = \sum_{i=1}^{n}\sum_{j=1}^{m} P(x_i, y_j) \log_2 \frac{1}{P(x_i)}$$
$$- \sum_{i=1}^{n}\sum_{j=1}^{m} P(x_i, y_j) \log_2 \frac{1}{P(x_i/y_j)}$$

From equation (1), above equation can be written as,

$$I(X;Y) = \sum_{i=1}^{n}\sum_{j=1}^{m} P(x_i, y_j) \log_2 \frac{1}{P(x_i)} - H(X/Y) \quad \text{..(2)}$$

Here let us use the standard probability relation which is given as follows:

$$\sum_{j=1}^{m} P(x_i, y_j) = P(x_i)$$

Hence equation (2) will be,

$$I(X;Y) = \sum_{i=1}^{n} P(x_i) \log_2 \frac{1}{P(x_i)} - H(X/Y)$$

First term of the above equation represents entropy. i.e.,

$$H(X) = \sum_{i=1}^{n} P(x_i) \log_2 \frac{1}{P(x_i)} \quad \text{...(3)}$$

Since above equation becomes

$$\boxed{I(X;Y) = H(X) - H\left(\frac{X}{Y}\right)}$$

(ii) From $H(X, Y) = H(X/Y) + H(Y)$ we know that

$$\therefore \quad H(X/Y) = H(X, Y) - H(Y) \quad \text{...(1)}$$

Mutual information is given by

$$I(X;Y) = H(X) - H(X/Y) \; i.e$$

Putting for $H(X/Y)$ in above equation from equation (1)

$$I(X;Y) = H(X) + H(Y) - H(X, Y) \quad \text{...(2)}$$

Thus the required relation is proved.

(iii) In the case of a noiseless channel, *i.e.* no interference or perturbation, the structure of the noise matrix is:

$$P(Y/X) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 0 \end{bmatrix} \quad \text{... (1)}$$

Having only 0 and 1 as elements; when we transmit the symbol x_i we know with certainty the received symbol. As a result:

$$\begin{cases} H(X/Y) = 0 \\ H(Y/X) = 0 \end{cases} \quad \text{... (2)}$$

We also know from $I(X:Y) = H(X) - H(X/Y)$ ... (3)

From (2) and (3) we obtain :

$$I(X;Y) = H(X) = H(Y)$$

Q.13 ***State and prove source coding theorem.*** *[R.T.U. 2016]*

OR

What is source coding theorem ? State its utility. *[R.T.U. 2018]*

Ans. Source Coding Theorem : Shanon's source coding theorem gives the range of the average code length $\bar{L}$ for a uniquely and instantaneously decodable source code.

The minimum value of $\bar{L}$ lies within the range

$$H(m) \le \bar{L} < H(m) + \epsilon$$

Where ϵ is a small positive quantity.

Proof :

Define $K = \sum_{i=1}^{N} 2^{-L_i}$ $\quad Q_i = \dfrac{2^{-L_i}}{K}$

Where L_i is the codeword length of the i^{th} symbol. It is obvious that

$$\sum_i Q_i = \frac{\sum_i 2^{-L_i}}{K} = 1 \quad \text{...(i)}$$

Now, consider the following expression

$$\sum_i P_i \log_2 \frac{Q_i}{P_i} = \sum_i P_i \left(\log_2 \frac{1}{P_i} - L_i - \log_2 K \right)$$
$$= H(m) - \bar{L} - \log_2 K \quad \text{...(ii)}$$

Where $\bar{L} = \sum_i L_i P_i$

Let us see the L.H.S. more elaborately.

$$\sum_i P_i \log_2 \frac{Q_i}{P_i} = \frac{1}{\ell n 2} \sum_i P_i \left(\ell n \frac{Q_i}{P_i} \right)$$

Put $\dfrac{Q_i}{P_i} = v$

Consider $f(v) = \ell n\, v - (v - 1)$

$f_{max} = f(1) = 0$ implies

$\ell n\, v \le v - 1$

Hence, $\sum_i P_i \log_2 \dfrac{Q_i}{P_i} \le \dfrac{1}{\ell n 2} \left[\sum_i P_i \left(\dfrac{Q_i}{P_i} - 1 \right) \right]$

$$\le \frac{1}{\ell n 2} \left[\sum_i Q_i - \sum_i P_i \right] \le \frac{1}{\ell n 2} \left[\sum_i Q_i - 1 \right]$$

Putting Eq. (i), one obtains,

$$\sum_i P_i \log_2 \frac{Q_i}{P_i} \le 0$$

Putting this value in Eq. (ii),

$$H(m) - \bar{L} - \log_2 K \le 0 \quad \text{...(iii)}$$

We state here, without proof, a constraint on the parameter K of the binary code to be uniquely and instantaneously decipherable. This constraint is also known as Kraft inequality.

The necessary and sufficient condition for a uniquely decipherable binary code is $K \le 1$.

It follows that for any decipherable code $\log_2 K \le 0$. So, we are left with the condition

$$H(m) \le \bar{L} \quad \text{...(iv)}$$

This proves the lower bound of Shanon's source coding theorem.

To prove the upper bound, consider a sub-optimum code. It is obvious that for same i, the length of the codeword for i^{th} symbol would be greater than or equal to the information content of the i^{th} symbol. Let our coding rule ensures that L_i is less than $I_i + 1$. So, we may write

$$I_i \le L_i < I_i + 1$$

Or, $\log \dfrac{1}{P_i} \le L_i < \log \dfrac{1}{P_i} + 1$

Multiplying by P_i and summing over i

$$\sum_i P_i \log \frac{1}{P_i} \le \sum_i P_i L_i < \sum_i \left(P_i \log \frac{1}{P_i} + P_i \right)$$

Or, $H(m) \le \bar{L} < H(m) + 1$

Thus, a code constituted with the coding rule

$$\log \frac{1}{P_i} \le L_i < \log \frac{1}{P_i} + 1$$

Will be reasonably efficient if either of the following two conditions hold:

(i) $H(m) >> 1$, in which case η will be quite high.

(ii) $L_i \approx \log \dfrac{1}{P_i}$ for all i, in which case the code would be almost optimum thereby making η almost 1.

If neither condition prevails, then also we can design a highly efficient code. The price we need to pay would be the increase in the complexity of the coding. This scheme is known as n^{th} extension coding. For an n^{th} extension code, n successive source symbols are grouped into blocks and the encoder operates on the blocks rather on the individual symbols. Each block consists of n statistically independent symbols, so block entropy is n H(m). So, in this case the coding rule turns out to be

$$nH(m) \le n\bar{L} < nH(m) + 1$$

Where $n\bar{L}$ is the average number of binary digits per block. Dividing by n,

$$H(m) \le \bar{L} < H(m) + \frac{1}{n} \quad \text{...(v)}$$

So, we can go to any small positive $\epsilon = 1/n$ by resorting to higher and higher value of n. Thus n-extension code comes arbitrarily close to optimum source code. This way we can always satisfy the upper bound of Shanon's source coding theorem. This completes the proof of the source coding theorem.

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Q.14 *What is entropy? Prove that*

$$H(X, Y) = H(X/Y) + H(Y) = H(Y/X) + H(X)$$

[R.T.U. 2017, 2011; Raj.Univ.2005, 2003, 2001]

OR

Prove that the entropy for a discrete source is maximum when the O/P symbols are equiprobable.
[R.T.U. 2012]

Ans. Average Information or Entropy : Suppose we have M different and independent messages m_1, m_2 .... with probabilities of occurrence p_1, p_2 .... . Suppose further during a long period of transmission of sequence of L messages has been generated thus if L is large then we expect that in L message sequence we transmit p_1L messages of m_1, p_2L messages of m_2, etc. The total in such a sequence will be

$$I_{total} = p_1 L \log_2 \frac{1}{p_1} + p_2 L \log_2 \frac{1}{p_2} \quad ...(1)$$

The average information per message interval, represented by symbol H will be

$$H = \frac{I_{total}}{L}$$

$$H = p_1 \log_2\left(\frac{1}{p_1}\right) + p_2 \log_2\left(\frac{1}{p_2}\right) + ...$$

$$H = \sum_{k=1}^{M} p_k \log_2\left(\frac{1}{p_k}\right)$$

This average information is termed as entropy. For example, we have two messages with probabilities p and (1 – p). Then average information per message interval is

$$H = p\log_2\left(\frac{1}{p}\right) + (1-p)\log_2\left(\frac{1}{1-p}\right)$$

A plot of H as a function of p is shown in given fig.

As shown H = 0 at p = 0 and p = 1. The maximum value of H may be located by setting the value of $\frac{dH}{dp} = 0$.

$$\therefore \quad H = p\log_2\frac{1}{p} + (1-p)\log_2\left(\frac{1}{1-p}\right) \quad .... (2)$$

Differentiating equation (2) with respect to p, put before this as we know that

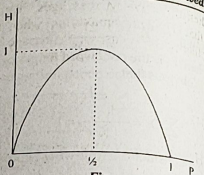

Fig.

$$\log_2 x = \frac{\log_e x}{\log_e 2}$$

After applying this, equation (2) look like

$$H = p\frac{\log_e\left(\frac{1}{p}\right)}{\log_e 2} + (1-p)\frac{\log_e\left(\frac{1}{1-p}\right)}{\log_e 2}$$

$$H = \frac{1}{\log_e 2}\left[p\log_e\frac{1}{p} + (1-p)\log_e\left(\frac{1}{1-p}\right)\right]$$

Now differentiating, we get

$$H = \frac{-1}{\log_e 2}[p\times\log_e p + (1-p)\log_e(1-p)]$$

$$\frac{dH}{dp} = \frac{-1}{\log_e 2}\left[p\times\frac{1}{p} + \log_e p + (1-p)\times\frac{-1}{(1-p)} + \log_e(1-p)(-1)\right]$$

$$= \frac{-1}{\log_e(2)}[1 + \log_e p - 1 - \log_e(1-p)]$$

$$= \frac{-1}{\log_e(2)}[\log_e p - \log_e(1-p)]$$

$$= -\log_2 p + \log_2(1-p)$$

$$= \log_2\left(\frac{1-p}{p}\right)$$

To find out the maximum value of p, equate $\frac{dH}{dp}$ zero

$$\therefore \quad \log_2\left(\frac{1-p}{p}\right) = 0$$

$$\left(\frac{1-p}{p}\right) = 2^0 \Rightarrow \left(\frac{1-p}{p}\right) = 1$$

$$\Rightarrow 1 - p = p \Rightarrow 2p = 1 \Rightarrow p = \frac{1}{2}$$

Hence, maximum value of H is

$$H_{max} = \frac{1}{2}\log_2 2 + \frac{1}{2}\log_2 2$$

or $H_{max} = \log_2 2 = 1$ bit / message

When there are M messages, it may likewise be proved that H becomes maximum when all messages are equally likely. In this case each message has probability,

$$p = \frac{1}{M} \text{ and}$$

$$H_{max} = \sum\frac{1}{M}\log_2$$

$$H_{max} = \frac{1}{M}\log_2 M + \frac{1}{M}\log_2 M + ...$$

$$= \frac{1}{M}[M\log_2 M]$$

$$\therefore H_{max} = \log_2 M$$

Now

H(X, Y) = H(X/Y) + H(Y) : *Refer to Q.12(a)(i).*

Q.15 *What is entropy? For a binary system prove that entropy is maximum when $P_1 = P_2 = 0.5$.*
[R.T.U. 2016]

Ans. Entropy : In a practical communication system, we usually transmit long sequences of symbols from an information source. Thus, we are more interested in the average information that a source produces than the information content of a single symbol.

The mean value of $I(x_i)$ over the alphabet of source X with m different symbols is given by

$$H(X) = E[I(x_i)] = \sum_{i=1}^{m} P(x_i)I(x_i)$$

$$= -\sum_{i=1}^{m} P(x_i)\log_2 I(x_i) \text{ bits / symbol}$$

The quantity H(X) is called the entropy of source X. It is a measure of the average information content per source symbol. The source entropy H(X) can be considered as the average amount of uncertainty within source X that is resolved by use of the alphabet.

Note that for a binary source X that generates independent symbol 0 and 1 with equal probability, the source entropy H (X) is :

$$H(X) = -\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{2}\log_2\frac{1}{2} = 1 \text{ b/symbol}$$

The source entropy H(X) satisfies the following relation:

$$0 \le H(X) \le \log_2 m$$

Where m is the size (number of symbols) of the alphabet of source X. The lower bound corresponds to no uncertainty, which occurs when one symbol has probability $P(x_i) = 1$ while $P(x_j) = 0$ for $j \neq i$ so X emits the same symbol x_i all the time. The upper bound corresponds to the maximum uncertainty which occurs when $P(x_i) = 1/m$ for all i, that is when all symbols are equally likely to be emitted by X.

Proof :

- Entropy is the measure of the average information content missing from a set of data when the value of the random variable is not known.
- Helps determine the average number of bits needed for storage or communication of a signal.
- As the number of possible outcomes for a random variable increases, entropy increases.
- As entropy increases, information decreases.

Mathematically entropy is expressed as:

$$H = -\sum p(x)\log p(x)$$

For a binary system (M=2), the entropy is :

$$= p_1 \log\frac{1}{p_1} + p_2 \log\frac{1}{p_2}$$

Let $p_1 = p$, then $p_2 = 1 - p_1 = 1 - p = q$

Hence,

$$H = p\log\frac{1}{p} + (1-p)\log\frac{1}{(1-p)} \quad ...(i)$$

$$= p\log\frac{1}{p} + q\log\frac{1}{q}$$

$$= H(p) = H(q)$$

A plot of H, as a function of p, as in Eq. (i) is shown in Fig. The condition for maximum entropy and its value can be found as follows:

Differentiating Eq. (i) w.r.t. p and equating it to zero yields.

$$\frac{dH}{dp} = 0 = -\ell n\, 2 - \log p + \ell n 2 + \log(1-p)$$

i.e. $\log p = \log(1-P)$

i.e. $p = 1 - p$

i.e. $p = 0.5$

This concludes that there is either a maxima or a minima at p = 0.5. If the second derivative of H is positive, then there is a minima and if it is negative, then it is a maxima. Now,

$$\frac{d^2H}{dp^2} = -\frac{1}{p} - \frac{1}{1-p} < 0$$

Hence, H has a maximum at 0.5.

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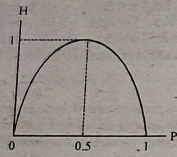

Fig.

The maximum value of H can be found from Eq. (i) by putting $p = 0.5$ in it. Thus,

$$H_{max} = H\big|_{p=0.5} = 0.5\log 2 + 0.5\log 2$$

$$= 1 \text{ bit/message}$$

Q.16 ***(a) Give the different properties of Entropies.***

(b) A discrete source emits one of six symbols once every m-sec. The symbol probabilities are $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$ and $\frac{1}{32}$ respectively. Find the source entropy and information rate. *[R.T.U. Dec. 2013]*

Ans.(a) Average Information or Entropy : *Refer to Q. 14.*

Information Rate : If the time rate at which source X emits symbols is r (symbols/s), the information, rate R of the source is given by

$$R = rH(X) \text{ bits/sec}$$

Properties of Information Content

This definition exhibits the following important properties that are intuitively satisfying :

(1) $I(s_k) > 0$ for $0 \le p_k \le 1$ ...(i)

That is to say, the occurrence of an event $S = s_k$ either provides some or no information, but never brings about a loss of information.

(2) $I(s_k) > I(s_i)$ for $p_k < p_i$ ...(ii)

That is, the less probable an event is, the more information we gain when it occurs.

(3) $I(s_k) = 0$ for $p_k = 1$ ...(iii)

Obviously if we are absolutely certain of the outcome of an event, even before it occurs, there is no information gained.

(4) The amount of information $I(s_k)$ produced by the source during an arbitrary signaling interval depends on the symbols s_k emitted by the source at that time. Indeed, $I(s_k)$ is a discrete random Variable that takes on the values $I(s_0)$, $I(s_1)$, ... $I(s_{k-1})$ with probabilities $p_0, p_1, \ldots p_{k-1}$ respectively. The mean of $I(s_k)$ over the source alphabet £ is given by

$$H(£) = E[I(s_k)]$$

$$= \sum_{k=0}^{k-1} p_k I(s_k)$$

$$= \sum_{k=0}^{k-1} p_k \log_2\left(\frac{1}{p_k}\right) \quad \ldots(iv)$$

The important quantity H (£) is called the entropy of a discrete memory less source with source alphabet £. It is measure of the average information content per source symbol.

(5) $I(s_k s_i) = I(s_k) + I(s_i)$ if s_k and s_i are statistically independent. ...(v)

The resulting unit of information is called the bit (a contraction of binary digit). We thus write

$$I(s_k) = \log_2\left(\frac{1}{p_k}\right)$$

$$= -\log_2 p_k \text{ for } k = 0, 1, \ldots\ldots k-1$$

When $p_k = 1/2$, we have $I(s_k) = 1$ bit hence, one bit is the amount of information that we gain when one of two possible and equally likely (i.e., equiprobable) events occurs. Note that the information $I(s_k)$ is positive, since the logarithm of a number less than one, such as a probability is negative.

Properties of Entropy

Consider a discrete memory less source whose mathematical model defined by equations. The entropy H(£) of such a source is bounded as follows:

$$0 \le H(£) \log_2 \le K$$

where K is the radix (number of symbols) of the alphabet £ of the source. Further more, we may make two statements:

(1) H(£)= 0, if and only if the probability $p_k = 1$ for some k and remaining probabilities in the set are all zero; this lower bound on entropy corresponds to maximum uncertainty.

(2) H(£) $\log_2$ k, if and only if $p_k = 1/k$ for all k (i.e, all the symbols in the alphabet £ are equiprobable); this upper bound on entropy corresponds to maximum uncertainty.

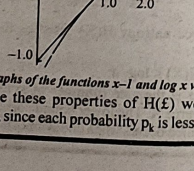

Fig. Graphs of the functions x–1 and log x versus x.

To prove these properties of H(£) we proceed as follows. First, since each probability p_k is less than or equal

to unity, it follows that each term $p_k \log_2(1/p_k)$ in equation (v) is always non–negative and so $H(£) \ge 0$. Next, we note that the product term $p_k \log_2(1/p_k)$ is zero, if, and only if, $p_k = 0$ or 1. We therefore deduce that $H(£) = 0$, if and only if , $p_k = 0$ or 1, that is $p_k = 1$ for some k and all the rest are zero.

This completes the proofs of the lower bound in eq (iv) and statement (2). To prove the upper bound in eq (iv) and statement (1) we make use of a property of the natural logarithm:

$$\log x \le x-1, \; x \ge 0$$

The inequality can be readily verified by plotting the functions log x and (x–1) versus x, as shown in figure. Here we see that the line y = x–1 always lies above the curve y = log x. The equality holds only at point x =1, where the line is tangential to the curve.

Then proceed with the proof, consider first any two probability distributions $\{p_0, p_1, \ldots p_{k-1}\}$ on the alphabet $£ = \{s_0, s_1, \ldots s_{k-1}\}$ of a discrete memory less source. Then changing to the natural logarithm, we may write

$$\sum_{k=0}^{k-1} p_k \log_2\left(\frac{q_k}{p_k}\right) = \frac{1}{\log 2}\sum_{k=0}^{k-1} p_k \left(\frac{q_k}{p_k}\right)$$

Hence using the inequality of above Equation, we get

$$\sum_{k=0}^{k-1} p_k \log_2\left(\frac{q_k}{p_k}\right) \le \frac{1}{\log 2}\sum_{k=0}^{k-1} p_k\left(\frac{q_k}{p_k} - 1\right)$$

$$\le \frac{1}{\log 2}\sum_{k=0}^{k-1}(q_k - p_k)$$

$$\le \frac{1}{\log 2}\left(\sum_{k=0}^{k-1} q_k - \sum_{k=0}^{k-1} p_k\right) = 0$$

we thus have the fundamental inequality

$$\sum_{k=0}^{k-1} p_k \log_2\left(\frac{q_k}{p_k}\right) \le 0 \quad \ldots(vi)$$

where the equality holds only if $q_k = p_k$ for all k.

Suppose we next put

$$q_k = \frac{1}{k}, k = 0, 1 \ldots k-1 \quad \ldots(vii)$$

which corresponds to an alphabet £ with equiprobable symbols. The entropy of a discrete memoryless source with such a characterization equals

$$\sum_{k=0}^{k-1} q_k \log_2\left(\frac{1}{q_k}\right) \log_2 k \quad \ldots(viii)$$

Also, the use of eq(vii) in eq(vi) yields

$$\sum_{k=0}^{k-1} p_k \log_2\left(\frac{1}{q_k}\right) \le \log_2 k$$

Equivalently, the entropy of a discrete memory less source with an arbitrary probability distribution for the symbols of its alphabet £ is bounded as

$$H(£) \ge \log_2 k$$

This H (£) is always less than or equal to $\log_2 k$. The equality holds only if the symbols in the alphabet £ are equiprobable, as in eq(vii).

Ans. (b) We know the source entropy is given by:

$$H(X) = \sum_{i=1}^{m} P(xi)\log\frac{1}{P(xi)} = \sum_{i=1}^{6} P(xi)\log\frac{1}{P(xi)} \text{ bits/symbol}$$

$$H(X) = \frac{1}{2}\log_2 2 + \frac{1}{4}\log_2 4 + \frac{1}{8}\log_2 8 + \frac{1}{16}\log_2 16 + \frac{1}{32}\log_2 32 + \frac{1}{32}\log_2 32$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{3}{8} + \frac{1}{4} + \frac{5}{32} + \frac{5}{32} = 1.9375 \text{ bits/symbol.}$$

The symbol rate $r = \frac{1}{T_b} = \frac{1}{10^{-3}} = 1000$ symbols/sec.

∴ the information rate is expressed as.

$$R = r\,H(x) = 1000 \times 1.9375$$

$$= 1937.5 \text{ bits/sec.}$$

Q.17 ***(a) Consider a DMS with the alphabet (S_0, S_1, S_2) with probabilities $P_0 = \frac{1}{2}$, $P_1 = \frac{1}{4}$, $P_2 = \frac{1}{2}$. Find out the entropy of the original source and second order extension entropy?***

(b) Two binary channels are connected in cascade as shown in fig.

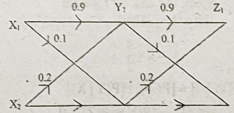

Fig.

(i) Find overall channel matrix and equivalent channel diagram.

(ii) Find $P(Z_1)$ and $P(Z_2)$ when $P(X_1) = P(X_2) = 0.5$ *[R.T.U. 2013]*

Ans.(a) Given $P_0 = \frac{1}{2}$, $P_1 = \frac{1}{4}$, $P_2 = \frac{1}{2}$

∴ Entropy of original source H(α)

$$H(\alpha) = \sum_{i=0}^{2} P_i \log_2 \frac{1}{P_i}$$

$$= \frac{1}{4}\log_2(4) + \frac{1}{2}\log_2(2) + \frac{1}{2}\log_2(2) = \frac{3}{2} \text{ bits}$$

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Now consider second order extension source. Source alphabet $\alpha = (S_0, S_1, S_2)$ consists of three symbols.

[k = no of symbols = 3]

Source alphabet of second order extension source consists of nine symbols

$$k^2 = 3^2 = 9$$

These symbols are

Symbols $\sigma_0\ \sigma_1\ \sigma_2\ \sigma_3\ \sigma_4\ \sigma_5\ \sigma_6\ \sigma_7\ \sigma_8$

$\alpha^2 = (S_0S_1, S_0S_0, S_0S_2, S_1S_0, S_1S_1, S_1S_2, S_2S_0, S_2S_1, S_2S_2)$

Probabilities of these in blocks that consists of 2 symbols are.

$$P(\sigma_i) = \left(\frac{1}{16}, \frac{1}{16}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{4}\right)$$

Entropy of second order extension

$$H(\alpha^2) = \sum_{i=0}^{8} P(\sigma i)\log_2\left(\frac{1}{P}\right)(\sigma i)$$

$$H(\alpha^2) = \frac{1}{16}\log_2(16) + \frac{1}{16}\log_2(16) + \frac{1}{8}\log_2(8) + \frac{1}{16}\log_2(16) + \frac{1}{16}\log_2(16) + \frac{1}{8}\log_2(8) + \frac{1}{8}\log_2(8) + \frac{1}{8}\log_2(8) + \frac{1}{4}\log_2(4)$$

$$= 3 \text{ bits}$$

Cross check the ans. by formula

$$H(\alpha^2) = 2H(\alpha)$$

$$3 = 2 \times \frac{3}{2}$$

Ans.(b)(i) $[P(Y)] = [P(X)]\ P[P(Y|X)]$ ...(1)

$[P(Z)] = [P(Y)][P(Z|Y)]$ ...(2)

from equation (1) and (2)

$$= [P(X)][P(Y|X)][P(Z|Y)]$$

$$P(Z|X) = [P(Y|X)][P(Z|Y)]$$

$$= \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}\begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$

$$= \begin{bmatrix} 0.83 & 0.17 \\ 0.34 & 0.66 \end{bmatrix}$$

$\therefore$ Resultant's Matrix Diagram

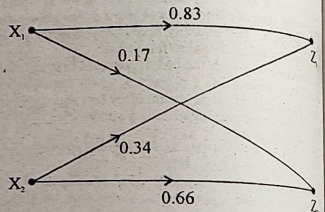


Fig.

(ii) $[P(Z)] = [P(X)][P(Z|X)]$

$$= [0.5 \quad 0.5]\begin{bmatrix} 0.83 & 0.17 \\ 0.34 & 0.66 \end{bmatrix}$$

$$= [0.585 \quad 0.415]$$

$P(Z_1) = 0.585$

$P(Z_2) = 0.415$

□□□

SOURCE CODING SCHEMES FOR DATA COMPACTION 2

PREVIOUS YEARS QUESTIONS

PART-A

Q.1 *Consider a source $S = [S_1, S_2]$ with probabilities ¼ and ¼ respectively. Obtain Shannon-fono code for source S, its 2nd and 3rd extensions. Calculate efficiency for each case.* *[R.T.U. 2018]*

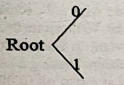
Ans. We can determine

$$l_1 = \log 21/S_1 = -0.41$$

$$l_2 = \log 21/S_2 = -2$$

and $l_{max} = -0.41$

Construct a binary tree of depth 1.



The source codewords are

$x_1 : 0\ 0$

Q.2 *Apply the Shannon – Fano coding and find code efficiency.*

$[x_i] = [x_1\ \ x_2\ \ x_3\ \ x_4\ \ x_5\ \ x_6]$

$[P(x_i)] = [0.30\ \ 0.25\ \ 0.20\ \ 0.12\ \ 0.08\ \ 0.05]$

[R.T.U. 2016]

Ans. Shannon – Fano coding

$[x_i]$	$[P(x_i)]$	1st Group	2nd Group	3rd Group	Codeword	L_i
x_1	0.30	0	0		00	2
x_2	0.25	0	1		01	2
x_3	0.20	1	0	0	100	3
x_4	0.12	1	0	1	101	3
x_5	0.08	1	1	0	110	3
x_6	0.05	1	1	1	111	3

Entropy

$$H(\zeta) = \sum_{k=0}^{5} P_k \log_2\left(\frac{1}{P_k}\right)$$

$$= 2.36 \text{ bits}$$

Average codeword length,

$$T = \sum_{k=0}^{5} P_k l_k$$

$$= 2.45 \text{ bits}$$

Code efficiency,

$$\eta = \frac{H(\zeta)}{T} \times 100$$

$$= \frac{2.36}{2.45} \times 100$$

$$\boxed{\eta = 96.33\%}$$

Q.3 *Consider a DMS with source probabilities {20, 20, 15, 15, 10, 10, 05, 05} Determine the Huffman code for this source.* *[R.T.U. 2012]*

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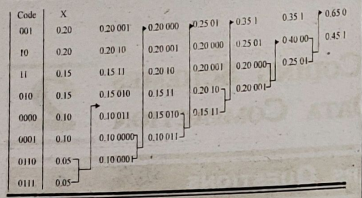
Ans. Huffman Code



Q.4 ***State Kraft Inequality Theorem.***

Ans. Kraft Inequality Theorem : A necessary and sufficient condition for the existence of a binary code with codewords having lenghts $n_1 \le n_2 \le ..n_L$ that satisfy the prefix condition is

$$\sum_{k=1}^{L} 2^{-n_k} \le 1$$

Q.5 ***Define Prefix Code.***

Ans. Prefix Code : This is variable length coding algorithm. It assigns binary digits to the messages as per their probabilities of occurrence. Prefix of the codeword means any sequence which is initial part of the codeword. In prefix code, no codeword is the prefix of any other codeword.

PART-B

Q.6 ***Write short notes on :***
(i) Noise Free channel
(ii) Shannon's theorem

[R.T.U 2018, Dec. 2013]

Ans. (i) Free Channel : The channel is called noise free or noiseless if it is both lossless and deterministic – *i.e.*

The channel matrix has only one element in each row and on each column.

It is show in following Fig.

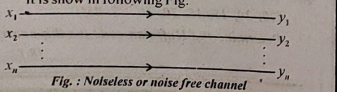


Fig. : Noiseless or noise free channel

It is channel matrix coil be like this

$$P(y/x) = \begin{matrix} \\ x_1 \\ x_2 \\ x_3 \\ x_n \end{matrix} \begin{matrix} y_1 & y_2 & y_3 \cdots y_n \\ \end{matrix} \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

The binary symmetric channels are also cons under noiseless channels.

$$[P(y/x)] = \begin{bmatrix} 1-P & P \\ P & 1-P \end{bmatrix}$$

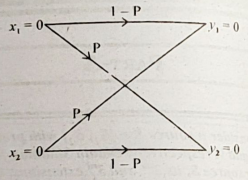


Fig. Binary symmetric channel

(ii) Shannon's Theorem : The Shannon-Hartley theore is of fundamental importance and has two import implications for communication systems engineers.

1. First, it gives us the upper limit that can be reache the way of reliable data transmission rate over Gauss channels. Thus, a system designer alway s tries to optimiz system to have a data rate as close to C given in equation as possible with an acceptable error rate.

2. The second implication of the Shannon-Hart theorem has to do with the exchange of signal-to-noi ratio for bandwidth. To illustrate this aspect of the theore suppose that we want to transmit data at a rate of 10,0 bits/sec. Over a channel having a bandwidth B = 3000 To transmit data at a rate of 10,000 bits/sec, we need channel with a capacity of at least 10,000 bits/sec. If t channel capacity is less than the data rate, then error transmission is not possible. So, with C = 10,000 bits/s we can obtain the (S/N) requirement of the channel a

$$(S/N) = 2^{(C/B)} - 1 = 2^{3.333} - 1 \approx 9$$

The Shannon-Hartley theorem indicates that noiseless channel has an infinite capacity. However, w noise is present the channel capacity does not approa infinity as the bandwidth is increased because the n power increases as the bandwidth increases. The chan capacity reaches a finite upper limit with increas bandwidth if the signal power is fixed. We can calcula

this limit as follows. With $N = \eta B$, where $\eta/2$ is the noise power spectral density, we have

$$C = B\log_2\left(1+\frac{S}{\eta B}\right) = \left(\frac{S}{\eta}\right)\left(\frac{\eta B}{S}\right)\log_2\left(1+\frac{S}{\eta B}\right) \quad ...(1)$$

$$= \frac{S}{\eta}\log_2\left(1+\frac{S}{\eta B}\right)$$

Recalling that $\lim_{x\to 0}(1+x)^{1/x} = e$ and letting $x = S/\eta B$ we have

$$\lim_{B\to\infty} C = \frac{S}{\eta}\log_2 = 1.44\frac{S}{\eta}$$

A communication system capable of transmitting information at a rate of $B \log_2 (1 + S/N)$ is called an ideal system. The ideal signalling scheme using noise like signals can convey information at a rate approaching the channel capacity only when $T \to \infty$. Only in the limiting case we have all the conditions satisfied. Under this limiting condition, the ideal system has the following characteristics:

(a) The information rate $\to B \log_2 (1 + S/N)$.
(b) The error rate $\to 0$.
(c) The transmitted and received signals have the characteristics of band limited Gaussian white noise.
(d) As $T \to \infty$, the number of signals $M \to \infty$ and coding delay also tends to ∞.

Q.7 ***Explain Shannon Theorem and Shannon Limit.***

[R.T.U. 2016]

Ans. Shannon's Theorem : *Refer to Q.6.*

Shannon Limit : There exists a limiting value of E_b/N_0 below which there can be no error-free communication at any information rate. Using the identity

$$\lim_{x\to 0}(1+x)^{1/x} = e$$

We can calculate the limiting value of E_b/N_0 as follows:

Let $$x = \frac{E_b}{N_0}\left(\frac{C}{W}\right)$$

Then, $$\frac{C}{W} = x\log_2(1+x)^{1/x}$$

and $$1 = \frac{E_b}{N_0}\log_2(1+x)^{1/x}$$

In the limit, as $C/W \to 0$, we get

$$\frac{E_b}{N_0} = \frac{1}{\log_2 e} = 0.693$$

Or, in decibels,

$$\frac{E_b}{N_0} = -1.6\text{dB}$$

This value of E_b/N_0 is called the Shannon limit.

Q.8 ***Derive the mathematical expression for channel capacity to transmit the information through it if the channel capacity is :***

$$C = B\log_2\left(1+\frac{S}{N}\right) b/s$$

[Note : Read B = ∞] *[R.T.U. 2016]*

Ans. The noise characteristics of channels encountered in practice is generally Gaussian (channels with Gaussian noise characteristic are known as Gaussian channels.) Moreover, the result obtained for a Gaussian channel often provide a lower bound on the performance of a system with the Gaussian channel. Thus, if a particular encoder-decoder is used with a Gaussian channel giving an error probability Pe, then, with a non-Gaussian channel, another encoder –decoder can be designed for which the error probability will be less than Pe. Hence, the study of a Gaussian channel is very important.

For a Gaussian channel,

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} \quad ...(1)$$

Hence,

$$H(x) = -\int_{-\infty}^{\infty} p(x)\log p(x)\,dx$$

But

$$-\log p(x) = \log\sqrt{2\pi\sigma^2} + \log e^{x^2/2\sigma^2}$$

From Eq. (1)

Hence,

$$H(x) = \int_{-\infty}^{\infty} p(x)\log\sqrt{2\pi\sigma^2}\,dx + \int_{-\infty}^{\infty} p(x)\log e^{x^2/2\sigma^2}$$

This may be evaluated to yield

$$H(x) = \log\sqrt{2\pi e\sigma^2} \text{ bits/message} \quad ...(2)$$

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Now, if the signal is band limited to ωHz, then it may be uniquely specified by taking 2ω samples per second (Sampling theorem). Hence, the rate of information transmission is

$$R(x) = 2\omega H(x)$$

$$= 2\omega \log\left(\sqrt{2\pi e\sigma^2}\right)$$

$$= \omega \log\left(\sqrt{2\pi e\sigma^2}\right)^2$$

or $$R(x) = \omega \log\left(2\pi e\sigma^2\right) \quad ...(3)$$

If p(x) is a band limited Gaussian noise with an average noise power N, then we have

$$R(n) = R(x) = \omega \log\left(2\pi eN\right)\left(\because \sigma^2 = N\right) \quad ...(4)$$

Now, consider a continuous source transmitting information over noisy channel. If the received signal is composed of a transmitted signal x and a noise n, then the joint entropy (bits/sec basis) of the source and noise is given by

$$R(x,n) = R(x) + R(n/x)$$

Assuming that the transmitted signal and noise are independent (as is the practical situation).

$$R(x,n) = R(x) + R(n) \quad ...(5)$$

Since the received signal y is the sum of the transmitted signal x and the noise n, we may equate

$$H(x,y) = H(x,n)$$

or $$H(y) + H(x/y) = H(x) + H(n)$$

or $$R(y) + R(x/y) = R(x) + R(n) \quad ...(6)$$

The rate at which the information is received from a noisy channel is

$$R = R(x) - R(x/y)$$

By using Eq. (6), we get

$$R = R(y) - R(n) \text{ bits/sec} \quad ...(7)$$

The channel capacity in bit/sec is

$$C = \text{Max } [(R)] \text{ bits/sec}$$

or $$C = \text{Max } [R(y) - R(n)] \text{ bits/sec} \quad ...(8)$$

Since R(n) is assumed to be independent of x(t), maximizing R requires maximizing R(y). Let a transmitted signal be limited to an average power S and the noise on the channel be white Gaussian with an average power N within the bandwidth ω of the channel. The received signal will now have an average power (S+N). R(y) is maximum when y(t) is also a Gaussian random process because noise is assumed to be Gaussian. Thus, the entropy from Eq. (4) on a per second basis is

$$R(y) = \omega \log [2\pi e(S+N)] \text{ bits/sec}$$

While the entropy of the noise is given by

$$R(n) = \omega \log [2\pi eN] \text{ bits/sec}$$

The channel capacity may now be obtained directly since R(y) has been maximized.

Thus, $$C = \max[R(y) - R(n)]$$

$$= \omega \log [2\pi e[S+N] - \omega \log [2\pi eN]$$

$$= \omega \log\left[\frac{S+N}{N}\right]$$

or $$C = \omega \log\left[1 + \frac{S}{N}\right] \text{ bits/sec} \quad ...(9)$$

Equation (9) is the famous Shannon-Hartley theorem, which is complementary to Shannon's theorem and applies to a Gaussian noise channel. The statement of the Shannon-Hartley theorem is given below:

The channel capacity of a white bandlimited Gaussian channel is

$$C = \omega \log\left[1 + \frac{S}{N}\right] \text{ bits/sec}$$

Where ω is the channel bandwidth, S the average signal power, and N the average noise power. If η/2 is the two-sided power spectral density of noise in watts/Hz then

$$N = \eta\omega$$

and $$C = \omega \log\left[1 + \frac{S}{\eta\omega}\right] \text{ bits/sec} \quad ...(10)$$

Q.9 ***Apply the Shannon - Fano coding procedure for the following message.***

[X] =	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
[P] =	1/4	1/8	1/16	1/16	1/16	1/4	1/16	1/8

[R.T.U. 2013]

Ans.

Message	Probability	Encoded message	Length (n_k)
X_1	0.25	0 0	2
X_6	0.25	0 1	2
X_2	0.125	1 0 0	3
X_8	0.125	1 0 1	3
X_3	0.0625	1 1 0 0	4
X_4	0.0625	1 1 0 1	4
X_5	0.0625	1 1 1 0	4
X_7	0.0625	1 1 1 1	4

$$\bar{N} = \sum_{k=1}^{8} P_k n_k = \left(\frac{1}{4}\times 2\right) + \left(\frac{1}{8}\times 3\right) + \left(\frac{1}{16}\times 4\right) + \left(\frac{1}{16}\times 4\right) + \left(\frac{1}{16}\times 4\right) + \left(\frac{1}{4}\times 2\right) + \left(\frac{1}{16}\times 4\right) + \left(\frac{1}{8}\times 3\right)$$

or $$\bar{N} = 2.75 \text{ letters/message}$$

$$H(X) = -\sum_{k=1}^{8} P_k \log P_k$$

$$= -\left[\frac{1}{4}\log\frac{1}{4} + \frac{1}{8}\log\frac{1}{8} + \frac{1}{16}\log\frac{1}{16} + \frac{1}{16}\log\frac{1}{16} + \frac{1}{16}\log\frac{1}{16} + \frac{1}{16}\log\frac{1}{16} + \frac{1}{4}\log\frac{1}{4} + \frac{1}{16}\log\frac{1}{16} + \frac{1}{8}\log\frac{1}{8}\right]$$

$$= 2.75 \text{ bits/message}$$

$$\log M = \log 2 = 1 \text{ bits/letter}$$

Hence $$\eta = \frac{H(X)}{\bar{N}\log M}\eta = \frac{2.75}{2.75\times 1} = 100\%$$

Q.10 ***The given string S = 0 0 1 2 1 2 1 2 1 2 1 0 2 1 0 1 2 1 0 1 1. Find the encoding and decoding process.*** *[R.T.U. 2013]*

Ans. The string S = 00121212102102101221011 is to be encoded. Fig. 1 shows the encoding process.

0	0	0	0	0	0		0		0			0						
0	0	1	2	1	2	1	2	1	0	2	1	0	1	2	1	0	1	1
1	2	1	2	1	2	1	2	1	0	2	1	0	1	2	1	0	1	1

#	Entry	Phrase	Output	(ternary)
1	0	0	0 0	(0 0)
2	1+1	01	1 1	(1 1)
3	2	2	0 2	(0 2)
4	1	1	0 1	(00 1)
5	3 + 1	21	3 1	(10 1)
6	5 + 0	210	5 0	(12 0)
7	6 + 1	2101	6 1	(20 1)
8	7 + 2	21012	7 2	(21 2)
9	7 + 1	21011	7 1	(21 1)

Fig. 1 : Encoding

(i) In the first step, 0 is encountered and added to the dictionary. The output is 00 because is no match (index 0) and the first non-matching character is 0. The encoder then proceeds to the second position, encountering 0, which is already in the dictionary. The following 1 is not yet in the dictionary, so the encoder adds the string 01 to the dictionary (a reference to the first entry plus the symbol 1) and outputs this pair. The next steps follow the same scheme until the end of the input is reached.

(ii) The decoding process is shown in fig. 2. The decoder receives the reference 00, with the index 0 indicating that a previously unknown symbol (0) needs to be added to the dictionary and to the uncompressed data. The next codeword is 11 resulting in the entry 01 (a reference to entry 1 plus the symbol 1) being added to the dictionary and the string 01 appended to the uncompressed data. The decoder continues this way until all codewords have been decoded.

0	0	0	0	0		0			0				0					0		
0	0	1	2	1	2	1	0	2	1	0	1	2	1	0	1	2	2	1	0	1	1

#	Entry	Phrase	Output	(ternary)
1	0	0	0 0	(0 0)
2	1+1	01	1 1	(1 1)
3	2	2	0 2	(0 2)
4	1	1	0 1	(00 1)
5	3 + 1	21	3 1	(10 1)
6	5 + 0	210	5 0	(12 0)
7	6 + 1	2101	6 1	(20 1)
8	7 + 2	21012	7 2	(21 2)
9	7 + 1	21011	7 1	(21 1)

Fig. 2 : Decoding

Q.11 ***Construct Huffman's code to the following set of messages. Also find the efficiency*** $p(x_1) = 0.49$, $p(x_2) = 0.14$, $p(x_3) = 0.14$, $p(x_4) = 0.07$, $p(x_5) = 0.07$, $p(x_6) = 0.04$, $p(x_7) = 0.02$, $p(x_8) = 0.02$, $p(x_9) = 0.01$. *[R.T.U. 2013]*

Ans.

Symbol	Probability	SS - I	SS - II	SS - III	SS - IV	SS - V	SS - VI	SS - VII	Code word	Length
x_1	0.49	0.49	0.49	0.49	0.49	0.49	0.49	0.51	0	1
x_2	0.14	0.14	0.14	0.14	0.14	0.23	0.28	0.49	100	3
x_3	0.14	0.14	0.14	0.14	0.14	0.14	0.23		101	3
x_4	0.07	0.07	0.07	0.09	0.14	0.14			1100	4
x_5	0.07	0.07	0.07	0.07	0.09				1101	4
x_6	0.04	0.04	0.05	0.07					1110	4
x_7	0.02	0.03	0.04						11110	5
x_8	0.02	0.02							111110	6
x_9	0.01								111111	6

Fig.

Average code word length

$$L = \sum_{k=1}^{K} P_k L_k$$

$$L = (0.49\times 1) + (0.14\times 3) + (0.14\times 3) + (0.07\times 4) + (0.07\times 4) + (0.04\times 4) + (0.02\times 5) + (0.02\times 6) + (0.01\times 6)$$

$$L = 2.33 \text{ bits/symbol}$$

$$H(X) = -\sum_{k=1}^{9} P_k \log_2 P_k$$

$$= -3.32\,[0.49\log_{10}0.49 + 0.14\log_{10}0.14 + 0.14\log_{10}0.14 + 0.07\log_{10}0.07 + 0.07\log_{10}0.07 + 0.04\log_{10}0.04 + 0.02\log_{10}0.02 + 0.02\log_{10}0.02 + 0.01\log_{10}0.01]$$

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$H(X) = 2.3122$ bits/symbol

$$\eta = \frac{H(X)}{L} = \frac{2.3122}{2.33} = 0.992$$

$\therefore$ Loading efficiency = 99.2%

$R = 1 - \eta$

$= 1 - 0.992$

$R = 0.00763$

PART-C

Q.12 ***Explain Huffman coding with help of suitable example.*** *[R.T.U. 2018]*

Ans. Huffman Coding Algorithm with Example

Huffman coding algorithm was invented by David Huffman in 1952. It is an algorithm which works with integer length codes. A Huffman tree represents Huffman codes for the character that might appear in a text file. Unlike to ASCII or Unicode, Huffman code uses different number of bits to encode letters. If the number of occurrence of any character is more, we use fewer numbers of bits. Huffman coding is a method for the construction of minimum redundancy codes.

Huffman tree can be achieved by using compression technique. Data compression has lot of advantages such as it minimizes cost, time, bandwidth, storage space for transmitting data from one place to another.

In regular text file each character would take up 1 byte (8 bits) i.e. there are 16 characters (including white spaces and punctuations) which normally take up 16 bytes. In the ASCII code there are 256 characters and this leads to the use of 8 bits to represent each character but in any test file we do not have to use all 256 characters. For example, in any English language text, generally the character 'e' appears more than the character 'z'. To achieve compression, we can often use a shorter bit string to represent more frequently occurring characters. We do not have to represent all 256 characters, unless they all appear in the document. The data encoding schemes are broadly categorized in two categories.

Fixed Length Encoding Scheme

Fixed length encoding scheme compresses our data by packing it into the minimum number of bits i.e. needed to represent all possible values of our data. The fixed length code can store maximum 224,000 bits data.

Variable Length Encoding Scheme

In variable length encoding scheme we map sin... symbol to variable number of bits. It allows source to b... compressed and decompressed with zero error.

Construction of Huffman Code

A greedy algorithm constructs an optimal pref... code called Huffman code. The algorithm builds the tr... T corresponding to the optimal code in a bottom-up manne... It begins with a set of |C| leaves (C is the number o... characters) and perform |C| – 1 'merging' operations t... create the final tree. In the Huffman algorithm 'n' denote... the number of sets of characters, z denotes the paren... node and x & y are the left & right child of z respectivel...

Huffman (C)

1. n = |C|
2. Q = C
3. for i = 1 to n-1
4. do
5. z = allocate_Node()
6. x = left[z] = Extract_Min(Q)
7. y = right[z] = Extract_Min(Q)
8. f[z] = f[x] + f[y]
9. Insert(Q,z)
10. returnExtract_Min(Q)

Analysis

The Q is initialized as a priority queue with th... character C.

Q = C can be performed by using Build_Heap in... O(n) time.

for loop takes (|n|-1) times because each heap... operation requires O(log n) time.

Hence, the total running time of Huffman code on... the set of n characters is O(n log n).

Method

The following general procedure is applied fo... construction a Huffman tree:

Search for the two nodes having the lowest... frequency, which are not yet assigned to a parent node.

Couple these nodes together to a new interior node.

Add both the frequencies and assign this value t... the new interior node.

The procedure has to be repeated until all nodes... are combined together in a root node.

Huffman Coding Algorithm Example

Construct a Huffman tree by using these nodes.

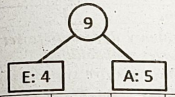
Value	A	B	C	D	E	F
Frequency	5	25	7	15	4	12

Solution:

Step 1 : According to the Huffman coding we arrange all the elements (values) in ascending order of the frequencies.

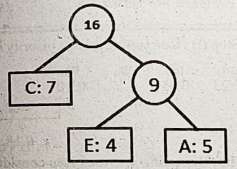
Value	E	A	C	F	D	B
Frequency	4	5	7	12	15	25

Step 2 : Insert first two elements which have smaller frequency.



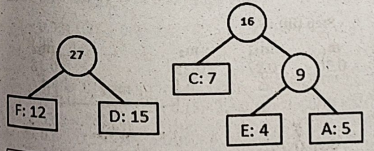
Value	C	EA	F	D	B
Frequency	7	9	12	15	25

Step 3 : Taking next smaller number and insert it at correct place.



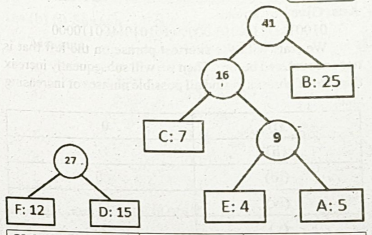
Value	F	D	CEA	B
Frequency	12	15	16	25

Step 4 : Next elements are F and D so we construct another subtree for F and D.



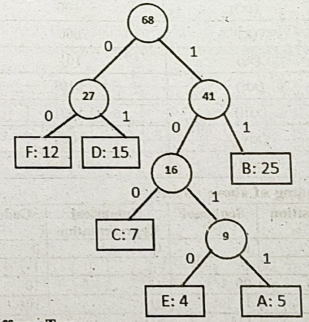
Value	CEA	B	FD
Frequency	16	25	27

Step 5 : Taking next value having smaller frequency then add it with CEA and insert it at correct place.



Value	FD	CEAB
Frequency	27	41

Step 6 : We have only two values hence we can combine by adding them.



Huffman Tree

Value	FDCEAB
Frequency	68

Now the list contains only one element i.e. FDCEAB having frequency 68 and this element (value) becomes the root of the Huffman tree.

Q.13 ***Determine the Lempel Ziv code for following bit stream. 0100111111100101 00000 1010101100110000 Recover the original sequence from encoded stream.*** *[R.T.U. 2017]*

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Ans. Given by stream :

0100111111001010 0000 1010101100110000

We start with the shortest phrase on the left that is not encountered before. Then we will subsequently increax the size of phrases to find all possible phrases of increasing size.

(i)	0
(ii)	1
(iii)	00
(iv)	11
(v)	111
(vi)	10
(vii)	01
(viii)	010
(ix)	100
(x)	000
(xi)	101
(xii)	0101
(xiii)	1001
(xiv)	1000
(xv)	0

Encoding of above stream

Position	Sequence	Numerical Representation	Code
1	0	0	0
2	1	–1	1
3	00	10	10
4	11	21	101
5	111	41	1001
6	10	20	0100
7	01	11	0011
8	010	70	1110
9	100	60	1100
10	000	30	0100
11	101	61	1101
12	0101	81	10001
13	1001	91	10011
14	1000	90	10010
15	0	1	00001

Compressed Sequence

0100101100101000011111011000100110110001100111001000001

Q.14 *(a) Construct a binary Huffman code of the following message. Find out its efficiency*

Message	Probability
m_1	.30
m_2	.25
m_3	.15
m_4	.12
m_5	.10
m_6	.08

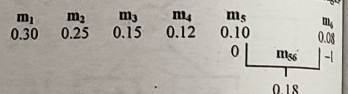
Use binary differentiate set for encoding messages probabilities.

(b) A DMS has five symbols x_1, x_2, x_3, x_4, x_5 with $P(x_1) = 0.4$, $P(x_2) = 1.19$, $P(x_3) = 0.16$, $P(x_4) = 0.15$, $P(x_5) = 0.1$

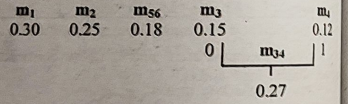
(i) Construct a shanon-fano code for y and calculate the efficiency of the code.

(ii) Repeat the Huffman code and compare the result. [R.T.U. 2017]

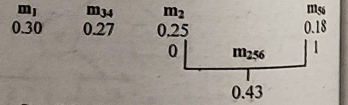
Ans.(a) Step (i) : Combining 2 least priority messages



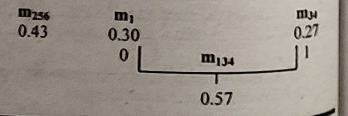
Step (ii) : Repeating previous step considering new combined message -



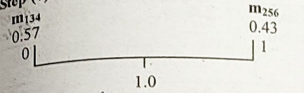
Step (iii) :



Step (iv) :



Step (v) :



for Huffman codes :

Creating binary tree,

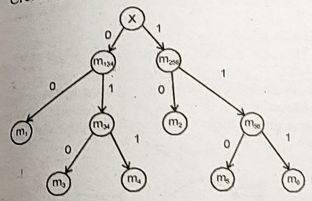


Fig.

Codes are as follows :

m_1	00
m_2	10
m_3	010
m_4	011
m_5	110
m_6	111

Entropy $H(z) = \sum_{i=1}^{6} p_i \log_2\left(\frac{1}{p_i}\right)$

$= 0.3 \times \log_2\left(\frac{1}{0.3}\right) + 0.25 \times \log_2\left(\frac{1}{0.25}\right)$

$+ 0.15 \times \log_2\left(\frac{1}{0.15}\right) + 0.12 \log_2\left(\frac{1}{0.12}\right)$

$+ 0.1 \times \log_2\left(\frac{1}{0.1}\right) + 0.08 \times \log\left(\frac{1}{0.08}\right)$

$= 0.3 \times 1.73 + 0.25 \times 2 + 0.15 \times 2.73 + 0.1 \times 3.32 + 0.08 \times 3.64$

$= 2.051$

Average Codeword length, $L(z) = \sum_{i=1}^{6} L(m_i) P(m_i)$

$= 0.3 \times 2 + 0.25 \times 2 + 0.15 \times 3 + 0.12 \times 3 + 0.10 \times 3 + 0.08 \times 3$

$= 0.6 + 0.5 + 0.45 + 0.36 + 0.3 + 0.24$

$= 2.45$

Coding Efficiency $= \frac{H(z)}{L(z)}$

$= \frac{2.051}{2.45}$

$= 0.8371$

$= 83.71\%$

Ans.(b) (i) Shanon - Fano code :

Message	Probability
x_1	0.4
x_2	0.19 ←Should be 0.19 as per logic
x_3	0.16
x_4	0.15
x_5	0.1

Step 1 - Arranging in order

x_1	0.4
x_2	0.19
x_3	0.16
x_4	0.15
x_5	0.10

Step 2 : Dividing in two partitions with nearly equal probabilities when combined until are partitioned with max size 1

x_1	0.4	0	00		00
x_2	0.19	0	01		01
x_3	0.16	1	10		10
x_4	0.15	1	11	110	110
x_5	0.10	1	11	111	111

Code entropy $H(z) = \sum_{i=1}^{5} p_i \log_2\left(\frac{1}{p_i}\right)$

$= 0.4 \log_2\left(\frac{1}{0.4}\right) + 0.19 \log_2\left(\frac{1}{0.19}\right)$

$+ 0.16 \log_2\left(\frac{1}{0.16}\right) + 0.15 \log_2\left(\frac{1}{0.15}\right)$

$+ 0.1 \log_2\left(\frac{1}{0.1}\right)$

$= 2.15$

Average Codeword Length $L(z) = \sum_{i=1}^{5} L(m_i) \times P(m_i)$

$= 0.4 \times 2 + 0.19 \times 2 + 0.16 \times 2 + 0.15 \times 3 + 0.11 \times 3$

$= 2.28$ bits/msg

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$$\text{Efficiency} = \frac{H(z)}{L(z)} = \frac{2.15}{2.28} = 0.943$$

$= 94.3\%$

(ii) Huffman code :

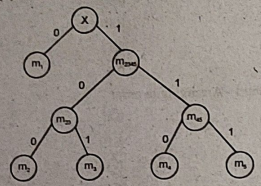


Fig.

Message	Huffman code
m_1	0
m_2	100
m_3	101
m_4	110
m_5	111

$$H(z) = \sum_{i=1}^{5} p_i \log_2\left(\frac{1}{p_i}\right)$$

$= 2.15$

$$L(z) = \sum_{i=1}^{5} L(m_i) \cdot P(m_i)$$

$= 1 \times 0.4 + 3 \times 0.19 + 3 \times 0.16 + 3 \times 0.15 + 3 \times 0.1$

$= 2.2$

$$\text{Efficiency} = \frac{H(z)}{L(z)} = \frac{2.15}{2.2} = 0.977$$

$= 97.7\%$

Efficiency of Huffman code is higher for the given set of messages.

Q.15 ***Explain prefix code with the help of an example and define its efficiency.*** ***[R.T.U. 2016]***

Ans. Prefix Code : *Refer to Q.5.*

Table shows four source symbols, their probabilities and the codewords and the codewords assigned to them by prefix coding.

Table : Prefix code

Source symbol	Probability of occurrence	Prefix code
s_0	0.5	0 ← Codeword
s_1	0.25	1 (prefix) 0 ← Codeword
s_2	0.125	11 (prefix) 0 ← Codeword
s_3	0.125	11 (prefix) 1 ← Codeword

In the table, observe that message s_0 has codeword '0'. Message s_1 has prefix of 1 and codeword of 1 Observe that no codeword is prefix of other codeword

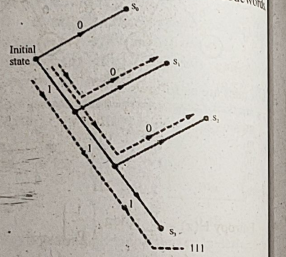


Fig. 1 : Decision tree for decoding prefix code

Fig. 1 shows the decision tree for decoding the prefix code of table.

As shown in fig.1 the tree has initial state. The decoder always starts from initial state. If the first received bit is '0' then the decoder decides in favour of message s_0 and goes to initial state. If the next received bit is 1, then the decoder goes one step down and waits for next bit. If next bit is '0' then decoder decides in favour of message $s_1(10)$ and goes to initial stage. This is how the decoder of prefix code works.

Prefix code efficiency : Prefix code is a type of code that let you decode the encoded text without special marker. For example, the map {a=0, b=10, c=11} is a prefix code as no marker is needed for decryption of any string. If you have '000101011' then it is clear when one list of bits for character ends and another begins you translate directly to 'aaabbc'. A counter example, the map {a=0, b=1, c=11} is not prefix code as the string '111' can be translated to 'bbb' or 'bc' i.e., we need some separator marker between list of bits in this code.

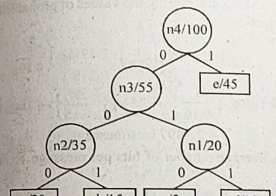
Coding efficiency : If the coding is represented by a map from a character c to a list of bits then we will define length (c) be the number of bits representing the character c and define frequency(c) to be the frequency that character c as appear in the text.

The code efficiency is calculated by

$$\sum_{c \in \text{Alphabet}} \text{length}(c) \times \text{frequency}(c)$$

i.e., the weight average of encoding lengths according to their frequencies.

Every Prefix code can also be represented as a binary tree where each edge is marked as '0' or '1' and the leaf are marked with a character so the list of edges to a leaf represent the character's code. Here is a picture that show this idea:



{a = 000, b = 001, c = 010, d = 011, e = 1}

Fig. 2

The depth of a leaf is actually the size of a character prefix code and so this representation provide an alternative to the efficient of the code in the language of trees i.e.,

$$B(T) = \sum_{c \in \text{Alphabet}} \text{depth}(c) \times \text{frequency}(c)$$

where depth (c) is the depth of the leaf c in the tree.

Q.16 ***(i) State and prove Kraft Inequality theorem.***
(ii) Consider a DMS with source probabilities {35, 25, 20, 15, 05}
(a) Determine the Shannon fano code for this source.
(b) Determine the average length $\bar{R}$ of the codewords.
(c) What is the efficiency η of the code ?
[R.T.U. 2012]

Ans. (i) Kraft Inequality Theorem : Kraft Inequality Theorem : A necessary and sufficient condition for the existence of a binary code with codewords having lenghts $n_1 \le n_2 \le \ldots n_L$ that satisfy the prefix condition is

$$\sum_{k=1}^{L} 2^{-n_k} \le 1 \qquad \ldots (1)$$

Proof : First we prove the sufficient condition. Consider a binary tree of order (depth) $n = n_L$. This tree has 2^n terminal nodes as depicted in Fig. Let us select any code of order n_1 as the first codeword c_1. Since no codeword is the prefix of any other codeword (the prefix condition), the choice eliminates 2^{n-n_1} terminal codes. This process continues until the last codeword is assigned at the terminal node $n = n_L$. Consider the node of order $j < L$. The fraction of number of terminal nodes eliminated is

$$\sum_{k=1}^{j} 2 - n_k < \sum_{k=1}^{L} 2^{-n_k} \le 1 \qquad \ldots (2)$$

Thus, we have been able to construct a prefix code that is embedded in the full tree of n_L nodes. The nodes that are eliminated are depicted by the dotted arrow lines leading on to them in fig.

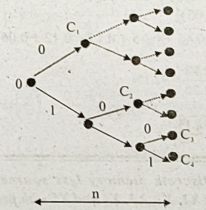


Fig : A binary tree of order n_L

We now prove the necessary condition. We observe that in the code tree of the order $n = n_L$, the number of terminal nodes eliminated from the total number of 2^n terminal nodes is

$$\sum_{k=1}^{L} 2^{n-n_k} \le 2^n. \qquad \ldots (3)$$

This leads to $\sum_{k=1}^{L} 2^{n-n_k} \le 1.$ $\qquad \ldots (4)$

We can easily extend this proof for prefix codes over an alphabet of size M. For the proof we will have to consider an array tree instead of a binary tree. The inequality in this case would become

$$\sum_{k=1}^{L} M^{-n_k} \le 1. \qquad \ldots (5)$$

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Ans. (ii) (a) By Shannon Fano Code

P(x)	Code	η
0.35	0 0	2
0.25	0 1	2
0.20	1 0	2
0.15	1 1 0	3
0.05	1 1 1	3

(b) $\bar{R} = \Sigma P_i(x_i).n$

$= 0.35 \times 2 + 0.25 \times 2 + 0.20 \times 2 + 0.15 \times 3 + 0.05 \times 3$

$= 0.70 + 0.50 + 0.40 + 0.45 + 0.15 = 2.20$

(c) $\eta = \dfrac{H(X)}{\bar{R}}$

$$H(X) = \sum_{i=1}^{5} P_i(x_i).\log\left[\frac{1}{P(x_i)}\right] = -[0.35 \log (0.35) + 0.25 \log (0.25) + 0.20 \log (0.20) + 0.15 \log (0.15) + 0.05 \log (0.05)]$$

$= +[0.15 + 0.15 + 0.13 + 0.12 + 0.06] = 2.02$

$$\eta = \frac{2.02}{2.20} = 92.1\%$$

$\boxed{\eta = 92.1\%}$

Q.17 ***(a) A discrete memory less source has five symbols X1, X2, X3, X4 and X5 with probabilities 0.4, 0.19, 0.16, 0.15 and 0.1 respectively attached to every symbol. Construct Shannon-Fano code for the source and calculate the code efficiency.***

(b) A channel has the following channel matrix

$$\left[P\left(\frac{y}{x}\right) = \begin{bmatrix} 1-p & p & 0 \\ 0 & p & 1-p \end{bmatrix}\right]$$

(i) Draw the channel diagram

(ii) If the source has equally likely outputs, compare the probabilities associated with the channel output for p = 0.4 and calculate $H(x), H(y)$ ***and*** $H\left(\frac{y}{x}\right)$. ***[R.T.U. 2011, 2010]***

Ans. (a) *Table : To obtain Shannon-Fano code*

Message	Probability of message	I	II	III	Code word for message	Number of bits per message n_k
x_1	0.4	0			0	1
x_2	0.19	1	0	0	100	3
x_3	0.16	1	0	1	101	3
x_4	0.15	1	1	0	110	3
x_5	0.1	1	1	1	111	3

The entropy (H) is given as,

$$H = \sum_{k=1}^{M} p_k \log_2\left(\frac{1}{p_k}\right)$$

Here M = 5 and putting the values of probabilities in above equation,

$$H = 0.4 \log_2\left(\frac{1}{0.4}\right) + 0.19 \log_2\left(\frac{1}{0.19}\right) + 0.16 \log_2\left(\frac{1}{0.16}\right) + 0.15 \log_2\left(\frac{1}{0.15}\right) + 0.1 \log_2\left(\frac{1}{0.1}\right)$$

$= 2.1497$ bits/message

The average number of bits per message $\bar{N}$ is

$$\bar{N} = \sum_{k=1}^{L} p_k n_k$$

Here p_k is the probabillty of k^{th} message and n_k is number of bits assigned to it. Putting the values in above equation.

$\bar{N} = 0.4(1) + 0.19(3) + 0.16(3) + 0.15(3) + 0.1(3)$

$= 2.2$

The code efficiency is given by equation i.e.

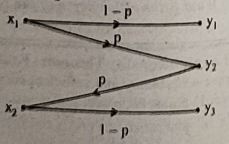
Code efficiency $\eta = \dfrac{H}{\bar{N}}$

$$= \frac{2.1497}{2.2} = 0.977$$

Ans. (b) Given

$$p\left(\frac{y}{x}\right) = \begin{matrix} \\ x_1 \\ x_2 \end{matrix} \begin{matrix} y_1 & y_2 & y_3 \end{matrix} \begin{bmatrix} 1-p & p & 0 \\ 0 & p & 1-p \end{bmatrix}$$

(i) Channel Diagram



(ii) Given that source has equal likely output. Hence

$$p(x_1) = p(x_2) = \frac{1}{2}$$

then the output probabilities are

$$\begin{bmatrix} p(y_1) \\ p(y_2) \\ p(y_3) \end{bmatrix} = [p(x_1)\, p(x_2)] \begin{bmatrix} 1-p & p & 0 \\ 0 & p & 1-p \end{bmatrix}$$

but given that p = 0.4

therefore

$$\begin{bmatrix} p(y_1) \\ p(y_2) \\ p(y_3) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1-0.4 & 0.4 & 0 \\ 0 & 0.4 & 1.04 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0.6 & 0.4 & 0 \\ 0 & 0.4 & 0.6 \end{bmatrix}$$

$$\begin{bmatrix} p(y_1) \\ p(y_2) \\ p(y_3) \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.4 \\ 0.3 \end{bmatrix}$$

Thus $p(y_1) = 0.3$

$p(y_2) = 0.4$

$p(y_3) = 0.3$

(a) $$H(x) = \sum_{k=1}^{2} p_k \log_2\left(\frac{1}{p_k}\right)$$

$$H(x) = p_1 \log_2\left(\frac{1}{p_1}\right) + p_2 \log_2\left(\frac{1}{p_2}\right)$$

$$H(x) = \frac{1}{2}\log_2(2) + \frac{1}{2}\log_2(2)$$

$H(x) = 1$ bit/message

(b) $$H(y) = \sum_{k=1}^{3} p_k \log_2\left(\frac{1}{p_k}\right)$$

$$H(y) = p_1 \log_2\left(\frac{1}{p_1}\right) + p_2 \log_2\left(\frac{1}{p_2}\right) + p_3 \log_3\left(\frac{1}{p_3}\right)$$

$$H(y) = 0.3 \log\left(\frac{1}{0.3}\right) + 0.4 \log\left(\frac{1}{0.4}\right) + 0.3 \log\left(\frac{1}{0.3}\right)$$

$$H(y) = \frac{0.313 + 0.159}{\log_2}$$

$H(y) = 1.56$ bit/message

(c) We know that

$$H\left(\frac{y}{x}\right) = H(x_i y) - H(x)$$

but $$p\left(\frac{y}{x}\right) = \begin{bmatrix} 0.6 & 0.4 & 0 \\ 0. & 0.4 & 0.6 \end{bmatrix}$$

$$H\left(\frac{y}{x}\right) = 0.6 \log_2\left(\frac{1}{0.6}\right) + 0.4 \log_2\left(\frac{1}{0.4}\right) + 0.4 \log_2\left(\frac{1}{0.4}\right) + 0.6 \log_2\left(\frac{1}{0.6}\right)$$

$$H\left(\frac{y}{x}\right) = \frac{0.266 + 0.318}{0.3010} \text{ bit/message}$$

$$H\left(\frac{y}{x}\right) = 1.94 \text{ bit/message}$$

□□□

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3 LINEAR BLOCK CODE

PREVIOUS YEARS QUESTIONS

PART-A

Q.1 ***Define code word.*** *[R.T.U. 2016]*

Ans. The channel encoder separates or segments the incoming bit stream (the output of the source encoder) into equal length blocks of L binary digits and maps each L-bit message block into an N-bit code word where N> L and the extra N – L check bits provide the required error protection. There are $M = 2^L$ message and thus 2^L code words of length N bits. The channel decoder maps the received N –bit word to the most likely code word and inversely maps the N –bit code word to the corresponding L-bit message.

Code Word : The encoded block of N bits is called a code word. It contains message bits and redundant bits.

Q.2 ***Define block length.*** *[R.T.U. 2016]*

Ans. Block Length : The number of bits N after coding is called the block length of the code.

Q.3 ***Define code rate.*** *[R.T.U. 2016]*

Ans. Code Rate: The ratio of message bits (K) and the encoder output bits (N) is called code rate. Code rate is defined by 'r' i.e.,

$$r = \frac{K}{N}$$

We find that $0 < r < 1$.

Q.4 ***Define channel data rate.*** *[R.T.U. 2016]*

Ans. Channel Data Rate : It is the bit rate at the output of encoder. If the bit rate at the input of encoder is R_s then channel data rate will be,

Channel data rate

$$(R_0) = \frac{N}{K} R_S$$

Q.5 ***What are content errors?***

Ans. Content Errors : The content errors are nothing but errors in the contents of a message i.e., a 0 may be received as 1 or vice-versa.

PART-B

Q.6 ***Consider a (6, 3) linear block code whose generator matrix is given by***

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

(a) Find the parity check matrix
(b) Find the minimum distance of the code.
(c) Draw the encoder and syndrome computation circuit. *[R.T.U. 2017]*

Ans.(a)

Generator Matrix = $\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$

Standard form $G = [I_k | A]$

Parity Check Matrix = $[-A^T | I_{n-k}]$

$$H = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

(b) To find minimum distance, we use the property that the minimum distance of a binary linear codes is equal to the smallest number of columns of the parity check matrix that sums up to zero.

Cleary all columns of H are non-zero, and they are all distinct

So, $d \geq 3$

Moreover, we can conclude that d = 3 by adding first 3 columns of H :-

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence, minimum distance of code is 3.

(c) Encoder circuit –

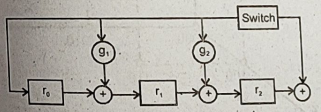

Fig.

Syndrome computation circuit

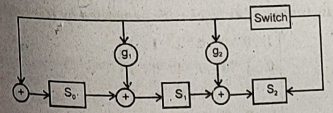

Fig.

Q.7 ***Differentiate between systematic and non-systematic codes. Give example of each.*** *[R.T.U. 2017]*

Ans. In coding theory, a systematic code is any error-correcting code in which the input data is embedded in the encoded output. Conversely, in a non-systematic code the output does not contain the input symbols.

Systematic codes have the advantage that the parity data can simply be appended to the source block, and receivers do not need to recover the original source symbols if received correctly - this is useful for example if error-correction coding is combined with a hash function for quickly determining the correctness of the received source symbols, or in cases where errors occur in erasures and a received symbol is thus always correct. Furthermore, for engineering purposes such as synchronization and monitoring, it is desirable to get reasonable good estimates of the received source symbols without going through the lengthy decoding process which may be carried out at a remote site at a later time.

Examples:

- Checksums and hash functions, combined with the input data, can be viewed as systematic error-detecting codes.
- Linear codes are usually implemented as systematic error-correcting codes (e.g., Reed-Solomon codes in CDs).
- Convolutional codes are implemented as either systematic or non-systematic codes. Non-systematic convolutional codes can provide better performance under maximum-likelihood (Viterbi) decoding.

Q.8 ***Given a (6,3) linear block code with the following parity check matrix H:***

$$H = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

(i) Find the generator matrix G.
(ii) Find the code word for data bit 110. *[R.T.U. 2016]*

Ans. (i) To obtain the generator matrix:

$$H = \left[P^T : I_q\right]_{q \times n}$$

$$H = \left[\begin{array}{ccc:ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

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$$P^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Hence $$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

The generator matrix is given as

$$G = [I_k : P_{kxq}]_{kxn}$$

$$\therefore \quad G = \begin{bmatrix} 1 & 0 & 0 & : & 1 & 0 & 1 \\ 0 & 1 & 0 & : & 0 & 1 & 1 \\ 0 & 0 & 1 & : & 1 & 1 & 1 \end{bmatrix}$$

(ii) To obtain the codeword for data bit 110 :

$$M = [110]$$

This is (6,3) code. The three check bits can be obtained by equation :

$$C = MP = [110]\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= [1\oplus 0\oplus 00\oplus 1\oplus 01\oplus 1\oplus 0]$$

$$= [110]$$

Code vector,

$$X = (m_1, m_2, m_3, c_1, c_2, c_3) = (110\ 110)$$

Code word = (110 110)

Q.9 ***Define minimum distanced d_{min} of Hamming code. Differentiate between Hamming distance and minimum distance. How minimum distance is related to error detection capability?***

[R.T.U. 2016]

Ans. Hamming Distance : The hamming distance between the two code vectors is equal to the number of elements in which they differ. For example, let X = (101) and Y = (110). The two code vectors differ in second and third bits. Therefore hamming distance between X and Y is 'two'. Hamming distance is denoted as d(X,Y) or simply 'd'. i.e.

$$d(X,Y) = d = 2$$

Thus we observe from Fig. that the hamming distance between (100) and (011) is maximum i.e. 3. This is indicated by the vector diagram also.

Minimum Distance (d_{min}) : It is the smallest hamming distance between the valid code vectors.

Error detection is possible if the received vector is not equal to some other code vector. This shows that the transmission errors in the received code vector should be less than minimum distance d_{min}. The table lists some of the requirements of error control capability of the code.

Table : Error control capabilities

Sr. No.	Name of errors detected/corrected	Distance requirement
1	Detect upto 's' errors per word	$d_{min} \geq s+1$
2	Correct upto 't' errors per word	$d_{min} \geq 2t+1$
3	Correct upto 't' errors and detect $s > t$ errors per word	$d_{min} \geq t+s+1$

For the (n,k) block code the minimum distance is given as,

$$d_{min} \leq n - k + 1$$

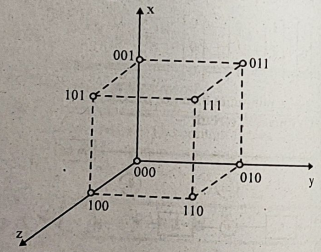

Fig. Code vectors representing 3-bit code words

Q.10 ***Given a (7,4) block code generated by (G) below:***

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

(i) Find all the code words of the code.

(ii) Find H, parity check matrix of the code.

[R.T.U. 2016]

Ans. (i) Codeword = DG

Message (D)				Codeword						
0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	1	1	1	0
0	0	1	0	0	0	1	0	0	1	1
0	0	1	1	0	0	1	1	1	0	1
0	1	0	0	0	1	0	0	1	0	1
0	1	0	1	0	1	0	1	0	1	1
0	1	1	0	0	1	1	0	1	1	0
0	1	1	1	0	1	1	1	0	0	0
1	0	0	0	1	0	0	0	1	1	1
1	0	0	1	1	0	0	1	0	0	1
1	0	1	0	1	0	1	0	1	0	0
1	0	1	1	1	0	1	1	0	1	0
1	1	0	0	1	1	0	0	0	1	0
1	1	0	1	1	1	0	1	1	0	0
1	1	1	0	1	1	1	0	0	0	1
1	1	1	1	1	1	1	1	1	1	1

(ii)
$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$= [I_4 : P]$$

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$P^T = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$H = [P^T : I_{7-4}]$$

$$H = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Q.11 ***The Parity check matrix of a (7, 4) Hamming code is as under***

$$H = \begin{bmatrix} 1101100 \\ 1110011 \\ 1011001 \end{bmatrix}$$

Calculate the syndrom vector for single bit errors.

[R.T.U. 2012]

Ans. n = 7

k = 4

q = n − k = 3

Error Pattern for Single Bit Error

Bit in error	Bits of error (E)						(non-zero bit)
1st	1	0	0	0	0	0	0
2nd	0	1	0	0	0	0	0
3rd	0	0	1	0	0	0	0
4th	0	0	0	1	0	0	0
5th	0	0	0	0	1	0	0
6th	0	0	0	0	0	1	0
7th	0	0	0	0	0	0	1

Syndrom Calculation

$$S = EH^T$$

$$H^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Syndrom for 1st Bit Error

$$S = EH^T$$

$$S = [1000000]\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$S = [1\ 0\ 1]$$

i.e. syndrom vector for 1st bit in error.

Similary syndrom table can be drawn as

S.No.	Error Vector (∈)	Syndrom (s)	Error
1.	0 0 0 0 0 0 0	0 0 0	← 1st row of H^T
2.	1 0 0 0 0 0 0	1 1 1	← 2nd row of H^T
3.	0 1 0 0 0 0 0	1 1 0	← 3rd row of H^T
4.	0 0 1 0 0 0 0	0 1 1	← 4th row of H^T
5.	0 0 0 1 0 0 0	1 0 1	← 5th row of H^T
6.	0 0 0 0 1 0 0	1 0 0	← 6th row of H^T
7.	0 0 0 0 0 1 0	0 1 0	← 7th row of H^T
8.	0 0 0 0 0 0 1	0 1 1	← 7th row of H^T

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PART-C

Q.12 *Explain the need of error correcting codes. How its Encoding/Decoding take place ? Explain with help of parity example.* *[R.T.U. 2018]*

Ans. Need of Error Correcting Codes : Applications that require low latency (such as telephone conversations) cannot use Automatic Repeat reQuest (ARQ); they must use forward error correction (FEC). By the time an ARQ system discovers an error and re-transmits it, the re-sent data will arrive too late to be any good.

Applications where the transmitter immediately forgets the information as soon as it is sent (such as most television cameras) cannot use ARQ; they must use FEC because when an error occurs, the original data is no longer available. (This is also why FEC is used in data storage systems such as RAID and distributed data store).

Applications that use ARQ must have a return channel; applications having no return channel cannot use ARQ. Applications that require extremely low error rates (such as digital money transfers) must use ARQ. Reliability and inspection engineering also make use of the theory of error-correcting codes.

Internet

In a typical TCP/IP stack, error control is performed at multiple levels:

- Each Ethernet-frame carries a CRC-32checksum. Frames received with incorrect checksums are discarded by the receiver hardware.
- The IPv4 header contains a checksum protecting the contents of the header. Packets with mismatching checksums are dropped within the network or at the receiver.
- The checksum was omitted from the IPv6 header in order to minimize processing costs in network routing and because current link layer technology is assumed to provide sufficient error detection (see also RFC 3819).
- UDP has an optional checksum covering the payload and addressing information from the UDP and IP headers. Packets with incorrect checksums are discarded by the operating system network stack. The checksum is optional under IPv4, only, because the Data-Link layer checksum may already provide the desired level of error protection.
- TCP provides a checksum for protecting the payload and addressing information from the TCP and IP headers. Packets with incorrect checksums are discarded within the network stack, or eventually get retransmitted using ARQ, either explicitly (such as through triple-ack) or implicitly due to a timeout.

Deep-space telecommunications

Development of error-correction codes was tightly coupled with the history of deep-space missions due to the extreme dilution of signal power over interplanetary distances and the limited power availability in aboard spacecraft probes. Whereas early missions sent their data uncoded, starting from 1968, digital error correction was implemented in the form of (sub-optimally decoded) convolutional codes and Reed–Muller codes. The Reed–Muller code was well suited to the noise, the spacecraft was subject to (approximately matching a bell curve), and was implemented at the Mariner spacecraft for missions between 1969 and 1977.

The Voyager 1 and Voyager 2 missions, which started in 1977, were designed to deliver color images amongst scientific information of Jupiter and Saturn. This resulted in increased coding requirements, and thus, the spacecraft were supported by (optimally Viterbi-decoded) convolutional codes that could be concatenated with an outer Golay (24,12,8) code.

The Voyager 2 craft additionally supported an implementation of a Reed–Solomon code: the concatenated Reed–Solomon–Viterbi (RSV) code allowed for very powerful error correction, and enabled the spacecraft's extended journey to Uranus and Neptune. Both crafts used V2 RSV coding due to ECC system upgraded after 1989.

The CCSDS currently recommends usage of error correction codes with performance similar to the Voyager 2 RSV code as a minimum. Concatenated codes are increasingly falling out of favor with space missions, and are replaced by more powerful codes such as Turbo codes or LDPC codes.

The different kinds of deep space and orbital missions that are conducted suggest that trying to find a "one size fits all" error correction system will be an ongoing problem for some time to come. For missions close to Earth, the nature of the noise in the communication channel is different from that which a spacecraft on an interplanetary mission experiences. Additionally, as a spacecraft increases its distance from Earth, the problem of correction of noise gets bigger.

Satellite broadcasting (DVB)

The demand for satellite transponder bandwidth continues to grow, fueled by the desire to deliver television (including new channels and high-definition television) and IP data. Transponder availability and bandwidth constraints

have limited this growth, because transponder capacity is determined by the selected modulation scheme and forward error correction (FEC) rate.

Overview

- QPSK coupled with traditional Reed-Solomon and Viterbi codes have been used for nearly 20 years for the delivery of digital satellite TV.
- Higher order modulation schemes such as 8PSK, 16QAM and 32QAM have enabled the satellite industry to increase transponder efficiency by several orders of magnitude.
- This increase in the information rate in a transponder comes at the expense of an increase in the carrier power to meet the threshold requirement for existing antennas.
- Tests conducted using the latest chipsets demonstrate that the performance achieved by using Turbo Codes may be even lower than the 0.8 dB figure assumed in early designs.

Data storage

Error detection and correction codes are often used to improve the reliability of data storage media. A "parity track" was present on the first magnetic tape data storage in 1951. The "Optimal Rectangular Code" used in group coded recording tapes not only detects but also corrects single-bit errors. Some file formats, particularly archive formats, include a checksum (most often CRC32) to detect corruption and truncation and can employ redundancy and/or parity files to recover portions of corrupted data. Reed-Solomon codes are used in compact discs to correct errors caused by scratches.

Modern hard drives use CRC codes to detect and Reed–Solomon codes to correct minor errors in sector reads, and to recover data from sectors that have "gone bad" and store that data in the spare sectors. RAID systems use a variety of error correction techniques to correct errors when a hard drive completely fails. Filesystems such as ZFS or Btrfs, as well as some RAID implementations, support data scrubbing and resilvering, which allows bad blocks to be detected and (hopefully) recovered before they are used. The recovered data may be re-written to exactly the same physical location, or to spare blocks elsewhere on the same piece of hardware, or the data may be rewritten onto replacement hardware.

Error correcting code using parity encoding and decoding:

Parity bit checking is used occasionally for transmitting ASCII characters, which have 7 bits, leaving the 8th bit as a parity bit.

For example, the parity bit can be computed as follows. Assume Alice and Bob are communicating and Alice wants to send Bob the simple 4-bit message 1001.

Type of bit parity	Successful transmission scenario
Even parity	Alice wants to transmit: 1001 Alice computes parity bit value: 1+0+0+1 (mod 2) = 0 Alice adds parity bit and sends: 10010 Bob receives: 10010 Bob computes parity: 1+0+0+1+0 (mod 2) = 0 Bob reports correct transmission after observing expected even result.
Odd parity	Alice wants to transmit: 1001 Alice computes parity bit value: 1+0+0+1 (mod 2) = 0 Alice adds parity bit and sends: 10011 Bob receives: 10011 Bob computes overall parity: 1+0+0+1+1 (mod 2) = 1 Bob reports correct transmission after observing expected odd result.

This mechanism enables the detection of single bit errors, because if one bit gets flipped due to line noise, there will be an incorrect number of ones in the received data. In the two examples above, Bob's calculated parity value matches the parity bit in its received value, indicating there are no single bit errors. Consider the following example with a transmission error in the second bit using XOR:

Type of bit parity error	Failed transmission scenario
Even parity error in the second bit	Alice wants to transmit: 1001 Alice computes parity bit value: 1 ^ 0 ^ 0 ^ 1 = 0 Alice adds parity bit and sends: 10010 **...TRANSMISSION ERROR...** Bob receives: 11010 Bob computes overall parity: 1 ^ 1 ^ 0 ^ 1 ^ 0 = 1 Bob reports incorrect transmission after observing unexpected odd result.
Even parity error in the parity bit	Alice wants to transmit: 1001 Alice computes even parity value: 1 ^ 0 ^ 0 ^ 1 = 0 Alice sends: 10010 **...TRANSMISSION ERROR...** Bob receives: 10011 Bob computes overall parity: 1 ^ 0 ^ 0 ^ 1 ^ 1 = 1 Bob reports incorrect transmission after observing unexpected odd result.

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There is a limitation to parity schemes. A parity bit is only guaranteed to detect an odd number of bit errors. If an even number of bits have errors, the parity bit records the correct number of ones, even though the data is corrupt. Consider the same example as before with an even number of corrupted bits:

Type of bit parity error	Failed transmission scenario
Even parity error in two corrupted bits	Alice wants to transmit: 1001 Alice computes even parity value: 1 ^ 0 ^ 0 ^ 1 = 0 Alice sends: 10010 ...TRANSMISSION ERROR... Bob receives: 11011 Bob computes overall parity: 1 ^ 1 ^ 0 ^ 1 ^ 1 = 0 Bob reports correct transmission though actually incorrect.

Bob observes even parity, as expected, thereby failing to catch the two bit errors.

Q.13 *Explain the type of errors and classification of codes.* ***[R.T.U. 2018, 2017, 2012, 2010]***

Ans. Type of errors

The errors introduced in the transmitted data during their transmission may be categorized as under

(i) Content errors

(ii) Flow integrity errors

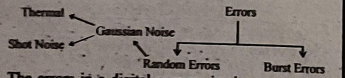
Content errors:

The content errors are nothing but errors in the contents of a message i.e., a 0 may be received as 1 or vice-versa.

Flow integrity errors:

Flow integrity errors meaning missing blocks of data. It is possible that a data block may be lost in the between as it has been delivered to a wrong destination.

Types of Errors



The errors in a digital communication system are caused by noise in the communication channel (Gaussian noise introduce in analog part of common channel).

Random errors due to white Gaussian noise are introduced. Gaussian noise had been our chief concern in designing and evaluating modulators and demodulators. Sources of Gaussian are :

(a) Thermal Noise : Due to vibration of individual molecules about their position of equilibrium in a crystal lattice, the conduction electron of metals wander randomly throughout the volume of metal, similarly molecule of an enclosed gas are in constant motion colliding with one another and colliding also with the walls of container. These agitations of molecules are called thermal agitations because they increase with temperature.

(b) Shot Noise : Result from a phenomenon associated with flow of current across semiconductor junctions. The charge carriers, electrons or holes enter the junction region from one side, drift or are accumulated at the junction and are collected on other side. The average junction current determines the average interval that elapses between time when two successive carriers enter the junction. The exact interval that elapses is subject to random fluctuations. This randomness give rise to shot noise. As we know that power spectral density of Gaussian noise at receiver input is white Gaussian noise. The transmission errors introduced during a particular interval by white Gaussian noise does not affect the performance of system during subsequent signalling interval.

(c) Burst Errors : Which is due to impulse noise by long quite interval's followed by high amplitude noise burst. This type of noise occurs from many natural and man-made causes such as lightning and switching transients. When such noise occurs, it affects more than one symbol or bit and there is usually a dependence of errors in successive transmitted symbols.

Error control schemes for dealing with random errors are random error correcting codes and coding scheme designed to correct burst errors are burst over correcting codes.

Shot Noise : Shot noise appears in active devices due to the random behaviour of charge carriers (electrons and holes). In electron tubes, shot noise is generated due to the random emission of electrons from cathodes; in semiconductor devices, it is caused due to the random diffusion of minority carriers or random generation and recombination of electron-hole pairs.

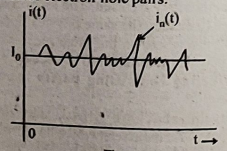


Fig.

Current in electron devices (tubes or solid state) flows in the form of discrete pulses, every time a charge carrier moves from one point to the other (e.g., cathode to plate).

Therefore, although current appears to be continuous, it is still a discrete phenomena. The nature of current variation with time is shown in fig.

The current fluctuates about a mean value I_0. This current $i_n(t)$ which wiggles around the mean value is known as shot noise. The wiggling nature of the current is not visualized by normal instruments and normally we think that the current is a constant equal to I_0. The wiggling nature of the current can be observed in a fast sweep oscilloscope.

The total current i(t) may be expressed as

$$i(t) = I_0 + i_n(t) \quad \ldots(1)$$

where I_0 is the constant (mean) and $i_n(t)$ is the fluctuating (noise) current.

Power Density Spectrum of Shot Noise in Diodes : The time varying component $i_n(t)$ of the current i(t) specified by eq.(1) is random in nature and it cannot be expressed as a function of time, i.e. it is an indeterministic function. However, this indeterministic stationary random $i_n(t)$ can be specified by its power density spectrum.

The number of electrons contributing to the random stationary current $i_n(t)$ are large. Assuming that the electrons do not interact with each other during their movement or emission (e.g., temperature limited diode current), the process may be considered statistically independent. According to central limit theorem, such a process has a Gaussian distribution. Hence, shot-noise is Gaussian-distribution with a zero mean.

The total diode current may be taken as the sum of the current pulses, each pulse being formed by the transit of an electron from cathode to anode. It can be seen that for all practical purposes the power density spectrum of the statistically independent non-interacting random noise current $i_n(t)$ is given by:

$$S_i(\omega) = qI_0 \quad \ldots(2)$$

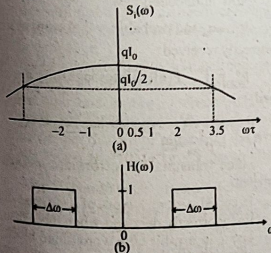


Fig. Shot Noise : (a) Power Density Spectrum (b) Bandwidth of Measuring System

where q is the electronic charge ($q = 1.59 \times 10^{-10}$) coulombs and I_0 is the mean value of the current in amperes. The power density spectrum in eq.(2) is frequency independent. This type of frequency independence is only up to a frequency range decided by the transit time of an electron to reach from anode to cathode. Beyond this frequency range, the power density varies with frequency as shown in fig.(a). The transit time of an electron, in a diode depends on anode voltage V and is given as

$$\tau = 3.36 \times \frac{d}{\sqrt{V}} \mu \text{sec.}$$

where d is spacing between anode and cathode.

For instance, in a diode with d = 2mm and V = 40 volts, we have $\tau \cong 10^{-3}$ μ sec. In fig.(a), the power density curve may be considered flat close to the origin, i.e. $|\omega\tau| \leq 0.5$. Therefore $S_i(\omega)$ can be considered constant up to $|\omega\tau| = 0.5$. For $\tau = 10^{-3}$ μ sec. The maximum frequency up to which power density $S_i(\omega)$ remains constant is given by

$$\omega = 0.5 \times 10^9 = 5 \times 10^8 \text{ rad/s}$$

This is equivalent to a linear frequency $(f) = \frac{\omega}{2\pi} = 80$ MHz.

Therefore for all practical purposes, the $S_i(\omega)$ may be considered to be frequency-independent below 100 MHz.

Resistor Noise : The noise arising due to random motion of free charged particles (usually electrons) in a conducting media, such as a resistor, is called resistor noise. This noise is also known as Jhonson noise, after J.B. Johnson who, investigated this type of noise in conductors. The random agitation is a universal phenomenon at atomic levels and is caused by the energy supplied through flow of heat. The intensity of random motion is proportional to thermal (heat) energy supplied (*i.e.* temperature) and is zero at a temperature of absolute zero. This noise is also known as thermal noise. The path of the electron motion is random because of their collisions with lattice structure. The net motion of all the electrons gives rise to an electric current to flow through the resistor, causing the noise.

Power Density Spectrum of Resistor Noise :

The free electrons contributing to resistor noise are large in number. If their random motion is assumed to be statistically independent, then the central limit theorem predicts the resistor noise to be, Gaussian, distributed with a zero mean. It can be shown that the power density spectrum of the current contributing the thermal noise is given by :

$$S_i(\omega) = \frac{2kTG}{1 + \left(\frac{\omega}{\alpha}\right)^2} \quad \ldots(1)$$

where T is ambient temperature in degree Kelvin, G is the conductance of the resistor in mhos, k is the Boltzmann constant and α is the average number of collisions per second per electron.

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The variation of power density spectrum with frequency is shown in Fig.

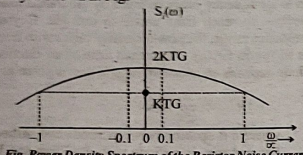

Fig. Power Density Spectrum of the Resistor Noise Current

It is obvious from the figure that the spectrum may be considered to be flat for $\frac{\omega}{\alpha} \le 0.1$. The power density spectrum $S_i(\omega)$ for this range of frequency is nearly constant and is given by

$$S_i(\omega) = 2kTG \quad ...(2)$$

The value of α is of the order of 10^{14} and hence the frequency corresponding to $\frac{\omega}{\alpha} < 0.1$ is of order of 10^{13} Hz. Therefore, the frequency independent expression of $S_i(\omega)$ given by eq. (2) holds up to a frequency range of 10^{13}Hz. This range covers almost all the practical applications in communication systems. Hence, for all practical purposes, the power density spectrum $S_i(\omega)$ is considered to be independent of frequency.

Classification of codes :

The codes are basically classified as under:

(i) Errors detecting codes: The error detecting codes are capable of may detecting the errors. They cannot connect errors.

(ii) Error correcting codes

(1) Block codes (2) Convolution codes

The error correcting codes are capable of detecting as well as correcting the errors. These codes can be classified into block codes and convolution codes or linear and non-linear codes.

For error- free transmission, following codes are used :

(A) Block Codes

(B) Burst and Random Error Correcting Codes

(C) Interleaving

(A) Block Codes

(i) For Error Correction

1. We compare the performance of system using block codes for error correction with systems (n,k) using no error control coding.
2. Two measures of performance are :
   (a) Problem of incorrectly decoding a message bit.
   (b) Problem of incorrectly decoding a block of message digits.
3. We will do the comparison on the condition that rate of information transmission is same for coded and uncoded systems and both systems are operating with average signal power and noise power spectral density.
4. Coded or uncoded a block of say k message bits must be transmitted in duration of time.

$$T_w = \frac{k}{r_b}$$

where r_b = message bit rate

5. $\therefore\ r_w = \frac{1}{T_w} = \frac{r_b}{k}$

if system uses an (n, k) block code, then bit rate going into channel

$$r_c = r_b\left(\frac{n}{k}\right) \text{ or } r_c > r_b$$

6. Now

r_b = Message bit rate.

r_c = Channel bit rate.

q_c = Channel bit error probability for coded system.

q_u = Channel bit error probability for uncoded system.

P_{be}^u = Probability of incorrectly decoding a message bit in uncoded system.

P_{be}^c = Probability of incorrectly decoding a message bit in coded system.

P_{we}^u = Probability of incorrectly decoding a word of message bits in uncoded system.

P_{we}^c = Probability of incorrrectly decoding a word if message bits in coded system.

7. Now in uncoded case

$P_{be}^u = q_u$ and probability that word of k message bit incorrectly received.

$P_{we}^u = 1 - P$(all k message bits are correctly received)

$$= 1 - (1 - q_u)\,k \text{ when } kq_u \le 1$$

$$= p^4{}_{we} = kq_u$$

since transmission errors are assumed to be independent.

8. In coded system, a word of k message digits will be incorrectly decoded when more than t errors occur in a n-bit codeword since block code is assumed to be able to correct upto t errors.

Thus

$P_{we}^c = P$(t + 1 or more errors in a codeword)

$$P_{we}^c = \sum_{i=t+1}^{n}\binom{n}{i} q_c^i (1-q_c)^{n-i} = \sum_{i=t+1}^{n} P(n,i)$$

where $P(n,i) = \binom{n}{i} q_c\,(1-q_c)^{n-i}$

$$\binom{n}{i} = \frac{n!}{i!(n-i)!};\ r_{bw} = \frac{r_b}{k}$$

$$r_b P_{be}^c = r_w\,(t+1)\frac{k}{n} P_{we}^c$$

If $nq_c \le 1, P(n,i+1) \le p(n,i)$

$$P_{we}^c = \binom{n}{t+1}(q_c)^{t+1}(1-q_c)^{n-t-1}$$

P_{be}^c = message bit error probability.

P_{be}^c implies that majority of decoding errors are due to (t + 1) bit errors in an n-bit codeword.

Out of (t + 1) error, the fraction k/n represent the erroneous message bits. Hence average message bit error rate.

$$r_b P_{be}^c = r_w\,(t+1)\frac{k}{n}\cdot P_{we}^c$$

$$\therefore\ P_{be}^c = \frac{(t+1)}{n} P_{we}^c$$

(ii) For Error Detection

1. We compare the performance of data transmission system using block codes for error detection with systems using direct transmission without error control coding.
2. We do the comparison under the assumed same assumption that S_{au}, η, r_b remain same for both coded and uncoded systems.
3. We use probability of incorrectly decoding a block of k message bits as our measure of comparative performance.
4. Here we assume that (n, k) block code is capable of detecting upto 2t errors per block.
5. Decoder checks the received codewords for errors and when error is deduced, the decoder may either discard or retransmit the message.
6. We know that data rate r_c over channel is $r_b\left(\frac{n}{k}\right)$ when an (n , k) error correcting block code is used. The data rate r_c will have to be higher than $r_b\left(\frac{n}{k}\right)$ for (n, k) error detecting block codes because of retransmission.

Stop and Wait Transmission Method :

1. The transmitter begins transmission at time say t_0 and completes the transmission of block of n bits at time $t_0 + t_n$.
2. Decoder starts receiving message block at time $+\Delta$ where Δ is propagation delay.
3. At time $t_0 + \Delta + t_n$ the decoder checks the n–bit block that was received and sends a positive acknowledgement (ACK) or a negative acknowledgement (NACK) to transmitter depending upon whether or not it detects an error in received block.
4. If ACK is positive, then next message block is transmitted if not previous is retransmitted. In either case transmitter is waiting from $t_0 + t_n$ to $t_0 + t_n + 2\Delta$ for acknowledgement from receiver.
5. To average channel all error rate blocks. Let us consider the transmission of N blocks of data over channel at rate r_c bits per second.
6. Total time needed to complete transmission we know r_c - bits per second rate.

$$T = \frac{1}{r_c}$$

Total n bits

$$\therefore\ T = \frac{n}{r_c}$$

Delay = 2Δ

$$T = \left(\frac{n}{r_c} + 2\Delta\right)$$

$$\therefore\ \text{for N block time} = N\left(\frac{n}{r_c} + 2\Delta\right)$$

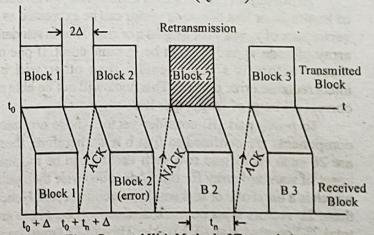

Fig. : Stop and Wait Method of Transmission

7. P_{we}^c = Probability of incorrectly decoding a message block in coded system for N blocks.

NP_{we}^c – Block in errors

$\therefore\ N(1 - P_{we}^c)$ – Blocks are correctly accepted by receiver block having k message bits

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∴ Time taken

$$\frac{Nk\left(1-P_{we}^{c}\right)}{r_b} = \frac{N(n+2\Delta r_c)}{t_c}$$

(B) Burst and Random Error Correcting Codes

As we know that in some coding techniques error occurs either at random or in bursts for channels in which both random or burst errors occur. It is better to design codes capable of correcting random errors and/ or single or multiple errors.

Burst of length q is defined as a vector whose non-zero components are confined to q consecutive digits position, the first and last of which is non-zero.

for example

v = (0010100001000)

Burst of length = 7

A code that is capable of correcting all burst errors of length q or less is called as q–burst error correcting code or code is said to have burst correcting ability.

Theorem : The number of parity check bits of a q burst error correcting code must have at least 2q is

$$n-k \geq 2q$$

Proof : We can proof this theorem by showing that following two statements are true.

(i) A necessary condition for a (n, k) linear code to be able to correct all burst of length q or less can be a code vector.

(ii) The number of parity check digits of a (n, k) linear code has no burst of length b or less as a code vector is at least b is $n-k \geq b$.

To prove part (i) consider a code vector v with burst of length 2q or less this code vector can be expressed as a vector sum of v_1 and v_2 of lengths q or less. Thus in standard array of code v_1 and v_2 must be in same coset. If one of these vectors is a coset leader, then other will be uncorrectable error pattern. This code will not be able to correct all burst of length q or less.

To prove part (ii) consider vector whose non-zero components are confined to first b bits, therefore, there are 2b such vectors no two such vectors can be in same coset of standard array for this code; otherwise their sum which is a burst of b or less would be a code vector. Hence these 2b vectors must be in 2b distinct cosets. There are total 2^{n-k} cosets for an (n, k) code. Thus $(n-k)$ must be at least equal to by combining (i) and (ii). Hence proved.

Burst error correcting capacity of (n, k) code is at most $\left(\frac{n-k}{2}\right)$

∴ upper bound of burst error correcting capability of (n, k) code

$$q \leq \frac{n-k}{2}$$

Burst correcting efficiency $y = \frac{2q}{n-k}$

If a code is needed for detecting a burst of length ≤ d then, number of check bits needed must satisfy.

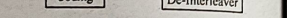
$$n-k \geq d$$

Decoding algorithm for correcting burst errors are similar to algorithm for cyclic codes designed to correct random errors.

(C) Interleaving

(i) Block Interleaving

1. A primary technique which is effective in overcoming burst errors is interleaving.

2. 

3. Before the data stream is applied to channel the data goes through a process of interleaving and error correction coding.

4. At the receiving side, data is decoded as data bits are evaluated in a manner to take advantage of error correcting and detecting features which result from coding and process of interleaving is undone.

5. Group of k*l* bits is loaded into shift registers which is organized into k rows with *l* bits/ row.

6. The data stream is entered into storage element at a_{11}. At each shift each bit moves one position to right while the bits in rightmost storage element moves to leftmost stage of next row.

7. When k*l* data bits have been entered, register is full, the first bit being in a_{kl} and last bit = a_{11}.

8. Now this data stream is diverted to the second shift register and process of coding is applied to data held stored in first register.

9. In this coding process, the information bits in column (eg., $a_{11}, a_{21} \ldots\ldots a_{k1}$) are viewed as the bits of uncoded word to which parity bits are added : code word.

$a_{11}, a_{22} \ldots\ldots\ldots a_{k1}, C_{11}, C_{21} \ldots\ldots C_{r1}$ is formed by information bits and r check bits.

b bits/row

$$\left.\begin{matrix} a_{11} & a_{12} & \cdots & a_{1\ell} \\ a_{21} & a_{22} & \cdots & a_{2\ell} \\ \vdots & \vdots & \vdots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{k\ell} \end{matrix}\right\} \text{k bits/ column}$$

$$\left.\begin{matrix} c_{11} & c_{12} & \cdots & c_{1\ell} \\ c_{21} & c_{22} & \cdots & c_{2\ell} \\ \vdots & \vdots & \vdots & \vdots \\ c_{r1} & c_{r2} & \cdots & c_{r\ell} \end{matrix}\right\} \text{r parity bits/ column}$$

Interleaved code = $(\lambda n, \lambda k)$

$= (5 \times 15, 5 \times 7) = (75, 35)$

Each row is a 15 bit code word

1	6		36	41			66	71
2	7		37	42			67	72
3	8	...	38	43	...	...	68	73
4	9		39	44			69	74
5	10		40	45			70	75

5 rows

with λ = 5 with a burst error correcting ability = 10. The arrangement of codewords by interleaving as shown in 4 rows and 5 columns.

Let us assume that errors occur at position 5, 37, 43 and 69. The decoder operates on column and in each row max 2 errors are possible.

∴ By this interleaving, these errors are corrected. Here 5, 69 are random errors and 37 and 43 are burst errors.

10. As we see that information bits in of particular codeword is *l* bits apart that original bits stream or we can say two adjacent bits are *l*-bits distance apart like.

$a_{11}, a_{21} \ldots\ldots a_{k1}$

11. When coding is complete, the entire content of k × *l* information bits as well as r × *l* parity bits are transmitted over channel.

Generally transmission carried out row by row in order.

$C_{r\ell} - C_{r1} \ldots\ldots C_{1\ell} \ldots\ldots C_{11} a_{k\ell} \ldots\ldots a_{k\ell} \ldots\ldots$

$a_{2\ell} \ldots\ldots a_{21} a_{1\ell} \ldots\ldots a_{12} a_{11}$

12. Here we see that parity bits are transmitted. The received data is again received in same order as in transmission and error correction is performed. The parity bits are discarded.

For example, we have (15, 7) BCH code generated by $g(x) = x^8 + x^4 + x^2 + x + 1$

Given $d_{min} = 5$

∴ q = error correcting ability of BCH

$$q \leq \frac{d_{min}-1}{2}$$

$$\leq \frac{5-1}{2} = \leq 2$$

∴ double error correcting ability

Here we have λ = 5.

(ii) Convolutional Interleaving

1. This is an another interleaving scheme.
2. In fig. we have transmitter and the receiver and four switches. There are total *l* lines on the both transmitter and receiver side.
3. On the transmitter side, we have 0 storage elements on line 1 and storage element increases by 5.

   As we move on transmitter side line to line or receiver side, first line consists of $(l-1)$S storage element, second line $(l-2)$S and so on. The sum of storage elements on a specific line of transmitter and receiver is constant and equal to $(l-1)$ S.
4. Here in fig. we have four switches that operates in step and more from line to line at rate of d(k). Thus each switch move from line1, then to line 2 and so on.
5. Let us consider a single line l_i on transmitter side. Suppose that during a particular bit interval of d(k) there is a switch contact at both input and output sides of line l_i.

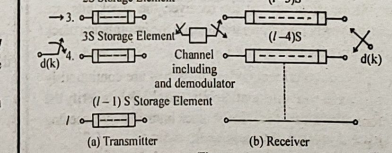


Fig.

6. At the end of bit interval, a clock signal causes the shift register of line l_i to enter the bit on input side at leftmost place and start moving content of each storage element one bit right.

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7. This process has started and synchronous clock advances the switch to next line $l_i + 1$.
8. When shift register response is completed there will be a new bit at output end of line l_i. But because of propagation delay through storage elements, switch has already lost contact with line l_i before new bit appeared at output.
9. In summary during interval of input d(k) during which switches were connected to line l_i, there is one bit shift of shift register on line l_i which accepts d(k) into register.
10. Such a shift is not noticed until the next time switch make contact with line l_i.
11. Now, let us consider that initially all shift registers are short circuit. (at both transmitter and receiver side). In this case received sequence is same as transmitted.
12. With shift register in transmitter and receiver each line has $(l-1)$ delay and therefore output segment will be interleaved.

Suppose that two successive input bit stream are transmitted d(k) and d (k + 1) then d(k + 1) will be (d + k + l).

$l = 5$, S = 3.

$lS = 15$ meaning 15 bits are interleaved between two bits.

Source Code : There are two types of source code. Source codes are often divided into two broad categories:

1 Block codes

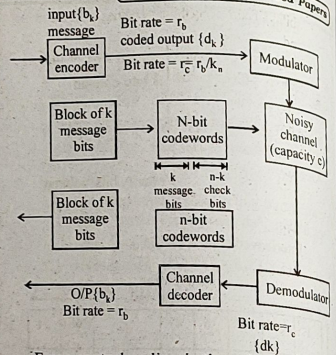
2 Convolutional codes

In block codes, a block of k information bits is followed by a group of r check bits that are derived from the block of information bits. At the receiver, the check bits, are used to verify the information bits in the information block preceding the bits.

In convolutional codes check bits are continuously interleaved with information bits; the check bits verify the information bits not only in the block immediately preceding them but in other blocks as well.

Error Control Coding

The practical alternative for reducing the probability of error is the use of error control coding, also known as channel coding.



Error control coding is the calculated use of redundancy.

The functional blocks that accomplish error control coding are the channel encoder and the channel decoder.

[Error control coding. Channel bit error probability is a/c = P {d ≠ k d_k} and message bit error probability is P_E = P {b ≠ k, b_k}

b_k - binary output r_b bits/sec.

The channel encoder and decoder are functional blocks in system that by acting together, reduce the overall probability of error. The encoder divides the input message bits into block of k message bits and replace each k bit message block D, with an n bit codeword Cw by acting n–k check bits to each message block. The mapping rules for coding and decoding are to be chosen such that error control coding lowers the overall probability of error. Important aspects of error controlling code:

1 It is possible to detect and correct errors by adding extra bits called check bits to the message stream.

2 It is not possible to detect and correct all errors.

Rate efficiency of a coding scheme is defined as r_b/r_c. There are various types of error correcting codes, some of them are given below-

1. Parity check bit coding for error detection
2. Block codes
3. BCH codes

Various applications of the error correcting codes-

1. They help in removing any type of errors from the message send.
2. They help in adding some extra bit to the sending message so that the received message will have

ability of correcting the code comes at the receiving end.

3. They provide accurate transmission of message from one place to other place.
4. They provide good efficiency of message sending.
5. They help in sending correct message to the receiver.

An another classification of codes are as follows:

Let us consider the following fuble where a source of size 4 has enceted in binary codes symbol 0 and 1.

Instantaneous codes:

A uniquelly decodable code is called an instantaneous code if the end of any codeward is recognizable without examining. Subsequent code symbols. Prefix free codes are sometimes known as instantaneous codes.

Optimal codes :

A code is said to be optinal if it is instantaneous and has minimum average L for a given source with a given probability assignment for the source symbol.

Q.14 (a) What is coding efficiency ? Show that the coding efficiency is maximum when P(0) = P (1). *[R.T.U. 2018, Dec. 2013]*

(b) Design (n, k) hamming code with a minimum distance of $d_{min} = 3$ and message length of 4 bits. *[R.T.U. Dec. 2013]*

Ans.(a) Coding efficiency : The code efficiency is defined as the ratio of message bits to the number of transimitted bits per blocks.

Let M be the number of symbols in an encoding alphabet. There messsage $[m_1\ m_2\ m_3\ ....\ m_N]$ with the probabilities $[P(m_1), P(m_2) ..... P(m_N)]$.

Let n_i be the number of symbols in the i^{th} message. The average length of the message or the average length per code word is than given by.

$$\tau = \sum_{i=1}^{n} n_i P(n_i) \text{ letters/message ... (1)}$$

I should be minimum to have an efficient transmission coding efficiency, then can be defined as

$$\eta = \frac{\tau_{min}}{\tau}$$

Prove of coding efficiency is maximum when P(0) = P(1)

Let H(x) be the entropy of the source in bits/message also let log m be the maximum average information associated with each letter in bits/letter.

Hence, the relation

$$\frac{H(x)}{\log m}$$

having a unit $\frac{\text{bits/message}}{\text{bits/letter}}$ or letter/message, gives the minimum average no. of letters per message $\frac{H(x)}{\log m} = \tau_{min}$

Hence the coding efficiency is

$$M = \frac{\tau_{min}}{L} = \frac{H(x)}{\tau \log m}$$

We know that H(x) will be maximum when symbols are equiproble.

And the coding efficiency will be maximum when H(x) will be maximum. So we can conclude that coding efficiency will be maximum when

$$P(0) = P(1)$$

Let us see an example to prove it

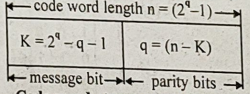
$$[M] = [m_1\ \ m_2]$$

$$P[M] = \left[\frac{1}{2}\ \ \frac{1}{2}\right]$$

$$\text{Efficiency } \eta = \frac{I(x;y)}{C} = \frac{H(x)}{\log_2 M}$$

$$= \left[\frac{\frac{1}{2}\log_2\frac{1}{\frac{1}{2}} + \frac{1}{2}\log_2\frac{1}{\frac{1}{2}}}{\log_2 2}\right] = \left(\frac{1}{1}\right) = 100\%$$

Ans.(b)



Code word structure of hamming code

here since message length is given as 4

$\therefore$ $4 = 2^q - q - 1$

$\Rightarrow$ $q = 3$

1. here, $n = 7 \Rightarrow$ length of code word is 7.
2. Number of message bits K = 4 (given)
3. Number of parity bits : (n –K) = 3 = q
4. Minimum distance $d_{min} = 3$
5. Code rate of code efficiency = $\frac{K}{n} = \frac{4}{7}$

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Q.15 *Consider a (7, 4) block code generated by*

$$G=\begin{bmatrix}1&0&0&0&1&1&1\\0&1&0&0&1&1&0\\0&0&1&0&1&0&1\\0&0&0&1&0&1&1\end{bmatrix}$$

Find out the error vector and suppose that the received vector R is 1001001. *[R.T.U. 2013]*

Ans.

Step : 1 $H=[P^T/I_{n-k}]_{m-k\times n}$

$$H=\underbrace{\left[\begin{array}{cccc}1&1&1&0\\1&1&0&1\\1&0&1&1\end{array}\right]}_{P^T}\;\underbrace{\left[\begin{array}{ccc}1&0&0\\0&1&0\\0&0&1\end{array}\right]}_{I_3}$$

Step : 2 We have K = 4

n = 7

$\therefore 2^k=2^4$ codewords for 2^4 messages (0 0 0 0) ....(1 1 1 1)

Step : 3 Choose a specific value of D from the 16 combinations for example 1011.

$\therefore$ C = DG

= 1 0 1 1 0 0 1

Step : 4 Calculate syndrome

$$S=\underset{(1011001)}{CH^T}\begin{bmatrix}1&1&1\\1&1&0\\1&0&1\\0&1&1\\1&0&0\\0&1&1\\0&0&1\end{bmatrix}=(1011001)\left[\begin{array}{ccc}1&1&1\\1&1&0\\1&0&1\\0&1&1\\1&0&0\\0&1&1\\0&0&1\end{array}\right]$$

Step : 5 If R = 1 0 0 1 0 0 1 is given find $S = RH^T$

$$=(1001001)\left[\begin{array}{ccc}1&1&1\\1&1&0\\1&0&1\\0&1&1\\1&0&0\\0&1&0\\0&0&1\end{array}\right]$$

= 101

Step : 6 Compare value of S by H^1. Now 101 is equal to third row of H^T $\therefore$ third bit is in error $\therefore$ the transmitted word

C = 1 0 1 1 0 0 1

Error vector E = R – C

E = 0 0 1 0 0 0 0

Q.16 *Consider (7, 4) linear code whose generator matrix is*

$$G=\left[\begin{array}{cccc|ccc}1&0&0&0&1&0&1\\0&1&0&0&1&1&1\\0&0&1&0&1&1&0\\0&0&0&1&0&1&1\end{array}\right]$$

(a) Find all the code vectors of this code.

(b) Find the parity check matrix for this code.

(c) Find the minimum weight of this code.

(d) Prove equation $CH^T = 0$.

[R. T. U. 2013, 2011; Raj. Univ. 2006, 2003]

Ans. Given (7, 4) linear code

$$G=\left[\begin{array}{cccc|ccc}1&0&0&0&1&0&1\\0&1&0&0&1&1&1\\0&0&1&0&1&1&0\\0&0&0&1&0&1&1\end{array}\right]$$

(a) Here K = 4, n = 7

K = number of bits per message.

There are 16 possible combinations of 4-bits ranging from 0000 - 1111. For each combination, the code word is formed by

C = DG

$$C=[1\,1\,1\,1]\begin{array}{c}\\D\end{array}\begin{array}{c}\begin{matrix}G&&&&G&&\end{matrix}\\\left[\begin{array}{cccc|ccc}1&0&0&0&1&0&1\\0&1&0&0&1&1&1\\0&0&1&0&1&1&0\\0&0&0&1&0&1&1\end{array}\right]\end{array}$$

C = 1 1 1 1 1 1 1 1

Similarly, for other combinations, codewords can be formed.

(b) Parity check matrix

$H = [P^T/I_{n-k}]$

$$H=\left[\begin{array}{cccc|ccc}1&1&1&0&1&0&0\\0&1&1&1&0&1&0\\1&1&0&1&0&0&1\end{array}\right]$$

(c) Minimum weight

First construct all codewords by C = DG, these are shown in given table.

Table

	Weight
0 0 0 0	0
0 0 0 1	0 1 1	3
0 0 1 0	1 1 0	3
0 0 1 1	1 0 1	4
0 1 0 0	1 1 1	4
0 1 0 1	1 0 0	3
0 1 1 0	0 0 1	3
0 1 1 1	0 1 0	4
1 0 0 0	1 0 1	3
1 0 0 1	0 1 1	4
1 0 1 1	0 0 0	3
1 1 0 0	0 1 0	3
1 1 0 1	0 0 1	4
1 1 1 0	1 0 0	4
1 1 1 1	1 1 1	7

(d) Equation - $CH^T = 0$

Choose any C = 0 0 1 0 1 1 1

$$H^T=\begin{bmatrix}1&0&1\\1&1&1\\1&1&0\\0&1&1\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}\quad\text{Now } CH^T=[0\,0\,1\,0\;1\,1\,1]$$

$$\begin{bmatrix}1&0&1\\1&1&1\\1&1&0\\0&1&1\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}=[000]$$ **Hence proved.**

Q.17 *The parity check matrix of a particular (7, 4) linear block code is given by*

$$H=\begin{bmatrix}1110100\\1101010\\1011001\end{bmatrix}$$

(a) Find the generator matrix G.

(b) List all the code vectors.

(c) What is the minimum distance between the code vectors ?

(d) How many errors can be detected and how many can be corrected ? *[R.T.U. 2012]*

Ans. $H=\left[\begin{array}{ccc:ccc}1&1&1&0&1&0&0\\1&1&0&1&0&1&0\\1&0&1&1&0&0&1\end{array}\right]$ ... (1)

$H = [P^T : I_3]$ ...(2)

Comparing matrix (1) & (2)

$$P^T=\left[\begin{array}{cccc}1&1&1&0\\1&1&0&1\\1&0&1&1\end{array}\right]$$

$$P=\left[P^T\right]^T$$

$$=\begin{bmatrix}1&1&1\\1&1&0\\1&0&1\\0&1&1\end{bmatrix}_{4\times3}$$

(a) Generator Matrix (G)

$G = [I_K : P_{k\times q}]_{k\times n}$

but k = 4, q = 7, n = 7

$G = [I_4 : P_{4\times3}]_{4\times7}$

so $$G=\left[\begin{array}{cccc:ccc}1&0&0&0&1&1&1\\0&1&0&0&1&1&0\\0&0&1&0&1&0&1\\0&0&0&1&0&1&1\end{array}\right]_{4\times7}$$

(b) Code Word

C = MP

$$[C_1\,C_2\,C_3,] = [M_1, M_2, M_3, M_4]\begin{bmatrix}1&1&1\\1&1&0\\1&0&1\\0&1&1\end{bmatrix}$$

$C_1 = M_1 \oplus M_2 \oplus M_3$

$C_2 = M_1 \oplus M_2 \oplus M_4$

$C_3 = M_1 \oplus M_3 \oplus M_4$

Code table

5.3.2/5-20.

(c) Minimum distance: It is equal to minimum weight of any non-zero code vector. So from the table

$d_{min} = [w(x)]_{min}$

$\boxed{d_{min} = 3}$

(d) Error Detection & Correction

$d_{min} \geq s+1$

$3 \geq s+1$

$s \leq 2$

two errors will be detected.

$d_{min} \geq 2t+1$

$3 \geq 2t+1$

$t \leq 1$

only one error will be corrected.

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4 CYCLIC CODE

PREVIOUS YEARS QUESTIONS

PART-A

Q.1 ***The intersection of cyclic codes is cyclic. Find the generator polynomial of $C_1 \cap C_2$.*** *[R.T.U. 2018]*

Ans. The generator polynomial of $C_1 \cap C_2$ is $g(x) = \text{lcm}(g_1(x), g_2(x))$.

Every codeword in the intersection of two cyclic codes is divisible by both generator polynomials and therefore by their least common multiple.

Conversely, every multiple of the least common multiple belongs to both codes, hence to their intersection. When $g_1(x)$ and $g_2(x)$ are relatively prime, their least common multiple is their product. In this case, the generator polynomial of the intersection of two cyclic codes is $g_1(x)g_2(x)$.

Q.2 ***The following polynomial f(x) and g(x) are defined over GF(3).***
$f(x) = 2 + x + x^2 + 2x^4$
$g(x) = 1 + 2x^2 + 2x^4 + x^5$
Calculate addition and multiplication of the above two polynomials. *[R.T.U. 2013]*

Ans. $f(x) + g(x) = (2 + 1) + x + (1 + 2)x^2 + (2 + 2)x^4 + x^5 = x + x^4 + x^5$

$f(x).g(x) = (2 + x + x^2 + 2x^4)(1 + 2x^2 + 2x^4 + x^5)$
$= 2 + x + (1 + 2.2)x^2 + 2x^3 + (2 + 2 + 2.2)x^4 + (2 + 2)x^5 + (1 + 2 + 1)x^6 + x^7 + 2.2x^8 + 2x^9$
$= 2 + x + (1 + 1)x^2 + 2x^3 + (2 + 2 + 1)x^4 + (2 + 2)x^5 + (1 + 2 + 1)x^6 + x^7 + x^8 + 2x^9$
$= 2 + x + 2x^2 + 2x^3 + 2x^4 + x^5 + x^6 + x^7 + x^8 + 2x^9$

Q.3 ***How RS code can be organized? Explain in short.***

Ans. RS code is organized on the basis of groups bits. Such groups of bits are referred to as symbols.

Q.4 ***Write one disadvantage of cyclic codes.***

Ans. The error detection in cyclic codes is simpler but error correction is little complicated since the combinational logic circuits in error detector are complex.

Q.5 ***Define parity-check polynomial.***

Ans. Parity-Check Polynomial : It is a polynomial that can be found as the remainder polynomial.

Q.6 ***What do you mean by cyclic codes.***

Ans. Cyclic Codes: It has the property that a cyclic shift of one codeword of the code forms another codeword.

PART-B

Q.7 ***Design a (4, 2) LBC :***
(i) Find the generator matrix for code vector set
(ii) Find the parity check matrix
(iii) Make an enconding ckt.
(iv) Draw the encoding ckt.
(v) Draw the syndrome calculation ckt. *[R.T.U. 2011]*

Ans. (i) Generator matrix of a (4, 2)

1011
1101

(ii)

0100
1001
0011

(iii)

Inputs				Outputs	
D_3	D_2	D_1	D_0	Q_1	Q_0
0	0	0	1	0	0
0	0	1	0	0	1
0	1	0	0	1	0
1	0	0	0	1	1
0	0	0	0	x	x

(iv)

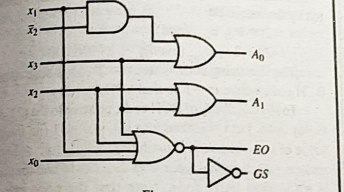

Fig.

(v)

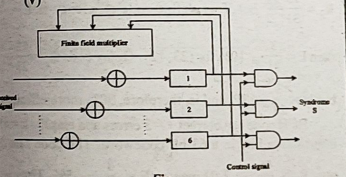

Fig.

Q.8 ***Write short notes on Cyclic codes.*** *[R.T.U. 2017]*

OR

What are the cyclic codes? Write the advantages and disadvantages of cyclic codes. *[R.T.U. 2016]*

Ans. Cyclic Codes : Cyclic code has the property that a cyclic shift of one codeword of the code forms another codeword.

Meaning of cyclic shift is explained from figure i.e. n bit word instead of being written out horizontally is written around a circle. Starting at any point A the 7-bit word encountered by a clockwise rotation is 1101001; starting at some other arbitrary point say B we would read 0111010. The two words are related such that one is derived from other by cyclic shift. There are seven possible starting plans as shown in fig. Order in which the words are generated depends on the direction, clockwise or counterclockwise of the shift, but the end result of the resultant collection of words is not affected by the shift direction.

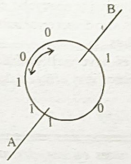

Fig.

A procedure for generating an (n, k) cyclic code is the following :

The bits of the uncoded word $\overline{A} = (A_0 A_1 .. A_{k-1})$ are written as the polynomial.

$$A(x) = A_0 \oplus A_1 x \oplus A_2 x^2 \oplus ... \oplus A_{k-1} X^{k-1} \quad ...(1)$$

The bits of the coded word $\overline{T} = [T_0 T_1 ... T_{n-1}]$ are written as the coefficients of the polynomial.

$$T(x) = T_0 \oplus T_1 x \oplus T_2 x^2 \oplus ... \oplus T_{n-1} X^{n-1} \quad ...(2)$$

We next form the "generating" polynomial g(x) of degree r = n – k.

$$g(x) = 1 \oplus g_1 x \oplus g_2 x^2 \oplus ... \oplus g_{r-1} x^{r-1} \oplus x^r \quad ...(3)$$

and we determine the values of the coefficient $g_1, g_2, ... g_{r-1}$ from the condition that g(x) be a factor of the polynomial

$$f(x) = x^n \oplus 1 \quad ...(4)$$

where n is the number of bits in the codeword. Finally, when g(x) is determined, T(x) is found from the equation-

$$T(x) = g(x) A(x) \quad ...(5)$$

As an example of the application of this procedure, let us generate a (7, 4) code since n = 7

$$f(x) = x^7 \oplus 1 \quad ... (6)$$

It can be verified that factors of f(x) are

$$f_1(x) = \lambda_1(x) . \lambda_2(x) . \lambda_3(x)$$
$$= (1 \oplus x)(1 \oplus x \oplus x^3)(1 \oplus x^2 \oplus x^3) \quad ...(7)$$

To generate a code with n = 7 bits, T(x) in equation (2) must be a polynomial of degree n – 1 = 6.

Advantages and Disadvantages of Cyclic Codes

As we have seen that cyclic codes are the subclass of linear block codes, they have some advantages over noncyclic block codes as given below-

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Advantages:

(1) The error correcting and decoding methods of cyclic codes are simpler and easy to implement. These methods eliminate the storage needed for lookup table decoding. Therefore the codes becomes powerful and efficient.

(2) The encoders and decoders for cyclic codes are simpler compared to noncyclic codes.

(3) Cyclic codes also detect error burst that span many successive bits.

(4) Cyclic codes have well defined mathematical structure. Hence very efficient decoding schemes are possible.

Disadvantages : *Refer to Q.4.*

Q.9 ***Write short notes on encoder and decoder for cyclic codes.*** *[R.T.U. 2017]*

Ans. Cyclic codes are an important class of linear block codes in which the cyclic shifting of the message bits results in another code vector, hence the name cyclic code. In other words a cyclic shift in a code word in C results in another code word in C. For Example C = {(110), (101), (011)} is cyclic, while C = {(000), (100), (111)} is not cyclic. A cyclic code is defined in terms of a generator polynomial g(X) of degree less than n-k , where k is the length of input message and n is the length of the encoded message. Cyclic codes are studied usually in polynomial form since it is easy to represent the code vectors as the coefficients of the polynomial. For example the message 110001 is represented as 1+ X + X?. Cyclic Codes are used both for encoding and decoding of the message bits. In an encoding process the message signal is divided by a certain sequence called generator number and corresponding to that the remainder bits form the parity check sequence which are concatenated to either front or back of the message signal to encode it. The process of encoding can be implemented using shift registers as shown in the Fig. 1, where g0,g1,..gn-k-1 is for generator bits. b0,b1,..bn-k-1 is for remainder bits and u(X) is for message bits. The decoding process is a more complex process and is followed by error detection and correction. In error detection the received vector is divided by the generator sequence and the computed remainder forms the syndrome. If the computed syndrome is identical to zero, then the received vector is error free, else the process advances to error correction. In error correction the computed syndrome is compared with the error pattern and accordingly the erroneous bit is XOR-ed with the error bit and the corrected received vector is obtained.

Q.10 ***Write short note on :***
(i) BCH code *[R.T.U. Dec. 2013, 2009]*
(ii) RS code *[R.T.U. Dec. 2013]*

Ans. (i) BCH Code : Bose, Chaudhuri and Hocquenghem codes form a large class of error correcting codes.

It employs k information bits, r parity check bits and therefore, the number of bits in a codeword is n = k + r. Furthermore, the number of errors t which can be corrected in an n-bit codeword is

$$t = r/m,$$

where m is integer to the number of bits n in the codeword by the formula-

$$n = 2^m - 1,$$

The minimum distance between BCH codes is related to t by the inequality.

$$2t + 1 \le d_{min} \text{ (odd)};$$

$$2t + 2 \le d_{min} \text{ (even)};$$

The Hamming code where $n = 2^r - 1$ is seen to be a BCH code where m = r so that t = 1.

To see capability of BCH code, relative to the (23, 12) Golay code. Let t = 3, then m = r/3, if n = 31 (no integer value of m) when m is substituted to $f(x) = x^n \oplus 1$ will make n = 23, then m = 5 & r = 15 thus k = 16 and the code is a (31, 16) BCH code. Note that the code rate is

$$R_c = \frac{16}{31} \simeq \frac{1}{2}$$ but if we define code efficiency as $E = \frac{k}{n}$

then $E = \frac{3}{31} \cong 10\%$ for (23, 12). Golay code $R_c = \frac{12}{23} \simeq \frac{1}{2}$

but $E = \frac{3}{23} \simeq 13\%$ and is, therefore, 30% more efficient.

BCH code can correct t = 3 errors and employ k = 45 information bits in a 63 bit codeword. The code rate is now $R_c = \frac{45}{63} \simeq 7$. However, $E = \frac{3}{63} \simeq 5\%$. Thus we see the tradeoff between the code rate and its capability to correct errors.

(ii) RS code : The RS code is organized on the basis of groups bits. Such groups of bits are referred to as symbols. If sequences of m individual bits which appear serially in a bit stream are stored, thereafter operate with m-bit sequence rather than individual bits.

The RS code has following characteristics:

(i) RS code has k information symbol (instead of bits)

(ii) r parity symbols making a total codeword length of n = k + r n.L. Number of symbols in codeword $n = 2^m - 1$

$m \rightarrow$ no. of symbols

(iii) RS code is able to correct errors in t symbols

$$t = \frac{r}{2}$$

(iv) Code Rate $= \frac{k}{n} = R_c$

(v) and no. of correctable bits $B = \frac{m}{t}$

(vi) RS code is not effective code for correcting Random errors.

It can correct only half parity symbols making a total codeword length of n = k +r. So it gives trade off in error correcting capacity than other codes as the code rate does not depend on the parity symbols. So independency on the parity symbol of the code rate gives another trade off of RS code.

Q.11 ***Design an encoder for (7, 4) BCC generated by $g(x) = 1 + x + x^3$ and verify its operation using message vector 0101.*** *[R.T.U. 2013]*

Ans. Here $g_0 = 1$

$g_1 = 1 \rightarrow$ closed path

$g_2 = 0 \rightarrow$ open path

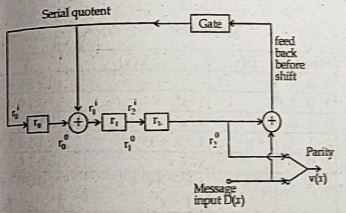

Fig.

Equation for $r_0^i = r_2^0 + d$

$r_1^i = r_0^0 + d$

$r_2^i = r_1^0$

Input bit	Register Inputs			Register Outputs		
d	r_0^i	r_1^i	r_2^i	r_0^0	r_1^0	r_2^0
-	0	0	0	0	0	0
1	1	1	0 shift 1	1	1	0
0	0	0	1 shift 2	0	0	1
1	0	0	1 shift 3	0	0	1
0	1	1	0 shift 4	1	1	0

The code vector for (0101) is (1100101).

Q.12 ***A (15, 5) linear cyclic code has a generator polynomial $g(x) = 1 + x + x^2 + x^4 + x^5 + x^8 + x^{10}$***
(i) Draw block diagram of an encoder and syndrome calculator for this code.
(ii) Find the code polynomial for the message polynomial.
$D(x) = 1 + x^2 + x^4$ (in a systematic - form). *[R.T.U. 2013]*

Ans. Given (15, 5) LBC

$$g(x) = 1 + x + x^2 + x^4 + x^5 + x^8 + x^{10}$$

(i) Block Diagram of Encoder

Here n – k = 10

∴ no of shift register is = 10 from $r_0 - r_9$ to calculate 10 Check bits

Here $g_0 = 1$ $g_1 = 1$ $g_2 = 1$ $g_3 = 0$ $g_4 = 1$ $g_5 = 1$ $g_6 = 0$ $g_7 = 0$ $g_8 = 1$ $g_9 = 0$ $g_{10} = 1$

According to these valves encoder is designed.

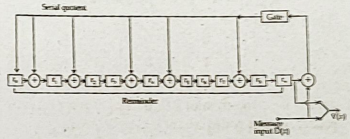

Fig. 1

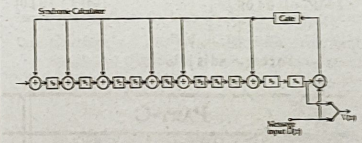

Fig. 2

(ii) Code Polynomial for Message Poly $D(x) = 1 + x^2 + x^4$ (in systematic form)

$$\frac{x^{n-k}D(x)}{g(x)}$$ Its remainder gives the values of parity check poly.

$$\frac{x^{15-5}D(x)}{g(x)}$$

$$\Rightarrow \frac{x^{10}(1 + x^2 + x^4)}{1 + x + x^2 + x^4 + x^5 + x^8 + x^{10}}$$

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$$1+x+x^2+x^4+x^5+x^8+x^{10}\overline{)\,x^{10}+x^{12}+x^{14}}\quad x^4+1$$
$$x^4+x^5+x^6+x^8+x^9+x^{12}+x^{14}$$
$$x^4+x^5+x^6+x^8+x^9+x^{10}$$
$$1+x+x^2+x^4+x^5+x^8+x^{10}$$

$r(x) = 1 + x + x^2 + x^6 + x^9$

$V = \underbrace{1110001001}_{r} + \underbrace{10101}_{D=15}$

Q.13 *Let C be a (7, 4) cyclic code with g (x) = 1 + x + x³. Find the generator matrix G for C and also find the codeword for d = (1010) ?* **[R.T.U. 2012]**

Ans. Since n = 7, k = 4, we have

$g(x) = 1 + x + x^3 \leftrightarrow 1101000$

$xg(x) = x + x^2 + x^4 \leftrightarrow 0110100$

$x^2g(x) = x^2 + x^3 + x^5 \leftrightarrow 0011010$

$x^3\,g(x) = x^3 + x^4 + x^6 \leftrightarrow 0001101$

Then we have

$$G = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \quad \ldots (1)$$

For d = [1 0 1 0]

$$c = dG = [1\ 0\ 1\ 0]\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} = [1\ 1\ 1\ 0\ 0\ 1\ 0]$$

So code word is 1110010.

PART-C

Q.14 ***(a) What do you understand by parity – check polynomial? Explain it in correspondence with generator polynomial.***

(b) The generator polynomial of a (6,3) cyclic code is g(x) = 1 + x². Find all the code words of the code. **[R.T.U. 2016]**

Ans.(a) Parity-Check Polynomial : The parity-check polynomial is a polynomial that can be found as the remainder polynomial.

When the message polynomial, shifted by (n − k) times, is divided by the generator polynomial g(x).

For an (n, k) cyclic code, the generator g(x) must divide $(x^n - 1)$ and the quotient $h(x) = (x^n - 1)/g(x)$, called the parity-check polynomial. For any codeword c(x), it follows that h(x) satisfies

$$h(x)\,c(x) \bmod x^n - 1 = 0$$

Since h(x) is given by dividing $x^n - 1$ by g(x), one can prove this statement by observing that $c(x) = m(x)$ g(x) for some m(x).

Explaination :

$$h(x)\,c(x) = m(x)\,g(x)\,h(x) = m(x)\,(x^n - 1)$$

Since $x^n - 1$ divides h(x) c(x), the remainder is zero.

The parity-check polynomial $h(x) = h_0 + h_1x + h_2x^2 + \ldots + h_kx^k$ of an (n, k) code has cyclic parity-check matrix of the form

$$H = \begin{bmatrix} h_k & h_{k-1} & \cdots & h_0 & 0 & 0 & 0 \\ 0 & h_k & h_{k-1} & \cdots & h_0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & h_k & \cdots & h_1 & h_0 \end{bmatrix}$$

In this case, one can use the fact that $g(x)h(x) = x^n - 1$ to verify that $GH^T = 0$.

Ans.(b) (6,3) cyclic code ⇒ no. of message bits = 3

No. of check bits = 6 − 3 = 3

$g(x) = 1 + x^2$

Message	D(x)	V(x)	V
0 0 1	0	0	0 0 0 0 0 1
0 0 1	1	$1+x^2$	1 0 1 0 0 1
0 1 0	x	$x+x^2$	0 1 1 0 0 0
0 1 1	1+ x	$1 + x + x^2 + x^3$	1 1 1 1 0 0
1 0 0	x^2	$x^2 + x^4$	0 0 1 0 1 0
1 0 1	$1 + x^2$	$1 + x^2 + x^2 + x^4 = 1 + x^4$	1 0 0 0 1 0
1 1 0	$x + x^2$	$x + x^2 + x^3 + x^4$	0 1 1 1 0 1
1 1 1	$1 + x + x^2$	$1 + x^2 + x + x^3 + x^2 + x^4 = 1 + x + x^3 + x^4$	1 1 0 1 1 0

$$D(x) = (D_0)1 + (D_1)x + (D_2)x^2$$

Code polynomial,

$$V(x) = D(x)\,g(x)$$
$$V(x) = (V_0)1 + (V_1)x + (V_2)x^2 + (V_3)x^3 + (V_4)x^4 + (V_5)x^5$$
$$V = [V_0\ V_1\ V_2\ V_3\ V_4\ V_5]$$

Q.15 ***What is Galois field? Explain properties of Galois fields.*** **[R.T.U. 2015]**

OR

Explain the construction of Galois Field (GF) along with its basic properties. **[R.T.U. 2016]**

Ans. Galois Field : A field can have a finite number m of elements in A. In this case, the field is called m degree finite field. The minimum number of elements is 2, namely the neutral elements of the two operations, so with the additive and multiplicative notations: 0 and 1. In this case, the second group contains a single element, the unit element 1. The operation tables for both elements are in Z_2:

⊕	0	1
0	0	1
1	1	0

⊗	0	1
0	0	0
1	0	1

This is the binary field, noted with **GF**(2), a very used one in digital processing. If p is a prime number, Z_p is a field, because {1, 2,…., p − 1} form a group with modulo p multiplication.

So the set {1, 2, …., p − 1} forms a field related to modulo p addition and multiplication.

This field is called *prime field* and is noted by GF(p). There is a generalisation which says that, for each positive integer m, we should extend the previous field into a field with p^m elements, called the extension of the field **GF**(p), noted by **GF**(p^m).

Finite fields are also called *Galois fields*, which justify the initials of the notation **GF** (Galois Field).

Field characteristic: We consider the finite field with q elements **GF**(q), where q is a natural number. If 1 is the neutral element for addition, be the summations :

$$\sum_{i=1}^{1} 1 = 1, \sum_{i=1}^{2} 1 = 1 + 1 = 2, \ldots\ldots, \sum_{i=1}^{k} 1 = 1 + 1 + k + 1$$

As the field is closed with respect to addition, these summations must be elements of the field.

The field having a finite number of elements, these summations cannot all be distinct, so they must repeat somewhere; there are two integers m and n (m < n), so that

$$\sum_{i=1}^{m} 1 = \sum_{i=1}^{n} 1 \Rightarrow \sum_{i=1}^{n-m} 1 = 0$$

There is the smallest integer λ so that $\sum_{i=1}^{\lambda} 1 = 0$.

This integer is called the characteristic of the field **GF**(q).

The characteristic of the binary field **GF**(2) is 2, because the smallest λ for which

$$\sum_{i=1}^{\lambda} 1 = 0 \text{ is 2, meaning } 1 + 1 = 0$$

The characteristic of the prime field **GF**(p) is p. It results that

1 the characteristic of a finite field is a prime number

2 for n, $m < \lambda$, $\sum_{i=1}^{n} 1 \neq \sum_{i=1}^{m} 1$

The summations: $1, \sum_{i=1}^{2} 1, \sum_{i=1}^{3} 1, \ldots, \sum_{i=1}^{\lambda-1} 1, \sum_{i=1}^{\lambda} 1 = 0$: are

λ distinct elements in **GF**(q), which form a field with λ elements **GF**(λ), called *subfield* of **GF**(q). Subsequently, any finite field **GF**(q) of characteristic λ contains a subfield with λ elements and it can be shown that if $q \neq \lambda$ then q is an exponent of λ .

Order of an element : We proceed a similar manner for multiplication: if a is a non zero element in **GF**(q), the smallest positive integer, n, so that $a^n = 1$ gives the order of the element a.

This means that a, a^2, ….., $a^n = 1$ are all distinct, so they form a multiplicative group in **GF**(q).

A group is called *cyclic group* if it contains an element whose successive exponents should give all the elements of the group. If in the multiplicative group, there are q−1 elements, we have $a^{q-1} = 1$ for any element, so the order n of the group divides q−1.

In a finite field **GF**(q) an element a is called *primitive element* if its order is q−1. The exponents of such an element generate all the non zero elements of **GF**(q). Any finite field has a primitive element.

Example : Let us consider the field **GF**(5), we have

$2^1 = 2$, $2^2 = 4$, $2^3 = 3$, $2^4 = 1$ so 2 is primitive

$3^1 = 3$, $3^2 = 4$, $3^3 = 2$, $3^4 = 1$ so 3 is primitive

$4^1 = 4$, $4^2 = 1$, so 4 is not primitive.

Q.16 ***Consider a (7, 4) cylic code with generator polynomial g(x) = 1 + x + x³ and let data word d = (1010).***

(a) Find corresponding systematic codeword.

(b) Find all the cyclic binary code of block length.

(c) Find the minimum distance of each code. **[R.T.U. 2012]**

Ans. Given

$g(x) = 1 + x + x^3$

n = 7

k = 4

q = n − k = 7 − 4 = 3

$2^4 = 16$ message vector

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(a) $M = (M_3\ M_2\ M_1\ M_0) = (1010)$

$M(x) = x^3 + x$

we know that

$$C(x) = rem\left[\frac{x^2 M(x)}{G(x)}\right]$$

$x^2 M(x) = x^3\ M(x)$

$= x^3 (x^3 + x)$

$= x^6 + 0.x^5 + x^4 + 0.x^3 + 0.x^2 + 0.x$

$G(x) = x^3 + x + 1$

$= x^3 + 0.x^2 + x + 1$

$$x^3 + 0.x^2 + x + 1 \overline{)x^6 + 0.x^5 + x^4 + 0.x^3 + 0.x^2 + 0.x + 0}$$

$$\underline{x^6 + 0.x^5 + x^4 + x^3}$$

$$x^3 + 0.x + 0 \overline{)x^3 + 0.x^2 + x + 1}$$

$$\underline{x^3 + 0.x + 0}$$

$0.x^2 + x + 1$ (remainder)

$$C(x) = rem\left[\frac{x^3 M(x)}{G(x)}\right]$$

$= x + 1$

$= \{011\}$

so the code word will be

$x = \{1010011\}$

(b) All the systematic cyclic code are tabulated below

Message bit				Code vector							Code length d_{min}
M_3	M_2	M_1	M_0	M_3	M_2	M_1	M_0	C_2	C_1	C_0	
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	1	0	1	1	3
0	0	1	0	0	0	1	0	1	1	0	3
0	0	1	1	0	0	1	1	1	0	1	4
0	1	0	0	0	1	0	0	1	1	1	4
0	1	0	1	0	1	0	1	1	0	0	3
0	1	1	0	0	1	1	0	0	0	1	3
0	1	1	1	0	1	1	1	0	1	0	4
1	0	0	0	1	0	0	0	1	0	1	3
1	0	0	1	1	0	0	1	1	1	0	4
1	0	1	0	1	0	1	0	0	1	1	4
1	0	1	1	1	0	1	1	0	0	0	4
1	1	0	0	1	1	0	0	0	1	0	3
1	1	0	1	1	1	0	1	0	0	1	4
1	1	1	0	1	1	1	0	1	0	0	4
1	1	1	1	1	1	1	1	1	1	1	7

(c) There is only $d_{min} = 3$, which has minimum non zero length of code.

so, $\boxed{d_{min} = 3}$

□□□

CONVOLUTIONAL CODE 5

PREVIOUS YEARS QUESTIONS

PART-A

Q.1 *Define Code Tree.* *[R.T.U. 2018, 2016]*

Ans. Code Tree

(a) Code tree indicate flow of the coded signal along the nodes of tree.
(b) Code tree is lengthy way of representing coding process.
(c) Decoding is very simple using code tree.
(d) It repeats after no. of stages used in encoder.
(e) It is complex to implement in programming.

Q.2 *Define Trellis.* *[R.T.U. 2016]*

Ans. Trellis :

(a) It indicates transitions from current to next state.
(b) It is shorter or compact way of representing coding process.
(c) Decoding is little complex using trellis diagram.
(d) It repeats in every stage in steady state, it has only one stage.
(e) It is simpler to implement in programming.

Q.3 *Define Constraint Length.* *[R.T.U. 2016]*

Ans. Constraint Length : Constraint length of a convolutional code is defined as the number of shifts over which a single information bit can influence the encoder output. The constraint lengths of the encoder form a vector whose length is the number of inputs in the encoder diagram.

Q.4 *Design an encoder for the (7, 4) binary cyclic code generated by g (x) 1+ x+ x³ and verify its operation using the message vectors (1001) and (1011).* *[R. T. U. Dec.2013]*

Ans. Given $g(x) = 1 + x + x^3$

The given generator polynomial can be written as.

$g(x) = 1 + x + 0.x^2 + x^3$

$g_1 = 1$

$g_2 = 0$

(Comparing with equation $x^3 + g_2x^2 + g_1x + 1$)

Encoder for generator polynomial $g(x) = 1 + x + x^3$ is given by figure

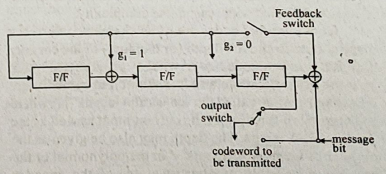

Fig.

Q.5 *For a (7, 4) cyclic code, the received vector Z (X) is 1110101 and g (x) = 1 + x + x³. Draw the syndrome calculation ckt and correct the single error in the received vector.* *[R.T.U.Dec. 2013]*

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Ans. Given generator polynomial is

$g(\lambda) = 1 + x + x^3$

$g(x) = x^3 + 0\,x^2 + 1.x + 1$ ... (i)

We know general form of generator polynomial is given by;

$g(x) = x^3 + g_2x^2 + g_1x + 1$ ... (ii)

Comparing (i) & (ii) we get

$g_2 = 0;\ g_1 = 1$

The syndrome calculator is shown in fig.

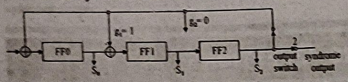

Fig. : Syndrome calculator for polynomial $g(x) = 1 + x + x^3$

PART-B

Q.6 ***Explain the coding and decoding in the convolution code.*** ***[R.T.U. 2017]***

OR

Decoding Probability of Convolution code. ***[R.T.U. 2018]***

Ans. In telecommunication, a convolutional code is a type of error-correcting code that generates parity symbols via the sliding application of a boolean polynomial function to a data stream. The sliding application represents the 'convolution' of the encoder over the data, which gives rise to the term 'convolutional coding.' The sliding nature of the convolutional codes facilitates trellis decoding using a time-invariant trellis. Time invariant trellis decoding allows convolutional codes to be maximum-likelihood soft-decision decoded with reasonable complexity.

Convolutional codes are often characterized by the base code rate and the depth (or memory) of the encoder [n,k,K]. The base code rate is typically given as n/k, where n is the input data rate and k is the output symbol rate. The depth is often called the "constraint length" 'K', where the output is a function of the current input as well as the previous K-1 inputs. The depth may also be given as the number of memory elements 'v' in the polynomial or the maximum possible number of states of the encoder (typically 2^v).

Convolutional codes are often described as continuous. However, it may also be said that convolutional codes have arbitrary block length, rather than being continuous, since most real-world convolutional encoding is performed on blocks of data. Convolutionally encoded block codes typically employ termination. The arbitrary block length of convolutional codes can also be contrasted to classic block codes, which generally have fixed block lengths that are determined by algebraic properties.

The code rate of a convolutional code is commonly modified via symbol puncturing. For example, a convolutional code with a 'mother' code rate n/k=1/2 may be punctured to a higher rate of, for example, 7/8 simply by not transmitting a portion of code symbols. The performance of a punctured convolutional code generally scales well with the amount of parity transmitted. The ability to perform economical soft decision decoding on convolutional codes, as well as the block length and code rate flexibility of convolutional codes, makes them very popular for digital communications.

Decoding in Convolution code:

Several algorithms exist for decoding convolutional codes. For relatively small values of k, the Viterbi algorithm is universally used as it provides maximum likelihood performance and is highly parallelizable. Viterbi decoders are thus easy to implement in VLSI hardware and in software on CPUs with SIMD instruction sets.

Longer constraint length codes are more practically decoded with any of several sequential decoding algorithms, of which the Fano algorithm is the best known. Unlike Viterbi decoding, sequential decoding is not maximum likelihood but its complexity increases only slightly with constraint length, allowing the use of strong, long-constraint-length codes. Such codes were used in the Pioneer program of the early 1970s to Jupiter and Saturn, but gave way to shorter, Viterbi-decoded codes, usually concatenated with large Reed-Solomon error correction codes that steepen the overall bit-error-rate curve and produce extremely low residual undetected error rates.

Both Viterbi and sequential decoding algorithms return hard decisions: the bits that form the most likely codeword. An approximate confidence measure can be added to each bit by use of the Soft output Viterbi algorithm. Maximum a posteriori(MAP) soft decisions for each bit can be obtained by use of the BCJR algorithm.

Q.7 ***A convolutional code is given by :***
$g_1 = [1\ 0\ 0]$, $g_2 = [1\ 0\ 1]$, $g_3 = [1\ 1\ 1]$
(i) Draw the encoder corresponding to this code.
(ii) Draw the state – transition diagram for this code.
(iii) Draw the trellis diagram for this code. ***[R.T.U. 2014]***

Ans.

(i)

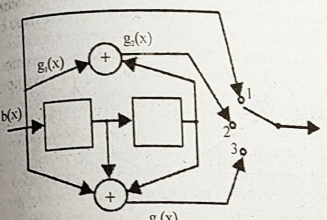

Fig.1

(ii)

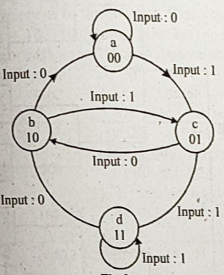

Fig.2

(iii) nth moment (n+1)st moment

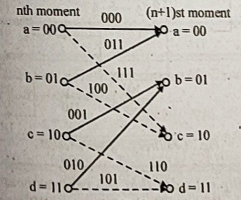

Fig.3

Q.8 ***Define Brust error.*** ***[R. T. U. Dec.2013]***

Ans. Burst-Error Detection and Correction

A Burst of length is defined as a sequence of digits in which the first digit and the b^{th} digit are in error, with the b-2 digit in between either in error or received correctly. For example an error vector e = 0010010100 has a burst length of 6.

It is easy to see that if a digit sequence of length b or less is in error, parity will be violated and the error will be detected (but not corrected) and the receiver can request retransmission of the digits lost.

One of the interesting properties of this code is that b, the number of parity check digits is independent of k(or n), which makes it a very useful code for such system as packet switching where the data digits may vary from packet to packet.

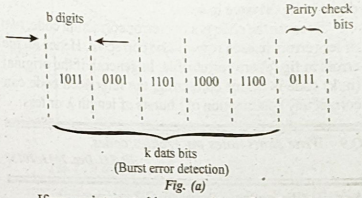

Fig. (a)

If we are interested in correcting rather than detecting burst errors, we require twice as many parity-check digits. A theorem says: In order to correct all burst errors of length b or less a linear block code must have at least 2b parity check digits.

Interlaced Codes for Burst Error Correction

In general, random-error correcting codes are not efficient for burst-error correcting and burst-error correcting codes are not efficient for random-error correcting. Out of the several methods proposed to simultaneoualy random and burst errors the interlaced code is the most effective.

For an (n,k) code, if we interlace λ code words, which is known as a $(\lambda n, \lambda k)$ interlaced code, instead of transmitting codewords one by one, we group λ code words and interlace them.

Consider for example, the case of $\lambda = 3$ and a two-error correcting (15,8) code. Each code word has 15 digits. We group code words to be transmitted in groups of three. Suppose that first three code words to be transmitted are $x(x_1, x_2, \ldots, x_{15})$, $y(y_1, y_2 \ldots y_{15})$ and $z(z_1, z_2 \ldots z_{15})$, respectively

x_1	x_2	x_3	⋯	x_{14}	x_{15}
y_1	y_2	y_3	⋯	y_{14}	y_{15}
z_1	z_2	z_3	⋯	z_{14}	z_{15}

Fig. (b) : Burst error correction

Instead of transmitting x y z in sequence as $x_1, x_2 \ldots x_{15}$, $y_1, y_2 \ldots y_{15}$, $z_1, z_2 \ldots z_{15}$ we transmit $x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3 \ldots\ldots x_{15}, y_{15}, z_{15}$.

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In usual transmission, we transmit one row after another. In the interlaced case, we transmit columns (of λ elements) in sequence. When all the 15(n) columns are transmitted , we repeat the procedure for the next λ code words to be transmitted.

To explain the error correcting capability of this code, we observe that the decoder will first remove the interlacing and regroup the received digits as $x_1, x_2 \ldots x_{15}, y_1, y_2 \ldots y_{15}, z_1, z_2 \ldots z_{15}$. Suppose that shaded digits in fig.(a) were in error.

Because the code is a two-error correcting code, two or less errors in each row will be corrected. Hence all the errors in fig.(a) are correctable. In general, if the original (n, k) code is t-error correcting, the interlaced code can correct any combination of t bursts of length λ or less.

Q.9 ***Write short notes on Trellies codes.***

[R.T.U. Dec. 2013, 2013]

Ans. Trellis codes : *Refer to Q.2.*

Trellis diagram

In shift register M_2 M_1, will indicate the state of encoder

So, let $M_2\ M_1 = 0\ \ 0$ State *a*
$M_2\ M_1 = 0\ \ 1$ State *b*
$M_2\ M_1 = 1\ \ 0$ State *c*
$M_2\ M_1 = 1\ \ 1$ State *d*

Then state transition table is

	Current State (M_2 M_1)	Input (M)	O/P X_1	O/P X_2	O/P X_3	Next State (M_1 M)
(a)	0 0	0	0	0	0	0 0 (a)
	0 0	1	1	1	1	0 1 (b)
(b)	0 1	0	0	1	0	1 0 (c)
	0 1	1	1	0	1	1 1 (d)
(c)	1 0	0	0	1	1	0 0 (a)
	1 0	1	1	0	0	0 1 (b)
(d)	1 1	0	0	0	1	1 0 (c)
	1 1	1	1	1	0	1 1 (d)

Trellis diagram

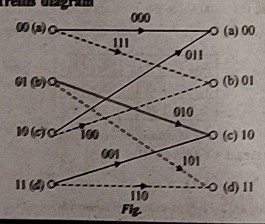

Fig.

Where Dotted line ⇒ 9/P M = 0
Solid line ⇒ 9/P M = 1

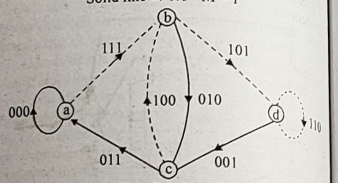

Fig. : State Diagram

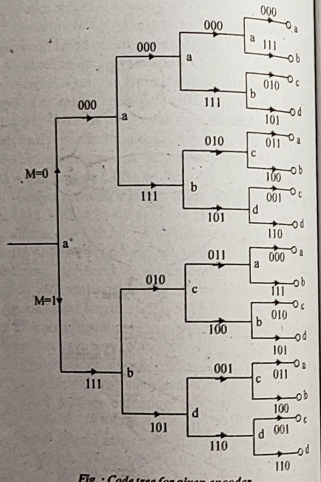

Fig. : Code tree for given encoder

Q.10 ***Initially consider that the register contains all zeroes. What will be the code sequence if the input data sequence is 1001110?***

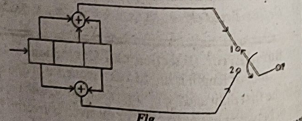

Fig.

[R.T.U. 2013]

Ans. The given figure illustrates a (2,1) convolutional encoder with constraint length k = 3. There are n = 2 modulo - 2 adders; thus the code rate k/n = 1/2.

We can approach the encoder in terms of its impulse response, *i.e*, the response of the encoder to a single one bit that moves through it. Consider the content of the register in given figure as a one moves through it.

Register contents	Branch word μ_1	Branch word μ_2
100	1	1
010	1	0
001	1	1

Input sequence : 1 0 0
Output sequence : 11 10 11

The output sequence for the input "one" is called the impulse response of the encoder. Then for the input sequence m = 100110, output may be found by the superposition or the linear addition of the time-shifted input "impulses" as follows :

Input (m)	Output							
1	11	10	11					
0		00	00	00				
0			00	00	00			
1				11	10	11		
1					11	10	11	
0						00	00	00
Modulo-2 Sum	11	10	11	11	01	01	11	00

PART-C

Q.11 ***Write short note on maximum likelihood decoding of convolutional codes.***

[R.T.U. 2018, 2013, 2010, 2008]

Ans. Like block codes, a subclass of convolutional codes can be decoded using majority logic decoding techniques. To illustrate this technique let us consider the convolutional code generated by the encoder shown in fig. 1 The (2, 1) convolutional encoder generates a check bit r_i for each message bit d_i according to

$$r_i = d_i + d_{i-2} + d_{i-4} \quad \ldots(1)$$

and the output consists of an interleaved sequence of r_i and d_i. Let $v_i^{(m)}$ and $v_i^{(c)}$ be the output of the channel (received bits) corresponding to input bit d_i and r_i, respectively.

If we denote the errors by $e_i^{(m)}$ and $e_i^{(c)}$, then

$v_i^{(m)} = d_i + e_i^{(m)}$ and $v_i^{(c)} = r_i + e_i^{(c)}$. The decoder for this code forms the i^{th} syndrome digit as

$$s_i = v_{i-4}^{(m)} + v_{i-2}^{(m)} + v_i^{m} + v_i^{(c)},\ i \geq 5$$

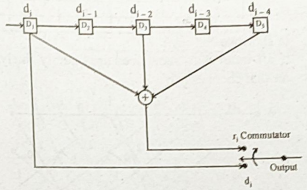

Fig. 1 : A convolutional encoder, N = 5, n = 2, k = 1

It follows from equation (1) that the preceding equation can be written as

$$s_i = e_{i-4}^{(m)} + e_{i-2}^{(m)} + e_i^{m} + e_i^{(c)},\ i \geq 5 \quad \ldots(2)$$

For the first four bits in a message, s_i is given by

$$s_1 = e_1^{(m)} + e_1^{(c)}$$

$$s_2 = e_2^{(m)} + e_2^{(c)}$$

$$s_3 = e_1^{(m)} + e_3^{(m)} + e_3^{(c)}$$

$$s_4 = e_2^{(m)} + e_4^{(m)} + e_4^{(c)} \quad \ldots(3)$$

Equations (2) and (3) reveal that any transmission error $e_i^{(m)}$ in the i^{th} message digit appears in syndrome bits s_i, s_{i+2} and s_{i+4}. Table shows the effect of error bits on various syndrome bits. It is apparent from what is shown in the table that if there are two or three 1s in s_1, s_3, s_5, then it is most likely that $e_i^{(m)} = 1$, meaning that the first message digit suffered a transmission error and hence it must be corrected. Similarly, we should correct the second message digit if there are more 1s than 0s in s_2, s_4, s_6 and so on. It can be verified that this majority logic decoding will correct up to four successive errors (in message bits and check bits) if the following eight digits are error free. Thus this convolution code can correct burst errors of length four or less.

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Table : Decoding procedure for the coder shown in fig. 2

	$c_1^{(m)}$	$c_1^{(c)}$	$c_2^{(m)}$	$c_2^{(c)}$	$c_3^{(m)}$	$c_3^{(c)}$	$c_4^{(m)}$	$c_4^{(c)}$	$c_5^{(m)}$	$c_5^{(c)}$	$c_6^{(m)}$	$c_6^{(c)}$
s_1	X	X										
s_2			X	X								
s_3	X				X	X						
s_4			X				X	X				
s_5	X				X				X	X		
s_6			X				X				X	X
s_7					X				X			
s_8							X				X	

In short we can say :

If all input data sequences are equally likely, a decoder that chooses $\hat{c}$ if

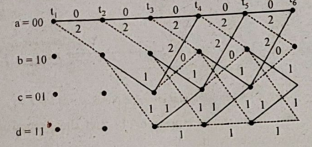

Fig. 2

$$P(r \mid \hat{c}) = \max P(rc_i) \text{ for all } c_i$$

where r is the received sequence and c_i is one of the possible transmitted sequences, is called the maximum *likelihood decoder*. The conditional probabilities $P(rc_i)$ are called the *likelihood functions*. Note that for the BSC (binary symmetrical channel) the maximum likelihood decoder reduces to a minimum distance decoder. The rule of the minimum distance decoding is as follows : Choose $\hat{c}$ that minimizes the Hamming distance between the received sequence r and the transmitted sequence c.

Q.12 ***Describe Viterbi Algorithm.*** **[R.T.U. 2018]**

OR

Explain the viterbi algorithm.

[R.T.U. 2017, 2016]

OR

Write short note on Viterbi Decoding.

[R.T.U.Dec. 2013]

Ans. Viterbi Algorithm :

1. We consider all possible paths through the coder from starting point to the end point. Each possible input bit sequence generates its own path.
2. For such path we determine the corresponding sequence of coder output bits and compare each of these output sequences with actual received sequence.
3. If received sequence is exactly, identical to a sequence corresponding to some particular path through coder, then we shall assume that corresponding input sequence is the one corresponding to some particular path.

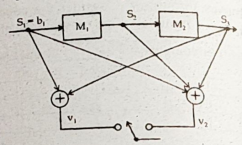

Fig. 1

4. If we find no exact corresponding, then we shall assume that input sequence to be one whose path generates the fewer bit discrepancies when compared to received sequence.
5. Now to illustrate viterbi algorithm let us use encoder

Here $v_1 = S_1 \oplus S_3$

$v_2 = S_1 \oplus S_2 \oplus S_3$

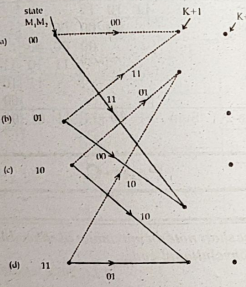

Fig. 2

6. Now initially encoder is clear

$M_1M_2 = 00$

Let there be presented at encoder a sequence of five information bits and let it be the corresponding sequence $V_{1R}V_{2R}$ bits are

$V_{1R}V_{2R} = 10\ 00\ 10\ 00\ 00$

7. Now trellis diagram

Here from state (a) is 00 if 0 is read, then received bits are 00 and if 1 is read then, received bits are 11. In either case $V_{1R}V_{2R} \neq 10$.

8. Now firstly without reference to received sequence let us trace the possible paths through encoder state as shown in trellis.
9. Starting from (a) in clock interval k = 1, a 0 will cause an output = 00 and will carry encoder in state (a). a 1 will generate output 11 and will carry encoder to state (c).
10. The number of discrepancies in each clock cycle between the bits associated with rallis in trellis diagram and actual received bits is shown in fig. 2
11. Thus if starting state is (a) at K = 1, a0 output generate an output = 00. Since, actual output is 10, the number of bit discrepancies = 1.
12. In next interval if input = 0 should again yield output = 00 and since the corresponding set of received bit is also 00 ∴ number discrepancies = 0. The cumulative discrepancies shown in circles.

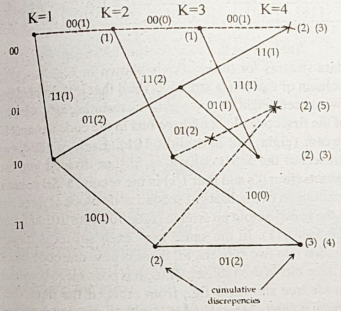

Fig. 3

To reach state (c) at K = 3 we go from (a) at K = 1 to (a) K = 2 and then from (a) at K = 2 to (c) at K = 3. In first path discrepancy = 1

In second path discrepancy = 2

∴ cumulative = 3

13. Suppose that we have to move from state (a) at K = 1 to state (a) K = L. How let us assume that path from state (b) at K = 4 to state (a) at K = L is fixed.

∴ We have to choose minimum discrepancy path from state (a) at K = 1 to state (b) at K = 4.

14. From trellis diagram we can notice that there are two paths to reach at K = 4 state (b) and cumulative discrepancies is written. Here we choose path with minimum discrepancies of (2) ∴ another path is discarded, the discarded path is shown by X.

∴ No paths surviving = no of states.

15. Surviving paths can be redrawn as shown : K = 1, K = 2, K = 3, K = 4

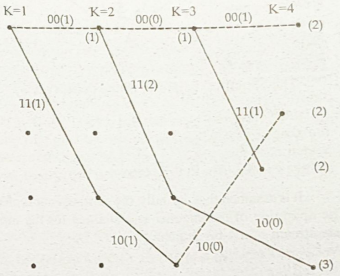

Fig. 4

16. Here we notice for output sequence of path through trellis corresponding to input bit stream consisting of all '0' yields minimum no of discrepancies. With such as result we would then decide that input sequence was all zero's, the received sequence as is readily verified should be all 0's.
17. Because of noise of received sequence = 10 00 10 00 00, the coder would have correct two errors.
18. As the number of paths are reduced ∴ memory required is reduced.

Q.13 ***Explain the operation of any convolutional encoder with the help of block diagram.***

[R.T.U. 2016]

Ans. Encoder for Convolutional Code : An encoder for a convolution code is shown in Fig. In this case.

K = no. of shift registers
= 3

v = no. of modulo-2 adders
= no. of bits in the code –block
= 3

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L = length of input data stream

= 4

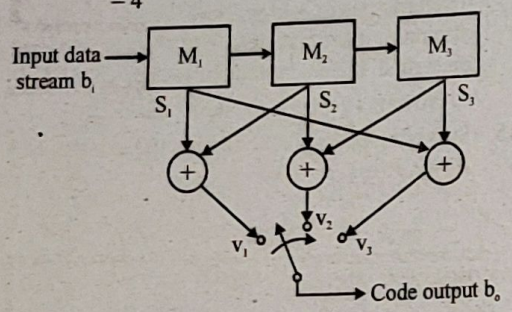

Fig. Encoder for Convolutional Code

The output v_1, v_2 and v_3 of the adders are

$$\left.\begin{aligned} v_1 &= S_1 \oplus S_2 \\ v_2 &= S_2 \oplus S_3 \\ v_3 &= S_1 \oplus S_3 \end{aligned}\right\}$$

It is assumed that, initially, the shift register is clear. The operation of the encoder is explained for the input data stream of a 4-bit sequence

$$m = 1101$$

This is entered in the shift register from MSB. Thus, at the first bit interval, $S_1 = 1, S_2 = 0, S_3 = 0$.

Now, v1, v2 and v3 can be found from Eqs. Thus

$$v_1 = 1 \oplus 0 = 1,\ v_2 = 0 \oplus 0 = 0,$$

$$v_3 = 1 \oplus 0 = 1$$

Hence, the output at the first-bit interval is 101. Similarly, at the second bit interval, $S_1 = 1, S_2 = 1, S_3 = 0$. Thus, $v_1 = 1 \oplus 1 = 0$, $v_2 = 1 \oplus 0 = 1$, $v_3 = 1 \oplus 0 = 1$. Hence, the output at second-bit interval is 011.

In the same manner, outputs at other bit intervals can be found out. Since $L = 4$ and $k = 3$, the register resets at seventh $(L + k = 4 + 3 = 7)$ bit interval. The output at each bit interval consists of v bits (in this case, $v = 3$). Thus, for each message, there are $v(L + k)$ bits in the output code word. Notice that each message bit remains in the shift register for k-bit intervals. Hence, each input bit has an influence on the k groups of v bits; i.e. on v-k output bits.

Table : Convolutional Code for Fig.

Input data stream	Coded output bit stream						
0000	000	000	000	000	000	000	000
0001	000	000	000	101	110	011	000
0010	000	000	101	110	011	000	000
0011	000	000	101	011	101	011	000
0100	000	101	110	011	000	000	000
0101	000	101	110	110	110	011	000
0110	000	101	011	101	011	000	000
0111	000	101	011	000	101	011	000
1000	101	110	011	000	000	000	000
1001	101	110	011	101	110	011	000
1010	101	110	110	110	011	000	000
1011	101	110	110	011	101	011	000
1100	101	011	101	011	000	000	000
1101	101	011	101	110	110	011	000
1110	101	011	000	101	011	000	000
1111	101	011	000	000	101	011	000

Table gives the coded output bit stream for all input data streams for the encoders shown in Fig. The MSB column of input data stream is such that it is divided into two subsets (eight 0's and eight 1's), resulting in two subsets of the first code block of three bits in the coded output bit stream (eight 000 and eight 101). Each of these two subsets of the MSB column is further divided into two subsets (four 0's and four 1's) in the second MSB column, resulting in two subsets of second code block of three bits in the coded output bit stream (four 000-four 101 and four 110-four 011). In the same way, each subset is further divided into two subsets, till there is only one code-block of three bits in each subset. Thus, it is possible to construct a code tree shown in fig., from table, if the input data stream is entered from the MSB in the convolutional code encoder. On the other hand, it is not possible to construct such code tree if the input data stream is entered from the LSB, as successive division in two subsets is not possible if we start from the LSB column. Hence, in the convolutional encoder, the input data stream is entered from the MSB and not from the LSB because the code tree structure is extremely useful for efficient decoding.

□□□

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