MA 202 : MATHEMATICS IV

NUMERICAL ANALYSIS OF FOUCAULT PENDULUM

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C Faheem Shanavas 19110145 | Chaitanya Guntoorkar 19110150

Manas Mulpuri 19110093 | Ishan Prayagi 19110194 | Thahir Naquash 19110034

IIT GANDHINAGAR

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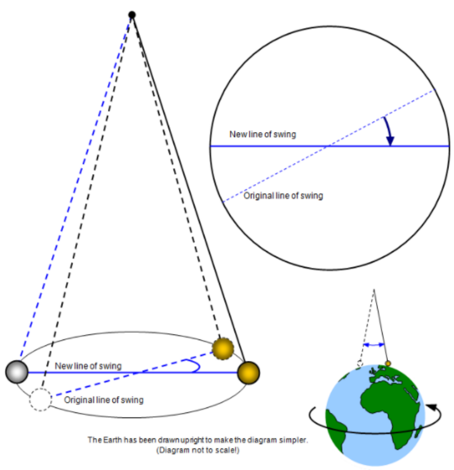
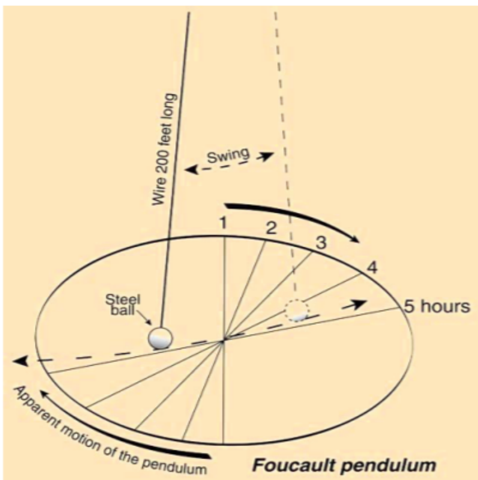
1. Problem Statement

A Foucault Pendulum is a simple but beneficial device. It was introduced by French physicist Sir Léon Foucault in 1851 as part of an experiment to demonstrate the theory of Earth's rotation. It consists of a large weight suspended on a cable attached to a point two to three floors above the weight. As opposed to a simple pendulum that oscillates in a single plane, the Foucault pendulum's plane of rotation changes with time. This phenomenon is primarily due to the rotation of the Earth. While the pendulum is oscillating in its plane, the Earth rotates beneath it. Hence relative motion exists between them.

The rate of rotation of the Foucault Pendulum is the rotation of Earth times the sine of the latitude's degrees where the pendulum is present. Thus, it is evident that the pendulum would rotate at the speed of Earth's rotation at the North pole. The results of the experiment successfully proved the phenomenon of Earth's rotation. Hence, it is crucial for us to understand the dynamics of the Foucault Pendulum.



(a) Image[2]



(b) Image[3]

Figure 1: Schematic representation of Foucault Pendulum

The dynamics of the Foucault Pendulum can be modelled mathematically using Newton's laws of motion. Reasonable assumptions that concur with the model in a practical sense are made for simplification of the governing equations. We employ various numerical methods, having different orders of accuracy, to conduct a comprehensive study of the dynamics involved with the pendulum's swinging motion. We will also compare and examine the results obtained by numerical methods with those obtained through analytical methods.

2. PHYSICAL MODEL

The goal of this project is to study the dynamics of the Foucault Pendulum which is subjected to perturbations due to the Coriolis effect. In doing so, we will consider the below mentioned physical model.

The entire problem will be looked at from the terrestrial spherical frame of reference. Let the pendulum be situated at an altitude of ϕ with respect to origin, which is at the center of the Earth. The position of the pendulum will be noted according to both the rectangular coordinate system and spherical coordinate system. The vectors U_x, U_y, U_z represent the base of the rectangular coordinate system. On the other hand, the vectors U_r, U_θ, U_ϕ are the base of the spherical coordinate system. Here, the bob of the pendulum has the weight of P, the wire has tension T, and l is the length of the wire.

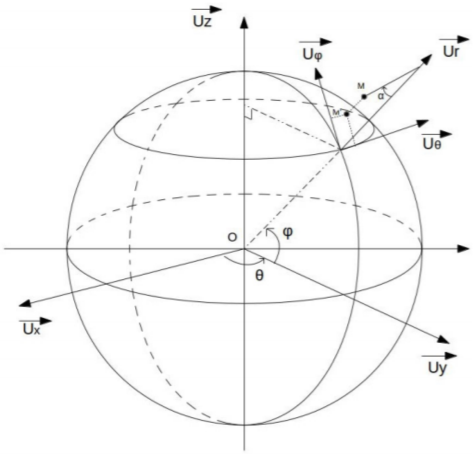


Figure 2: Terrestrial Spherical Frame of Reference

The pendulum oscillates in the plane which if considered fixed in space can be used to compute the rotation of Earth with respect to it. While figuring out the mathematical governing equations for the dynamics of the pendulum, we will consider the coriolis force which is a result of Earth's rotation. The perturbations created by the coriolis forces reduce the accuracy of the pendulum. This will be considered in the 'Governing Equations' section.

Notations

1. $\mathbf{U}_i$: Position vector of pendulum

2. ϕ: Latitude of pendulum

3. Ω: Earth's rotational vector

4. $\mathbf{P}$: Weight of the bob of pendulum

5. $\mathbf{T}$: Tension in the wire of pendulum

6. $\mathbf{l}$: Length of the wire of pendulum

7. Γ: Bob's acceleration vector

8. $\mathbf{m}$: Mass of the pendulum bob

9. $\mathbf{g}$: Gravitational acceleration

3. Assumptions

We will make the following assumptions to study the dynamics of the Foucault Pendulum.

1. Perturbations due to Coriolis Effect are considered, but perturbations due to other forces are not.

2. Earth is considered to be a perfect sphere.

3. Earth is assumed to be an inertial frame of reference.

4. Latitude of the pendulum is constant.

5. Rotation speed of Earth is constant.

6. Variation in the position vector of bob is negligible in comparison with length of the pendulum.

4. Governing Equations

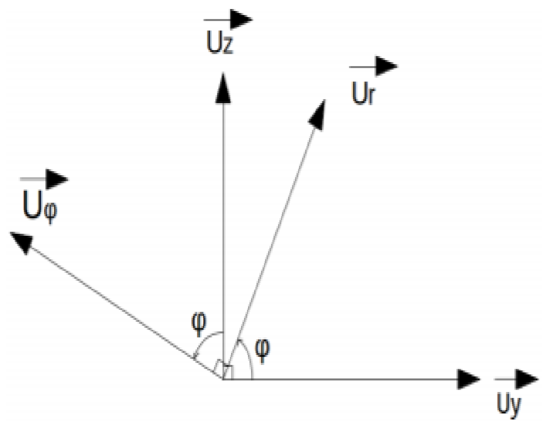


Figure 3: Plane depicting the projection of Earth's rotational vector onto local frame of reference

According to our physical model, we consider the Earth to rotate about an axis $\overrightarrow{U_z}$.

For our convenience, we will study the dynamics of the Foucault Pendulum on a local scale. In order to do so, we project the Earth's rotation on to the local frame of reference as depicted in Figure 3.

Now $\overrightarrow{U_z}$ is expressed as

$$\overrightarrow{U_z} = \sin\phi\overrightarrow{U_r} + \cos\phi\overrightarrow{U_\phi} \tag{1}$$

Earth's rotational vector $\overrightarrow{\Omega}$, can be expressed in the local frame of reference as follows:

$$\begin{Bmatrix} 0 \\ 0 \\ \Omega \end{Bmatrix}_{(x,y,z)} = \begin{Bmatrix} \Omega\sin\phi \\ 0 \\ \Omega\cos\phi \end{Bmatrix}_{(r,\theta,\phi)} \tag{2}$$

The local frame of reference in which the pendulum swings is depicted below:

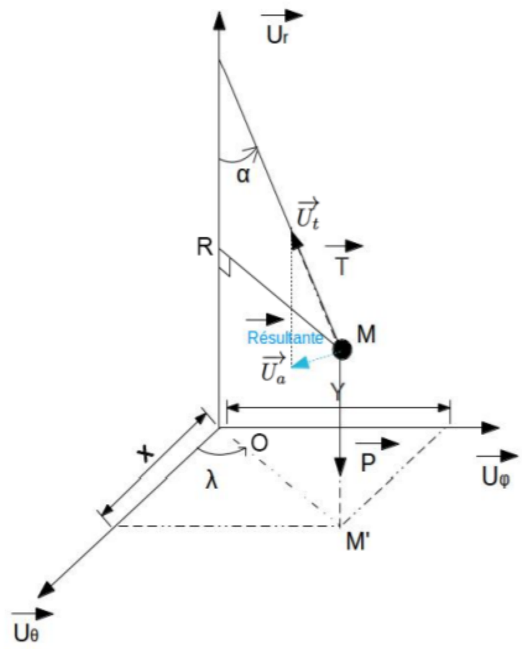


Figure 4: Local frame of reference of pendulum

Let M denote the instantaneous position of the bob during its swinging motion. Let R denote the projection of point M on the axis $\overrightarrow{U_r}$.

$$r = Radius\ of\ Earth\ +\ |\overrightarrow{OR}| \tag{3}$$

x , y - positions in projection to vectors $\overrightarrow{U_\theta}$ and $\overrightarrow{U_\phi}$ respectively.

$$\overrightarrow{OM} = \overrightarrow{OR} + \overrightarrow{RM} = r\overrightarrow{U_r} + x\overrightarrow{U_\theta} + y\overrightarrow{U_\phi} \tag{4}$$

Given that the position vector of the bob is $\overrightarrow{OM}$,

Velocity of the bob,

$$\vec{V}(\overrightarrow{OM})_R = \frac{d(\overrightarrow{OM})}{dt_{(R)}} = \frac{d}{dt_{(R)}}(r\overrightarrow{U_r}+x\overrightarrow{U_\theta}+y\overrightarrow{U_\phi}) = \dot{r}\overrightarrow{U_r}+\dot{x}\overrightarrow{U_\theta}+\dot{y}\overrightarrow{U_\phi}+r\frac{d(\overrightarrow{U_r})}{dt_{(R)}}+x\frac{d(\overrightarrow{U_\theta})}{dt_{(R)}}+y\frac{d(\overrightarrow{U_\phi})}{dt_{(R)}} \tag{5}$$

Since, the vectors $\overrightarrow{U_r}$, $\overrightarrow{U_\theta}$ and $\overrightarrow{U_\phi}$ are vectors of the local frame of reference which is non-inertial, the time derivatives of these vectors are dependent on the inertial frame of reference.

In our experiment, the Earth is assumed to be immobile or stationary. Hence, Earth is the inertial frame of reference. The time derivative of a vector $\overrightarrow{U_i}$ in the non-inertial frame of reference, with respect to the inertial frame of reference is expressed as:

$$\frac{d(\overrightarrow{U_i})}{dt_{(R)}} = \frac{d(\overrightarrow{U_r})}{dt_{(R')}} + \overrightarrow{\Omega_{(R'/R)}} \times \overrightarrow{U_i} \tag{6}$$

where $\overrightarrow{U_i}$ is the position vector of the pendulum, R is the terrestrial frame of reference or Earth, R' is the local frame of reference. As the local frame of reference does not accelerate with respect to its own reference,

$$\frac{d(\overrightarrow{U_i})}{dt_{(R')}} = 0 \tag{7}$$

The time derivative of local frame vector with respect to its own frame will be zero. Hence, the derivatives of the local plane vectors $\overrightarrow{U_r}$, $\overrightarrow{U_\theta}$ and $\overrightarrow{U_\phi}$ with respect to time are as follows:

$$\begin{aligned}
\frac{d(\overrightarrow{U_r})}{dt_{(R)}} &= \frac{d(\overrightarrow{U_r})}{dt_{(R')}} + \begin{bmatrix} \Omega\sin\phi \\ 0 \\ \Omega\cos\phi \end{bmatrix}_{(r,\theta,\phi)} \times \begin{bmatrix} U_r \\ 0 \\ 0 \end{bmatrix}_{(r,\theta,\phi)} = \begin{bmatrix} 0 \\ \Omega\cos\phi \\ 0 \end{bmatrix}_{(r,\theta,\phi)} \\
\frac{d(\overrightarrow{U_\theta})}{dt_{(R)}} &= \frac{d(\overrightarrow{U_\theta})}{dt_{(R')}} + \begin{bmatrix} \Omega\sin\phi \\ 0 \\ \Omega\cos\phi \end{bmatrix}_{(r,\theta,\phi)} \times \begin{bmatrix} 0 \\ U_\theta \\ 0 \end{bmatrix}_{(r,\theta,\phi)} = \begin{bmatrix} -\Omega\cos\phi \\ 0 \\ \Omega\sin\phi \end{bmatrix}_{(r,\theta,\phi)} \\
\frac{d(\overrightarrow{U_\phi})}{dt_{(R)}} &= \frac{d(\overrightarrow{U_\phi})}{dt_{(R')}} + \begin{bmatrix} \Omega\sin\phi \\ 0 \\ \Omega\cos\phi \end{bmatrix}_{(r,\theta,\phi)} \times \begin{bmatrix} 0 \\ 0 \\ U_\phi \end{bmatrix}_{(r,\theta,\phi)} = \begin{bmatrix} 0 \\ -\Omega\sin\phi \\ 0 \end{bmatrix}_{(r,\theta,\phi)}
\end{aligned} \tag{8}$$

where $\overrightarrow{U_r}$, $\overrightarrow{U_\theta}$ and $\overrightarrow{U_\phi}$ are unit vectors.

The variation in 'r' with respect to the pendulum length and oscillation can be neglected.

Hence,

$$\frac{dr}{dt} = \dot{r} = 0 \tag{9}$$

The velocity of the bob is given by:

$$\begin{aligned}\overrightarrow{V}(\overrightarrow{OM})_R &= 0 + r\Omega\cos\phi\overrightarrow{U_\theta} + \dot{x}\overrightarrow{U_\theta} + x\Omega(\sin\phi\overrightarrow{U_\phi} - \cos\phi\overrightarrow{U_r}) + \dot{y}\overrightarrow{U_\phi} - y\Omega\sin\phi\overrightarrow{U_\theta} \\ &= (-x\Omega\cos\phi)\overrightarrow{U_r} + (r\Omega\cos\phi - y\Omega\sin\phi + \dot{x})\overrightarrow{U_\theta} + (x\Omega\sin\phi + \dot{y})\overrightarrow{U_\phi}\end{aligned} \tag{10}$$

We will also consider some key assumptions in this model.

1. The latitude of the pendulum is constant. ($\dot{\phi} = 0$)

2. The rotation speed of Earth is constant. ($\dot{\Omega} = 0$)

Using these assumptions, the expression for the final acceleration of the bob can be obtained as follows :

$$\begin{aligned}\overrightarrow{\Gamma}(\overrightarrow{OM}) &= \frac{\overrightarrow{V}(\overrightarrow{OM})}{dt_{(R)}} \\ &= (\ddot{x} - 2\dot{y}\Omega\sin\phi - x\Omega^2)\overrightarrow{U_\theta} + (\ddot{y} + 2\dot{x}\Omega\sin\phi - y\Omega^2\sin^2\phi + r\Omega^2\cos\phi\sin\phi)\overrightarrow{U_\phi}\end{aligned} \tag{11}$$

Now, let us consider the forces which are acting on the pendulum's bob.

The weight of the bob 'P'.

The tension in wire 'T'.

The net force acting on the bob is given by the following expression:

$$\begin{aligned}\Sigma(\overrightarrow{F^{ext/b}}) &= \overrightarrow{T} + \overrightarrow{P} \\ &= T\overrightarrow{U_t} - P\cos\alpha\overrightarrow{U_t} + P\sin\alpha\overrightarrow{U_a}\end{aligned} \tag{12}$$

We project the force vectors onto vectors that are parallel and orthogonal to the pendulum's swinging movement.

α- Instantaneous angle of pendulum's movement

The projection onto vector $\overrightarrow{U_t}$ gives

$$T - P\cos\alpha = 0 \tag{13}$$

Therefore, the only term of the net force acting on the bob is P $\sin\alpha\overrightarrow{U_a}$.

Consider the projection of the net force vector $\overrightarrow{U_a}$ onto the vectors $\overrightarrow{U_\theta}$ and $\overrightarrow{U_\phi}$.

P = mg

m - mass of the bob

g - gravitational acceleration at location where pendulum is suspended

M - instantaneous position of bob

R - projection of point M onto axis Ur

l - length of pendulum's wire

$$\begin{aligned}\Sigma(\overrightarrow{F^{ext/b}}) &= mg\sin\alpha\overrightarrow{U_a} \\ &= mg\frac{RM}{l}\overrightarrow{U_a} \\ &= \frac{mg}{l}(x\overrightarrow{U_\theta} + y\overrightarrow{U_\phi})\end{aligned} \tag{14}$$

Employing Newton's Second Law of motion,

$$\Sigma(\overrightarrow{F^{ext/b}}) = m\overrightarrow{\Gamma} \tag{15}$$

Using earlier expression of net force, we can rewrite the second law as,

$$\frac{mg}{l}(x\overrightarrow{U_\theta} + y\overrightarrow{U_\phi}) = m\Gamma\overrightarrow{U_a} \tag{16}$$

The pendulum's period is given by the expression ω_o= $\sqrt{g/l}$

By replacing the acceleration using the expression given by Eq 11.

$$m\omega_o^2(x\overrightarrow{U_\theta}+y\overrightarrow{U_\phi}) = m(\ddot{x}-2\Omega\sin\phi\dot{y}-x\Omega^2)\overrightarrow{U_\theta}+m(\ddot{y}+2\Omega\sin\phi\dot{x}-y\Omega^2\sin^2\phi+r\Omega^2\cos\phi\sin\phi)\overrightarrow{U_\phi} \tag{17}$$

To simplify the expression, we will project the acceleration on their respective vectors

$$\begin{aligned}\overrightarrow{U_\theta} : \ddot{x}-2\Omega\sin\phi\dot{y}-x\Omega^2 &= -\omega_o^2 x\\ \overrightarrow{U_\phi} : \ddot{y}+2\Omega\sin\phi\dot{x}-y\Omega^2\sin^2\phi+r\Omega^2\cos\phi\sin\phi &= -\omega_o^2 y\end{aligned} \tag{18}$$

We may ignore the terms xΩ^2 and y$\Omega^2\sin^2\phi$ as they have a value of the order 10^{-10} m/s^2 and can be neglected with respect to other terms in the equation.
Hence, the final equations which describe the dynamics of the Foucault Pendulum have been derived.

$$\begin{aligned}&\ddot{x}-2\Omega\sin\phi\dot{y}+\omega_o^2 x = 0\\ &\ddot{y}+2\Omega\sin\phi\dot{x}+\omega_o^2 y = -r\Omega^2\cos\phi\sin\phi\end{aligned} \tag{19}$$

5. Analytical Solution

The governing equations obtained are a set of non-homogeneous second order linear equations. Here ω=$\sqrt{g/l}$; where g is taken as 9.81m/s^{2}and l = 8 m.
Latitude of laboratory(ϕ) is 0.872665 rad and angular rotation speed of Earth(Ω) is taken as 7.2921150 x 10^{-5} rad/s.
r = l + Earth's radius = 6.371397 x 10^6m.

All these measurements are the values according to the position of the laboratory. For solving the non-homogeneous system we first consider the system of linear homogeneous equation given below

$$\begin{aligned} \ddot{x_o} - 2\Omega \sin\phi \dot{y_o} + \omega_o^2 x_o = 0 \\ \ddot{y_o} + 2\Omega \sin\phi \dot{x_o} + \omega_o^2 y_o = 0 \end{aligned} \tag{20}$$

Multiplying the second equation by ι and adding and with first gives the following single differential equation where z is a complex variable assuming the form $z_0=x_0+\iota y_0$

$$\ddot{z_o} + 2\iota\Omega \sin\phi \dot{z_o} + \omega_o^2 z_o = 0 \tag{21}$$

The characteristic equation for the differential equation yields the following values:

$$r_{1,2} = -\iota\Omega \sin\phi \pm \iota\sqrt{(\Omega \sin\phi)^2 + \omega_o^2} \tag{22}$$

This gives the solution of (21) as,

$$z_o(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} \tag{23}$$

C_1 and C_2 can be obtained from the initial conditions at time t=0. The initial conditions are obtained from the following two conditions:

$$\begin{aligned} z_o(t=0) = C_1 + C_2 \\ \dot{z_o}(t=0) = C_1 r_1 + C_2 r_2 \end{aligned} \tag{24}$$

$$C_{1,2} = 0.5 \pm \frac{\Omega \sin\phi}{\sqrt{(\Omega \sin\phi)^2 + \omega_o^2}} \tag{25}$$

The calculated values are:

ω_o=1.10736173

r_1=1.107304141

r_2= -1.107415862

C_1=0.5000504451

C_2=0.4999495549

Here the homogeneous solution takes the form:

$$z_o(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} \tag{26}$$

Now to obtain the analytical solution of the general governing equation.

For obtaining the particular solution we take the equilibrium position and at the equilibrium position acceleration and velocity both in x and y directions are zero hence we obtain x_{eq}=0 and y_{eq}=$-r\Omega^2 \cos\phi \sin\phi / \omega_o^2$ (y_{eq}=-0.01360454653m). Here $y_{eq}$is a particular solution.

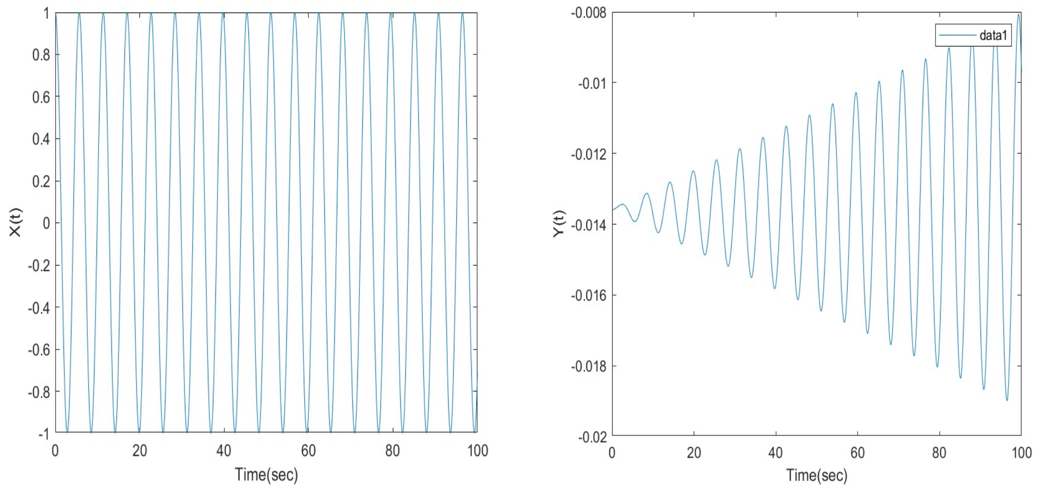
Thus the final solution obtained can be given as:

$$z(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + i\, y_{eq} \tag{27}$$

We can split the real and imaginary part of z(t) to obtain the functions of x(t) and y(t)respectively. This gives:

$$\begin{aligned} x(t) &= C_1 \cos(r_1 t) + C_2 \cos(r_2 t) \\ y(t) &= C_1 \sin(r_1 t) + C_2 \sin(r_2 t) + y_{eq} \end{aligned} \tag{28}$$

The y_{eq} and x_{eq} are the new positions of equilibrium, since y_{eq} changes the apparent vertical changes by an angle, $\alpha = \sin\alpha = \frac{y_{eq}}{l}$=-1.700568316x$10^{-3}$ rad.



(a) Scale for x-axis is 1 sec

(b) Scale for x-axis is 1 sec

Figure 5: Plots obtained from Analytical Solution

6. Numerical Methods

The length of the string of the pendulum we have considered is l = 8 m. The Earth's radius + the height of the pendulum bob from the ground, r = 6.371397 x 10^6 m. Let us assume the angular velocity of Earth's rotation to be Ω = 7.292 x 10^{-5} rad/s. Let us assume the latitude of the place where this experiment was performed be ϕ = 0.872665 rad (50^o). Let us assume that the acceleration due to gravity has a value g = 9.81 m/sec^2. We know that $\omega_o = \sqrt{(g/l)}$. We have performed the simulation for a duration of 2 hrs. Assuming a step size of h = 0.1 sec.

We convert Eq (19) to a system of first order ODE's.

First we consider,

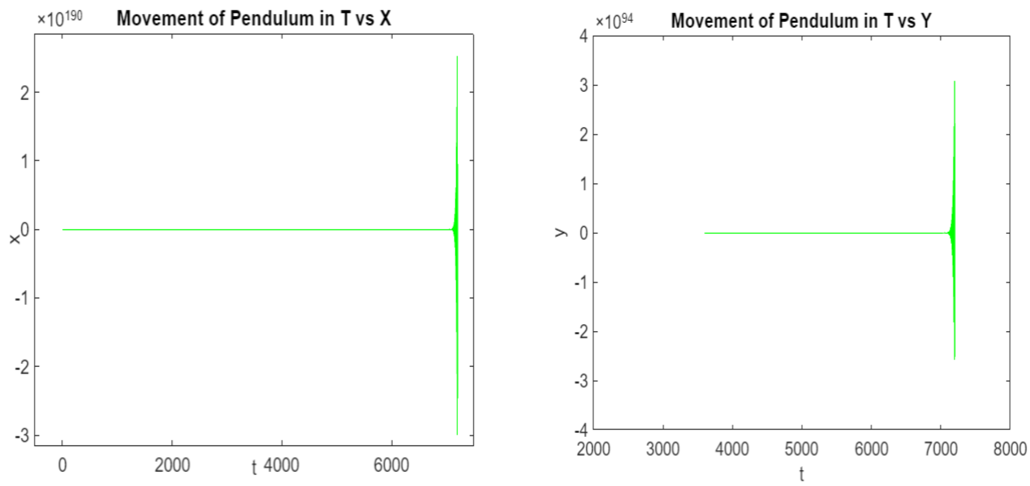
$$\begin{aligned} \frac{dx}{dt} &= m \\ \frac{dy}{dt} &= n \end{aligned} \tag{29}$$

Substituting the above we get the following system of first order ODE's,

$$\begin{aligned}
\frac{dx}{dt} &= m \\
\frac{dy}{dt} &= n \\
\frac{dm}{dt} &= 2\Omega n \sin(\phi) - \omega_o^2 x \\
\frac{dn}{dt} &= -2\Omega m \sin(\phi) - \omega_o^2 y - r\Omega^2 \sin(\phi)\cos(\phi)
\end{aligned} \tag{30}$$

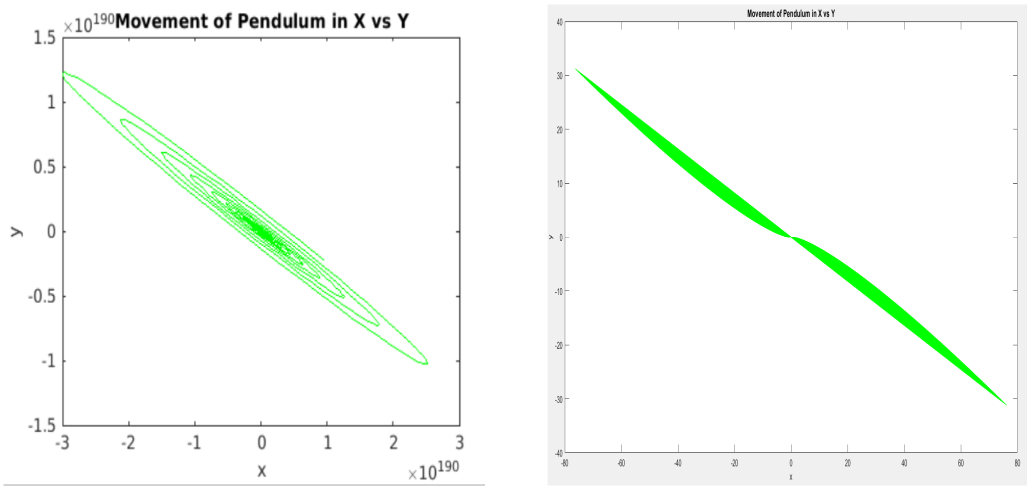
We have developed MATLAB codes for the following five methods and using them we simulate the above ODEs. We then visualise the pendulum's movement using the figures obtained by running the simulation.

6.1. Euler's Method



(a) Scale for X-axis is 1 sec **(b)** Scale for X-axis is 1 sec

Figure 6: x vs t and y vs t for Euler's method



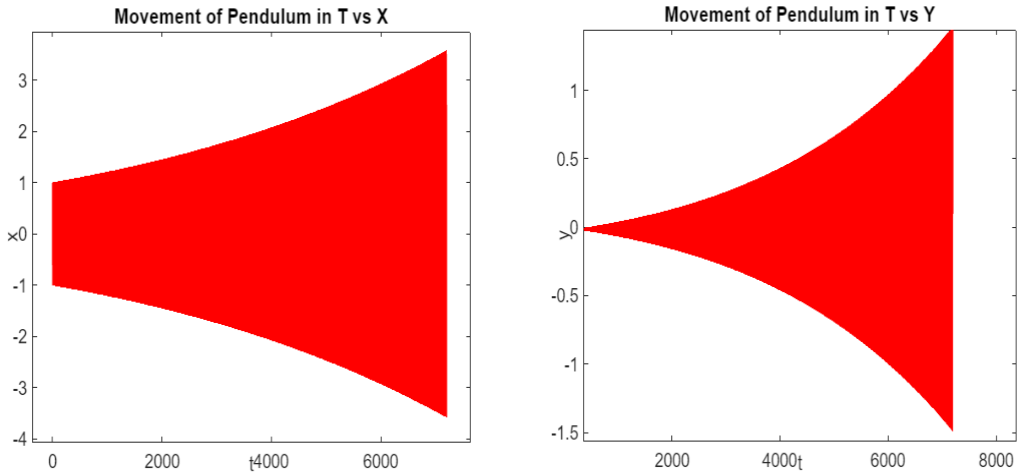
(a) Scale for X-axis is 1 m

(b) Scale for X-axis is 1 m and h =0.001

Figure 7: x vs y plots for Euler's method

By using Euler's method we obtained the curves as seen in Figures 6 and 7. It is clearly visible that results obtained in figure 6.a, 6.b and 7.a are absurd as the Y-axis is of the order $10^{94}/10^{190}$. This scale is practically impossible for a pendulum's movement. Hence we tried the simulation of x vs y by reducing the step size h to 0.001 sec. The result obtained by this reduction can be seen in figure 7.b. Further reduction in step size did not seem feasible as the number of iterations shot up to extremely large numbers. Thus we observe that Euler's method is not appropriate for studying the dynamics of Foucault's pendulum numerically.

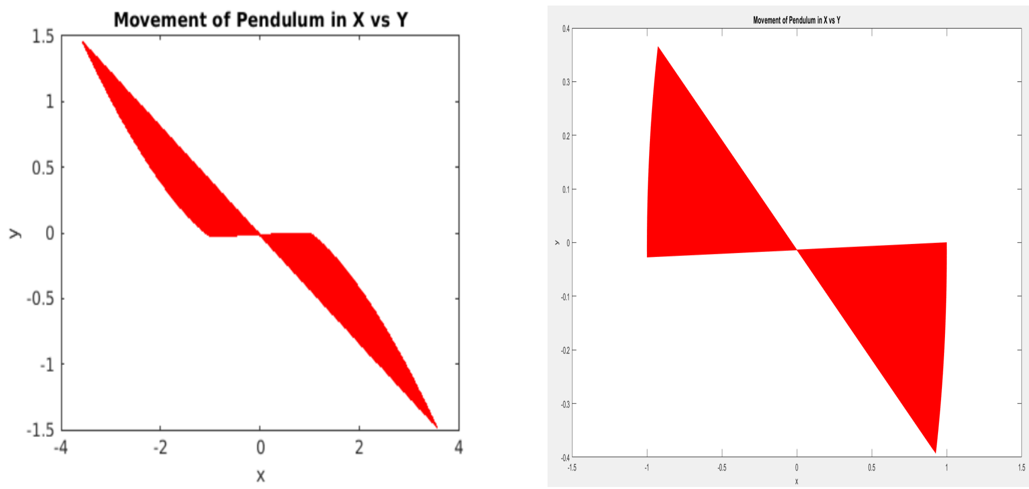
6.2. Heun's Method



(a) Scale for X-axis is 1 sec

(b) Scale for X-axis is 1 sec

Figure 8: x vs t and y vs t plots for Heun's method



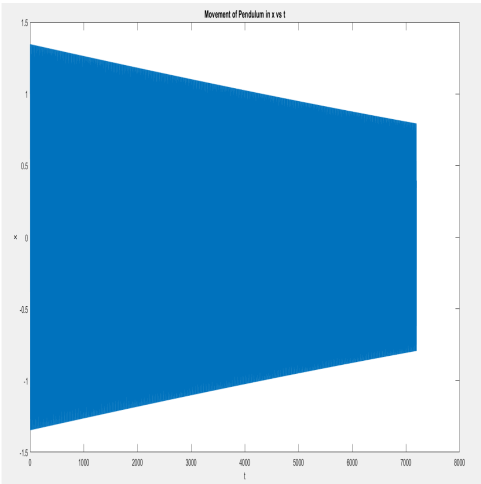
(a) Scale for X-axis is 1 m

(b) Scale for X-axis is 1 m and h = 0.01

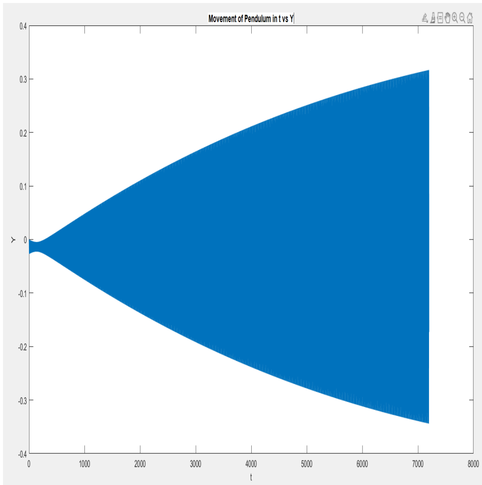
Figure 9: x vs y plots for Heun's method

By using Heun's method we obtained the curves as seen in Figures 8 and 9. There is a clear distinction between the results obtained by Euler's method and the ones obtained by Heun's method. Heun's method can be seen clearly outperforming Euler's method. Yet, figure 9.a does not seem to very closely resemble the actual solution. The reason behind it might be the step size chosen. Hence we reduce the step size to 0.01 to obtain figure 9.b. This curve resembles very closely the actual analytical solution. Hence we observe that Heun's method is preferable over Euler's method for better accuracy though the complexity of implementation is more in this case.

6.3. 3rd Order RK Method



(a) Scale for X-axis is 1 sec



(b) Scale for X-axis is 1 sec

Figure 10: x vs t and y vs t plots for 3rd order Runge Kutta method

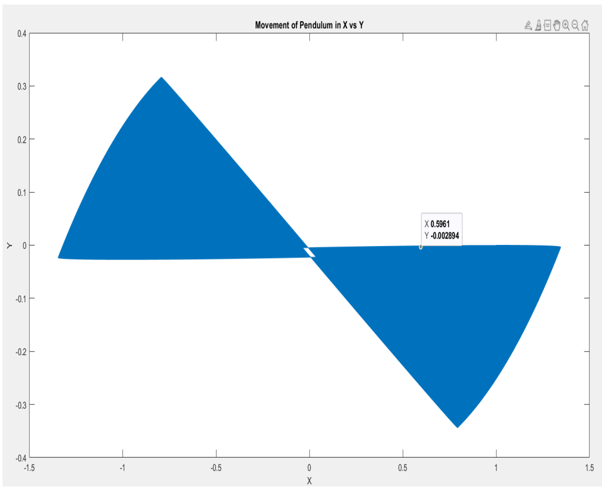
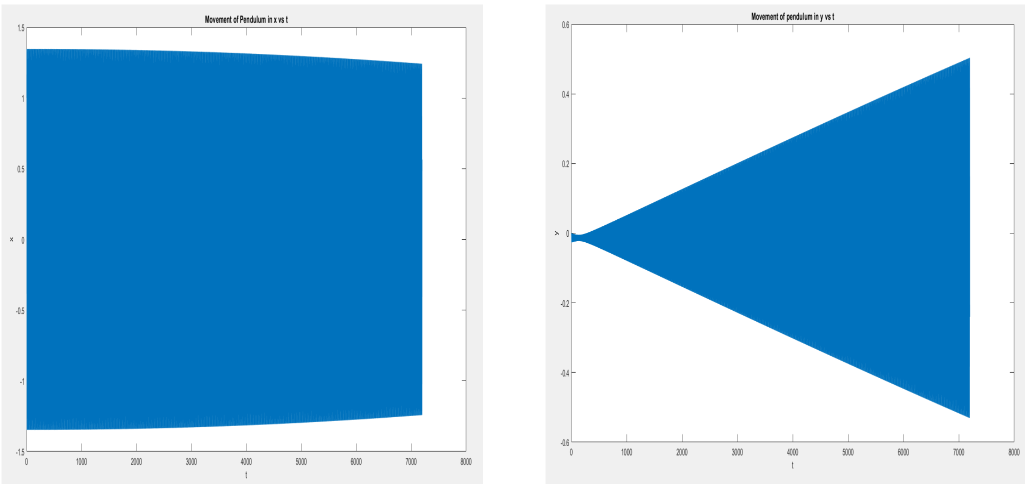


Figure 11: x vs y plot for 3rd order RK method

By using the third-order Runge-Kutta method we obtained the curves as seen in Figures 10 and 11. Figure 11 clearly indicates that the accuracy of this method is way better than the two previously used methods. But at the same time, the implementation complexity of this method is also quite high. So the trade-off between accuracy and complexity holds. Also, there is no need to change the step size in this method as the results obtained resemble the analytical solutions to a good extent.

6.4. 4th Order RK Method



(a) Scale for X-axis is 1 sec

(b) Scale for X-axis is 1 sec

Figure 12: x vs t and y vs t plots for 4th order RK method

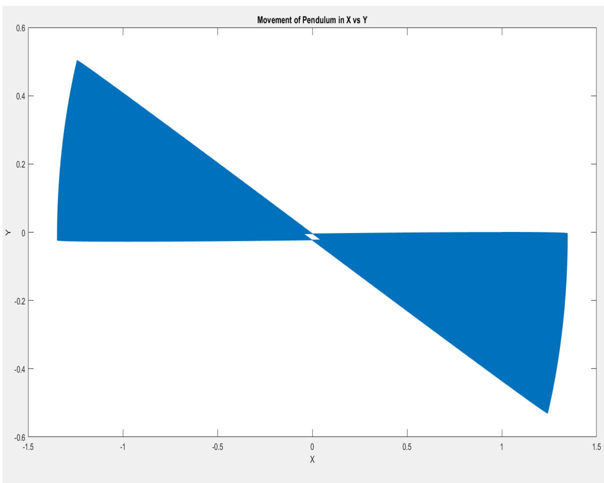
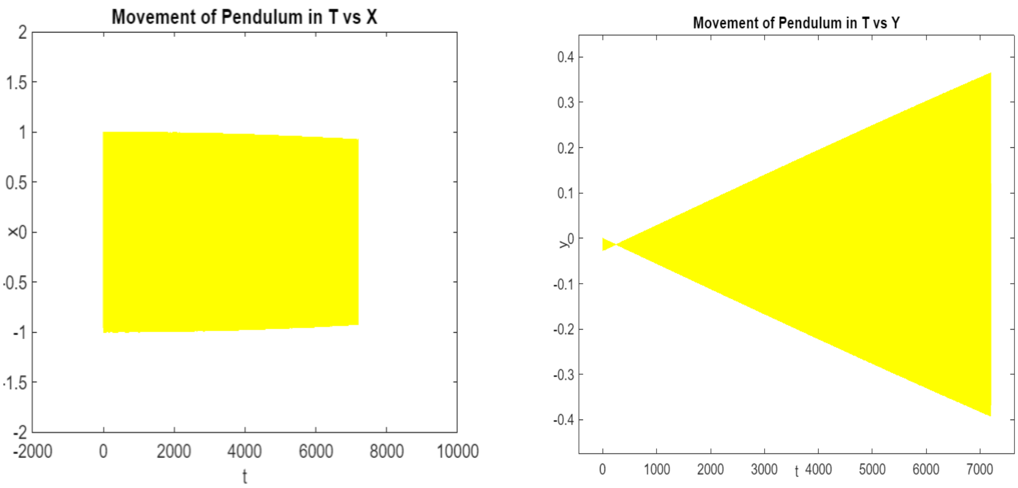


Figure 13: x vs y plot for 4th order RK method

By using the fourth-order Runge-Kutta method we obtained the curves as seen in Figures 12 and 13. Figure 13 clearly indicates that the accuracy of this method is much better than the three previously used methods. But at the same time, the implementation complexity of this method is also very high. In fact, it is higher than that of all the previously used methods. The results obtained resemble the analytical solutions to a very good extent.

6.5. 5th Order RK Method



(a) Scale for X-axis is 1 sec **(b)** Scale for X-axis is 1 sec

Figure 14: x vs t and y vs t plots for 5th order RK method

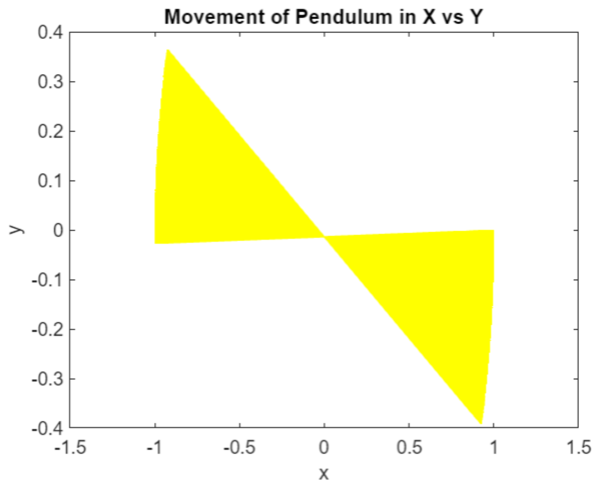
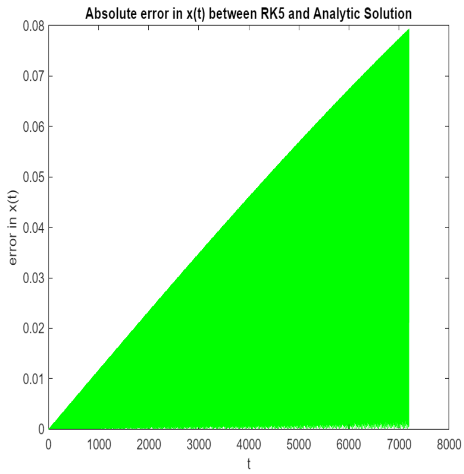


Figure 15: x vs y plots for 5th order RK method

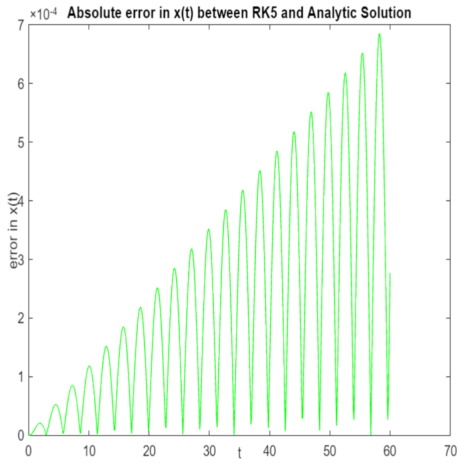
By using the fifth-order Runge-Kutta method we obtained the curves as seen in Figures 14 and 15. Figure 15 clearly states that the accuracy of this method is the best amongst all the analytical methods used. But at the same time, the implementation complexity of this method is also the highest. The results obtained resemble the analytical solutions to the best extent. Hence it is safe to assume that the most appropriate method to numerically analyse the above ODEs is the fifth-order Runge-Kutta Method. Since, in our experiment, a great deal of precision is required in the analysis of the dynamics of the pendulum, we shall prefer the accuracy of the numerical method over its implementation complexity. Hence we shall finally consider the fifth-order Runge-Kutta method for our study.

6.6. Error Analysis

We have calculated the absolute difference of x(t) and y(t) vs time from the solutions obtained from the 5th order RK method and the analytical solution.



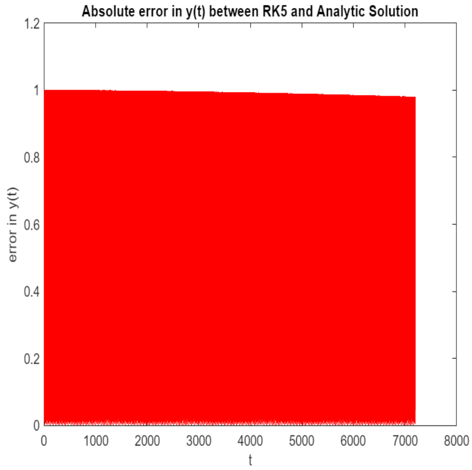
(a) Scale for X-axis is 1 sec



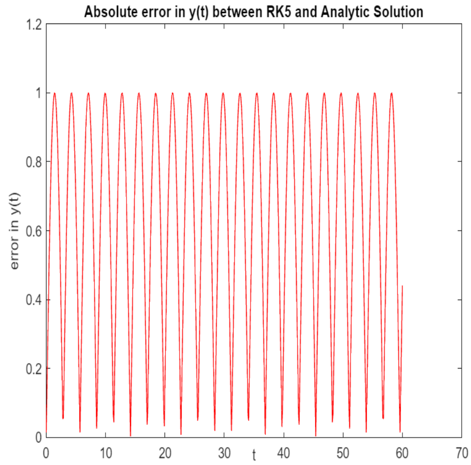
(b) Scale for X-axis is 1 sec

Figure 16: Absolute error in numerical and analytical x vs time

As we can see above the amplitude of the absolute error of x(t) increases over time.



(a) Scale for X-axis is 1 sec



(b) Scale for X-axis is 1 sec

Figure 17: Absolute error in analytic and numerical y vs time

Here there is periodic variation of the error but amplitude is almost constant.

7. ALGORITHM USED

To solve the given ODEs numerically, we need to use the following algorithm by converting it to a MATLAB code:

1. Write down all the constants.

2. Convert the two second-order ODEs into four first-order derivatives by denoting the first derivative of x and y by two new variables.

3. Provide the initial conditions for the four first-order ODEs.

4. Initialise a loop that runs until the simulation time of two hours is reached.

5. Within the loop, there will be six different constants for each of the four variables whose values we have to determine.

6. These constants will be the function of either the four variables alone or will also include the previously calculated constants.

7. So, if our variable is ‘z’ then:

$$z_{i+1} = z_i + \frac{h}{90}(7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6) \tag{31}$$

Where:

$$\begin{aligned}
&k_1 = f(t_i, z_i)\\
&k_2 = f(t_i + h/4, z_i + k_1 h/4)\\
&k_3 = f(t_i + h/4, z_i + k_1 h/8 + k_2 h/8)\\
&k_4 = f(t_i + h/2, z_i - k_2 h/2 + k_3 h)\\
&k_5 = f(t_i + 3h/4, z_i + 3k_1 h/16 + 9k_4 h/16)\\
&k_6 = f(t_i + h, z_i - 3k_1 h/7 + 2k_2 h/7 + 12k_3 h/7 - 12k_4 h/7 + 8k_5 h/7)
\end{aligned} \tag{32}$$

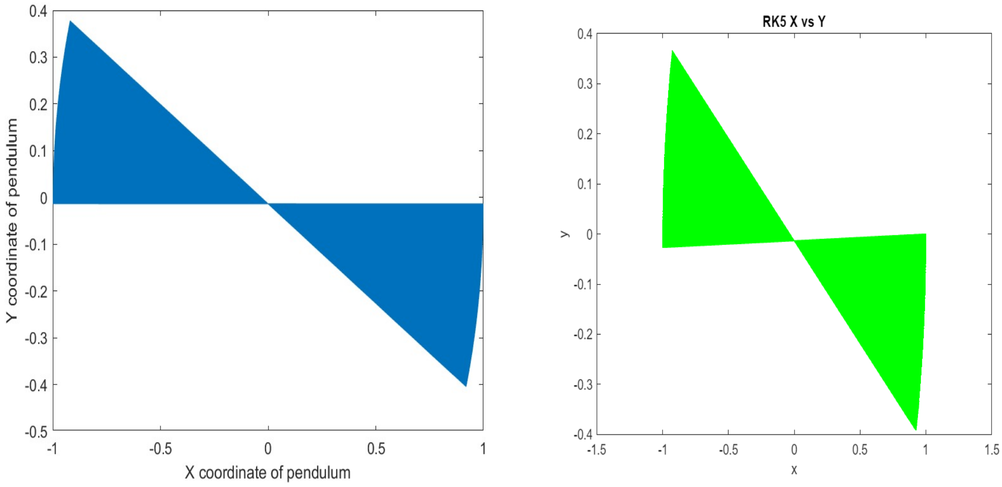
And ‘t’ is the variable ‘z’ is dependent on.

8. Update the above constants and the variables in each ith iteration and at the same time also update the time taken by the step size.

8. Finally, use the ‘plot’ function of MATLAB to obtain the various plots required.

8. Results and Discussions

We have successfully analysed the dynamics of Foucault’s pendulum for a simulation time of 2 hrs. The motion of the pendulum in the domain is obtained both analytically and numerically and is shown in the following graphs.



(a) Scale of X-axis is in 1 m

(b) Scale of X-axis is in 1 m

Figure 18: Analytic and RK 5: x vs y plots

9. CONCLUSIONS

It is evident from the plots of the motion of the pendulum that the plane of oscillation of the pendulum changes its position with time. Also, we have seen that the analytical solution is very cumbersome to obtain. On the contrary, numerical analysis is easy to implement and saves a lot of time and effort. But it must be noted that numerical techniques introduce small errors. Hence in order to minimise the errors, we use the most optimal numerical method which is the fifth-order Runge-Kutta method. The error introduced by this method periodically varies time but it never increases beyond a certain limit which is of the order 10^{-3} which is very less practically and is safe to neglect. Thus we prove the rotation of Earth by using the concept of Foucault's pendulum.

10. References

[1] Portier, Thomas Shukor, Ridhuan. 2013. "A Numerical and Experimental Study of the Dynamics of the Foucault Pendulum Subject to Perturbations". Laboratoire des Écoulements Géophysiques et Industriels.

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For images,

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