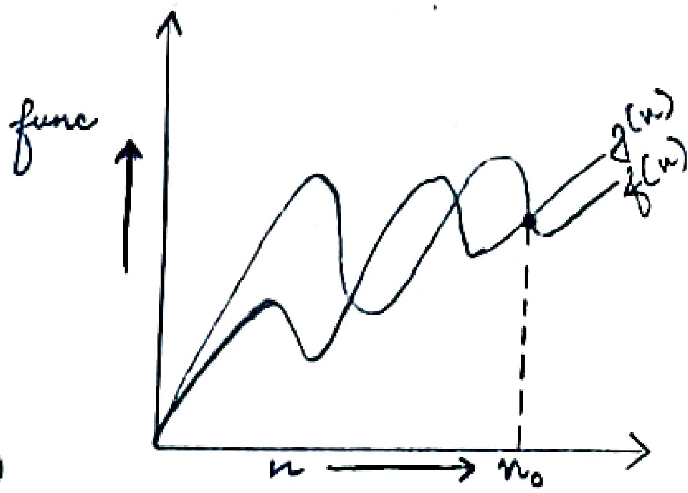
Q1. What do you understand by Asymptotic notation, define different asymptotic notation with example.

i) <u>Big O(n)</u>



$f(n) \Rightarrow O(g(n))$

if $f(n) \leq g(n) \times c \;\; \forall \; n \geq n_0$

for some constant, $c > 0$

$g(n)$ is 'tight' upper bound of $f(n)$

<u>eg</u>:- $f(n) \Rightarrow n^2 + n$

$g(n) \Rightarrow n^3$

$n^2 + n \leq c * n^3$

$n^2 + n = O(n^3)$

ii) <u>Big Omega (Ω)</u>

When $f(n) = \Omega(g(n))$

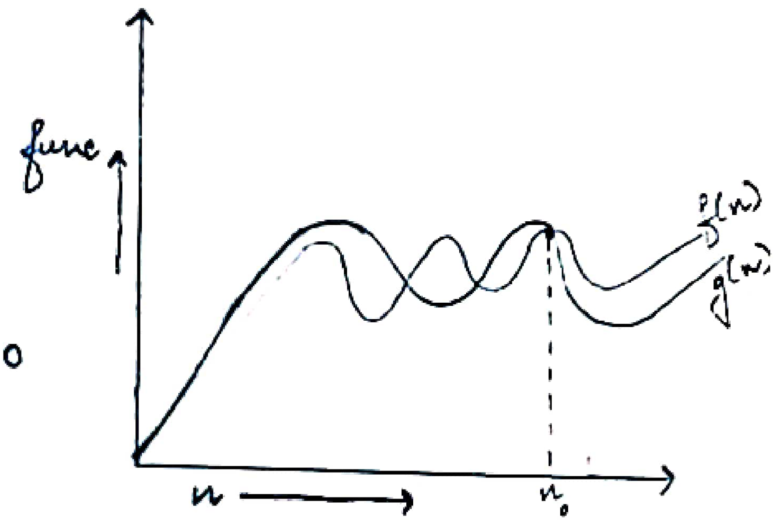
means $g(n)$ is "tight" lowerbound of $f(n)$ ie $f(n)$ can go beyond $g(n)$

ie $f(x) = \Omega \, g(n)$

if and only if

$f(n) \geq c.g(n)$

$\forall \; n_2 > n_0$ and c = constant > 0



<u>Ex</u>: $f(n) \Rightarrow n^3 + 4n^2$

$g(n) \Rightarrow n^2$

ie $f(n) \geq c * g(n)$

$n^3 + 4n^2 = \Omega(n^2)$

iii) <u>Big Theta (θ)</u>

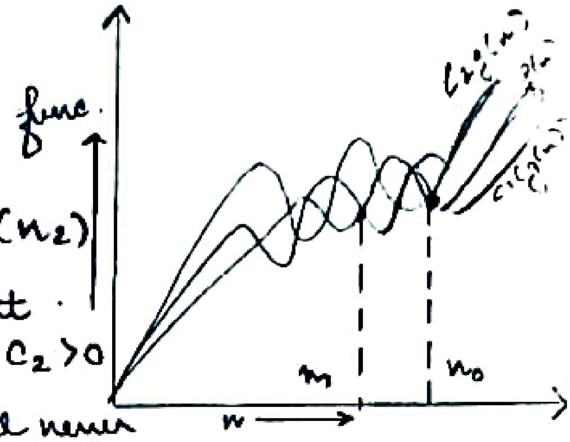
When $f(n) = \theta(g(n))$ gives the tight upperbound and lowerbound both.

ie $f(n) = \theta(g(n))$

if and only if

$$c_1 * g(n_1) \leq f(n) \leq c_2 * g(n_2)$$

for all $n \geq \max(n_1, n_2)$, some constant $c_1 > 0$ & $c_2 > 0$



ie. $f(n)$ can never go beyond $c_2 g(n)$ and will never come down of $c_1 g(n)$.

<u>Ex:-</u> $3n+2 = \theta(n)$ as $3n+2 \geq 3n$ &

$3n+2 \leq 4n$ for n, $c_1 = 3$, $c_2 = 4$ & $n_0 = 2$

iv) <u>Small o (θ)</u>

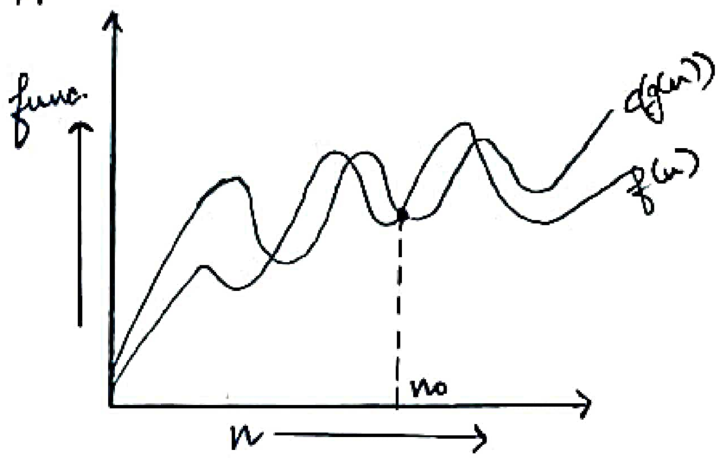
when $f(n) = o\, g(n)$ gives the upper bound

ie $f(n) = o\, g(n)$

if and only if

$$f(n) < c * g(n)$$

$\forall\ n > n_0$ & $n > 0$



<u>Ex:-</u> $f(n) = n^2$; $g(n) = n^3$

$$f(n) < c * g(n)$$

$$n^2 = o(n^3)$$

v) <u>Small Omega (ω)</u>

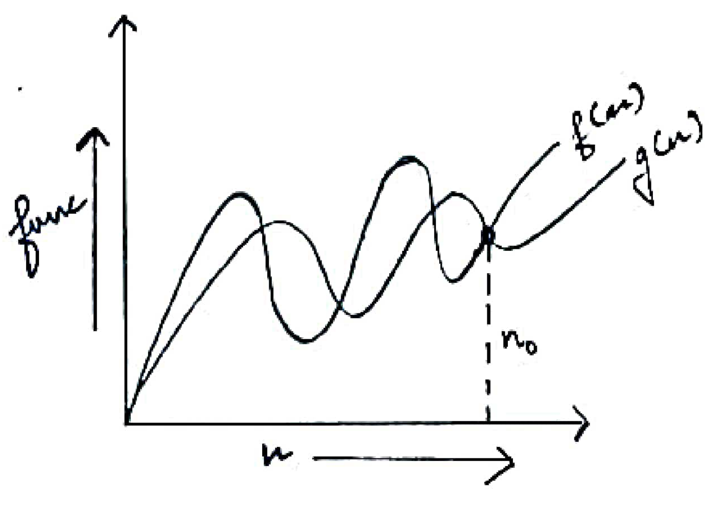
It gives the 'lower bound' ie

$$f(n) = \omega(g(n))$$

where $g(n)$ is lower bound of $f(n)$

if and only if $f(n) > c * g(n)$

$\forall\ n > n_0$ & some constant, $c > 0$



Q2 What should be Time complexity of:

```
for (int i = 1 to n)
  {
     i = i * 2;   → O(1)
  }
```

↳ for $i \Rightarrow 1, 2, 4, 6, 8 \dots$ n times

ie Series is a GP

So $a = 1$, $r = 2/1$

k^{th} value of GP:

$$t_k = a r^{k-1}$$

$$t_k = 1(2)^{k-1}$$

$$2n = 2^k$$

$$\log_2(2n) = k \log 2$$

$$\log_2 2 + \log_2 n = k$$

$$\log_2 n + 1 = k \quad (\text{Neglecting '1'})$$

So, Time Complexity $T(n) \Rightarrow \underline{O(\log_2 n)}$ → Ans.

Q3. $T(n) = \{ 3T(n-1)$ if $n > 0$
otherwise 1
}

↳ ie $T(n) \Rightarrow 3T(n-1)$ — (1)

$T(n) \Rightarrow 1$

put $n \Rightarrow n-1$ in (1)

$T(n-1) \Rightarrow 3T(n-2)$ — (2)

put (2) in (1)

$T(n) \Rightarrow 3 \times 3T(n-2)$

$T(n) \Rightarrow 9T(n-2)$ → (3)

put $n \Rightarrow n-2$ in (1)

$T(n-2) = 3T(n-3)$

put in (3).

$T(n) = 27T(n-3)$ → (4)

Generalising series,

$$T(k) = 3^k T(n-k) \quad — (5)$$

for k^{th} terms, let $n-k=1$ <u>Base case</u>

$$k = n-1$$

put in (5)

$$T(n) = 3^{n-1} T(1)$$

$$T(n) = 3^{n-1} \qquad (\underline{\text{neglecting } 3'})$$

$$\underline{T(n) = O(3^n)}$$

Q4. $T(n) = \{ \; 2T(n-1)-1$ if $n>0$,
otherwise 1
}

$$T(n) = 2T(n-1) - 1 \quad \rightarrow (1)$$

put $n = n-1$

$$T(n-1) = 2T(n-2) - 1 \quad \rightarrow (2)$$

put in (1)

$$T(n) = 2 \times (2T(n-2)-1) - 1$$

$$= 4T(n-2) - 2 - 1 \quad — (3)$$

put $n = n-2$ in (1)

$$T(n-2) = 2T(n-3) - 1$$

Put in (1)

$$T(n) = 8T(n-3) - 4 - 2 - 1 \quad — (4)$$

Generalising series

$$T(n) = 2^k T(n-k) - 2^{k-1} - 2^{k-2} \ldots 2^0$$

* <u>k^{th} term</u> Let $n-k=1$
$k = n-1$

$$T(n) = 2^{n-1} T(1) - 2^k \left(\frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^k} \right)$$

$$= 2^{n-1} - 2^{n-1} \left(\frac{1}{2} + \frac{1}{2^2} + \cdots \frac{1}{2^{n-1}} \right)$$

ie Series in GP.

$$a = 1/2 \quad , \quad r = 1/2.$$

So,

$$T(n) = 2^{n-1}\left(1 - \left(\frac{1}{2}\left(\frac{1-(1/2)^{n-1}}{1-1/2}\right)\right)\right)$$

$$= 2^{n-1}\left(1 - 1 + (1/2)^{n-1}\right)$$

$$= \frac{2^{n-1}}{2^{n-1}}$$

$$\underline{T(n) = O(1)} \text{ Ans}$$

Q5. What should be time complexity of

```
int i=1, s=1;
while (s<=n)
  {
    i++;
    s = s+i;
    printf("#");
  }
```

→ $i = 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \ldots$

$s = 1 + 3 + 6 + 10 + 15 + \ldots$

Sum of $s = 1 + 3 + 6 + 10 + \ldots + n \rightarrow 1)$

Also $S = 1 + 3 + 6 + 10 + \ldots T_{n-1} + T_n \rightarrow 2)$

$0 = 1 + 2 + 3 + 4 + \ldots n - T_n$

$T_k = 1 + 2 + 3 + 4 + \ldots + k$

$T_k = \frac{1}{2} k (k+1)$

for k iterations

$1 + 2 + 3 + \ldots k <= n$

$\frac{k(k+1)}{2} <= n$

$\frac{k^2 + k}{2} <= n$

$O(k^2) <= n$

$k = O(\sqrt{n})$

$\boxed{T(n) = O(\sqrt{n})}$ Ans.

Q6. Time Complexity of

```
void f(int n)
{
    int i, count=0;
    for(i=1; i*i<=n; ++i)
}
```

↳ As $i^2 = n$

$i = \sqrt{n}$

$i = 1, 2, 3, 4, \ldots \sqrt{n}$

$\sum_{i=1} 1+2+3+4+\ldots+\sqrt{n}$

$T(n) = \frac{\sqrt{n} * (\sqrt{n}+1)}{2}$

$T(n) = \frac{n * \sqrt{n}}{2}$

$\underline{T(n) = O(n)}$ → Ans.

Q7. Time Complexity of

```
void f(int n)
{
    int i, j, k, count=0;
    for(int i = n/2 ; i<=n ; ++i)
    for (j=1; j<=n; j=j*2)
    for ( k=1; k<=n; k=k*2)
        count++;
}
```

↳ Since, for $k = k^2$

$k = 1, 2, 4, 8, \ldots n$

∵ Series is in GP

So, $a=1$, $r=2$

$\frac{a(r^n-1)}{r-1}$

$= \frac{1(2^k-1)}{1}$

$n = 2^k - 1$

$n+1 = 2^k$

$\log_2(n) = k$

i	j	k
1	log(n)	log(n) * log(n)
2	log(n)	log(n) * log(n)
⋮	⋮	⋮
n	log(n)	log(n) * log(n)

$T.C \Rightarrow O(n * \log n * \log n)$

$\Rightarrow \underline{O(n \log^2(n))} \rightarrow$ Ans

Q8. Time Complexity of

```
void function (int n)
{
    if (n==1) return;
    for (i=1 to n) {
    for (j=1 to n) {
        printf ("*");
    }
}
    function (n-3);
}
```

↳ for (i=1 to n)

we get j→n times every turn

$\therefore i * j = n^2$

k^{th}, Now,

$T(n) = n^2 + T(n-3);$

$T(n-3) = (n^2 3)^2 + T(n-6);$

$T(n-6) = (n^3 6)^2 + T(n-9);$

and $T(1) = 1;$

Now, substitute each value in T(n)

$T(n) = n^2 + (n-3)^2 + (n-6)^2 + \dots + 1$

Let

$n - 3k = 1$

$k = (n-1)/3$ total terms $= k+1$

$T(n) = n^2 + (n-3)^2 + (n-6)^2 + \dots + 1$

$T(n) \simeq k n^2$

$T(n) \simeq (k-1)/3 * n^2$

So, $\underline{T(n) = O(n^3)} \rightarrow$ Ans

Q9. Time Complexity of :-

```
void function (int n)
{
    for (int i = 1 to n) {
        for (int j = 1; j <= n; j = j + i) {
            printf ("*");
        }
    }
}
```

↳ for $i = 1$ $\quad j = 1 + 2 + \dots$ $(n \geq j + i)$

$i = 2$ $\quad j = 1 + 3 + 5 \dots$ $(n \geq j + i)$

$i = 3$ $\quad j = 1 + 4 + 7 \dots$ $(n \geq j + i)$

n^{th} term of AP is

$T(n) = a + d \times m$

$T(m) = 1 + d \times m$

$(n-1)/d = m$

for $i = 1$ $\quad (n-1)/1$ times

$i = 2$ $\quad (n-1)/2$ times

$\vdots$

$i = n-1$

we get,

$$T(n) = i_1 j_1 + i_2 j_2 + \dots i_{n-1} j_{n-1}$$

$$= \frac{(n-1)}{2} + \frac{(n-2)}{2} + \frac{(n-3)}{3} + \dots 1$$

$$= n + n/2 + n/3 + \dots n/n-1 - n \times 1$$

$$= n\left[1 + 1/2 + 1/3 + \dots 1/n-1\right] - n * 1$$

$$= n \times \log n - n + 1$$

Since $\int 1/x = \log x$

$\underline{T(n) = O(n \log n)}$ → Ans.

Q10

For the Function $n^{-1}k$ & C^n, what is the asymptotic Relationship b/w these functions?

Assume that $k >= 1$ & $C > 1$ are constants. Find out the value of c & n_0 of which relationship holds.

→ As given n^k and c^n.

Relationship b/w n^k & c^n is

$$n^k = O(c^n)$$

$$n^k \leq a(c^n)$$

$\forall$ $n \geq n_0$ & constant, $a > 0$

for $n_0 = 1$; $C = 2$

$\Rightarrow$ $1^k < a^2$

$\Rightarrow$ $\underline{n_0 = 1 \ \& \ c = 2}$ → Ans