CT4 – P MAS – 16

Mock Exam A Solutions

ActEd Study Materials: 2016 Examinations

Subject CT4

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ISBN 178-1-4727-0082-5

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Mock Exam A – Solutions

Markers: This document sets out one approach to answering each of the questions. Please give credit for other valid approaches.

Solution 1

This topic is covered in Chapter 9.

If we choose the baseline hazard function to be:

$$\mu_x(0) = \mu_x(NF) \qquad [½]$$

the model would be:

$$\mu_x(ij) = \mu_x(0)\exp\left[\beta_1 z_1 + \beta_2 z_2 + \beta_{12} z_1 z_2\right] \qquad [1]$$

where:

$$z_1 = \begin{cases} 1 & \text{if } i = S \\ 0 & \text{if } i = N \end{cases}$$

$$z_2 = \begin{cases} 1 & \text{if } j = M \\ 0 & \text{if } j = F \end{cases} \qquad [½]$$

So:

$$\frac{\mu_x(SF)}{\mu_x(NF)} = 2.32 = e^{\beta_1} \Rightarrow \beta_1 = \ln 2.32 = 0.841567 \qquad [½]$$

$$\frac{\mu_x(NM)}{\mu_x(NF)} = 1.45 = e^{\beta_2} \Rightarrow \beta_2 = \ln 1.45 = 0.371564 \qquad [½]$$

$$\frac{\mu_x(SM)}{\mu_x(NM)} = 1.86 = \frac{e^{\beta_1+\beta_2+\beta_{12}}}{e^{\beta_2}} = e^{\beta_1+\beta_{12}} \Rightarrow 0.841567 + \beta_{12} = \ln 1.86$$

$$\Rightarrow \beta_{12} = -0.220991 \qquad [1]$$

[Total 4]

Alternatively, you could have chosen:

- $\mu_x(0) = \mu_x(NM) \Rightarrow \beta_1 = 0.620576, \beta_2 = -0.371564, \beta_{12} = 0.220991$
- $\mu_x(0) = \mu_x(SF) \Rightarrow \beta_1 = -0.841567, \beta_2 = 0.150573, \beta_{12} = 0.220991$
- $\mu_x(0) = \mu_x(SM) \Rightarrow \beta_1 = -0.620576, \beta_2 = -0.150573, \beta_{12} = -0.220991$

Solution 2

This topic is covered in Chapter 1.

When assessing the suitability of a model for a particular exercise, it is important to consider the following:

- The objectives of the modelling exercise. [½]
- The validity of the model for the purpose to which it is to be put. [½]
- The validity of the data to be used. [½]
- The possible errors associated with the model or parameters used not being a perfect representation of the real world situation being modelled. [½]
- The impact of correlations between the random variables that "drive" the model. [½]
- The extent of correlations between the various results produced from the model. [½]
- The current relevance of models written and used in the past. [½]
- The credibility of the data input. [½]
- The credibility of the results output. [½]
- The dangers of spurious accuracy. [½]
- The ease with which the model and its results can be communicated. [½]
- Short-term and long-term properties of a model. [½]

[Maximum 4]

Solution 3

Poisson processes are covered in Chapter 2 and Chapter 5. White noise processes are covered in Chapter 2.

(i)(a) ***Poisson process***

A Poisson process with rate λ is a continuous-time, integer-valued process, $\{N_t : t \geq 0\}$, such that:

1. $N_0 = 0$

2. N_t has independent increments

3. $N_t - N_s \sim Poi\left[\lambda(t-s)\right]$ for $t > s$, *ie*:

$$P\left(N_t - N_s = n\right) = \frac{\left[\lambda\left(t-s\right)\right]^n e^{-\lambda(t-s)}}{n!} \quad \text{for } t > s \text{ and } n = 0,1,2,...$$

[2]

We could also define a Poisson process with rate λ in one of the following ways.

Alternative 1

An integer-valued process $\left\{N_t, t \geq 0\right\}$, with filtration $\left\{F_t, t \geq 0\right\}$, is a Poisson process if:

$$P\left(N_{t+h} - N_t = 1 \mid F_t\right) = \lambda h + o(h)$$

$$P\left(N_{t+h} - N_t = 0 \mid F_t\right) = 1 - \lambda h + o(h)$$

$$P\left(N_{t+h} - N_t \neq 0,1 \mid F_t\right) = o(h)$$

[2]

Alternative 2

$\left\{N_t, t \geq 0\right\}$ *is a Poisson process with rate* λ if the holding times, $T_0, T_1, \ldots$ of $\left\{N_t, t \geq 0\right\}$ are independent exponential random variables with parameter λ and $N_{T_0+T_1+\cdots+T_{n-1}} = n$. *[2]*

Alternative 3

$\{N_t, t \geq 0\}$ *is a Poisson process with rate* λ *if it is a Markov jump process with independent increments and transition rates given by:*

$$\mu_{ij} = \begin{cases} -\lambda & \text{if } j=i \\ \lambda & \text{if } j=i+1 \\ 0 & \text{otherwise} \end{cases}$$ *[2]*

(i)(b) ***White noise process***

A white noise process $\{Z_t : t \in J\}$ is a set of independent and identically distributed random variables (where J denotes the time set of the process). [1]

[Total 3]

(ii)(a) ***Time set***

A Poisson process has a continuous time set. [½]

The time set of a white noise process can be either discrete or continuous. [½]

(ii)(b) ***State space***

A Poisson process has a discrete state space. [½]

A Poisson process is often used to model the number of events up to a given point in time. The number of events must be a non-negative integer.

The state space of a white noise process can be either discrete or continuous. [½]

It is determined by the distribution of the random variables Z_t.

[Total 2]

Solution 4

This topic is covered in Chapter 5.

The process can be shown in the following diagram:

We can derive an expression for $p_{AA}(t)$ by solving Kolmogorov's forward differential equation for this probability. This differential equation is:

$$\frac{d}{dt}p_{AA}(t) = p_{AA}(t)\times(-0.4) + p_{AB}(t)\times 0.1 \qquad [1]$$

Since $p_{AB}(t) = 1 - p_{AA}(t)$:

$$\begin{aligned}\frac{d}{dt}p_{AA}(t) &= -0.4p_{AA}(t) + 0.1\left(1 - p_{AA}(t)\right)\\ &= 0.1 - 0.5p_{AA}(t) \qquad [1]\end{aligned}$$

We can solve this using the integrating factor method, which is given on Page 4 of the *Tables*. We first have to put all of the $p_{AA}(t)$ terms on the LHS of the equation:

$$\frac{d}{dt}p_{AA}(t) + 0.5p_{AA}(t) = 0.1 \qquad (*)$$

The integrating factor is:

$$\exp\left(\int 0.5\,dt\right) = e^{0.5t} \qquad [½]$$

Multiplying each term in (*) by the integrating factor gives:

$$e^{0.5t}\frac{d}{dt}p_{AA}(t) + 0.5e^{0.5t}p_{AA}(t) = 0.1e^{0.5t} \qquad [½]$$

Now integrating with respect to t we get:

$$e^{0.5t} p_{AA}(t) = \frac{1}{5} e^{0.5t} + C \qquad [1]$$

where C is a constant of integration.

We can determine the value of C using the initial condition $p_{AA}(0) = 1$. This gives:

$$1 = \frac{1}{5} + C$$

So $C = \frac{4}{5}$. [1]

Hence:

$$e^{0.5t} p_{AA}(t) = \frac{1}{5} e^{0.5t} + \frac{4}{5}$$

and:

$$p_{AA}(t) = \frac{1}{5}\left(1 + 4e^{-0.5t}\right) \qquad [½]$$

So the probability that the process is in State A at time 1 given that it was in State A at time 0 is:

$$p_{AA}(1) = \frac{1}{5}\left(1 + 4e^{-0.5}\right) = 0.68522 \qquad [½]$$

[Total 6]

Solution 5

This topic is covered in Chapter 7.

(i) ***Force of mortality***

The force of mortality at age $x+t$ can be obtained from the result:

$$\mu_{x+t} = -\frac{\partial}{\partial t}\ln S_x(t)$$

So under the Weibull model:

$$\mu_{x+t} = \frac{\partial}{\partial t}\left(\alpha t^{\beta}\right) = \alpha\beta t^{\beta-1}$$ [1]

(ii) ***Log-likelihood function***

The general form of the likelihood function is:

$$L = \prod_{i=1}^{m} f(t_i) \prod_{j=m+1}^{n} S\left(t_j\right)$$ [1]

where:

$$f(t) = {}_t p_x \, \mu_{x+t}$$

is the PDF of the future lifetime random variable.

Under the Weibull model:

$$f(t) = \alpha\beta t^{\beta-1}\exp\left(-\alpha t^{\beta}\right)$$

So the likelihood function is:

$$L = \prod_{i=1}^{m} \alpha\beta t_i^{\beta-1}\exp\left(-\alpha t_i^{\beta}\right) \prod_{j=m+1}^{n} \exp\left(-\alpha t_j^{\beta}\right)$$

$$= \alpha^m \beta^m \left(\prod_{i=1}^{m} t_i^{\beta-1}\right)\left(\exp\left(-\alpha\sum_{j=1}^{n} t_j^{\beta}\right)\right)$$ [1]

Taking logs, we get:

$$\ln L = m\ln\alpha + m\ln\beta + (\beta - 1)\sum_{i=1}^{m}\ln t_i - \alpha\sum_{j=1}^{n} t_j^{\beta}$$ [1]

[Total 3]

(iii)(a) ***Estimated value of α***

Given that $\beta = 1.5$, $m = 10$, $n = 20$ and $\sum_{j=1}^{20} t_j^{1.5} = 1,648.42$, the log-likelihood function becomes:

$$\ln L = 10\ln\alpha + 10\ln 1.5 + 0.5\sum_{i=1}^{10}\ln t_i - 1,648.42\alpha$$

$$= 10\ln\alpha - 1,648.42\alpha + C$$

where C is a constant.

Differentiating with respect to α:

$$\frac{d\ln L}{d\alpha} = \frac{10}{\alpha} - 1,648.42$$ [½]

Setting the derivative equal to 0, we get:

$$\hat{\alpha} = \frac{10}{1,648.42} = 0.006066$$ [½]

Differentiating again:

$$\frac{d^2\ln L}{d\alpha^2} = -\frac{10}{\alpha^2} < 0 \Rightarrow \max$$ [½]

So $\hat{\alpha} = 0.006066$.

(iii)(b) ***Estimated survival probability***

Using the estimated value of α from part (iii)(a), the survival function is estimated by:

$$\hat{S}_x(t) = \exp\left(-0.006066t^{1.5}\right)$$

At time 10, this is:

$$\hat{S}_x(10) = \exp\left(-0.006066 \times 10^{1.5}\right) = e^{-0.19184} = 0.82544$$ [½]

[Total 2]

Solution 6

This topic is covered in Chapter 10.

(i)(a) ***Data items required to estimate the probability of dying***

To estimate q_x, we would just need to know the number of people who died out of the 10,000 over the year of age, d. [½]

The estimate is then:

$$\hat{q}_x = \frac{d}{10,000}$$ [½]

(i)(b) ***Data items required to estimate the force of mortality***

To estimate μ, we would additionally need to know, for each death, the exact age at death. [½]

The estimate is then:

$$\hat{\mu} = \frac{d}{\sum_{i=1}^{d} t_i + 10,000 - d}$$ [1]

where $x + t_i$ is the age at death of the ith death, $0 < t_i < 1$. [½]

[Total 3]

(ii)(a) ***Formula and asymptotic distribution of the estimator***

The estimator of μ is:

$$\tilde{\mu} = \frac{D}{V}$$ [½]

where:

- D is the random variable number of deaths amongst the 10,000 lives over the year of age $(x, x+1)$ [½]
- V is the waiting time, *ie* the total time these lives are observed during the year. [½]

The asymptotic distribution of $\tilde{\mu}$ is:

$$N\left(\mu, \frac{\mu}{E(V)}\right)$$ [½]

Markers: Give full credit to students who say that the asymptotic variance is given by the Cramér-Rao lower bound, which is $-1 \Big/ E\left[\dfrac{\partial^2 \ln L}{\partial \mu^2}\right]$.

(ii)(b) ***Total expected waiting time***

There are several alternative approaches we could use here.

Method 1

The expected waiting time over $(x, x+1)$ is the expected number of years of life lived by the 10,000 lives currently aged exactly x. So this can be calculated as:

$$10,000 \int_0^1 {}_t p_x \, dt = 10,000 \int_0^1 e^{-0.05t} \, dt = -\frac{10,000}{0.05}\left[e^{-0.05t}\right]_0^1 = 10,000\left(\frac{1-e^{-0.05}}{0.05}\right)$$

$$= 9,754.115 \text{ years}$$

Method 2

We could also use the equation $E\left[D\right]=\mu E\left[V\right]$ from Section 4 of Chapter 4. The expected number of deaths is $E[D]=10,000q_x=10,000\left(1-e^{-\mu}\right)$ and $\mu=0.05$.

So we can deduce that $E\left[V\right]=\dfrac{E\left[D\right]}{\mu}=\dfrac{10,000\left(1-e^{-0.05}\right)}{0.05}=9,754.115$ years.

Method 3

Let V_i denote the random variable waiting time for the i th life and let $V=\sum_{i=1}^{10,000}V_i$ denote the total waiting time for the 10,000 lives. The expected waiting time for the i th life is:

$$E\left(V_i\right)=\int_0^1 v_i\, f\left(v_i\right)dv_i+1\times p_x$$

Since the force of mortality is constant over the age range $\left(x,x+1\right)$:

$$f\left(v_i\right)=\mu e^{-\mu v_i}=0.05e^{-0.05v_i}$$

and: $\quad p_x=e^{-\mu}=e^{-0.05}$

So:

$$E\left(V_i\right)=\int_0^1 0.05v_i\, e^{-0.05v_i}\,dv_i+e^{-0.05}$$

Integrating by parts gives:

$$E\left(V_i\right)=\left[-v_ie^{-0.05v_i}\right]_0^1+\int_0^1 e^{-0.05v_i}\,dv_i+e^{-0.05}$$

$$=\cancel{-e^{-0.05}}+\int_0^1 e^{-0.05v_i}\,dv_i+\cancel{e^{-0.05}}$$

$$=\left[-\frac{1}{0.05}e^{-0.05v_i}\right]_0^1$$

$$=\frac{1}{0.05}\left[1-e^{-0.05}\right]$$

$$=0.9754115 \text{ years}$$

So:

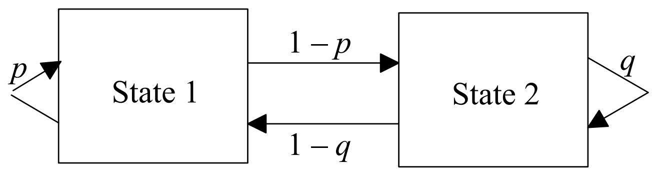
$$E(V) = 10,000 \times 0.9754115 = 9,754.115 \text{ years}$$

[2 for any valid approach]
[Total 4]

Solution 7

This topic is covered in Chapter 3.

(i) ***Transition diagram***



[1]

(ii) ***Irreducible and/or aperiodic***

A chain is irreducible if every state can eventually be reached from every other state. [½]

We can see from the diagram that it is possible to reach State 2 from State 1 and to reach State 1 from State 2. So the chain is irreducible. [½]

A state is aperiodic with period $d > 1$ if a return to that state is possible only in a number of steps that is a multiple of d. If there is no such $d > 1$, then the state is aperiodic. If all the states are aperiodic, then the chain is aperiodic. [½]

The diagram shows that a return to State 1 is possible in any number of states (as it has an arrow back to itself). So State 1 is aperiodic. State 2 is similar, so the chain is aperiodic. [½]

Alternatively, we could say that, since the chain is irreducible, each state has the same periodicity.

[Total 2]

(iii) ***Determine the stationary distribution***

Suppose that the stationary distribution is represented by the vector $(\pi_1 \quad \pi_2)$. Then:

$$(\pi_1 \quad \pi_2)\begin{pmatrix} p & 1-p \\ 1-q & q \end{pmatrix} = (\pi_1 \quad \pi_2)$$

and: $\pi_1 + \pi_2 = 1$ [½ for both equations]

From the matrix equation, we have:

$$p\pi_1 + (1-q)\pi_2 = \pi_1$$

and: $(1-p)\pi_1 + q\pi_2 = \pi_2$

These two equations are not independent, and only one of them is required.

From the first of these equations, we have:

$$(1-q)\pi_2 = (1-p)\pi_1$$

So:

$$\pi_2 = \left(\frac{1-p}{1-q}\right)\pi_1$$ [½]

and:

$$\pi_1 + \pi_2 = 1 \Rightarrow \left(1 + \frac{1-p}{1-q}\right)\pi_1 = 1$$

$$\Rightarrow \left(\frac{1-q+1-p}{1-q}\right)\pi_1 = 1$$

$$\Rightarrow \pi_1 = \frac{1-q}{2-p-q}$$ [½]

and: $\pi_2 = \left(\frac{1-p}{1-q}\right)\pi_1 = \left(\frac{1-p}{1-q}\right)\left(\frac{1-q}{2-p-q}\right) = \frac{1-p}{2-p-q}$ [½]

[Total 2]

(iv) ***Estimation***

We have:

$$\hat{p} = \frac{n_{11}}{n_{11} + n_{12}} = \frac{62}{62 + 59} = \frac{62}{121} = 0.5124 \qquad [½]$$

$$\hat{q} = \frac{n_{22}}{n_{21} + n_{22}} = \frac{69}{59 + 69} = \frac{69}{128} = 0.5391 \qquad [½]$$

So:

$$\hat{\pi}_1 = \frac{1 - \hat{q}}{2 - \hat{p} - \hat{q}} = 0.4859 \qquad [½]$$

and:

$$\hat{\pi}_2 = 0.5141 \qquad [½]$$

[Total 2]

Solution 8

This topic is covered in Chapter 8.

(i) ***Types of censoring***

Right censoring is present where visitors left the shop without making a purchase, as we cannot say how long they would have taken to buy something if they had stayed. [1]

Random censoring is present for those who left the shop, as their exclusion could not be predicted in advance. [1]

This may not be true for the lady with the dog who was asked to leave.

Non-informative censoring is present where the decision to leave the shop was not linked to the waiting time (*eg* for the businessman who received the phone call). [1]

Note that Type I censoring is not present here, as observations did not stop at 12:00.

[Total 3]

(ii) ***Tables / diagrams***

We need to split up the family group (*) into two entries, as the mother was censored, but the father was not. This leads to the following table of values of t.

Time joined queue	*Time stopped observing*	*Time t (minutes)*	*Censored?*
11:01	11:23	22	
11:06	11:25	19	
11:06	11:20	14	Censored
11:08	11:26	18	
11:12	11:27	15	
11:25	11:28	3	
11:25	11:30	5	Censored
11:27	11:48	21	Censored
11:36	11:50	14	
11:38	11:55	17	
11:40	11:45	5	Censored
11:50	12:05	15	
11:55	12:10	15	Censored
11:58	12:00	2	Censored
11:58	12:15	17	

Arranging these in order of the values of t, and using "+" to denote a censored visitor, we have:

2+ 3 5+ 5+ 14 14+ 15 15 15+ 17 17 18 19 21+ 22

Alternatively, this can be represented in diagram form with P (say) denoting purchases and C denoting censored times:

C		CC	C	C				C	
	P		P	PP	PP	P	P		P
2	3	5	14	15	17	18	19	21	22

We can then construct the usual table for Kaplan-Meier and Nelson-Aalen calculations. Where purchases and censored times coincide, we assume that the purchases occurred first.

Row (j)	Time of purchase (t_j)	Number of purchases made (d_j)	Number "at risk" of making a purchase (n_j)
1	3	1	14
2	14	1	11
3	15	2	9
4	17	2	6
5	18	1	4
6	19	1	3
7	22	1	1

Students do not need to present all the tables and diagrams shown. Award marks as follows:

– *for calculating the values of t correctly (as in the first table)* *[1]*

– *for correctly distinguishing the censored entries* *[½]*

– *for dealing appropriately with the family group* *[½]*

– *for correctly identifying the t_j's (as in the second table)* *[½]*

– *for calculating the d_j's correctly* *[½]*

– *for calculating the n_j's correctly* *[1]*

If students have not explicitly stated all these values, but their answers to part (iii) imply that they have calculated them correctly, they should be awarded the relevant marks.

[Total 4]

(iii) ***Estimates and assumptions***

The estimate of $S(20)$ using the Kaplan-Meier model is:

$$\hat{S}(20) = \prod_{t_j \le 20} \left(1 - \frac{d_j}{n_j}\right)$$

$$= \left(1 - \frac{1}{14}\right)\left(1 - \frac{1}{11}\right)\left(1 - \frac{2}{9}\right)\left(1 - \frac{2}{6}\right)\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{3}\right) = 0.21886 \qquad [1½]$$

The estimate of the cumulative hazard Λ_{20} using the Nelson-Aalen model is:

$$\hat{\Lambda}_{20} = \sum_{t_j \le 20} \frac{d_j}{n_j} = \frac{1}{14} + \frac{1}{11} + \frac{2}{9} + \frac{2}{6} + \frac{1}{4} + \frac{1}{3} = 1.3012 \qquad [1]$$

and the estimate of $S(20)$ using the Nelson-Aalen model is:

$$\hat{S}(20) = \exp\left(-\hat{\Lambda}_{20}\right) = e^{-1.3012} = 0.27220 \qquad [1]$$

The models assume that the distribution of the waiting times is the same for all visitors, *ie* it forms a homogeneous group. This may not be true (*eg* for the old lady in the wheelchair who was given special treatment). [½]

The models assume that the waiting times are independent, which is probably true here. [½]

The models assume that all censoring is non-informative, which also might not be the case here. [½]

[Total 5]

Solution 9

This topic is covered in Chapter 5 and Chapter 11.

(i) ***Limitations of the model***

The model may not allow easily for some of the policy features, *eg* a waiting period. [½]

The model assumes the same mortality rate applies for both sick and healthy individuals: this would seem to be unrealistic. [½]

The model requires the Markov assumption: that the transition rates depend only on the current state (and age) of the policyholder. In real life this may not be true. [½]

For example, additionally:

- recovery and mortality rates from the sick state are likely to depend on the duration in the sick state
- the rates of transition from healthy to sick (*sickness inception rates*) may depend on how many times a currently healthy person has been sick in the past, and for how long they have been sick
- sickness inception and mortality rates of healthy people may depend on how long a policy has been in force
- all transition rates may depend on other factors, such as sex and smoker status.

[1½]

Markers: award ½ mark each for any three appropriate examples.

[Total 3]

(ii) ***Rate interval***

The rate interval for the age definition "50 nearest birthday" is the life year between exact ages 49½ and 50½. [1]

Markers: only award ½ a mark if a student only states $x-½$ *to* $x+½$.

(iii) ***Deriving census formulae***

We will need the central exposed to risk (or observed waiting time) separately for healthy lives aged 50 nearest birthday, for sick lives aged 50 nearest birthday, and for both healthy and sick lives combined aged 50 nearest birthday. These will be denoted by ${}^{H}E_{50}^{c}$, ${}^{S}E_{50}^{c}$ and E_{50}^{c} respectively. [½]

Let ${}^{j}P_{50}(t)$ be the number of policyholders alive at time t (where $t=0$ is 1.1.2013) who are at that time aged 50 nearest birthday and in state j ($j=H,S$). [½]

Then:

$${}^{j}E_{50}^{c}=\int_0^2 {}^{j}P_{50}(t)\,dt$$ [½]

Using the trapezium rule:

$${}^{j}E_{50}^{c}=\tfrac{1}{2}\left(\frac{{}^{j}P_{50}(0)+{}^{j}P_{50}(½)}{2}+\frac{{}^{j}P_{50}(½)+{}^{j}P_{50}(1)}{2}+\frac{{}^{j}P_{50}(1)+{}^{j}P_{50}(1½)}{2}+\frac{{}^{j}P_{50}(1½)+{}^{j}P_{50}(2)}{2}\right)$$

$$=\tfrac{1}{2}\left(\frac{{}^{j}P_{50}(0)}{2}+{}^{j}P_{50}(½)+{}^{j}P_{50}(1)+{}^{j}P_{50}(1½)+\frac{{}^{j}P_{50}(2)}{2}\right)$$ [1]

assuming ${}^{j}P_{50}(t)$ varies linearly over the separate periods between the census dates (*ie* over 6-month intervals). [½]

We need to use census data grouped by age last birthday to obtain approximate values of ${}^{j}P_{50}(t)$ for $t=0,½,...,2$.

Assuming birthdays are uniformly distributed over the calendar year: [½]

$${}^{j}P_{50}(t)=\tfrac{1}{2}\left[{}^{j}N_{49}(t)+{}^{j}N_{50}(t)\right]$$ [½]

where ${}^{j}N_{x}(t)$ is the number of lives in state j at time t aged x last birthday. [½]

We also need:

$$E_{50}^{c}={}^{H}E_{50}^{c}+{}^{S}E_{50}^{c}$$ [½]

Now let:

$n_{50}^{ij}=$ observed number of transitions from state i to state j while aged 50 nearest birthday [½]

Then the estimates of the transition rates are:

$$\hat{\sigma}_{50+f} = \frac{n_{50}^{HS}}{{}^{H}E_{50}^{c}}; \qquad \hat{\rho}_{50+f} = \frac{n_{50}^{SH}}{{}^{S}E_{50}^{c}}; \qquad \hat{\mu}_{50+f} = \frac{n_{50}^{HD} + n_{50}^{SD}}{E_{50}^{c}}$$ [1½]

These all estimate the respective transition rate at exact age 50 (*ie* $f = 0$), as the age for the transition rate estimates is the exact age in the middle of the rate interval defined in the answer to part (ii). [½]

The estimates also assume that the transition rates are constant over the year of age $(49\frac{1}{2}, 50\frac{1}{2})$. [½]

We need to assume that the force of transition is constant over the rate interval to derive the above formula for the MLE. However, in practice, the force is not constant. So we are really estimating the average force of transition over the rate interval, which we assume to be the force of transition at the middle age in the rate interval.

[Total 8]

(iv) ***Transition rate from sick to healthy at exact age 50***

The estimate is:

$$\hat{\rho}_{50} = \frac{n_{50}^{SH}}{{}^{S}E_{50}^{c}}$$

where $n_{50}^{SH} = 390$ and:

$${}^{S}E_{50}^{c} = \tfrac{1}{2}\left(\frac{\tfrac{1}{2}[54+72]}{2} + \tfrac{1}{2}[65+67] + \tfrac{1}{2}[39+95] + \tfrac{1}{2}[42+70] + \frac{\tfrac{1}{2}[60+81]}{2}\right)$$

$$= \tfrac{1}{2}\left(\frac{63}{2} + 66 + 67 + 56 + \frac{70.5}{2}\right)$$

$$= 127.875$$ [½]

Therefore:

$$\hat{\rho}_{50} = \frac{390}{127.875} = 3.050 \, pa$$ [½]

[Total 1]

Solution 10

This topic is covered in Chapter 12.

(i) ***Why the insurer would not use the crude mortality rates***

The crude rates calculated from the mortality investigation will contain sampling error and would be unlikely to progress smoothly with age, whereas the true underlying rates would be expected to be smooth. [½]

If the insurer calculated the premiums using the crude mortality rates, this may produce irregularities in the premium rates and these would be hard to justify in practice. [½]

In addition, the insurer would always consider the suitability of the rates it uses for their required purpose. [½]

For example, it may wish to take account of changes occurring in mortality since the investigation period, or to build in some caution regarding the expected future experience (*eg* so as not to underestimate future mortality for life assurance premium calculations). [½]

Markers award credit for valid alternative reasons and/or examples. [Total 2]

(ii) ***Why graduated rates have to pass tests before they can be used***

Every graduation is a trade-off between smoothness and adherence to data. We want the graduated rates to be smooth, but we also want them to be representative of the data from which they were derived. We need to make sure that the graduated rates exhibit both of these characteristics before we can use them in financial calculations. [1]

(iii) ***Rationale behind the cumulative deviations test***

Suppose that D_x denotes the number of deaths at age x last birthday. If the graduated rates are the true mortality rates underlying the data, then, according to the Poisson model:

$$D_x \sim Poi\left(E_x^c \mathring{\mu}_x\right) \qquad [½]$$

and this distribution can be approximated by a normal distribution, *ie*:

$$D_x \sim N\left(E_x^c \mathring{\mu}_x, E_x^c \mathring{\mu}_x\right) \text{ approximately} \qquad [½]$$

Assuming that the random variables D_x are independent:

$$\sum_{x=50}^{58} D_x \sim N\left(\sum_{x=50}^{58} E_x^c \mathring{\mu}_x, \sum_{x=50}^{58} E_x^c \mathring{\mu}_x\right) \text{ approximately} \qquad [1]$$

So:

$$\frac{\sum_{x=50}^{58} D_x - \sum_{x=50}^{58} E_x^c \mathring{\mu}_x}{\sqrt{\sum_{x=50}^{58} E_x^c \mathring{\mu}_x}} \sim N(0,1) \text{ approximately} \qquad [½]$$

Hence we can test the null hypothesis:

H_0 : the graduated rates are the true mortality rates underlying the data [½]

by calculating the value of:

$$\frac{\sum_{x=50}^{58} D_x - \sum_{x=50}^{58} E_x^c \mathring{\mu}_x}{\sqrt{\sum_{x=50}^{58} E_x^c \mathring{\mu}_x}}$$

(which is the test statistic for this test) and comparing it with the standard normal distribution. [½]

The test is two-tailed. A large positive test statistic tells us that, overall, the graduated rates are too low. A large negative test statistic tells us that, overall, the graduated rates are too high. [½]

[Total 4]

(iv) ***Cumulative deviations test***

We have:

Age, x	Observed deaths	Expected deaths, $E_x^c \mathring{\mu}_x$
50	18	15.476
51	16	19.380
52	14	24.745
53	30	26.858
54	30	33.291
55	50	40.997
56	64	59.103
57	67	61.760
58	90	77.283
Total	**379**	**358.893**

[1½]

So the value of the test statistic is:

$$\frac{379-358.893}{\sqrt{358.893}}=1.061$$ [½]

Using a 5% level of significance, the critical values are ± 1.96. Since 1.061 lies between –1.96 and 1.96, we have insufficient evidence to reject the null hypothesis at the 5% significance level and we conclude that there is no significant overall bias in the graduated rates. [1]

[Total 3]

(v)(a) ***Chi-squared goodness-of-fit test***

The test statistic for this test is:

$$\sum_{x=50}^{58} z_x^2$$

where:

$$z_x = \frac{D_x - E_x^c \mathring{\mu}_x}{\sqrt{E_x^c \mathring{\mu}_x}}$$

We have:

Age, x	z_x	z_x^2
50	0.6415	0.4115
51	–0.7679	0.5896
52	–2.1601	4.6660
53	0.6064	0.3677
54	–0.5703	0.3252
55	1.4060	1.9770
56	0.6370	0.4057
57	0.6668	0.4447
58	1.4466	2.0926

[2]

So the value of the test statistic is:

$$\sum_{x=50}^{58} z_x^2 = 11.280$$ [1]

We compare this with a χ^2 distribution. The number of degrees of freedom depends on the method of graduation (which we are not told), but cannot be more than 9 because we have 9 age groups. [½]

This is a one-tailed test. From Page 169 of the *Tables*, we see that:

Degrees of freedom	Upper 5% point of χ^2 distribution
5	11.07
6	12.59
7	14.07
8	15.51
9	16.92

So we would only reject the null hypothesis at the 5% level if we lost more than 3 degrees of freedom owing to the method of graduation, which is unlikely. [1]

(v)(b) ***Grouping of signs test***

We have:

n_1 = number of positive deviations = 6

n_2 = number of negative deviations = 3

G = number of groups of positive deviations = 3

This is a one-tailed test. From page 189 of the *Tables*, the critical value is 1. Since G is greater than the critical value, we have insufficient evidence to reject the null hypothesis at the 5% significance level. [1]

So we conclude that, overall, the graduated rates are a good fit to the data and there is no problem with clumping of deviations of the same sign. [½]

[Total 6]

Solution 11

This topic is covered in Chapter 5 and Chapter 6.

(i) ***Suitability of Poisson process***

There is an assumption that the process is homogeneous over time. [½]

However, it is likely that boat traffic will vary on a seasonal basis, *ie* by time of year, day of the week, and by time of day. [½]

It is also likely that boat traffic will vary across a broader timescale, *ie* it may vary from one year to another (*eg* as a result of changes in the economy). [½]
There is an assumption that all arrival times are independent of each other (and hence that the Markov property applies). [½]

This is unlikely in practice, particularly for the big and medium sized boats. This is because any that would arrive at more or less the same time would have to wait for each other and take their turn, as there may not be sufficient space to take the ships side by side in the canal. [1]
[Total 3]

(ii) ***Time until the next boat arrives at the bridge***

The arrival of any boat at the bridge is Poisson with parameter:

$$20+600+46,000=46,620 \text{ per year}$$

or: $$\frac{46,620}{365.25}=127.6386 \text{ per day.}$$ [½]

This results from the fact that the combination of a number of independent Poisson processes is itself a Poisson process, with rate parameter equal to the sum of all the individual rate parameters.

The time to the next arrival is therefore exponentially distributed with the same parameter value.

The expected time to the next arrival is then:

$$\frac{1}{127.6386} = 0.0078346 \text{ days}$$

or: $0.0078346 \times 24 \times 60 = 11.28$ minutes [½]

[Total 1]

(iii) ***Probability the bridge is open for more than 30 minutes on any single day***

First we will convert the rates we need into rates per day. These are:

$$\lambda_B = \frac{20}{365.25} = 0.05476$$

$$\lambda_M = \frac{600}{365.25} = 1.64271$$ [½]

The bridge will be open for more than 30 minutes per day if:

(1) 2 or more big boats arrive in one day

(2) 1 big boat and at least 1 medium-sized boat arrive in one day

(3) no big boat and more than 2 medium sized boats arrive in one day. [½]

Let X_i be the number of boats arriving of size $i = B, M, S$ per day.

The probability of event (1) is:

$$P_1 = P(X_B > 1) = 1 - P(X_B = 0) - P(X_B = 1)$$

$$= 1 - e^{-0.05476}[1 + 0.05476]$$

$$= 0.001446$$ [1]

The probability of event (2) is:

$$P_2 = P(X_B = 1)P(X_M > 0) = 0.05476e^{-0.05476}\left(1 - e^{-1.64271}\right) = 0.041811$$ [1]

The probability of event (3) is:

$$P_3 = P(X_B = 0)P(X_M > 2)$$
$$= P(X_B = 0)\left[1 - P(X_M = 0) - P(X_M = 1) - P(X_M = 2)\right]$$
$$= e^{-0.05476}\left[1 - e^{-1.64271}\left(1 + 1.64271 + \frac{1.64271^2}{2}\right)\right]$$
$$= 0.946715 \times 0.227736$$
$$= 0.215601 \qquad [1½]$$

Therefore, the probability of any of the three events occurring is:

$$0.001446 + 0.041811 + 0.215601 = 0.25886 \qquad [½]$$

[Total 5]

Alternatively, you could calculate this probability as one minus the probability that the bridge is open for 30 minutes or less. The bridge is open for 30 minutes or less if:

(1) no big boat and 0, 1, or 2 medium sized boats arrive in one day, or

(2) 1 big boat and no medium sized boat arrive in one day.

The probabilities of these are:

$$P_1' = e^{-0.05476}e^{-1.64271}\left(1 + 1.64271 + \frac{1.64271^2}{2}\right) = 0.73111$$

$$P_2' = 0.05476\, e^{-0.05476}e^{-1.64271} = 0.01003$$

and so the required probability is:

$$1 - P_1' - P_2' = 1 - 0.73111 - 0.01003 = 0.25886$$

(iv) ***Transition graph and generator matrix***

The state space is $\{0, 1, 2, ...\}$

Medium-sized boats are arriving at the port (*ie* arriving from one direction) at half of the rate of λ_M (because λ_M is the rate at which the boats pass a point travelling in both directions).

So the rate at which the process changes from state n to state $n+1$ is:

$$\lambda = \frac{300}{365.25 \times 24} = 0.034223 \text{ per hour}$$

for all n.

The transition rate $\mu_{n,n-1}$ from state n to state $n-1$ depends on the value of n, as follows:

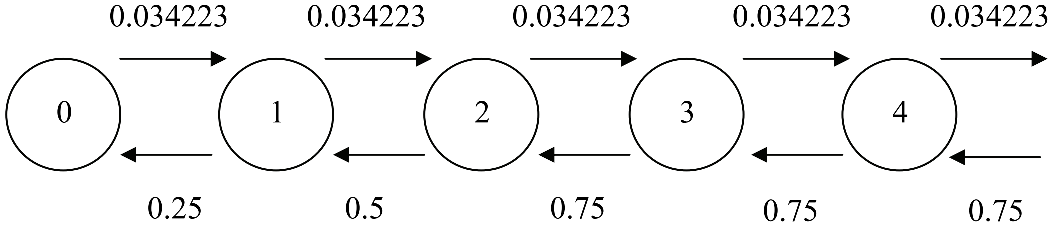
$$\mu_{n,n-1} = \begin{cases} \frac{1}{4} = 0.25 & n=1 \\ 2 \times 0.25 = 0.5 & n=2 \\ 3 \times 0.25 = 0.75 & n \geq 3 \end{cases}$$

When $n=1$ the transition rate is the parameter of a Poisson process equal to the reciprocal of the expected waiting time.

When $n=2$ the transition rate is the parameter of a Poisson process formed from combining two Poisson processes, each with parameter 0.25 – which are therefore added together. This gives the rate at which a (either) dock becomes free when two docks are currently occupied. (A similar logic applies to $n=3$.)

When $n>3$ the rate at which a dock becomes free is the same as when $n=3$. This is because only three can be occupied at any one time and the other boats are waiting outside the port.

This gives us the following transition graph:



[2]

The generator matrix is:

$$A = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 & \dots \\ 0.25 & -\lambda - 0.25 & \lambda & 0 & 0 & 0 & \dots \\ 0 & 0.5 & -\lambda - 0.5 & \lambda & 0 & 0 & \dots \\ 0 & 0 & 0.75 & -\lambda - 0.75 & \lambda & 0 & \dots \\ 0 & 0 & 0 & 0.75 & -\lambda - 0.75 & \lambda & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

$$= \begin{pmatrix} -0.03422 & 0.03422 & 0 & 0 & 0 & 0 & \dots \\ 0.25 & -0.28422 & 0.03422 & 0 & 0 & 0 & \dots \\ 0 & 0.5 & -0.53422 & 0.03422 & 0 & 0 & \dots \\ 0 & 0 & 0.75 & -0.78422 & 0.03422 & 0 & \dots \\ 0 & 0 & 0 & 0.75 & -0.78422 & 0.03422 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

[1]

Markers: deduct 1 mark in total if students have calculated λ *as:*

$$600/(365.25 \times 24) = 0.068446$$

but have done everything else correctly.

[Total 3]

(v) ***Kolmogorov backward differential equation***

The differential equation can be found from:

$$\left[A\,P(t)\right]_{1,3}$$

ie by multiplying the second row of the generator matrix A *by the fourth column of the matrix of transition probabilities* $P(t)$ *(that have elements of the form* $p_{ij}(t)$*).*

So, the differential equation is:

$$\frac{d}{dt} p_{13}(t) = 0.25\, p_{03}(t) - (\lambda + 0.25)\, p_{13}(t) + \lambda\, p_{23}(t)$$

$$= 0.25\, p_{03}(t) - 0.284223\, p_{13}(t) + 0.034223\, p_{23}(t) \qquad [2]$$

Markers: award no marks if students give the forward equation instead.

(vi) ***Integral expressions***

(a) In backward form:

$$p_{13}(2) = \int_0^2 p_{\overline{11}}(w)\left[0.25\, p_{03}(2-w) + \lambda\, p_{23}(2-w)\right]dw$$

$$= \int_0^2 e^{-0.284223w}\left[0.25\, p_{03}(2-w) + 0.034223\, p_{23}(2-w)\right]dw \qquad [2]$$

Alternatively this could be written as:

$$p_{13}(2) = \int_0^2 e^{-0.284223(2-w)}\left[0.25\, p_{03}(w) + 0.034223\, p_{23}(w)\right]dw$$

(b) In forward form:

$$p_{13}(2) = \int_0^2 \left[p_{12}(w)\,\lambda + p_{14}(w)\times 0.75\right] p_{\overline{33}}(2-w)\,dw$$

$$= \int_0^2 \left[0.034223\, p_{12}(w) + 0.75\, p_{14}(w)\right] e^{-0.784223(2-w)}\,dw \qquad [2]$$

Alternatively this could be written as:

$$p_{13}(2) = \int_0^2 \left[0.034223\, p_{12}(2-w) + 0.75\, p_{14}(2-w)\right] e^{-0.784223w}\,dw$$

[Total 4]

(vii) ***Probability that docks will empty before next boat arrives***

The process is currently in state 3. In order for the required event to occur, the next three jumps in the process must be to states 2, 1 and 0 respectively.

So, at the next jump, the probability that the transition is to state 2 is:

$$p_{32} = \frac{0.75}{\lambda + 0.75} = \frac{0.75}{0.784223} = 0.956360 \qquad [½]$$

Once in state 2, the probability that the next transition is to state 1 is:

$$p_{21} = \frac{0.5}{\lambda + 0.5} = \frac{0.5}{0.534223} = 0.935939$$ [½]

Once in state 1, the probability that the next transition is to state 0 is:

$$p_{10} = \frac{0.25}{\lambda + 0.25} = \frac{0.25}{0.284223} = 0.879591$$ [½]

The overall probability is then:

$$0.956360 \times 0.935939 \times 0.879591 = 0.787317$$ [½]

[Total 2]