CT5 – P C – 16

Combined Materials Pack

ActEd Study Materials: 2016 Examinations

Subject CT5

Contents

Study Guide for the 2016 exams
Course Notes
Question and Answer Bank
Series X Assignments*

***Note:** The Series X Assignment Solutions should also be supplied with this pack unless you chose not to receive them with your study material.

If you think that any pages are missing from this pack, please contact ActEd's admin team by email at **ActEd@bpp.com**.

How to use the Combined Materials Pack

Guidance on how and when to use the Combined Materials Pack is set out in the *Study Guide for the 2016 exams*.

Important: Copyright Agreement

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2016 Study Guide

Subject CT5

Introduction

This Study Guide contains all the information that you will need before starting to study Subject CT5 for the 2016 exams. **Please read this Study Guide carefully before reading the Course Notes,** even if you have studied for some actuarial exams before.

When studying for the UK actuarial exams, you will need:

- a copy of the **Formulae and Tables for Examinations of the Faculty of Actuaries and the Institute of Actuaries, 2nd Edition (2002)** – these are often referred to as simply the "Yellow Tables"

- a "permitted" **scientific calculator** – you will find the list of permitted calculators on the profession's website. Please check the list carefully, since it is reviewed each year.

These are both available from the Institute and Faculty of Actuaries' eShop. Please visit **www.actuaries.org.uk**.

Contents

1 The Subject CT5 course structure

There are four parts to the Subject CT5 course. The parts cover related topics and have broadly equal lengths. The parts are broken down into chapters.

The following table shows how the parts, the chapters and the syllabus items relate to each other. The end columns show how the chapters relate to the days of the regular tutorials. This table should help you plan your progress across the study session.

Part	Chapter	Title	No of pages	Syllabus objectives	2 full days	3 full days
1	1	Life assurance contracts	52	(i) 1-5	1	1
	2	Life annuity contracts	42	(i) 1, 3-6		
	3	The life table	42	(i) 2, 5, (ii) 1-3		
	4	Evaluation of assurances and annuities	28	(i) 5, 7, (ii) 4-5		
2	5	Net premiums and reserves	66	(i) 9, (ii) 6, (iii)		
	6	Variable benefits and with-profit policies	60	(iv)		2
	7	Gross premiums and reserves for fixed and variable benefit contracts	62	(v)		
3	8	Simple annuities and assurances involving two lives	32	(vi)	2	
	9	Contingent and reversionary benefits	58	(vi)		
	10	Competing risks	54	(vii), (viii) 1-4, 6		3
	11	Pension funds	60	(viii) 5-6		
4	12	Profit testing	50	(ix) 1-4		
	13	Profit testing and reserves	38	(ix) 5		
	14	Mortality, selection and standardisation	62	(x)		

2 ActEd study support

Successful students tend to undertake three main study activities:

1. *Learning* – initial study and understanding of subject material
2. *Revision* – learning subject material and preparing to tackle exam-style questions
3. *Rehearsal* – answering exam-style questions, culminating in answering questions at exam speed without notes.

Different approaches suit different people. For example, you may like to learn material gradually over the months running up to the exams or you may do your revision in a shorter period just before the exams. Also, these three activities will almost certainly overlap.

We offer a flexible range of products to suit you and let you control your own learning and exam preparation. The following table shows the products that we produce. Note that not all products are available for all subjects.

LEARNING	LEARNING & REVISION	REVISION	REVISION & REHEARSAL	REHEARSAL
Course Notes	Q&A Bank	Flashcards	Revision Notes	Mock Exam A
	X Assignments		ASET	Additional Mock Pack (AMP)
	Combined Materials Pack (CMP)		Revision Tutorials	Mock / AMP Marking
	X Assignment Marking			
	Tutorials			
	Online Classroom			

The products and services available for Subject CT5 are described below.

"Learning" products

Course Notes

The Course Notes will help you develop the basic knowledge and understanding of principles needed to pass the exam. They incorporate the complete Core Reading and include full explanation of all the syllabus objectives, with worked examples and short questions to test your understanding.

Each chapter includes the relevant syllabus objectives, a chapter summary and, where appropriate, a page of important formulae or definitions.

"Learning & revision" products

Question and Answer Bank

The Question and Answer Bank provides a comprehensive bank of questions (including some past exam questions) with full solutions and comments.

The Question and Answer Bank is divided into five parts. The first four parts include a range of short and long questions to test your understanding of the corresponding part of the Course Notes. Part five consists of 100 marks of exam-style questions.

X Assignments

The four Series X Assignments (X1 to X4) cover the material in Parts 1 to 4 respectively. Assignments X1 and X2 are 80-mark tests and should take you two and a half hours to complete. Assignments X3 and X4 are 100-mark tests and should take you three hours to complete. The actual Subject CT5 examination will have a total of 100 marks.

Combined Materials Pack (CMP)

The Combined Materials Pack (CMP) comprises the Course Notes, the Question and Answer Bank and the Series X Assignments.

The CMP is available in **eBook** format for viewing on a range of electronic devices. eBooks can be ordered separately or as an addition to paper products. Visit **www.ActEd.co.uk** for full details about the eBooks that are available, compatibility with different devices, software requirements and printing restrictions.

CMP Upgrade

The purpose of the CMP Upgrade is to enable you to amend last year's study material to make it suitable for study for this year. In most cases, it lists all significant changes to the Core Reading and ActEd material so that you can manually amend your notes. The upgrade includes replacement pages and additional pages where appropriate.

However, if a large proportion of the material has changed significantly, making it inappropriate to include *all* changes, the upgrade will only *outline* what has changed. In this case, we recommend that you purchase a replacement CMP (printed copy or eBook) or Course Notes at a significantly reduced price.

The CMP Upgrade can be downloaded free of charge from our website at **www.ActEd.co.uk**. Alternatively, if the upgrade contains a large number of pages, you may prefer to purchase a hard copy from us at a minimal price to cover production and handling costs.

A separate upgrade for eBooks is not produced but a significant discount is available for retakers wishing to re-purchase the latest eBook.

X Assignment Marking

We are happy to mark your attempts at the X assignments. Marking is not included with the Assignments or the CMP and you need to order it separately. We recommend that you submit your script by email. Your script will be marked electronically and you will be able to download your marked script via a secure link on the internet.

Don't underestimate the benefits of doing and submitting assignments:

- Question practice during this phase of your study gives an early focus on the end goal of answering exam-style questions.
- You're incentivised to keep up with your study plan and get a regular, realistic assessment of progress.
- Objective, personalised feedback from a high quality marker will highlight areas on which to work and help with exam technique.

In a recent study, we found that students who attempt more than half the assignments have significantly higher pass rates.

Series Marking

Series Marking applies to a specified subject, session and student. If you purchase Series Marking, you will **not** be able to defer the marking to a future exam sitting or transfer it to a different subject or student.

We typically send out full solutions with the Series X Assignments. However, if you order Series Marking at the same time as you order the Series X Assignments, you can choose whether or not to receive a copy of the solutions in advance. If you choose not to receive them with the study material, you will be able to download the solutions via a secure link on the internet when your marked script is returned (or following the final deadline date if you do not submit a script).

If you are having your attempts at the assignments marked by ActEd, you should submit your scripts regularly throughout the session, in accordance with the schedule of recommended dates set out in information provided with the assignments. This will help you to pace your study throughout the session and leave an adequate amount of time for revision and question practice.

The recommended submission dates are realistic targets for the majority of students. Your scripts will be returned more quickly if you submit them well before the final deadline dates.

Any script submitted *after* the relevant final deadline date will not be marked. It is your responsibility to ensure that we receive scripts in good time.

Marking Vouchers

Marking Vouchers give the holder the right to submit a script for marking at any time, irrespective of the individual assignment deadlines, study session, subject or person.

Marking Vouchers can be used for any assignment. They are valid for four years from the date of purchase and can be refunded at any time up to the expiry date.

Although you may submit your script with a Marking Voucher at any time, you will need to adhere to the explicit Marking Voucher deadline dates to ensure that your script is returned before the date of the exam. The deadline dates are provided with the assignments.

Tutorials

Our tutorials are specifically designed to develop the knowledge that you will acquire from the course material into the higher-level understanding that is needed to pass the exam.

We run a range of different tutorials including face-to-face tutorials at various locations, and Live Online tutorials. Full details are set out in our *Tuition Bulletin*, which is available from our website at **www.ActEd.co.uk**.

Regular and Block Tutorials

In preparation for these tutorials, we expect you to have read the relevant part(s) of the Course Notes before attending the tutorial so that the group can spend time on exam questions and discussion to develop understanding rather than basic bookwork.

You can choose ***one*** of the following types of tutorial:

- **Regular Tutorials** (two or three days) spread over the session.
- **A Block Tutorial** (two or three consecutive days) held two to eight weeks before the exam.

Online Classroom

The Online Classroom acts as either a valuable add-on or a great alternative to a face-to-face or Live Online tutorial.

At the heart of the Online Classroom in each subject is a comprehensive, easily-searched collection of over 100 tutorial units. These are a mix of:

- teaching units, helping you to really get to grips with the course material, and
- guided questions, enabling you to learn the most efficient ways to answer questions and avoid common exam pitfalls.

The best way to discover the Online Classroom is to see it in action. You can watch a sample of the Online Classroom tutorial units on our website at **www.ActEd.co.uk**.

Flashcards

Flashcards are a set of A6-sized cards that cover the key points of the subject that most students want to commit to memory. Each flashcard has questions on one side and the answers on the reverse. We recommend that you use the cards actively and test yourself as you go.

Flashcards are available in **eBook** format for viewing on a range of electronic devices. eBooks can be ordered separately or as an addition to paper products. Visit **www.ActEd.co.uk** for full details about the eBooks that are available, compatibility with different devices, software requirements and printing restrictions.

"Revision & rehearsal" products

Revision Notes

Our Revision Notes have been designed with input from students to help you revise efficiently. They are suitable for first-time sitters who have worked through the ActEd Course Notes or for retakers (who should find them much more useful and challenging than simply reading through the course again).

The Revision Notes are a set of nine A5 booklets – perfect for revising on the train or tube to work. Each booklet covers one main theme or a set of related topics from the course and includes:

- Core Reading with a set of integrated short questions to develop your bookwork knowledge
- relevant past exam questions with concise solutions from the last ten years
- detailed analysis of key past exam questions (selected for their difficulty), and
- other useful revision aids.

ActEd Solutions with Exam Technique (ASET)

The ActEd Solutions with Exam Technique (ASET) contains our solutions to the previous four years' exam papers, *ie* eight papers, plus comment and explanation. In particular it will highlight how questions might have been analysed and interpreted so as to produce a good solution with a wide range of relevant points. This will be valuable in approaching questions in subsequent examinations.

A "Mini-ASET" will also be available in the summer session covering the April Exam only.

Revision Tutorials

Revision Tutorials are intensive one-day face-to-face or Live Online tutorials in the final run-up to the exam.

They give you the opportunity to practise interpreting and answering past exam questions and to raise any outstanding queries with an ActEd tutor. These courses are most suitable if you have previously attended Regular Tutorials or a Block Tutorial in the same subject.

Details of how to apply for our tutorials are set out in our *Tuition Bulletin*, which is available from our website at **www.ActEd.co.uk**.

"Rehearsal" products

Mock Exam A

Mock Exam A is a 100-mark mock exam paper and is a realistic test of your exam preparation. It is based on Mock Exam A from last year but it has been updated to reflect any changes to the Syllabus and Core Reading.

Additional Mock Pack (AMP)

The Additional Mock Pack (AMP) consists of two further 100-mark mock exam papers – Mock Exam B and Mock Exam C. This is ideal if you are retaking and have already sat Mock Exam A, or if you just want some extra question practice.

Mock / AMP Marking

We are happy to mark your attempts at Mock Exam A or the mock exams included within the AMP. The same general principles apply as for the X Assignment Marking. In particular:

- Mock Exam Marking is available for Mock Exam A and it applies to a specified subject, session and student
- Marking Vouchers can be used for Mock Exam A or the mock exams contained within the AMP; please note that attempts at the AMP can **only** be marked using Marking Vouchers.

Recall that:

- marking is not included with the products themselves and you need to order it separately
- we recommend that you submit your script by email
- your script will be marked electronically and you will be able to download your marked script via a secure link on the internet.

Queries and feedback

From time to time you may come across something in the study material that is unclear to you. The easiest way to solve such problems is often through discussion with friends, colleagues and peers – they will probably have had similar experiences whilst studying. If there's no-one at work to talk to then use our discussion forum at **www.ActEd.co.uk/forums** (or use the link from our home page at **www.ActEd.co.uk**).

Our online forum is dedicated to actuarial students so that you can get help from fellow students on any aspect of your studies from technical issues to study advice. You could also use it to get ideas for revision or for further reading around the subject that you are studying. ActEd tutors will visit the site from time to time to ensure that you are not being led astray and we also post other frequently asked questions from students on the forum as they arise.

If you are still stuck, then you can send queries by email to **CT5@bpp.com** (but we recommend that you try the forum first). We will endeavour to contact you as soon as possible after receiving your query but you should be aware that it may take some time to reply to queries, particularly when tutors are away from the office running tutorials. At the busiest teaching times of year, it may take us more than a week to get back to you.

If you have many queries on the course material, you should raise them at a tutorial or book a personal tuition session with an ActEd tutor. Information about personal tuition is set out in our current brochure. Please email **ActEd@bpp.com** for more details.

If you find an error in the course, please check the corrections page of our website (**www.ActEd.co.uk/Html/paper_corrections.htm**) to see if the correction has already been dealt with. Otherwise please send details via email to **CT5@bpp.com** or send a fax to **01235 550085**.

Each year our tutors work hard to improve the quality of the study material and to ensure that the courses are as clear as possible and free from errors. We are always happy to receive feedback from students, particularly details concerning any errors, contradictions or unclear statements in the courses. If you have any comments on this course please email them to **CT5@bpp.com** or fax them to **01235 550085**.

Our tutors also work with the profession to suggest developments and improvements to the Syllabus and Core Reading. If you have any comments or concerns about the Syllabus or Core Reading, these can be passed on via ActEd. Alternatively, you can send them directly to the Institute and Faculty of Actuaries' Examination Team by email to **education.services@actuaries.org.uk**.

3 How to study to pass the exams

The CT Subject exams

The Core Reading and exam papers for these subjects tend to be very technical. The exams themselves have many calculation and manipulation questions. The emphasis in the exam will therefore be on *understanding* the mathematical techniques and applying them to various, frequently unfamiliar, situations. It is important to have a feel for what the numerical answer should be by having a deep understanding of the material and by doing reasonableness checks.

Subjects CT2 and CT7 are more "wordy" than the other subjects, including an "essay-style" question or two in Subject CT7.

As a high level of mathematics is required in the courses it is important that your mathematical skills are extremely good. If you are a little rusty you may wish to consider buying the Foundation ActEd Course (FAC). This covers all of the mathematical techniques that are required for the CT Subjects, some of which are beyond A-Level (or Higher) standard. It is a reference document to which you can refer when you need help on a particular topic.

You will have sat many exams before and will have mastered the exam and revision techniques that suit you. However it is important to note that due to the high volume of work involved in the CT Subjects it is not possible to leave all your revision to the last minute. Students who prepare well in advance have a better chance of passing their exams on the first sitting.

Unprepared students find that they are under time pressure in the exam. Therefore it is important to find ways of maximising your score in the shortest possible time. Part of your preparation should be to practise a large number of exam-style questions under timed exam conditions as soon as possible. This will:

- help you to develop the necessary understanding of the techniques required
- highlight the key topics, which crop up regularly in many different contexts and questions
- help you to practise the specific skills that you will need to pass the exam.

There are many sources of exam-style questions. You can use past exam papers, the Question and Answer Bank (which includes many past exam questions), assignments, mock exams, the Revision Notes and ASET.

Overall study plan

We suggest that you develop a realistic study plan, building in time for relaxation and allowing some time for contingencies. Be aware of busy times at work, when you may not be able to take as much study leave as you would like. Once you have set your plan, be determined to stick to it. You don't have to be too prescriptive at this stage about what precisely you do on each study day. The main thing is to be clear that you will cover all the important activities in an appropriate manner and leave plenty of time for revision and question practice.

Aim to manage your study so as to allow plenty of time for the concepts you meet in this course to "bed down" in your mind. Most successful students will probably aim to complete the course at least a month before the exam, thereby leaving a sufficient amount of time for revision. By finishing the course as quickly as possible, you will have a much clearer view of the big picture. It will also allow you to structure your revision so that you can concentrate on the important and difficult areas of the course.

A sample CT subject study plan is available on our website at:

www.ActEd.co.uk/Html/help_and_advice_study_plans.htm

It includes details of useful dates, including assignment deadlines and tutorial finalisation dates.

Study sessions

Only do activities that will increase your chance of passing. Try to avoid including activities for the sake of it and don't spend time reviewing material that you already understand. You will only improve your chances of passing the exam by getting on top of the material that you currently find difficult.

Ideally, each study session should have a specific purpose and be based on a specific task, *eg* "*Finish reading Chapter 3 and attempt Questions 1.4, 1.7 and 1.12 from the Question and Answer Bank*", as opposed to a specific amount of time, *eg "Three hours studying the material in Chapter 3*".

Try to study somewhere quiet and free from distractions (*eg* a library or a desk at home dedicated to study). Find out when you operate at your peak, and endeavour to study at those times of the day. This might be between 8*am* and 10*am* or could be in the evening. Take short breaks during your study to remain focused – it's definitely time for a short break if you find that your brain is tired and that your concentration has started to drift from the information in front of you.

Order of study

We suggest that you work through each of the chapters in turn. To get the maximum benefit from each chapter you should proceed in the following order:

1. Read the Syllabus Objectives. These are set out in the box on page 1 of each chapter.

2. Read the Chapter Summary at the end of each chapter. This will give you a useful overview of the material that you are about to study and help you to appreciate the context of the ideas that you meet.

3. Study the Course Notes in detail, annotating them and possibly making your own notes. Try the self-assessment questions as you come to them. Our suggested solutions are at the end of each chapter. As you study, pay particular attention to the listing of the Syllabus Objectives and to the Core Reading.

4. Read the Chapter Summary again carefully. If there are any ideas that you can't remember covering in the Course Notes, read the relevant section of the notes again to refresh your memory.

It's a fact that people are more likely to remember something if they review it several times. So, do look over the chapters you have studied so far from time to time. It is useful to re-read the Chapter Summaries or to try the self-assessment questions again a few days after reading the chapter itself.

You may like to attempt some questions from the Question and Answer Bank when you have completed a part of the course. It's a good idea to annotate the questions with details of when you attempted each one. This makes it easier to ensure that you try all of the questions as part of your revision without repeating any that you got right first time.

Once you've read the relevant part of the notes and tried a selection of questions from the Question and Answer Bank (and attended a tutorial, if appropriate) you should attempt the corresponding assignment. If you submit your assignment for marking, spend some time looking through it carefully when it is returned. It can seem a bit depressing to analyse the errors you made, but you will increase your chances of passing the exam by learning from your mistakes. The markers will try their best to provide practical comments to help you to improve.

To be really prepared for the exam, you should not only know and understand the Core Reading but also be aware of what the examiners will expect. Your revision programme should include plenty of question practice so that you are aware of the typical style, content and marking structure of exam questions. You should attempt as many questions as you can from the Question and Answer Bank and past exam papers.

Active study

Here are some techniques that may help you to study actively.

1. Don't believe everything you read! Good students tend to question everything that they read. They will ask "why, how, what for, when?" when confronted with a new concept, and they will apply their own judgement. This contrasts with those who unquestioningly believe what they are told, learn it thoroughly, and reproduce it (unquestioningly?) in response to exam questions.

2. Another useful technique as you read the Course Notes is to think of possible questions that the examiners could ask. This will help you to understand the examiners' point of view and should mean that there are fewer nasty surprises in the exam room! Use the Syllabus to help you make up questions.

3. Annotate your notes with your own ideas and questions. This will make you study more actively and will help when you come to review and revise the material. Do not simply copy out the notes without thinking about the issues.

4. Attempt the questions in the notes as you work through the course. Write down your answer before you refer to the solution.

5. Attempt other questions and assignments on a similar basis, *ie* write down your answer before looking at the solution provided. Attempting the assignments under exam conditions has some particular benefits:

   - It forces you to think and act in a way that is similar to how you will behave in the exam.
   - When you have your assignments marked it is *much* more useful if the marker's comments can show you how to improve your performance under exam conditions than your performance when you have access to the notes and are under no time pressure.
   - The knowledge that you are going to do an assignment under exam conditions and then submit it (however good or bad) for marking can act as a powerful incentive to make you study each part as well as possible.
   - It is also quicker than trying to write perfect answers.

6. Sit a mock exam four to six weeks before the real exam to identify your weaknesses and work to improve them. You could use a mock exam written by ActEd or a past exam paper.

4 Frequently asked questions

Q: What knowledge of earlier subjects should I have?

A: The Course Notes are written on the assumption that students have studied Subjects CT1 and CT3.

Q: What level of mathematics is required?

A: The level of maths you need for this course is broadly A-level standard. However, there may be some symbols (*eg* the gamma function) that are not usually included on A-level syllabuses. You will find the course (and the exam!) much easier if you feel comfortable with the mathematical techniques used in the course and you feel confident in applying them yourself. If you feel that you need to brush up on your mathematical skills before starting the course, you may find it useful to study the Foundation ActEd Course (FAC) or read an appropriate textbook. The full Syllabus for FAC, a sample of the Course Notes and an Initial Assessment to test your mathematical skills can be found on our website at **www.ActEd.co.uk**.

Q: What calculators am I allowed to use in the exam?

A: Please refer to **www.actuaries.org.uk** for the latest advice.

Q: What should I do if I discover an error in the course?

A: If you find an error in the course, please check our website at:

www.acted.co.uk/Html/paper_corrections.htm

to see if the correction has already been dealt with. Otherwise please send details via email to **CT5@bpp.com** or send a fax to **01235 550085**.

Q: What do I need to do to pass the exam?

A: Work extremely hard and practise as many questions as you can.

Q: Which past exam questions are relevant for the exam?

A: With a few exceptions (detailed below), all the past exam questions in Subject CT5 dating back to 2005 are relevant to the current syllabus. The past exam papers can be found on the Institute and Faculty of Actuaries website at **www.actuaries.org.uk**. However, there were a number of material changes that affected the syllabus from the 2015 exam sitting onwards. As a result, the following past questions should be ignored as they do not relate to the current syllabus:

- April 2006 Q7
- September 2006 Q8
- September 2009 Q2
- September 2010 Q5
- September 2013 Q21 part (i).

As a consequence of changes that were made to the multiple decrements section of the syllabus, the following questions are either not relevant in their original form, or the answers expected from them under the current syllabus would be different from those shown in the published examiners' reports:

- September 2008 Q7
- April 2009 Q5 and Q13
- April 2010 Q10
- September 2010 Q4 and Q9
- April 2011 Q11
- September 2011 Q9
- September 2013 Q20
- April 2014 Q8
- September 2014 Q13.

You will, however, be able to find many of these questions in ASET and the Revision Notes, for which the questions and/or their solutions have been suitably revised and adapted to be consistent with the current syllabus. If you wish to use any of these past questions in your studies, we recommend you buy ASET and/or the Revision Notes to be sure you are using appropriate materials.

5 Core Reading and the Syllabus

Core Reading

The Syllabus for Subject CT5, and the Core Reading that supplements it, has been produced by the Institute and Faculty of Actuaries. The relevant individual Syllabus Objectives are included at the start of each course chapter and a complete copy of the Syllabus is included in Section 6 of this Study Guide. We recommend that you use the Syllabus as an important part of your study. The purpose of Core Reading is to assist in ensuring that tutors, students and examiners have a clear, shared appreciation of the requirements of the Syllabus. The Core Reading supports coverage of the Syllabus in helping to ensure that both depth and breadth are reinforced. It is therefore important that students have a good understanding of the concepts covered by the Core Reading.

Core Reading deals with each syllabus objective. Core Reading covers what is needed to pass the exam but the tuition material that has been written by ActEd enhances it by giving examples and further explanation of key points. The Subject CT5 Course Notes include the Core Reading in full, integrated throughout the course. Here is an excerpt from some ActEd Course Notes to show you how to identify Core Reading and the ActEd material. **Core Reading is shown in this bold font.**

Note that in the example given above, the index *will* fall if the actual share price goes below the theoretical ex-rights share price. Again, this is consistent with what would happen to an underlying portfolio.

This is ActEd text

After allowing for chain-linking, **the formula for the investment index becomes:**

This is Core Reading

$$\boldsymbol{I(t) = \frac{\sum_i N_{i,t} P_{i,t}}{B(t)}}$$

where $N_{i,t}$ is the number of shares issued for the *i*th constituent at time *t*;

$B(t)$ is the base value, or divisor, at time *t*.

Core Reading accreditation

The Institute and Faculty of Actuaries would like to thank the numerous people who have helped in the development of this material and in the previous versions of Core Reading.

Changes to the Syllabus and Core Reading

The Syllabus and Core Reading are updated as at 31 May each year. The exams in April and September/October 2016 will be based on the Syllabus and Core Reading as at 31 May 2015.

We recommend that you always use the up-to-date Core Reading to prepare for the exams.

The Institute and Faculty of Actuaries' Copyright

Past exam papers

You can download some past exam papers and Examiners' Reports from the profession's website at **www.actuaries.org.uk**.

Further reading

The exam will be based on the relevant Syllabus and Core Reading and the ActEd course material will be the main source of tuition for students.

However, some students may find it useful to obtain a different viewpoint on a particular topic covered in Subject CT5. The following list of further reading for Subject CT5 has been prepared by the Institute and Faculty of Actuaries. This list is not exhaustive and other useful material may be available.

Actuarial mathematics. Bowers, Newton L *et al*. – 2nd ed. – Society of Actuaries, 1997. xxvi, 753 pages. ISBN: 0 938959 46 8.

The analysis of mortality and other actuarial statistics. Benjamin, Bernard; Pollard, John H. – 3rd ed. – Faculty and Institute of Actuaries, 1993. 519 pages. ISBN 0 90106626 5.

Life contingencies. Neill, Alistair. – Heinemann, 1977. vii, 452 pages. ISBN 0 434 91440 1.

(This text is no longer in print, but has been used as a textbook under earlier education strategies. You should find it relatively easy to borrow a copy from a colleague. Alternatively, you can borrow it from the library at Napier House.)

Life insurance mathematics. Gerber, Hans U. – 3rd ed. – Springer. Swiss Association of Actuaries, 1997. 217 pages. ISBN 3 540 62242 X.

Modern actuarial theory and practice. Booth, Philip M *et al*. – Chapman & Hall, 1999. xiii, 716 pages. ISBN 0 8493 0388 5.

6 Syllabus

The full Syllabus for Subject CT5 is given here. To the right of each objective are the chapter numbers in which the objective is covered in the ActEd course.

Aim

The aim of the Contingencies subject is to provide a grounding in the mathematical techniques which can be used to model and value cashflows dependent on death, survival, or other uncertain risks.

Links to other subjects

- Subjects CT1 – Finance and Financial Reporting, CT3 – Probability and Mathematical Statistics and CT4 – Models: introduce techniques that will be drawn upon and used in the development of this subject.

- Subject ST1 – Health and Care Specialist Technical, ST2 – Life Insurance Specialist Technical and ST4 – Pensions and other Benefits Specialist Technical: use the principles introduced in this subject.

Objectives

On completion of the course the candidate will be able to:

(i) Define simple assurance and annuity contracts, and develop formulae for the means and variances of the present values of the payments under these contracts, assuming constant deterministic interest. (Chapters 1 to 4)

1. Define the following terms:
   - whole life assurance
   - term assurance
   - pure endowment
   - endowment assurance
   - whole life level annuity
   - temporary level annuity
   - guaranteed level annuity
   - premium
   - benefit

   including assurance and annuity contracts where the benefits are deferred.

2. Define the following probabilities: ${}_{n|m}q_x$, ${}_{n|}q_x$ and their select equivalents ${}_{n|m}q_{[x]+r}$, ${}_{n|}q_{[x]+r}$.

3. Obtain expressions in the form of sums for the mean and variance of the present value of benefit payments under each contract above, in terms of the curtate random future lifetime, assuming that death benefits are payable at the end of the year of death and that annuities are paid annually in advance or in arrear, and, where appropriate, simplify these expressions into a form suitable for evaluation by table look-up or other means.

4. Obtain expressions in the form of integrals for the mean and variance of the present value of benefit payments under each contract above, in terms of the random future lifetime, assuming that death benefits are payable at the moment of death and that annuities are paid continuously, and, where appropriate, simplify these expressions into a form suitable for evaluation by table look-up or other means.

5. Define the symbols A_x , $A_{x:\overline{n}|}$, $A^{1}_{x:\overline{n}|}$, $A_{x:\overset{1}{\overline{n}|}}$, a_x , $a_{x:\overline{n}|}$, ${}_{m|}a_{x:\overline{n}|}$, $\ddot{a}_x$, $\ddot{a}_{x:\overline{n}|}$, ${}_{m|}\ddot{a}_{x:\overline{n}|}$ and their select and continuous equivalents. Extend the annuity factors to allow for the possibility that payments are more frequent than annual but less frequent than continuous.

6. Understand and use the relations between annuities payable in advance and in arrear, and between temporary, deferred and whole life annuities.

7. Understand and use the relations $A_x = 1 - d\ddot{a}_x$, $A_{x:\overline{n}|} = 1 - d\ddot{a}_{x:\overline{n}|}$, and their select and continuous equivalents.

8. Define the expected accumulation of the benefits in 1., and obtain expressions for them corresponding to the expected present values in 3. and 4. (note: expected values only).

(ii) Describe and use practical methods of evaluating expected values and variances of the simple contracts defined in objective (i). (Chapters 1 to 4)

1. Describe the life table functions l_x and d_x and their select equivalents $l_{[x]+r}$ and $d_{[x]+r}$.

2. Express the following life table probabilities in terms of the functions in 1.: ${}_np_x$, ${}_nq_x$, ${}_{n|m}q_x$ and their select equivalents ${}_np_{[x]+r}$, ${}_nq_{[x]+r}$, ${}_{n|m}q_{[x]+r}$.

3. Express the expected values and variances in objective (i) 3. in terms of the functions in 1. and 2.

4. Evaluate the expected values and variances in objective (i) 3. by table look-up or other means, including the use of the relationships in objectives (i) 6. and 7.

5. Derive approximations for, and hence evaluate, the expected values and variances in objective (i) 4. in terms of those in objective (i) 3.

6. Evaluate the expected accumulations in objective (i) 8.

(iii) Describe and calculate, using ultimate or select mortality, net premiums and net premium reserves of simple insurance contracts. (Chapter 5)

1. Define the net random future loss under an insurance contract, and state the principle of equivalence.

2. Define and calculate net premiums for the insurance contract benefits in objective (i) 1. Regular premiums and annuity benefits may be payable annually, more frequently than annually, or continuously. Death benefits may be payable at the end of the year of death, or immediately on death.

3. State why an insurance company will set up reserves.

4. Describe prospective and retrospective reserves.

5. Define and evaluate prospective and retrospective net premium reserves in respect of the contracts in objective (i) 1., with premiums as in (iii) 2.

6. Show that prospective and retrospective reserves are equal when calculated on the same basis.

7. Obtain recursive relationships between net premium reserves at annual intervals, for contracts with death benefits paid at the end of the year of death, and annual premiums.

8. Define and calculate, for a single policy or a portfolio of policies (as appropriate):
   - death strain at risk
   - expected death strain
   - actual death strain
   - mortality profit

   for policies with death benefits payable immediately on death or at the end of the year of death; for policies paying annuity benefits at the start of the year or on survival to the end of the year; and for policies where single or annual premiums are payable.

(iv) Describe and calculate, using ultimate or select mortality, net premiums and net premium reserves for increasing and decreasing benefits and annuities.

(Chapter 6)

1. Extend the techniques of (ii) to calculate the expected present value of an annuity, premium, or benefit payable on death, which increases or decreases by a constant compound rate. Calculate net premiums and net premium reserves for contracts with premiums and benefits which vary as described.

2. Define the symbols $(IA)_x$, $(I\bar{A})_x$, $(I\ddot{a})_x$, $(Ia)_x$ and $(\bar{I}\bar{a})_x$ and their select equivalents.

3. Calculate the expected present value of an annuity, premium or benefit payable on death, which increases or decreases by a constant monetary amount. Calculate net premiums and net premium reserves for contracts with premiums and benefits which vary as described.

4. Describe the operation of conventional with-profits contracts, in which profits are distributed by the use of regular reversionary bonuses, and by terminal bonuses.

5. Calculate net premiums and net premium reserves for the conventional with-profits contracts defined in 4.

6. Describe the operation of accumulating with-profits contracts, in which benefits take the form of an accumulating fund of premiums, where either:

   - the fund is defined in monetary terms, has no explicit charges, and is increased by the addition of regular guaranteed and bonus interest payments plus a terminal bonus; or
   - the fund is defined in terms of the value of a unit fund, is subject to explicit charges, and is increased by regular bonus additions plus a terminal bonus ("Unitised with-profits")

   In the case of unitised with-profits, the regular additions can take the form of (a) unit price increases (guaranteed and/or discretionary), or (b) allocations of additional units.

   In either case a guaranteed minimum monetary death benefit may be applied.

(v) Describe and calculate gross premiums and reserves of assurance and annuity contracts. (Chapter 7)

1. List the types of expenses incurred in writing a life insurance contract.

2. Describe the influence of inflation on the expenses listed in 1.

3. Define the gross future loss random variable for the benefits and annuities listed in (i) 1. and (iv) 1.–5., and calculate gross premiums and reserves that satisfy probabilities involving the future loss random variable. Regular premiums and annuity benefits may be payable annually or continuously. Death benefits may be payable at the end of the year of death or immediately on death.

4. Calculate gross premiums using the equivalence principle. Regular premiums and annuity benefits may be payable annually, more frequently than annually, or continuously. Death benefits may be payable at the end of the year of death or immediately on death.

5. Calculate gross premiums using simple criteria other than the principles described in 3. and 4.

6. Define and calculate the gross premium prospective reserve.

7. Define and calculate the gross premium retrospective reserve.

8. State the conditions under which, in general, the prospective reserve is equal to the retrospective reserve allowing for expenses.

9. Prove that, under the appropriate conditions, the prospective reserve is equal to the retrospective reserve, with or without allowance for expenses, for all standard fixed benefit and increasing/decreasing benefit contracts.

10. Obtain a recursive relation between successive annual reserves for an annual premium contract, with allowance for expenses, for standard fixed benefit contracts, and use this relation to calculate the profit earned from a contract during a year.

(vi) Define and use functions involving two lives. (Chapters 8 and 9)

1. Extend the techniques of objectives (i)–(v) to deal with cashflows dependent upon the death or survival of either or both of two lives.

2. Extend the techniques of 1. to deal with functions dependent upon a fixed term as well as age.

(vii) Describe and illustrate methods of valuing cashflows that are contingent upon multiple transition events. (Chapter 10)

1. Define health insurance, and describe simple health insurance premium and benefit structures.

2. Explain how a cashflow, contingent upon multiple transition events, may be valued using a multiple-state Markov Model, in terms of the forces and probabilities of transition.

3. Construct formulae for the expected present values of cashflows that are contingent upon multiple transition events, including simple health insurance premiums and benefits, and calculate these in simple cases. Regular premiums and sickness benefits are payable continuously and assurance benefits are payable immediately on transition.

(viii) Describe and use methods of projecting and valuing expected cashflows that are contingent upon multiple decrement events. (Chapters 10 and 11)

1. Define a multiple decrement model as a special case of multiple-state Markov model.

2. Derive dependent probabilities for a multiple decrement model in terms of given forces of transition, assuming forces of transition are constant over single years of age.

3. Derive forces of transition from given dependent probabilities, assuming forces of transition are constant over single years of age.

4. Describe the construction and use of multiple decrement tables.

5. Describe the typical benefit and contribution structures of pension schemes, including:

   - defined contribution schemes
   - defined benefit (final salary) schemes

6. Use multiple decrement tables to evaluate expected present values of cashflows dependent upon more than one decrement, including those of pension schemes.

(ix) Describe and use projected cashflow techniques, where and as appropriate for use in pricing, reserving, and assessing profitability. (Chapters 12 and 13)

1. Define a unit-linked contract.

2. Project expected future cashflows for whole life, endowment and term assurances, annuities, unit-linked contracts, and unitised with-profits contracts, incorporating multiple decrement models as appropriate.

3. Profit test simple annual premium contracts of the types listed in 2. and determine the profit vector, the profit signature, the net present value, and the profit margin.

4. Show how a profit test may be used to price a product, and use a profit test to calculate a premium for a conventional (without profits) life insurance contract.

5. Show how, for unit-linked contracts, non-unit reserves can be established to eliminate ("zeroise") future negative cashflows, using a profit test model.

(x) Describe the principal forms of heterogeneity within a population and the ways in which selection can occur. (Chapter 14)

1. Explain why it is necessary to have different mortality tables for different classes of lives.

2. Explain the theoretical basis of the use of risk classification in life insurance.

3. State the principal factors which contribute to the variation in mortality and morbidity by region and according to the social and economic environment, specifically:
   - occupation
   - nutrition
   - housing
   - climate/geography
   - education
   - genetics

4. Define and give examples of the main forms of selection:
   - temporary initial selection
   - class selection
   - time selection
   - spurious selection
   - adverse selection

5. Explain how selection can be expected to occur amongst individuals or groups taking out each of the main types of life insurance contracts, or amongst members of large pension schemes.

6. Explain the concept of mortality convergence.

7. Explain how decrements can have a selective effect.

8. Explain the concept of a single figure index and its advantages and disadvantages for summarising and comparing actual experience.

9. Define the terms crude mortality rate, directly standardised and indirectly standardised mortality rate, standardised mortality ratio, and illustrate their use.

CT5 Index

Chapter 1

Life assurance contracts

Syllabus objectives

(i) Define simple assurance contracts, and develop formulae for the means and variances of the present values of the payments under these contracts, assuming constant deterministic interest.

1. Define the following terms:

- *whole life assurance*
- *term assurance*
- *pure endowment*
- *endowment assurance*
- *premium*
- *benefit*

including assurance contracts where the benefits are deferred.

2. Define the probability ${}_{n|}q_x$.

3. Obtain expressions in the form of sums for the mean and variance of the present value of benefit payments under each contract above, in terms of the curtate random future lifetime, assuming that death benefits are payable at the end of the year of death and,… where appropriate, simplify these expressions into a form suitable for evaluation by table look-up or other means.

4. Obtain expressions in the form of integrals for the mean and variance of the present value of benefit payments under each contract above, in terms of the random future lifetime, assuming that death benefits are payable at the moment of death,… and, where appropriate, simplify these expressions into a form suitable for evaluation by table look-up or other means.

Continued…

5. *Define the symbols* A_x, $A_{x:\overline{n}|}$, $A^{1}_{x:\overline{n}|}$, $A_{x:\overline{n}|}^{\ \ 1}$ *and their continuous equivalents.*

0 Introduction

Life insurance contracts (also called policies) are made between a life insurance company and one or more persons called the policyholders.

The policyholder(s) will agree to pay an amount or a series of amounts to the life insurance company, called premiums.

The premiums may be paid:

- on a regular basis (known as regular premiums), typically paid monthly, quarterly or annually, or
- as one single payment (known as a single premium).

In return the life insurance company agrees to pay an amount or amounts called the benefit(s), to the policyholder(s) on the occurrence of a specified event.

In this subject we first consider contracts with a single policyholder and then later show how to extend the theory to two policyholders.

The benefits payable under simple life insurance contracts are of two main types.

(a) **The benefit may be payable on or following the death of the policyholder.** For example, a term assurance contract, where the insurance company will make a payment (the sum assured) to the policyholder's estate if the policyholder dies during the term of the policy.

(b) **The benefit(s) may be payable provided the life survives for a given term. An example of this type of contract is an annuity, under which amounts are payable at regular intervals as long as the policyholder is still alive.** We will look at life annuities in Chapter 2.

More generally, the theory of this subject may be applied to "near-life" contingencies – such as state of health of a policyholder – or to "non-life" contingencies – such as the cost of replacing a machine at the time of failure. The theory assumes only that the payment is of known amount. Subject ST3, General Insurance Specialist Technical, examines the valuation of non-life contingencies where the future payment is of unknown amount.

The simplest life insurance contract is the *whole life assurance*. The benefit under such a contract is an amount, called the *sum assured*, which will be paid on the policyholder's death.

A *term assurance* contract is a contract to pay a sum assured on or after death, provided death occurs during a specified period, called the term of the contract.

A *pure endowment* contract provides a sum assured at the end of a fixed term, provided the policyholder is then alive.

An *endowment assurance* is a combination of:

(i) a term assurance, and

(ii) a pure endowment assurance.

That is, a sum assured is payable either on death during the term or on survival to the end of the term. The sums assured payable on death or survival need not be the same, although they often are.

1 Pricing of life insurance contracts

1.1 Equations of value

To calculate life assurance premiums we use a similar *equation of value* to the one we used in Subject CT1:

Equation of value

Expected present value of money in = Expected present value of money out

Note that, when we are applying this equation to calculate an insurance premium, we are considering the money into and out of the insurance company. So, in the situations we will look at in this course, the items that we will be considering are:

- premiums coming in
- expected payments to policyholders (*ie* benefits) going out
- expenses that are incurred by the company.

Much actuarial work is concerned with finding a fair price for a life insurance contract. In such calculations we must consider:

(a) the time value of money, and

(b) the uncertainty attached to payments to be made in the future, depending on the death or survival of a given life.

Therefore this subject requires us to bring together the topics covered in Subject CT1, Financial Mathematics, in particular compound interest, and the topics covered in Subject CT4, Models, in particular the unknown future lifetime and its associated probabilities.

1.2 Allowance for investment income

Premiums are usually paid in advance (*eg* at the start of each month or year) and benefits will be paid sometime later (if at all). So the insurer will need to allow for the time value of money when calculating the appropriate premium to charge.

Question 1.1

Why are premiums paid in advance?

In this subject we will usually assume that money can be invested or borrowed at some given rate of interest. We will always assume that the rate of interest is *known*, that is, deterministic. The mathematics of finance includes several useful stochastic models of the behaviour of interest rates, but we will not use them. We will not always assume that the rate of interest is *constant*, however. When the rate of interest *is* constant, we denote the effective compound rate of interest per annum by i and define $v = (1+i)^{-1}$, and we will use these without further comment.

In practice, life assurance companies may price products by any of the three approaches mentioned above, *ie* they may:

- assume a constant interest rate for each future year
- use a deterministic approach where interest rates are assumed to change in a predetermined way
- use a stochastic approach, where future interest rates are random and follow a certain statistical distribution.

As stated in the Core Reading, we will not consider the stochastic approach for modelling interest rates in this subject.

1.3 Other assumptions

We will also assume knowledge of the basic probabilities introduced in Subject CT4, Models, namely $_tp_x$ and $_tq_x$ and their degenerate quantities, when $t=1$, p_x and q_x. Further, we assume knowledge of the key survival model formulae, the fundamental ones being that:

$$_tq_x = \int_0^t {}_sp_x\,\mu_{x+s}\,ds$$

and:

$$_tp_x = \exp\left\{-\int_0^t \mu_{x+s}\,ds\right\}$$

Recall that $_tp_x$ is the probability that a life aged x survives for at least another t years and $_tq_x$ is the probability that a life aged x dies within the next t years.

μ_{x+s} is the annual rate of transfer between alive and dead at exact age $x+s$, *ie* it's the annual rate at which living people are dying at that exact age. It is probably most helpful to think of this as:

$\mu_{x+s}.ds \simeq$ probability of a life aged $x+s$ dying over the very small time interval $x+s, x+s+ds$

For example:

$_sp_x\,\mu_{x+s}\,ds \simeq$ probabilty of a life aged x living s years and then dying in the next instant of time

so that:

$_tq_x = \int_0^t {}_sp_x\,\mu_{x+s}\,ds =$ probability that a life aged x dies at any of the possible moments over the next t years

remembering that an integral is just the continuous version of a summation. μ_{x+s} is also known as the *force of mortality* at exact age $x+s$.

Using the two basic building blocks described above in Section 1.1, and the assumptions made in this section and the last, we will develop formulae for the means and variances of the present value of contingent benefits.

In Chapter 3 we will consider ways of assigning probability values to the unknown future lifetime, so as to evaluate the formulae. Two different assumptions are typically used in practice when determining the probability values. The first is to assume that the underlying probability depends on age only. The second is to assume that the underlying probability depends on age plus duration since some specific event. For example, considering mortality, the assumption in the second case can allow for the likely lower level of mortality that might result from a requirement that prospective policyholders pass a medical test before the insurer agrees to accept them for life insurance.

Consider the case of life insurance policies. Under the first assumption, all lives aged x are assumed to have the same mortality, regardless of when they took out their policies. Under the second assumption, this is no longer the case. The mortality of the policyholder is now assumed also to depend on the time that has elapsed since the policy was issued.

The first assumption is described as assuming *ultimate mortality*, and the second as assuming *select mortality*. We will return later to discussing select mortality but assume for the moment that ultimate mortality applies.

2 Whole life assurance contracts

We begin by looking at the simplest assurance contract, the whole life assurance, which pays the sum assured on the policyholder's death. For the moment we ignore the premiums which the policyholder might pay.

We will use this simple contract to introduce important concepts, in particular the expected present value (EPV) of a payment contingent on an uncertain future event.

So, in the premium equation mentioned above, we are calculating the expected present value of the amount paid when the policyholder dies.

We will then apply these concepts to other types of life insurance contracts.

In mathematics of finance, the present value at time 0 of a payment of 1 to be made at time t is v^t.

Note that in the case mentioned above, we know precisely when the payment will be made, *ie* at time *t*.

Suppose, however, that the time of payment is not certain but is a random variable, say H. Then the present value of the payment is v^H, which is also a random variable. A whole life assurance benefit is a payment of this type.

Q

Question 1.2

Why is the time to payment a random variable?

2.1 Present value random variable

For the moment we will introduce two conventions, which can be relaxed later on.

Convention 1: **we suppose that we are considering a benefit payable to a life who is currently aged x, where x is an integer.**

Convention 2: **we suppose that the sum assured is payable, not *on* death, but at the end of the year of death (based on policy year).**

These will not always hold in practice, of course, but they simplify the application of life table functions.

Under these conventions we see that the whole life sum assured will be paid at time $K_x + 1$, where K_x denotes the curtate random future lifetime of a life currently aged x.

Note that we will often use the notational convention "(x)" to indicate the phrase "a life who is currently aged x".

Recall from Subject CT4 that K_x is the integer part of T_x, the complete future lifetime random variable.

> ***Question 1.3***
>
> A person is aged exactly 40 years old. Suppose that she dies when she is aged 76 years and 197 days old. What are the values of T_{40} and K_{40} for this person?

So we see that, for our assurance, duration $K_x + 1$ takes us to the *end* of the year in which the life aged x dies.

Let the sum assured be S, then the present value of the benefit is Sv^{K_x+1}, a random variable. Obvious questions are, what are the expected value and the variance of Sv^{K_x+1}?

These are not only obvious questions, but are also very important practical questions for insurance companies.

2.2 *The expected present value*

Since K_x is a discrete random variable taking whole number values $\left(K_x = 0, 1, 2, ...\right)$:

$$E\left(K_x\right) = \sum_{k=0}^{\infty} k\,P\left(K_x = k\right)$$

and:

$$E\left[g\left(K_x\right)\right] = \sum_{k=0}^{\infty} g(k)\,P\left(K_x = k\right)$$

for any function $g(\cdot)$.

In particular, for $g(k) = v^{k+1}$, we have:

$$E\left[v^{K_x+1}\right] = \sum_{k=0}^{\infty} v^{k+1}\,P\left(K_x = k\right)$$

We can also express this in terms of p and q (survival and death probability) notation.

We define a new probability $_{n|}q_x$:

$$_{n|}q_x = {}_n p_x\, q_{x+n} \qquad (n = 0, 1, 2, ...)$$

By the definition of $_t p_x$ and $_t q_x$ it follows that

$$P\left[K_x = k\right] = {}_k p_x\, q_{x+k} = {}_{k|}q_x \quad (k = 0, 1, 2, \dots)$$

So $_{n|}q_x$ is the probability that a life aged x survives for n years, but dies before time $n+1$.

Therefore:

$$E[v^{K_x+1}] = \sum_{k=0}^{\infty} v^{k+1}\,{}_k p_x\, q_{x+k} = \sum_{k=0}^{\infty} v^{k+1}\,{}_{k|}q_x$$

Question 1.4

Explain the formula given above, by general reasoning.

Question 1.5

What is $E\left[v^{K_x+1}\right]$ at an interest rate of 0%?

Sometimes we use the symbol ω (omega) to indicate the maximum possible age for human life. More precisely, ω is the smallest value of t for which ${}_tp_0 = 0$. So we sometimes see written:

$$E\left(v^{K_x+1}\right) = \sum_{k=0}^{\omega-x-1} v^{k+1}\ {}_{k|}q_x$$

but it means exactly the same as if you were summing to infinity.

Actuarial notation for the expected present value

$E[v^{K_x+1}]$ is the *expected present value* (EPV) of a sum assured of 1, payable at the end of the year of death. Such functions play a central role in life insurance mathematics and are included in the standard actuarial notation. We define

$$A_x = E[v^{K_x+1}] = \sum_{k=0}^{\infty} v^{k+1}\ {}_{k|}q_x$$

[Note that, for brevity, we have written the sum as $\sum_{k=0}^{\infty}$ instead of $\sum_{k=0}^{\omega-x-1}$. This should cause no confusion, since ${}_kp_x = 0$ for $k \geq \omega - x$.]

An explanation of the actuarial notation is called for.

A big A function denotes the expected present value of a single payment of 1 unit.

The subscript x (in the A_x symbol) indicates that the payment is contingent (= depends on) the status of "x" in some way.

Now the status "*x*" is called a *life status*, and relates to a person who is aged *x* years old exactly at the valuation date. The status remains in existence (*ie* remains "active") for as long as the person stays alive into the future. And the status ceases to exist (or "fails") at the point in the future when this person, currently aged *x*, dies. In summary:

The life status x

- remains active for as long as (x) remains alive in the future
- ceases at the moment at which (x) dies in the future.

Coming back to A_x, we can now say that:

- A_x is the EPV of a single payment of 1 unit paid out *at the end of the year* in which the status *x* fails

or, in other words, is paid out at the end of the year in which a person, currently aged *x*, dies. (We will see a little later in this chapter how we vary the notation when the payment is made immediately on death, rather than at the end of the year.)

Note that $E\left[S.v^{K_x+1}\right]=S.E\left[v^{K_x+1}\right]$. So, **if the sum assured is S, then the EPV of the benefit is $S\,A_x$. Values of A_x at various rates of interest are tabulated in (for example) AM92, which can be found in "Formulae and Tables for Examinations".**

Example

Find A_{40} (AM92 at 6%).

Solution

0.12313

Question 1.6

What are A_{30} and A_{70}, based on the mortality of AM92 (Ultimate) at 6% *pa* interest?

Comment on the relative values of these two figures.

2.3 *Variance of the present value random variable*

Recall from Subject CT3 that:

$$\text{var}[X] = E\left[X^2\right] - \left[E(X)\right]^2$$

for any random variable X, and:

$$\text{var}\left[g(X)\right] = E\left[\left(g(X)\right)^2\right] - \left[E\left(g(X)\right)\right]^2$$

for any random variable X and any function $g(\cdot)$.

Turning now to the variance of v^{K_x+1}, we have:

$$\text{var}\left[v^{K_x+1}\right] = E\left[v^{(K_x+1)2}\right] - \left[E\left(v^{K_x+1}\right)\right]^2$$

This can also be written as:

$$\textbf{var}[v^{K_x+1}] = \sum_{k=0}^{\infty} (v^{k+1})^2 \, {}_{k|}q_x - \left(A_x\right)^2$$

But since $\left(v^{k+1}\right)^2 = \left(v^2\right)^{k+1}$, the first term is just ${}^2A_x$ where the "2" prefix denotes an EPV calculated at a rate of interest $(1+i)^2 - 1$.

So:

$$\text{var}\left[v^{K_x+1}\right] = {}^2A_x - \left(A_x\right)^2$$

The "trick" used here is to notice that $(v^2)^{k+1}$ is the same as v^{k+1}, except that we have replaced v by v^2. So we are effectively using a new interest rate (i^*, say) for which $v^* = v^2$, *ie* $\dfrac{1}{1+i^*} = \left(\dfrac{1}{1+i}\right)^2$. If we invert this equation and subtract 1, we find that the new interest rate required is $i^* = (1+i)^2 - 1$.

So provided we can calculate EPVs at any rates of interest, it is easy to find the variance of a whole life benefit.

Note that:

$$\text{var}\left[Sv^{K_x+1}\right] = S^2 \ \text{var}\left[v^{K_x+1}\right]$$

Values of ^2A_x at various rates of interest are tabulated in (for example) AM92 "Formulae and Tables for Examinations".

Question 1.7

Claire, aged exactly 30, buys a whole life assurance with a sum assured of £50,000 payable at the end of the year of her death. Calculate the standard deviation of the present value of this benefit using AM92 Ultimate mortality and 6% *pa* interest.

3 Term assurance contracts

A term assurance contract is a contract to pay a sum assured on or after death, provided death occurs during a specified period, called the term of the contract.

If the life survives to the end of the term of the contract there will be no payment made by the life insurer to the policyholder.

3.1 Present value random variable

Consider such a contract, which is to pay a sum assured at the end of the year of death of a life aged x, provided this occurs during the next n years. We assume that n is an integer.

Let F denote the present value of this payment. F is a random variable.

Note that we are just using the letter F here for notational convenience. It is not standard notation.

If the policyholder dies within the n-year term, then $F = v^{K_x+1}$. If the policyholder is still alive at the end of the n-year term, then $F = 0$. We can express this more mathematically as follows:

$$F = \begin{cases} v^{K_x+1} & \text{if } K_x < n \\ 0 & \text{if } K_x \geq n \end{cases}$$

3.2 Expected present value

The expected present value of the term assurance is:

$$\begin{aligned} E\left[F\right] &= \sum_{k=0}^{n-1} v^{k+1} P\left(K_x = k\right) + \sum_{k=n}^{\infty} 0.P\left(K_x = k\right) \\ &= \sum_{k=0}^{n-1} v^{k+1} P\left(K_x = k\right) + 0\sum_{k=n}^{\infty} P\left(K_x = k\right) \end{aligned}$$

Writing this using actuarial notation:

$$E[F] = \sum_{k=0}^{n-1} v^{k+1} {}_{k|}q_x + 0 \times {}_n p_x$$
$$= \sum_{k=0}^{n-1} v^{k+1} {}_{k|}q_x$$

Actuarial notation for the expected present value

Before we look at this, we need to introduce a second kind of status, the duration status $\overline{n|}$.

This is easily distinguished from the life status because the number (n) is enclosed by the corner symbol (some might call this a rabbit hutch or other such flippant name). You should be familiar with this status from Subject CT1.

The status $\overline{n|}$ remains active for as long as the duration of time from the valuation date does not exceed n years. The status fails at the moment at which the elapsed time reaches n years in duration.

Question 1.8

What do you think $A_{\overline{10|}}$ means, and give a formula for it.

Describe in words the difference in meaning between $A_{\overline{10|}}$ and A_{10}.

In actuarial notation, we define:

$$A^{1}_{x:\overline{n|}} = E[F] = \sum_{k=0}^{n-1} v^{k+1} {}_{k|}q_x$$

to be the EPV of a term assurance benefit of 1, payable at the end of the year of death of a life x, provided this occurs during the next n years.

Question 1.9

Explain the formula above by general reasoning.

Now let's explain the more complex notation used here ($A^{1}_{x:\overline{n}|}$).

It is still the EPV of a single payment, because it is a "big A" symbol. However, the payment is now contingent on what happens to *two* statuses in some way (multiple statuses are shown in the subscript separated by a colon). The exact condition is identified by the number that is placed above the statuses (the 1), and according to where it is placed:

- the number is positioned above the life status x: this indicates that the payment is made *only* when the life status x fails (*ie* dies)
- the number over the x is a 1: this tells us that the life status x has to fail *first* out of the two statuses involved, in order for the payment to be made.

As the $\overline{n}|$ status will fail after n years, then (x) has to die within n years for the payment to be made. In other words:

- $A^{1}_{x:\overline{n}|}$ is the EPV of 1 unit paid only on the death of (x), provided that occurs within n years.

As before, the symbol also indicates that the payment is made at the end of the year of death: this aspect of the notation will become clearer later in this chapter.

3.3 *Variance of the present value random variable*

Along the same lines as for the whole life assurance:

$$\text{var}[F] = {}^{2}A^{1}_{x:\overline{n}|} - \left(A^{1}_{x:\overline{n}|}\right)^{2}$$

where the "2" prefix means that the EPV is calculated at rate of interest $(1+i)^{2}-1$.

4 Pure endowment contracts

A pure endowment contract provides a sum assured at the end of a fixed term, provided the policyholder is then alive.

4.1 Present value random variable

Consider a pure endowment contract to pay a sum assured of 1 after n years, provided a life aged x is still alive. We assume that n is an integer.

Let G denote the present value of the payment.

Again, we are using the letter G for notational convenience. This is not standard notation.

The payment is equal to zero if the policyholder dies within n years, but 1 if the policyholder survives to the end of the term. So, if the policyholder dies within the n-year term, then $G = 0$. On the other hand, if the policyholder is still alive at the end of the n-year term, then $G = v^n$.

This can be written mathematically as:

$$G = \begin{cases} 0 & \text{if } K_x < n \\ v^n & \text{if } K_x \geq n \end{cases}$$

4.2 Expected present value

The expected present value of the pure endowment is:

$$\begin{aligned} E[G] &= \sum_{k=0}^{n-1} 0.P(K_x = k) + \sum_{k=n}^{\infty} v^n P(K_x = k) \\ &= 0 \times \sum_{k=0}^{n-1} P(K_x = k) + v^n \sum_{k=n}^{\infty} P(K_x = k) \\ &= 0 \times P(K_x < n) + v^n P(K_x \geq n) \end{aligned}$$

In actuarial notation (and reversing the order of the two components), this is:

$$E[G] = v^n \, {}_n p_x + (0 \times {}_n q_x)$$

Question 1.10

At a certain company, the probability of each employee leaving during any given year is 5%, independent of the other employees. Those who remain with the company for 25 years are given \$1,000,000. What is the expected present value of this payment to a new starter, assuming an interest rate of 7% *pa* and ignoring the possibility of death?

Actuarial notation for the expected present value

Question 1.11

Without looking ahead, see if you can write down the actuarial notation (*ie* the "big A" symbol) that represents $E(G)$.

In actuarial notation, we define:

$$A_{x:\overline{n}|}^{\;\;1} = E[G] = v^n \, {}_n p_x$$

to be the EPV of a pure endowment benefit of 1, payable after n years to a life aged x.

4.3 Variance of the present value random variable

Following the same lines as before:

$$\text{var}[G] = {}^2A_{x:\overline{n}|}^{\;\;1} - \left(A_{x:\overline{n}|}^{\;\;1}\right)^2$$

where the "2" prefix denotes an EPV calculated at a rate of interest of $(1+i)^2 - 1$.

Question 1.12

Calculate the standard deviation of the present value defined in Question 1.10.

5 Endowment assurance contracts

An endowment assurance is a combination of:

- a term assurance and
- a pure endowment assurance.

That is, a sum assured is payable either on death during the term or on survival to the end of the term. The sums assured payable on death or survival need not be the same, although they often are.

5.1 Present value random variable

Consider an endowment assurance contract to pay a sum assured of 1 to a life now aged x at the end of the year of death, if death occurs during the next n years, or after n years if the life is then alive. We assume that n is an integer.

Let H be the present value of this payment.

The endowment is paid on death or survival, hence its value must be the sum of the values of a benefit paid on death and a benefit paid on survival.

In terms of the present values already defined, $H = F + G$.

Hence:

$$H = \begin{cases} v^{K_x+1} & \text{if } K_x < n \\ v^n & \text{if } K_x \geq n \end{cases}$$

This can also be written as:

$$H = v^{\min(K_x+1,\, n)}$$

5.2 Expected present value

The expected present value of the endowment assurance is:

$$E[H] = \sum_{k=0}^{n-1} v^{k+1} P(K_x = k) + \sum_{k=n}^{\infty} v^n P(K_x = k)$$
$$= \sum_{k=0}^{n-1} v^{k+1} P(K_x = k) + v^n P(K_x \geq n)$$

This can also be written as:

$$\boldsymbol{E[H] = E[F] + E[G]}$$

$$\boldsymbol{= \sum_{k=0}^{n-1} v^{k+1} {}_{k|}q_x + v^n {}_n p_x}$$

$$\boldsymbol{= \sum_{k=0}^{n-2} v^{k+1} {}_{k|}q_x + v^n {}_{n-1} p_x}$$

The last expression holds because payment at time n is certain if the life survives to age $x+n-1$.

Actuarial notation for the expected present value

In actuarial notation we define:

$$\boldsymbol{A_{x:\overline{n}|} = E[H]}$$

$$\boldsymbol{= E[F] + E[G]}$$

$$\boldsymbol{= A^{1}_{x:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}}}$$

The interpretation of the $A_{x:\overline{n}|}$ symbol is to be seen in the above result, *ie* that it is equal to $A^{1}_{x:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}}$.

So, where we have no number above either status, it implies that the payment of 1 unit is made on the *first* of the two statuses to fail, regardless of order.

So $A_{x:\overline{n}|}$ is the EPV of 1 unit, paid after n years, or at the end of the year of death of (x), whichever occurs first.

Question 1.13

Rewrite the last expression in the above Core Reading, assuming that the amount of the death benefit is 3 units and the amount of the survival benefit is 1 unit.

Question 1.14

Allan, aged exactly 40, has just bought a 20-year endowment assurance policy. The sum assured is 1 and is payable on survival to age 60 or at the end of the year of earlier death. Determine the expected present value of the benefit paid to Allan, assuming AM92 Ultimate mortality and 4% *pa* interest.

5.3 Variance of the present value random variable

Note that F and G are not independent random variables (one must be zero and the other non-zero).

This is because the life will either survive or die during the n-year period.

Therefore:

$$\text{var}[H] \neq \text{var}[F] + \text{var}[G]$$

We must find $\text{var}[H]$ from first principles. As before, we find that:

$$\text{var}[H] = {}^2A_{x:\overline{n}|} - \left(A_{x:\overline{n}|}\right)^2$$

where the "2" prefix denotes an EPV calculated at rate of interest $(1+i)^2 - 1$.

Question 1.15

Derive $\text{var}[H]$ from first principles.

6 Deferred assurance benefits

Although not as common as deferred annuities, which you will meet in Chapter 2, **deferred assurance benefits can be defined in a similar way.**

6.1 Present value random variable

A whole life assurance with sum assured 1, payable to a life aged x but deferred n years is a contract to pay a death benefit of 1 provided death occurs after age $x+n$.

If we let J denote the present value of this benefit, then:

$$J = \begin{cases} 0 & \text{if } K_x < n \\ v^{K_x+1} & \text{if } K_x \geq n \end{cases}$$

6.2 Expected present value

If the benefit is payable at the end of the year of death (if at all), the EPV of this assurance is denoted $_{n|}A_x$.

As usual, the subscript of $n\,|$ to the left of the symbol indicates that the event is deferred for n years.

The subscript x to the right of the symbol means that the payment will be made on the failure of the life status (but now we have to "wait" at least n years before anything can be paid).

It is easily shown that

$$_{n|}A_x = A_x - A^{1}_{x:\overline{n|}} = v^n \, _np_x \, A_{x+n}$$

Note the appearance of $v^n \, _np_x$. The factor $v^n \, _np_x$ is important and useful in developing EPVs. It plays the role of the pure interest discount factor v^n, where now the payment or present value being discounted depends on the survival of a life aged x.

Question 1.16

Prove that ${}_{n|}A_x = A_x - A^1_{x:\overline{n|}} = v^n \, {}_n p_x \, A_{x+n}$.

6.3 Variance of the present value random variable

One useful feature of deferred assurances is that it is easier to find their variances directly than is the case for (deferred) annuities. For example, let X be the present value of a whole life assurance and Y the present value of a temporary assurance with term n years, both for a sum assured of 1 payable at the end of the year of death of a life aged x. Then $E[X] = A_x$, $E[Y] = A^1_{x:\overline{n|}}$ and:

$$E[X - Y] = {}_{n|}A_x = A_x - A^1_{x:\overline{n|}}$$

Moreover:

$$\text{var}[X - Y] = \text{var}[X] + \text{var}[Y] - 2\,\text{cov}[X, Y]$$

and it can be shown by considering the distributions of X and Y, that

$$\text{cov}[X, Y] = {}^2A^1_{x:\overline{n|}} - A_x \, A^1_{x:\overline{n|}}$$

where the "2" superscript has its usual meaning.

Question 1.17

Show that $\text{cov}(X, Y) = {}^2A^1_{x:\overline{n|}} - A_x A^1_{x:\overline{n|}}$.

Hence:

$$\mathbf{var}[X-Y] = {}^2A_x - (A_x)^2 + {}^2A^{1}_{x:\overline{n}|} - \left(A^{1}_{x:\overline{n}|}\right)^2 - 2\left({}^2A^{1}_{x:\overline{n}|} - A_x\, A^{1}_{x:\overline{n}|}\right)$$

$$= {}^2A_x - \left(A_x - A^{1}_{x:\overline{n}|}\right)^2 - {}^2A^{1}_{x:\overline{n}|}$$

$$= {}^2A_x - \left({}_{n|}A_x\right)^2 - {}^2A^{1}_{x:\overline{n}|}$$

$$= {}_{n|}^{2}A_x - \left({}_{n|}A_x\right)^2$$

This can also be calculated as:

$$\text{var}[J] = E\left[J^2\right] - \left(E[J]\right)^2$$

Question 1.18

Derive the variance of the deferred assurance in the same way as above.

7 Benefits payable immediately on death

So far we have assumed that assurance (death) benefits have been paid at the end of the year of death. In practice, death benefits are paid a short time after death, as soon as the validity of the claim can be verified.

People's deaths will occur throughout the policy year and claims will be paid shortly afterwards.

Assuming a delay until the end of the year of death is therefore not a prudent approximation, but assuming that there is no delay and that the sum assured is paid immediately on death is a prudent approximation.

This is due to the fact that if we delay payment until the end of the year, the "value of the benefits" part of the equation of value will be smaller. Thus the premium will be lower. Likewise, assuming that claims are paid immediately on death would give a higher (*ie* prudent) premium.

The present values of such death benefits payable immediately on the death of the policyholder can be expressed in terms of the policyholder's complete future lifetime, T_x.

Related to such assurance benefits are annuities under which payment is made in a continuous stream instead of at discrete intervals (which we will look at in Chapter 2). **Of course this does not happen in practice, but such an assumption is reasonable if payments are very frequent, say weekly or daily. Later in the course we will consider annuities with a payment frequency between continuous and annual.**

7.1 Whole life assurance

Consider a whole life assurance with sum assured 1, payable immediately on the death of a life aged x.

Present value random variable

The payments will be made exactly T_x years from now (as T_x is the time from now until the exact moment of death for a life currently aged x). So:

The present value of this benefit is v^{T_x}.

Expected present value

As shown in Subject CT3, for any continuous random variable Y, with density function $F_Y(y)$, the expectation is:

$$E(Y) = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy$$

Also, for any function $g(.)$:

$$E\left(g(Y)\right) = \int_{-\infty}^{\infty} g(y)\, f_Y(y)\, dy$$

Since T_x is a continuous random variable taking any positive value and, from Subject CT4, **the density function of T_x is $_t p_x\, \mu_{x+t}$, the EPV of the benefit, denoted by $\bar{A}_x$ is given by:**

$$\bar{A}_x = E\left[v^{T_x}\right] = \int_0^{\infty} v^t\, {}_t p_x \mu_{x+t} dt$$

and its variance is easily seen to be:

$$\text{var}\left[v^{T_x}\right] = {}^2\bar{A}_x - (\bar{A}_x)^2$$

where the "2" superscript means an EPV calculated at rate of interest $(1+i)^2 - 1$.

As usual, this result is derived from the variance formula $\text{var}[X] = E[X^2] - E[X]^2$.

> ***Question 1.19***
>
> Explain the formula shown above for the expected value by general reasoning.

> ***Question 1.20***
>
> Derive the formula for the variance shown above.

The standard actuarial notation for the EPVs of assurances with a death benefit payable immediately on death, or of annuities payable continuously, is a bar added above the symbol for the EPV of an assurance with a death benefit payable at the end of the year of death, or an immediate annuity with annual payments, respectively. We will see this in Chapter 2.

Term assurance contracts with a death benefit payable immediately on death can be defined in a similar way, with the obvious notation for their EPVs and deferred assurance benefits likewise.

7.2 *Term assurance*

Consider a term assurance with a sum assured of 1 payable immediately upon the death of a life now aged x, provided that this life dies within n years.

Present value random variable

Let $\overline{F}$ denote the present value of this benefit. Then:

$$\overline{F} = \begin{cases} v^{T_x} & \text{if } T_x < n \\ 0 & \text{if } T_x \geq n \end{cases}$$

Expected present value

The expected present value is:

$$E\left(\overline{F}\right) = \int_0^n v^t f_{T_x}(t)\, dt = \int_0^n v^t \, {}_t p_x \, \mu_{x+t} \, dt$$

Actuarial notation for the expected present value

The EPV of the benefit is denoted $\overline{A}^{1}_{x:\overline{n|}}$.

Variance of the present value random variable

Its variance is ${}^2\overline{A}^{1}_{x:\overline{n|}} - (\overline{A}^{1}_{x:\overline{n|}})^2$.

7.3 Endowment assurance

Consider an endowment assurance with a sum assured of 1 payable after n years or immediately upon the earlier death of a life now aged x.

Present value random variable

Let $\bar{H}$ denote the present value of this benefit. Then:

$$\bar{H} = \bar{F} + G = \begin{cases} v^{T_x} & \text{if } T_x < n \\ v^n & \text{if } T_x \geq n \end{cases}$$

Expected present value

The expected present value is:

$$\begin{aligned} E\left(\bar{H}\right) &= \int_0^n v^t f_{T_x}(t)\,dt + \int_n^\infty v^n f_{T_x}(t)\,dt \\ &= \int_0^n v^t\,{}_tp_x\,\mu_{x+t}\,dt + v^n \int_n^\infty f_{T_x}(t)\,dt \\ &= \int_0^n v^t\,{}_tp_x\,\mu_{x+t}\,dt + v^n P\left(T_x > n\right) \\ &= \int_0^n v^t\,{}_tp_x\,\mu_{x+t}\,dt + v^n\,{}_np_x \end{aligned}$$

Actuarial notation for the expected present value

The EPV of the benefit is denoted $\bar{A}_{x:\overline{n}|}$.

Variance of the present value random variable

Its variance is ${}^2\bar{A}_{x:\overline{n}|} - (\bar{A}_{x:\overline{n}|})^2$.

Question 1.21

Why isn't there a $\bar{G}$ random variable?

7.4 Other relationships

We leave the reader to supply the obvious definitions and proofs of the following:

$$\bar{A}_x = \bar{A}^{\,1}_{x:\overline{n|}} + {}_{n|}\bar{A}_x$$

$$\bar{A}_{x:\overline{n|}} = \bar{A}^{\,1}_{x:\overline{n|}} + A_{x:\overline{n|}}^{\;\;\;\;\,1}$$

$${}_{n|}\bar{A}_x = v^n \, {}_n p_x \, \bar{A}_{x+n}$$

Note in the second of these that it is only death benefits that are affected by the changed time of payment. Survival benefits such as a pure endowment are not affected.

Question 1.22

Explain each of the formulae shown above by general reasoning.

7.5 Claim acceleration approximation

It is convenient to be able to estimate $\bar{A}_x$, $\bar{A}_{x:\overline{n|}}$ and so on in terms of commonly tabulated functions. One simple approximation is *claims acceleration*. Of deaths occurring between ages $x+k$ and $x+k+1$, say, ($k = 0, 1, 2, \ldots$) roughly speaking the average age at death will be $x+k+½$. Under this assumption claims are paid on average 6 months before the end of the year of death.

So:

$$\begin{aligned}\bar{A}_x &= v^{½} q_x + v^{1½}\, p_x\, q_{x+1} + v^{2½}\, {}_2p_x\, q_{x+2} + \cdots \\ &= (1+i)^{½}\left(v\,q_x + v^2\, p_x\, q_{x+1} + v^3\, {}_2p_x\, q_{x+2} + \cdots\right) \\ &= (1+i)^{½} A_x\end{aligned}$$

and a similar result holds for term assurances.

Relationships between benefits payable immediately and benefits payable at the end of the year of death

We have the approximate EPVs:

$$\bar{A}_x \cong (1+i)^{1/2} A_x$$

$$\bar{A}^{1}_{x:\overline{n}|} \cong (1+i)^{1/2} A^{1}_{x:\overline{n}|}$$

$$\bar{A}_{x:\overline{n}|} \cong (1+i)^{1/2} A^{1}_{x:\overline{n}|} + A_{x:\overline{n}|}^{\ \ 1}$$

Note again that, in the case of the endowment assurance, only the death benefit is affected by the claims acceleration.

7.6 Further approximation

A second approximation is obtained by considering a whole life or term assurance to be a sum of deferred term assurances, each for a term of one year. Then, taking the whole life case as an example:

$$\bar{A}_x = {}_{0|}\bar{A}^{1}_{x:\overline{1}|} + {}_{1|}\bar{A}^{1}_{x:\overline{1}|} + {}_{2|}\bar{A}^{1}_{x:\overline{1}|} + \ldots$$

$$= \bar{A}^{1}_{x:\overline{1}|} + vp_x \bar{A}^{\ 1}_{x+1:\overline{1}|} + v^2 {}_2p_x \bar{A}^{\ 1}_{x+2:\overline{1}|} + \ldots$$

Now:

$$\bar{A}^{\ 1}_{x+k:\overline{1}|} = \int_0^1 v^t \, {}_tp_{x+k} \mu_{x+k+t} dt$$

We are now going to make the assumption that deaths are uniformly distributed between integer ages, which means that the PDF of the complete future lifetime random variable is constant between integer ages. So for an integer age $x+k$, we have:

$$f_{T_{x+k}}(t) = {}_tp_{x+k}\,\mu_{x+k+t} = \text{constant} \qquad (0 \le t \le 1)$$

But:

$$q_{x+k} = \int_0^1 {}_t p_{x+k}\, \mu_{x+k+t}\, dt = {}_t p_{x+k}\, \mu_{x+k+t} \qquad (0 \le t \le 1)$$

So, **if we assume that deaths are uniformly distributed between integer ages, such that:**

$$\boldsymbol{{}_t p_{x+k}\, \mu_{x+k+t} = q_{x+k} \quad (0 \le t < 1)}$$

then:

$$\boldsymbol{\bar{A}^{\,1}_{x+k:\overline{1|}} \cong q_{x+k} \int_0^1 v^t\, dt = q_{x+k}\, \frac{iv}{\delta}}$$

We will consider the uniform distribution of deaths assumption in more depth in Section 2 of Chapter 3.

Question 1.23

Prove that $\int_0^1 v^t\, dt = \dfrac{iv}{\delta}$.

Hence:

$$\boldsymbol{\bar{A}_x \cong \frac{i}{\delta}\left(vq_x + v^2 p_x\, q_{x+1} + v^3\, {}_2p_x\, q_{x+2} + \ldots\right)}$$

$$\boldsymbol{= \frac{i}{\delta} A_x}$$

Similarly:

$$\boldsymbol{\bar{A}^{\,1}_{x:\overline{n|}} \cong \frac{i}{\delta} A^{1}_{x:\overline{n|}}}$$

You may also come across the approximation $(1+\frac{1}{2}i)$ instead of $(1+i)^{½}$ or $\dfrac{i}{\delta}$.

Question 1.24

Evaluate A_{40} and $\bar{A}_{40}$ based on AM92 Ultimate mortality at 4% *pa* interest.

8 Specimen exam questions

At the end of some of the chapters in this course we have included an exam-style question for you to attempt. The first of these are coming up at the end of this chapter; you'll find that they appear in more and more chapters as you cover an increasing proportion of the course material.

We suggest two different ways that you might wish to use these questions to help you progress through the course:

(1) You could attempt the questions as soon as you reach them in your studies. You may find them quite difficult on the first attempt, and we would expect you to refer back to the notes in order to answer them. However, by tackling them as you go through the course, you will get to know more quickly the level you need to be aiming for in order to pass the Subject CT5 exam. But you should not be worried if your answers appear far from perfect on these first attempts.

(2) Alternatively, you could miss them out until you get to the end of each part. At this point you should be aiming to tackle a good sample of questions from the Question and Answer Bank prior to attempting the relevant assignment. Immediately before the assignment, you could go back and attempt all the exam-style questions from the relevant part, which should help your preparation for tackling the assignment.

Whichever of these you follow, you are likely to benefit from a fresh second attempt at these questions as part of your revision. On these second attempts you should be looking to do the questions under exam conditions, and strictly within the time available – remember that 1 mark = 1.8 minutes of exam time! We suggest that you don't try all the questions in one sitting, but tackle them one at a time once you have fully revised the topics involved.

Of course, you may wish to use the questions in other ways. These are just suggestions.

Question 1.25 ***(Subject 104, April 2000, Question 7)***

Let X be a random variable representing the present value of the benefits of a whole of life assurance, and Y be a random variable representing the present value of the benefits of a temporary assurance with a term of n years. Both assurances have a sum assured of 1 payable at the end of the year of death and were issued to the same life aged x.

(i) Describe the benefits provided by the contract which has a present value represented by the random variable $X - Y$. [1]

(ii) Show that:

$$\text{cov}(X,Y) = {}^2A^{1}_{x:\overline{n}|} - A_x\, A^{1}_{x:\overline{n}|}$$

and hence or otherwise that:

$$\text{var}(X-Y) = {}^2A_x - ({}_{n|}A_x)^2 - {}^2A^{1}_{x:\overline{n}|}$$

where the functions A are determined using an interest rate of i, and functions ${}^2A$ are determined using an interest rate of $i^2 + 2i$. [7]

[Total 8]

Question 1.26 ***(Subject CT5, April 2005, Question 12, amended)***

(i) By considering a term assurance policy as a series of one-year deferred term assurance policies, show that:

$$\bar{A}^{1}_{x:\overline{n}|} = \frac{i}{\delta} A^{1}_{x:\overline{n}|} \qquad [5]$$

(ii) Calculate the expected present value and variance of the present value of a term assurance of 1 payable immediately on death for a life aged 40 exact, if death occurs within 30 years.

Basis: 4% per annum interest, AM92 mortality, no expenses. [6]

[Total 11]

Note: where required, ${}_tp_x$ can be calculated for integers t and x using the formula ${}_tp_x = l_{x+t}/l_x$, where the l_s functions can be found on pages 74 and 75 of the Tables.

Chapter 1 Summary

Types of contracts

Assurance contracts are ones where the insurer makes a payment on death. These might be term assurances, whole life assurances or endowment assurances. The benefits can be payable at the end of the year of death or at the time of death.

Pure endowment contracts make a payment on survival to a given age.

For each type of contract we can write down expressions for:

- the present value of the benefits, which is a random variable
- the expected present value of the benefits
- the variance of the present value of the benefits.

For each of the following benefits we assume that the sum assured is 1 and the policyholder is aged x at the outset.

Whole life assurance with benefit payable at the end of the year of death

Present value: v^{K_x+1}

Expected present value: $E\left[v^{K_x+1}\right] = A_x$

Variance of present value: $\text{var}\left[v^{K_x+1}\right] = {}^2A_x - \left(A_x\right)^2$

Whole life assurance with benefit payable immediately on death

Present value: v^{T_x}

Expected present value: $E\left[v^{T_x}\right] = \bar{A}_x$

Variance of present value: $\text{var}\left[v^{T_x}\right] = {}^2\bar{A}_x - \left(\bar{A}_x\right)^2$

Term assurance (n-year term) with benefit payable at the end of the year of death

Present value: $F = \begin{cases} v^{K_x+1} & \text{if } K_x < n \\ 0 & \text{if } K_x \geq n \end{cases}$

Expected present value: $E[F] = A^{1}_{x:\overline{n}|}$

Variance of present value: $\text{var}[F] = {}^2A^{1}_{x:\overline{n}|} - \left(A^{1}_{x:\overline{n}|}\right)^2$

Term assurance (n-year term) with benefit payable immediately on death

Present value: $\bar{F} = \begin{cases} v^{T_x} & \text{if } T_x < n \\ 0 & \text{if } T_x \geq n \end{cases}$

Expected present value: $E\left[\bar{F}\right] = \bar{A}^{1}_{x:\overline{n}|}$

Variance of present value: $\text{var}\left[\bar{F}\right] = {}^2\bar{A}^{1}_{x:\overline{n}|} - \left(\bar{A}^{1}_{x:\overline{n}|}\right)^2$

Pure endowment (n-year term)

Present value: $G = \begin{cases} 0 & \text{if } K_x < n \\ v^n & \text{if } K_x \geq n \end{cases}$

Expected present value: $E[G] = A_{x:\overset{1}{\overline{n}|}}$

Variance of present value: $\text{var}[G] = {}^2A_{x:\overset{1}{\overline{n}|}} - \left(A_{x:\overset{1}{\overline{n}|}}\right)^2$

Endowment assurance (n-year term) with benefit payable on survival to the maturity date or at the end of the year of earlier death

Present value: $$H = F + G = \begin{cases} v^{K_x+1} & \text{if } K_x < n \\ v^n & \text{if } K_x \geq n \end{cases}$$

Expected present value: $$E[H] = A_{x:\overline{n|}}$$

Variance of present value: $$\text{var}[H] = {}^2A_{x:\overline{n|}} - \left(A_{x:\overline{n|}}\right)^2$$

Endowment assurance (n-year term) with benefit payable on survival to the maturity date or immediately on earlier death

Present value: $$\bar{H} = \bar{F} + G = \begin{cases} v^{T_x} & \text{if } T_x < n \\ v^n & \text{if } T_x \geq n \end{cases}$$

Expected present value: $$E\left[\bar{H}\right] = \bar{A}_{x:\overline{n|}}$$

Variance of present value: $$\text{var}\left[\bar{H}\right] = {}^2\bar{A}_{x:\overline{n|}} - \left(\bar{A}_{x:\overline{n|}}\right)^2$$

Deferred whole life assurance with benefit paid at end of year of death

Present value: $$J = v^{K_x+1} - F = \begin{cases} 0 & \text{if } K_x < n \\ v^{K_x+1} & \text{if } K_x \geq n \end{cases}$$

Expected present value: $$E[J] = {}_{n|}A_x = A_x - A^1_{x:\overline{n|}}$$

Variance of present value: $$\text{var}[J] = {}^2_{n|}A_x - \left({}_{n|}A_x\right)^2$$

The letters F, G, H and J have simply been used for notational convenience. They do not represent standard notation.

This page has been left blank so that you can keep the chapter summaries together for revision purposes.

Chapter 1 Solutions

Solution 1.1

We might argue that higher premiums would be payable if paid in arrears, and that these would cover the expected claims just as well as if premiums were paid in advance.

However, the problem is that policyholders can lapse their contracts (*ie* stop paying their premiums), and might be very tempted to do so at the end of the first year, so receiving one year's insurance protection without paying any premiums at all! Some policyholders will die in the first policy year, and the cost of paying the death benefits could quickly create large losses and potentially insolvency for the insurer in this situation.

Solution 1.2

Because we do not know when the policyholder will die.

Solution 1.3

T_{40} is the exact number of years lived from the current age (40 exact) to the age at death. This is therefore 36 years and 197 days, or:

$$T_{40} = 36 + \frac{197}{365.25} = 36.539 \text{ years}$$

K_{40} is the integer value of this, and therefore equals 36 years.

Solution 1.4

We are taking each possible outcome for the present value of the benefits (which would be the present value *if* death occurred in that particular year), multiplying by the probability of that outcome (*ie* the probability of dying in that particular year), and summing over all possible outcomes for that variable. (As death could occur in *any* future year, then this involves summing to infinity.)

This is the formula for expectation, and so gives us an expression for the expectation of the present value of the benefit (or just "expected present value" for the benefits, for short).

Solution 1.5

$$E\left(v^{K_x+1}\right)=E\left(1^{K_x+1}\right)=\sum_{k=0}^{\infty}1^{k+1}P\left(K_x=k\right)=\sum_{k=0}^{\infty}P\left(K_x=k\right)=P\left(K_x\geq 0\right)=1$$

ie the probability that the life will die eventually. (Remember that at 0%, $v=1$.)

Solution 1.6

$A_{30}=0.07328$, $A_{70}=0.48265$

The figure is much higher for the 70 year-old, since this life is more likely to die in the near future, hence the benefit has a higher present value (because we are discounting for a shorter period).

Solution 1.7

The standard deviation of the present value is:

$$\sqrt{50{,}000^2\left[{}^2A_{30}-\left(A_{30}\right)^2\right]}=50{,}000\sqrt{0.01210-0.07328^2}=£4{,}102$$

Solution 1.8

$A_{\overline{10}|}$ is the present value of 1 unit paid in exactly 10 years time (*ie* the moment at which the 10-year duration status fails).

This is a certain payment, so we don't use the phrase "expected present value" for this.

The amount is therefore simply $A_{\overline{10}|}=v^{10}$.

We don't usually have much call to use the $A_{\overline{n}|}$ notation, as it is just as easy to write v^n every time we need it. However, it is useful when we need to combine duration and life statuses, as we will do in a moment.

A_{10}, on the other hand, is the expected present value of a payment of 1 unit, paid at the end of the year of death of a currently 10-year old person.

Solution 1.9

This is similar to Question 1.4. However the benefit is only paid if the life dies within n years, so the summation is only for n years.

Solution 1.10

The expected present value of the benefit is equal to:

$$1,000,000 \times (0.95)^{25} \times \frac{1}{1.07^{25}} = \$51,109$$

Solution 1.11

This is $A_{x:\overline{n|}}^{\;\;1}$.

This is because:

- the benefit is paid *only* when n years expires, *ie* at the moment at which the $\overline{n|}$ status fails, so the number (whatever it may be) needs to be placed above the $\overline{n|}$
- the benefit will only be paid (at time n) if the person is still alive at that time: this requires the life status x to fail (die) *after* time n, in other words $\overline{n|}$ has to be the first of the two statuses to fail, and hence we need the number to be a "1".

Solution 1.12

The standard deviation is:

$$SD(PV) = \sqrt{1,000,000^2 \times \left(\frac{1}{1.07^2}\right)^{25} {}_{25}p_x - \left[E\left(PV\right)\right]^2}$$

$$= \sqrt{1,000,000^2 \left(\frac{1}{1.07}\right)^{50} \left(0.95\right)^{25} - 51,109^2}$$

$$= \$82,490$$

Solution 1.13

$$E[H] = 3\sum_{k=0}^{n-1} v^{k+1}\,{}_{k|}q_x + v^n\,{}_n p_x = 3A^1_{x:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}}$$

Solution 1.14

The expected present value is:

$$A_{40:\overline{20}|}\ @\ 4\% = 0.46433$$

You will find this value on page 100 of the Tables. You will need to become very familiar with the functions that are tabulated and where they can be found in the Tables, so make sure you can find this one for a start!

Solution 1.15

$$\begin{aligned}\text{var}\left[H\right] &= \text{var}\left[v^{\min\left(K_x+1,\, n\right)}\right]\\ &= E\left[\left(v^{\min\left(K_x+1,\, n\right)}\right)^2\right] - \left(E\left[v^{\min\left(K_x+1,\, n\right)}\right]\right)^2\\ &= E\left[\left(v^2\right)^{\min\left(K_x+1,\, n\right)}\right] - \left(A_{x:\overline{n}|}\right)^2\\ &= {}^2A_{x:\overline{n}|} - \left(A_{x:\overline{n}|}\right)^2\end{aligned}$$

where ${}^2A_{x:\overline{n}|}$ is calculated at rate of interest $(1+i)^2 - 1$.

Solution 1.16

The first result is proved as follows:

$$_{n|}A_x = E[J]$$

where:

$$J = \begin{cases} 0 & \text{if } K_x < n \\ v^{K_x+1} & \text{if } K_x \ge n \end{cases}$$

Now $A_x = E(X)$, say, where:

$$X = \begin{cases} v^{K_x+1} & \text{if } K_x < n \\ v^{K_x+1} & \text{if } K_x \ge n \end{cases}$$

and $A^1_{x:\overline{n|}} = E(Y)$, say, where:

$$Y = \begin{cases} v^{K_x+1} & \text{if } K_x < n \\ 0 & \text{if } K_x \ge n \end{cases}$$

Now:

$$X - Y = \begin{cases} v^{K_x+1} - v^{K_x+1} & = 0 & \text{if } K_x < n \\ v^{K_x+1} - 0 & = v^{K_x+1} & \text{if } K_x \ge n \end{cases}$$
$$= J$$

Therefore:

$$E(J) = E(X - Y) = E(X) - E(Y) = A_x - A^1_{x:\overline{n|}}$$

Furthermore:

$$\begin{aligned} A_x - A^{1}_{x:\overline{n}|} &= \sum_{k=0}^{\infty} v^{k+1} P\left(K_x = k\right) - \sum_{k=0}^{n-1} v^{k+1} P\left(K_x = k\right) \\ &= \sum_{k=n}^{\infty} v^{k+1} P\left(K_x = k\right) \\ &= \sum_{k=n}^{\infty} v^{k+1}\ {}_k p_x\, q_{x+k} \\ &= v^n\ {}_n p_x \sum_{k=n}^{\infty} v^{k+1-n}\ {}_{k-n} p_{x+n}\, q_{x+k} \end{aligned}$$

If we then let $j = k - n$, we can write:

$$\begin{aligned} A_x - A^{1}_{x:\overline{n}|} &= v^n\ {}_n p_x \sum_{j=0}^{\infty} v^{j+1}\ {}_j p_{x+n}\, q_{x+n+j} \\ &= v^n\ {}_n p_x \sum_{j=0}^{\infty} v^{j+1} P\left(K_{x+n} = j\right) \\ &= v^n\ {}_n p_x\, A_{x+n} \end{aligned}$$

Solution 1.17

The covariance of two random variables X and Y is defined by the equation:

$$\text{cov}(X,Y) = E(XY) - E(X)E(Y)$$

(Note that $\text{cov}(X,X) = \text{var}(X)$.)

For a whole life assurance with a sum assured of 1 payable at the end of the year of death, the present value random variable is:

$$X = v^{K+1}$$

Furthermore, for an n-year term assurance, the present value random variable is:

$$Y = \begin{cases} v^{K+1} & \text{if } K < n \\ 0 & \text{otherwise} \end{cases}$$

Multiplying X and Y gives:

$$XY = \begin{cases} v^{2(K+1)} & \text{if } K < n \\ 0 & \text{otherwise} \end{cases}$$

From the above, we see that XY is just the same as Y, except for the fact that v has been replaced by v^2. Hence $E(XY) = {}^2A^{1}_{x:\overline{n}|}$.

The superscript of 2 to the left of the assurance symbol indicates that v has been replaced by v^2, or in other words that we are valuing the benefit using twice the standard force of interest.
By the definition of covariance (from Subject CT3), we now have:

$$\text{cov}(X,Y) = E(XY) - E(X)E(Y) = {}^2A^{1}_{x:\overline{n}|} - A_x A^{1}_{x:\overline{n}|}$$

Solution 1.18

$$\text{var}[J] = E\left[J^2\right] - \left(E[J]\right)^2$$

where:

$$J = \begin{cases} 0 & \text{if } K_x < n \\ v^{K_x+1} & \text{if } K_x \geq n \end{cases}$$

$$E[J] = \sum_{k=n}^{\infty} v^{k+1}\ {}_{k|}q_x = {}_{n|}A_x$$

So:

$$\text{var}[J] = \sum_{k=n}^{\infty} \left(v^2\right)^{k+1}\ {}_{k|}q_x - \left({}_{n|}A_x\right)^2$$

$$= {}_{n|}^{2}A_x - \left({}_{n|}A_x\right)^2$$

Solution 1.19

The expression ${}_tp_x\mu_{x+t}\,dt$ is the probability that the life survives to time t and then dies during the short interval $(t, t+dt)$. The factor v^t gives the present value of the payment and the integral sums this over all future time periods. As before, this sum of all the possible values multiplied by their probabilities will give us the expectation.

Solution 1.20

$$\begin{aligned}\text{var}\left[v^{T_x}\right] &= E\left[\left(v^{T_x}\right)^2\right]-\left(E\left[v^{T_x}\right]\right)^2 \\ &= E\left[\left(v^2\right)^{T_x}\right]-\left(\bar{A}_x\right)^2 \\ &= \int_0^\infty v^{2t}\,{}_tp_x\,\mu_{x+t}\,dt-\left(\bar{A}_x\right)^2 \\ &= {}^2\bar{A}_x-\left(\bar{A}_x\right)^2\end{aligned}$$

where ${}^2\bar{A}_x$ is calculated at rate of interest $(1+i)^2-1$

Solution 1.21

The random variable G is equal to the present value of the pure endowment. As this can never be paid earlier than the maturity date, there is no need for a random variable $\bar{G}$.

Solution 1.22

For the first equation, a whole life benefit is equal to a term assurance for n years (which pays out on death in the first n years) plus a benefit covering the whole of the remainder of the policyholder's life, provided (s)he survives for n years.

For the second equation, an endowment assurance is equal to a term assurance paid immediately on death if it occurs within n years plus a pure endowment benefit paid if the policyholder survives for the n year period.

For the third equation the deferred whole life assurance is paid on death, but only if death happens after n years. Therefore the benefit is equal to a whole life assurance paid to a life n years older, but only if the life aged x survives for n years (and discounted to allow for interest).

Solution 1.23

$$\int_0^1 v^t\, dt = \int_0^1 e^{-\delta t}\, dt = \left[-\frac{e^{-\delta t}}{\delta} \right]_0^1 = \frac{1-e^{-\delta}}{\delta} = \frac{1-v}{\delta} = \frac{d}{\delta} = \frac{iv}{\delta}$$

Solution 1.24

$$A_{40} = 0.23056$$

$$\bar{A}_{40} \approx 1.04^{\frac{1}{2}}\, A_{40} = 0.23513$$

Using the $\dfrac{i}{\delta}$ approximation gives $\bar{A}_{40} \approx 0.23514$.

Solution 1.25

(i) ***Benefit represented by $X - Y$***

$X - Y$ represents a deferred assurance that pays 1 unit at the end of the year of death, provided this occurs after n years.

(ii) ***Covariance and variance***

Mathematically, X and Y are given by:

$$X = v^{K+1} \quad \text{and} \quad Y = \begin{cases} v^{K+1} & \textit{if } K < n \\ 0 & \textit{if } K \geq n \end{cases}$$

So $X-Y$ is given by:

$$X-Y=\begin{cases} 0 & \text{if } K<n \\ v^{K+1} & \text{if } K\geq n \end{cases}$$

The covariance of X and Y is:

$$\text{cov}(X,Y)=E(XY)-E(X)E(Y)$$

The simpler expectations correspond to standard actuarial symbols:

$$E(X)=A_x \quad \text{and} \quad E(Y)=A^1_{x:\overline{n|}}$$

The expectation of the product is:

$$E(XY)=\sum_{k=0}^{\infty} xyP(K=k)$$

Since Y is zero when $K\geq n$, we get:

$$E(XY)=\sum_{k=0}^{n-1} v^{k+1}v^{k+1}{}_{k|}q_x=\sum_{k=0}^{n-1}(v^2)^{k+1}{}_{k|}q_x$$

But this corresponds to the function $A^1_{x:\overline{n|}}$ evaluated at a rate of interest $i'=(i+1)^2-1$. So:

$$E(XY)={}^2A^1_{x:\overline{n|}}$$

Putting these together gives the required result:

$$\text{cov}(X,Y)={}^2A^1_{x:\overline{n|}}-A_xA^1_{x:\overline{n|}}$$

We can then work out the variance of the deferred assurance $X-Y$ using the formula:

$$\text{var}(X-Y)=\text{var}(X)+\text{var}(Y)-2\,\text{cov}(XY)$$

We know that the variances of X and Y are:

$$\text{var}(X) = {}^2A_x - (A_x)^2 \quad \text{and} \quad \text{var}(Y) = {}^2A^1_{x:\overline{n}|} - (A^1_{x:\overline{n}|})^2$$

So we get:

$$\begin{aligned}\text{var}(X-Y) &= {}^2A_x - (A_x)^2 + {}^2A^1_{x:\overline{n}|} - (A^1_{x:\overline{n}|})^2 - 2({}^2A^1_{x:\overline{n}|} - A_x A^1_{x:\overline{n}|}) \\ &= {}^2A_x - (A_x)^2 - {}^2A^1_{x:\overline{n}|} - (A^1_{x:\overline{n}|})^2 + 2A_x A^1_{x:\overline{n}|} \\ &= {}^2A_x - {}^2A^1_{x:\overline{n}|} - [(A_x)^2 - 2A_x A^1_{x:\overline{n}|} + (A^1_{x:\overline{n}|})^2] \\ &= {}^2A_x - {}^2A^1_{x:\overline{n}|} - (A_x - A^1_{x:\overline{n}|})^2 \\ &= {}^2A_x - {}^2A^1_{x:\overline{n}|} - ({}_{n|}A_x)^2\end{aligned}$$

In fact this formula for the variance of a deferred assurance can be simplified slightly to give $\text{var}(X-Y) = {}^2_{n|}A_x - ({}_{n|}A_x)^2$*, which has the same form as the variance formulae for other assurances.*

Solution 1.26

(i) ***Proof***

An n-year term assurance payable immediately on death can be written as a series of one-year term assurance policies as follows:

$$\bar{A}^1_{x:\overline{n}|} = \bar{A}^1_{x:\overline{1}|} + v\,p_x\,\bar{A}^1_{x+1:\overline{1}|} + v^2\,{}_2p_x\,\bar{A}^1_{x+2:\overline{1}|} + \cdots + v^{n-1}\,{}_{n-1}p_x\,\bar{A}^1_{x+n-1:\overline{1}|}$$

Now:

$$\bar{A}^1_{x:\overline{1}|} = \int_0^1 v^t\,{}_tp_x\mu_{x+t}\,dt$$

But assuming that deaths are uniformly distributed over each year of age:

$$f_{T_x}(t) = {}_tp_x\,\mu_{x+t} = \text{constant} \qquad \text{for integer } x \text{ and } 0 \le t \le 1$$

In fact, since:

$$q_x = \int_0^1 {}_t p_x \, \mu_{x+t} \, dt$$

and the integrand is constant, it follows that ${}_t p_x \, \mu_{x+t} = q_x$ for integer x and $0 \le t \le 1$.

So:

$$\bar{A}^{1}_{x:\overline{1|}} = q_x \int_0^1 v^t \, dt = q_x \int_0^1 e^{-\delta t} \, dt = q_x \left[-\frac{1}{\delta} e^{-\delta t} \right]_0^1$$

$$= q_x \left(\frac{1-e^{-\delta}}{\delta} \right) = q_x \left(\frac{1-v}{\delta} \right) = q_x \left(\frac{d}{\delta} \right) = q_x \left(\frac{iv}{\delta} \right)$$

Similarly:

$$\bar{A}^{\;1}_{x+1:\overline{1|}} = q_{x+1} \left(\frac{iv}{\delta} \right)$$

and so on. Substituting back into the first equation gives:

$$\bar{A}^{1}_{x:\overline{n|}} = \frac{i}{\delta} \left(v q_x + v^2 p_x \, q_{x+1} + v^3 {}_2 p_x \, q_{x+2} + \cdots + v^n {}_{n-1} p_x \, q_{x+n-1} \right)$$

$$= \frac{i}{\delta} A^{1}_{x:\overline{n|}}$$

(ii) ***Expected present value and variance of the present value***

The expected present value is:

$$\bar{A}^{1}_{40:\overline{30|}} = \frac{i}{\delta} A^{1}_{40:\overline{30|}} = \frac{i}{\delta} \left(A_{40} - v^{30} {}_{30}p_{40} A_{70} \right) = \frac{i}{\delta} \left(A_{40} - v^{30} \frac{l_{70}}{l_{40}} A_{70} \right)$$

$$= \frac{0.04}{\ln 1.04} \left(0.23056 - 1.04^{-30} \times \frac{8,054.0544}{9,856.2863} \times 0.60097 \right)$$

$$= 0.08072$$

The variance of the present value is:

$$ {}^{2}\bar{A}^{\,1}_{40:\overline{30|}} - \left(\bar{A}^{\,1}_{40:\overline{30|}}\right)^2 $$

The superscript of 2 means we use twice the standard force of interest in the evaluation of the assurance. This is equivalent to using an interest rate:

$$ i' = (1+i)^2 - 1 = 1.04^2 - 1 = 0.0816 $$

Be careful to make this adjustment in the $\frac{i}{\delta}$ *term too. It is now* $\frac{i'}{\delta'} = \frac{0.0816}{2\ln 1.04}$. *Also remember to replace* v *by* v^2.

So the variance of the present value is:

$$ \frac{0.0816}{2\ln 1.04}\left({}^{2}A_{40} - 1.04^{-60}\,{}_{30}p_{40}\,{}^{2}A_{70}\right) - 0.08072^2 $$

$$ = \frac{0.0816}{2\ln 1.04}\left(0.06792 - 1.04^{-60} \times \frac{8,054.0544}{9,856.2863} \times 0.38975\right) - 0.08072^2 $$

$$ = 0.03264 $$

Chapter 2

Life annuity contracts

Syllabus objectives

(i) *Define simple annuity contracts, and develop formulae for the means and variances of the present values of the payments under these contracts, assuming constant deterministic interest.*

1. *Define the following terms:*
   - *whole life level annuity*
   - *temporary level annuity*
   - *guaranteed level annuity*

   including annuity contracts where the benefits are deferred.

3. *Obtain expressions in the form of sums for the mean and variance of the present value of benefit payments under each contract above, in terms of the curtate random future lifetime, assuming that… annuities are paid annually in advance or in arrear, and, where appropriate, simplify these expressions into a form suitable for evaluation by table look-up or other means.*

4. *Obtain expressions in the form of integrals for the mean and variance of the present value of benefit payments under each contract above, in terms of the random future lifetime, assuming that… annuities are paid continuously, and, where appropriate, simplify these expressions into a form suitable for evaluation by table look-up or other means.*

5. *Define the symbols* a_x, $a_{x:\overline{n}|}$, ${}_{m|}a_{x:\overline{n}|}$, $\ddot{a}_x$, $\ddot{a}_{x:\overline{n}|}$, ${}_{m|}\ddot{a}_{x:\overline{n}|}$ *and their… continuous equivalents.*

6. *Understand and use the relations between annual annuities payable in advance and in arrear, and between temporary, deferred and whole life annuities.*

0 Introduction

At the beginning of the previous chapter we stated that simple life assurance contracts were of two main types. We described assurances in Chapter 1. In this chapter we describe the other main type, *ie* annuities.

1 Life annuity contracts

An annuity contract is one where the insurer pays a regular income to the policyholder or another specified person.

A *life annuity contract* provides payments of amounts, which might be level or variable, at stated times, provided a life is still then alive.

Here we consider four varieties of life annuity contract:

(1) Annuities under which payments are made for the whole of life, with level payments, called a *whole life level annuity* or, more usually, an immediate annuity.

(2) Annuities under which level payments are made only during a limited term, called a *temporary level annuity* or, more usually, just a temporary annuity.

(3) Annuities under which the start of payment is deferred for a given term, called a *deferred annuity*.

(4) Annuities under which payments are made for the whole of life, or for a given term if longer, called a *guaranteed annuity*.

So, the income may be:

- level, *eg* £X per annum
- increasing, *eg* starting at £Y, but increasing at 5% per annum
- paid for the whole of a person's life, *eg* until Mrs A dies
- paid for a limited term, *eg* for at most 5 years
- paid for a minimum term, *eg* for at least 5 years
- deferred, *eg* £Z per annum paid from your 60th birthday

or a combination of the above.

Further, we consider the possibilities that payments are made in advance or in arrear. For the moment we only consider contracts under which level payments are made at yearly intervals.

2 *Whole life annuities payable annually in arrears*

An *immediate* annuity is one under which the first payment is made within the first year. For the purposes of this section we will assume that payments are made in arrear.

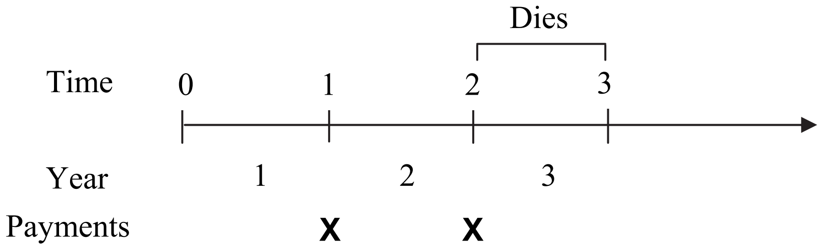
The word *immediate* is used to distinguish these from *deferred* annuities, where the payments start some years in the future (*eg* on reaching age 65).

2.1 *Present value random variable*

Consider an annuity contract to pay 1 at the end of each future year (this includes the first year)**, provided a life now aged x is then alive.** For example, this could be a pension of £1 per annum paid to a life until death. The first payment would be paid one year after the start.

If the life dies between ages $x+k$ and $x+k+1$ ($k=0,\ldots,\omega-x-1$) which is to say, $K_x = k$, the present value at time 0 of the annuity payments which are made is $a_{\overline{k|}}$. (We define $a_{\overline{0|}}=0$.) Therefore the present value at time 0 of the annuity payments is $a_{\overline{K_x|}}$.

For example, suppose the person dies in the third year after taking out the annuity. That means (s)he dies between times 2 and 3:



Because the complete lifetime T_x is two-and-a-bit years, then the curtate lifetime (its integer value K_x), is 2. (You can think of K_x as being the duration of life from age x until the *start* of the year in which (x) dies.)

What is the present value of the annuity in this case? Payments are made at the end of each year but only as long as the person is living (as shown in the diagram). So, exactly 2 payments are made if the life dies in year 3, and so the present value must be $a_{\overline{2|}}$.

That is, when $K_x = 2$, $PV = a_{\overline{2|}}$. Or, in general:

$$PV = a_{\overline{K_x|}}$$

Since we know the distribution of K_x, we can compute moments of $a_{\overline{K_x|}}$.

2.2 *Expected present value*

The expected value of $a_{\overline{K_x|}}$ is:

$$E\left[a_{\overline{K_x|}}\right] = \sum_{k=0}^{\infty} a_{\overline{k|}} P(K_x = k)$$

where we are again using the general formula for expectation:

$$E[g(x)] = \sum_x g(x).P(X = x)$$

Actuarial notation for the expected present value

We'll now explain the annuity notation. The symbol $a_{\overline{n|}}$ is introduced in Subject CT1. It can be interpreted as:

$a_{\overline{n|}} =$ *EPV* of an immediate annuity of 1 unit *pa* paid in arrear *for as long as the status* $\overline{n|}$ *remains active.*

In other words, this implies an annually-in-arrear annuity paid until the failure of the $\overline{n|}$ status, that is, until *n* years have elapsed.

We are now going to use a_x, which has the same meaning except pertaining to the life status *x*. So:

$a_x =$ *EPV* of an immediate annuity of 1 unit *pa* paid in arrear for as long as (*x*) remains alive in the future.

So the payments stop from the moment that (*x*) dies.

So, the expectation of $a_{\overline{K_x}|}$ defines the expected present value (EPV) a_x, and:

$$a_x = E[a_{\overline{K_x}|}] = \sum_{k=0}^{\infty} a_{\overline{k}|}\ {}_{k|}q_x$$

We can write this in a form that is easier to calculate.

We begin by writing:

$$a_x = \sum_{k=0}^{\infty} a_{\overline{k}|}\ {}_{k|}q_x = \sum_{k=1}^{\infty} \left(\sum_{j=0}^{k-1} v^{j+1} \right) {}_{k|}q_x$$

This result holds since $a_{\overline{0}|} = 0$ and $a_{\overline{k}|} = v + v^2 + \cdots + v^k = \sum_{j=0}^{k-1} v^{j+1}$.

If we write out the sum more fully:

$$\begin{aligned} a_x = {} & 0 \times {}_{0|}q_x \\ & + v \times {}_{1|}q_x \\ & + \left(v + v^2 \right) \times {}_{2|}q_x \\ & + \left(v + v^2 + v^3 \right) \times {}_{3|}q_x \\ & + \ldots \end{aligned}$$

Now, reversing the order of summation:

$$\begin{aligned} a_x &= v \left[{}_{1|}q_x + {}_{2|}q_x + \ldots \right] + v^2 \left[{}_{2|}q_x + {}_{3|}q_x + \ldots \right] + \ldots \\ &= v \sum_{k=1}^{\infty} {}_{k|}q_x + v^2 \sum_{k=2}^{\infty} {}_{k|}q_x + \ldots \\ &= \sum_{j=1}^{\infty} \left(\sum_{k=j}^{\infty} {}_{k|}q_x \right) v^j \end{aligned}$$

Or:

$$\boldsymbol{a_x = \sum_{j=0}^{\infty}\left(\sum_{k=j+1}^{\infty} {}_{k|}q_x\right)v^{j+1}}$$

But, for example, $\sum_{k=1}^{\infty} {}_{k|}q_x$ is the probability that the life dies after 1 year (*ie* is still alive after 1 year), and which we can therefore write as ${}_1p_x$. So $\sum_{k=j+1}^{\infty} {}_{k|}q_x$ equal to ${}_{j+1}p_x$.

Hence:

$$\boldsymbol{a_x = \sum_{j=0}^{\infty} {}_{j+1}p_x\, v^{j+1} = \sum_{j=1}^{\infty} {}_{j}p_x\, v^{j}}$$

Although this result is very useful, it is important to realise that it is *not* the *definition* of a_x.

> ***Question 2.1***
>
> Give a general reasoning explanation of the last formula given above.

2.3 *Variance of the present value random variable*

The relationships derived in this chapter provide the easiest approach to finding the variances of the present values of annuity benefits.

Recall that:

$$a_{\overline{n}|} = \frac{1-v^n}{i}$$

Using this result and the properties of the variance, we can write:

$$\text{var}\left[a_{\overline{K_x}|}\right] = \text{var}\left[\frac{1-v^{K_x}}{i}\right] = \frac{1}{i^2}\text{var}\left[v^{K_x}\right]$$

In Chapter 1 we showed that:

$$\text{var}\left[v^{K_x+1}\right] = {}^2A_x - \left(A_x\right)^2$$

where the 2 to the top left hand corner of the first A indicates that the function is valued using the interest rate $i' = (1+i)^2 - 1$.

So:

$$\begin{aligned}\text{var}\left[a_{\overline{K_x}|}\right] &= \frac{1}{i^2}\text{var}\left[\frac{v^{K_x+1}}{v}\right] \\ &= \frac{1}{i^2 v^2}\text{var}\left[v^{K_x+1}\right] \\ &= \frac{1}{d^2}\left[{}^2A_x - \left(A_x\right)^2\right]\end{aligned}$$

3 Whole life annuities payable annually in advance

An *annuity-due* is one under which payments are made in advance.

3.1 Present value random variable

Consider an annuity contract to pay 1 at the start of each future year, provided a life now aged x is then alive. By similar reasoning to that above, we see that the present value of these payments is $\ddot{a}_{\overline{K_x+1|}}$.

The annuity-due provides the following payments:

- one at age x, which is certain to be paid, and
- a further payment at the end of each policy year as long as the policyholder is still alive.

The second of these alone results in the same number of payments as under the whole life annuity paid in arrear, *ie* K_x payments. For the annuity in advance, therefore, there will be one additional payment at the start, so $K_x + 1$ payments altogether.

Actuarial notation for the expected present value

In actuarial notation we denote the EPV, $E\left[\ddot{a}_{\overline{K_x+1|}}\right]$, by $\ddot{a}_x$.

$\ddot{a}_x$ has identical meaning to a_x – in that payments continue for as long as (x) survives – except that the payments are made in advance.

We can again write down $\ddot{a}_x$ in a form that is simple to calculate:

$$\ddot{a}_x = E\left[\ddot{a}_{\overline{K_x+1|}}\right]$$

$$= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} v^j\right){}_{k|}q_x$$

$$= \sum_{j=0}^{\infty}\left(\sum_{k=j}^{\infty} {}_{k|}q_x\right)v^j$$

$$= \sum_{j=0}^{\infty} {}_j p_x v^j$$

Question 2.2

What is $\ddot{a}_x - a_x$ equal to? Explain your answer by general reasoning.

3.2 Variance of the present value random variable

The variance is:

$$\text{var}\left[\ddot{a}_{\overline{K_x+1}|}\right] = \text{var}\left[\frac{1-v^{K_x+1}}{d}\right]$$

$$= \frac{1}{d^2}\text{var}\left[v^{K_x+1}\right]$$

$$= \frac{1}{d^2}\left[{}^2A_x - (A_x)^2\right]$$

where the "2" superscript denotes an assurance function calculated at a rate of interest of $(1+i)^2 - 1$.

It is simple to prove the formula for the variance of an immediate whole life annuity starting from this result.

And, as shown earlier, **for an immediate annuity payable annually in arrears, we obtain:**

$$\text{var}\left[a_{\overline{K_x}|}\right] = \text{var}\left[\ddot{a}_{\overline{K_x+1}|} - 1\right]$$

$$= \text{var}\left[\ddot{a}_{\overline{K_x+1}|}\right]$$

$$= \frac{1}{d^2}\left[{}^2A_x - (A_x)^2\right]$$

4 Temporary annuities payable annually in arrears

A *temporary annuity* differs from a whole life annuity in that the payments are limited to a specified term.

4.1 Present value random variable

Consider a temporary annuity contract to pay 1 at the end of each of the next n years, provided a life now aged x is then alive.

This contract differs from the last one since the payments stop after n years, even if the life is still alive.

If we let X denote the present value of the temporary annuity, then:

$$X = \begin{cases} a_{\overline{K_x}|} & \text{if } K_x < n \\ a_{\overline{n}|} & \text{if } K_x \geq n \end{cases}$$

From this we can see that the number of payments made is K_x or n, whichever is the smaller. So, alternatively we can write:

The present value of this benefit is $a_{\overline{\min[K_x, n]}|}$.

4.2 Expected present value

The expected present value is:

$$E\left[a_{\overline{\min\{K_x,n\}}|}\right] = \sum_{k=0}^{\infty} a_{\overline{\min\{k,n\}}|} P\left(K_x = k\right)$$

$$= \sum_{k=0}^{n-1} a_{\overline{k}|} P\left(K_x = k\right) + a_{\overline{n}|} \sum_{k=n}^{\infty} P\left(K_x = k\right)$$

$$= \sum_{k=0}^{n-1} a_{\overline{k}|} P\left(K_x = k\right) + a_{\overline{n}|} P\left(K_x \geq n\right)$$

$$= \sum_{k=0}^{n-1} a_{\overline{k}|}\, {}_{k|}q_x + a_{\overline{n}|}\, {}_n p_x$$

Looking at the formula above, we see that the benefit paid is:

- 0 if death occurs in the first year
- a payment of 1 (with present value $a_{\overline{1}|} = v$) if death occurs in the second year
- two payments of 1 (with present value $a_{\overline{2}|} = v + v^2$) if death occurs in the third year
- and so on, up until a payment of 1 every year for *n* years (present value $a_{\overline{n}|}$) if the life survives for *n* years, with no further payments.

Actuarial notation for the expected present value

In actuarial notation, $E\left[a_{\overline{\min\{K_x,n\}}|}\right]$ is denoted $a_{x:\overline{n}|}$.

Compare this symbol $a_{x:\overline{n}|}$ with the symbol $A_{x:\overline{n}|}$ we described earlier. So, whereas $A_{x:\overline{n}|}$ relates to a *single* payment made *when* the first out of x or $\overline{n}|$ fails, $a_{x:\overline{n}|}$ relates to a *stream* of payments that continues *until* the first out of x or $\overline{n}|$ to fail.

As for a whole life annuity, we would like to develop a formula to simplify the calculation of the expected present value. Writing the EPV as a summation, we have:

$$\begin{aligned} a_{x:\overline{n}|} &= E\left[a_{\overline{\min\{K_x,n\}}|}\right] \\ &= \sum_{k=1}^{n-1} a_{\overline{k}|}\ {}_{k|}q_x + a_{\overline{n}|}\ {}_n p_x = \sum_{k=1}^{n-1}\left(\sum_{j=0}^{k-1} v^{j+1}\right){}_{k|}q_x + \left(\sum_{j=0}^{n-1} v^{j+1}\right){}_n p_x \end{aligned}$$

Reversing the order of summation then gives:

$$a_{x:\overline{n}|} = \sum_{j=0}^{n-2}\left(\sum_{k=j+1}^{n-1} {}_{k|}q_x\right)v^{j+1} + \left(\sum_{j=0}^{n-1} v^{j+1}\right){}_n p_x$$

Note that ${}_n p_x = {}_{n|}q_x + {}_{n+1|}q_x + \cdots$, so:

$$\sum_{k=j+1}^{n-1} {}_{k|}q_x = {}_{j+1|}q_x + {}_{j+2|}q_x + \cdots + {}_{n-1|}q_x = {}_{j+1}p_x - {}_n p_x$$

Substituting this into the previous equation gives:

$$a_{x:\overline{n}|} = \sum_{j=0}^{n-2} \left({}_{j+1}p_x - {}_n p_x \right) v^{j+1} + \left(\sum_{j=0}^{n-1} v^{j+1} \right) {}_n p_x$$

$$= \sum_{j=0}^{n-2} v^{j+1} \, {}_{j+1}p_x + {}_n p_x \left(\sum_{j=0}^{n-1} v^{j+1} - \sum_{j=0}^{n-2} v^{j+1} \right)$$

$$= \sum_{j=0}^{n-2} v^{j+1} \, {}_{j+1}p_x + v^n \, {}_n p_x$$

$$= \sum_{j=0}^{n-1} v^{j+1} \, {}_{j+1}p_x$$

So, to calculate $a_{x:\overline{n}|}$, we use the following:

$$a_{x:\overline{n}|} = \sum_{j=0}^{n-1} {}_{j+1}p_x v^{j+1} = \sum_{j=1}^{n} {}_j p_x \, v^j$$

This expression seems logical given that $a_x = \sum_{j=1}^{\infty} {}_j p_x v^j$. $a_{x:\overline{n}|}$ is the same summation but allows only for the first n payments. *It is this final result that is most useful when calculating expected present values in practice.*

4.3 Variance of the present value random variable

For a temporary annuity payable annually in arrears we have:

$$\text{var}\left[a_{\overline{\min\{K_x, n\}}|} \right] = \frac{1}{d^2} \left[{}^2A_{x:\overline{n+1}|} - (A_{x:\overline{n+1}|})^2 \right]$$

The easiest way to prove this formula is to use the result for a temporary annuity due. So we will defer the proof until the end of the next section.

5 Temporary annuities payable annually in advance

A *temporary annuity-due* has payments that are made in advance and are limited to a specified term.

5.1 Present value random variable

Consider a temporary annuity-due contract to pay 1 at the start of each of the next n years, provided a life now aged x is then alive.

The present value of the benefit is $\ddot{a}_{\overline{\min[K_x+1,n]}|}$.

Another way to write this is as follows. If we let Y denote the present value of the temporary annuity-due, then:

$$Y = \begin{cases} \ddot{a}_{\overline{K_x+1}|} & \text{if } K_x < n \\ \ddot{a}_{\overline{n}|} & \text{if } K_x \geq n \end{cases}$$

and so we can see that the number of payments is the smaller of $K_x + 1$ and n.

5.2 Expected present value

Question 2.3

Before reading any further, write down an expression for $E\left[\ddot{a}_{\overline{\min[K_x+1,n]}|}\right]$ analogous to that for $E\left[a_{\overline{\min[K_x,n]}|}\right]$ at the bottom of Page 11.

Actuarial notation for the expected present value

In actuarial notation, $E\left[\ddot{a}_{\overline{\min[K_x+1,n]}|}\right]$ is denoted $\ddot{a}_{x:\overline{n}|}$.

Then:

$$\ddot{a}_{x:\overline{n}|} = E\left[\ddot{a}_{\overline{\min[K_x+1,n]}|}\right]$$

$$= \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|}\ {}_{k|}q_x + \ddot{a}_{\overline{n}|}\ {}_n p_x$$

$$= \sum_{k=0}^{n-1} \left(\sum_{j=0}^{k} v^j \right) {}_{k|}q_x + \left(\sum_{j=0}^{n-1} v^j \right) {}_n p_x$$

$$= \sum_{j=0}^{n-1} \left(\sum_{k=j}^{n-1} {}_{k|}q_x + {}_n p_x \right) v^j$$

$$= \sum_{j=0}^{n-1} {}_j p_x v^j$$

Again, this seems logical. It is similar to a whole life annuity-due, but with payments continuing only up to time $n-1$ (making n payments in total).

Question 2.4

Explain why $\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n}|} \neq 1$.

Question 2.5

Write down a single actuarial symbol for the value of $a_{x:\overline{n}|} + 1$.

5.3 Variance of the present value random variable

For a temporary annuity-due:

$$\text{var}\left[\ddot{a}_{\overline{\min[K_x+1,n]}|}\right] = \frac{1}{d^2}\left[{}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2 \right]$$

Question 2.6

Prove this result.

We can now use this to prove the corresponding result for a temporary immediate annuity.

For a temporary immediate annuity we have:

$$\text{var}\left[a_{\overline{\min[K_x,n]}|}\right] = \text{var}\left[\ddot{a}_{\overline{\min[K_x+1,n+1]}|} - 1\right]$$

$$= \text{var}\left[\ddot{a}_{\overline{\min[K_x+1,n+1]}|}\right]$$

$$= \frac{1}{d^2}\left[\,{}^2A_{x:\overline{n+1}|} - (A_{x:\overline{n+1}|})^2\right]$$

It is an excellent test of understanding to see why this last result is correct and why:

$$\text{var}\left[a_{\overline{\min[K_x,n]}|}\right] = \text{var}\left[vp_x\,\ddot{a}_{\overline{\min[K_{x+1}+1,n]}|}\right]$$

$$= \frac{v^2\,p_x{}^2}{d^2}\left[\,{}^2A_{x+1:\overline{n}|} - (A_{x+1:\overline{n}|})^2\right]$$

is *not* correct, although:

$$E\left[a_{\overline{\min[K_x,n]}|}\right] = a_{x:\overline{n}|} = E\left[v\,p_x\,\ddot{a}_{\overline{\min[K_{x+1}+1,n]}|}\right] = v\,p_x\,\ddot{a}_{x+1:\overline{n}|}$$

***is* correct.**

Question 2.7

Explain why the first variance formula given above is correct but the second one isn't, even though $E\left(a_{\overline{\min\{K_x,n\}}|}\right) = E\left(v\,p_x\,\ddot{a}_{\overline{\min\{K_{x+1}+1,n\}}|}\right)$.

6 Deferred annuities

Deferred annuities are annuities under which payment does not begin immediately but is deferred for one or more years. For example, a child in a motor accident has been awarded \$100,000 *pa* from his 18th birthday, payable annually in advance for the rest of his lifetime.

6.1 Present value random variable

Consider, for example, an annuity of 1 per annum payable annually in arrears to a life now aged x, deferred for n years. Payment will be at ages $x+n+1$, $x+n+2$, ... , provided that the life survives to these ages, instead of at ages $x+1$, $x+2$, ...

Let X represent the (random) present value of the annuity.

Here are three different ways of representing X.

$$X = \begin{cases} 0 & \text{if } K_x \le n \\ v^n a_{\overline{K_x-n}|} & \text{if } K_x > n \end{cases}$$

$$= \begin{cases} 0 & \text{if } K_x \le n \\ a_{\overline{K_x}|} - a_{\overline{n}|} & \text{if } K_x > n \end{cases}$$

$$= v^n a_{\overline{\max(K_x-n,\,0)}|}$$

6.2 Expected present value

Then, by considering the distribution of X, and noting that $v^n a_{\overline{k-n}|} = {}_{n|}a_{\overline{k-n}|}$, **we have that:**

$$E(X) = \sum_{k=0}^{n} 0 \times P(K_x = k) + \sum_{k=n+1}^{\infty} {}_{n|}a_{\overline{k-n}|} \times P(K_x = k)$$

Now adding in the terms $\sum_{k=0}^{n} a_{\overline{k}|} P(K_x = k)$ and $a_{\overline{n}|} P(K_x > n)$ and taking them out again, we can write:

$$\boldsymbol{E(X) = \sum_{k=0}^{n} a_{\overline{k}|} P(K_x = k) + a_{\overline{n}|} P(K_x > n) - \sum_{k=0}^{n} a_{\overline{k}|} P(K_x = k)}$$

$$\boldsymbol{-a_{\overline{n}|} P(K_x > n) + \sum_{k=n+1}^{\infty} {}_{n|}a_{\overline{k-n}|} P(K_x = k)}$$

We can combine the first, second and last terms in the line above as follows:

$$\sum_{k=0}^{n} a_{\overline{k}|} P(K_x = k) + a_{\overline{n}|} P(K_x > n) + \sum_{k=n+1}^{\infty} {}_{n|}a_{\overline{k-n}|} P(K_x = k)$$

$$= \sum_{k=0}^{n} a_{\overline{k}|} P(K_x = k) + \sum_{k=n+1}^{\infty} \left(a_{\overline{n}|} + v^n a_{\overline{k-n}|}\right) P(K_x = k)$$

$$= \sum_{k=0}^{n} a_{\overline{k}|} P(K_x = k) + \sum_{k=n+1}^{\infty} a_{\overline{k}|} P(K_x = k)$$

$$= \sum_{k=0}^{\infty} a_{\overline{k}|} P(K_x = k)$$

Hence:

$$\boldsymbol{E(X) = \sum_{k=0}^{\infty} a_{\overline{k}|} P(K_x = k) - \sum_{k=0}^{n} a_{\overline{k}|} P(K_x = k) - a_{\overline{n}|} P(K_x > n)}$$

$$\boldsymbol{= a_x - a_{x:\overline{n}|}}$$

Alternatively, we could consider the random variables. So we have:

$$a_x = E(Y) \qquad \text{where } Y = a_{\overline{K_x}|} = \begin{cases} a_{\overline{K_x}|} & \text{if } K_x \le n \\ a_{\overline{K_x}|} & \text{if } K_x > n \end{cases}$$

$$a_{x:\overline{n}|} = E(Z) \qquad \text{where } Z = \begin{cases} a_{\overline{K_x}|} & \text{if } K_x \le n \\ a_{\overline{n}|} & \text{if } K_x > n \end{cases}$$

It is then easy to see that:

$$Y - Z = \begin{cases} a_{\overline{K_x}|} - a_{\overline{K_x}|} = 0 & \text{if } K_x \le n \\ a_{\overline{K_x}|} - a_{\overline{n}|} & \text{if } K_x > n \end{cases}$$

$$= X$$

Therefore:

$$E(X) = E(Y - Z) = E(Y) - E(Z) = a_x - a_{x:\overline{n}|}$$

This formula $E(X) = a_x - a_{x:\overline{n}|}$ is intuitively correct, the value of the deferred annuity being equal to a stream of payments paid for the whole of your lifetime, less the value of the payments that will not be made for the first n years.

Actuarial notation for the expected present value

In actuarial notation, the EPV of this deferred annuity is denoted ${}_{n|}a_x$, so:

$$\boldsymbol{{}_{n|}a_x = a_x - a_{x:\overline{n}|}}$$

So we read ${}_{n|}a_x$ notation as:

- the expected present value of an annuity of 1 unit *pa* payable in arrear until the failure (death) of life status *x*, with a waiting period of *n* years before payments can begin.

Note that the subscript to the right of the symbol always denotes the *current* age, *not* the age of the policyholder when payments begin.

Similarly, expressions can be derived for ${}_{m|}a_{x:\overline{n}|}$, the present value of an n-year temporary annuity deferred for m years (assuming survival to that point).

Alternative approach to evaluating the expected present value

An alternative way to evaluate $_{n|}a_x$ follows from:

$$
\begin{aligned}
{}_{n|}a_x &= a_x - a_{x:\overline{n|}} \\
&= \sum_{k=1}^{\infty} v^k \, {}_k p_x - \sum_{k=1}^{n} v^k \, {}_k p_x \\
&= \sum_{k=n+1}^{\infty} v^k \, {}_k p_x
\end{aligned}
$$

Pulling out a factor of $v^n \, {}_n p_x$ gives:

$$ {}_{n|}a_x = v^n \, {}_n p_x \sum_{k-n=1}^{\infty} v^{k-n} \, {}_{k-n} p_{x+n} $$

We now have a sum based on successive values of $k-n$. If we rename this variable as "k", we get:

$$ {}_{n|}a_x = v^n \, {}_n p_x \sum_{k=1}^{\infty} v^k \, {}_k p_{x+n} = v^n \, {}_n p_x \, a_{x+n} $$

Again, this final result is intuitively obvious. The EPV of the benefit is equal to the expected present value of the annuity as for a *survivor* to age $x+n$, discounted back for n years to allow for interest, multiplied by the probability that the policyholder survives to age $x+n$.

Note the appearance once more of the "discount factor" $v^n \, {}_n p_x$.

7 *Deferred annuities-due*

Deferred annuities-due can be defined similarly, with the corresponding formulae such as:

$$\boldsymbol{{}_{n|}\ddot{a}_x = \ddot{a}_x - \ddot{a}_{x:\overline{n|}} = v^n\,{}_np_x\,\ddot{a}_{x+n}}$$

Note that $\boldsymbol{a_x = {}_{1|}\ddot{a}_x}$.

In Chapter 1 we showed that the variance of a deferred whole life assurance is:

$${}^2A_x - ({}_{n|}A_x)^2 - {}^2A^{1}_{x:\overline{n|}} = {}^2_{n|}A_x - ({}_{n|}A_x)^2$$

This can be used to find the variance of the corresponding deferred annuity-due.

However, it is much easier to proceed using a first principles approach.

Question 2.8

Write down a single term expression to represent the present value of a deferred annuity-due of 1 *pa* payable to a life now aged x, with a deferment period of n years.

Hence derive a formula for the variance of the present value of this contract (*hard!*).

8 Guaranteed annuities payable annually in advance

A *guaranteed annuity-due* has payments that are made in advance and have a minimum specified term.

Question 2.9

Suggest a reason why guaranteed annuities are commonplace.

8.1 Present value random variable

Consider a guaranteed annuity contract to pay 1 at the start of each future year for the next n years, and at the start of each subsequent future year provided a life now aged x is then alive.

The present value of this benefit is $\ddot{a}_{\overline{\max[K_x+1,n]}|}$.

If (x) dies within n years, then exactly n years payments will be made. If (x) lives for longer than n years, then K_x+1 payments will be made. So alternatively the present value can be written:

$$\begin{cases} \ddot{a}_{\overline{n}|} & \text{if } K_x < n \\ \ddot{a}_{\overline{K_x+1}|} & \text{if } K_x \geq n \end{cases}$$

8.2 Expected present value

In actuarial notation, $E[\ddot{a}_{\overline{\max[K_x+1,n]}|}]$ is denoted $\ddot{a}_{\overline{x:\overline{n}|}}$.

The combined status $\overline{u:v}$ (*ie* with a bar over) means a status that is active while *either or both* of the individual statuses u and v are active (it is known as the *last survivor* status, and we will meet it again when we study multiple lives in a later chapter). When applied to an annuity, $\ddot{a}_{\overline{u:v}}$ implies that payments continue until the last surviving status fails, so in the case of $\ddot{a}_{\overline{x:\overline{n}|}}$ this means paying until the *later* of the death of (x), or the expiry of n years.

To calculate $\ddot{a}_{x:\overline{n}|}$, we use the following:

$$\ddot{a}_{x:\overline{n}|} = E[\ddot{a}_{\overline{\max[K_x+1,n]}|}] = \sum_{k=0}^{n-1} \ddot{a}_{\overline{n}|}\ {}_{k|}q_x + \sum_{k=n}^{\infty} \ddot{a}_{\overline{k+1}|}\ {}_{k|}q_x$$

This is because the present value is $\ddot{a}_{\overline{n}|}$ if (x) dies in any of the first n years (*ie* for $0 \le k < n$), and $\ddot{a}_{\overline{K_x+1}|}$ if (x) dies in any year thereafter (*ie* for $K_x \ge n$).

Using $\ddot{a}_{\overline{t}|} = v^0 + v^1 + \cdots + v^{t-1}$, we obtain:

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1}\left(\sum_{j=0}^{n-1} v^j\right){}_{k|}q_x + \sum_{k=n}^{\infty}\left(\sum_{j=0}^{k} v^j\right){}_{k|}q_x$$

$$= \left(v^0 + v^1 + \cdots + v^{n-1}\right) \times {}_{0|}q_x + \left(v^0 + v^1 + \cdots + v^{n-1}\right) \times {}_{1|}q_x + \cdots + \left(v^0 + v^1 + \cdots + v^{n-1}\right) \times {}_{n-1|}q_x$$

$$+ \left(v^0 + v^1 + \cdots + v^{n-1} + v^n\right) \times {}_{n|}q_x$$

$$+ \left(v^0 + v^1 + \cdots + v^{n-1} + v^n + v^{n+1}\right) \times {}_{n+1|}q_x$$

$$+ \cdots$$

$$= v^0 \sum_{k=0}^{\infty} {}_{k|}q_x + v^1 \sum_{k=0}^{\infty} {}_{k|}q_x + \cdots + v^{n-1} \sum_{k=0}^{\infty} {}_{k|}q_x$$

$$+ v^n \sum_{k=n}^{\infty} {}_{k|}q_x + v^{n+1} \sum_{k=n+1}^{\infty} {}_{k|}q_x + \cdots$$

$$= \sum_{j=0}^{n-1}\left(\sum_{k=0}^{\infty} {}_{k|}q_x\right) v^j + \sum_{j=n}^{\infty}\left(\sum_{k=j}^{\infty} {}_{k|}q_x\right) v^j$$

Now $\sum_{k=0}^{\infty} {}_{k|}q_x = 1$ and $\sum_{k=j}^{\infty} {}_{k|}q_x = {}_j p_x$, and so:

$$\ddot{a}_{x:\overline{n}|} = \sum_{j=0}^{n-1} 1.v^j + \sum_{j=n}^{\infty} {}_j p_x \, v^j$$

$$= \ddot{a}_{\overline{n}|} + \sum_{j=n}^{\infty} {}_j p_x \, v^j$$

$$= \ddot{a}_{\overline{n}|} + {}_{n|}\ddot{a}_x$$

8.3 *Variance of the present value random variable*

The variance of this benefit is:

$$\text{var}[\ddot{a}_{\overline{\max[K_x+1,n]}|}] = \text{var}\left[\frac{1 - v^{\max[K_x+1,n]}}{d}\right]$$

$$= \frac{1}{d^2}\text{var}\left[v^{\max[K_x+1,n]}\right]$$

$$= \frac{1}{d^2}\left(E\left[\left(v^{\max[K_x+1,n]}\right)^2\right] - \left(E\left[v^{\max[K_x+1,n]}\right]\right)^2\right)$$

$$= \frac{1}{d^2}\left(v^{2n} {}_n q_x + {}_{n|}{}^2A_x - \left(v^n {}_n q_x + {}_{n|}A_x\right)^2\right)$$

We obtain the last step as follows. First consider:

$$v^{\max[K_x+1,\, n]} = \begin{cases} v^n & \text{if } K_x < n \\ v^{K_x+1} & \text{if } K_x \geq n \end{cases}$$

So:

$$E\left[v^{\max[K_x+1,\,n]}\right] = v^n \times P(K_x < n) + v^{n+1} \times \Pr(K_x = n)$$

$$+ v^{n+2} \times \Pr(K_x = n+1) + \cdots$$

$$= v^n \, {}_nq_x + \sum_{k=n}^{\infty} v^{k+1} \, {}_{k|}q_x$$

$$= v^n \, {}_nq_x + {}_{n|}A_x$$

Similarly:

$$E\left[\left(v^{\max[K_x+1,\,n]}\right)^2\right] = E\left[\left(v^2\right)^{\max[K_x+1,\,n]}\right] = v^{2n} \, {}_nq_x + {}_{n|}{}^2A_x$$

9 Guaranteed annuities payable annually in arrears

A *guaranteed annuity* differs from a whole life annuity in that the payments have a minimum specified term.

It differs from a guaranteed annuity-due by having payments in arrears.

9.1 Present value random variable

Consider a guaranteed annuity contract to pay 1 at the end of each future year for the next n years, and at the end of each subsequent future year provided a life now aged x is then alive.

The present value of this benefit is $a_{\overline{\max[K_x,n]}|}$

which can alternatively be written:

$$\begin{cases} a_{\overline{n}|} & \text{if } K_x \le n \\ a_{\overline{K_x}|} & \text{if } K_x > n \end{cases} \quad = \quad \begin{cases} a_{\overline{n}|} & \text{if } K_x < n \\ a_{\overline{K_x}|} & \text{if } K_x \ge n \end{cases}$$

9.2 Expected present value

In actuarial notation, $E[a_{\overline{\max[K_x,n]}|}]$ is denoted $a_{\overline{x:\overline{n}|}}$.

Question 2.10

Define what $a_{\overline{x:\overline{n}|}}$ means.

Following similar logic to Section 8.2, **to calculate $a_{\overline{x:\overline{n}|}}$, we use the following:**

$$a_{\overline{x:\overline{n}|}} = E[a_{\overline{\max[K_x,n]}|}] = \sum_{k=0}^{n-1} a_{\overline{n}|}\ {}_{k|}q_x + \sum_{k=n}^{\infty} a_{\overline{k}|}\ {}_{k|}q_x$$

$$= \sum_{k=0}^{n-1}\left(\sum_{j=1}^{n} v^j\right){}_{k|}q_x + \sum_{k=n}^{\infty}\left(\sum_{j=1}^{k} v^j\right){}_{k|}q_x$$

$$= \sum_{j=1}^{n}\left(\sum_{k=0}^{\infty} {}_{k|}q_x\right)v^j + \sum_{j=n+1}^{\infty}\left(\sum_{k=j}^{\infty} {}_{k|}q_x\right)v^j$$

$$= a_{\overline{n}|} + \sum_{j=n+1}^{\infty} {}_j p_x\, v^j$$

$$= a_{\overline{n}|} + {}_{n|}a_x$$

9.3 Variance of the present value random variable

The variance of this benefit is:

$$\text{var}[a_{\overline{\max[K_x,n]}|}] = \text{var}[\ddot{a}_{\overline{\max[K_x+1,n+1]}|} - 1]$$

$$= \text{var}[\ddot{a}_{\overline{\max[K_x+1,n+1]}|}]$$

$$= \frac{1}{d^2}\left(v^{2(n+1)}\ {}_{n+1}q_x + {}_{n+1|}{}^2A_x - \left(v^{n+1}\ {}_{n+1}q_x + {}_{n+1|}A_x\right)^2\right)$$

10 Continuous annuities

Related to assurance benefits that are payable immediately on the death of the policyholder **are annuities under which payment is made in a continuous stream instead of at discrete intervals. Of course this does not happen in practice, but such an assumption is reasonable if payments are very frequent, say weekly or daily.**

10.1 Immediate annuity

Consider an immediate annuity of 1 per annum payable continuously during the lifetime of a life now aged x.

Present value random variable

The present value of this annuity is $\bar{a}_{\overline{T_x}|}$.

Expected present value

Since T_x is a continuous random variable with pdf $f_{T_x}(t) = {}_t p_x \, \mu_{x+t}$, **the EPV, denoted $\bar{a}_x$, is:**

$$\bar{a}_x = E\left[\bar{a}_{\overline{T_x}|}\right] = \int_0^\infty \bar{a}_{\overline{t}|} \, {}_t p_x \, \mu_{x+t} \, dt$$

By general reasoning, this is saying that the expected value is an annuity paid to time t, multiplied by the probability of survival to t, multiplied by the probability of death in the next instant. The integral is "summing" this over all future instants at which death could occur.

We can derive a more useful formula if we **note that $\bar{a}_{\overline{t}|} = \int_0^t e^{-\delta s} ds$, so that $\frac{d}{dt}\bar{a}_{\overline{t}|} = e^{-\delta t} = v^t$, and then integrate by parts.**

The formula for integrating by parts (given on Page 3 of the *Tables*) is:

$$\int_0^\infty u \frac{dw}{dt} dt = \left[uw\right]_0^\infty - \int_0^\infty w \frac{du}{dt} dt$$

In this case, we can set:

$$u = \bar{a}_{\overline{t}|} \text{ and } \frac{dw}{dt} = {}_t p_x \, \mu_{x+t}$$

so that $\dfrac{du}{dt} = v^t$ and $w = -\, {}_t p_x$.

Putting this together gives:

$$\boldsymbol{\bar{a}_x = -\left[\bar{a}_{\overline{t}|} \; {}_t p_x\right]_0^\infty + \int_0^\infty v^t \, {}_t p_x \, dt = \int_0^\infty v^t \, {}_t p_x \, dt}$$

Again this final result follows by general reasoning. The annuity is the value of a payment at each future instant assuming that the life is alive at each future instant.

Another way to prove this result, which is analogous to the proof of the discrete annuity result, is to write $\bar{a}_{\overline{t}|} = \int_0^t v^s \, ds$ and reverse the order of integration. Then:

$$\bar{a}_x = \int_0^\infty \bar{a}_{\overline{t}|} \, f_{T_x}(t) \, dt = \int_0^\infty \left(\int_0^t v^s \, ds \right) f_{T_x}(t) \, dt$$

$$= \int_0^\infty \left(\int_s^\infty f_{T_x}(t) \, dt \right) v^s \, ds = \int_0^\infty v^s P\left(T_x > s\right) ds$$

$$= \int_0^\infty v^s \, {}_s p_x \, ds$$

Variance of the present value random variable

The variance of $\bar{a}_{\overline{T_x}|}$ is:

$$\text{var}\left(\bar{a}_{\overline{T_x}|}\right) = \frac{1}{\delta^2}\left[{}^2\bar{A}_x - \left(\bar{A}_x\right)^2 \right]$$

Question 2.11

Prove this result.

10.2 Other annuities

Temporary, deferred and guaranteed continuous annuities can be defined, and their EPVs calculated, in a similar way. Using the obvious notation, for example:

$$\bar{a}_{x:\overline{n|}} = E\left[\bar{a}_{\overline{\min[T_x,n]|}}\right] = \int_0^n \bar{a}_{\overline{t|}}\ {}_tp_x\mu_{x+t}dt + \bar{a}_{\overline{n|}}\ {}_np_x = \int_0^n v^t\ {}_tp_x dt$$

$$\bar{a}_x = \bar{a}_{x:\overline{n|}} + {}_{n|}\bar{a}_x$$

$$\bar{a}_{\overline{x:\overline{n|}}} = \bar{a}_{\overline{n|}} + {}_{n|}\bar{a}_x$$

Question 2.12

Explain the relationship

$$\bar{a}_x = \bar{a}_{x:\overline{n|}} + {}_{n|}\bar{a}_x$$

by general reasoning.

10.3 Approximations

To evaluate these annuities, use the approximation:

$$\bar{a}_x \cong \ddot{a}_x - ½ \quad \text{or} \quad \bar{a}_x \cong a_x + ½$$

Question 2.13

Can you see any rationale for these approximations?

For temporary annuities:

$$\bar{a}_{x:\overline{n|}} \cong \ddot{a}_{x:\overline{n|}} - ½(1 - v^n\ {}_np_x)$$

Question 2.14

Prove this formula.

Question 2.15

A level annuity of 1 *pa* is to be paid continuously to a 40 year-old male. On the basis of 4% *pa* interest and AM92 Ultimate mortality, calculate the expected present value of this annuity.

Chapter 2 Summary

Annuities

Annuity contracts pay a regular income to the policyholder. The income might be deferred to a future date and could be paid in advance or in arrears.

For each type of contract we can write down expressions for:

- the present value of the benefits, which is a random variable
- the expected present value of the benefits
- the variance of the present value of the benefits.

Whole life immediate annuity

Present value: $a_{\overline{K_x}|}$

Expected present value: $E\left(a_{\overline{K_x}|}\right) = a_x$

Variance of present value: $\text{var}\left(a_{\overline{K_x}|}\right) = \dfrac{1}{d^2}\left[\,{}^2A_x - \left(A_x\right)^2\,\right]$

Whole life annuity-due

Present value: $\ddot{a}_{\overline{K_x+1}|}$

Expected present value: $E\left(\ddot{a}_{\overline{K_x+1}|}\right) = \ddot{a}_x$

Variance of present value: $\text{var}\left(\ddot{a}_{\overline{K_x+1}|}\right) = \dfrac{1}{d^2}\left[\,{}^2A_x - \left(A_x\right)^2\,\right]$

Continuously payable whole life annuity

Present value: $\bar{a}_{\overline{T_x}|}$

Expected present value: $E\left(\bar{a}_{\overline{T_x}|}\right) = \bar{a}_x$

Variance of present value: $\text{var}\left(\bar{a}_{\overline{T_x}|}\right) = \dfrac{1}{\delta^2}\left[\,{}^2\bar{A}_x - \left(\bar{A}_x\right)^2\right]$

Temporary immediate annuity

Present value: $a_{\overline{\min\{K_x,n\}}|}$

Expected present value: $E\left(a_{\overline{\min\{K_x,n\}}|}\right) = a_{x:\overline{n}|}$

Variance of present value: $\text{var}\left(a_{\overline{\min\{K_x,n\}}|}\right) = \dfrac{1}{d^2}\left[\,{}^2A_{x:\overline{n+1}|} - \left(A_{x:\overline{n+1}|}\right)^2\right]$

Temporary annuity-due

Present value: $\ddot{a}_{\overline{\min\{K_x+1,n\}}|}$

Expected present value: $E\left(\ddot{a}_{\overline{\min\{K_x+1,n\}}|}\right) = \ddot{a}_{x:\overline{n}|}$

Variance of present value: $\text{var}\left(\ddot{a}_{\overline{\min\{K_x+1,n\}}|}\right) = \dfrac{1}{d^2}\left[\,{}^2A_{x:\overline{n}|} - \left(A_{x:\overline{n}|}\right)^2\right]$

Deferred annuity

Present value: $X = \begin{cases} 0 & \text{if } K_x < n \\ v^n a_{\overline{K_x-n}|} & \text{if } K_x \geq n \end{cases}$

Expected present value: $E(X) = a_x - a_{x:\overline{n}|}$

Guaranteed annuity-due

Present value: $\ddot{a}_{\overline{\max\{K_x+1,n\}}|}$

Expected present value: $E\left(\ddot{a}_{\overline{\max\{K_x+1,n\}}|}\right) = \ddot{a}_{\overline{x:n}|}$

Variance of present value:

$$\text{var}\left(\ddot{a}_{\overline{\max\{K_x+1,n\}}|}\right) = \frac{1}{d^2}\left[v^{2n}\,{}_nq_x + {}_{n|}{}^2A_x - \left(v^n\,{}_nq_x + {}_{n|}A_x\right)^2\right]$$

Guaranteed annuity

Present value: $a_{\overline{\max\{K_x,n\}}|}$

Expected present value: $E\left(a_{\overline{\max\{K_x,n\}}|}\right) = a_{\overline{x:n}|}$

Variance of present value:

$$\text{var}\left(a_{\overline{\max\{K_x,n\}}|}\right) = \frac{1}{d^2}\left[v^{2(n+1)}\,{}_{n+1}q_x + {}_{n+1|}{}^2A_x - \left(v^{n+1}\,{}_{n+1}q_x + {}_{n+1|}A_x\right)^2\right]$$

This page has been left blank so that you can keep the chapter summaries together for revision purposes.

Chapter 2 Solutions

Solution 2.1

The present value of each payment is conditional on whether the policyholder is alive or not at the time the payment is due. If alive the present value is v^j, if not it is zero. The expected present value of this payment is then just v^j multiplied by the probability of being alive at this point, ${}_jp_x$.

Summing over all future years gives the expected present value of all the future benefit paymemts.

Solution 2.2

$$\ddot{a}_x - a_x = \sum_{j=0}^{\infty} {}_jp_x\, v^j - \sum_{j=1}^{\infty} {}_jp_x\, v^j = {}_0p_x\, v^0 = 1$$

The annuity-due is the same as the immediate annuity, except for the additional payment of 1 unit made now. This payment has present value 1, and will definitely be paid, so its expected present value must also be 1.

Solution 2.3

$$\begin{aligned} E\left[\ddot{a}_{\overline{\min\{K_x+1,n\}}|}\right] &= \sum_{k=0}^{\infty} \ddot{a}_{\overline{\min\{k+1,n\}}|} P(K_x = k) \\ &= \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|} P(K_x = k) + \ddot{a}_{\overline{n}|} \sum_{k=n}^{\infty} P(K_x = k) \\ &= \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|}\; {}_{k|}q_x + \ddot{a}_{\overline{n}|}\; {}_np_x \end{aligned}$$

Solution 2.4

The identity $\ddot{a}_x - a_x = 1$ works only for whole life annuities. If you write out the terms for $a_{x:\overline{n}|}$, the expected present value of an n-year temporary annuity payable to a life aged x, you have:

$$a_{x:\overline{n}|} = v\,p_x + v^2\,{}_2p_x + \cdots + v^n\,{}_np_x$$

If you put a double dot over the annuity symbol, it means that the payments are made at the beginning of each year instead of at the end. The payments are still contingent on the survival of the policyholder, and there is still a maximum of n payments. So:

$$\ddot{a}_{x:\overline{n}|} = 1 + v\,p_x + \cdots + v^{n-1}\,{}_{n-1}p_x$$

Comparing this to $a_{x:\overline{n}|}$, we see that the first and the last terms are different. In fact:

$$\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n}|} = 1 - v^n\,{}_np_x$$

Solution 2.5

$$a_{x:\overline{n}|} + 1 = v\,p_x + v^2\,{}_2p_x + \ldots + v^n\,{}_np_x + 1$$

$$= 1 + v\,p_x + v^2\,{}_2p_x + \ldots + v^n\,{}_np_x$$

$$= \ddot{a}_{x:\overline{n+1}|}$$

In other words $\ddot{a}_{x:\overline{n+1}|} - a_{x:\overline{n}|} = 1$

This also holds for random variables. For example:

$$\ddot{a}_{\overline{\min(K_x+1,\, n+1)}|} - a_{\overline{\min(K_x,\, n)}|} = 1$$

because the only difference between them is the certain payment of 1 unit paid at the start of the annuity-due.

Solution 2.6

$$\text{var}\left[\ddot{a}_{\overline{\min\{K_x+1,n\}}|}\right] = \text{var}\left[\frac{1-v^{\min\{K_x+1,n\}}}{d}\right]$$

$$= \frac{1}{d^2}\text{var}\left[v^{\min\{K_x+1,n\}}\right]$$

$$= \frac{1}{d^2}\left({}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2\right)$$

Solution 2.7

It is true to say that:

$$a_{\overline{\min\{K_x,n\}}|} = \ddot{a}_{\overline{\min\{K_x+1,n+1\}}|} - 1$$

since $a_{\overline{m}|} = \ddot{a}_{\overline{m+1}|} - 1$. Here we are replacing m by $\min\{K_x, n\}$. Taking the variance of both sides of the equation above gives:

$$\text{var}\left(a_{\overline{\min\{K_x,n\}}|}\right) = \text{var}\left(\ddot{a}_{\overline{\min\{K_x+1,n+1\}}|}\right)$$

since subtracting 1 from a random variable does not alter its variance (or spread). Then using the formula proved in Question 2.6 (with n replaced by $n+1$) gives the required result.

We have also proved in Sections 4.2 and 5.2 that:

$$E\left(a_{\overline{\min\{K_x,n\}}|}\right) = v\,p_x + v^2\,{}_2p_x + v^3\,{}_3p_x + \cdots + v^n\,{}_np_x$$

and: $$E\left(\ddot{a}_{\overline{\min\{K_x+1,n\}}|}\right) = 1 + v\,p_x + v^2\,{}_2p_x + \cdots + v^{n-1}\,{}_{n-1}p_x$$

Also, we have defined:

$$a_{x:\overline{n}|} = E\left(a_{\overline{\min\{K_x,n\}}|}\right)$$

and: $$\ddot{a}_{x:\overline{n}|} = E\left(\ddot{a}_{\overline{\min\{K_x+1,n\}}|}\right)$$

So we have:

$$\begin{aligned} a_{x:\overline{n}|} &= v\,p_x + v^2\,{}_2p_x + v^3\,{}_3p_x + \cdots + v^n\,{}_np_x \\ &= v\,p_x\left(1 + v\,p_{x+1} + v^2\,{}_2p_{x+1} + \cdots + v^{n-1}\,{}_{n-1}p_{x+1}\right) \\ &= v\,p_x\,\ddot{a}_{x+1:\overline{n}|} \end{aligned}$$

or equivalently:

$$E\left(a_{\overline{\min\{K_x,n\}}|}\right) = E\left(v\,p_x\,\ddot{a}_{\overline{\min\{K_{x+1}+1,n\}}|}\right)$$

This proves the result for expected values.

However, just because $E(X) = E(Y)$, it does not necessarily follow that $X = Y$. Indeed:

$$a_{\overline{\min\{K_x,n\}}|} \neq v\,p_x\,\ddot{a}_{\overline{\min\{K_{x+1}+1,n\}}|}$$

Instead we have:

$$a_{\overline{\min\{K_x,n\}}|} = \begin{cases} 0 & \text{if } K_x = 0 \\ a_{\overline{1}|} & \text{if } K_x = 1 \\ a_{\overline{2}|} & \text{if } K_x = 2 \\ \vdots & \vdots \\ a_{\overline{n-1}|} & \text{if } K_x = n-1 \\ a_{\overline{n}|} & \text{if } K_x \geq n \end{cases} = \begin{cases} 0 & \text{if } K_x = 0 \\ v\,\ddot{a}_{\overline{1}|} & \text{if } K_x = 1 \\ v\,\ddot{a}_{\overline{2}|} & \text{if } K_x = 2 \\ \vdots & \vdots \\ v\,\ddot{a}_{\overline{n-1}|} & \text{if } K_x = n-1 \\ v\,\ddot{a}_{\overline{n}|} & \text{if } K_x \geq n \end{cases}$$

$$= \begin{cases} 0 & \text{if } K_x = 0 \\ v\,\ddot{a}_{\overline{\min\{K_x,n\}}|} & \text{if } K_x \geq 1 \end{cases}$$

If $K_x \geq 1$, then K_{x+1} is defined and $K_{x+1} = K_x - 1$. So, in this case, $K_x = K_{x+1} + 1$ and:

$$a_{\overline{\min\{K_x,n\}}|} = \begin{cases} 0 & \text{if } K_x = 0 \\ v\,\ddot{a}_{\overline{\min\{K_{x+1}+1,n\}}|} & \text{if } K_x \geq 1 \end{cases}$$

If we incorrectly assumed that $a_{\overline{\min\{K_x,n\}}|}$ was equal to $v\,p_x\,\ddot{a}_{\overline{\min\{K_{x+1}+1,n\}}|}$, we would get the second (incorrect) variance result.

Solution 2.8

The present value random variable can be written as:

$$v^n\,\ddot{a}_{\overline{\max\{K_x+1-n,0\}}|}$$

We can check this as follows.

If $K_x < n$, *the present value is* $v^n\,\ddot{a}_{\overline{0}|} = 0$.

If $K_x = n$, *the present value is* $v^n\,\ddot{a}_{\overline{1}|} = v^n$, *as it should be.*

If $K_x = n+1$, *the present value is* $v^n\,\ddot{a}_{\overline{2}|} = v^n\left(1+v\right) = v^n + v^{n+1}$, *again as it should be.*

The formula for larger values of K_x *can be checked in a similar way.*

Note that it is always necessary to express our random variable using a single term if we wish to derive a variance.

The variance is:

$$\begin{aligned}
\operatorname{var}\left(v^n\,\ddot{a}_{\overline{\max\{K_x+1-n,0\}}|}\right) &= v^{2n}\operatorname{var}\left(\ddot{a}_{\overline{\max\{K_x+1-n,0\}}|}\right)\\
&= v^{2n}\operatorname{var}\left(\frac{1-v^{\max\{K_x+1-n,0\}}}{d}\right)\\
&= \frac{v^{2n}}{d^2}\operatorname{var}\left(v^{\max\{K_x+1-n,0\}}\right)\\
&= \frac{v^{2n}}{d^2}\operatorname{var}\left(\frac{v^{\max\{K_x+1,n\}}}{v^n}\right)\\
&= \frac{1}{d^2}\operatorname{var}\left(v^{\max\{K_x+1,n\}}\right)\\
&= \frac{1}{d^2}\left[E\left(v^{2\max\{K_x+1,n\}}\right)-\left(E\left(v^{\max\{K_x+1,n\}}\right)\right)^2\right]
\end{aligned}$$

Now:

$$v^{\max\{K_x+1,n\}} = \begin{cases} v^n & \text{if } K_x < n \\ v^{K_x+1} & \text{if } K_x \ge n \end{cases}$$

In other words, 1 is paid at time n if the life dies before time n and 1 is paid at the end of the year of death if the life dies after time n.

So:

$$E\left(v^{\max\{K_x+1,n\}}\right) = v^n \; {}_nq_x + v^n \; {}_np_x \; A_{x+n}$$

and:

$$E\left(v^{2\max\{K_x+1,n\}}\right) = v^{2n} \; {}_nq_x + v^{2n} \; {}_np_x \; {}^2A_{x+n}$$

Hence, the variance of the present value of the deferred annuity is:

$$\begin{aligned} &\operatorname{var}\left(v^n \, \ddot{a}_{\overline{\max\{K_x+1-n,0\}}|}\right) \\ &= \frac{1}{d^2}\left[v^{2n} \; {}_nq_x + v^{2n} \; {}_np_x \; {}^2A_{x+n} - \left(v^n \; {}_nq_x + v^n \; {}_np_x \; A_{x+n}\right)^2\right] \\ &= \frac{v^{2n}}{d^2}\left[{}_nq_x + {}_np_x \; {}^2A_{x+n} - \left({}_nq_x + {}_np_x \; A_{x+n}\right)^2\right] \end{aligned}$$

Solution 2.9

When people buy annuities they are often investing large amounts of their life savings. Should they die soon after purchasing the annuity, and the annuity is not guaranteed, they would effectively lose nearly all of their life savings. Relatives of the deceased are likely to find this distressing, in addition to the emotional distress they would be experiencing at the time. The insurer could also experience bad publicity and suffer reputational damage as a result.

Issuing annuities with guaranteed payment periods (typically of five or ten years) reduces the financial loss on early death and so goes a long way in mitigating these problems.

Solution 2.10

It is the EPV of an annuity of 1 *pa* paid annually in arrears for n years, or until the death of a life who is currently aged x, if this occurs after n years.

Solution 2.11

Using the formula for continuous annuities-certain:

$$\text{var}\left(\bar{a}_{\overline{T_x}|}\right) = \text{var}\left(\frac{1-v^{T_x}}{\delta}\right)$$

and using the properties of variance:

$$\text{var}\left(\bar{a}_{\overline{T_x}|}\right) = \frac{1}{\delta^2}\text{var}\left(v^{T_x}\right)$$

In Chapter 1 we saw that:

$$\text{var}\left(v^{T_x}\right) = {}^2\bar{A}_x - \left(\bar{A}_x\right)^2$$

Hence:

$$\text{var}\left(\bar{a}_{\overline{T_x}|}\right) = \frac{1}{\delta^2}\left[{}^2\bar{A}_x - \left(\bar{A}_x\right)^2\right]$$

Solution 2.12

An annuity paid for life is equal to one paid up to death or n years if earlier, plus another paid for life if you survive the n-year period.

Solution 2.13

$\ddot{a}_x$ and a_x represent two extremes, in which the payments are made at the beginning and at the end of each year, respectively. With $\bar{a}_x$, the payments are spread uniformly over the year, so we might expect its value to lie roughly midway between $\ddot{a}_x$ and a_x (and since $\ddot{a}_x$ differs from a_x by 1, $\bar{a}_x$ differs from them both by ½).

Solution 2.14

We can prove this using the result $\bar{a}_{x:\overline{n}|} = \bar{a}_x - {}_{n|}\bar{a}_x = \bar{a}_x - v^n \, {}_n p_x \, \bar{a}_{x+n}$ from Section 10.2. Since the RHS involves whole life annuities, we can apply the approximations we have just discussed. These give:

$$\begin{aligned}
\bar{a}_{x:\overline{n}|} &= \bar{a}_x - v^n \, {}_n p_x \, \bar{a}_{x+n} \\
&\approx \ddot{a}_x - \tfrac{1}{2} - v^n \, {}_n p_x \, (\ddot{a}_{x+n} - \tfrac{1}{2}) \\
&= \ddot{a}_x - v^n \, {}_n p_x \, \ddot{a}_{x+n} - \tfrac{1}{2}(1 - v^n \, {}_n p_x)
\end{aligned}$$

But $\ddot{a}_x - v^n \, {}_n p_x \, \ddot{a}_{x+n} = \ddot{a}_{x:\overline{n}|}$. So we get:

$$\bar{a}_{x:\overline{n}|} \approx \ddot{a}_{x:\overline{n}|} - \tfrac{1}{2}(1 - v^n \, {}_n p_x)$$

Solution 2.15

The expected present value of the annuity is:

$$\bar{a}_{40} \cong \ddot{a}_{40} - \tfrac{1}{2} = 20.005 - \tfrac{1}{2} = 19.505$$

Chapter 3

The life table

Syllabus objectives

(i) Define simple assurance and annuity contracts, and develop formulae for the means and variances of the present values of the payments under these contracts, assuming constant deterministic interest.

2. Define the following probabilities: ${}_{n|m}q_x$, ${}_{n|}q_x$ and their select equivalents ${}_{n|m}q_{[x]+r}$, ${}_{n|}q_{[x]+r}$.

5. Define the symbols A_x, $A_{x:\overline{n|}}$, $A^{1}_{x:\overline{n|}}$, $A_{x:\overline{n|}}^{\ \ \ 1}$, a_x, $a_{x:\overline{n|}}$, ${}_{m|}a_{x:\overline{n|}}$, $\ddot{a}_x$, $\ddot{a}_{x:\overline{n|}}$, ${}_{m|}\ddot{a}_{x:\overline{n|}}$ and their select and continuous equivalents. [Note: Select equivalents only are covered in this chapter; the remainder was covered in Chapters 1 and 2.]

(ii) Describe and use practical methods of evaluating expected values and variances of the simple contracts defined in objective (i).

1. Describe the life table functions l_x and d_x and their select equivalents $l_{[x]+r}$ and $d_{[x]+r}$.

2. Express the following life table probabilities in terms of the functions in (ii) 1: ${}_np_x$, ${}_nq_x$, ${}_{n|m}q_x$ and their select equivalents ${}_np_{[x]+r}$, ${}_nq_{[x]+r}$, ${}_{n|m}q_{[x]+r}$.

3. Express the expected values and variances in objective (i) 3 in terms of the functions in (ii) 1 and 2.

0 Introduction

In Chapters 1 and 2, expressions were established for the expected values and variances of benefits under simple contracts.

The evaluation of these expressions depends upon:

- **the assumed value of the constant deterministic interest rate; and**
- **the assumed probability distribution of the unknown future lifetime (or similar contingency).**

The interest rate can be specified as a particular number. This chapter will describe practical ways of assigning numerical probabilities of death.

We will first look at probabilities continuing to use our ultimate mortality assumption from Chapters 1 and 2. Then we will see how the same formulae of Chapters 1 and 2 may be applied with the probabilities replaced with the equivalents that stem from the select mortality assumption mentioned in Section 1.3 of Chapter 1.

The most practical means of evaluation is to assume that mortality follows a probability distribution defined by tracing the number of lives alive at each age in a population, and summarising the results in a "life table".

1 The life table

1.1 Introduction

We first introduce the life table making the ultimate mortality assumption.

In actuarial applications, we carry out many calculations using the probabilities ${}_tp_x$ and ${}_tq_x$, so it is useful to have some way of tabulating these functions, say for integer values of t and x. However, this would result in very large tables. The life table is a device for calculating all such probabilities from a smaller table, one whose entries depend on age only. The key to the definition of a life table is the relationship:

$$ {}_{t+s}p_x = {}_tp_x \times {}_sp_{x+t} = {}_sp_x \times {}_tp_{x+s} $$

1.2 Constructing a life table

Choose a starting age, which will be the lowest age in the table. We denote this lowest age α. This choice of α will often depend on the data that are available. For example, in studies of pensioners' mortality it is unusual to observe anyone younger than (say) 50, so 50 might be a suitable choice for α in a life table that is to represent pensioners' mortality.

Choose an arbitrary positive number and denote it l_α. We call l_α the *radix* of the life table. It is convenient to interpret l_α as being the number of lives starting out at age α in a homogeneous population, but the mathematics does not depend on this interpretation.

As we will see, the choice of the radix (l_α) is not important. When setting the radix, it is common to choose a large round number such as 100,000. This is purely for presentational purposes. In the definition below, ω denotes the maximum or *limiting* age of the population.

Definition

For $\alpha \le x \le \omega$, define the function l_x by

$$ l_x = l_\alpha \times {}_{x-\alpha}p_\alpha $$

We assume that the probabilities ${}_{x-\alpha}p_\alpha$ are all known. Obviously, $l_\omega = 0$.

Now we see that, for $\alpha \le x \le \omega$ and for $t \ge 0$:

$$_t p_x = \frac{_{t+x-\alpha} p_\alpha}{_{x-\alpha} p_\alpha} = \frac{l_{x+t}}{l_\alpha} \times \frac{l_\alpha}{l_x} = \frac{l_{x+t}}{l_x}$$

Hence, if we know the function l_x for $\alpha \le x \le \omega$, we can find any probability $_t p_x$ or $_t q_x$.

The choice of radix is unimportant since the l_α terms cancel. The formula for $_t p_x$ involves a ratio of two entries from the life table, so the answer would be the same if the radix (and hence all the other figures in the life table) were multiplied by 2, say.

Question 3.1

Write down an expression for $_t q_x$ in terms of the function l_x.

The function l_x is called the *life table*. It depends on age only, so it is more easily tabulated than the probabilities that we need in calculations, although the importance of this has diminished with the widespread use of computers.

The life table is an important tool that enables actuaries to calculate a wide range of useful figures from a single set of tabulated factors. A clear understanding of the underlying principles is still important today since life tables are at the heart of much actuarial valuation software.

1.3 The force of mortality

In Subject CT4 (or Subject 104), the force of mortality was defined by the equation:

$$\mu_x = \lim_{h \to 0+} \frac{1}{h} \times P[T \le x+h \,|\, T > x]$$

It is useful to see how this definition can be expressed in terms of life table functions. The definition above can be expressed as:

$$\mu_x = \lim_{h \to 0+} \frac{1}{h} \times {_h q_x}$$

In terms of life table functions, this is:

$$\mu_x = \lim_{h \to 0+} \frac{1}{h} \times \frac{l_x - l_{x+h}}{l_x} = -\frac{1}{l_x} \times \lim_{h \to 0+} \frac{l_{x+h} - l_x}{h}$$

The limit in this last expression matches the definition of a derivative. So we get:

$$\mu_x = -\frac{1}{l_x} \times \frac{d}{dx} l_x$$

which can also be written as:

$$\mu_x = -\frac{d}{dx} \ln l_x$$

1.4 *Interpretation*

If we interpret l_α to be the number of lives known to be alive at age α (in which case l_α has to be an integer) then we can interpret l_x $(x > \alpha)$ as the *expected* number of those lives who survive to age x.

After all, l_x $(x > \alpha)$ is defined as l_α multiplied by ${}_{x-\alpha}p_\alpha$, the probability of a life aged α surviving to age x.

A life table is sometimes given a deterministic interpretation. That is, l_α is interpreted as above, and l_x $(x > \alpha)$ is interpreted as the number of lives who will survive to age x, as if this were a fixed quantity.

Then the symbol ${}_tp_x = l_{x+t}/l_x$ is taken to be the proportion of the l_x lives alive at age x who survive to age $x+t$. This is the so-called "deterministic model of mortality". It is not such a fruitful approach as the stochastic model that we have outlined, and we will not use it. In particular, while it is useful in computing quantities like premium rates, it is of no use when we must analyse mortality data.

This deterministic model of mortality is too inflexible for our needs. After all, lives rarely survive to later ages in the exact proportions dictated by the life table. Our stochastic approach allows us to model the inherent variability.

Another point to note is that the probabilities, ${}_t p_x$ *etc*, that are used to construct a life table can only be *estimates* of the true underlying probabilities. So the life table does not *define* the survival probabilities and mortality rates – it is merely one representation of them.

1.5 Using the life table

We now introduce a further life table function d_x.

> ***Definition***
>
> **For $\alpha \le x \le \omega - 1$, define:**
>
> $$d_x = l_x - l_{x+1}$$
>
> **We interpret d_x as the expected number of lives who die between age x and age $x+1$, out of the l_α lives alive at age α.**

Note that:

$$q_x = 1 - p_x = \frac{l_x}{l_x} - \frac{l_{x+1}}{l_x} = \frac{d_x}{l_x}$$

It is trivial to see that:

$$d_x + d_{x+1} + \cdots + d_{x+n-1} = l_x - l_{x+n}$$

and that (if x and ω are integers):

$$d_x + d_{x+1} + \cdots + d_{\omega-1} = l_x$$

> ***Question 3.2***
>
> Express in words the two results:
>
> (i) $d_x + d_{x+1} + \cdots + d_{x+n-1} = l_x - l_{x+n}$
>
> (ii) $d_x + d_{x+1} + \cdots + d_{\omega-1} = l_x$

It is usual to tabulate values of l_x and d_x at integer ages, and often other functions such as μ_x, p_x or q_x as well. For an example, see the English Life Table No. 15 (Males) in the "Formulae and Tables for Examinations".

The following is an extract from that table.

Age	l_x	d_x	q_x	μ_x
0	**100,000**	**814**	**0.00814**	
1	**99,186**	**62**	**0.00062**	**0.00080**
2	**99,124**	**38**	**0.00038**	**0.00043**
3	**99,086**	**30**	**0.00030**	**0.00033**
4	**99,056**	**24**	**0.00024**	**0.00027**
5	**99,032**	**22**	**0.00022**	**0.00023**

(No value is given for μ_0 because of the difficulty of calculating a reasonable estimate from observed data.) It is easy to check the relationships:

$$ {}_tp_x = \frac{l_{x+t}}{l_x} \qquad d_x = l_x - l_{x+1} \qquad q_x = \frac{d_x}{l_x} $$

Question 3.3

Using the extract from ELT15 (Males) given above, calculate the values of:

(i) p_2

(ii) ${}_2p_3$

(iii) ${}_4q_1$

(iv) l_6

1.6 *The pattern of human mortality*

Notice that $\mu_x > q_x$ at all ages in this part of the table. At some higher ages it is found that $\mu_x < q_x$. In fact, since:

$$q_x = \int_0^1 {}_tp_x\,\mu_{x+t}\,dt$$

we see that if ${}_tp_x\mu_{x+t}$ is increasing for $0 \le t \le 1$:

$$q_x = \int_0^1 {}_tp_x\,\mu_{x+t}\,dt > {}_0p_x\mu_{x+0} = \mu_x$$

while if ${}_tp_x\mu_{x+t}$ is decreasing for $0 \le t \le 1$:

$$q_x = \int_0^1 {}_tp_x\,\mu_{x+t}\,dt < {}_0p_x\mu_{x+0} = \mu_x$$

It is therefore of interest to note the behaviour of the function ${}_tp_x\mu_{x+t}$ for $0 \le t < \omega - x$ (recall that this function is the probability density function of T_x).

Figure 1 shows ${}_tp_0\mu_t$ (*ie* the density of $T = T_0$) for the English Life Table No. 15 (Males).

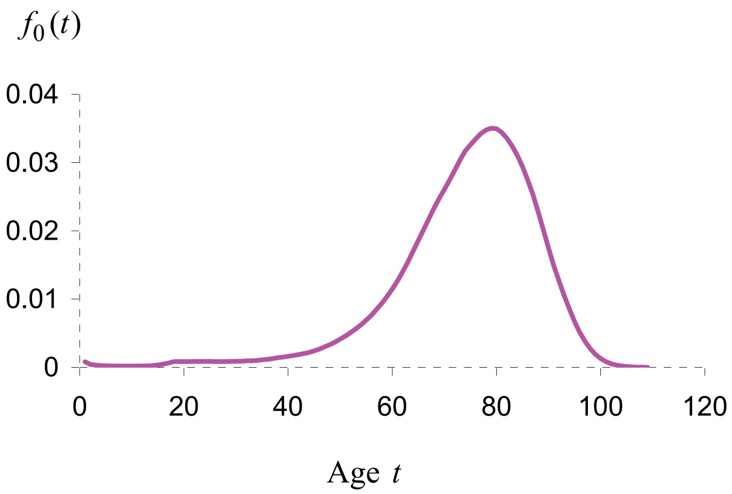


Figure 1

$f_0(t) = {}_tp_0\mu_t$ (ELT15 (Males) Mortality Table)

The graph has the following features, which are typical of life tables based on human mortality in modern times:

(1) Mortality just after birth ("infant mortality") is very high.

(2) Mortality falls during the first few years of life.

(3) There is a distinct "hump" in the function at ages around 18–25. This is often attributed to a rise in accidental deaths during young adulthood, and is called the "accident hump".

(4) From middle age onwards there is a steep increase in mortality, reaching a peak at about age 80.

(5) The probability of death at higher ages falls again (even though q_x continues to increase) since the probabilities of surviving to these ages are small.

We shall return to the pattern of mortality again in the last chapter of this course, when we will look at graphs of other quantities of interest including l_x and q_x.

1.7 More notation

We will now introduce some more actuarial notation for probabilities of death, and give formulae for them in terms of the life table l_x.

Definition

$$ {}_{n|m}q_x = P[\, n < T_x \le n+m \,] $$

In words, ${}_{n|m}q_x$ is the probability that a life age x will survive for n years but die during the subsequent m years.

It is easy to see that:

$$ {}_{n|m}q_x = \frac{l_{x+n} - l_{x+n+m}}{l_x} $$

or alternatively that:

$$ {}_{n|m}q_x = {}_np_x \times {}_mq_{x+n} $$

Question 3.4

Show that these two results are equivalent.

Question 3.5

Using the ELT15 (Males) life table in the *Tables*, calculate the probability of a 37-year old dying between ages 65 and 75.

An important special case for actuarial calculations is $m = 1$, since we often use probabilities of death over one year of age. By convention, we drop the "*m*" and write:

$$_{n|1}q_x = {}_{n|}q_x$$

In words, $_{n|}q_x$ is the probability that a life aged x will survive for n years but die during the subsequent year, *ie* die between ages $x+n$ and $x+n+1$.

It is easy to see that:

$$_{n|}q_x = {}_n p_x \times q_{x+n} = \frac{l_{x+n} - l_{x+n+1}}{l_x} = \frac{d_{x+n}}{l_x}$$

Recall the definition of the curtate future lifetime, K_x.

K_x is the integer part of the complete future lifetime random variable, T_x.

We now see that the probability function of K_x can be written:

$$P[K_x = k] = {}_{k|}q_x$$

Question 3.6

Using the ELT15 (Males) life table in the *Tables*, calculate:

(i) $P(K_{30} = 40)$

(ii) $P(T_{30} > 40)$

2 Life table functions at non-integer ages

2.1 Introduction

Life table functions such as l_x, p_x or μ_x are usually tabulated at integer ages only, but sometimes we need to compute probabilities involving non-integer ages or durations, such as ${}_{2.5}p_{37.5}$. We can do so using approximate methods. We will show two methods.

In both cases, we suppose that we split up the required probability so that we need only approximate over single years of age. For example, we would write

$${}_3p_{55.5} \text{ as } {}_{0.5}p_{55.5} \times {}_2p_{56} \times {}_{0.5}p_{58}$$

The middle factor can be found from the life table. To approximate the other two factors we need only consider single years of age.

> ***Question 3.7***
>
> Why do you think we split up the required probability so that we need to approximate only over single years of age?

2.2 Method 1 – uniform distribution of deaths (UDD)

> ***Assumption***
>
> **The first method is based on the assumption that, for integer x and $0 \le t \le 1$, the function ${}_tp_x\mu_{x+t}$ is a constant.**

Since this is the density (PDF) **of the time to death from age x, it is seen that this assumption is equivalent to a uniform distribution of the time to death, conditional on death falling between these two ages. Hence it is called the *Uniform Distribution of Deaths* (or UDD) assumption.**

In other words, for an individual aged exactly x, the probability of dying on one particular day over the next year is the same as that of dying on any other day over the next year.

Question 3.8

What does the UDD assumption implicitly assume about μ_x over the year of age?

In order to calculate that probability, we need to determine the constant ${}_t p_x \mu_{x+t}$.

Since ${}_s q_x = \int_0^s {}_t p_x \, \mu_{x+t} \, dt$, by putting $s = 1$ we must have:

$$ {}_t p_x . \mu_{x+t} = q_x \qquad (0 \le t \le 1) $$

Remember that we are assuming that ${}_t p_x \mu_{x+t}$ is constant over the year.

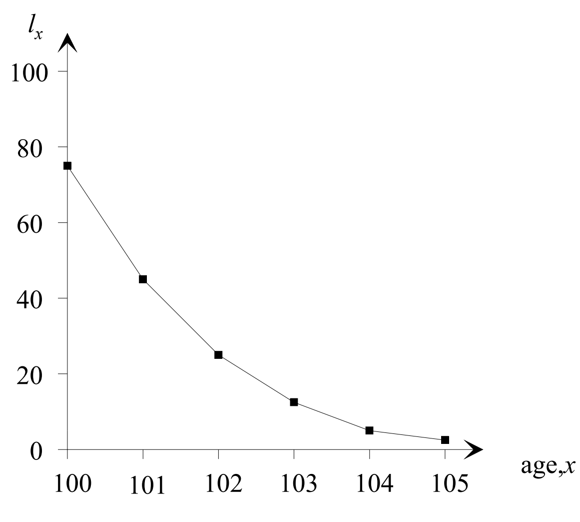
Therefore:

$$ {}_s q_x = \int_0^s q_x \, dt = s . q_x $$

This is sometimes taken as the definition of the UDD assumption.

Since q_x can be found from the life table, we can use this to approximate any ${}_s q_x$ or ${}_s p_x$ $(0 \le s \le 1)$.

Under the UDD assumption l_x is made up of straight-line segments between ages, as shown in the following graph of l_x between ages 100 and 105:



Note that we must have an integer age x in the above formula, so (in our example) we can now estimate ${}_{0.5}p_{58}$ but not ${}_{0.5}p_{55.5}$. (Not using the ${}_sq_x = s.q_x$ formula, at any rate.)

It is easy to show that for $0 \le s < t \le 1$:

$${}_{t-s}q_{x+s} = \frac{(t-s)q_x}{1-s\,q_x} \qquad \text{[Hint: } {}_tp_x = {}_sp_x \times {}_{t-s}p_{x+s}\text{]}$$

This result, with s = 0.5, t = 1, can be used to estimate probabilities of the form ${}_{0.5}p_{55.5}$.

Question 3.9

Using the hint that ${}_tp_x = {}_sp_x \times {}_{t-s}p_{x+s}$, prove that ${}_{t-s}q_{x+s} = \dfrac{(t-s)q_x}{1-s\,q_x}$.

Question 3.10

Show that, under the UDD assumption over each year of age:

(i) $l_{x+t} = l_x - t\,d_x$

(ii) $l_{x+t} = (1-t)\,l_x + t\,l_{x+1}$

for $x = 0, 1, 2, ..., \omega - 1$ and for $0 \le t < 1$.

2.3 Method 2 – constant force of mortality (CFM)

Assumption

The second method of approximation is based on the assumption of a *constant force of mortality*. That is, for integer x and $0 \le t \le 1$, we suppose that:

$$\mu_{x+t} = \mu = \text{constant}$$

Then using the formula:

$${}_t p_x = \exp\left\{-\int_0^t \mu_{x+s}\, ds\right\} = e^{-t\mu}$$

we can find the required probabilities.

We first have to find μ.

Note that $p_x = e^{-\mu}$ so $\mu = -\log p_x$, which we can find from the life table.

Next, note that matters are rather simpler than under the UDD assumption since for $0 \le s < t < 1$ we have:

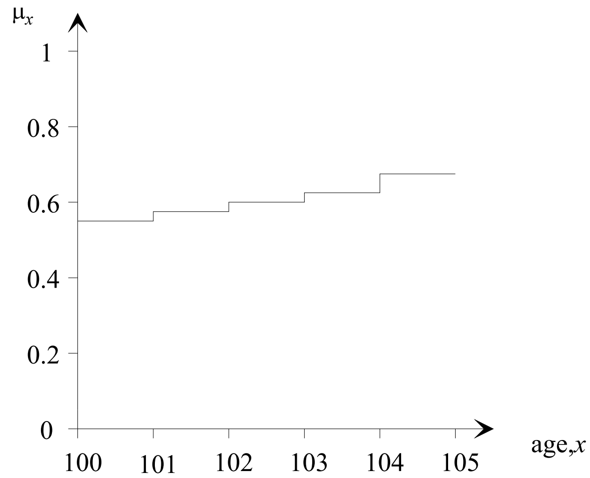
$${}_{t-s} p_{x+s} = \exp\left\{-\int_s^t \mu_{x+r}\, dr\right\} = e^{-(t-s)\mu}$$

Hence we can easily calculate any required probability.

Note that we don't actually have to calculate μ since we can write:

$${}_{t-s} p_{x+s} = \left(p_x\right)^{t-s} \qquad \text{for } 0 \le s < t < 1$$

Under this assumption μ_x has a stepped shape, as shown in the following graph of μ_x between ages 100 and 105:



Question 3.11 ***(CT5, September 2005, Question 5, part)***

A population is subject to a constant force of mortality of 0.015.

Calculate the probability that a life aged 20 exact will die before age 21.25 exact. [2]

Question 3.12

Calculate ${}_3p_{62½}$ based on the PFA92C20 table in the *Tables* using:

(i) the UDD assumption

(ii) the CFM assumption.

3 Using the life table to evaluate means and variances

All of the formulae developed in Chapters 1 and 2, such as:

$$A_x = \sum_{k=0}^{\infty} v^{k+1} {}_{k|}q_x$$

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k {}_k p_x$$

can (obviously) be evaluated by direct calculation given only the life table probabilities, and with the widespread use of computers this is often the simplest method.

To save the time involved in the direct calculation using the underlying probabilities, the *Tables* book tabulates a selection of annuity and assurance functions on certain interest rates (including higher interest rates for variance calculations) and various mortality tables.

For functions dependent upon age as well as term, tabulations are restricted in order to save space. For example, in AM92, $\ddot{a}_{x:\overline{n}|}$ is tabulated for $x+n=60$ and for $x+n=65$. Other values can be calculated using formulae like:

$$\ddot{a}_{x:\overline{n}|} = \ddot{a}_x - v^n {}_n p_x \ddot{a}_{x+n}$$

Assurance functions may be calculated directly, or by calculation of a corresponding annuity value and then use of a premium conversion formula like those you will meet in Chapter 4.

When dealing with annuities paid continuously or with death benefits payable immediately on death, it is straightforward to calculate functions as above and use the relevant approximations for the type of insurance concerned. For example, for whole life contracts, the formulae would be:

$$\bar{a}_x \cong \ddot{a}_x - ½$$

$$\bar{A}_x \cong (1+i)^{½} A_x$$

Retrospective accumulations can be calculated along similar lines, so for example:

$$\ddot{s}_{x:\overline{n}|} = \frac{(1+i)^n \, l_x \, (1+vp_x + \ldots + v^{n-1} \, {}_{n-1}p_x)}{l_{x+n}}$$

$$= \frac{l_x (1+i)^n + l_{x+1}(1+i)^{n-1} + \ldots + l_{x+n-1}(1+i)}{l_{x+n}}$$

We will study retrospective accumulations in more detail in Chapter 5.

4 Evaluating means and variances without use of the life table

The main alternative to using a life table is to postulate a formula and parameter values for the probability ${}_tp_x$ in the equations of Chapters 1 and 2, and then evaluate the expressions directly.

This means that we hypothesise a formula for ${}_tp_x$, estimate the values of the parameters in the formula, and then calculate the probabilities for different values of x and t.

Equivalently a formula for ${}_tq_x$ or μ_{x+t} could be postulated.

The difficulty of adopting this approach is that the postulation would need to be valid across the whole age range for which the formulae might be applied. As can be seen from the discussion in Section 1.6 above, the shape of human mortality may make a simple postulation difficult. Simple formulae may, however, be more appropriate and convenient for non-life contingencies.

The following example shows how mortality and survival probabilities can be evaluated without using a life table.

Example

In a certain population, the force of mortality equals 0.025 at all ages.

So the "formula" we are assuming for μ_{x+t} here is a very simple one, simply that it takes a constant value at all ages. (We are also assuming that there is no upper limit to age.)

Calculate:

(i) the probability that a new-born baby will survive to age 5

(ii) the probability that a life aged exactly 10 will die before age 12

(iii) the probability that a life aged exactly 5 will die between ages 10 and 12.

Solution

(i) $${}_5p_0 = \exp\left(-\int_0^5 0.025\,dt\right) = e^{-0.125} = 0.88250$$

(ii) $${}_2q_{10} = 1 - {}_2p_{10} = 1 - \exp\left(-\int_0^2 0.025\,dt\right) = 1 - e^{-0.05} = 0.04877$$

(iii) $${}_{5|2}q_5 = {}_5p_5 \cdot {}_2q_{10} = e^{-0.125} \times 0.04877 = 0.88250 \times 0.04877 = 0.04304$$

5 Select mortality

5.1 Introduction

So far, we have made an assumption of ultimate mortality, that is, that mortality varies by age only. As discussed further in the last chapter of this course, many factors other than just age might affect observed mortality rates. In practice, therefore, the evaluation of assurance and annuity benefits is often modified to allow for factors other than just age, which affect the survival probabilities.

Many factors can be allowed for by segregating the population, the assumption being that an age pattern of mortality can be discerned in the sub-population. For example, a population may well be segregated by sex and then sex-specific mortality rates (and mortality tables) can be used directly by using the techniques so far described.

Where, however, the pattern of mortality is assumed to depend not just on age, then slightly more complicated survival probabilities are employed. The most important, in the case of human mortality, is where the mortality rates depend upon duration as well as age, called *select rates*.

Question 3.13

Apart from age and duration, what other factors would you expect to affect observed mortality rates? Give at least three.

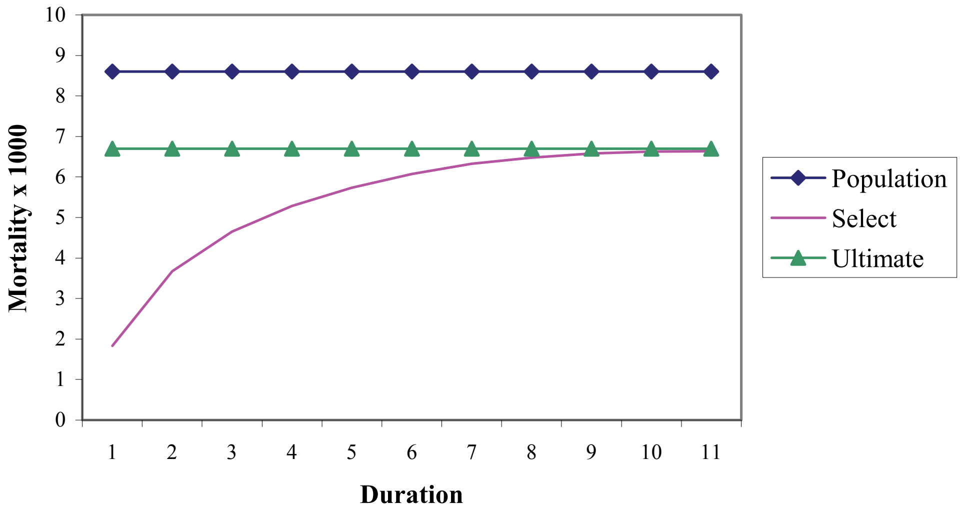
Why does the concept of duration in the population matter so much? The most important example is that of a life insurance company, where we consider the population of "policyholders with assurance policies" (rather than annuities, which we shall consider later). So "date of joining the population" here will be the "date of policy inception".

In order to take out the policy, the policyholder will have been subject to some *medical underwriting*. This means that they provide evidence about their current and recent state of health, answering simple questions on the policy proposal form, and perhaps also attending a medical examination if the sum insured is very high. The aim of this underwriting is to allow the company to screen out very bad risks, and to charge appropriately higher premiums for worse than average risks.

So if someone is a policyholder with duration of one year, say, then this means that they recently satisfied the company about their state of health. We would therefore expect their mortality to be better than that of someone with duration of, say, 3 years who passed the medical underwriting hurdle several years ago. And we would expect both of these policyholders to evince better mortality than policyholders of the same age with duration 10 or 20 years, for instance. The mortality of the recently joined policyholders is called *select* mortality, and we expect it to be better than that of longer duration policyholders, whose mortality we shall call *ultimate* mortality.

However, it is important to realise that even ultimate mortality may be good when compared with population mortality. We shall consider this again later in the course.

If we were to plot mortality for any given age x by varying duration, and compare it also against population mortality, we might expect something like:



Thinking about the underlying ideas, in theory we would expect select mortality always to be lower than ultimate; or ultimate mortality should not exist, and we should always consider select rates even with durations of 30 years! However, in practice:

- intuitively, we would not expect a difference in mortality between someone who answered some health questions on a policy proposal form 10 years ago and someone else who answered such questions 20 years ago; and,
- it would be very difficult to measure mortality rates if we want to split the population up by every year of duration in addition to age and sex because in each group we might see only 50 to 100 policyholders, even for a very large company.

So the actuary adopts a pragmatic approach and looks for the duration beyond which there is no significant change in mortality with further increases in duration. This is called the *select period*. The AM92 table has a 2-year select period. However, if you look at this table, you will see that the functions are almost equal at ages $[x-1]+1$ and x. We examine the choice of the select period in Section 5.3.

We have discussed select mortality in the context of life assurance policyholders. In fact the effect of selection is also found with annuity business: the mortality of policyholders who have recently bought an annuity policy is lighter than that of policyholders who took out their policy some time ago.

5.2 Mortality rates that depend on both age and duration

Select rates are usually studied by modelling the force of mortality μ as a function of the age at joining the population and the duration since joining the population. The usual notation is:

$[x]+r$	**age at date of transition**

where:

$[x]$	**age at date of joining population**
r	**duration from date of joining the population until date of transition**

The transition intensity is written $\mu_{[x],r}$.

So $\mu_{[30],4}$ would refer to the force of mortality for a life now aged 34, who entered the population at age 30.

It is more usual to use the notation $\mu_{[x]+r}$, so the above would be $\mu_{[30]+4}$. In addition:

$l_{[x]+r}$	**number of lives alive at duration r having joined the population at age $[x]$, based on some assumed radix**

The important principle to grasp with this notation is that the term in square brackets [] denotes the age at joining the population, so the age "now" (the moment of transition, or when exposed to possible transition) will be $x+r$. In some circumstances we may be given the age "now" y and the duration r; in this case the transition intensity "now" would be expressed as $\mu_{[y-r]+r}$.

In effect a model showing how μ varies with r is constructed for each value of $[x]$. Instead of the single life age-specific life table described above we have a series of life tables, one for each value of $[x]$.

5.3 *Displaying select rates*

Once select rates have been estimated, it is conventional to display estimated rates for each age at entry into the population, $[x]$, by age attained at the date of transition *ie* $[x]+r$. This can be done in an array

....	....	....	....	....
$\mu_{[x]}$	$\mu_{[x-1]+1}$	$\mu_{[x-2]+2}$	$\mu_{[x-3]+3}$	....
$\mu_{[x+1]}$	$\mu_{[x]+1}$	$\mu_{[x-1]+2}$	$\mu_{[x-2]+3}$	....
$\mu_{[x+2]}$	$\mu_{[x+1]+1}$	$\mu_{[x]+2}$	$\mu_{[x-1]+3}$	....
....	....	....	....	....

Question 3.14

How would you expect $\mu_{[x+1]}$ to compare with $\mu_{[x]+1}$? And $\mu_{[x+2]+2}$ compared with $\mu_{[x+4]}$? And $\mu_{[x+4]}$ compared with $\mu_{[x+1]+2}$? Assume throughout that the force of mortality increases with both age and policy duration.

Each diagonal (↘) of the array represents a model of how rates vary with duration since joining the population for a particular age at the date of joining, *ie* each is a set of life table mortality rates. The rates displayed on the rows of the array are rates for lives who have a common age attained at the time of transition, but different ages at the date of joining the population.

The diagonals referred to here are those going from top left to bottom right (for those of you holding the page in the normal manner!).

Considering the rows, the second such row in the table above contains $\mu_{[x+1]}$ and $\mu_{[x]+1}$. These both apply to lives aged $x+1$, but with differing durations. Similarly the next row relates to lives age $x+2$ but with differing durations, and so on.

If the rates did not depend on the duration since the date of joining the population, then apart from sampling error the rates on each row would be equal.

If we are presenting crude estimates of the rates then we shall find sampling errors. These will be removed by the process of graduation, which was described in Subject CT4.

Usually it is the case that rates depend on duration until duration s, and after s they are independent of duration. This phenomenon is termed Temporary Initial Selection (see the last chapter of the course) and s is called the length of the select period. In any investigation s is determined empirically by considering the statistical significance of the differences in transition rates along each row and the substantive impact of the different possible values of s.

We examine rows to ensure that we consider different durations but equal ages.

Typical select periods seen in life company investigations range from one to five years.

Example

For instance, the AM92 tables (UK assured lives 1991-94) in the *Tables* are based on a two-year select period.

The UK 1980 assured lives table AM80 used a five-year select period, but grouping together durations 2 to 4 where no significant differences were seen. Thus a typical row would show the rates

Duration 0	Duration 1	Duration 2-4	Duration 5+
$\mu_{[34]}$	$\mu_{[33]+1}$	$\mu_{[32]+2} = \mu_{[31]+3} = \mu_{[30]+4}$	μ_{34}

Question 3.15

Given the graph of select rates shown on Page 21, what select period might be appropriate?

When a value of s has been determined, then the estimates of the rates for duration $\geq s$ are pooled to obtain a common estimated value that is used in all the life tables in which it is needed. The array can be written:

....	....	....	....	....	....
$\mu_{[x]}$	$\mu_{[x-1]+1}$	$\mu_{[x-2]+2}$	....	$\mu_{[x-s+1]+s-1}$	μ_x
$\mu_{[x+1]}$	$\mu_{[x]+1}$	$\mu_{[x-1]+2}$	....	$\mu_{[x-s+2]+s-1}$	μ_{x+1}
$\mu_{[x+2]}$	$\mu_{[x+1]+1}$	$\mu_{[x]+2}$	....	$\mu_{[x-s+3]+s-1}$	μ_{x+2}
....	....	....	....	....	....

The right hand column of the above array represents a set of rates that are common to all the constituent life tables in the model of rates by age and duration. It is called an ultimate table.

Once we have decided on an appropriate select period, we shall want to graduate the results before presenting and using them. You are not required to know any details of graduating select rates.

The initial q-type select probabilities can be displayed in a similar way.

The AM92 Select mortality rates are tabulated in the way shown above, in the *Tables*.

5.4 *Constructing select and ultimate life tables*

The first step in constructing life tables is to refine the crude estimated rates (μ or q) into a smooth set of rates that statistically represent the true underlying mortality rates. This refinement process, called "graduation", is dealt with in Subject CT4, Models, and we now assume that we have a set of graduated mortality rates.

Using the graduated values of the initial transition probabilities displayed in the array:

$$\begin{array}{llllll}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
q_{[x]} & q_{[x-1]+1} & q_{[x-2]+2} & \cdots & q_{[x-s+1]+s-1} & q_x \\
q_{[x+1]} & q_{[x]+1} & q_{[x-1]+2} & \cdots & q_{[x-s+2]+s-1} & q_{x+1} \\
q_{[x+2]} & q_{[x+1]+1} & q_{[x]+2} & \cdots & q_{[x-s+3]+s-1} & q_{x+2} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

a table representing the select and ultimate experience can be constructed.

This is achieved by first constructing the ultimate life table based on the final column of the array and the formula described in Section 1.2 above:

- **choose the starting age of the table, k**
- **choose an arbitrary radix for the table, l_k**
- **recursively calculate the values of l_x using $l_{x+1} = l_x\left(1-q_x\right)$**

Beginning with the appropriate ultimate value in the final column, the select life table functions for each row of the array are determined. This is achieved by "working backwards" up each diagonal using:

$$l_{[x]+t} = \frac{l_{[x]+t+1}}{(1-q_{[x]+t})}$$

for $t = s-1, ..., 1, 0$

and noting in the first iteration that $l_{[x]+s-1+1} = l_{x+s}$.

Here is one way of dealing with this.

Example

Given the following select and ultimate mortality rates from AF80, construct the corresponding l_x's for ages x from 45 and 46.

Age	Duration 0	Duration 1	Duration 2+
45	0.000838		
46	0.000924	0.001158	
47	0.001018	0.001284	0.001415
48		0.001423	0.001564
49			0.001729

Solution

First choose some arbitrary radix for l_{47} (*ie* the lives aged 45 at entry with duration 2+ gives age 47 ultimate). Say $l_{47} = 1,000$.

Then working back diagonally upwards to the left from l_{47} to $l_{[45]}$:

$$l_{[45]+1} = \frac{l_{47}}{\left(1-q_{[45]+1}\right)} = \frac{1,000}{(1-0.001158)} = 1,001.16$$

$$l_{[45]} = \frac{l_{[45]+1}}{\left(1-q_{[45]}\right)} = \frac{1,001.16}{(1-0.000838)} = 1,002.00$$

And, working down the column from l_{47} to l_{48}:

$$l_{48} = l_{47}\left(1-q_{47}\right) = 1,000(1-0.001415) = 998.59$$

which then allows us to calculate $l_{[46]}$ and $l_{[46]+1}$:

$$l_{[46]+1} = \frac{l_{48}}{\left(1-q_{[46]+1}\right)} = \frac{998.59}{(1-0.001284)} = 999.87$$

$$l_{[46]} = \frac{l_{[46]+1}}{\left(1-q_{[46]}\right)} = \frac{999.87}{(1-0.000924)} = 1,000.79$$

Question 3.16

Continue the above procedure to calculate l_{49}, $l_{[47]+1}$ and $l_{[47]}$.

5.5 *Using tabulated select life table functions*

Some probabilities are particularly useful in life contingencies calculations. We have already defined:

$$ {}_{n|m}q_x = \frac{l_{x+n} - l_{x+n+m}}{l_x} $$

$$ {}_{n|}q_x = \frac{l_{x+n} - l_{x+n+1}}{l_x} $$

representing the m year and 1 year probabilities of transition when the event of transition is deferred for n years.

Similar probabilities can be defined for each select mortality table:

$$ {}_{n|m}q_{[x]+r} = \frac{l_{[x]+r+n} - l_{[x]+r+n+m}}{l_{[x]+r}} $$

$$ {}_{n|}q_{[x]+r} = \frac{l_{[x]+r+n} - l_{[x]+r+n+1}}{l_{[x]+r}} $$

with the special case of $n = 0$ and $m = n$ being of particular interest:

$$ {}_nq_{[x]+r} = \frac{l_{[x]+r} - l_{[x]+r+n}}{l_{[x]+r}} $$

and the complement of this n year transition probability, the n year survival probability, is:

$$ {}_np_{[x]+r} = \frac{l_{[x]+r+n}}{l_{[x]+r}} $$

The above probabilities may also be expressed in terms of the number of deaths in the select mortality table by defining:

$$d_{[x]+r} = l_{[x]+r} - l_{[x]+r+1}$$

> ***Question 3.17***
>
> Using the mortality table in the previous example, calculate the probability that a life selected at age 46 dies between ages 47 and 48.

5.6 *Evaluating means and variances using select mortality*

Corresponding to the assurances and annuities defined in Chapters 1 and 2 are select equivalents defined as before but assumed to be issued to a select life denoted $[x]$ rather than x.

So, for example, $A_{[x]} = \sum_{k=0}^{k=\infty} v^{k+1}\,{}_{k|}q_{[x]}$ can be used to calculate the EPV of benefits of a whole life assurance issued to a select life aged $[x]$ at entry.

Similarly, $\ddot{a}_{[x]} = \sum_{k=0}^{k=\infty} {}_kp_{[x]}.v^k$ can be used to calculate the EPV of benefits of a whole life annuity due, with level annual payments, issued to a select life aged $[x]$ at entry.

The variance formulae established in Chapters 1 and 2 also apply replacing x with $[x]$.

6 Exam-style questions

We now conclude this chapter with two past exam questions:

Question 3.18 ***(CT5, September 2005, Question 3)***

A graph of $f_0(t)$, the probability density function for the random future lifetime, T_0, is plotted on the vertical axis, with t plotted on the horizontal axis, for data taken from the English Life Table No. 15 (Males).

You are given that $f_0(t) = {}_t p_0 \, \mu_t$. You observe that the graph rises to a peak at around $t = 80$ and then falls.

Explain why the graph falls at around $t = 80$. [3]

Question 3.19 ***(CT5, September 2005, Question 4)***

Calculate the value of ${}_{1.75}p_{45.5}$ on the basis of mortality of AM92 Ultimate and assuming that deaths are uniformly distributed between integral ages. [3]

Chapter 3 Summary

Modelling mortality

We can model mortality by assuming that future lifetime is a continuous random variable taking values between 0 and some limiting age ω. From this starting point, we can calculate probabilities of survival $({}_tp_x)$ and death $({}_tq_x)$ for an individual aged x over a period of t years.

Definitions of probabilities of death and survival

$$ {}_tq_x = F_x(t) = P[\,T_x \le t\,] $$

$$ {}_tp_x = 1 - {}_tq_x = S_x(t) = 1 - F_x(t) = P[\,T_x > t\,] $$

$$ {}_{t+s}p_x = {}_tp_x \times {}_sp_{x+t} = {}_sp_x \times {}_tp_{x+s} $$

Force of mortality

The force of mortality μ_x is the *instantaneous rate of mortality* at age x. It is defined by the equation:

$$ \mu_x = \lim_{h \to 0^+} \frac{1}{h} \times P[\,T \le x+h \,|\, T > x\,] $$

We also have the following results about μ_x:

$$ \mu_x = \lim_{h \to 0+} \frac{1}{h} \times {}_hq_x $$

$$ \mu_x = -\frac{1}{l_x} \times \frac{d}{dx} l_x = -\frac{d}{dx} \ln l_x $$

$$ {}_tq_x = \int_0^t {}_sp_x \, \mu_{x+s} \, ds \qquad {}_tp_x = \exp\left\{ -\int_0^t \mu_{x+s} \, ds \right\} $$

Using a life table

We can tabulate survival probabilities and other quantities for each year of age in a *life table*. Simple formulae allow us to combine the entries in the life table to calculate a wide range of useful quantities efficiently:

$$l_x = l_\alpha \times {}_{x-\alpha}p_\alpha \qquad\qquad {}_t p_x = \frac{l_{x+t}}{l_x}$$

$$d_x = l_x - l_{x+1} \qquad\qquad q_x = \frac{d_x}{l_x}$$

$${}_{n|m}q_x = P[\,n < T_x \le n+m\,] = {}_n p_x \times {}_m q_{x+n} = \frac{l_{x+n} - l_{x+n+m}}{l_x}$$

Dealing with non-integer ages

By making an assumption about the rate or force of mortality within a year of age, we can calculate the probability of survival for individuals at non-integer ages and for periods other than a whole number of years.

Uniform distribution of deaths

Assumption: ${}_t p_x \mu_{x+t}$ is a constant for integer x and $0 \le t \le 1$

$${}_s q_x = s\, q_x \quad (0 \le s \le 1) \qquad\qquad {}_{t-s} q_{x+s} = \frac{(t-s)\, q_x}{1 - s\, q_x} \quad (0 \le s \le t \le 1)$$

Constant force of mortality

Assumption: μ_{x+t} is a constant for integer x and $0 \le t \le 1$

$${}_t p_x = e^{-t\mu} \qquad\qquad {}_{t-s} p_{x+s} = e^{-(t-s)\mu} \quad (0 \le s \le t \le 1)$$

Select mortality

Lives who are 'selected' from a larger group of people, in some non-random way with respect to their mortality, will experience different mortality at any given age from the group as a whole. They are then referred to as having *select mortality*.

An example occurs when applicants for life assurance are underwritten, so that only the 'better' risks are accepted for cover at the insurer's standard premium rates.

Lives subject to select mortality are denoted by $[x]+r$, where x is the age at selection and r is the number of years since selection, meaning that $x+r$ is the current age.

A select mortality table (such AM92) has select functions (*eg* $q_{[x]+r}$, $l_{[x]+r}$ and $\ddot{a}_{[x]+r}$) for $r=0,1,\ldots s-1$, where s is the *select period* of the table. The select period is the number of years since selection during which mortality rates are assumed to be dependent upon the duration since selection as well as on current age.

For $r \geq s$, mortality is assumed to be a function of age only (called *ultimate mortality*), *eg*:

$$q_x = q_{[x-r]+r} \quad \text{for } r \geq s$$

The version of the AM92 table quoted in the *Tables* has a 2-year select period. This means that, for example, the expected progression of survivors through the table is:

$$l_{[x]} \to l_{[x]+1} \to l_{x+2} \to l_{x+3} \to l_{x+4} \to \cdots$$

This page has been left blank so that you can keep the chapter summaries together for revision purposes.

Chapter 3 Solutions

Solution 3.1

The required expression is:

$${}_tq_x = 1 - {}_tp_x = 1 - \frac{l_{x+t}}{l_x} = \frac{l_x - l_{x+t}}{l_x}$$

Solution 3.2

(i) The difference between the number of lives expected to be alive at age x and the number of lives expected to be alive at age $x+n$ (*ie* the RHS) is equal to the total of the number of lives expected to die between those two ages (*ie* the LHS).

(ii) The expected number of lives alive at age x equals the expected number of deaths at each age from x onwards, *ie* all those lives alive at age x must die at some point in the future.

Solution 3.3

(i) $p_2 = \dfrac{l_3}{l_2} = \dfrac{99{,}086}{99{,}124} = 0.99962$

(ii) ${}_2p_3 = \dfrac{l_5}{l_3} = \dfrac{99{,}032}{99{,}086} = 0.99946$

(iii) ${}_4q_1 = 1 - {}_4p_1 = 1 - \dfrac{l_5}{l_1} = 1 - \dfrac{99{,}032}{99{,}186} = 0.00155$

(iv) $l_6 = l_5 - d_5 = 99{,}032 - 22 = 99{,}010$

Solution 3.4

$${}_np_x \times {}_mq_{x+n} = \frac{l_{x+n}}{l_x} \times \frac{l_{x+n} - l_{x+n+m}}{l_{x+n}} = \frac{l_{x+n} - l_{x+n+m}}{l_x}$$

Solution 3.5

The required probability is:

$$ {}_{28|10}q_{37} = \frac{l_{65} - l_{75}}{l_{37}} = \frac{79,293 - 53,266}{96,933} = 0.26851 $$

Solution 3.6

(i) $P(K_{30} = 40) = {}_{40|}q_{30} = \dfrac{d_{70}}{l_{30}} = \dfrac{2,674}{97,645} = 0.02738$

(ii) $P(T_{30} > 40)$ is the probability that a life aged exactly 30 lives for at least another 40 more years.

So:

$$ P(T_{30} > 40) = {}_{40}p_{30} = \frac{l_{70}}{l_{30}} = \frac{68,055}{97,645} = 0.69696 $$

Solution 3.7

In order to approximate the mortality rate for a period of less than one year, we will need to make an assumption about the behaviour of the mortality rate over the year of age. Whilst the underlying force of mortality can vary greatly at different ages, it should not change significantly within a single year of age. By splitting up the required probability in this way, our approximations should be reasonably accurate.

Solution 3.8

The quantity ${}_tp_x$ is a decreasing function of t. Since the function ${}_tp_x\,\mu_{x+t}$ is constant, the UDD assumption implies that μ_{x+t} is an increasing function of t.

So, μ_x increases over the year of age.

Solution 3.9

We can write:

$$
{}_{t-s}q_{x+s} = 1 - {}_{t-s}p_{x+s} = 1 - \frac{{}_t p_x}{{}_s p_x} = 1 - \frac{1 - {}_t q_x}{1 - {}_s q_x} = 1 - \frac{1 - t\,q_x}{1 - s\,q_x}
$$

$$
= \frac{(1 - s\,q_x) - (1 - t\,q_x)}{1 - s\,q_x} = \frac{(t-s)q_x}{1 - s\,q_x}
$$

Solution 3.10

We have:

$$
\begin{aligned}
l_{x+t} &= l_x \times {}_t p_x \\
&= l_x(1 - {}_t q_x) \\
&= l_x(1 - t\,q_x) \\
&= l_x - t\,(l_x q_x) \\
&= l_x - t\,d_x && \text{result (i)} \\
&= l_x - t\,(l_x - l_{x+1}) \\
&= (1-t)l_x + t\,l_{x+1} && \text{result (ii)}
\end{aligned}
$$

Solution 3.11

This is:

$$ {}_{1.25}q_{20} = 1 - {}_{1.25}p_{20} = 1 - e^{-1.25\mu} = 1 - e^{-1.25\times 0.015} = 1 - e^{-0.01875} = 0.018575 $$

Solution 3.12

We'll split up the probability as follows:

$$ {}_{3}p_{62½} = {}_{½}p_{62½} \times {}_{2}p_{63} \times {}_{½}p_{65} $$

From the PFA92C20 table:

$$ {}_{2}p_{63} = \frac{l_{65}}{l_{63}} = \frac{9,703.708}{9,775.888} = 0.992617 $$

(i) Under the UDD assumption:

$$ {}_{½}p_{62½} = \frac{p_{62}}{{}_{½}p_{62}} = \frac{1-q_{62}}{1-½q_{62}} = \frac{1-0.002885}{1-½\times 0.002885} = 0.998555 $$

$$ {}_{½}p_{65} = 1 - {}_{½}q_{65} = 1 - ½q_{65} = 1 - ½\times 0.004681 = 0.997660 $$

which gives ${}_{3}p_{62½} = 0.998555\times 0.992617\times 0.997660 = 0.988863$

(ii) Under the constant force of mortality assumption:

$$ {}_{½}p_{62½} = \exp(-½\mu) = \left(e^{-\mu}\right)^{½} = \left(p_{62}\right)^{½} = \left(1-0.002885\right)^{½} = 0.998556 $$

$$ {}_{½}p_{65} = \left(p_{65}\right)^{½} = \left(1-0.004681\right)^{½} = 0.997657 $$

which gives ${}_{3}p_{62½} = 0.998556\times 0.992617\times 0.997657 = 0.988861$

Solution 3.13

There are a number of possible answers here. However the three perhaps most likely to affect mortality are sex, smoker status and occupation. We will discuss this in much more detail later in the course.

Solution 3.14

We have:

$$\mu_{[x+1]} < \mu_{[x]+1}$$

because the rates relate to the same age ($x+1$), but $\mu_{[x]+1}$ relates to a later duration (1 instead of 0).

Secondly:

$$\mu_{[x+2]+2} > \mu_{[x+4]}$$

again because they relate to the same age but the first term is for duration 2 rather than 0.

We might also expect the difference to be higher than in the previous case, because:

- it involves twice the duration difference (2 instead of 1)
- the current age is three years older ($x+4$ instead of $x+1$), which should increase the absolute size of the effect of duration differences on mortality

How $\mu_{[x+4]}$ compares with $\mu_{[x+1]+2}$ will depend on how duration and age affect mortality, because the current age has decreased (reducing mortality) while duration has increased (increasing mortality). It is therefore impossible to give a generalised (correct!) answer in this case.

Solution 3.15

In theory, the graph would suggest a select period of 9 to 10 years, since after that time differences in duration seem to have no effect on mortality. However, in practice we might choose a lower select period, perhaps 5 or 6 years, if we think that the difference in mortality between durations 5 and 10 is very slight, in particular when compared with the variance of estimated rates given the size of our investigation.

Solution 3.16

The required values are:

$$l_{49} = l_{48}\left(1-q_{48}\right) = 998.59\left(1-0.001564\right) = 997.02$$

$$l_{[47]+1} = \frac{l_{49}}{\left(1-q_{[47]+1}\right)} = \frac{997.02}{\left(1-0.001423\right)} = 998.44$$

$$l_{[47]} = \frac{l_{[47]+1}}{\left(1-q_{[47]}\right)} = \frac{998.44}{\left(1-0.001018\right)} = 999.46$$

Solution 3.17

We want:

$$\frac{l_{[46]+1} - l_{48}}{l_{[46]}} = \frac{999.87 - 998.59}{1000.79} = 0.001279$$

Solution 3.18

From middle age onwards the force of mortality increases exponentially. This dominates (*ie* has a stronger effect than) the ${}_t p_0$ term for values of t up to 80 or so. As a result, the pdf increases to about age 80. After age 80, the ${}_t p_0$ term dominates the μ_t term. So although the force of mortality continues to increase after age 80, the probability of surviving to these older ages is small and eventually becomes 0.

Solution 3.19

To calculate the value of ${}_{1.75}p_{45.5}$, we first have to split the age range up into single years of age:

$${}_{1.75}p_{45.5} = {}_{0.5}p_{45.5} \times p_{46} \times {}_{0.25}p_{47}$$

Now:

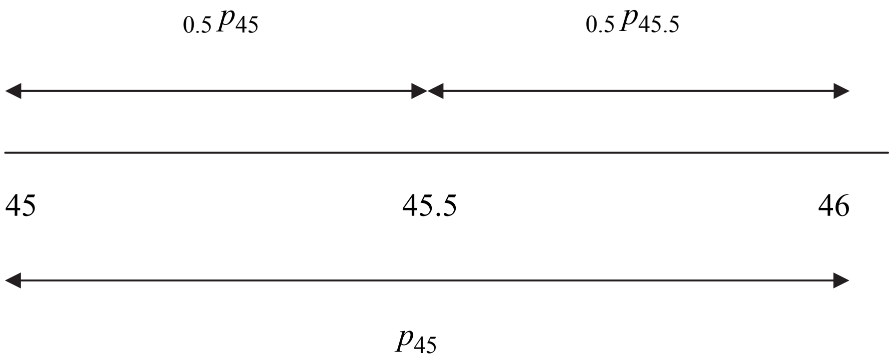
$$p_{46} = 1 - q_{46} = 1 - 0.001622 = 0.998378$$

$$_{0.25}p_{47} = 1 - {}_{0.25}q_{47} \overset{UDD}{=} 1 - 0.25\,q_{47} = 1 - 0.25 \times 0.001802 = 0.999550$$

and:

$$_{0.5}p_{45.5} = \frac{p_{45}}{_{0.5}p_{45}} \overset{UDD}{=} \frac{1 - q_{45}}{1 - 0.5\,q_{45}} = \frac{1 - 0.001465}{1 - 0.5 \times 0.001465} = 0.999267$$

Note that we cannot apply the result $_tq_x = t\,q_x$ *to this last case directly because 45.5 is not an integer age. We first have to write* $_{0.5}p_{45.5}$ *in terms of* p_{45}*. From Section 1.1 of Chapter 3, we know that* $_{0.5}p_{45} \times {}_{0.5}p_{45.5} = p_{45}$*. This can also be easily seen from the following diagram:*



Dividing through by $_{0.5}p_{45}$ *gives the expression above.*

So we have:

$$_{1.75}p_{45.5} = 0.999267 \times 0.998378 \times 0.999550 = 0.997197$$

Chapter 4

Evaluation of assurances and annuities

Syllabus objectives

(i) Define simple assurance and annuity contracts, and develop formulae for the means and variances of the present values of the payments under these contracts, assuming constant deterministic interest.

5. Define the symbols a_x, $a_{x:\overline{n}|}$, $_{m|}a_{x:\overline{n}|}$, $\ddot{a}_x$, $\ddot{a}_{x:\overline{n}|}$, $_{m|}\ddot{a}_{x:\overline{n}|}$. Extend the annuity factors to allow for the possibility that payments are more frequent than annual but less frequent than continuous. [Note: the symbols have already been defined in Chapter 2.]

7. Understand and use the relations $A_x = 1 - d\ddot{a}_x$ and $A_{x:\overline{n}|} = 1 - d\ddot{a}_{x:\overline{n}|}$ and their select and continuous equivalents.

(ii) Describe and use practical methods of evaluating expected values and variances of the simple contracts defined in (i).

4. Evaluate the expected values and variances in (i) 3 by table look-up or other means, including the use of the relationships in (i) 6 and 7.

5. Derive approximations for, and hence evaluate, the expected values and variances in (i) 4 in terms of those in (i) 3.

0 Introduction

We have introduced the basic functions of life insurance mathematics – EPVs of simple assurance and annuity contracts. The next step is to explore useful relationships among these EPVs, and then we can apply the same ideas to other types of life insurance contracts.

A selection of the above functions are tabulated in, for example, the Formulae and Tables for Examinations. Later in this chapter we will discuss ways of calculating a greater range of functions.

Example

Find:

(i) $\ddot{a}_{30}$ **(AM92 at 4%)**

(ii) $\ddot{a}_{75}$ **(PMA92C20 at 4%)**

Solution

(i) **21.834**

(ii) **9.456**

The formulae we have derived for EPVs can be interpreted in a simple way, which is often useful in practice. Consider, for example, the results:

$$A_x = \sum_{k=0}^{\infty} v^{k+1} {}_{k|}q_x \quad \text{or} \quad \ddot{a}_x = \sum_{k=0}^{\infty} v^k {}_k p_x$$

Each term of these sums can be interpreted as:

Expected present values

An amount payable at time k

$\times$ the probability that a payment will be made at time k

$\times$ a discount factor for k years.

This last comment is very important and should always be borne in mind. It is also important to remember that the equations above do not define A_x and $\ddot{a}_x$. The definition of any assurance or annuity must be given in terms of a future lifetime random variable.

The first term in each case (*ie* the benefit paid) is just 1, but it should be easy to see that this interpretation can be applied to any benefit, level or not, payable on death or survival. This makes it easy to write down formulae for EPVs.

For example, consider an annuity-due, under which an amount k will be payable at the start of the kth year provided a life aged x is then alive (an increasing annuity-due). With this interpretation of EPVs, we can write down the EPV of this benefit (which is denoted $(I\ddot{a})_x$):

$$(I\ddot{a})_x = \sum_{k=0}^{\infty} (k+1)v^k \, {}_kp_x$$

Increasing benefits will be covered in Chapter 6.

In this chapter we will discuss different ways of evaluating assurance and annuity benefits and consider the relationships that exist between the various functions.

1 Evaluating assurance benefits

It is important that you become familiar with the *Tables* and know which annuity and assurance functions are included in them. The AM92 table, for example, contains the values of A_x at 4% and 6% *pa* interest. It also contains the values of $A_{x:\overline{n}|}$ (at 4% and 6% *pa* interest) for ages x and terms n such that $x+n=60$ and $x+n=65$. But how can you determine the value of assurance functions, *eg* $A_{30:\overline{25}|}$ and $A^{1}_{40:\overline{25}|}$, that aren't listed in the *Tables*?

There are, in fact, several ways to proceed. One possibility is to use the relationships that exist between the assurance functions in order to write the required function in terms of functions listed in the *Tables*.

We saw in Chapter 1 that:

$$_{n|}A_x = A_x - A^{1}_{x:\overline{n}|} = v^n \, {}_np_x \, A_{x+n}$$

Rearranging this gives:

$$A^{1}_{x:\overline{n}|} = A_x - v^n \, {}_np_x \, A_{x+n}$$

So we can calculate the value of a term assurance by writing it in terms of whole life assurances.

Question 4.1

Calculate the values of $A^{1}_{40:\overline{25}|}$, $A_{30:\overline{25}|}$ and $\bar{A}_{30:\overline{25}|}$ using AM92 mortality and 4% *pa* interest.

Question 4.2

A life office has just sold a 25-year term assurance policy to a life aged 40. The sum assured is £50,000 and is payable at the end of the year of death. Calculate the variance of the present value of this benefit. Assume AM92 Ultimate mortality and 4% *pa* interest.

2 Evaluating annuity benefits

The following relationships are easy to prove.

$$\ddot{a}_x = 1 + a_x \qquad \ddot{a}_{x:\overline{n}|} = 1 + a_{x:\overline{n-1}|}$$

$$a_x = v\,p_x\,\ddot{a}_{x+1} \qquad a_{x:\overline{n}|} = vp_x\,\ddot{a}_{x+1:\overline{n}|}$$

Question 4.3

Prove that $a_{x:\overline{n}|} = vp_x \ddot{a}_{x+1:\overline{n}|}$.

Here's a very easy example, which illustrates the first of the formulae above. In this example, we use PFA92C20 mortality. The values of annuities-due are listed there, assuming an effective annual rate of interest of 4%.

Example

Find a_{65} **(PFA92C20 at 4%)**

Solution

$$a_{65} = \ddot{a}_{65} - 1 = 13.871$$

The AM92 table contains the values of $\ddot{a}_x$ at 4% and 6% *pa* interest. It also contains the values of $\ddot{a}_{x:\overline{n}|}$ (at 4% and 6% *pa* interest) for ages x and terms n such that $x+n=60$ and $x+n=65$. However, we also want to be able to calculate functions such as $\ddot{a}_{30:\overline{25}|}$ and $a_{[40]:\overline{25}|}$, which aren't listed in the *Tables*.

Again we can proceed by writing the required function in terms of the functions that are given in the *Tables*.

Question 4.4

Write down a formula for $\ddot{a}_{x:\overline{n}|}$ in terms of whole life annuities-due.

Similar formulae hold for temporary annuities payable annually in arrears and temporary annuities payable continuously. Alternatively, we could calculate $a_{x:\overline{n}|}$ using the formula:

$$a_{x:\overline{n}|} = \ddot{a}_{x:\overline{n}|} - 1 + v^n \, {}_n p_x$$

Question 4.5

Prove this result.

Another way to write the factor $v^n \, {}_n p_x$ is as $\dfrac{D_{x+n}}{D_x}$. D_x is an example of a commutation function, and is defined as follows:

$$D_x = v^x l_x$$

Commutation functions used to be widely used in the calculation of annuities and assurances. You will find values of the functions D_x, N_x, S_x, C_x, M_x and R_x listed in the AM92 Table at 4% *pa* interest. The only one of these that we will use in this course is D_x; we use it purely because it is quicker to calculate $\dfrac{D_{x+n}}{D_x}$ than $v^n \, {}_n p_x$. However, if the assumed rate of interest is not 4% *pa*, commutation functions are not available and we have to use $v^n \, {}_n p_x$.

Now let's look at an example involving the results given above.

Example

Calculate the values of $\ddot{a}_{30:\overline{25}|}$ and $a_{[40]:\overline{25}|}$ using AM92 mortality and 4% *pa* interest.

Solution

We can write:

$$\ddot{a}_{30:\overline{25}|} = \ddot{a}_{30} - v^{25}\,{}_{25}p_{30}\,\ddot{a}_{55} = \ddot{a}_{30} - \frac{D_{55}}{D_{30}}\ddot{a}_{55}$$

$$= 21.834 - \frac{1{,}105.41}{3{,}060.13} \times 15.873 = 16.100$$

Also:

$$a_{[40]:\overline{25}|} = \ddot{a}_{[40]:\overline{25}|} - 1 + v^{25}\,{}_{25}p_{[40]} = \ddot{a}_{[40]:\overline{25}|} - 1 + \frac{D_{65}}{D_{[40]}}$$

$$= 15.887 - 1 + \frac{689.23}{2{,}052.54} = 15.223$$

Try these ones yourself.

Question 4.6

Calculate the values of $\ddot{a}_{60:\overline{10}|}$ and $\bar{a}_{60:\overline{10}|}$ using AM92 mortality and 6% *pa* interest.

Question 4.7

Calculate a_{30} and ${}_{10|}a_{30}$, and also a_{70} and ${}_{10|}a_{70}$, based on AM92 Ultimate mortality at 4% *pa* interest. Comment on your results.

Question 4.8

Calculate the value of $\ddot{a}_{[55]+1:\overline{4}|}$ using AM92 mortality and 4% *pa* interest.

Question 4.9

A male pension policyholder is aged 50 and he will retire at age 65, from which age a pension of £5,000 *pa* will be paid annually in advance. Before retirement he is assumed to experience mortality in line with AM92 Ultimate and after retirement in line with PMA92C20. Calculate the expected present value of the benefits assuming interest of 4% *pa*.

3 Premium conversion equations

There are both discrete and continuous versions of the premium conversion equations.

3.1 Discrete version

There is a simple and very useful relationship between the EPVs of certain assurance contracts and the EPVs of annuities-due:

$$\ddot{a}_x = E\left[\ddot{a}_{\overline{K_x+1}|}\right] = E\left[\frac{1-v^{K_x+1}}{d}\right] = \frac{1-E\left[v^{K_x+1}\right]}{d} = \frac{1-A_x}{d}$$

Hence $A_x = 1 - d\ddot{a}_x$ **.**

> ***Question 4.10***
>
> Using the AM92 mortality table, look up A_{65} and $\ddot{a}_{65}$ at 4% *pa* interest. Hence verify that $A_{65} = 1 - d\ddot{a}_{65}$.

Along similar lines, we find that:

$$A_{x:\overline{n}|} = 1 - d\ddot{a}_{x:\overline{n}|}$$

and as we shall see, similar relationships hold for all of the whole life and endowment contracts that we consider.

These relationships also apply when we replace x **with** $[x]$**, *ie*:**

$$A_{[x]} = 1 - d\ddot{a}_{[x]}$$

and:

$$A_{[x]:\overline{n}|} = 1 - d\ddot{a}_{[x]:\overline{n}|}$$

3.2 Continuous version

The premium conversion relationships also work for the "bar" functions, but we have to use δ instead of d.

There is an important relationship between level annuities payable continuously and assurance contracts with death benefits payable immediately on death. For whole life benefits:

$$\bar{a}_x = E\left[\bar{a}_{\overline{T_x}|}\right] = E\left[\frac{1-v^{T_x}}{\delta}\right] = \frac{1}{\delta}(1-\bar{A}_x)$$

Hence $\bar{A}_x = 1-\delta\,\bar{a}_x$**.**

For temporary benefits:

$$\bar{a}_{x:\overline{n}|} = E\left[\bar{a}_{\overline{\min[T_x,n]}|}\right] = E\left[\frac{1-v^{\min[T_x,n]}}{\delta}\right] = \frac{1}{\delta}(1-\bar{A}_{x:\overline{n}|})$$

Hence $\bar{A}_{x:\overline{n}|} = 1-\delta\,\bar{a}_{x:\overline{n}|}$**.**

These formulae also apply replacing x with $[x]$.

The premium conversion formulae are given on Page 37 of the *Tables*.

3.3 Variance of benefits

We can use these relationships to express the variances of annuities payable continuously. For example:

$$\text{var}\left[\bar{a}_{\overline{T_x}|}\right] = \text{var}\left[\frac{1-v^{T_x}}{\delta}\right] = \frac{1}{\delta^2}\text{var}\left[v^{T_x}\right] = \frac{1}{\delta^2}\left({}^2\bar{A}_x - (\bar{A}_x)^2\right)$$

where the "2" superscript indicates an EPV calculated at rate of interest $(1+i)^2-1$.

We have already seen this result in Section 8.1 of Chapter 2.

4 Expected present values of annuities payable m times each year

Very often, premiums are not paid annually but with some other frequency: for instance every quarter, or every month. This is for the convenience of the policyholder. Likewise, we may want to value annuity benefits payable more frequently than annually.

We now consider the question of how annuities, with payments made more than once each year but less than continuously, may be evaluated.

We define the expected present value of an immediate annuity of 1 per annum, payable *m* times each year to a life aged *x*, as $a_x^{(m)}$.

This comprises payments, each of $\frac{1}{m}$, at ages:

$$x+\frac{1}{m},\; x+\frac{2}{m},\; x+\frac{3}{m} \text{ and so on.}$$

The expected present value may be written:

$$a_x^{(m)} = \frac{1}{m}\sum_{t=1}^{\infty} v^{t/m}\frac{l_{x+t/m}}{l_x}.$$

If a mathematical formula for l_x is known, this expression may be evaluated directly.

Question 4.11

If $l_x = 1{,}000 - 10x$ for $90 \le x \le 100$, write down an expression for $a_{90}^{(4)}$. Leave your answer in the form of a summation formula.

More often, an approximation will be needed to evaluate the expression.

An expression for $\ddot{a}_x^{(m)}$ can be valued as a series of deferred annuities with annual payments of $\frac{1}{m}$ and deferred period of $\frac{t}{m}$; $t = 0, 1, 2, \ldots, m-1$:

$$\sum_{t=0}^{m-1}\frac{1}{m}\;{}_{\frac{t}{m}|}\ddot{a}_x$$

Using the approximation that a sum of £1 payable a proportion k $(0 < k < 1)$ through the year is equivalent to £$(1-k)$ paid at the start of the year and £k at the end of the year, we can write:

$$_{\frac{t}{m}|}\ddot{a}_x \cong \ddot{a}_x - \frac{t}{m}$$

Question 4.12

Explain this result.

So the expected present value of the *m*thly annuity is approximately:

$$\sum_{t=0}^{t=m-1} \frac{1}{m}\left(\ddot{a}_x - \frac{t}{m}\right) = m.\frac{1}{m}\ddot{a}_x - \frac{1}{m}.\frac{1}{2}\frac{(m-1)m}{m}$$

This is because:

$$\frac{1}{m}\sum_{t=0}^{m-1}\frac{t}{m} = \frac{1}{m}\times\frac{0+1+2+\cdots+(m-1)}{m} = \frac{1}{m}\times\frac{½\, m(m-1)}{m}$$

That is:

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{(m-1)}{2m}$$

This formula is given on Page 36 of the *Tables.*

The corresponding expression for $a_x^{(m)}$ then follows from the relationship:

$$\ddot{a}_x^{(m)} = \frac{1}{m} + a_x^{(m)}$$

that is:

$$a_x^{(m)} \approx a_x + \frac{m-1}{2m}$$

(Note that, letting $m \to \infty$, we obtain the expression $\bar{a}_x \approx \ddot{a}_x - ½$, as referred to in Chapter 2.)

These approximations may be used to develop expressions for temporary and deferred annuities.

The formula for $\ddot{a}^{(m)}_{x:\overline{n}|}$ is also given on Page 36 of the *Tables*.

Question 4.13

Calculate values for $\ddot{a}^{(2)}_{60}$, $a^{(12)}_{60}$ and $\ddot{a}^{(4)}_{50:\overline{15}|}$ using AM92 mortality and 4% *pa* interest.

5 Exam-style questions

We finish off this chapter with a couple of exam-style questions.

Question 4.14

An impaired life aged 40 experiences 5 times the force of mortality of a life of the same age subject to standard mortality. A two-year term assurance policy is sold to this impaired life, and another two-year term assurance is sold to a standard life aged 40. Both policies have a sum assured of £10,000 payable at the end of the year of death. Calculate the expected present value of the benefits payable to each life assuming that standard mortality is AM92 Ultimate and interest is 4% *pa*.

Question 4.15

Assuming that the force of mortality between consecutive integer ages is constant in the AM92 Ultimate table, calculate the exact value of $\bar{A}_{50:\overline{2|}}$ using a rate of interest of 4% *pa*.

6 End of Part 1

What next?

1. Briefly **review** the key areas of Part 1 and/or re-read the **summaries** at the end of Chapters 1 to 4.
2. Attempt some of the questions in Part 1 of the **Question and Answer Bank**. If you don't have time to do them all, you could save the remainder for use as part of your revision.
3. Attempt **Assignment X1**.

This page has been left blank so that you can keep the chapter summaries together for revision purposes.

Chapter 4 Summary

Relationships between assurances

$$A^{1}_{x:\overline{n}|} = A_x - v^n \, {}_n p_x \, A_{x+n}$$

$$_{n|}A_x = A_x - A^{1}_{x:\overline{n}|} = v^n \, {}_n p_x \, A_{x+n}$$

Relationships between annuities

$$a_x = \ddot{a}_x - 1 \qquad \bar{a}_x = \ddot{a}_x - \tfrac{1}{2}$$

$$\ddot{a}_{x:\overline{n}|} = \ddot{a}_x - v^n \, {}_n p_x \, \ddot{a}_{x+n} \qquad a_{x:\overline{n}|} = a_x - v^n \, {}_n p_x \, a_{x+n}$$

$$a_{x:\overline{n}|} = \ddot{a}_{x:\overline{n}|} - 1 + v^n \, {}_n p_x \qquad \ddot{a}_{x:\overline{n}|} = 1 + a_{x:\overline{n-1}|}$$

$$a_x = v \, p_x \, \ddot{a}_{x+1} \qquad a_{x:\overline{n}|} = v \, p_x \, \ddot{a}_{x+1:\overline{n}|}$$

Premium conversion formulae

$$A_x = 1 - d\ddot{a}_x \qquad A_{x:\overline{n}|} = 1 - d\ddot{a}_{x:\overline{n}|}$$

$$\bar{A}_x = 1 - \delta\bar{a}_x \qquad \bar{A}_{x:\overline{n}|} = 1 - \delta\bar{a}_{x:\overline{n}|}$$

Annuities payable m times a year

The expected present value of an immediate life annuity of 1 *pa*, payable m times a year to a life aged x is:

$$a_x^{(m)} = \frac{1}{m}\sum_{t=1}^{\infty} v^{t/m} \, {}_{t/m}p_x \approx a_x + \frac{m-1}{2m}$$

The corresponding annuity-due has expected present value:

$$\ddot{a}_x^{(m)} = \frac{1}{m}\sum_{t=0}^{\infty} v^{t/m} \, {}_{t/m}p_x \approx \ddot{a}_x - \frac{m-1}{2m}$$

This page has been left blank so that you can keep the chapter summaries together for revision purposes.

Chapter 4 Solutions

Solution 4.1

The value of the term assurance is:

$$A^{1}_{40:\overline{25|}} = A_{40} - v^{25}\ {}_{25}p_{40}\ A_{65}$$

$$= 0.23056 - \frac{1}{1.04^{25}} \times \frac{8,821.2612}{9,856.2863} \times 0.52786$$

$$= 0.05334$$

The value of the endowment assurance is:

$$A_{30:\overline{25|}} = A^{1}_{30:\overline{25|}} + A_{30:\overline{25|}}^{\ \ \ \ \ 1}$$

$$= A_{30} - v^{25}\ {}_{25}p_{30}\ A_{55} + v^{25}\ {}_{25}p_{30}$$

$$= 0.16023 + \frac{1}{1.04^{25}} \times \frac{9,557.8179}{9,925.2094} \times (1 - 0.38950)$$

$$= 0.38076$$

The value of the endowment assurance with the death benefit payable immediately on death is:

$$\bar{A}_{30:\overline{25|}} \approx 1.04^{½}\, A_{30:\overline{25|}} + \left(1 - 1.04^{½}\right) A_{30:\overline{25|}}^{\ \ \ \ \ 1}$$

$$= 1.04^{½} \times 0.38076 + \left(1 - 1.04^{½}\right) \times \frac{1}{1.04^{25}} \times \frac{9,557.8179}{9,925.2094}$$

$$= 0.38115$$

Remember that it is only the death benefit part that gets multiplied by the acceleration factor. We have used the $(1+i)^{½}$ *approximation here, but the* $1 + ½ i$ *or* $\frac{i}{\delta}$ *approximation would have been acceptable too.*

Another way to calculate the value of the benefit is as follows:

$$\begin{aligned}\bar{A}_{30:\overline{25|}} &= \bar{A}^{1}_{30:\overline{25|}} + A_{30:\overset{1}{\overline{25|}}} \\ &\approx 1.04^{1/2} A^{1}_{30:\overline{25|}} + A_{30:\overset{1}{\overline{25|}}} \\ &= 1.04^{1/2}\left(0.16023 - \frac{1}{1.04^{25}} \times \frac{9,557.8179}{9,925.2094} \times 0.38950\right) \\ &\quad + \frac{1}{1.04^{25}} \times \frac{9,557.8179}{9,925.2094} \\ &= 0.38115\end{aligned}$$

Solution 4.2

The variance of the present value is:

$$50,000^2\left[{}^2A^{1}_{40:\overline{25|}} - \left(A^{1}_{40:\overline{25|}}\right)^2\right]$$

From Solution 4.1 we know that $A^{1}_{40:\overline{25|}} = 0.05334$.

We can calculate ${}^2A^{1}_{40:\overline{25|}}$ as follows:

$$\begin{aligned}{}^2A^{1}_{40:\overline{25|}} &= {}^2A_{40} - \left(v^2\right)^{25} \times {}_{25}p_{40} \times {}^2A_{65} \\ &= 0.06792 - \frac{1}{1.04^{50}} \times \frac{8,821.2612}{9,856.2863} \times 0.30855 \\ &= 0.02906\end{aligned}$$

So the variance of the present value is:

$$50,000^2\left[0.02906 - 0.05334^2\right] = 65,543,115 = (£8,096)^2$$

Solution 4.3

Starting with the definition of $a_{x:\overline{n}|}$, we have:

$$a_{x:\overline{n}|} = v\,p_x + v^2{}_2p_x + \ldots + v^n{}_np_x$$

Now take out the factor $v\,p_x$. For example:

$$v^2{}_2p_x = \left(v\,p_x\right)\left(v\,p_{x+1}\right)$$

$$v^3{}_3p_x = \left(v\,p_x\right)\left(v^2\,{}_2p_{x+1}\right)$$

and so on.

Therefore:

$$\begin{aligned} a_{x:\overline{n}|} &= v\,p_x\left(1 + v\,{}_1p_{x+1} + v^2{}_2p_{x+1} + \ldots + v^{n-1}{}_{n-1}p_{x+1}\right) \\ &= v\,p_x\,\ddot{a}_{x+1:\overline{n}|} \end{aligned}$$

Solution 4.4

In terms of whole life annuities-due:

$$\ddot{a}_{x:\overline{n}|} = \ddot{a}_x - v^n\,{}_np_x\,\ddot{a}_{x+n}$$

Solution 4.5

In Chapter 2 we saw that:

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k\,{}_kp_x = 1 + v\,p_x + v^2\,{}_2p_x + \cdots + v^{n-1}\,{}_{n-1}p_x$$

and:

$$a_{x:\overline{n}|} = \sum_{k=1}^{n} v^k\,{}_kp_x = v\,p_x + v^2\,{}_2p_x + \cdots + v^n\,{}_np_x$$

So:

$$\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n}|} = 1 - v^n \,{}_n p_x$$

and the result follows.

Solution 4.6

The value of the temporary annuity-due is:

$$\begin{aligned}\ddot{a}_{60:\overline{10}|} &= \ddot{a}_{60} - v^{10}\,{}_{10}p_{60}\,\ddot{a}_{70} \\ &= 11.891 - \frac{1}{1.06^{10}} \times \frac{8,054.0544}{9,287.2164} \times 9.140 \\ &= 7.465\end{aligned}$$

and the value of the temporary continuous annuity is:

$$\begin{aligned}\bar{a}_{60:\overline{10}|} &= \bar{a}_{60} - v^{10}\,{}_{10}p_{60}\,\bar{a}_{70} \\ &= \left(\ddot{a}_{60} - ½\right) - v^{10}\,{}_{10}p_{60}\left(\ddot{a}_{70} - ½\right) \\ &= \ddot{a}_{60:\overline{10}|} - ½\left(1 - v^{10}\,{}_{10}p_{60}\right) \\ &= 7.465 - ½\left(1 - \frac{1}{1.06^{10}} \times \frac{8,054.0544}{9,287.2164}\right) \\ &= 7.207\end{aligned}$$

Solution 4.7

$a_{30} = \ddot{a}_{30} - 1 = 20.834$

$${}_{10|}a_{30} = \frac{D_{40}}{D_{30}} a_{40} = \frac{2,052.96}{3,060.13} \times 19.005 = 12.750$$

$a_{70} = 9.375$

$${}_{10|}a_{70} = \frac{D_{80}}{D_{70}} a_{80} = \frac{228.48}{517.23} \times 5.818 = 2.570$$

Comments

a_{30} is much bigger than a_{70} since the income is expected to be paid for a much longer period.

${}_{10|}a_{30}$ is lower than a_{30}, because no payments are made during the first 10 years, and since these payments are most likely to made (and are discounted least), the difference is quite close to 10.

${}_{10|}a_{70}$ is lower than a_{70} for the same reason. The difference is less here ($9.375 - 2.570 < 20.834 - 12.750$) since the "missing" payments are less likely to be made.

Solution 4.8

$$\ddot{a}_{[55]+1:\overline{4}|} = 1 + v\, p_{[55]+1}\, \ddot{a}_{57:\overline{3}|} = 1 + \frac{1}{1.04}(1 - 0.004903) \times 2.870 = 3.746$$

The value of $\ddot{a}_{57:\overline{3}|}$ *is given on Page 100 of the Tables.*

Solution 4.9

The expected present value of the benefit is:

$$EPV = 5{,}000 \times \frac{D'_{65}}{D'_{50}} \times \ddot{a}''_{65}$$

where $\dfrac{D'_{65}}{D'_{50}}$ is calculated using AM92 Ultimate mortality, and $\ddot{a}''_{65}$ uses PMA92C20 mortality. Therefore:

$$EPV = 5{,}000 \times \frac{689.23}{1{,}366.61} \times 13.666 = £34{,}461$$

Solution 4.10

From the AM92 table with 4% *pa* interest:

$$A_{65} = 0.52786$$

$$\ddot{a}_{65} = 12.276$$

The RHS of the premium conversion formula is:

$$1 - d\ddot{a}_{65} = 1 - \frac{0.04}{1.04} \times 12.276 = 0.52785$$

The slight difference between this and the *Tables* value of $A_{65} = 0.52786$ has occurred because of a rounding error.

Solution 4.11

We have:

$$a_{90}^{(4)} = \frac{1}{4}\sum_{t=1}^{\infty} v^{t/4} \frac{l_{90+t/4}}{l_{90}} = \frac{1}{4}\sum_{t=1}^{\infty} v^{t/4} \left(\frac{100 - 2.5t}{100}\right)$$

Solution 4.12

Let's think about an example. Suppose that $t = 3$ and $m = 4$. Then the annuity-due is deferred for ¾ of a year. So there are payments of £1 at times ¾, 1¾, 2¾, *etc*. In symbols we have:

$$_{\frac{3}{4}|}\ddot{a}_x = v^{\frac{3}{4}}\ {}_{\frac{3}{4}}p_x + v^{1\frac{3}{4}}\ {}_{1\frac{3}{4}}p_x + v^{2\frac{3}{4}}\ {}_{2\frac{3}{4}}p_x + \cdots$$

Now the EPV of £1 at time ¾ is very similar to the EPV of £¼ paid at time 0, plus the EPV of ¾ paid at time 1, *ie*:

$$v^{\frac{3}{4}}\ {}_{\frac{3}{4}}p_x \approx \tfrac{1}{4} + \tfrac{3}{4} v\, p_x$$

Similarly, the EPV of £1 at time 1¾ is very similar to the EPV of £¼ paid at time 1, plus the EPV of ¾ paid at time 2, *ie*:

$$v^{1\frac{3}{4}}\ {}_{1\frac{3}{4}}p_x \approx \tfrac{1}{4} v\, p_x + \tfrac{3}{4} v^2\ {}_2p_x$$

And so on. Adding up all these payments we get:

$$\begin{aligned} {}_{\frac{3}{4}|}\ddot{a}_x &\approx \left(\tfrac{1}{4} + \tfrac{3}{4} v\, p_x\right) + \left(\tfrac{1}{4} v\, p_x + \tfrac{3}{4} v^2\ {}_2p_x\right) + \cdots \\ &= \tfrac{1}{4} + v\, p_x + v^2\ {}_2p_x + \cdots \\ &= \ddot{a}_x - \tfrac{3}{4} \end{aligned}$$

In general, we have:

$$_{t/m|}\ddot{a}_x \cong \ddot{a}_x - \frac{t}{m}$$

Solution 4.13

We have:

$$\ddot{a}_{60}^{(2)} = \ddot{a}_{60} - \frac{1}{4} = 14.134 - 0.25 = 13.884$$

$$a_{60}^{(12)} = a_{60} + \frac{11}{24} = 13.134 + \frac{11}{24} = 13.592$$

and:

$$\begin{aligned}
\ddot{a}_{50:\overline{15}|}^{(4)} &= \ddot{a}_{50}^{(4)} - v^{15}\,{}_{15}p_{50}\,\ddot{a}_{65}^{(4)} \\
&= \left(\ddot{a}_{50} - \frac{3}{8}\right) - v^{15}\,{}_{15}p_{50}\left(\ddot{a}_{65} - \frac{3}{8}\right) \\
&= \ddot{a}_{50:\overline{15}|} - \frac{3}{8}\left(1 - \frac{D_{65}}{D_{50}}\right) \\
&= 11.253 - \frac{3}{8}\left(1 - \frac{689.23}{1,366.61}\right) \\
&= 11.067
\end{aligned}$$

Solution 4.14

Standard life

The expected present value of the benefit is:

$$10{,}000 A^{1}_{40:\overline{2}|} = 10{,}000\left(A_{40} - \frac{D_{42}}{D_{40}} A_{42} \right)$$

$$= 10{,}000\left(0.23056 - \frac{1{,}894.37}{2{,}052.96} \times 0.24787 \right)$$

$$= £18.38$$

Impaired life

From first principles, we can write the expected present value of the term assurance as:

$$10{,}000\left(v q^{*}_{40} + v^2 p^{*}_{40} q^{*}_{41} \right)$$

where * denotes impaired mortality. Now:

$$p^{*}_{40} = \exp\left(-\int_0^1 \mu^{*}_{40+t}\, dt \right) = \exp\left(-\int_0^1 5\mu_{40+t}\, dt \right) = \left(p_{40} \right)^5$$

So the expected present value of the benefit payable to the impaired life is:

$$10{,}000\left(v\left[1 - \left(p_{40} \right)^5 \right] + v^2 \left(p_{40} \right)^5 \left[1 - \left(p_{41} \right)^5 \right] \right)$$

$$= 10{,}000\left(\frac{1}{1.04} \times \left(1 - 0.999063^5 \right) + \frac{1}{1.04^2} \times \left(0.999063^5 \right)\left(1 - 0.998986^5 \right) \right)$$

$$= £91.53$$

Solution 4.15

We can write:

$$\bar{A}_{50:\overline{2|}} = \int_0^2 v^t \, {}_t p_{50} \mu_{50+t} \, dt + v^2 \, {}_2 p_{50}$$

$$= \int_0^1 v^t \, {}_t p_{50} \mu_{\overline{50}} \, dt + v p_{50} \int_0^1 v^t \, {}_t p_{51} \mu_{\overline{51}} \, dt + v^2 \, {}_2 p_{50}$$

where we have written $\mu_{\bar{x}}$ to indicate the assumed constant force of mortality operating between integer ages x and $x+1$.

Now:

$$\mu_{\overline{50}} = -\ln p_{50} = -\ln 0.997492 = 0.00251115$$

$$\mu_{\overline{51}} = -\ln p_{51} = -\ln 0.997191 = 0.00281295$$

$$\delta = \ln 1.04$$

$$\int_0^1 v^t \, {}_t p_{50} \mu_{\overline{50}} \, dt = \mu_{\overline{50}} \int_0^1 e^{-(\delta + \mu_{\overline{50}})t} \, dt = \frac{\mu_{\overline{50}}}{\delta + \mu_{\overline{50}}} \left[1 - e^{-(\delta + \mu_{\overline{50}})} \right] = 0.00245947$$

$$\int_0^1 v^t \, {}_t p_{51} \mu_{\overline{51}} \, dt = \frac{\mu_{\overline{51}}}{\delta + \mu_{\overline{51}}} \left[1 - e^{-(\delta + \mu_{\overline{51}})} \right] = 0.00275465$$

So:

$$\bar{A}_{50:\overline{2|}} = 0.00245947 + \frac{1}{1.04} \times 0.997492 \times 0.00275465$$

$$+ \frac{1}{1.04^2} \times 0.997492 \times 0.997191$$

$$= 0.924748$$

Chapter 5

Net premiums and reserves

Syllabus objectives

(i) Define simple assurance and annuity contracts, and develop formulae for the means and variances of the present values of the payments under these contracts, assuming constant deterministic interest.

9. Define the expected accumulation of the benefits in objective (i) 1, and obtain expressions for them corresponding to the expected present values in objectives (i) 3 and 4. [Note: expected values only.]

(ii) Describe and use practical methods of evaluating expected values and variances of the simple contracts defined in objective (i).

6. Evaluate the expected accumulations in objective (i) 9.

(iii) Describe and calculate, using ultimate or select mortality, net premiums and net premium reserves of simple insurance contracts.

1. Define the net random future loss under an insurance contract, and state the principle of equivalence.

2. Define and calculate net premiums for the insurance contract benefits in objective (i) 1. Regular premiums and annuity benefits may be payable annually, more frequently than annually, or continuously. Death benefits may be payable at the end of the year of death, or immediately on death.

3. State why an insurance company will set up reserves.

4. Describe prospective and retrospective reserves.

Continued…

5. *Define and evaluate prospective and retrospective net premium policy reserves in respect of the contracts in (i) 1, with premiums as in (iii)2.*

6. *Show that prospective and retrospective reserves are equal when calculated on the same basis.*

7. *Obtain recursive relationships between net premium reserves at annual intervals, for contracts with death benefits paid at the end of the year of death, and annual premiums.*

8. *Define and calculate, for a single policy or a portfolio of policies (as appropriate):*

   – *death strain at risk*

   – *expected death strain*

   – *actual death strain*

   – *mortality profit*

   for policies with death benefits payable immediately on death or at the end of the year of death; for policies paying annuity benefits at the start of the year or on survival to the end of the year; and for policies where single or annual premiums are payable.

0 Introduction

In Subject CT1, Financial Mathematics, payments were generally treated as *certain* to be paid. The equation of value where payments are certain has already been introduced there. In most actuarial contexts some or all of the cashflows in a contract are uncertain, depending on the death or survival (or possibly the state of health) of a life.

We therefore extend the concept of the equation of value to deal with this uncertainty, by equating expected present values of uncertain cashflows. The equation of expected present values for a contract, usually referred to as the equation of value, is as follows:

Equation of value

The expected present value of the income

= The expected present value of the outgo

Alternatively, this is referred to as the *principle of equivalence*.

The income to a life insurer comes from the payments made by policyholders, called the *premiums*. The outgo arises from benefits paid to policyholders and expenses of the insurer.

Given a suitable set of assumptions, which we call the *basis*, we may use the equation of value to calculate the premium or premiums, which a policyholder must make in return for a given benefit.

This is sometimes referred to as calculating a premium using the principle of equivalence.

We may also calculate the amount of benefit payable for a given premium.

A basis is a set of assumptions regarding expected future experience, *eg*:

- mortality experience
- investment returns
- future expenses
- bonus rates.

We will see later that there is an alternative method of determining the appropriate premium, called "profit testing". This approach also requires a similar set of assumptions.

We may also use the equation of value when a policyholder wishes to adjust the terms of a contract after it is effected – for example by changing the contract term.

Extreme examples of such adjustments are if the policyholder *surrenders* the policy (*ie* they elect to terminate the policy by "cashing it in") or if the policyholder decides to stop paying premiums and accept a reduction in benefits, which is known as making the policy *paid-up*.

In the remainder of this chapter we will concentrate on net premiums and the associated net premium reserves (*ie* we will ignore actual premiums and the associated gross premium reserves). Although the relevance of the net premium valuation in the UK has diminished in recent years, primarily on account of regulatory changes, it continues to play a significant role in territories other than the UK and therefore remains an important part of this subject. Gross premium valuations, on which much greater emphasis is now placed in the UK, are covered in Chapter 7.

1 The basis

The basis for applying the equation of value for a life insurance contract will specify the mortality and interest rates to be assumed.

Usually the assumptions will not be the best estimates we can find of the individual basis elements, but will be more cautious than the best estimates.

For example, if we expect to earn a rate of interest of 8% *pa* on the invested premiums, we may calculate the premiums assuming we earn only 6% *pa*. As we are assuming that we earn less interest than we really expect, then the premiums calculated will be higher than they would need to be if the expected rate of 8% were actually earned.

The size of the appropriate margin to incorporate will depend upon a number of factors, such as:

- The level of risk that the insurer is undertaking. The higher the risk the bigger the margin.
- The purpose of the investigation. Different purposes will have different levels of caution. For example, when setting statutory reserves (*ie* those required by legal regulations) the basis will have more of a margin than if the purpose is to calculate the purchase price of the insurer (*ie* the amount that another company should pay if it wants to take over the insurance company).

Question 5.1

Why will the margin be higher for statutory purposes? What margin should be incorporated if another insurance company is buying this one?

Two reasons for an element of caution in the basis are:

1. **To allow a contingency margin, to ensure a high probability that the premiums plus interest income meet the cost of benefits, allowing for random variation. In other words, to ensure a high probability of making a profit.**

2. **To allow for uncertainty in the estimates themselves.**

2 Premiums

2.1 Frequency of payment

The premium payment arrangement will commonly be one of the following:

- **A *single premium* contract, under which benefits are paid for by a single lump sum premium paid at the time the contract is effected. This payment is certain, so that the left hand side of the equation of value is the certain payment, not the expected value of a payment.**
- **An *annual premium* contract, under which benefits are paid for by a regular annual payment of a level amount, the first premium being due at the time the contract is effected. Premiums continue to be paid until the end of some agreed maximum premium term, often the same as the contract term, or until the life dies if this is sooner. Therefore, there would not usually be a premium payment at the end of the contract term.**
- **A *true* m*thly premium* contract, under which benefits are paid for by m level payments made every $1/m$ years. As in the annual premium case, premiums continue to be paid until the end of some agreed maximum premium term or until the life dies if this is sooner. Again, there would not usually be a premium paid at the end of the contract term. Often the premium is paid monthly (that is, $m = 12$). For some types of contract, weekly premiums are possible.**

The use of the word "true" here refers to a technical distinction (between so-called "true" and "instalment" premiums), which we needn't worry about here.

***Premiums are always paid in advance*, so the first payment is always due at the time the policy is effected.**

Given a basis specifying mortality and interest to be assumed, and given details of the benefits to be purchased, we can use the equation of value to calculate the premium payable.

2.2 Examples

Example

To calculate the purchase price of an annuity we need a mortality assumption, which would be based on the age and sex of the policyholder and a suitable mortality table.

We would also need an interest rate to discount the future payments and an assumption about the level of future expenses.

Example

Calculate the annual premium for a term assurance with a term of 10 years to a male aged 30, with a sum assured of £500,000, assuming AM92 Ultimate mortality and interest of 4% *pa*. Assume that the death benefit is paid at the end of the year of death.

Solution

$$\text{EPV benefits} = 500{,}000\, A^{1}_{30:\overline{10|}}$$

$$= 500{,}000\left(A_{30} - v^{10}\, {}_{10}p_{30}\, A_{40}\right)$$

$$= 500{,}000\left(0.16023 - 1.04^{-10} \times \frac{9{,}856.2863}{9{,}925.2094} \times 0.23056\right)$$

$$= 2{,}776.77$$

$$\text{EPV premiums} = P\ddot{a}_{30:\overline{10|}}$$

$$= P\left(\ddot{a}_{30} - v^{10}\, {}_{10}p_{30}\, \ddot{a}_{40}\right)$$

$$= P\left(21.834 - 1.04^{-10} \times \frac{9{,}856.2863}{9{,}925.2094} \times 20.005\right)$$

$$= 8.41319P$$

So:

$$P = \frac{2{,}776.77}{8.41319} = £330.05$$

3 The net premium

3.1 Definition

The *net premium* is the amount of premium required to meet the expected cost of the assurance or annuity benefits under a contract, given mortality and interest assumptions.

So, in the second example on the previous page, we calculated the "net premium" because we did not incorporate any allowance for the insurance company's expenses.

The net premium is also sometimes referred to as the pure premium or the risk premium.

The net premium for a contract, given suitable mortality and interest assumptions, is found from the equation of expected present value:

Equation for net premium

The expected present value of the net premium income

= The expected present value of the outgo on benefits

If you're wondering why we would *want* to calculate a net premium, when it doesn't allow for one of the elements of outgo (the expenses), we'll be discussing this in Section 7.

3.2 Notation

We have some standard notation for the net premiums of common life insurance contracts, related to the notation for the expected present value of the assurance benefits, which the net premium is to pay for:

$P_{x:\overline{n|}}$ **is the net premium payable annually in advance throughout the duration of the contract for an endowment assurance issued to a life aged x with term n years, under which the sum assured is 1, payable at the end of the year of death or at maturity.**

From the net premium definition:

$$P_{x:\overline{n}|} = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}$$

Question 5.2

Calculate $P_{30:\overline{10}|}$ and $P_{30:\overline{15}|}$, based on AM92 Ultimate mortality at 4% *pa* interest.

Explain the difference between the two figures.

If the death benefit is instead payable immediately on death then the net premium payable annually in advance is denoted by:

$$P\left(\bar{A}_{x:\overline{n}|}\right)$$

and:

$$P\left(\bar{A}_{x:\overline{n}|}\right) = \frac{\bar{A}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}$$

If the premium is instead payable m-thly per annum in advance, this would be denoted by placing a superscript (m) above the P and a functions. So, assuming again that the death benefit is payable at the end of the year of death, for example, we would have:

$$P^{(m)}_{x:\overline{n}|} = \frac{A_{x:\overline{n}|}}{\ddot{a}^{(m)}_{x:\overline{n}|}}$$

Note that $P^{(m)}_{x:\overline{n}|}$ denotes the sum of the m annual payments. This should become clearer by working through the following example.

Example

A life aged exactly 50 buys a 15-year endowment assurance policy with a sum assured of £50,000 payable on maturity or at the end of the year of earlier death. Level premiums are payable monthly in advance. Calculate the monthly premium assuming AM92 Ultimate mortality and 4% *pa* interest. Ignore expenses.

Solution

Let P denote the monthly premium. In the notation given above, $P = \frac{1}{12} P^{(12)}_{50:\overline{15}|}$.

The expected present value of the premiums is:

$$12P\ddot{a}^{(12)}_{50:\overline{15}|} = 12P\left[\ddot{a}_{50:\overline{15}|} - \frac{11}{24}\left(1 - \frac{D_{65}}{D_{50}}\right)\right]$$

$$= 12P\left[11.253 - \frac{11}{24}\left(1 - \frac{689.23}{1,366.61}\right)\right]$$

$$= 132.310P$$

We have used the formula on Page 36 of the Tables in this calculation.

The expected present value of the benefits is:

$$50,000A_{50:\overline{15}|} = 50,000 \times 0.56719 = 28,359.5$$

So the monthly premium is:

$$P = \frac{28,359.5}{132.310} = £214.34$$

If the premium is payable continuously then, assuming again that the death benefit is payable at the end of the year of death, for example, we would have:

$$\bar{P}_{x:\overline{n}|}$$

and:

$$\bar{P}_{x:\overline{n}|} = \frac{A_{x:\overline{n}|}}{\bar{a}_{x:\overline{n}|}}$$

The final main variation would be to assume select, rather than ultimate, mortality. Then the above expressions apply but with x replaced by $[x]$.

$$P_{[x]:\overline{n}|}$$

is the net premium payable annually in advance throughout the duration of the contract for an endowment assurance issued to a life aged x with term n years, under which the sum assured is 1, payable at the end of the year of death or at maturity, where select mortality is assumed:

$$P_{[x]:\overline{n}|} = \frac{A_{[x]:\overline{n}|}}{\ddot{a}_{[x]:\overline{n}|}}$$

The above notation is flexible enough for the relevant symbols to be unambiguously combined in different ways when the benefits and premiums have alternative combinations of timings to those shown above. A full description of the approach to this notation can be found, for example, in the "International Notation Section" of the *Formulae and Tables for Examinations* book.

Without loss of generality, we now show the corresponding notation for other simple insurance contracts assuming ultimate mortality, annual premiums in advance, and death benefits payable at the end of the year of death. We leave it to the reader to provide the alternative combinations in a similar manner to those outlined above.

$P^{1}_{x:\overline{n}|}$ is the net premium payable annually in advance throughout the duration of the contract for a term assurance issued to a life aged x with term n years, under which the sum assured is 1, payable at the end of the year of death.

From the net premium definition:

$$P^{1}_{x:\overline{n|}} = \frac{A^{1}_{x:\overline{n|}}}{\ddot{a}_{x:\overline{n|}}}$$

P_x is the net premium payable annually in advance throughout the duration of the contract for a whole life assurance issued to a life aged x, under which the sum assured is 1, payable at the end of the year of death, and where ultimate mortality is assumed.

From the net premium definition:

$$P_x = \frac{A_x}{\ddot{a}_x}$$

For some contracts, premiums are only paid during the first few years of the policy.

${}_tP_x$ is the net premium payable annually in advance for a maximum of t years (or until earlier death) for a whole life assurance issued to a life aged x, under which the sum assured is 1, payable at the end of the year of death, and where ultimate mortality is assumed.

Note that the P's in these symbols are capital letters. Don't mix this symbol up with the probability ${}_tp_x$, which is written with a small p.

From the net premium definition:

$${}_tP_x = \frac{A_x}{\ddot{a}_{x:\overline{t|}}}$$

Question 5.3

Without working them out:

(i) Which is bigger, ${}_5P_{40}$ or ${}_{10}P_{40}$, and why?

(ii) Which is bigger , ${}_5p_{40}$ or ${}_{10}p_{40}$, and why?

4 The insurer's loss random variable

The loss incurred by the insurer on a particular policy will depend on the future lifetime of the policyholder. From the equation of value, we can calculate the premium that sets the expected present value of the insurer's loss equal to 0. However, given any level of premium, we can derive formulae for the expectation and variance of the present value of the loss. Since:

$$PV \text{ loss} = PV \text{ outgo} - PV \text{ income}$$

it follows that:

$$EPV \text{ profit } = EPV \text{ income } - EPV \text{ outgo}$$

Finding an expression for the expected present value of the insurer's loss is straightforward since the premiums and benefits can be dealt with separately. Deriving a formula for the variance requires more work, as we see in the example below.

Example

Derive a formula for the variance of the profit earned by an insurance company offering an n-year endowment assurance policy to lives aged x. Assume that premiums are payable annually in advance and death benefits are payable at the end of the year of death.

Solution

Let P denote the premium and let S denote the sum assured.

The actual profit to the insurer will be:

$$X = \text{PV premiums } - \text{PV benefits}$$

If the curtate future lifetime of a policyholder is K years, then the present value of the premiums received is $P\ddot{a}_{\overline{\min\{K+1,n\}}|}$.

In addition, the present value of the benefits paid is $S\,v^{\min\{K+1,n\}}$.

Since $\ddot{a}_{\overline{\min\{K+1,n\}}|} = \dfrac{1-v^{\min\{K+1,n\}}}{d}$, it follows that:

$$X = P\left(\frac{1-v^{\min\{K+1,n\}}}{d}\right) - S\,v^{\min\{K+1,n\}} = \frac{P}{d} - \left(\frac{P}{d}+S\right)v^{\min\{K+1,n\}}$$

So:

$$\text{var}(X) = \left(\frac{P}{d}+S\right)^2 \text{var}\left(v^{\min\{K+1,n\}}\right)$$

From Chapter 1, we know that:

$$\text{var}\left(v^{\min\{K+1,n\}}\right) = {}^2A_{x:\overline{n}|} - \left(A_{x:\overline{n}|}\right)^2$$

This gives us the formula:

$$\text{var}(X) = \left(\frac{P}{d}+S\right)^2 \left[{}^2A_{x:\overline{n}|} - \left(A_{x:\overline{n}|}\right)^2\right]$$

The derivation is similar for different types of benefit.

Question 5.4

Derive a formula for the variance of the insurer's profit on a whole life assurance policy issued to a life aged x. Assume that level premiums are payable annually in advance and the sum assured is payable at the end of the year of death.

Question 5.5

A life aged exactly 33 purchases a whole life assurance policy with a sum assured of £40,000 payable at the end of the year of death. Premiums of £520 are payable annually in advance. Calculate the variance of the insurer's profit on this contract, assuming AM92 Ultimate mortality and 4% *pa* interest.

5 Retrospective accumulations

This section will provide us with a valuable grounding in the distinction between the terms prospective and retrospective. This will be useful when we come to look at retrospective reserves in Section 6.4.

In mathematics of finance there are two common viewpoints from which a stream of cashflows may be considered.

(1) Prospectively, leading to the calculation of present values

(2) Retrospectively, leading to the calculation of accumulations.

In this section we discuss the latter approach allowing for the presence of mortality.

The "retrospective accumulation" can be thought of as the "pot of money" accumulated in respect of a policy, *ie* premiums paid, plus interest, less expenses, less the cost of life cover. The amount is often referred to as the "asset share" of a policy.

The basic idea is that we consider a *group* of lives, who are regarded as identical and stochastically independent as far as mortality is concerned. At age x, each life transacts an identical life insurance contract. Under these contracts, payments will be made (the direction of the payments is immaterial), depending on the experience of the members of the group. We imagine these payments being accumulated in a fund at rate of interest i. After n years, we divide this fund equally among the surviving members of the group. (If the fund is negative we imagine charging the survivors in equal shares.) The question is, what is the *expected share of the fund per survivor*?

Question 5.6

A fund of £1,000,000 has 10,000 members aged 40. The fund accumulates at an interest rate of 4% per annum and will be divided by all members who survive to age 60.

Based on AM92 Ultimate mortality, what is the expected payout for each survivor?

In the solution to this question we divided the projected fund (\$2,191,123) by the *expected* number of survivors (9,422.63). However, the number of survivors, N say, is actually a random variable. So logically, we should be looking at the expected value of $\dfrac{2{,}191{,}123}{N}$, not $\dfrac{2{,}191{,}123}{E(N)}$.

From Subject CT3, you should know that, in general, $E\left(\dfrac{2{,}191{,}123}{N}\right) \neq \dfrac{2{,}191{,}123}{E(N)}$. So we need to justify this approach. The example below shows that, if we are dealing with large numbers of people, this approximation is valid.

5.1 Pure endowment

We illustrate with the simplest example of a pure endowment contract. Suppose we begin with *N* people in the group. After *n* years, a payment of 1 is made to each survivor. The number of survivors is, we suppose, a random variable. If there are N^1 survivors ($N^1 > 0$) then the fund is N^1 and the share of the fund per survivor is simply 1. We deal with the awkward possibility that $N^1 = 0$ by supposing N to be large enough that the probability that $N^1 = 0$ vanishes.

Note that N^1 denotes the number of survivors. It is not N raised to the power 1.

Let us now formalise the above. Suppose that there are N^1 survivors at age $x+n$ out of *N* "starters" at age *x*, and that the accumulated fund at age $x+n$ is $F(N)$. The retrospective accumulation of the benefit under consideration is defined to be:

$$\lim_{N\to\infty} \frac{F(N)}{N^1}$$

Clearly N^1 and $F(N)$ are random variables. The process of taking the limit eliminates the probability that $N^1 = 0$, but also since:

$$\lim_{N\to\infty} \frac{F(N)}{N} = E[F(1)]$$

$$\lim_{N\to\infty} \frac{N^1}{N} = {}_np_x$$

and so **(by the law of large numbers) the limit** of $\dfrac{F(N)}{N^1}$ **is equal to** $\dfrac{E[F(1)]}{{}_np_x}$.

This proof justifies the method we used in Question 5.6.

5.2 Term assurance

For an example less trivial than the pure endowment, consider a term assurance with term n years. We need only consider a single life and calculate $E[F(1)]$. It is easy to see that $F(1)$ has the following distribution:

$$F(1) = (1+i)^{n-(k+1)} \quad \text{if } K_x = k \qquad (k = 0,1,\ldots,n-1)$$

$$F(1) = 0 \quad \text{if } K_x \geq n$$

Note that $F(1) = 0$ if you survive since there is no survival benefit for a term assurance contract. Here we've accumulated the death benefit (paid at time $k+1$) to the end of the term (time n). So:

$$E[F(1)] = \sum_{k=0}^{n-1} (1+i)^{n-(k+1)} {}_{k|}q_x = (1+i)^n A^{1}_{x:\overline{n}|}$$

Hence the accumulation of the term assurance benefit is:

$$\frac{(1+i)^n A^{1}_{x:\overline{n}|}}{{}_np_x}$$

This can also be written as $A^{1}_{x:\overline{n}|} \dfrac{D_x}{D_{x+n}}$.

Question 5.7

What would the accumulation be after t years (where $t < n$)?

Question 5.8

John, aged exactly 35, buys a term assurance policy that pays a benefit of £100,000 at the end of the year of his death if he dies before age 65. What is the expected accumulated value of the benefits at time 10?

Basis: AM92 Ultimate, 6% *pa* interest

We now look at one more example of an accumulation, this time for an annuity.

5.3 Annuity

Consider an annuity-due with a term of n years. $F(1)$ has the following distribution:

$$F(1) = \begin{cases} (1+i)^{n-(k+1)}\,\ddot{s}_{\overline{k+1|}} & \text{if } K_x = k \quad (k = 0, 1, \ldots, n-1) \\ \ddot{s}_{\overline{n|}} & \text{if } K_x \geq n \end{cases}$$

Hence:

$$E[F(1)] = \sum_{k=0}^{n-1} (1+i)^{n-(k+1)}\,\ddot{s}_{\overline{k+1|}}\;{}_{k|}q_x \;+\; \ddot{s}_{\overline{n|}}\;{}_n p_x$$

$$= (1+i)^n \left(\sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1|}}\;{}_{k|}q_x \;+\; \ddot{a}_{\overline{n|}}\;{}_n p_x \right)$$

$$= (1+i)^n\,\ddot{a}_{x:\overline{n|}}$$

Therefore the accumulation of the annuity-due is:

$$\frac{(1+i)^n\,\ddot{a}_{x:\overline{n|}}}{{}_n p_x}$$

To each annuity EPV there corresponds an accumulation denoted by an "*s*" symbol instead of an "*a*" symbol. Thus in the example above we define:

$$\ddot{s}_{x:\overline{n|}} = \frac{(1+i)^n\,\ddot{a}_{x:\overline{n|}}}{{}_n p_x}$$

and we define symbols for the accumulation of other annuities similarly. There is no actuarial notation for the accumulation of assurance benefits.

Q

Question 5.9

Consider an annuity of 1 *pa* that has expected present value, at the start of the contract, of $\ddot{a}_{x:\overline{n|}}$. Write down an expression for the accumulation to time t $(t < n)$, of the payments made by that date.

It is easy to see that we can write down the accumulation of any benefit after n years in the same way – simply multiply its EPV by $\dfrac{(1+i)^n}{{}_np_x}$. If $i = 4\%$, the accumulation can be obtained more efficiently by multiplying the EPV by the commutation function expression $\dfrac{D_x}{D_{x+n}}$.

6 Reserves

6.1 What is a reserve?

A reserve is money that an insurer sets aside to meet its future payments, *ie* benefits to policyholders, and expenses.

6.2 Prospective reserve

The prospective reserve for a life insurance contract that is in force (that is, has been written but has not yet expired through claim or reaching the end of the term) is defined to be, for a given basis:

Prospective reserve

The prospective reserve is given by:

The expected present value of the future outgo

less

the expected present value of the future income

This is the prospective reserve because it looks forward to the future cashflows of the contract. The prospective reserve is important because if the company holds funds equal to the reserve, and the future experience follows the reserve basis, then, averaging over many policies, the combination of reserve and future income will be sufficient to pay the future liabilities.

This last sentence is very important and forms the foundation of life insurance reserving and hence the rest of this chapter. That is to say that the money an insurer needs to set aside to meet its future payments is the present value of future outgo (benefits plus expenses) less the present value of future income (premiums).

The reserve therefore gives the office a measure of the minimum funds it needs to hold at any point *during* the term of a contract. The process of calculating a reserve is called the *valuation* of the policy.

Reserves are also calculated for other reasons, such as the calculation of surrender values. The insurer may set the surrender value (*ie* the amount paid to the policyholder) by reference to the reserve.

Question 5.10

An annual premium endowment policy is surrendered one year before the maturity date. Explain why the reserve is a sensible surrender value for the insurer to pay.

Another view of the prospective reserve is gained by considering the *net random future loss* (or just " net loss") from a policy that is in force – where the loss, L, is defined to be:

Net future random loss

$L =$ present value of the future outgo – present value of the future income

Now L is a random variable, since both terms are random variables that depend on the policyholder's future lifetime. (If premiums are not being paid, the second term is zero, the first term is a random variable, so L is still a random variable). The reserve may now be defined as:

Relationship between prospective reserve and net future random loss

Prospective reserve $= E(L)$

Example

Consider an n-year endowment assurance with sum assured S and premium P, originally issued to a life aged x. The benefit is payable on maturity or at the end of the year of earlier death. At an integer time t (just before the premium is paid) the net future loss random variable is:

$$L = S\, v^{\min\{K_{x+t}+1,n-t\}} - P\ddot{a}_{\overline{\min\{K_{x+t}+1,n-t\}}|}$$

and the prospective reserve is:

$$E[L] = SA_{x+t:\overline{n-t}|} - P\ddot{a}_{x+t:\overline{n-t}|}$$

6.3 Why hold reserves?

In many life insurance contracts, the expected cost of paying benefits increases over the term of the contract – consider an endowment assurance, for example. The probability that the benefit will be paid in the first few years is small – the life is young and in good health. Later the expected cost increases as the life ages and the probability of a claim by death increases. In the final year the probability of payment is large, since the payment will be made if the life survives the term, and for most contracts (which often mature when the policyholder is in his or her 50s or 60s) the probability of survival is large.

Although on average the cost of the benefit is increasing over the term, the premiums that pay for these benefits are level. This means that the premiums received in the early years of a contract are more than enough to pay the benefits that fall due in those early years. But in the later years, and particularly in the last year of an endowment assurance policy, the premiums are too small to pay for the benefits.

It is therefore prudent for the premiums that are not required in the early years of a contract to be set aside, or reserved, to fund the shortfall in the later years of the contract. While funds are reserved, they are invested so that interest also contributes to the cost of benefits.

If the life insurance company were to spend all the premiums received, perhaps by distributing to shareholders the premiums that were not required to pay benefits, then later in the contract the company may not be able to find the money to pay for the excess of the cost of benefits over the premiums received. The company sets up reserves to ensure (as far as possible) that this does not happen, and that the company remains solvent.

Example

Consider the case of a term assurance sold to a 30-year old British male, with a 10-year term and a sum assured of £500,000. The premium is £48 per month, paid for the 10-year term. The premium is level (£48 per month) but the cost of benefits is increasing, since the policyholder is getting older and UK mortality rates increase after age 30.

For example, at age 30 the expected cost is $500,000\,q_{30}$, but at age 39 it is $500,000\,q_{39}$, which is bigger than $500,000\,q_{30}$.

When designing the products sold by the insurer, the actuary will always try to ensure that in the later years the benefit payments are greater than the premiums payable.

Suppose now that the product is redesigned so that the sum assured is falling over the term of the contract, say by £50,000 per year. The premium is recalculated to be £23 per month.

In that case the average cost of benefits may well be falling over time (not rising as in the previous contract) since the sum assured is falling. For example, at age 30 the expected cost is $500,000\,q_{30}$, but at age 39 it is $50,000\,q_{39}$.

If the premiums are greater than benefits, say in the final few years, then the policyholder may stop paying premiums. The result is that the policyholder will have held the contract for the time period for which the benefit is greater than the premium but not for the subsequent period when premiums exceed benefits.

Question 5.11

Consider the situation described in the last paragraph of the example above. What consequence would this have for the insurer?

6.4 Retrospective reserves

In Section 5 we learnt about retrospective accumulations. It is useful to bear those concepts in mind now as we study retrospective reserves.

The retrospective reserve for a life insurance contract that is in force is defined to be, for a given basis:

Retrospective reserve

The retrospective reserve is given by:

The accumulated value allowing for interest and survivorship of the premiums received to date

less

the accumulated value allowing for interest and survivorship of the benefits and expenses paid to date

Note that this reserve has been calculated by "looking backwards" as "accumulated money in, less accumulated money out".

The accumulated value of benefits, sometimes called the *cost of assurance*, for a life who purchased a life insurance policy at age x, and is now age $x+t$, where the sum assured for the t years of past cover was S, payable at the end of the year of death, is:

$$S\,A^{1}_{x:\overline{t}|}\,(1+i)^{t}\,\frac{l_x}{l_{x+t}}$$

The accumulated premiums for the life, assuming premiums of P per annum payable annually in advance, is:

$$P\ddot{a}_{x:\overline{t}|}\,(1+i)^{t}\,\frac{l_x}{l_{x+t}} = P\ddot{s}_{x:\overline{t}|}$$

The retrospective reserve on a given basis tells us how much the premiums less expenses and claims have accumulated to, averaging over a large number of policies.

Question 5.12

A 10-year term assurance with a sum assured of £500,000 payable at the end of the year of death, is issued to a male aged 30 for a level annual premium of £330.05. Calculate the prospective and retrospective reserves at the end of the fifth year, *ie* just before the sixth premium has been paid, assuming AM92 Ultimate mortality and 4% *pa* interest. Ignore expenses.

6.5 Conditions for equality of prospective and retrospective reserves

Conditions for equality

If:

1. **the retrospective and prospective reserves are calculated on the same basis, and**

2. **this basis is the same as the basis used to calculate the premiums used in the reserve calculation,**

then the retrospective reserve will be equal to the prospective reserve.

In practice, these conditions rarely hold, since the assumptions that are appropriate for the retrospective calculation (for which we use the experienced conditions over the duration of the contract up to the valuation date) are not generally appropriate for the prospective calculation. The prospective reserve will be calculated using assumptions considered suitable for the remainder of the policy term. Also, the assumptions that were considered appropriate at the time the premium was calculated may not be appropriate for the retrospective or prospective reserve some years later.

This last point is important and is worth reiterating. The retrospective reserve is based on actual experience, in terms of interest, expenses, mortality and so on. This will *not* usually be the same as that assumed when the premiums were set. For example:

- The number of people who actually die will differ from the number expected.
- Investment returns will not exactly match what was assumed.
- We may have had margins in our assumptions (as suggested in the Core Reading earlier in this chapter). For example, we may have been expecting 8% interest but assumed only 6%. If we actually did earn 8% then the prospective and retrospective reserves would not be the same.
- The actuary's expectation of future interest rates may have changed, so that now the expected return on money held is 5%.

Question 5.13

How would the situation in the last bullet point above affect each of the retrospective and prospective reserves?

We demonstrate the equality of prospective and retrospective reserves with an example, ${}_tV_x$, the reserve at duration t years for a whole life assurance to a life age x, assuming death benefits payable at the end of the year of death and ultimate mortality.

We are also assuming that the sum assured is 1.

The prospective reserve is:

$$A_{x+t} - P_x\, \ddot{a}_{x+t}$$

and the retrospective reserve is:

$$P_x\, \ddot{s}_{x:\overline{t}|} - \frac{(1+i)^t}{{}_tp_x} A^1_{x:\overline{t}|}$$

The first term in the retrospective reserve is interpreted as the expected accumulation of premiums received, and the second term as the expected accumulated cost of benefits paid.

Start with the prospective reserve:

$$\boldsymbol{{}_tV_x = A_{x+t} - P_x\,\ddot{a}_{x+t}}$$
$$\boldsymbol{= A_{x+t} - P_x\,\ddot{a}_{x+t} - P_x\,\ddot{s}_{x:\overline{t}|} + P_x\,\ddot{s}_{x:\overline{t}|}}$$

Now:

$$\boldsymbol{{}_{t|}\ddot{a}_x = v^t\,{}_tp_x\,\ddot{a}_{x+t}}$$

and:

$$\boldsymbol{\ddot{s}_{x:\overline{t}|} = \frac{(1+i)^t}{{}_tp_x}\ddot{a}_{x:\overline{t}|}}$$

So replacing $\ddot{a}_{x+t}$ by $\dfrac{{}_{t|}\ddot{a}_x}{v^t\,{}_tp_x} = \dfrac{(1+i)^t}{{}_tp_x}\,{}_{t|}\ddot{a}_x$ and $\ddot{s}_{x:\overline{t}|}$ by $\dfrac{(1+i)^t}{{}_tp_x}\ddot{a}_{x:\overline{t}|}$ in the equation for ${}_tV_x$ gives:

$$\boldsymbol{{}_tV_x = P_x\,\ddot{s}_{x:\overline{t}|} - \frac{(1+i)^t}{{}_tp_x}P_x\left({}_{t|}\ddot{a}_x + \ddot{a}_{x:\overline{t}|}\right) + A_{x+t}}$$

Since ${}_{t|}\ddot{a}_x + \ddot{a}_{x:\overline{t}|} = \ddot{a}_x$, it follows that:

$$\boldsymbol{{}_tV_x = P_x\,\ddot{s}_{x:\overline{t}|} - \frac{(1+i)^t}{{}_tp_x}P_x\,\ddot{a}_x + A_{x+t}}$$

From the equation of value, we have $P_x\ddot{a}_x = A_x$. So:

$$\boldsymbol{{}_tV_x = P_x\,\ddot{s}_{x:\overline{t}|} - \frac{(1+i)^t}{{}_tp_x}A_x + A_{x+t}}$$

$$= P_x\,\ddot{s}_{x:\overline{t}|} - \frac{(1+i)^t}{{}_tp_x}\left[A_x - v^t\,{}_tp_x\,A_{x+t}\right]$$

Finally, since $A_x - v^t\ {}_tp_x\ A_{x+t} = A^1_{x:\overline{t}|}$, we obtain:

$$\boldsymbol{{}_tV_x = P_x\ \ddot{s}_{x:\overline{t}|} - \frac{(1+i)^t}{{}_tp_x} A^1_{x:\overline{t}|}}$$

which is the formula for the retrospective reserve.

The details of this proof can easily be adjusted for other types of contract.

Alternatively, you could prove the equality of the prospective and retrospective reserves as follows. The premium equation is:

$$P_x\ \ddot{a}_x = A_x$$

which can also be written as:

$$P_x\left(\ddot{a}_{x:\overline{t}|} + v^t\ {}_tp_x\ \ddot{a}_{x+t}\right) = A^1_{x:\overline{t}|} + v^t\ {}_tp_x\ A_{x+t}$$

Rearranging gives:

$$P_x\ \ddot{a}_{x:\overline{t}|} - A^1_{x:\overline{t}|} = v^t\ {}_tp_x\left(A_{x+t} - P_x\ \ddot{a}_{x+t}\right)$$

Dividing both sides by $v^t\ {}_tp_x$ gives:

$$\left(P_x\ \ddot{a}_{x:\overline{t}|} - A^1_{x:\overline{t}|}\right)\frac{(1+i)^t}{{}_tp_x} = A_{x+t} - P_x\ \ddot{a}_{x+t}$$

ie:

$${}_tV^{retro} = {}_tV^{pro}$$

Q

Question 5.14

A whole life annuity is issued to a life aged x. The annuity is purchased by a single premium and a benefit of 1 is payable at the beginning of every year throughout life. Show that the net prospective and retrospective reserves are equal.

6.6 Reserve conventions

We often calculate reserves at integer durations. In this case, we calculate the reserve just before any payment of premium due on that date, and just after any payment of annuity payable in arrear due on that date.

You don't have to worry about the timing of *death* benefits because reserves are only required for policies that are still in force.

The general rule is, for valuation on the tth policy anniversary, payments in respect of the year $t-1$ to t payable in arrear (*ie* on the tth anniversary) are assumed to have been paid, payments in respect of the year t to $t+1$ payable in advance (and so are also due on the tth anniversary) are assumed not yet to have been paid.

7 Net premium reserves

The definition of a net premium reserve is given in the box below.

> ***Definition***
>
> **The net premium reserve is the prospective reserve, where we make no allowance for future expenses, and where the premium used in the calculation is a notional premium, calculated using the reserve basis.**

The net premium reserve is calculated as the expected present value of the future benefits less the expected present value of the future net premiums. No explicit allowance is made for either expenses or the actual premium received from the policyholder (the office premium).

This may appear very artificial. In fact, the net premium valuation has been an important feature in life insurance for many years.

The key reason why this is the case is that the reserve is simple to calculate. The net premium method was used before computers, spreadsheets or calculators were available.

The notional net premium calculated and valued as the *future income* element of the reserve is generally considerably smaller than the actual premium being paid. It is considered that the excess of the actual premium over the notional premium will be sufficient to cover the expenses, which are not specifically valued.

7.1 Endowment assurance policies

${}_tV_{x:\overline{n}|}$ is the net premium reserve at duration t for an n-year endowment assurance policy, with sum assured of 1, payable at the end of the year of death or at maturity, and with level annual premiums payable during the duration of the policy.

From the definitions of the net premium reserve and using the net premium notation from above, we have:

$$ {}_tV_{x:\overline{n}|} = A_{x+t:\overline{n-t}|} - P_{x:\overline{n}|}\,\ddot{a}_{x+t:\overline{n-t}|} $$

But $P_{x:\overline{n}|} = \dfrac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}$ **, so we have:**

$$
\begin{aligned}
{}_tV_{x:\overline{n}|} &= A_{x+t:\overline{n-t}|} - \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} \times \ddot{a}_{x+t:\overline{n-t}|} \\
&= (1 - d\,\ddot{a}_{x+t:\overline{n-t}|}) - (1 - d\,\ddot{a}_{x:\overline{n}|})\,\frac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x:\overline{n}|}}
\end{aligned}
$$

using the premium conversion formula for endowment assurances that we met in Section 3 of Chapter 4.

Since the net premium and the net premium reserve are both calculated using the same basis, the formula for the net premium reserve simplifies to:

$$
{}_tV_{x:\overline{n}|} = 1 - \frac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x:\overline{n}|}}
$$

This is an important and useful result.

7.2 Whole life assurance policies

We find a similar result for ${}_tV_x$**, the net premium reserve at duration** t **of an annual premium whole life assurance, issued to a life aged** x**, with premiums payable throughout the contract, and with the sum assured of 1 payable at the end of the year of death:**

$$
\begin{aligned}
{}_tV_x &= A_{x+t} - P_x\,\ddot{a}_{x+t} \\
&= A_{x+t} - \frac{A_x}{\ddot{a}_x}\,\ddot{a}_{x+t} \\
&= (1 - d\,\ddot{a}_{x+t}) \;-\; (1 - d\,\ddot{a}_x)\,\frac{\ddot{a}_{x+t}}{\ddot{a}_x} \\
&= 1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x}
\end{aligned}
$$

Again, the simplification is possible because the net premium and the net premium reserve are calculated using the same basis. This result is given on Page 37 of the *Tables*.

7.3 Endowment assurance and whole life assurance policies with continuously payable premiums and benefits payable immediately on death

In the continuous time case, the corresponding definition and relationships are, for endowment assurances:

$$ {}_t\bar{V}_{x:\overline{n}|} = E(L) = \bar{A}_{x+t:\overline{n-t}|} - \bar{P}_{x:\overline{n}|}\ \bar{a}_{x+t:\overline{n-t}|} = 1 - \frac{\bar{a}_{x+t:\overline{n-t}|}}{\bar{a}_{x:\overline{n}|}} $$

and for whole life assurances:

$$ {}_t\bar{V}_x = E(L) = \bar{A}_{x+t} - \bar{P}_x\ \bar{a}_{x+t} = 1 - \frac{\bar{a}_{x+t}}{\bar{a}_x} $$

We leave the proofs to the reader.

The proofs come from the fact that $\bar{P}_{x:\overline{n}|} = \dfrac{\bar{A}_{x:\overline{n}|}}{\bar{a}_{x:\overline{n}|}}$ and $\bar{P}_x = \dfrac{\bar{A}_x}{\bar{a}_x}$, and the premium conversion relationships $\bar{A}_{x:\overline{n}|} = 1 - \delta\,\bar{a}_{x:\overline{n}|}$ and $\bar{A}_x = 1 - \delta\,\bar{a}_x$.

The net premium reserve formulae for whole life assurance policies are given on Page 37 of the *Tables*.

7.4 Non-annual premiums

In $m$thly premium cases, such straightforward equalities as given in Sections 7.1 and 7.2 would not hold, and reserves would need to be calculated from the basic formulae.

The notation for such reserves follows the notation approach described above in Section 0. The reserve, for a unit benefit, is denoted V and subscripts and superscripts may be added. The right subscript corresponds to the subscript of the assurance benefit function for the contract; the right superscript (if any) indicates the premium frequency per annum of the contract; the left subscript indicates the duration at which the reserve is taken. Hence, for example, ${}_tV^{(m)\,1}_{x:\overline{n}|}$ is the net premium reserve at duration t of an n-year pure endowment, sum assured 1, premiums payable $m$thly.

The reserve calculation, for a whole life assurance payable by true m thly premiums and assuming a benefit paid at the end of the year of death, for example, at *integral* duration t would be based on the formula:

$$ {}_tV_x^{(m)} = A_{x+t} - P_x^{(m)}\ddot{a}_{x+t}^{(m)} $$

which could be evaluated using the $m$thly annuity approximation $\ddot{a}_{x+t}^{(m)} \approx \ddot{a}_{x+t} - \dfrac{m-1}{2m}$.

You'll find this formula on Page 36 of the *Tables.*

7.5 Term assurances

${}_tV^{1}_{x:\overline{n}|}$ is the net premium reserve at duration t for an n-year term assurance policy, with sum assured of 1 payable at the end of the year of death, and with level annual premiums payable during the duration of the policy.

This may be evaluated from first principles as:

$$ {}_tV^{1}_{x:\overline{n}|} = A^{\;1}_{x+t:\overline{n-t}|} - P^{1}_{x:\overline{n}|}\,\ddot{a}_{x+t:\overline{n-t}|} $$

and there is no neat simplification as in the whole life and endowment assurance cases.

This is because there is no premium conversion formula for term assurances.

Note that it is easy to show that:

$$ {}_tV_{x:\overline{n}|}^{\;\;\;1} + {}_tV^{1}_{x:\overline{n}|} = {}_tV_{x:\overline{n}|} $$

7.6 Other contracts

The above discussion can be simply extended to annuity contracts, as well as to assurance contracts.

8 Recursive calculation of reserves

8.1 Conditions for recursive calculations

We will develop relationships (sometimes referred to as equations of equilibrium) **between the reserves at successive integer durations for net premium reserves. For these relationships to hold, the reserves at successive durations and the net premiums must be calculated on the same basis.**

8.2 The equation of equilibrium for a whole life assurance

Let i be the valuation interest rate, and let q_y and p_y be the valuation rates of mortality and survival at age y.

For a whole life assurance issued to a life aged x with sum assured 1 payable at the end of the year of death and annual premiums, the net premium reserves at times t and $t+1$ are related by the formula:

$$({}_tV_x + P_x)(1+i) = q_{x+t} + p_{x+t}\,{}_{t+1}V_x$$

Proof

From the definition of ${}_tV_x$, we have:

$$({}_tV_x + P_x) = (A_{x+t} - P_x\ddot{a}_{x+t}) + P_x$$

But:

$$A_{x+t} = vq_{x+t} + vp_{x+t}A_{x+t+1}$$

and:

$$\ddot{a}_{x+t} = 1 + vp_{x+t}\ddot{a}_{x+t+1}$$

So:

$$\begin{aligned}({}_tV_x + P_x) &= vq_{x+t} + vp_{x+t}A_{x+t+1} - P_x(1 + vp_{x+t}\ddot{a}_{x+t+1}) + P_x \\ &= v(q_{x+t} + p_{x+t}(A_{x+t+1} - P_x\ddot{a}_{x+t+1})) \\ &= v(q_{x+t} + p_{x+t}\,{}_{t+1}V_x)\end{aligned}$$

$$\Rightarrow ({}_tV_x + P_x)(1+i) = q_{x+t} + p_{x+t}\,{}_{t+1}V_x$$

8.3 General reasoning

It is very helpful to consider a verbal explanation of this result. If the insurer holds reserves at the start of the year equal to the policy reserve, then the total at the year end of the reserve brought forward, plus the premium income, plus the interest earned on these, must be equal to the expected cost of the death benefits at the year end (here the benefit is 1 payable with probability q_{x+t}) plus the expected cost of setting up the reserve at the year end of the year end reserve (here the reserve of ${}_{t+1}V_x$ is required with probability p_{x+t}).

In the case above, the relationship states that the year-end value of the policy income for the year equals the expected value of the policy outgo at the year end.

Income in this case includes the reserves available for a policy in force at the start of the year, and outgo includes the reserves you need to set aside at the end of the year for those who survive the year.

The equation of equilibrium can also be written as:

$$P_x(1+i) = q_{x+t} + \left[p_{x+t} \ {}_{t+1}V_x - {}_tV_x(1+i) \right]$$

This shows that the premium accumulated with interest to the end of the policy year covers the benefits payable in respect of each death plus the cost of increasing the reserves for the policies still in force at the end of the year. The expected cost of increasing the reserve at time $t+1$ for a single policy known to be in force at time t is the term in brackets on the RHS of the equation above, namely:

$$p_{x+t} \ {}_{t+1}V_x - {}_tV_x(1+i)$$

You can think about this in the following way:

- At time t we have a reserve of ${}_tV_x$ for each policy in force. This grows with interest over the year $(t, t+1)$ to become ${}_tV_x(1+i)$.
- At time $t+1$ we need a reserve of ${}_{t+1}V_x$ per policy still in force. The probability that a policyholder is alive at time $t+1$, given that he is alive at time t, is p_{x+t}. So the expected reserve needed at time $t+1$ is $p_{x+t} \ {}_{t+1}V_x$.
- The difference between what we need at time $t+1$ and what we have accumulated to time $t+1$ is the expected cost of increasing the reserve.

The expected cost of increasing the reserve can also be expressed as:

$$p_{x+t}\ {}_{t+1}V_x - {}_tV_x\,(1+i) = \left[\,p_{x+t}\ {}_{t+1}V_x - {}_tV_x\,\right] - i\ {}_tV_x$$

Some people prefer to consider the two components $p_{x+t}\ {}_{t+1}V_x - {}_tV_x$ and $i\ {}_tV_x$ separately.

If the income were a random variable (for example, if the premiums were payable $m$thly, so that payment depended on survival), then the relationship would be that the expected income, accumulated to the year end, is equal to the expected value of the year-end outgo.

8.4 Further examples of equations of equilibrium

Some further relationships are given. It is left to the reader to verify these.

$$\left({}_tV_{x:\overline{n}|} + P_{x:\overline{n}|}\right)(1+i) \quad = \quad q_{x+t} + p_{x+t}\ {}_{t+1}V_{x:\overline{n}|} \qquad \textbf{(1)}$$

$$\left({}_tV^{1}_{x:\overline{n}|} + P^{1}_{x:\overline{n}|}\right)(1+i) \quad = \quad q_{x+t} + p_{x+t}\ {}_{t+1}V^{1}_{x:\overline{n}|} \qquad \textbf{(2)}$$

$$\left({}_tV_{x:\overset{\ \ 1}{\overline{n}|}} + P_{x:\overset{\ \ 1}{\overline{n}|}}\right)(1+i) \quad = \quad p_{x+t}\ {}_{t+1}V_{x:\overset{\ \ 1}{\overline{n}|}} \qquad \textbf{(3)}$$

Question 5.15

Verify these three results algebraically.

9 Mortality profit

In the last section it was shown that, if the experience exactly follows the reserve basis, then, on average, the income and outgo in each policy year are equal.

In this case note that "outgo" includes the increase in reserves. When talking about outgo from the insurance company, we consider reserves as money for policyholders. Hence increase in reserves is a form of outgo.

If the experience does not follow the assumptions then there will either be an excess of income over outgo (a profit, or surplus) or an excess of outgo over income (a loss or negative profit). Profits and losses may arise from any element of the reserve basis. For example:

1. **If the interest earned is greater than that assumed in the reserve, then the premium and reserve at the start of the year will accumulate to more than the sum required to cover the cost of the benefits and the year-end reserve, giving an interest surplus.**

2. **If the policyholder decides to surrender his or her policy (that is, to cease paying premiums, and take some lump sum in respect of the future benefits already paid for) then the year end outgo is not as assumed. The lump sum will be required in place of the year end reserve and there will be a surrender profit (positive if the lump sum is less than the reserve).**

3. **If the experienced mortality is heavier than that assumed in the basis, then there will be a profit or loss from mortality.**

So, if experience is not as assumed, profits or losses will arise. Exactly the same principle applies in pension schemes; surpluses and deficits arise because experience is not in line with the actuary's view of future experience.

We consider here mortality profit only. We assume, therefore, that in all elements other than mortality, experienced rates follow the assumed rates exactly.

In practice, they do not; and each of the above will give rise to profits or losses. The impact of each element may be quantified. This procedure is known as analysis of surplus.

9.1 Death strain at risk (DSAR)

Consider a policy issued t years ago to a life then aged x, with sum assured of S payable at the end of the year of death. Also, assume that no survival benefit is due if the life survives to $t+1$. (We will extend these ideas to death benefits payable immediately on death, and to survival benefits, in later sections).

Let ${}_tV$ be the reserve at time t. Then we define the death strain in the policy year t to $t+1$ to be the random variable, DS, say,

$$\text{DS} = \begin{cases} 0 & \text{if the life survives to } t+1 \\ \left(S - {}_{t+1}V\right) & \text{if the life dies in the year } t \to t+1 \end{cases}$$

The maximum death strain, $\left(S - {}_{t+1}V\right)$ is called the death strain at risk or DSAR.

If you think of the reserve as money already set aside for the policyholder, the death strain at risk for the current policy year is the amount of extra money that the company would need to pay if the policyholder died during that policy year. The death strain at risk is sometimes also called the sum at risk.

The word *strain* is used loosely to mean a cost to the company. The reasoning behind the DSAR definition is seen more clearly if we rearrange the recursive relationship between ${}_tV$ and ${}_{t+1}V$, assuming level premiums for simplicity:

$$\begin{aligned}\left({}_tV + P\right)(1+i) &= q_{x+t}\, S + p_{x+t}\; {}_{t+1}V \\ &= q_{x+t}\, S + (1 - q_{x+t})\, {}_{t+1}V \\ &= {}_{t+1}V + q_{x+t}(S - {}_{t+1}V)\end{aligned}$$

In words, the reasoning is that for each policy we must pay out at least ${}_{t+1}V$ at the end of the year. In addition, if the policy becomes a claim during the year, with probability q_{x+t}, then we must pay out an extra sum of $\left(S - {}_{t+1}V\right)$ which is the DSAR. Note that q_{x+t} is the probability of dying in the year t to $t+1$, and therefore $x+t$ is the age at the start of the year.

9.2 Expected death strain (EDS) for a single policy

The expected amount of the death strain for a single policy is called the expected death strain (EDS) for that policy. This is the amount that the life insurance company expects to pay extra to the year-end reserve for the policy. The probability of claiming in the policy year t to $t+1$ is q_{x+t}, so that:

$$EDS = q_{x+t}\left(S - {}_{t+1}V\right)$$

9.3 Actual death strain (ADS) for a single policy

The actual death strain for a single policy is simply the observed value at $t+1$ of the death strain random variable, that is:

$$ADS = \begin{cases} 0 & \text{if the life survived to } t+1 \\ \left(S - {}_{t+1}V\right) & \text{if the life died in the year } t \to t+1 \end{cases}$$

9.4 Mortality profit

The mortality profit is defined as:

Mortality profit

Mortality profit = Expected Death Strain – Actual Death Strain

The EDS is the amount the company *expects* to pay out, in addition to the year end reserve for a policy. The ADS is the amount it actually pays out, in addition to the year-end reserve. If it actually pays out less than it expected to pay, there will be a profit. If the actual strain is greater than the expected strain, there will be a loss.

9.5 *Mortality profit on a portfolio of policies*

We are often interested in analysing the experience of a group of similar policies. We use the term *portfolio of policies* to mean any group of policies. In this case we simply sum the EDS and the ADS over all the relevant policies. If all lives are the same age, and subject to the same mortality table, this gives:

$$\textbf{Total DSAR} = \sum_{\text{all policies}} (S - {}_{t+1}V)$$

$$\textbf{Total EDS} = \sum_{\text{all policies}} q_{x+t}(S - {}_{t+1}V)$$

$$= q_{x+t}\left(\sum_{\text{all policies}} (S - {}_{t+1}V)\right)$$

$$= q_{x+t} \times \textbf{Total DSAR}$$

$$\textbf{Total ADS} = \sum_{\text{death claims}} (S - {}_{t+1}V)$$

$$\textbf{Mortality Profit} = \textbf{EDS} - \textbf{ADS}$$

Note that if the policies are identical, then:

Total EDS = expected number of deaths $\times$ DSAR

Total ADS = actual number of deaths $\times$ DSAR

In many situations the DSAR of each of the individual policies is not known, but the total DSAR is simply the total sum assured less the total year end reserve, and the EDS is $q_{x+t} \times$DSAR .

In the above equations, it is important to note that:

- the summation is over all policies that are in force at the *start* of the year
- the mortality rate q_{x+t} relates to the age of the policyholder(s) at the *start* of the year
- the reserves ${}_{t+1}V$ are calculated as at the *end* of the year.

Example

A life insurance company has a portfolio of 10,000 single premium one-year term assurances. For each policy, there is a sum assured of \$50,000 payable at the end of the year if the policyholder dies during the year. The company assumes that mortality will be 1% *pa*.

(i) Calculate the expected death strain for this portfolio.

(ii) Given that 89 people die during the year, calculate the actual death strain and hence the mortality profit or loss for this portfolio.

Solution

Since this is a one-year policy, no reserves will be required at the end of the year. The death strain at risk for each policy is therefore \$50,000.

(i) The expected number of deaths is $10,000 \times 0.01 = 100$ and the expected death strain is $100 \times 50,000 = \$5m$.

(ii) The actual death strain is $89 \times 50,000 = \$4.45m$.

The mortality profit is the difference between these, *ie* \$550,000.

Note that the insurance company has made a profit here because the actual number of deaths (89) was less than the expected number (100).

9.6 *Allowing for death benefits payable immediately*

Where death benefits are payable immediately on death, in the calculation of the death strain we allow for interest between the time of payment and the end of the year of death. In this case, the death strain defined in Section 9.1 would become:

$$\mathbf{DS} = \begin{cases} \mathbf{0} & \textbf{if the life survives to } \boldsymbol{t+1} \\ \left(\boldsymbol{S}(\mathbf{1}+\boldsymbol{i})^{\frac{1}{2}} - {}_{\boldsymbol{t+1}}\boldsymbol{V}\right) & \textbf{if the life dies in the year } \boldsymbol{t \to t+1} \end{cases}$$

The death strain formula requires the value of the death benefit payment as at the *end* of the year of death. As the sum assured is paid out during the year, the value of this payment will have increased with interest between the date of death and the end of the year of death. The above formula therefore assumes that death occurs half way through the year, on average.

Question 5.16

How would you adjust the above formula for the death strain, if the sum assured was paid, on average, two months after the actual date of death?

Similar adjustments would be applied to the formulae in Sections 9.2, 9.3 and 9.5.

Question 5.17

You are given the following details about a particular group of whole life assurance policies:

year of issue: 2007
number in force at the policy anniversary in 2012: 1,900
number in force at the policy anniversary in 2013: 1,867
exact age at the policy anniversary in 2012: 70
sum assured: 60,000, payable immediately on death
level premiums are payable annually in advance for the whole of life

Calculate the mortality profit for the policy year commencing at the policy anniversary in 2012, assuming death is the only cause of policy termination, and that the insurer holds net premium reserves for these contracts calculated on the following assumptions:

Mortality: AM92 Ultimate
Interest: 4% *pa*

9.7 *Allowing for survival benefits*

Suppose the contract provides for a benefit at the end of a policy year t to $t+1$. By convention, the present value of this will have been included in ${}_tV$ but will fall outside the computation of ${}_{t+1}V$. So, the survival benefit needs to be allowed for as an additional payment from the reserve.

Let R be the benefit payable at the end of the policy year $t \to t+1$ contingent on the survival of the policyholder. Assuming death benefits are paid at the end of the year of death, the recursive relationship between successive reserves is now:

$$\left({}_tV + P\right)(1+i) = q_{x+t}S + p_{x+t}\left({}_{t+1}V + R\right)$$

$$= q_{x+t}S + (1 - q_{x+t})\left({}_{t+1}V + R\right)$$

$$= {}_{t+1}V + R + q_{x+t}\left(S - \left({}_{t+1}V + R\right)\right)$$

The DS is now:

$$DS = \begin{cases} 0 & \text{if the life survives to } t+1 \\ S - \left({}_{t+1}V + R\right) & \text{if the life dies in the year } t \to t+1 \end{cases}$$

as the office must pay out:

- **$\left({}_{t+1}V + R\right)$ for all policyholders, and an additional**
- **$S - \left({}_{t+1}V + R\right)$ for policies becoming claims by death.**

So the DSAR for a single policy is $S - \left({}_{t+1}V + R\right)$.

The expected death strain is then $q_{x+t}\left[S - \left({}_{t+1}V + R\right)\right]$; the actual death strain is 0 if the life survived the year and $S - \left({}_{t+1}V + R\right)$ if the life died during the year; the mortality profit is EDS – ADS.

> ***Question 5.18***
>
> Repeat the calculations given in the last example (in Section 9.5), assuming that the survivors are paid a lump sum benefit of $20,000 at the end of the year.

Question 5.19

On 1 January 1997 a life insurance company issued a number of 30-year pure endowment contracts to lives then aged 35. Level premiums are payable annually in advance throughout the term of the contract or until earlier death. In each case, the only benefit is a sum assured of £20,000, payable on survival to the end of the term.

On 1 January 2005, 600 policies were still in force. During 2005, 3 policyholders died. Assuming that the company uses net premium policy reserves, calculate the profit or loss from mortality for calendar year 2005 in respect of this group of policies.

Basis:
mortality: AM92 Ultimate
interest: 4% *pa*

9.8 Annuities

In the case of an annuity of R *pa*, payable annually in arrear, with no death benefit, the DSAR would be $-\left({}_{t+1}V + R\right)$ (so each death causes a negative strain or *release of reserves*).

Example

At the start of a particular year a life insurance company had a portfolio of 5,000 female pensioners, all aged exactly 60, who each receive an income of £10,000 per annum, paid annually in arrears.

The company holds net premium reserves, calculated using PFA92C20 mortality and 4% *pa* interest.

During that year, 9 pensioners died. Calculate the mortality profit or loss.

Solution

For policyholders who die during the year, no funds are required at the end of the year.

The reserve required at the end of the year for each surviving policy plus the annuity payment due at that time is:

$$10,000a_{61} + 10,000 = 10,000 \times 16.311 = £163,110$$

So the death strain at risk is $0-163{,}110=-£163{,}110$.

From the *Tables*, $q_{60}=0.002058$. So the expected number of deaths during the year is $5{,}000\,q_{60}=10.29$.

So the EDS is $-£163{,}110\times 10.29=-£1{,}678{,}402$.

The ADS is $-£163{,}110\times 9=-£1{,}467{,}990$.

So the mortality profit is $EDS-ADS=-1{,}678{,}402-(-1{,}467{,}990)=-£210{,}412$

ie a loss of about £210,400. (The loss arises because fewer people died than expected.)

In the case of an annuity of *R pa*, payable annually in advance, with no death benefit, the DSAR would be $-{}_{t+1}V$ only (because the annuity payment is made by all policies in force at the start of the year, and is not affected by whether or not the policyholder survives the year).

9.9 Allowing for different premium or annuity payment frequencies

The above formulae for the death strain and mortality profit are appropriate where premiums are either paid annually in advance or as a single payment at the outset of the policy.

Where premiums (or annuities) are paid more frequently than annually, the formulae for the death strain and mortality profit will be different, because the death or survival of the policyholder during the year will affect how many premiums are actually received, or how many annuity payments are actually made. These variations are not covered by the Subject CT5 syllabus.

10 Exam-style questions

We finish this chapter with a couple of past exam questions. These have been adapted slightly, so that mortality assumptions and terminology are consistent with those used in Subject CT5.

Question 5.20 ***(Subject 104, September 2000, Question 2, adapted)***

A life insurance company sells whole-life assurance policies with a sum assured of £20,000, payable at the end of the year of death. The premium is £420 payable annually in advance until the death of the policyholder.

A life now aged 50 purchased a policy exactly one year ago, and is now due to pay the second annual premium.

(i) Find the expected present value of the future loss to the company arising from this policy. [2]

(ii) Show that the variance of the present value of the future loss from this policy can be expressed as:

$$b.A'_{50} + c$$

Determine the numerical values of b and c, and the rate of interest used to evaluate A'_{50}. [3]

Basis: mortality AM92 Ultimate, interest 4% *pa*. Ignore expenses. [Total 5]

Question 5.21 (Subject 104, September 2000, Question 13, adapted)

A life insurance company issues 20-year temporary assurance policies to lives aged 45. The sum assured, which is payable immediately on death, is £400,000 for the first 10 years, and £100,000 thereafter. Level annual premiums are payable in advance for 20 years, or until earlier death.

The premium basis is:

Mortality: AM92 Ultimate
Interest: 4% per annum
Expenses: nil.

(i) Show that the premium payable is approximately £870.25 per annum. [4]

(ii) Find the net premium reserve ten years after the commencement of the policy, immediately before the payment of the eleventh premium, assuming the reserving basis is the same as the premium basis. [4]

(iii) Give an explanation of your numerical answer to part (ii). Describe the disadvantages to the insurance company of issuing this policy. [3]

(iv) How could the terms of the policy be altered, so as to remove the disadvantages described in part (iii)? [2]
[Total 13]

Chapter 5 Summary

Principle of equivalence

The equation of value is:

Expected present value of income = Expected present value of outgo.

This equation is used to calculate premiums and reserves.

A basis is a set of assumptions regarding future experience, used in a calculation.

Net premiums for endowment assurance contracts

The net premium reflects the cost of the benefits and the premium receipts only, ignoring any expenses.

For an endowment assurance policy with a sum assured of 1 payable at the end of the year of death, the net premium is:

$$P_{x:\overline{n|}} = \frac{A_{x:\overline{n|}}}{\ddot{a}_{x:\overline{n|}}}$$

If the death benefit is payable immediately on death, then the net premium payable annually in advance is:

$$P\left(\bar{A}_{x:\overline{n|}}\right) = \frac{\bar{A}_{x:\overline{n|}}}{\ddot{a}_{x:\overline{n|}}}$$

If the premium is payable m -thly per annum in advance, and the death benefit is payable at the end of the year of death, then the total amount of premium paid in a year is:

$$P^{(m)}_{x:\overline{n|}} = \frac{A_{x:\overline{n|}}}{\ddot{a}^{(m)}_{x:\overline{n|}}}$$

If the premium is payable continuously and the death benefit is payable at the end of the year of death, then the total amount of premium paid in a year is:

$$\bar{P}_{x:\overline{n|}} = \frac{A_{x:\overline{n|}}}{\bar{a}_{x:\overline{n|}}}$$

If mortality is select, rather than ultimate, then the above expressions apply but with x replaced by $[x]$.

Net premiums for other contracts

For a term assurance policy, the net premium is: $P^1_{x:\overline{n|}} = \dfrac{A^1_{x:\overline{n|}}}{\ddot{a}_{x:\overline{n|}}}$

For a whole life assurance policy, the net premium is: $P_x = \dfrac{A_x}{\ddot{a}_x}$

For a whole life policy with premiums limited to t years, the net premium is:

$${}_tP_x = \frac{A_x}{\ddot{a}_{x:\overline{t|}}}$$

In each of the above expressions we are assuming that death benefits are payable at the end of the year of death and premiums are paid annually in advance.

Retrospective accumulations

It is sometimes necessary to find the accumulated value of a benefit at the current date or a future date. The expected value can be calculated by modifying the commutation formulae. Similarly we can accumulate the value of premiums received.

The accumulated value at time n of an annuity-due is:

$$\ddot{s}_{x:\overline{n|}} = \ddot{a}_{x:\overline{n|}} \times \frac{(1+i)^n}{{}_np_x} = \ddot{a}_{x:\overline{n|}} \times \frac{D_x}{D_{x+n}}$$

Reserves

A reserve is money set aside by the insurer, for the policyholder, to pay policyholders' benefits and, where appropriate, future expenses.

Reserves may be calculated by looking forward (prospectively) or looking backwards (retrospectively). If experience is exactly as assumed in the pricing basis and the reserving basis then these will be the same.

Net premium reserves

Prospectively, net premium reserves are calculated as the EPV of future benefits, less the EPV of future net premiums.

Retrospectively, net premium reserves are calculated as the accumulated value of past net premiums, less the accumulated value of past claims.

The net premiums used in these formulae are calculated from the premium equation of value, with the same interest and mortality assumptions as used for the reserves.

Special "quick" formulae apply for certain endowment and whole life assurances. The net premium reserve at time t for an n-year endowment assurance policy is given by:

$$ {}_tV_{x:\overline{n}|} = A_{x+t:\overline{n-t}|} - P_{x:\overline{n}|} \times \ddot{a}_{x+t:\overline{n-t}|} = 1 - \frac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x:\overline{n}|}} $$

The net premium reserve at time t for a whole life assurance policy is:

$$ {}_tV_x = A_{x+t} - P_x\, \ddot{a}_{x+t} = 1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x} $$

In the two formulae given above, we are assuming that death benefits are payable at the end of the year of death and premiums are paid annually in advance.

If death benefits are payable immediately on death and premiums are payable continuously, then the formulae become:

$$ {}_t\overline{V}_{x:\overline{n}|} = \overline{A}_{x+t:\overline{n-t}|} - \overline{P}_{x:\overline{n}|}\, \overline{a}_{x+t:\overline{n-t}|} = 1 - \frac{\overline{a}_{x+t:\overline{n-t}|}}{\overline{a}_{x:\overline{n}|}} $$

$$ {}_t\overline{V}_x = \overline{A}_{x+t} - \overline{P}_x\, \overline{a}_{x+t} = 1 - \frac{\overline{a}_{x+t}}{\overline{a}_x} $$

Equation of equilibrium

The reserve at time $t+1$ can be calculated from the reserve at time t using the equation of equilibrium (the recursive relation for reserves).

For an annual premium whole life assurance with sum assured S payable at the end of the year of death, the equation of equilibrium is:

$$\left({}_tV + P\right)(1+i) = {}_{t+1}V\, p_{x+t} + S\, q_{x+t}$$

or: $$\left({}_tV + P\right)(1+i) = {}_{t+1}V + q_{x+t}\left(S - {}_{t+1}V\right)$$

The quantity $S - {}_{t+1}V$ is called the *death strain at risk*.

Mortality profit

Profits and losses arise because experience is different from assumed. The effect can be quantified. In general:

$$\text{Mortality profit } = \text{EDS} - \text{ADS}$$

$$\text{Total EDS} = \sum_{\text{all policies}} q_{x+t}(S - X - {}_{t+1}V)$$

$$\text{Total ADS} = \sum_{\text{all deaths}} (S - X - {}_{t+1}V)$$

where S is the sum assured paid on death during the year (revalued to the end of the year), and X is the sum paid on survival to the end of the year.

Chapter 5 Solutions

Solution 5.1

For demonstrating statutory solvency you are showing that you can meet the liabilities to the policyholder. Therefore the basis will be prudent, *ie* you will show that even on a pessimistic set of assumptions the company has got enough money to meet its liabilities. A prudent basis may be enforced by the authorities, or the life company may have chosen such a basis for reasons of prudence.

For buying a company we want to pay a fair price, so the basis will be realistic, *ie* having no (or only a small) margin in the assumptions.

Solution 5.2

$$P_{30:\overline{10}|} = \frac{A_{30:\overline{10}|}}{\ddot{a}_{30:\overline{10}|}} = \frac{0.67643}{8.4129} = 0.0804$$

$$P_{30:\overline{15}|} = \frac{A_{30:\overline{15}|}}{\ddot{a}_{30:\overline{15}|}} = \frac{0.55720}{11.513} = 0.0484$$

Most of the premium paid will go to cover the pure endowment. For the longer-term policy, the money will normally be invested for longer and there are 5 more premiums received. Hence the premium is lower. (With endowment assurance policies most policyholders get the maturity payment payable at the end of the term.)

Solution 5.3

${}_5P_{40}$ or ${}_{10}P_{40}$ are the premium rates if premiums are paid over 5 years and 10 years, respectively. So ${}_5P_{40}$ will be bigger.

${}_5p_{40}$ or ${}_{10}p_{40}$ are the probabilities that a 40 year old will survive for 5 and 10 years, respectively. So ${}_5p_{40}$ will be bigger.

Solution 5.4

Let P denote the premium and let S denote the sum assured.

The actual profit to the insurer will be:

$$X = PV \text{ premiums } - PV \text{ benefits}$$

If the curtate future lifetime of a policyholder is K years, then the present value of the premiums received is $P\ddot{a}_{\overline{K+1|}}$ and the present value of the benefits paid is $S\,v^{K+1}$.

Since $\ddot{a}_{\overline{K+1|}} = \dfrac{1-v^{K+1}}{d}$, it follows that:

$$X = P\left(\frac{1-v^{K+1}}{d}\right) - Sv^{K+1} = \frac{P}{d} - \left(\frac{P}{d}+S\right)v^{K+1}$$

So the variance of the insurer's profit is:

$$\text{var}(X) = \left(\frac{P}{d}+S\right)^2 \text{var}\left(v^{K+1}\right) = \left(\frac{P}{d}+S\right)^2 \left[{}^2A_x - \left(A_x\right)^2\right]$$

Solution 5.5

Let X denote the present value of the insurer's profit. Using the result from the previous question, we have:

$$\text{var}(X) = \left(\frac{P}{d}+S\right)^2 \left[{}^2A_{33} - \left(A_{33}\right)^2\right]$$

$$= \left(\frac{520\times 1.04}{0.04} + 40{,}000\right)^2 \left[0.04276 - 0.17868^2\right]$$

$$= 31{,}031{,}251$$

$$= (£5{,}571)^2$$

Solution 5.6

The accumulated fund will be:

$$1{,}000{,}000 \times 1.04^{20} = 2{,}191{,}123$$

The expected number of survivors will be:

$$10{,}000 \times \frac{l_{60}}{l_{40}} = 10{,}000 \times \frac{9{,}287.2164}{9{,}856.2863} = 9{,}422.63$$

So the expected payout per survivor is £232.54.

Solution 5.7

The accumulation after t years is:

$$\frac{D_x}{D_{x+t}} A^{1}_{x:\overline{t|}}$$

ie you just change the n's to t's.

Solution 5.8

The accumulated value after 10 years is:

$$\begin{aligned}
100{,}000\, A^{1}_{35:\overline{10|}} \times \frac{(1+i)^{10}}{{}_{10}p_{35}} &= 100{,}000\left(A_{35} - v^{10}\ {}_{10}p_{35}\ A_{45}\right) \times \frac{(1+i)^{10}}{{}_{10}p_{35}} \\
&= 100{,}000\left(A_{35} \times (1+i)^{10} \times \frac{l_{35}}{l_{45}} - A_{45}\right) \\
&= 100{,}000\left(0.09488 \times 1.06^{10} \times \frac{9{,}894.4299}{9{,}801.3123} - 0.15943\right) \\
&= 1{,}210
\end{aligned}$$

Solution 5.9

The accumulation after t years is:

$$\frac{D_x}{D_{x+t}}\ddot{a}_{x:\overline{t}|}$$

ie you just change the n's to t's.

Solution 5.10

In one year's time the policyholder would have received the maturity value (or slightly earlier if they die) but only if they had paid the next premium. Also the company has saved one year of expenses.

Therefore it is reasonable to give the value of the maturity benefit less the premium that they owe the company plus the expenses saved.

Solution 5.11

If the policyholder surrenders the policy, then the insurer will make a loss from having sold the contract.

Solution 5.12

Prospective calculation

The expected present value of future premiums is:

$$\begin{aligned}330.05\,\ddot{a}_{35:\overline{5}|} &= 330.05\left(\ddot{a}_{35} - v^5\,{}_5p_{35}\,\ddot{a}_{40}\right)\\ &= 330.05\left(21.003 - 1.04^{-5} \times \frac{9,856.2863}{9,894.4299} \times 20.005\right)\\ &= 330.05 \times 4.6237\\ &= 1,526.06\end{aligned}$$

The expected present value of future benefits is:

$$500,000\,A^{1}_{35:\overline{5|}} = 500,000\left(A_{35} - v^{5}\,{}_{5}p_{35}\,A_{40}\right)$$

$$= 500,000\left(0.19219 - 1.04^{-5} \times \frac{9,856.2863}{9,894.4299} \times 0.23056\right)$$

$$= 500,000 \times 0.00341703$$

$$= 1,708.52$$

Hence the prospective reserve is:

$$1,708.52 - 1,526.06 = £182 \text{ to the nearest } £1.$$

Retrospective calculation

The accumulated value of the past premiums is:

$$330.05\,\ddot{a}_{30:\overline{5|}} \times \frac{(1+i)^{5}}{{}_{5}p_{30}} = 330.05\left(\ddot{a}_{30} - v^{5}\,{}_{5}p_{30}\,\ddot{a}_{35}\right) \times \frac{(1+i)^{5}}{{}_{5}p_{30}}$$

$$= 330.05\left(21.834 \times 1.04^{5} \times \frac{9,925.2094}{9,894.4299} - 21.003\right)$$

$$= 330.05 \times 5.64404$$

$$= 1,862.81$$

The accumulated value of the past benefits is:

$$500,000\,A^{1}_{30:\overline{5|}} \times \frac{(1+i)^{5}}{{}_{5}p_{30}} = 500,000\left(A_{30} - v^{5}\,{}_{5}p_{30}\,A_{35}\right) \times \frac{(1+i)^{5}}{{}_{5}p_{30}}$$

$$= 500,000\left(0.16023 \times 1.04^{5} \times \frac{9,925.2094}{9,894.4299} - 0.19219\right)$$

$$= 500,000 \times 0.0033607$$

$$= 1,680.36$$

Hence the retrospective reserve is $1,862.81 - 1,680.36 = £182$, as before.

Solution 5.13

The retrospective reserve would be unchanged (because it is based on the past experience). The prospective reserve would increase. (If we only expect to earn the lower rate of 5% interest on future premiums, we will need to have more money "saved up" now in order to meet the cost of the benefits.)

Solution 5.14

The prospective reserve at time t is:

$${}_tV^{pro} = \ddot{a}_{x+t}$$

and the retrospective reserve at time t is:

$${}_tV^{retro} = \left(P - \ddot{a}_{x:\overline{t}|}\right)\frac{(1+i)^t}{{}_tp_x}$$

where P is the single premium for this policy and is given by:

$$P = \ddot{a}_x = \ddot{a}_{x:\overline{t}|} + v^t \ {}_tp_x \, \ddot{a}_{x+t}$$

Rearranging gives:

$$P - \ddot{a}_{x:\overline{t}|} = v^t \ {}_tp_x \, \ddot{a}_{x+t}$$

and dividing both sides by $v^t \ {}_tp_x$ gives:

$$\left(P - \ddot{a}_{x:\overline{t}|}\right)\frac{(1+i)^t}{{}_tp_x} = \ddot{a}_{x+t}$$

The LHS is the retrospective reserve at time t and the RHS is the prospective reserve at time t.

Solution 5.15

Endowment assurance

We have:

$$ {}_tV_{x:\overline{n}|} = A_{x+t:\overline{n-t}|} - P_{x:\overline{n}|}\, \ddot{a}_{x+t:\overline{n-t}|} $$

$$ A_{x+t:\overline{n-t}|} = v\, q_{x+t} + v\, p_{x+t}\, A_{x+t+1:\overline{n-t-1}|} $$

and:

$$ \ddot{a}_{x+t:\overline{n-t}|} = 1 + v\, p_{x+t}\, \ddot{a}_{x+t+1:\overline{n-t-1}|} $$

So we can write:

$$ \begin{aligned} {}_tV_{x:\overline{n}|} + P_{x:\overline{n}|} &= A_{x+t:\overline{n-t}|} - P_{x:\overline{n}|}\, \ddot{a}_{x+t:\overline{n-t}|} + P_{x:\overline{n}|} \\ &= v\, q_{x+t} + v\, p_{x+t}\, A_{x+t+1:\overline{n-t-1}|} - P_{x:\overline{n}|}\left(1 + v\, p_{x+t}\, \ddot{a}_{x+t+1:\overline{n-t-1}|}\right) + P_{x:\overline{n}|} \\ &= v\left[q_{x+t} + p_{x+t}\left(A_{x+t+1:\overline{n-t-1}|} - P_{x:\overline{n}|}\, \ddot{a}_{x+t+1:\overline{n-t-1}|}\right)\right] \\ &= v\left[q_{x+t} + p_{x+t}\; {}_{t+1}V_{x:\overline{n}|}\right] \end{aligned} $$

Multiplying through by $1+i$ gives:

$$ \left({}_tV_{x:\overline{n}|} + P_{x:\overline{n}|}\right)(1+i) = q_{x+t} + p_{x+t}\; {}_{t+1}V_{x:\overline{n}|} $$

Term assurance

We have:

$${}_tV^1_{x:\overline{n}|} = A^{\ 1}_{x+t:\overline{n-t}|} - P^1_{x:\overline{n}|}\,\ddot{a}_{x+t:\overline{n-t}|}$$

$$A^{\ 1}_{x+t:\overline{n-t}|} = v\,q_{x+t} + v\,p_{x+t}\ A^{\ 1}_{x+t+1:\overline{n-t-1}|}$$

and:

$$\ddot{a}_{x+t:\overline{n-t}|} = 1 + v\,p_{x+t}\ \ddot{a}_{x+t+1:\overline{n-t-1}|}$$

So:

$$\begin{aligned}
{}_tV^1_{x:\overline{n}|} + P^1_{x:\overline{n}|} &= A^{\ 1}_{x+t:\overline{n-t}|} - P^1_{x:\overline{n}|}\,\ddot{a}_{x+t:\overline{n-t}|} + P^1_{x:\overline{n}|} \\
&= v\,q_{x+t} + v\,p_{x+t}\ A^{\ 1}_{x+t+1:\overline{n-t-1}|} - P^1_{x:\overline{n}|}\left(1 + v\,p_{x+t}\ \ddot{a}_{x+t+1:\overline{n-t-1}|}\right) + P^1_{x:\overline{n}|} \\
&= v\left[q_{x+t} + p_{x+t}\left(\ A^{\ 1}_{x+t+1:\overline{n-t-1}|} - P^1_{x:\overline{n}|}\,\ddot{a}_{x+t+1:\overline{n-t-1}|}\right)\right] \\
&= v\left[q_{x+t} + p_{x+t}\ {}_{t+1}V^1_{x:\overline{n}|}\right]
\end{aligned}$$

Multiplying through by $1+i$ then gives:

$$\left({}_tV^1_{x:\overline{n}|} + P^1_{x:\overline{n}|}\right)(1+i) = q_{x+t} + p_{x+t}\ {}_{t+1}V^1_{x:\overline{n}|}$$

Pure Endowment

We have:

$$ {}_tV_{x:\overline{n}|}^{\;\;1} = A_{x+t:\overline{n-t}|}^{\;\;\;\;\;\;1} - P_{x:\overline{n}|}^{\;\;1}\,\ddot{a}_{x+t:\overline{n-t}|} $$

$$ A_{x+t:\overline{n-t}|}^{\;\;\;\;\;\;1} = v\,p_{x+t}\;A_{x+t+1:\overline{n-t-1}|}^{\;\;\;\;\;\;\;\;\;1} $$

and:

$$ \ddot{a}_{x+t:\overline{n-t}|} = 1 + v\,p_{x+t}\;\;\ddot{a}_{x+t+1:\overline{n-t-1}|} $$

So:

$$ \begin{aligned} {}_tV_{x:\overline{n}|}^{\;\;1} + P_{x:\overline{n}|}^{\;\;1} &= A_{x+t:\overline{n-t}|}^{\;\;\;\;\;\;1} - P_{x:\overline{n}|}^{\;\;1}\,\ddot{a}_{x+t:\overline{n-t}|} + P_{x:\overline{n}|}^{\;\;1} \\ &= v\,p_{x+t}\;A_{x+t+1:\overline{n-t-1}|}^{\;\;\;\;\;\;\;\;\;1} - P_{x:\overline{n}|}^{\;\;1}\left(1 + v\,p_{x+t}\;\;\ddot{a}_{x+t+1:\overline{n-t-1}|}\right) + P_{x:\overline{n}|}^{\;\;1} \\ &= vp_{x+t}\left(\;A_{x+t+1:\overline{n-t-1}|}^{\;\;\;\;\;\;\;\;\;1} - P_{x:\overline{n}|}^{\;\;1}\,\ddot{a}_{x+t+1:\overline{n-t-1}|}\right) \\ &= vp_{x+t}\;\;{}_{t+1}V_{x:\overline{n}|}^{\;\;1} \end{aligned} $$

Multiplying through by $1+i$ then gives:

$$ \left({}_tV_{x:\overline{n}|}^{\;\;1} + P_{x:\overline{n}|}^{\;\;1}\right)(1+i) = p_{x+t}\;\;{}_{t+1}V_{x:\overline{n}|}^{\;\;1} $$

Solution 5.16

The death strain formula would change to:

$$ DS = \begin{cases} 0 & \text{if the life survives to } t+1 \\ S\left(1+i\right)^{4/12} - {}_{t+1}V & \text{if the life dies in the year } t \rightarrow t+1 \end{cases} $$

This is because (assuming deaths occur half way through the year on average) the payment of the sum assured occurs 8/12 through the year, on average. So, in order to revalue the payment to the end of the year of death, we need to accumulate the payment with 4 months interest.

Solution 5.17

The death strain at risk is calculated as:

$$DSAR = 60,000 \times (1+i)^{1/2} - {}_6V$$

where ${}_6V$ is the reserve held at the policy anniversary in 2013 (which is at exact duration 6).

Now:

$${}_6V = 60,000\bar{A}_{71} - P\,\ddot{a}_{71}$$

where:

$$P = \frac{60,000\bar{A}_{65}}{\ddot{a}_{65}} \simeq \frac{60,000 \times 1.04^{1/2} \times 0.52786}{12.276} = 2,631.05$$

using $\bar{A}_x \simeq (1+i)^{1/2} A_x$.

So:

$${}_6V = 60,000 \times 1.04^{1/2} \times A_{71} - 2,631.05\,\ddot{a}_{71}$$

$$= 60,000 \times 1.04^{1/2} \times 0.61548 - 2,631.05 \times 9.998 = 11,354.90$$

and the death strain at risk is:

$$DSAR = 60,000 \times 1.04^{1/2} - 11,354.90 = 49,833.33$$

The expected death strain for this policy was:

$$EDS = 1,900 \times q_{70} \times DSAR = 1,900 \times 0.024783 \times 49,833.33$$

$$= 2,346,537$$

During the policy year, 33 people died, so the actual death strain was:

$$ADS = 33 \times DSAR = 33 \times 49,833.33 = 1,644,500$$

This gives a mortality profit of:

$$EDS - ADS = £702,037$$

Solution 5.18

For policyholders who die during the year we need the sum assured of \$50,000 at the end of the year.

For policyholders who survive we need the endowment payment of \$20,000.

So the death strain at risk is now $50{,}000 - 20{,}000 = \$30{,}000$.

Therefore the mortality profit is $(0.01 \times 10000 - 89) \times 30{,}000 = \$330{,}000$.

Solution 5.19

The reserve at 31 December 2005, *ie* at time 9, is:

$$\begin{aligned}
{}_9V &= \text{EPV future benefits} - \text{EPV future premiums} \\
&= 20,000 v^{21} \, {}_{21}p_{44} - P\ddot{a}_{44:\overline{21|}} \\
&= 20,000 \times 1.04^{-21} \times \frac{8,821.2612}{9,814.3359} \\
&\qquad - P\left(19.075 - 1.04^{-21} \times \frac{8,821.2612}{9,814.3359} \times 12.276\right) \\
&= 7,888.59 - 14.233P
\end{aligned}$$

where:

$$P = \frac{20,000 v^{30} \, {}_{30}p_{35}}{\ddot{a}_{35:\overline{30|}}} = \frac{20,000 \times 1.04^{-30} \times \dfrac{8,821.2612}{9,894.4299}}{17.629} = £311.85$$

So:

$${}_9V = £3,450.07$$

Since there is no benefit payable on death during the year, or on survival to the end of the year:

$$DSAR = -{}_9V = -£3,450.07$$

The expected death strain is:

$$EDS = 600q_{43} \times DSAR = -600 \times 0.001208 \times 3,450.07 = -£2,500.61$$

The actual death strain is:

$$ADS = 3 \times DSAR = -£10,350.21$$

So the mortality profit is:

$$-2,500.61 - (-10,350.21) = £7,850$$

Solution 5.20

(i) ***EPV of future loss***

The expected present value of the future loss is the expected present value of the future outgo less the expected present value of the future income (*ie* the policy value immediately before the next premium is paid). This is:

$$20,000A_{50} - 420\ddot{a}_{50} = 20,000 \times 0.32907 - 420 \times 17.444 = -£745.08$$

(*ie* an expected profit of £745.08).

(ii) ***Proof***

Let X denote the present value of the loss that results from the death of (50). Then:

$$X = 20,000v^{K_{50}+1} - 420 \times \ddot{a}_{\overline{K_{50}+1}|}$$

where K_{50} is the curtate future lifetime of (50).

The variance of the loss random variable is:

$$\text{var}(X) = \text{var}\left\{20,000v^{K_{50}+1} - 420\frac{\left(1-v^{K_{50}+1}\right)}{d}\right\}$$

$$= \text{var}\left\{v^{K_{50}+1}\left(20,000+\frac{420}{d}\right) - \frac{420}{d}\right\}$$

$$= \left(20,000+\frac{420}{d}\right)^2 \text{var}\left(v^{K_{50}+1}\right)$$

Now:

$$\text{var}\left(v^{K_{50}+1}\right) = E\left(v^{(K_{50}+1)\times 2}\right) - \left\{E\left(v^{K_{50}+1}\right)\right\}^2 = {}^2A_{50} - \left(A_{50}\right)^2$$

where ${}^2A_{50}$ is calculated at the rate of interest $i^* = 1.04^2 - 1 = 8.16\%$.

So:

$$\text{var}(X) = \left(20,000+\frac{420}{d}\right)^2 \left({}^2A_{50} - \left(A_{50}\right)^2\right)$$

which is in the required form with:

$$b = \left(20,000+\frac{420}{d}\right)^2 = \left(20,000+\frac{420}{0.0384615}\right)^2$$

$$= 30920^2 = 956,046,400$$

$$c = -\left(20,000+\frac{420}{d}\right)^2 \left(A_{50}\right)^2$$

$$= -30,920^2 \times 0.32907^2 = -103,527,459$$

and $A'_{50} = {}^2A_{50}$.

Solution 5.21

(i) ***Calculating the premium***

The equation of value is:

$$P\ddot{a}_{45:\overline{20|}} = 100,000\bar{A}^{1}_{45:\overline{20|}} + 300,000\bar{A}^{1}_{45:\overline{10|}}$$

Now:

$$\bar{A}^{1}_{45:\overline{20|}} = \sqrt{1.04}\left(A_{45} - \frac{D_{65}}{D_{45}}A_{65}\right)$$

$$= \sqrt{1.04}\left(0.27605 - \frac{689.23}{1,677.97} \times 0.52786\right)$$

$$= 0.06040$$

Similarly:

$$\bar{A}^{1}_{45:\overline{10|}} = \sqrt{1.04}\left(A_{45} - \frac{D_{55}}{D_{45}}A_{55}\right)$$

$$= \sqrt{1.04}\left(0.27605 - \frac{1,105.41}{1,677.97} \times 0.38950\right)$$

$$= 0.01984$$

So:

$$P = \frac{11,992}{13.780} = £870.25$$

(ii) ***Net premium reserve at time 10***

The net premium reserve at time 10 is:

$$_{10}V = 100{,}000\bar{A}^{1}_{55:\overline{10}|} - 870.25\,\ddot{a}_{55:\overline{10}|}$$

Now:

$$\bar{A}^{1}_{55:\overline{10}|} = \sqrt{1.04}\left(A_{55} - \frac{D_{65}}{D_{55}}A_{65}\right)$$

$$= \sqrt{1.04}\left(0.38950 - \frac{689.23}{1{,}105.41}\times 0.52786\right)$$

$$= 0.06157$$

So:

$$_{10}V = 100{,}000\times 0.06157 - 870.25\times 8.219 = -£995.58$$

(iii) ***Explanation of negative reserve***

The negative reserve results from the expected cost of the benefit in the second ten years being lower than the expected value of the remaining premiums. This in turn is because a level premium is being paid for a reducing benefit over the whole 20-year period.

When the reserve is negative, it means that the policyholder "owes" that amount of money to the company. Should the policy lapse at such a time, then the debt becomes unrecoverable and the company makes a loss. The company can't impose a surrender penalty with term assurance because the surrender value is usually 0.

(iv) ***Altering the policy terms***

The terms should be altered so that, at any time during the 20-year term of the policy, the reserve cannot be negative. In other words, we need to ensure that:

$$\left[\text{EPV of future benefits @ time } t\right] \geq \left[\text{EPV of future premiums @ time } t\right]$$

for $0 < t \leq 20$.

The following could achieve this, either alone or in combination:

(1) Reduce the sum assured payable in the first ten years relative to the sum assured payable in the second ten years.

(2) Increase the premiums payable in the first ten years relative to the premiums paid in the second ten years.

(3) Reduce the total premium paying term (which effectively achieves (2) above).

(4) Incorporate an endowment benefit payable on survival to the end of the term. This way, the company could impose a surrender penalty.

Chapter 6

Variable benefits and with-profits policies

Syllabus objectives:

(iv) *Describe and calculate, using ultimate or select mortality, net premiums and net premium reserves for increasing and decreasing benefits and annuities.*

1. *Extend the techniques of (ii) to calculate the expected present value of an annuity, premium, or benefit payable on death, which increases or decreases by a constant compound rate. Calculate net premiums and net premium reserves for contracts with premiums and benefits which vary as described.*

2. *Define the symbols $(IA)_x$, $(I\bar{A})_x$, $(I\ddot{a})_x$, $(Ia)_x$ and $(I\bar{a})_x$ and their select equivalents.*

3. *Calculate the expected present value of an annuity, premium or benefit payable on death, which increases or decreases by a constant monetary amount. Calculate net premiums and net premium reserves for contracts with premiums and benefits which vary as described.*

4. *Describe the operation of conventional with-profits contracts, in which profits are distributed by the use of regular reversionary bonuses, and by terminal bonuses.*

5. *Calculate net premiums and net premium reserves for the conventional with-profits contracts defined in 4.*

Continued…

6. *Describe the operation of accumulating with-profits contracts, in which benefits take the form of an accumulating fund of premiums, where either:*

   – *the fund is defined in monetary terms, has no explicit charges, and is increased by the addition of regular guaranteed and bonus interest payments plus a terminal bonus; or*

   – *the fund is defined in terms of the value of a unit fund, is subject to explicit charges, and is increased by regular bonus additions plus a terminal bonus ("Unitised with-profits")*

   In the case of unitised with-profits, the regular additions can take the form of (a) unit price increases (guaranteed and/or discretionary), or (b) allocations of additional units.

   In either case a guaranteed minimum monetary death benefit may be applied.

0 Introduction

In this chapter we look at policies for which the benefit amount varies over time, including conventional and accumulating with-profits contracts. In particular, we examine how to calculate the expected present value of the benefits when they increase by a constant amount or constant percentage each year. This will enable us to calculate premiums for with-profits policies (which is covered in Chapter 7).

1 Variable payments

The simple contracts discussed so far have had constant payments of the form:

- **a benefit of 1 payable on death; or**
- **a benefit of 1 payable, at some frequency, on survival to the date of payment.**

We consider now the possibility that the payment amount varies.

In the case of a benefit payable on death, let the payment be Y_x if death occurs in the year of age $(x, x+1)$.

Assume first that the benefit is payable at the end of the year of death. The EPV of this death benefit, to a life currently aged x, will be given by, using life table notation,

$$Y_x v \frac{d_x}{l_x} + Y_{x+1} v^2 \frac{d_{x+1}}{l_x} + \ldots + Y_{x+t} v^{t+1} \frac{d_{x+t}}{l_x} + \ldots$$

This can also be written using sigma notation as:

$$\sum_{j=0}^{\infty} Y_{x+j} v^{j+1} \frac{d_{x+j}}{l_x} = \sum_{j=0}^{\infty} Y_{x+j} v^{j+1} {}_j p_x q_{x+j}$$

Equivalent integral expressions apply if the benefit is payable immediately on death.

In this case the expected present value of the benefit would be:

$$\int_0^1 Y_x v^t {}_t p_x \mu_{x+t} dt + \int_1^2 Y_{x+1} v^t {}_t p_x \mu_{x+t} dt + \int_2^3 Y_{x+2} v^t {}_t p_x \mu_{x+t} dt + \cdots$$

$$= \sum_{j=0}^{\infty} Y_{x+j} \int_j^{j+1} v^t {}_t p_x \mu_{x+t} dt$$

The above expression may be evaluated directly, and this would be the usual approach when computer power is available. This direct evaluation would be the only approach when no set pattern to the Y_x is imposed.

This chapter will describe ways of evaluating the EPV when Y_x is:

- **constant – this evaluation has already been dealt with in Chapter 1, and the function of interest is A_x.**
- **changing by a constant compound rate – we will use A values at an adjusted interest rate.**
- **changing by a constant monetary amount – we will use tabulated factors which increase by 1 per annum.**

Corresponding to the assurance evaluation above we will discuss a similar approach to evaluating annuity benefits where the variation follows one of the patterns just described.

In general, the EPV of an annuity of amount F_{x+t} payable on survival to age $x+t$ to a life currently aged x, assuming, for example, immediate annual payments in arrear, would be evaluated directly from:

$$F_{x+1}\, v \frac{l_{x+1}}{l_x} + F_{x+2}\, v^2 \frac{l_{x+2}}{l_x} + \ldots + F_{x+t}\, v^t \frac{l_{x+t}}{l_x} + \ldots$$

Having evaluated the appropriate assurance and annuity factors, the equivalence principle may then be used to calculate the required premiums and reserves.

The notation in this chapter for age x can be taken to mean ultimate mortality is being assumed. The algebra and definitions are identical if we assume select mortality, in which case x will be replaced with $[x]$.

2 Payments varying at a constant compound rate

Consider first a whole life assurance issued to a life aged x where the benefit, payable at the end of the year of death, is $(1+b)^k$ if death occurs in the year of age $(x+k, x+k+1)$, $k=0,1,\ldots$.

Then the EPV of these benefits is:

$$\sum_{k=0}^{\infty} (1+b)^k v^{k+1} {}_{k|}q_x = \frac{1}{1+b} A_x^j$$

where the assurance function is determined on the normal mortality basis but using an interest rate, j, where:

$$j = \frac{(1+i)}{(1+b)} - 1$$

A similar approach may be derived for other types of assurance.

This approach may be used to allow for compound-decreasing benefits when b is negative.

Example

Calculate the expected present value of a whole life assurance taken out by a life aged 50, where:

- the basic sum assured is £100,000
- the sum assured increases by 1.9231% at the start of each year excluding the first
- the benefits are payable at the end of the year of death

Assume AM92 Ultimate mortality and 6% *pa* interest.

Solution

The expected present value of the benefit is:

$$100,000\left(vq_{50} + 1.019231v^2 \;_{1|}q_{50} + 1.019231^2 v^3 \;_{2|}q_{50} + \cdots\right)$$

$$= \frac{100,000}{1.019231}\left(\frac{1.019231}{1.06}q_{50} + \frac{1.019231^2}{1.06^2}\;_{1|}q_{50} + \frac{1.019231^3}{1.06^3}\;_{2|}q_{50} + \cdots\right)$$

$$= \frac{100,000}{1.019231}A_{50} \text{ @ } 4\%$$

$$= \frac{100,000}{1.019231} \times 0.32907$$

$$= £32,286$$

You will often see increases of 1.9231% and $i = 6\%$ in questions because it means we end up evaluating the benefit at 4% interest. However, do take care with questions like these, as you will often have to pull a factor out of the EPV to obtain a standard assurance function. (In the example above, we took out 1.019231^{-1}.)

To consider the evaluation of compound-varying survival benefits, consider, for example, an immediate annuity payable annually in arrear, with the benefit payable on survival to age $x+k$ being $(1+c)^k$, $k = 1,2,\ldots$.

Then the EPV of the annuity is:

$$\sum_{k=1}^{\infty} {}_k p_x \, (1+c)^k \, v^k = a_x^j$$

where the annuity function a_x^j is determined on the normal mortality basis but using an interest rate j, where $j = \dfrac{1+i}{1+c} - 1$.

This approach may be used to allow for compound-decreasing annuities by using a negative value for c.

3 Payments changing by a constant monetary amount

3.1 Whole life assurance

Consider, for example, a whole life assurance issued to a life aged x where the benefit, payable at the end of the year of death, is $k+1$ if death occurs in the year of age $(x+k, x+k+1)$, $k = 0,1,\ldots$.

The EPV of this assurance benefit is:

$$\sum_{k=0}^{\infty} (k+1) v^{k+1} \, {}_{k|}q_x$$

which is denoted by the actuarial symbol $(IA)_x$.

Values of this function are tabulated in, for example, the *Tables*.

Where tabulations are available, this provides the quickest way of evaluating such functions.

It is not logical to define a whole life assurance with constant decreases. If we did, it would be possible to have negative benefits!

3.2 Term assurance

An increasing temporary assurance can now be evaluated using the formula:

$$(IA)^{1}_{x:\overline{n}|} = (IA)_x - v^n \frac{l_{x+n}}{l_x}\left[nA_{x+n} + (IA)_{x+n}\right]$$

Question 6.1

A man aged exactly 40 buys a special 25-year endowment policy that pays £30,000 on maturity. If the policyholder dies before age 65, then all premiums paid so far are returned without interest at the end of the year of death. Level premiums are payable annually in advance for 25 years or until earlier death. Calculate the annual premium assuming AM92 Select mortality and 4% *pa* interest. Ignore expenses.

3.3 Endowment assurance

An increasing endowment assurance, with term *n* years, can be defined. An example is a payment of k paid at the end of the year of death of (x) if the life dies in policy year k $(k \le n)$, or a payment of n if (x) is still alive at the end of the term. The EPV of this can be evaluated using:

$$(IA)_{x:\overline{n}|} = (IA)^{1}_{x:\overline{n}|} + n\,A_{x:\overline{n}|}^{\;\;\;1}$$

3.4 Decreasing term assurance

A decreasing temporary assurance with a term of *n* years can also be defined. For example, suppose the benefit is *n* in the first year, and decreases by 1 per subsequent year. Then the EPV can be evaluated using the formula:

$$(n+1)A^{1}_{x:\overline{n}|} - (IA)^{1}_{x:\overline{n}|}$$

3.5 Increasing assurances payable immediately on death

Increasing assurances payable immediately on death can also be defined. For example, $(I\bar{A})_x$ is the expected present value of payments of $k+1$ paid immediately on death occurring in the year of age $(x+k, x+k+1), k = 0, 1, \ldots$. It can be calculated using the usual approximations, for example:

$$(I\bar{A})_x \simeq (1+i)^{½}\,(IA)_x$$

The formulae in Sections 3.2, 3.3 and 3.4 can be adjusted in a similar way to allow for immediate payment of the benefit on death.

Question 6.2

Calculate the value of $(I\bar{A})_{50:\overline{10}|}$ assuming AM92 Ultimate mortality and 4% *pa* interest.

3.6 *Whole life annuity payable annually in arrears*

In the case of an annuity that increases by a constant amount each year consider, for example, an immediate annuity payable annually in arrear, with the benefit payable on survival to age $x+k+1$ being $1+k$, $k=0,1,\ldots$.

The EPV of this annuity benefit is:

$$\sum_{k=1}^{\infty} k\, v^k\ {}_k p_x$$

which is given the actuarial symbol $(Ia)_x$.

3.7 *Whole life annuity payable annually in advance*

Similarly we can define the actuarial symbol:

$$(I\ddot{a})_x$$

to represent the EPV of an annuity due with the first payment being 1 and subsequent payments increasing by 1 per annum.

Values of this function are tabulated in, for example, AM92 in the "Formulae and Tables for Examinations".

It is not logical to define an immediate annuity with constant decreases.

Question 6.3

Calculate the value of $(Ia)_{50}$ assuming AM92 mortality and 4% *pa* interest.

3.8 *Temporary annuities*

Increasing temporary annuities can now be evaluated. For example, an increasing temporary annuity-due has an EPV given by:

$$(I\ddot{a})_{x:\overline{n|}} = (I\ddot{a})_x - v^n \frac{l_{x+n}}{l_x}\left[n\ddot{a}_{x+n} + (I\ddot{a})_{x+n}\right]$$

A decreasing temporary annuity with a term of n years can also be defined. For example, suppose we have an annuity-due with a payment of n in the first year, and decreasing by 1 per subsequent year. Then the EPV can be evaluated using the formula:

$$(n+1)\ddot{a}_{x:\overline{n}|} - (I\ddot{a})_{x:\overline{n}|}$$

3.9 *Annuities payable continuously*

Increasing annuities payable continuously can also be defined. For example, $(I\bar{a})_x$ is the EPV of an immediate annuity payable continuously, with the (level) benefit payable over the year of age $(x+k, x+k+1)$ being $1+k$, $k=0,1,\ldots$. The approximate calculation of this is:

$$(I\bar{a})_x \simeq (I\ddot{a})_x - \tfrac{1}{2}\ddot{a}_x$$

The formulae in Section 3.8 can be adjusted in a similar way to allow for continuous annuity payments.

Assurances and annuities that increase (or decrease) continuously at constant monetary rates are not covered by the CT5 syllabus.

Question 6.4

Calculate the value of $(I\bar{a})_x$ assuming that (x) is subject to a constant force of mortality of 0.02 *pa* and that $\delta = 0.04$ *pa*.

4 *Calculating net premiums and net premium reserves for contracts with variable benefits*

4.1 *Net future loss random variable*

Here we are concerned with the future loss that relates purely to benefit payments (including all future benefit increases), and the premiums that would be required to pay for these.

We will give an example of the determination of the net future loss random variable assuming compound or simple increases. We will calculate the net future loss random variable at integral policy duration t. By convention, this is taken to be just before any premium is paid, but just after any increase has been made, and any claims have been paid relating to year t.

Whole life assurance policies with compound increases

We assume the same contract as described at the start of Section 2 above.

This is a whole life assurance issued to a life aged x where the benefit, payable at the end of the year of death, is $(1+b)^k$ if death occurs in the year of age beginning at exact age $x+k$. With this contract, the sum assured increases (by multiplying the previous sum assured by $(1+b)$) at the *end* of each year.

Premiums of P are payable annually in advance throughout life. The total sum assured as at (integral) **duration t will be $(1+b)^t$.**

The net future loss random variable is then:

$$(1+b)^{t+K_{x+t}}\, v^{K_{x+t}+1} - P\ddot{a}_{\overline{K_{x+t}+1}|}$$

where v is the discount function determined at the required rate of interest of i *pa*.

The first term is the present value of the payment of the sum assured, including the existing increases, and all future increases expected to be added over the remaining lifetime of the policyholder. There have been t years of past increases so far, and as the policyholder is aged $x+t$ at time t, so K_{x+t} is the number of future annual increases that occur up to and including the start of the year in which the policyholder dies.

The total sum assured paid on death (in $K_{x+t}+1$ years time) will therefore be:

$$(1+b)^t \times (1+b)^{K_{x+t}} = (1+b)^{t+K_{x+t}} .$$

To obtain the present value, we need to discount this from the end of the year of death, that is by $K_{x+t}+1$ years, in the usual way.

The second term is simply the present value of the future net premiums, paid in advance for $K_{x+t}+1$ years. (We describe the calculation of these premiums in the next section.)

If we assume that increases are made continuously, and premiums are payable continuously, then the expression becomes:

$$(1+b)^{t+T_{x+t}}\, v^{T_{x+t}} - P\bar{a}_{\overline{T_{x+t}}|}$$

Question 6.5

Why is the $(1+b)^t$ term unaltered?

Question 6.6

Write down the corresponding formula if premiums are paid continuously but increases are made at the end of the year and benefits are paid immediately on death.

Whole life assurance policies with simple increases

Now consider a whole life assurance issued to a life aged x where the benefit, payable at the end of the year of death, is $(1+kb)$ if death occurs in the year of age $(x+k, x+k+1), k=0,1,2,\dots$. Premiums of P are payable annually in advance throughout life. The total sum assured as at duration t will be $(1+tb)$.

The net future loss random variable (at duration t) **is then:**

$$\left[1+(t+K_{x+t})b\right] v^{K_{x+t}+1} - P\ddot{a}_{\overline{K_{x+t}+1}|}$$

4.2 *Net premiums*

We determine the net annual premium, P, for a contract by determining the expected value of the future loss random variable at the date of issue of the policy $(t=0)$, setting its expected value equal to zero and solving for P.

Example: Whole life assurance policy with compound increases

Consider again a whole life assurance issued to a life aged x where the benefit, payable at the end of the year of death, is $(1+b)^k$ if death occurs in the year of age $(x+k,\, x+k+1)$, $k=0, 1, 2, \ldots$, and premiums of P are payable annually in advance for the whole of life.

For this contract, the net future loss random variable at time $t=0$ is:

$$(1+b)^{K_x}\ v^{K_x+1} - P\ddot{a}_{\overline{K_x+1|}}$$

which has an expected value of:

$$\sum_{k=0}^{\infty} (1+b)^k\ v^{k+1}\ {}_{k|}q_x - P\ddot{a}_x = \frac{1}{1+b} A_x^j - P\ddot{a}_x$$

Here the assurance function is determined on the normal mortality basis but using an interest rate, j, where:

$$j = \frac{(1+i)}{(1+b)} - 1$$

So the net premium P is found by solving the equation of value:

$$P\ddot{a}_x = \frac{1}{1+b} A_x^j$$

Similar expressions may be developed for policies where payments are made immediately on death, and for endowment assurance policies.

Question 6.7

Calculate the annual premium that should be paid annually in advance by a 40-year old policyholder, for a 20-year endowment assurance with a starting sum assured of £40,000, and where death benefits are payable immediately on death. The sum assured will increase at a guaranteed rate of 1.92% *pa* compound at the end of each policy year.

Assume an interest rate of 6% *pa* and AM92 Ultimate mortality.

Example: whole life policy with simple increases

Now consider again a whole life assurance issued to a life aged x where the benefit, payable at the end of the year of death, is $(1+kb)$ if death occurs in the year of age $(x+k, x+k+1)$, $k=0,1,2,\ldots$, and premiums of P are payable annually in advance for the whole of life.

The net future loss random variable at time $t=0$ is:

$$(1+bK_x)v^{K_x+1} - P\ddot{a}_{\overline{K_x+1|}}$$

This has an expected value of:

$$\sum_{k=0}^{\infty} v^{k+1}\,{}_{k|}q_x + b\left\{\sum_{k=0}^{\infty}(k+1)v^{k+1}\,{}_{k|}q_x - \sum_{k=0}^{\infty} v^{k+1}\,{}_{k|}q_x\right\} - P\sum_{k=0}^{\infty} v^k\,{}_kp_x$$

$$= A_x + b(IA)_x - bA_x - P\ddot{a}_x$$

So the net premium P is found by solving:

$$P\ddot{a}_x = (1-b)A_x + b(IA)_x$$

Similar expressions may also be developed for policies where payments are made immediately on death and for endowment assurance policies.

4.3 *Net premium reserves*

The net premium prospective reserve at policy duration t is obtained by determining the expected value of the net future loss random variable at policy duration t, and evaluating this expression when the premium is the net premium determined using the methods described above.

Example

Suppose we have a whole life policy, with initial sum assured 10,000 payable at the end of the year of death. The sum assured increases at a guaranteed rate of 2% *pa* simple at the end of each year. How do we calculate the net premium reserve for this contract?

We need to follow the principles of Chapter 5, but allowing fully for the future sum assured increases, both in the net premium calculation and in the valuation of the expected future loss.

First we calculate the net premium. For a policyholder aged x exact at entry, this is found by solving:

$$P\ddot{a}_x = 9{,}800A_x + 200(IA)_x$$

$$\Rightarrow P = \frac{9{,}800A_x + 200(IA)_x}{\ddot{a}_x}$$

The factor of $200(IA)_x$ will value a payment of 200 on death in the first year, 400 on death in the second year, and so on. However, as the increases apply only at the end of the year, we need to add only $9{,}800A_x$ to this in order to ensure that the total death benefit in year 1 is 10,000, increasing by 200 in each future year.

The net premium reserve in after exactly t future years $(t = 1, 2, ...)$ is:

$${}_tV = (9{,}800 + 200t)\,A_{x+t} + 200\,(IA)_{x+t} - P\ddot{a}_{x+t}$$

Question 6.8

Calculate the net premium reserve after exactly 6 years, for the contract described in the previous example, according to the following additional information:

- entry age 45
- level annual premiums are payable for the whole of life
- interest 4% *pa*
- mortality AM92 Ultimate.

Now try a slightly different example in this next question.

Question 6.9

An index-linked annuity payable annually in arrears is sold to a 60-year old male. The first payment is £8,160 and the annuity is assumed to increase at 2% *pa*. Calculate the net premium reserve at time 4 assuming PMA92C20 mortality and an interest rate of 6.1% *pa*.

5 Conventional with-profits contracts

In this section we see how with-profits business operates. These ideas will be extended in the life insurance course, ST2.

A conventional whole life or endowment policy can be issued on a without-profit or a with-profits basis. On a without-profit basis, both the premiums and benefits under the policy are usually fixed and guaranteed at the date of issue.

When the Core Reading says that the benefits are fixed, it does not mean they are constant. The benefits amount may be different from one year to the next, but the amount of the benefit in each year is known from the outset. All the contracts with increasing or decreasing benefits described in Section 4, for example, are examples of conventional without-profits contracts.

On a with-profits basis the premiums and/or the benefits can be varied to give an additional benefit to the policyholder in respect of any emerging surplus of assets over liabilities following a valuation. For example, surplus might be used to reduce the premium payable for the same benefit or to increase the sum assured without any additional premium becoming payable.

An alternative way of distributing surplus is to make a cash payment to policyholders.

Where surplus is distributed so as to increase benefits, additions to the sum assured are called bonuses.

At first glance it might seem undesirable from the perspective of both the policyholder and the life insurance company to have contracts whose final benefit is very uncertain. So why has with-profits business arisen? Consider a life company deciding on the premium basis for an endowment assurance contract. The future interest rate assumed will be a critical parameter. If the company calculates premiums assuming an interest rate of, say, 6% over the next twenty years, but actual interest rates fall below that level, then the company will almost certainly make a loss on those policies. On the other hand, if interest rates over the term are significantly in excess of 6%, then the policyholders will feel hard done by in comparison with the benefits that they might have received from other mediums such as bank savings.

So it makes sense for both the company and the policyholder to assume a low rate of interest in determining the premiums that are required to meet the initial sum insured, and then to distribute bonuses to policyholders when investment returns exceed those assumed in the premium basis.

So far we have discussed only interest. However, the same principles also apply to mortality and expenses; under with-profits business, the company can make slightly pessimistic assumptions about mortality and future expenses when setting premiums, and then pay bonuses to policyholders when mortality and expense experience prove better than that assumed in the premium basis.

The major types of contract for which a with-profits treatment is suitable are:

- endowment assurances,
- whole life,
- deferred annuity, and
- immediate annuity.

Note that with-profits contracts can be regular or single premium.

One type of with-profits contract in the UK is an endowment assurance with bonuses added to the sum assured.

Example

A policyholder, wanting to provide for his pet gerbil colony on retirement, has just taken out a with-profits endowment assurance policy with Abingdon Life. The sum insured is £10,000. At the end of the first year, Abingdon Life declares a bonus of 4% of sum insured. The sum insured is now increased to £10,400. If there were to be no more bonus declarations, the sum insured would remain £10,400 until eventual maturity.

When the company receives a premium from a policyholder with a with-profits contract, that premium (or part of it) will be invested. At that moment, the value of assets will equal the value of liabilities, as far as that premium is concerned (if the company uses the same assumptions in determining reserves as it uses in determining the premiums). Now over the course of the year we expect experience to be slightly better than assumed, because our assumptions – for instance, the investment return we assumed – were prudent.

Thus we would normally expect some of the following events to have occurred:

- investment returns on assets are greater than assumed,
- number of death claims is lower than assumed, and
- the amount of expenses is lower than assumed.

So by the end of the year, the assets in respect of the contract will have grown by more than the value of the liabilities. This is what we mean by the creation of surplus.
What next? Well, now that the company has this surplus, it will want to distribute it to the policyholder as a bonus. If the company wants to distribute all of the surplus there and then to the policyholder, it will calculate an increase to the sum insured such that the new value of the liabilities (calculated using the increased sum insured) is equal to the value of the assets.

In practice, companies will determine the surplus generated and bonus to be distributed using very broad groupings (*eg* all endowment assurances of term 10 years issued 4 years ago) rather than on a policy-by-policy basis.

Example

A life company has calculated that the amount of surplus attributable to a with-profits single premium whole life contract issued nine years ago to a 44-year old male with guaranteed sum insured of \$90,000 is \$2,800. What bonus should be declared?

Solution

The company will need to calculate an additional amount of sum insured S' such that

$$S' A_{53} = 2,800$$

As we shall see below, this extra sum insured will normally be presented as a percentage increase to the sum insured.

Question 6.10

Bonuses are usually funded by distributing part of the surplus. Why might the company distribute just part of the surplus rather than all of it?

5.1 Types of bonus

Various methods of allocating bonuses have been developed, each intended to provide a way of matching the surplus emerging over the duration of the policy. In the past the choice of methods was restricted by the difficulties of completing a valuation quickly and cheaply and by the difficulties of allocating complex bonuses to individual policies. Modern record keeping systems have largely removed these difficulties, and current systems are chosen to match the bonus distribution philosophy conveyed to the policyholders when the policy was issued (in the UK this is called policyholders' reasonable expectations).

By bonus distribution philosophy, we mean matters such as:

- what form bonuses take (of the various forms described below)
- what portion of surplus the company distributes to policyholders
- what degree of smoothing the company operates (if investment returns in one year are very good, do we distribute all of the resultant surplus immediately or hold some back to compensate in later years of poorer investment returns?) and
- to some extent, the broad investment strategy of the company (will policyholders expect bonuses in line with equity market performance, for example?).

Clearly what the company says to potential policyholders at the policy sale stage about these aspects will create justifiable expectations, which the company should then try to meet over the lifetime of the policy.

Bonuses are usually allocated annually, which is likely to tie in with the minimum required frequency of valuation for each insurer. Once added to the sum assured, bonuses become guaranteed benefits, which then need to be reserved for. The implication of this is that the sooner, and greater, the rate at which bonuses are added, the more conservative the insurer is likely to be in choosing its investments.

When the company declares a reversionary bonus of, say, 6% of sum insured then it is not actually paying out any money in cash right now. It is merely promising that the amount of money paid on claim or maturity will be 6% greater than was previously the case. So cashflows are unchanged until the moment when the policies in question terminate, *ie* when the *claims* are paid out.

Bonuses can be distributed more slowly, or at a lower rate, which may allow the insurer to choose investments that are more volatile in the short term, but are expected to be more profitable in the long term. A highly effective example of this is choosing to distribute part of the available surplus as a terminal bonus, rather than as an annual bonus. Terminal bonuses are allocated when a policy matures or becomes a claim as a result of the death of the life assured.

Terminal bonuses are usually allocated as a percentage of the basic sum assured and the bonuses allocated prior to termination. The percentage rate will vary with the term of the policy at the date of payment. Because the policy is being terminated, the terminal bonus rate is usually chosen so as to distribute all the surplus available to the policy, based on asset share.

The asset share of a policy is the accumulation of premiums at actual investment returns, less expenses and the cost of cover. We can quantify the surplus in respect of any policy by comparing the asset share at maturity with the sum assured plus attaching bonuses payable at maturity. Asset share is critical for with-profits policies and can be seen as the level consistent with policyholders' reasonable expectations.

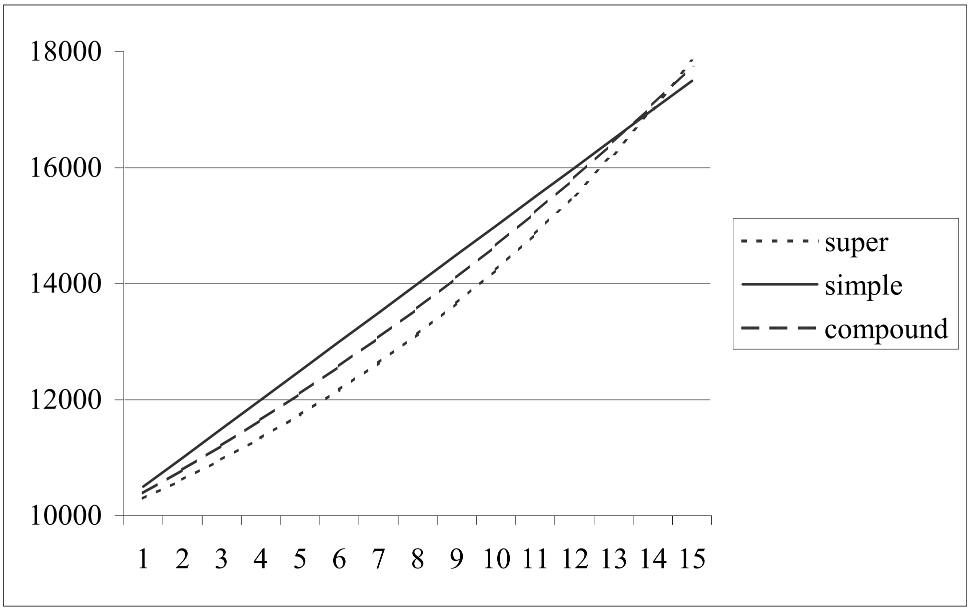
Typically, bonuses are added by a mixture of annual and terminal components. The annual bonuses will be at variable rates determined from time to time by the insurer based on actual arising surpluses. Typically, annual bonuses are added according to one of the following methods:

- **Simple – the rate of bonus each year is a percentage of the initial (basic) sum assured under the policy. The sum assured will increase linearly over the term of the policy.**
- **Compound – the rate of bonus each year is a percentage of the basic sum assured and the bonuses added in the past. The sum assured increases exponentially over the term of the policy.**
- **Super compound – two compound bonus rates are declared each year. The first rate (usually the lower) is applied to the basic sum assured. The second rate is applied to the bonuses added to the policy in the past. The sum assured increases exponentially over the term of the policy. The sum assured including bonuses increases more slowly than under a compound allocation in the earlier years, but faster in the later years.**

Example

Compare the build up of sum assured over the lifetime of a 15-year policy with the following alternative methods, assuming an initial sum assured of £10,000:

- simple bonus at 5% *pa*
- compound bonus at 3.9% *pa*
- super compound bonus at 3% *pa* on basic sum assured, 7.5% *pa* on bonuses



We see from the graph that:

- the sum assured under a simple bonus arrangement is a straight line
- under a compound bonus arrangement, the sum assured increases slowly at first and more quickly later in the policy
- under a super compound arrangement, this feature becomes more pronounced, with lower increases earlier on being compensated for by greater increases later on in the policy.

The rates have been chosen so that the final value of the sum assured is approximately the same at the end of 15 years under each of the three different arrangements.

Question 6.11

For the three levels of bonus allocation specified in the above example, given an initial guaranteed sum insured of £10,000, calculate the sum insured at the end of Years 1, 2 and 3.

Annual bonuses are an allocation in arrear to reflect the growth in the available surplus since the last valuation. At each valuation it is usual to declare a rate of interim bonus which will be applied to policies becoming claims before the next valuation so as to provide an allocation of bonus for the period from the last valuation to the date of the claim. This rate is applied in the same way as that used for annual bonuses.

An anticipated bonus rate is usually loaded in premium rates by choosing (conservative) rates of bonus allocation and valuing these as benefits in determining the premium to be charged for the policy. The additional premium (above that that would be charged for a without-profit contract with the same (basic) sum assured) is termed the bonus loading.

We saw earlier how with-profits business arose because it allowed companies to price products using conservative interest rate assumptions, which did not hurt the policyholder because as experience proved better than these assumptions the policyholders would see their benefit increased accordingly. Many companies now go a step further than this. They want to be reasonably sure of being able to pay significant bonuses, which are of similar size to the bonuses which competitors will be paying.

So rather than choose a prudent interest rate, which in reality we expect to exceed by, say 1 to 2% *pa* over the policy's lifetime, giving an equivalent amount of bonus, companies will load explicitly for some expected level of reversionary bonus. When doing this it is common to allow for terminal bonus implicitly via a slightly conservative interest rate assumption.

Example

A company expects an investment return of 8.5% on the assets backing its with-profits business over the foreseeable future. Instead of just pricing with-profits policies on 8.5% less some margin for prudence to give, say, a premium basis interest rate of 7% (in which case we would expect to see bonuses of about 1.5% *pa*), the company wants to price to ensure that it can support reversionary bonuses of 3% *pa* (added at the start of each year, using the simple distribution system) and some eventual terminal bonus.

The company therefore assumes a slightly prudent interest rate, *eg* 7.5%, so that the margin of 1% *pa* will build up to a reasonable amount of terminal bonus. The company then calculates premiums assuming this interest rate of 7.5% but valuing as an explicit benefit a simple future reversionary bonus of 3% *pa*.

Thus, for an endowment assurance issued to a life aged x, with a term of n years and a sum assured S payable at the end of the year of death, the premiums will need to meet a benefit with expected present value:

$$S\,A_{x:\overline{n|}} + 0.03\,S\,(IA)_{x:\overline{n|}} \quad \text{with } i = 7.5\%$$

Question 6.12

In the above example, how might we allow for a reversionary bonus of 3% *pa* if we declare bonuses using the compound system?

5.2 Net premium reserves for with-profits policies

These are calculated using the following principles.

For reasons explained below, the net premium used to calculate net premium reserves for with-profits policies, at any duration $t \geq 0$, is always a net premium calculated at outset of the policy, based only on the guaranteed benefit at the outset. There is no explicit allowance for any bonuses expected to be earned over the lifetime of the contract.

The determination of the expected values at policy duration t uses the same methodology as that used to determine the expected values at policy duration 0.

The expressions for net premium reserves at duration t can be determined by direct evaluation using life table and discount functions.

Finally, it should be noted that the calculation of prospective net premium reserves will be done according to mortality and interest assumptions specifically chosen for the purpose, and these are referred to as the net premium valuation basis. These assumptions will normally be different from the underlying basis used to calculate the gross premiums (as described later in Chapter 7).

For with-profits policies, the eventual form of the net premium prospective reserve is (A) – (B):

The expected present value of benefits, allowing for bonuses to date (but not future bonuses) (A)

minus

the expected present value of net premiums, where these are calculated at the outset of the policy on the basic sum assured (ignoring all bonuses). (B)

Using the terminology developed above, the net premium prospective reserve at time t for a whole life policy is, in the terminology of the life table:

$$(S + B_t)A_{x+t} - P\ddot{a}_{x+t}$$

where:

$$P = \frac{S\,A_x}{\ddot{a}_x}$$

and B_t represents the total amount of bonuses added up to and including time t, and is of known amount.

Example

A life insurance company issues a 25-year with-profits endowment assurance to a life aged 40. The basic sum assured of £50,000 and any attaching bonuses are payable on maturity or at the end of the year of earlier death. Bonuses are expected to be 2% *pa* simple, vesting at the end of each policy year. Premiums are payable annually in advance throughout the term of the policy.

Calculate the net premium prospective reserve for this policy at time 10, assuming that bonuses have been in line with expectations, mortality is AM92 Ultimate, and interest is 4% *pa*.

Solution

The net premium P, making no allowance for any bonuses, is given by:

$$P\ddot{a}_{40:\overline{25}|} = 50,000A_{40:\overline{25}|} \Rightarrow P = \frac{50,000 \times 0.38907}{15.884} = £1,224.72$$

We are told that bonuses vest, or are added, at the end of each policy year. Assuming that bonuses are in line with expectations by the end of Year 10, 10 simple bonuses of $0.02 \times 50,000 = £1,000$ will have been added. So the sum assured will have risen to £60,000, and the net premium prospective reserve at time 10 is:

$$\begin{aligned}{}_{10}V^{pro} &= 60,000A_{50:\overline{15}|} - 1,224.72\,\ddot{a}_{50:\overline{15}|} \\ &= 60,000 \times 0.56719 - 1,224.72 \times 11.253 \\ &= £20,249.63\end{aligned}$$

Alternatively we could have produced the net premium reserve ignoring all bonuses using the ${}_tV_{x:\overline{n}|}$ *formula, and adding the EPV of the declared bonuses in afterwards. This would give (with differences due to rounding errors):*

$${}_{10}V^{pro} = 50,000\,{}_{10}V_{40:\overline{25}|} + 10,000A_{50:\overline{15}|} = 50,000\left(1 - \frac{\ddot{a}_{50:\overline{15}|}}{\ddot{a}_{40:\overline{25}|}}\right) + 10,000 \times 0.56719$$

$$= 50,000\left(1 - \frac{11.253}{15.884}\right) + 5,671.90 = £20,249.46$$

Question 6.13

A 20-year with-profits endowment with a basic sum assured of \$15,000 was issued to a man aged 45 exact at entry. The policy has been in force for 7 years.

The policy is subject to simple reversionary bonuses that vest at the start of each year, and death benefits are paid at the end of the policy year of death. In addition, a terminal bonus equal to 25% of the sum assured plus existing bonus, is payable on a death or maturity claim.

Simple bonuses were declared at the rate of 4% for the first year of the policy, and 3.75% *pa* thereafter to date. The company expects to pay future bonuses at a rate of 3.5% *pa* from now on.

Calculate the net premium reserve for this policy as at the present time. Assume AM92 Ultimate mortality and 4% *pa* interest.

Rationale for with-profits policies

In this section we are explaining the logic behind the method; there is no new methodology described here.

The form of the net premium reserve, where benefits allow for some bonuses and the premium for none, may appear to be somewhat artificial and lacking in logic. However, there is some underlying rationale.

Recall that the net premium reserve takes the form of $(\mathrm{A})-(\mathrm{B})$ where:

Part (A) values the total guaranteed sum assured, allowing for bonuses to the point of calculation of the reserve, because this is the amount of sum assured to which policyholders have become entitled at that point. There is no contractual entitlement to any future bonus, and so they are not valued as part of the reserve.

Part (B) values the net premium, ignoring all bonuses, because this gives a higher (more prudent) reserve than would be obtained by loading the premium for any bonuses. Crucially in practice, as will be seen in Chapter 7, the life insurance company will charge a much higher gross premium which is loaded for bonuses and expenses. Therefore, as the with-profits contract progresses each year, the difference between gross and net premium will emerge as an item of valuation surplus each year. So, by ignoring the expense and bonus loadings in the calculation of the net premium, we have made an implicit prudent allowance for the cost of meeting future bonuses and expenses. These are key considerations under policyholders' reasonable expectations.

6 Accumulating with-profits contracts

These contracts originated in the UK, and now form almost all of the new with-profits business sold by UK insurers at the present time.

6.1 Definition

Under an accumulating with-profits (AWP) contract, the basic benefit takes the form of an accumulating fund of premiums (rather like a unit-linked policy, which is described in a later chapter). If the accumulating fund at time t is denoted by F_t, the simplest form of an AWP contract follows the following recursive formula:

$$F_t = \left(F_{t-1} + P\right)(1 + b_t)$$

This example assumes that annual premiums of P are payable at the start of each year. b_t is the annual bonus interest declared for year t. As in the case of the regular reversionary bonus described for conventional with-profits contracts, this is a discretionary amount determined by the insurance company each year. It will reflect both the returns achieved on the underlying assets over the period plus any additional profits made on the contract in this time. As it is discretionary, it does not exactly reflect these amounts, and in practice the insurer tends to smooth out the variations in returns and profits achieved from year to year to produce a bonus interest rate that is more stable over time than the underlying asset returns, for example. A key feature of the regular bonus interest is that it cannot be negative, whereas for certain asset types (*eg* equity portfolios) actual returns can be negative.

Question 6.14

A man pays a premium of £7,000 at the start of each year into an accumulating with-profits contract. Calculate the policy fund value after 3 years if the insurer declares annual bonus interest rates as follows:

year 1: 2.3%

year 2: 2.6%

year 3: 2.5%

Sometimes, as was often the case in the UK in the past, part of the bonus interest would be guaranteed. One way of including a guaranteed bonus interest rate of g per annum is shown in the following recursive formula:

$$F_t = (F_{t-1} + P)(1+g)(1+b_t)$$

An alternative way of incorporating a guaranteed bonus interest rate is simply to ensure that the total bonus interest of b_t cannot be lower than the guaranteed rate of g each year.

It is unusual for any guaranteed rate to be applied to AWP in modern conditions (other than the degenerate case where $g = 0$).

This is because investment returns in general are currently too low to enable insurance companies to offer higher guaranteed rates without risking significant losses. However, interest rates have been much higher historically, and many of the early AWP contracts included significant guarantees: in the 1980s guarantees of 3% and 4% *pa* were not uncommon.

As with conventional with-profits, the regular bonuses under AWP can be reduced so as to retain profit for subsequent deferred payment as a terminal bonus. The contractual benefit under an AWP policy (payable on death or maturity as appropriate) could then be defined as:

$$B_t = F_t + T_t$$

where T_t is the amount of terminal bonus payable on a claim at time t. The purpose and rationale for paying terminal bonus is the same under AWP as it is for conventional with-profits.

Apart from the terminal bonus component, these simple AWP contracts operate in a very similar way to a deposit account administered by a bank.

Question 6.15

A man currently aged 42 exact wishes to provide himself with a pension of around £25,000 *pa* on his retirement at age 67. He intends to purchase an accumulating with-profits endowment policy that will mature on his 67th birthday, and which he hopes will provide enough funds at retirement to purchase the required pension.

Calculate the annual premium that would provide the required expected amount of pension based on the following assumptions:

– premiums are level and are paid at the start of each year throughout the duration of the AWP contract, which has no explicit charges

– the insurance company declares an annual bonus interest rate of 3.5% *pa* throughout the duration of the AWP contract

– terminal bonus is ignored

– the annuity is to be paid monthly in advance for the whole of life from age 67, without guarantee

– the insurance company projects that it will use the following annuity basis to convert cash into annuity payments at the time of retirement:

Interest:	4% *pa*
Mortality:	PMA92C20 with a 7-year deduction from the age
Expenses:	Ignored

6.2 Unitised (accumulating) with-profits contracts

Many companies that sell AWP administer the contract in unitised form (called "unitised with-profits" (UWP)). The policyholder is allocated units and the fund value at any time for any policy is equal to the number of units held multiplied by the current price (or value) of each unit at that time.

In this way, UWP operates in a very similar way to unit-linked contracts, which we describe in a later Chapter. A key difference is the way the unit price is calculated. Two example possibilities are:

Method (1) the unit price allows for guaranteed bonus interest increases only; the discretionary bonus is credited to the policy by awarding additional (bonus) units from time to time

Method (2) the unit price allows for both guaranteed and bonus interest increases.

In both cases, it would be normal for unit prices to be changing on an effectively continuous basis (*eg* daily). The company would declare its regular bonus interest rate in advance, so that interest would accrue to policies at the equivalent daily rate.

Example

The unit price for an insurer's UWP contracts at the start of a year is £1.60. There is a guaranteed bonus interest rate of 2% per annum, and for the coming year the discretionary bonus interest is to be 1.5% per annum. The insurer operates its UWP contracts using method (2).

A policyholder holds 2,000 units at the start of the year, immediately prior to paying a monthly premium of £400. Calculate the value of this policyholder's fund in 16 days time assuming there are 365 days in a year and that no deductions are made from the policyholder's fund during this time.

Solution

The number of new units allocated as a result of the premium payment is:

$$\frac{400}{1.6} = 250$$

so that the policyholder now has 2,250 units in total.

Over the next year, the unit price is increasing at a daily compound rate of:

$$(1.02 \times 1.015)^{1/365} - 1 = 9.5049 \times 10^{-5}$$

The unit price of the contract in 16 days time is therefore:

$$1.6 \times 1.000095049^{16} = 1.602435$$

Therefore the policyholder's fund value at this time is:

$$2,250 \times 1.602435 = £3,605.48$$

Notice that to perform this calculation we do not need to calculate the unit prices and the number of units separately. We can arrive at the same figure by using unit fund values directly.

Therefore, at the start of the year, the unit fund value is $2{,}000 \times 1.6 = £3{,}200$. The calculation then proceeds using the recursive formula:

$$(3{,}200 + 400) \times (1.02 \times 1.015)^{16/365} = 3{,}600 \times 1.000095049^{16} = £3{,}605.48$$

6.3 *Charges and benefits under UWP*

The unitised nature of UWP means that it is easy to make allocations or deductions from the policyholder's fund at any time. As a result, much more complex product designs have been developed (often mirroring the unit linked products that might be issued by the same insurance company). For UWP, insurers typically make explicit deductions for expense (and other) charges, as appropriate.

For example, in the example given in Section 6.2 an expense charge of £4 might be made at the start of the period. In that case, the unit fund value at time 16 would have been:

$$(3{,}200 + 400 - 4) \times 1.000095049^{16} = £3{,}601.47$$

Question 6.16

Continuing the example given in Section 6.2, a charge of £4 is actually deducted on 17th January (*ie* 16 days after the start of the year), rather than at the start of the period. A second premium of £400 is paid on 1st February.

Calculate:

(a) the number of units cancelled on 17th January to meet the expense charge

(b) the number of units created on 1st February as a result of the premium payment

(c) the value of the fund on 14th February.

A disadvantage of the AWP designs that we have described so far is that the benefit paid on early death, if it is just the fund value plus a terminal bonus, is likely to be very small. Any policyholder requiring life insurance cover under AWP therefore needs an additional death benefit to be incorporated into the contract.

This is straightforward to include under a UWP contract. If a minimum death benefit of *S* was to be applied, then the death benefit payable on death at time *t* would be calculated as:

$$\boldsymbol{D_t = \max\left[S, B_t\right] = \max\left[S, F_t + T_t\right]}$$

A minimum sum assured could also be applied at maturity in theory, but this is uncommon in practice.

A regular charge would then typically be deducted from the policyholder's fund to pay for the cost of providing the additional death (and/or maturity) benefit.

The charge for the minimum death benefit would be taken regularly, possibly annually but more likely on a monthly basis, and it would be proportional to the expected cost of providing the extra death benefit over the year (or month).

So the charge at policy duration t might be calculated as:

$$q'_{x+t} \times \max\left\{S - F_t, 0\right\}$$

where q'_{x+t} is the probability of the policyholder (aged $x+t$ at duration t) dying over the next year (or month), S is the guaranteed sum assured payable on death, and F_t is the fund value at time t.

Question 6.17

Two UWP contracts (*A* and *B*) were issued on the same day, both for the same annual premium of £5,000 and for the same term to maturity of 20 years. Policy *A* had a minimum death benefit of £50,000, while Policy *B* had no minimum death benefit. The policies were otherwise identical.

Explain the differences you would expect between the two policies in terms of the likely amounts of their:

(a) death benefits in 5 years time

(b) death benefits in 15 years time

(c) maturity benefits.

6.4 *Comparison between UWP and the simple AWP designs*

With the simple AWP design (described in Section 6.1), the bonus interest would distribute profits net of all expenses and other costs incurred. In this way it is similar to the with-profits approach that is embodied in conventional with-profits contacts.

In the case of UWP designs (described in Sections 6.2 and 6.3), explicit charges are made to cover the various expense and other costs incurred for the policy. The bonus rates declared would then be closely related to the rates of return obtained on the underlying assets only, smoothed (and possibly deferred) over time as usual. These contracts then fit in well with the unit-linked products that the same insurers might be offering.

It should be noted that AWP (and UWP) essentially provides an accumulating fund approach to with-profits. Many individual variations on the basic design are possible and it is therefore impossible to document them all in this Course. Students should be aware of the basic approach and main variations described above.

7 Exam-style questions

We finish this chapter with two exam-style questions. Both of these questions have been included deliberately at this stage in the notes because they involve variable benefits. However, they both also involve the evaluation of expenses and "gross premium reserves". Gross premium reserves are covered in the next chapter, so you may find it helpful to read Chapter 7 before returning to these questions. Alternatively, if you're feeling brave(!), you could have a go now:

Hint (for the brave!)

Gross premiums allow additionally for expenses and future bonuses. Gross premium reserves also allow additionally for expenses and future bonuses, and include gross premiums!

Question 6.18

A life insurance company issues a whole life assurance to a life aged x exact. The benefit is 1 if death occurs in the first year, 2 if death occurs in the second year, 3 if death occurs in the third year, and so on. The death benefit is payable at the end of the year of death. Level premiums are payable annually in advance. There is an initial expense of I and a renewal expense of e at the start of each year, including the first.

(i) Write down:

(a) the equation of value

(b) an expression for the gross premium prospective reserve at integer time t

(c) an expression for the gross premium retrospective reserve at integer time t. [3]

(ii) Show that the expressions in (i)(b) and (i)(c) are equal if the premium, the prospective reserve and the retrospective reserve are calculated using the same basis. [4]

[Total 7]

Question 6.19

On 1 May 1998, a life insurance company issued a conventional whole life with-profits policy to a life then aged exactly 45. The basic sum assured was £50,000. The sum assured and attaching bonuses are payable 3 months after the death of the policyholder. Level monthly premiums are payable in advance for the whole of life. The company calculated the premium on the following basis:

Mortality:	AM92 Select	
Interest:	6% *pa*	
Bonus loading:	1.9231% *pa* compound, vesting at the end of each policy year	
Expenses:	initial:	£300
	renewal:	5% of each premium, excluding the first
	termination:	£200 payable at the same time as the death benefit

(i) Show that the monthly premium is £85.65. [7]

The company holds gross premium retrospective reserves for the policy, calculated on the following basis:

Mortality:	AM92 Select	
Interest:	4% *pa*	
Past bonuses:	4% *pa* compound, vesting at the end of each policy year	
Expenses:	initial:	£300
	renewal:	£5 at the start of each month, excluding the first
	termination:	£100 payable at the same time as the death benefit

(ii) Calculate the reserve for the policy on 30 April 2005. [8]

[Total 15]

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Chapter 6 Summary

Variable benefits

In this chapter we have developed ways of calculating the expected present value of benefits that vary from one year to another according to some predetermined pattern.

Payments varying at a constant compound rate

Assurances

The expected present value of a whole life assurance issued to a life aged x where the benefit, payable at the end of the year of death, is $(1+b)^k$ if death occurs in the year of age $(x+k, x+k+1)$, $k=0,1,\ldots$, is:

$$\sum_{k=0}^{\infty}(1+b)^k v^{k+1} {}_{k|}q_x = \frac{1}{1+b} A_x^j$$

where A_x^j is a whole life assurance function determined using the normal mortality basis, but using an interest rate j such that $j = \dfrac{1+i}{1+b} - 1$.

A similar approach can be used for other types of assurance.

Annuities

The expected present value of a whole life annuity payable annually in arrears to a life aged x, with the benefit payable on survival to age $x+k$ being $(1+c)^k$, $k=1,2,\ldots$, is:

$$\sum_{k=1}^{\infty}(1+c)^k v^k {}_k p_x = a_x^j$$

where the annuity function a_x^j is evaluated using the normal mortality basis, but using an interest rate j, such that $j = \dfrac{1+i}{1+c} - 1$.

Payments changing by a constant monetary amount

Whole life assurance

The expected present value of a whole life assurance issued to a life aged x where the benefit, payable at the end of the year of death, is $k+1$ if death occurs in the year of age $\left(x+k, x+k+1\right)$, $k = 0,1,\ldots$, is:

$$\sum_{k=0}^{\infty} \left(k+1\right) v^{k+1} \;_{k|}q_x = \left(IA\right)_x$$

Term assurance

An increasing term assurance can be evaluated using the formula:

$$(IA)^1_{x:\overline{n}|} = (IA)_x - v^n \frac{l_{x+n}}{l_x}\left[nA_{x+n} + (IA)_{x+n}\right]$$

The expected present value of a decreasing term assurance with a term of n years, where the benefit is n in the first year and decreases by 1 *pa*, can be evaluated using the formula:

$$(n+1)A^1_{x:\overline{n}|} - (IA)^1_{x:\overline{n}|}$$

Endowment assurance

An increasing endowment assurance can be evaluated using the formula:

$$(IA)_{x:\overline{n}|} = (IA)^1_{x:\overline{n}|} + n\,(IA)_{x:\overline{n}|}^{\;\;\,1} = (IA)^1_{x:\overline{n}|} + n\frac{D_{x+n}}{D_x}$$

Assurances with death benefits payable immediately

The expected present value for an increasing whole life assurance, which pays immediately on death, is:

$$\left(I\bar{A}\right)_x \simeq (1+i)^{1/2}\left(IA\right)_x$$

Similar formulae can be used for the other assurance types.

Whole life annuity

The expected present value of a whole life annuity payable annually in arrears to a life aged x, with the benefit payable on survival to age $x+k$ being k, $k=1,2,\ldots$, is:

$$\sum_{k=1}^{\infty} k\, v^k \,{}_k p_x = (Ia)_x$$

The symbol $(I\ddot{a})_x$ denotes the expected present value of the corresponding annuity-due, where the first payment is 1 and payments increase at the rate of 1 *pa*, and:

$$(I\ddot{a})_x = (Ia)_x + \ddot{a}_x$$

Temporary annuity

An increasing temporary annuity-due can be evaluated using the formula:

$$(I\ddot{a})_{x:\overline{n}|} = (I\ddot{a})_x - v^n \frac{l_{x+n}}{l_x}\left[n\ddot{a}_{x+n} + (I\ddot{a})_{x+n}\right]$$

The expected present value of a decreasing temporary annuity-due with a term of n years, where the benefit is n in the first year and decreases by 1 *pa*, can be evaluated using the formula:

$$(n+1)\ddot{a}_{x:\overline{n}|} - (I\ddot{a})_{x:\overline{n}|}$$

Annuities payable continuously

The expected present value of a whole life annuity payable continuously to a life aged x, which pays at the rate of k in year k, $k=1,2,\ldots$, is:

$$(I\bar{a})_x \simeq (I\ddot{a})_x - \tfrac{1}{2}\ddot{a}_x$$

Similar adjustments can used for temporary annuities.

Conventional with-profits contracts

Conventional with-profits contracts are priced on a prudent basis. As experience proves better than the prudent assumptions, surplus arises. This surplus is distributed to policyholders as an increase to the benefit.

This increase can be done as a guaranteed reversionary bonus during the lifetime of the policy, using one of the following methods:

- simple
- compound
- super-compound

and also at claim or maturity as a terminal bonus.

The more a company defers the distribution of surplus as bonus, the greater working capital it keeps for itself with which to permit a more risky investment strategy and thus greater expected eventual investment returns. Terminal bonus can be viewed as an extreme example of such deferment.

Reversionary bonuses are often allowed for explicitly when determining the premium basis for a contract.

Accumulating with-profits (AWP)

The basic benefit is an accumulating fund of premiums with discretionary interest rates. The accumulation follows the recursive relation:

$$F_t = \left(F_{t-1} + p\right)(1+g)(1+b_t)$$

where g is the guaranteed annual interest and b_t is the bonus annual interest for year t.

On death or survival, the benefits can be further increased by a terminal bonus.

AWP contracts can be unitised (UWP) or non-unitised. For UWP, guaranteed interest is factored into the unit price. The bonus interest can be factored into the unit price also, or can be allocated by creating new units. Policies may include a guaranteed minimum death benefit. UWP is usually subject to explicit charges, to cover expenses and any additional death benefit cost.

Premiums and reserves

Premiums and reserves can be determined by consideration of the appropriate future loss random variable.

Net premium reserves for conventional with-profits contracts

When calculating a net premium prospective reserve for a conventional with-profits policy, remember that:

- the premium used in the reserve calculation:
  - is calculated using mortality and interest given in the reserving basis
  - makes no allowance for any expenses or bonuses
- the EPV of the future benefits used in the reserve calculation takes account of bonuses added so far, but makes no allowance for any future bonuses
- expenses are ignored.

Net premium reserves for conventional without-profit contracts

In this case:

- the premium used in the reserve calculation:
  - is calculated using mortality and interest given in the reserving basis
  - includes all future guaranteed benefit payments, including future increases or decreases in those benefits
  - makes no allowance for expenses
- the EPV of the future benefits used in the reserve calculation takes account of all benefit increases or decreases, both those that have already occurred and those that are contracted to occur in the future.
- expenses are ignored.

Where future benefit increases are of uncertain amount, but still non-discretionary (such as index-linked benefits), then a suitable assumption for the future rate of change in benefits would need to be made (*eg* an inflation assumption).

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Chapter 6 Solutions

Solution 6.1

Let P denote the annual premium. Then:

$$\text{EPV premiums} = P\ddot{a}_{[40]:\overline{25}|} = 15.887P$$

$$\text{EPV maturity benefit } = 30,000 \times \frac{D_{65}}{D_{[40]}} = 30,000 \times \frac{689.23}{2,052.54} = 10,073.81$$

$$\begin{aligned}\text{EPV death benefit } &= P\left(IA\right)^{1}_{[40]:\overline{25}|} \\ &= P\left[\left(IA\right)_{[40]} - \frac{D_{65}}{D_{[40]}}\left[\left(IA\right)_{65} + 25A_{65}\right]\right] \\ &= P\left[7.95835 - \frac{689.23}{2,052.54}\left(7.89442 + 25 \times 0.52786\right)\right] \\ &= 0.87615P\end{aligned}$$

So the equation of value is:

$$15.887P = 10,073.81 + 0.87615P$$

Solving this gives:

$$P = £671.10$$

Solution 6.2

We want:

$$\left(I\bar{A}\right)_{50\,\overline{10}|} = \left(I\bar{A}\right)^{1}_{50\,\overline{10}|} + 10A_{50\,\overline{10}|}^{\;\;1}$$

This is the EPV of:

- k paid immediately on death if the life dies in policy year k $\left(k \le 10\right)$
- 10 on surviving to the end of 10 years.

The acceleration of the payment only applies to the death benefit, which is why we need to split the payment into the death benefit and survival benefit components.

So:

$$\begin{aligned}\left(I\bar{A}\right)_{50\,\overline{10}|} &= (1+i)^{½}\left(IA\right)^{1}_{50\,\overline{10}|} + 10A_{50\,\overline{10}|}^{\;\;1} \\ &= (1+i)^{½}\left\{(IA)_{50} - \frac{D_{60}}{D_{50}}\left[(IA)_{60} + 10\,A_{60}\right]\right\} + 10\frac{D_{60}}{D_{50}} \\ &= 1.04^{½} \times \left\{8.55929 - \frac{882.85}{1,366.61} \times \left[8.36234 + 10 \times 0.45640\right]\right\} \\ &\qquad + 10 \times \frac{882.85}{1,366.61} \\ &= 6.673\end{aligned}$$

Solution 6.3

We want:

$$\left(Ia\right)_{50} = vp_{50} + 2v^2\,{}_2p_{50} + 3v^3\,{}_3p_{50} \cdots$$

But we have:

$$\left(I\ddot{a}\right)_{50} = 1 + 2vp_{50} + 3v^2\,{}_2p_{50} + \ldots$$

So:

$$(I\ddot{a})_{50} - (Ia)_{50} = 1 + vp_{50} + v^2\,{}_2p_{50} + \ldots$$

$$= \ddot{a}_{50}$$

Therefore:

$$(Ia)_{50} = (I\ddot{a})_{50} - \ddot{a}_{50} = 231.007 - 17.444 = 213.563$$

Solution 6.4

$(I\bar{a})_x$ is the expected present value of a whole life annuity payable continuously to a life now aged exactly x, where the rate of payment is 1 *pa* in the first year, 2 *pa* in the second year, 3 *pa* in the third year, and so on. So:

$$(I\bar{a})_x = \int_0^1 v^t\,{}_tp_x\,dt + \int_1^2 2v^t\,{}_tp_x\,dt + \int_2^3 3v^t\,{}_tp_x\,dt + \cdots$$

Assuming a constant force of mortality of 0.02 *pa*, we have ${}_tp_x = e^{-0.02t}$. Also since $v = e^{-\delta} = e^{-0.04}$, it follows that:

$$(I\bar{a})_x = \int_0^1 e^{-0.06t}\,dt + \int_1^2 2e^{-0.06t}\,dt + \int_2^3 3e^{-0.06t}\,dt + \cdots$$

$$= \left[-\frac{1}{0.06}e^{-0.06t}\right]_0^1 + \left[-\frac{2}{0.06}e^{-0.06t}\right]_1^2 + \left[-\frac{3}{0.06}e^{-0.06t}\right]_2^3 + \cdots$$

$$= \frac{1}{0.06}\left[1 - e^{-0.06}\right] + \frac{2}{0.06}\left[e^{-0.06} - e^{-0.06\times 2}\right]$$

$$+ \frac{3}{0.06}\left[e^{-0.06\times 2} - e^{-0.06\times 3}\right] + \cdots$$

$$= \frac{1}{0.06}\left[1 - e^{-0.06} + 2e^{-0.06} - 2e^{-0.06\times 2} + 3e^{-0.06\times 2} - 3e^{-0.06\times 3} + \cdots\right]$$

$$= \frac{1}{0.06}\left[1 + e^{-0.06} + e^{-0.06\times 2} + e^{-0.06\times 3} + \cdots\right]$$

The expression in brackets in the line above is a geometric series. Using the formula for the sum to infinity of a geometric series (which is $\dfrac{a}{1-r}$ where a is the first term and r is the common ratio), we have:

$$(I\bar{a})_x = \frac{1}{0.06}\left[\frac{1}{1-e^{-0.06}}\right] = 286.19$$

Solution 6.5

The $(1+b)^t$ term represents the total sum assured as at duration t. So whether the additions have been made continuously or at the end of the year is not important. We are only interested in the current value of the sum assured plus the increases made to time t.

Solution 6.6

If premiums are paid continuously, but increases are made at the end of the year, then we have:

$$(1+b)^{t+K_{x+t}}\, v^{T_{x+t}} - P\,\bar{a}_{\overline{T_{x+t}}|}$$

Solution 6.7

Assuming that deaths occur halfway between birthdays, the expected present value of the future benefits is:

$$EPV = 40{,}000 \frac{1}{l_{40}} \left\{ \frac{1}{1.06^{0.5}} d_{40} + \frac{1.0192}{1.06^{1.5}} d_{41} + \frac{1.0192^2}{1.06^{2.5}} d_{42} + \cdots + \frac{1.0192^{19}}{1.06^{19.5}} d_{59} + \frac{1.0192^{20}}{1.06^{20}} l_{60} \right\}$$

$$= 40{,}000 \frac{1}{l_{40}} \left(\frac{1.06^{0.5}}{1.0192} \right) \left\{ \left(\frac{1.0192}{1.06} \right) d_{40} + \left(\frac{1.0192}{1.06} \right)^2 d_{41} + \cdots + \left(\frac{1.0192}{1.06} \right)^{20} d_{59} \right\}$$

$$+ 40{,}000 \frac{1.0192^{20}}{1.06^{20}} \frac{l_{60}}{l_{40}}$$

$$= 40{,}000 \frac{1}{l_{40}} \left(\frac{1.06^{0.5}}{1.0192} \right) \left\{ \left(\frac{1}{1.04} \right) d_{40} + \left(\frac{1}{1.04} \right)^2 d_{41} + \cdots + \left(\frac{1}{1.04} \right)^{20} d_{59} \right\}$$

$$+ 40{,}000 \left(\frac{1}{1.04} \right)^{20} \frac{l_{60}}{l_{40}}$$

$$= 40{,}000 \left(\frac{1.06^{0.5}}{1.0192} \right) A^{1}_{40:\overline{20|}}{}^{@4\%} + 40{,}000 \left(\frac{1}{1.04} \right)^{20} \frac{l_{60}}{l_{40}}$$

$$= 40{,}000 \left(\frac{1.06^{0.5}}{1.0192} \right) \left(A_{40} - \left(\frac{1}{1.04} \right)^{20} \frac{l_{60}}{l_{40}} A_{60} \right) + 40{,}000 \left(\frac{1}{1.04} \right)^{20} \frac{l_{60}}{l_{40}}$$

$$= 40{,}000 \left(\frac{1.06^{0.5}}{1.0192} \right) \left(0.23056 - \left(\frac{1}{1.04} \right)^{20} \times \frac{9{,}287.2164}{9{,}856.2863} \times 0.45640 \right)$$

$$+ 40{,}000 \times \left(\frac{1}{1.04} \right)^{20} \times \frac{9{,}287.2164}{9{,}856.2863}$$

$$= 1{,}385.60 + 17{,}201.47$$

$$= 18{,}587.06$$

So equating this to $P\ddot{a}_{40:\overline{20|}}$ valued at 6% gives:

$$P = \frac{18{,}587.06}{11.998} = £1{,}549.18$$

Solution 6.8

Using the formula for the net premium reserve, we have:

$$P = \frac{9,800 A_{45} + 200 (IA)_{45}}{\ddot{a}_{45}}$$

$$= \frac{9,800 \times 0.27605 + 200 \times 8.33628}{18.823}$$

$$= 232.30$$

By the end of year 6, the sum assured has increased to:

$$10,000 + 6 \times 200 = 11,200$$

Therefore the net premium reserve (calculated prospectively) at this point will be:

$$11,000 A_{51} + 200 (IA)_{51} - 232.30 \ddot{a}_{51}$$

$$= 11,000 \times 0.34058 + 200 \times 8.58095 - 232.30 \times 17.145$$

$$= 1,479.79$$

Solution 6.9

The payments under this annuity are:

- 8,160 at time 1
- $8{,}160 \times 1.02$ at time 2
- $8{,}160 \times 1.02^2$ at time 3, and so on.

All payments are contingent on the survival of the policyholder.

Assuming the policyholder is still alive at time 4 and the annuity payment at time 4 has just been made, the EPV at time 4 of the future benefits is:

$$8{,}160 \left(1.02^4 \, v \, p_{64} + 1.02^5 \, v^2 \, {}_2p_{64} + 1.02^6 \, v^3 \, {}_3p_{64} + \cdots \right)$$

$$= 8{,}160 \times 1.02^3 \left(1.02 \, v \, p_{64} + 1.02^2 \, v^2 \, {}_2p_{64} + 1.02^3 \, v^3 \, {}_3p_{64} + \cdots \right)$$

$$= 8{,}160 \times 1.02^3 \, a_{64}^{@i'}$$

where $i' = \dfrac{1.061}{1.02} - 1 = 4\%$.

Since there are no future premiums, the reserve at time 4 is:

$$_4V = 8{,}160 \times 1.02^3 \times 13.073 = £113{,}205$$

Solution 6.10

A proprietary company will want to make a profit on the business. A common procedure both in the UK and in many overseas markets is to give 90% of surplus to policyholders, 10% to shareholders.

In addition, the company may want to deliberately under-distribute surplus now in order

- to defer eventual surplus distribution (we consider the reasons for this later), and
- to allow smoothing of payouts (the company deliberately pays out less than the surplus generated when investment markets are booming but deliberately pays out more when investment markets are declining). The smoother the bonus payments, the "safer" the policyholders feel. Ultimately, this can drive up the share price.

Solution 6.11

Simple:	Year 1:	$10{,}000 \times 1.05 = 10{,}500$
	Year 2:	$10{,}000 \times 1.10 = 11{,}000$
	Year 3:	$10{,}000 \times 1.15 = 11{,}500$
Compound:	Year 1:	$10{,}000 \times 1.039 = 10{,}390$
	Year 2:	$10{,}390 \times 1.039 = 10{,}795$
	Year 3:	$10{,}795 \times 1.039 = 11{,}216$
Super-compound:	Year 1:	$10{,}000 \times 1.03 = 10{,}300$
	Year 2:	$10{,}000 \times 1.03 + (10{,}300 - 10{,}000) \times 1.075 = 10{,}623$
	Year 3:	$10{,}000 \times 1.03 + (10{,}623 - 10{,}000) \times 1.075 = 10{,}969$

Solution 6.12

The EPV of the benefits would now be:

$$1.03S\,v\,q_x + 1.03^2\,S\,v^2\;{}_{1|}q_x + \cdots + 1.03^n\,S\,v^n\;{}_{n-1|}q_x + 1.03^n\,S\,v^n\;{}_{n}p_x = S\,A^{@i'}_{x:\overline{n}|}$$

where $i' = \dfrac{1.075}{1.03} - 1 \approx 4.5\%$.

Note that we are assuming that the bonuses are added at the start of each policy year.

Solution 6.13

First we need to work out the net premium, which is calculated based on the basic sum assured only. This is:

$$P = 15{,}000\frac{A_{45:\overline{20}|}}{\ddot{a}_{45:\overline{20}|}} = 15{,}000 \times \frac{0.46998}{13.780} = 511.59$$

The net premium reserve at time 7 includes the past reversionary bonuses, but excludes all future reversionary and terminal bonuses. The past reversionary bonuses are equal to:

$$15{,}000 \times (0.04 + 0.0375 \times 6) = 3{,}975$$

Therefore:

$$\begin{aligned} {}_7V^{net} &= (15{,}000 + 3{,}975)\,A_{52:\overline{13}|} - 511.59\ddot{a}_{52:\overline{13}|} \\ &= 18{,}975 \times 0.61130 - 511.59 \times 10.106 \\ &= 6{,}429.30 \end{aligned}$$

Alternatively, we could calculate:

$$
\begin{aligned}
{}_7V^{net} &= 15{,}000\,{}_7V_{45:\overline{20}|} + 3{,}975 \times A_{52:\overline{13}|} \\
&= 15{,}000\left(1 - \frac{\ddot{a}_{52:\overline{13}|}}{\ddot{a}_{45:\overline{20}|}}\right) + 3{,}975 \times 0.61130 \\
&= 15{,}000\left(1 - \frac{10.106}{13.780}\right) + 2{,}429.92 \\
&= 6{,}429.19
\end{aligned}
$$

the slight difference being due to the rounding used in the tabulated values.

Note that this alternative can be a useful time saver if you have several reserve calculations to perform on the same policy at different durations. This can happen in the exam, for example when doing profit testing questions (described in Chapter 12).

Solution 6.14

The fund value after 3 years is built up recursively as follows:

$$F_1 = 7{,}000 \times 1.023 = 7{,}161$$

$$F_2 = (7{,}161 + 7{,}000) \times 1.026 = 14{,}529.19$$

$$F_3 = (14{,}529.19 + 7{,}000) \times 1.025 = 22{,}067.42$$

Solution 6.15

The fund required at age 67 to produce an annuity of 25,000 *pa*, payable monthly for the whole of life, would be:

$$
\begin{aligned}
25{,}000\,\ddot{a}^{(12)}_{67-7} &= 25{,}000 \times \left(\ddot{a}_{60} - \tfrac{11}{24}\right) \\
&= 25{,}000 \times \left(15.632 - \tfrac{11}{24}\right) \\
&= 379{,}342
\end{aligned}
$$

The premium P required to provide this fund under the AWP policy, using the given assumptions, satisfies:

$$P\,\ddot{s}_{\overline{25}|}^{@3.5\%} = 379{,}342$$

where:

$$\ddot{s}_{\overline{25}|}^{@3.5\%} = \frac{1.035^{25} - 1}{0.035/1.035} = 40.31310$$

Hence:

$$P = \frac{379{,}342}{40.31310} = £9{,}410\ pa$$

Solution 6.16

(a) *Number of units cancelled to pay for the expense charge*

The unit price on 17th January is £1.602435 from before. The number of units cancelled to cover the £4 expense charge would then be:

$$\frac{4}{1.602435} = 2.50 \text{ units}$$

which leaves the policyholder with $2{,}250 - 2.50 = 2{,}247.50$ units

(b) *Number of units created on 1st February by the premium payment*

The price of the units on 1st February (which is 15 days after 17th January) is:

$$1.602435 \times 1.000095049^{15} = 1.604721$$

So the number of new units created by the premium of £400 paid on that day is:

$$\frac{400}{1.604721} = 249.26 \text{ units}$$

which means that the policyholder now has $2{,}247.50 + 249.26 = 2{,}496.76$ units.

(c) *Fund value at 14th February*

On 14th February (*ie* 13 days later), the unit price will be:

$$1.604721 \times 1.000095049^{13} = 1.606705$$

and so the unit fund value on this day will be:

$$2,496.76 \times 1.606705 = £4,011.56$$

Solution 6.17

(a) *Death benefits in 5 years time*

For Policy *A*, in 5 years time the fund value will have accumulated due to the addition of 5 premiums of £5,000 each, plus some bonus interest less some charges. There might also be a small terminal bonus component at this stage. The total is therefore likely to be somewhat higher than $5 \times £5,000 = £25,000$, but almost certainly nowhere near as high as the minimum sum assured of £50,000. The death benefit under Policy *A* will therefore be £50,000 at this time.

As Policy *B* pays out the fund plus any terminal bonus at the time of death, the death benefit under this policy will be significantly smaller than for Policy *A* after 5 years.

(b) *Death benefits in 15 years time*

Each policy will by now have received 15 premiums of £5,000 each – amounting to £75,000 (ignoring interest and charges) – so the fund value itself will be greater than the minimum death benefit. The death benefit under Policy *A* should therefore be somewhat higher than £75,000, after bonus interest has been added, charges deducted, and any terminal bonus added at the time of claim.

At first sight the death benefit under policy *B* should be the same, as the minimum death benefit does not apply. However, it is probable that the insurer will have made appropriate charges on Policy *A* to cover the additional cost of providing the minimum death benefit during the early years. These additional charges will have caused the fund value to grow slightly more slowly in the case of Policy *A*, and so by time 15 the fund value (and therefore the death benefit) for Policy *B* is likely to be slightly higher than for Policy *A*.

(c) *Maturity benefits*

The policies mature after 20 years. There is no minimum maturity value, so both will equal their respective fund values plus any terminal bonuses paid. The situation will therefore be similar to (b): if Policy A has incurred charges for the minimum death benefit, then the maturity value for Policy B will be slightly greater than that for Policy A.

Solution 6.18

(i)(a) ***Equation of value***

If P denotes the annual premium, then the equation of value is:

$$P\ddot{a}_x = (IA)_x + I + e\ddot{a}_x$$

(i)(b) ***Gross premium prospective reserve at time t***

The gross premium prospective reserve at time t is:

$${}_tV^{pro} = (IA)_{x+t} + tA_{x+t} + e\ddot{a}_{x+t} - P\ddot{a}_{x+t}$$

Note that if the life dies in the year $(t, t+1)$, the death benefit is $t+1$, and this increases by 1 every year. So we have split the benefit into a simple increasing assurance from age $x+t$ plus a level assurance of t from age $x+t$.

(i)(c) ***Gross premium retrospective reserve at time t***

The gross premium retrospective reserve at time t is:

$${}_tV^{retro} = \left[P\ddot{a}_{x:\overline{t}|} - (IA)^1_{x:\overline{t}|} - I - e\ddot{a}_{x:\overline{t}|}\right]\frac{(1+i)^t}{{}_tp_x}$$

(ii) ***Equality of reserves***

The equation of value from (i)(a) can be written as:

$$P\left[\ddot{a}_{x:\overline{t}|} + v^t\,{}_tp_x\,\ddot{a}_{x+t}\right] = (IA)^1_{x:\overline{t}|} + v^t\,{}_tp_x\left[(IA)_{x+t} + tA_{x+t}\right] + I$$
$$+ e\left[\ddot{a}_{x:\overline{t}|} + v^t\,{}_tp_x\,\ddot{a}_{x+t}\right]$$

Rearranging this so that all the terms containing $v^t \ {}_t p_x$ are on the RHS, we get:

$$(P-e)\ddot{a}_{x:\overline{t}|} - (IA)^1_{x:\overline{t}|} - I - e\ddot{a}_{x:\overline{t}|} = v^t \ {}_t p_x \left\{ \left[(IA)_{x+t} + tA_{x+t} \right] - (P-e)\ddot{a}_{x+t} \right\}$$

Now dividing through by $v^t \ {}_t p_x$, we have:

$$\left[(P-e)\ddot{a}_{x:\overline{t}|} - (IA)^1_{x:\overline{t}|} - I \right] \frac{(1+i)^t}{{}_t p_x} = (IA)_{x+t} + tA_{x+t} - (P-e)\ddot{a}_{x+t}$$

Provided that the prospective and retrospective reserves are both calculated on the same basis as the premium, the equation above shows that they are equal. The LHS is the retrospective reserve at time t and the RHS is the prospective reserve at time t.

Solution 6.19

(i) ***Monthly premium***

Let P denote the monthly premium. Then:

$$\begin{aligned} \text{EPV premiums} &= 12P\ddot{a}^{(12)}_{[45]} \ @ 6\% \\ &= 12P\left(\ddot{a}_{[45]} - \frac{11}{24} \right) \\ &= 12P\left(14.855 - \frac{11}{24} \right) \\ &= 172.76P \end{aligned}$$

Assuming that deaths occur halfway through each year of age and benefits are payable 3 months after death:

$$\begin{aligned} \text{EPV benefits} = 50,000\Big(& v^{9/12} \, q_{[45]} + 1.019231 v^{1\,9/12} \ {}_{1|}q_{[45]} \\ & + 1.019231^2 \, v^{2\,9/12} \ {}_{2|}q_{[45]} + \cdots \Big) \end{aligned}$$

The trick is to make the power on the v *term match with the power on the 1.019231 term, with the first term containing* $1.019231v$ *(to the power 1). So we will multiply everything inside the brackets by* $1.019231v^{3/12}$ *and divide the 50,000 by* $1.019231v^{3/12}$.

So:

$$\text{EPV benefits} = \frac{50{,}000\,(1+i)^{3/12}}{1.019231}\Big(1.019231\,v\,q_{[45]} + 1.019231^2\,v^2\ {}_{1|}q_{[45]}$$

$$+1.019231^3\,v^3\ {}_{2|}q_{[45]} + \cdots\Big)$$

$$= \frac{50{,}000\,(1+i)^{3/12}}{1.019231}\Big(v'\,q_{[45]} + (v')^2\ {}_{1|}q_{[45]} + (v')^3\ {}_{2|}q_{[45]} + \cdots\Big)$$

where:

$$v' = 1.019231\,v = \frac{1.019231}{1.06} = \frac{1}{1.04}$$

Hence:

$$\text{EPV benefits} = \frac{50{,}000\,(1.06)^{3/12}}{1.019231}\,A_{[45]}\ @\,4\%$$

$$= \frac{50{,}000\,(1.06)^{3/12}}{1.019231} \times 0.27583$$

$$= 13{,}729.84$$

Finally:

$$\text{EPV expenses} = 300 + 0.05 \times 12P\left(\ddot{a}^{(12)}_{[45]} - \frac{1}{12}\right) + 200\,v^{3/12}\,\bar{A}_{[45]}\ @\,6\%$$

Now:

$$v^{3/12}\,\bar{A}_{[45]} = v^{3/12} \times (1+i)^{1/2}\,A_{[45]} = (1+i)^{3/12}\,A_{[45]}$$

So:

$$\text{EPV expenses} = 300 + 0.05P(172.76 - 1) + 200(1.06)^{3/12} \times 0.15918$$

$$= 332.30 + 8.588P$$

The equation of value is then:

$$172.76P = 13,729.84 + 332.30 + 8.588P$$

and solving for P we get:

$$P = £85.65$$

(ii) ***Reserve on 30 April 2005***

If we take the date of issue (1 May 1998) to be time 0, then 30 April 2005 corresponds to time 7. The retrospective gross premium reserve at time 7 is:

$${}_7V^{retro} = \left(\text{EPV past premiums} - \text{EPV past benefits and expenses}\right) \times \frac{D_{[45]}}{D_{52}}$$

Now:

$$\text{EPV past premiums} = 12 \times 85.65\, \ddot{a}^{(12)}_{[45]:\overline{7|}}$$

$$\text{EPV past expenses} = 295 + 12 \times 5\, \ddot{a}^{(12)}_{[45]:\overline{7|}} + 100(1+i)^{3/12} A^{\;1}_{[45]:\overline{7|}}$$

and:

$$\text{EPV past benefits} = 50,000\Big(v^{9/12}\, q_{[45]} + 1.04\, v^{1\,9/12}\; {}_{1|}q_{[45]} + 1.04^2\, v^{2\,9/12}\; {}_{2|}q_{[45]}$$
$$\cdots + 1.04^6\, v^{6\,9/12}\; {}_{6|}q_{[45]}\Big)$$
$$= 50,000\, v^{9/12}\left(q_{[45]} + {}_{1|}q_{[45]} + \cdots + {}_{6|}q_{[45]}\right)$$

since $v = 1.04^{-1}$. Also:

$$q_{[45]} + {}_{1|}q_{[45]} + \cdots + {}_{6|}q_{[45]} = \left(1 - p_{[45]}\right) + \left(p_{[45]} - {}_2p_{[45]}\right) +$$
$$\left({}_2p_{[45]} - {}_3p_{[45]}\right) + \cdots + \left({}_6p_{[45]} - {}_7p_{[45]}\right)$$
$$= 1 - {}_7p_{[45]}$$
$$= {}_7q_{[45]}$$

So:

$$\text{EPV past benefits} = 50,000\, v^{9/12}\ {}_7q_{[45]}$$

Putting these terms together gives:

$${}_7V^{retro} = \left(967.8\,\ddot{a}^{(12)}_{[45]:\overline{7}|} - 295 - 100(1+i)^{3/12}\, A^{\ 1}_{[45]:\overline{7}|} - 50,000\, v^{9/12}\ {}_7q_{[45]}\right)\frac{D_{[45]}}{D_{52}}$$

Now using the formula on Page 36 of the *Tables*:

$$\ddot{a}^{(12)}_{[45]:\overline{7}|} = \ddot{a}_{[45]:\overline{7}|} - \frac{11}{24}\left(1 - \frac{D_{52}}{D_{[45]}}\right)$$

Noting that the valuation rate of interest is 4% *pa*, we have:

$$\frac{D_{52}}{D_{[45]}} = \frac{1,256.80}{1,677.42} = 0.74925$$

$$\ddot{a}_{[45]:\overline{7}|} = \ddot{a}_{[45]} - \frac{D_{52}}{D_{[45]}}\ddot{a}_{52} = 18.829 - 0.74924 \times 16.838 = 6.213$$

$$\ddot{a}^{(12)}_{[45]:\overline{7}|} = 6.213 - \frac{11}{24}(1 - 0.74925) = 6.098$$

$$A^{\ 1}_{[45]:\overline{7}|} = A_{[45]} - \frac{D_{52}}{D_{[45]}}A_{52} = 0.27583 - 0.74925 \times 0.35238 = 0.01181$$

$${}_7q_{[45]} = 1 - \frac{9,660.5021}{9,798.0837} = 0.014042$$

So:

$$\begin{aligned}{}_7V^{retro} &= \Big[967.8 \times 6.098 - 295 - 100(1.04)^{3/12} \times 0.01181 \\ &\qquad - 50,000(1.04)^{-9/12} \times 0.014042\Big] \times \frac{1}{0.74925} \\ &= 6,571.91\end{aligned}$$

Chapter 7

Gross premiums and reserves for fixed and variable benefit contracts

Syllabus objectives:

(v) *Describe and calculate gross premiums and reserves of assurance and annuity contracts.*

1. *List the types of expenses incurred in writing a life insurance contract.*

2. *Describe the influence of inflation on the expenses listed in 1.*

3. *Define the gross future loss random variable for the benefits and annuities listed in (i) 1 and (iv) 1-5, and calculate gross premiums and reserves that satisfy probabilities involving the future loss random variable. Regular premiums and annuity benefits may be payable annually or continuously. Death benefits may be payable at the end of the year of death or immediately on death.*

4. *Calculate the gross premium using the equivalence principle. Regular premiums and annuity benefits may be payable annually, more frequently than annually or continuously. Death benefits may be payable at the end of the year of death or immediately on death.*

5. *Calculate the gross premium using simple criteria other than the principles described in 3 and 4.*

6. *Define and calculate the gross premium prospective reserve.*

7. *Define and calculate the gross premium retrospective reserve.*

8. *State the conditions under which, in general, the prospective reserve is equal to the retrospective reserve allowing for expenses.*

Continued…

9. *Prove that, under the appropriate conditions, the prospective reserve is equal to the retrospective reserve, with or without allowance for expenses, for all standard fixed benefit and increasing/decreasing benefit contracts.*

10. *Obtain a recursive relation between successive annual reserves for an annual premium contract, with allowance for expenses, for standard fixed benefit contracts, and use this relation to calculate the profit earned from a contract during a year.*

0 Introduction

In this chapter we see how the concepts of net premiums and reserves studied earlier change when we introduce expenses.

The net premium was calculated as the premium that the life insurance company should charge to ensure that the expected present value of premiums is equal to the expected present value of benefits, on the basis of some set of assumptions. In reality the company will have expenses to meet, and will therefore need to calculate premiums that cover both future benefits and future expenses. Such premiums are called gross, or office, premiums.

Likewise, when the company calculates reserves, it should allow for the expenses that will be involved in administering the policies. Reserves that allow explicitly for expenses are called gross premium reserves, as opposed to net premium reserves that do not allow explicitly for expenses.

For both premium and reserve calculations we start by considering the generic random variable case and then see how that translates into a deterministic life table approach.

Note that in this context, net and gross have nothing to do with tax. We are referring to expenses here, and whether or not we make explicit allowance for them.

We begin, though, by considering the types of expenses that will be incurred by a company writing life insurance business.

1 Types of expenses incurred in writing a life insurance contract

Suppose that a life insurance company wants to sell a new policy. In order to run the business at a profit, it will need to:

- determine what the expenses attributable to that policy will be, on average, and
- calculate a premium for the policy that allows for those expenses.

One complication is that, given the long time for which life insurance contracts stay on the books, the expenses that the company will incur in respect of this policy will increase due to inflation. If the policy is a conventional contract, where normally the premiums cannot be varied (as opposed to a unit-linked contract, where normally the charges can be varied throughout the lifetime of the contract), it is vital that the company allows correctly for such inflation.

1.1 Measuring and allocating costs

In addition to the costs of providing benefits an insurance company will incur other costs in running its businesses. These costs must be recovered by a loading for expenses in the premiums charged to policyholders. A mechanism to measure costs and to allocate them as expenses to individual contracts is required.

Note that this does not mean that the company would work out the precise costs attributable to each individual policy on a policy-by-policy basis and then charge each of the individual policies accordingly. Rather, we need to determine the average costs that we expect policies to incur, given the recent experience of the company, and then price new contracts appropriately. For this we will split policies into broad groupings, so for instance we might end up with expense figures for "temporary life insurance policies of term 20 years".

The term for this process is "expense investigation".

The measurement of costs is part of the accounting function within the company. Costs are usually allocated to business functions within the company *eg* policy servicing, computing, new business, buildings and their maintenance, cost of selling/salesforce.

These categories of costs can be divided into overhead expenses and direct expenses:

- **Overhead expenses are those that in the short term do not vary with the amount of business written.**
- **Direct expenses are those that do vary with amount of business written.**

Example

The costs of the company's premises are an overhead, because the sale of an extra policy next week will have no impact on these costs.

The medical underwriting costs of the company (for instance, the cost of arranging medicals for certain policyholders) are a direct cost, because writing extra policies will increase the number of medicals *etc* that the company needs to pay for.

In the long term, large changes in the amount of business written will result in all expenses being direct.

For instance, the cost of head office premises is an overhead in that it will not change if we double next month's new business figures. However, if new business levels over the next three years are doubled, then the company will need to physically expand in order to cope with the extra staff required to process and administer the business.

Question 7.1

Classify the following costs as overheads or direct:

(i) Board directors' remuneration

(ii) Commission payments to brokers

(iii) £10,000 bonus payable to sales manager on completion of target new business levels

(iv) Head office canteen workers' salaries

(v) New business administration department's salary costs.

Once we have determined total overhead and direct expenses, we need to determine the average expenses per policy.

Overhead expenses are usually allocated on a per policy basis.

Suppose that the accounting department has determined that the company has expenses deemed "overhead" that amount to £14.2 million over the year. What is a fair share of this for any policy? A fair share would be the total £14.2 million divided by the number of policies in force over that year. So no policy is deemed to require a greater share of, say, building costs, than any other policy.

Direct expenses are allocated according to their "drivers". Drivers are the factors (*eg* number of policies, size of premium and size of benefit) changes in which are observed to lead to changes in direct expenses. For example, underwriting costs are directly related to the size of the sum assured. The majority of direct expenses are driven by the number of policies in force.

Question 7.2

State the most appropriate drivers for the following direct expenses:

(i) Commission payment to direct salesman

(ii) Medical examinations for temporary life policies

(iii) New business administration department's salary costs.

Further costs may result from commissions paid as part of obtaining the sale of a policy (initial commission) and its renewal (renewal commission).

Commission costs are, by definition, direct in nature rather than overhead, and their "drivers" are normally the amount of annual or single premium concerned. If advice were fee based, the number of policies would be the driver.

Commissions are normally referred to separately from expenses because, from the point of view of the life insurance company:

- **the level of commissions may be defined by the company itself, and**
- **the level may be changed at the will of the company.**

The level of expenses, by comparison, cannot be changed at will, or as quickly as commissions could be changed, because complex factors are at play in the structure and human resource requirements of the whole company.

Question 7.3

In the most recent expense investigation, a life company has determined that the total direct expenses attributable to whole life contracts is £1.51 million. This total breaks down as follows:

Expense varies by	Expenses (£ million)
Number of policies	0.32
Premium	1.02
Sum assured	0.17
Total	1.51

The policies underlying the investigation were 7,620 in number and totalling £4,435,000 in annual premium and £211.7 million in sum assured.

Calculate the average costs for the period, expressed in terms of per policy, per £ of premium and per £ of sum assured.

1.2 *Charging for expenses*

Once a life insurance company has established the average expenses that it thinks its new policies will incur, it needs to ensure that premium rates reflect these expenses. In practice, companies may adopt different ways of pricing policies to recoup expenses. Here is one, theoretically valid, approach.

Overhead expenses are charged as renewal expenses on a per policy basis.

Direct expenses are divided into:

- **initial expenses – those arising when policies are issued,**
- **renewal expenses – those arising regularly during the policy term,**
- **termination expenses – those arising when the policy terminates as a result of an insured contingency** (*eg* a death claim for a temporary life insurance policy)**.**

These are allocated to policies according to their appropriate driver in the form of cashflows that have to be met at the specified time in the policy's life.

Example

Suppose we want to price a regular premium endowment insurance contract with term n years. From our expense investigations, we think the expenses for this contract will be:

Initial expenses	£175 per policy + 86% of annual premium P
Renewal expenses	£65 + 6% of annual premium P per policy in force
Claim expenses	£50 per death in the event of a death claim £25 per maturity payment in the event of maturity

The value of these expenses is:

$$175+0.86P+65a_{x:\overline{n-1}|}+0.06Pa_{x:\overline{n-1}|}+25A^1_{x:\overline{n}|}+25A_{x:\overline{n}|}$$

We usually assume that renewal expenses are incurred at the *start* of each policy year, but excluding the first year (for which the initial expenses apply). This is why the term of the annuity functions is $n-1$ rather than n.

If you had to calculate these annuity functions in the exam, the easiest way would be to think of $a_{x:\overline{n-1}|}$ as $\ddot{a}_{x:\overline{n}|}-1$.

The valuation of expenses uses the same methodology as that used for the valuation of benefits. This means that if we determine the value of benefits deterministically, for instance, then we should apply the same method to valuing the expenses.

2 *The influence of inflation on expenses*

Inflation affects underlying costs, which in turn influence the level of expenses allocated to policies.

The main categories of costs are:

- **salaries and salary-related expenses,**
- **buildings and other property costs,**
- **computing and associated costs, and**
- **costs associated with the investment of funds.**

Note that this is a different split from the overhead/direct split we discussed earlier.

Many of these items are directly or indirectly linked to wage and salary levels. Other items are influenced by the general level of prices or by the prices of particular commodities. Publicly available data *eg* retail price index, national average earnings index, and similar data internal to the insurance company can be used to forecast the expected general price, future wage or specific price inflation.

These forecasts enable the expenses being valued to be adjusted for expected inflation.

Question 7.4

In the previous example, how would we allow for expected average future expense inflation of, say, 4% *pa*?

3 Gross future loss random variable for standard contracts

The gross future loss random variable is an extension of the net future loss random variable concept described in Chapter 5. Gross implies that the definition of loss now includes an allowance for the expenses associated with the contract. Bonuses will also be incorporated as appropriate in the case of with-profit contracts.

We illustrate the approach of the gross future loss random variable using the example of a whole life assurance. This approach can be extended to any of the standard contract types for a given allocation of expenses and, in the case of with-profit contracts, method of bonuses.

Example: Whole life assurance

Suppose we can allocate expenses as:

- I **initial expenses in excess of those occurring regularly each year**
- e **level annual expenses**
- f **additional expenses incurred when the contract terminates**

Note that we are assuming here that the level expenses of e are incurred every year *including* the first, *ie* that I represents the amount by which the total initial expense amount *exceeds* the subsequent regular expense amount.

You need to be careful when dealing with examination questions that talk about "regular expenses" or "renewal expenses". You need to be very clear as to whether the expense amount applies at the start of every year, or only from the start of Year 2 onwards. If the renewal expenses start in Year 1, it's easier to think of them as starting in Year 1 and then change I to $I-e$. The question should always make it clear what is required. If it doesn't, make sure to state the assumption you are making in your solution.

The gross future loss random variable when a policy is issued to a life aged x is:

$$Sv^{T_x} + I + e\bar{a}_{\overline{T_x}|} + fv^{T_x} - G\bar{a}_{\overline{T_x}|}$$

where a gross premium of G secures a sum assured of S, the sum assured is paid immediately on death and the premium is payable continuously.

Question 7.5

What does the equation above assume about the payment of renewal expenses?

Question 7.6

Write down the gross future loss random variable for an n year endowment assurance. Use the symbols defined in the above example and state any assumptions that you make.

Question 7.7

Write down an expression for the gross future loss random variable for a deferred life annuity, with a deferred period of n years. Premiums of amount G are paid annually in advance for a maximum of n years, the annuity benefit of B is paid for life annually in advance starting in n years time, and no benefit is paid if the life does not survive to time n. State any other assumptions that you make, and define any other symbols used.

3.1 Calculating premiums and reserves that satisfy probabilities

Premiums and reserves can be calculated that satisfy probabilities involving the gross future loss random variable.

Example

A whole life assurance pays a sum assured of 10,000 at the end of the year of death of a life aged 50 exact at entry. Assuming 3% per annum interest, AM92 Ultimate mortality and expenses of 4% of every premium, calculate the smallest level annual premium payable at the start of each year that will ensure the probability of making a loss under this contract is not greater than 5%.

Solution

If the annual premium is G, the future loss (random variable) of the policy at outset is:

$$L_0 = 10{,}000\, v^{K_{50}+1} - 0.96\, G\, \ddot{a}_{\overline{K_{50}+1}|}$$

We need to find the smallest value of *G* such that:

$$\boldsymbol{P\left(L_0 > 0\right) \leq 0.05}$$

***ie* such that**

$$\boldsymbol{P\left(L_0 \leq 0\right) \geq 0.95}$$

Define G_n to be the annual premium that ensures $L_0 = 0$ for $K_{50} = n$, for $n = 0, 1, 2, \ldots$. That is, G_n is the premium we need to charge to cover the benefits *exactly*, if the policyholder were to die between times n and $n+1$. **This means:**

$$\boldsymbol{G_n = \frac{10{,}000\, v^{n+1}}{0.96\, \ddot{a}_{\overline{n+1|}}}}$$

Now, suppose we *do* charge a premium of G_n, and the policyholder dies at a later duration than time n (*ie* $K_{50} > n$). In this case, the present value of the benefit will be smaller (because it occurs later), and the present value of the premium income will be greater (because more premiums are received). So, if $K_{50} \geq n$, and G_n is the premium, we cannot make a loss on the contract (*ie* $L_0 \leq 0$). Representing this in symbols we have:

$$\boldsymbol{P\left(L_0 \leq 0 \middle| G = G_n\right) = P\left(K_{50} \geq n\right)}$$

which reads as "the probability of not making a loss when the premium is G_n, equals the probability of surviving for at least n years".

However, this probability needs to be at least 0.95. Now look at the formula for G_n, and you will see that as n increases, the premium G_n decreases. **We therefore find the largest value of n that satisfies this condition** (*ie* that $P\left(K_{50} \geq n\right) \geq 0.95$), **and the corresponding value of G_n is then the minimum premium required.**

So

$$\boldsymbol{P\left(K_{50} \geq n\right) \geq 0.95} \quad \Rightarrow \quad {}_n p_{50} \geq 0.95 \quad \Rightarrow \quad \frac{l_{50+n}}{l_{50}} \geq 0.95$$

$$\boldsymbol{\Rightarrow l_{50+n} \geq 0.95 \times l_{50} = 0.95 \times 9{,}712.0728 = 9{,}226.4692}$$

From the Tables, $l_{60} = 9,287.2164$ and $l_{61} = 9,212.7143$, and so the largest value of n that satisfies the required probability is $n = 10$. Hence, the smallest premium that satisfies the required probability is

$$G_{10} = \frac{10,000\, v^{11}}{0.96\, \ddot{a}_{\overline{11}|}} = \frac{10,000 \times 0.72242}{0.96 \times 9.5302} = 790$$

A premium calculated in this way is sometimes called a *percentile premium*.

A reserve at policy duration t can be calculated in a similar way, to satisfy a probability specified in terms of the gross future loss random variable at time t.

Question 7.8

An insurance company is carrying out a valuation of its in-force business. The following details relate to one particular policy that is in force on the valuation date:

- Policy type: Whole life assurance
- Benefit: £75,000 payable at the end of the year of death
- Entry age: 50 exact
- Current duration in force: 8 years exact
- Annual premium (paid at the start of each year): £1,500

Calculate the smallest reserve that could be held at the valuation date, which will ensure that the insurance company can cover the liability under this contract with a probability of at least 97.5%, assuming interest of 3% *pa* and that mortality follows the AM92 Ultimate table. Ignore expenses.

Question 7.9

Suppose that the insurer in Question 7.8 has two policies in force on independent lives with identical details as listed in that question.

The insurer now wishes to calculate the smallest total reserve that would ensure it could meet its liabilities under both policies with a probability of at least 97.5%.

Without performing any more calculations, explain whether the total reserve would be:

I less than double

II exactly double, or

III more than double

the reserve that was required for one policy when considered on its own.

Question 7.10

Hubert, aged 60, is applying to buy a whole life immediate annuity from an insurance company, with his life savings of £200,000. Calculate the largest amount of level annuity, payable annually in arrear, that the insurer could pay if it requires a probability of loss from the contract of no more than 10%.

Assume PMA92C20 mortality, interest of 5% *pa*, and expenses of 1% of each annuity payment.

4 Determining gross premiums using the equivalence principle

The *gross premium* is the premium required to meet all the costs under an insurance contract, and is the premium that the policyholder pays. When we talk of "the premium" for a contract, we mean the gross premium. It is also sometimes referred to as the *office premium*.

The net premium already discussed in this subject makes no explicit allowance for expenses. The gross premium for a contract, given suitable mortality, interest and expense assumptions would be found from the equation of expected present value:

Equation of value

The expected present value of the gross premium income
=
The expected present value of the outgo on benefits
+
the expected present value of the outgo on expenses

The equivalence principle states that:

$$E[\text{gross future loss}] = 0$$

which implies that:

$$\text{EPV benefits} + \text{EPV expenses} = \text{EPV premiums}$$

so that in an expected present value sense the premiums are equal (equivalent) in value to the expenses and the benefits. This relationship is usually called an equation of (gross expected present) value.

Previously, we determined net premiums by equating the expected present value of benefits with the expected present value of premiums. Here we have merely introduced expenses as another item that the premiums need to cover.

4.1 Annual premium contracts

We will illustrate the methodology by using the example of a whole life assurance. The same method can be used for other standard contracts.

In particular, the same method can also be used for single premium contracts with the simplification that the expected present value of future premiums will just be equal to the single premium that the company receives on day 1.

If we have premiums of G payable annually in advance and a benefit of S payable at the end of the year of death, **the equation of value is:**

$$S A_x + I + e\,\ddot{a}_x + f A_x = G\,\ddot{a}_x$$

and, with a basis (to determine the annuity and assurance values) and an expense allocation, the value of G can be determined.

Example

Gordon, aged 45, buys a whole life assurance policy with sum assured £10,000, payable immediately on death. Calculate the gross premium payable by Gordon annually in advance for ten years or until earlier death, allowing for the following expenses:

> Initial expenses of £160 plus 75% of the annual premium
>
> Renewal expenses of £50 (incurred throughout life from year 2 onwards) plus 4% of the annual premium (incurred at the time of payment of each premium from year 2 onwards)
>
> Claim expenses of 2.5% of sum insured

Basis: AM92 Ultimate mortality, 4% *pa* interest

Solution

The equation of value is:

> EPV premiums = EPV benefits + EPV expenses

Now:

$$\text{EPV premiums} = P\ddot{a}_{45:\overline{10|}} = P\left(\ddot{a}_{45} - \frac{D_{55}}{D_{45}}\ddot{a}_{55}\right)$$

$$= \left(18.823 - \frac{1{,}105.41}{1{,}677.97} \times 15.873\right)P = 8.366P$$

$$\text{EPV benefits} = 10{,}000\bar{A}_{45} \approx 10{,}000 \times (1+i)^{\frac{1}{2}} A_{45}$$

$$= 10{,}000 \times 1.04^{\frac{1}{2}} \times 0.27605 = 2{,}815.17$$

and:

$$\text{EPV expenses} = 160 + 0.75P + 50a_{45} + 0.04P\left(\ddot{a}_{45:\overline{10|}} - 1\right) + 0.025 \times 10{,}000\bar{A}_{45}$$

$$= 160 + 0.75P + 50 \times 17.823 + 0.04 \times 7.366P + 0.025 \times 2{,}815.17$$

$$= 1.045P + 1{,}121.53$$

So:

$$8.366P = 2{,}815.17 + 1.045P + 1{,}121.53$$

$$\Rightarrow P = £537.73$$

If the sum assured is paid immediately on death and the premium is paid continuously we can use functions defined in Chapters 1 and 2 to write:

$$S\,\bar{A}_x + I + e\,\bar{a}_x + f\,\bar{A}_x = G\,\bar{a}_x$$

which can be solved for G as before.

We can also start to tackle gross premium calculations involving select mortality.

Question 7.11

A 25-year endowment assurance policy provides a payment of £75,000 on maturity or at the end of the year of earlier death. Calculate the annual premium payable for a policyholder who effects this insurance at exact age 45.

Expenses are 75% of the first premium and 5% of each subsequent premium, plus an initial expense of £250.

Assume AM92 Select mortality and 4% *pa* interest.

4.2 *Conventional with-profits contracts*

Gross premiums for with-profit contracts will include not only loadings for expenses, but also for future bonuses.

Extending the preceding example, suppose the contract is now with-profit, and a compound bonus of b per annum (to be added to basic sum assured and existing bonuses) is assumed. Then the equation of value is, assuming annual premiums and the first bonus in one year's time from outset, (so that no bonus is available to deaths in the first year):

$$Svq_x + \left\{ S(1+b)v^2 \, {}_{1|}q_x + S(1+b)^2 v^3 \, {}_{2|}q_x + \cdots \right\} + I + e\ddot{a}_x + fA_x = G\ddot{a}_x$$

or:

$$S\frac{1}{1+b}\left\{ (1+b)vq_x + (1+b)^2 v^2 \, {}_{1|}q_x + (1+b)^3 v^3 \, {}_{2|}q_x + \cdots \right\} + I + e\ddot{a}_x + fA_x = G\ddot{a}_x$$

Here we are defining b as a proportion (of the sum assured plus previously added bonuses).

This can then be simplified to:

$$S\frac{1}{1+b}A_x^j + I + e\,\ddot{a}_x + f\,A_x = G\,\ddot{a}_x$$

where the assurance function is determined on the normal mortality basis but using an interest rate j where $j = \dfrac{(1+i)}{(1+b)} - 1$.

What we have done here is to take out a factor of $\dfrac{1}{1+b}$ so that the powers of v match the powers of $1+b$ in all the terms in the big bracket. So we can now think of $(1+b)v$ as the "v" for some different interest rate, and so the expression reduces to an A_x term, but calculated at a new rate of interest, in this case at $j=\dfrac{1+i}{1+b}-1$.

We say here that the bonus *vests* at the end of the year. This refers to the date when bonuses are added to the policy value. If bonuses vested at the start of the year, then the value of the benefit if the policyholder had died in the first year would be $S(1+b)$, in the second year it would be $S(1+b)^2$ and so on. This is another area where you need to read the question very carefully, to ensure that you get the right level of benefit at the right time.

Simple bonus cases may be valued using the increasing assurance function $(IA)_x$ defined in Chapter 6.

Example

A man aged 45 buys a 15-year with-profit endowment assurance with a basic sum assured of £25,000. Determine the single premium to be paid for this assurance, assuming that simple reversionary bonuses of 6% *pa* vest at the end of each policy year and that death benefits are payable at the end of the year of death. Assume AM92 Ultimate mortality and 4% *pa* interest. Initial expenses are £200 and renewal expenses are £30 at the start of each policy year, excluding the first.

Solution

The single premium is equal to the expected present value of the benefits and the expenses.

On death during the first year, the benefit is just the basic sum assured of £25,000. The benefit on death in subsequent years will increase by $0.06\times 25{,}000=£1{,}500$ because of the bonuses. So:

$$\begin{aligned}\text{EPV benefits} &= 23{,}500A^{1}_{45:\overline{15}|}+1{,}500\left(IA\right)^{1}_{45:\overline{15}|}+47{,}500A_{45:\overline{15}|}^{\;\;\;\;\;1}\\ &= 23{,}500A_{45:\overline{15}|}+1{,}500\left(IA\right)^{1}_{45:\overline{15}|}+24{,}000A_{45:\overline{15}|}^{\;\;\;\;\;1}\end{aligned}$$

Now:

$$A_{45:\overline{15}|} = 0.56206$$

$$A_{45:\overline{15}|}^{\ \ \ \ 1} = \frac{D_{60}}{D_{45}} = \frac{882.85}{1,677.97} = 0.52614$$

$$(IA)^1_{45:\overline{15}|} = (IA)_{45} - \frac{D_{60}}{D_{45}}\left[(IA)_{60} + 15A_{60}\right]$$

$$= 8.33628 - 0.52614[8.36234 + 15 \times 0.45640]$$

$$= 0.33454$$

So:

$$\text{EPV benefits} = 23,500 \times 0.56206 + 1,500 \times 0.33454 + 24,000 \times 0.52614$$
$$= 26,337.62$$

Also:

$$\text{EPV expenses} = 200 + 30\left(\ddot{a}_{45:\overline{15}|} - 1\right) = 200 + 30 \times 10.386 = 511.58$$

So the single premium for the policy is:

$$26,337.62 + 511.58 = £26,849.20$$

Reserves for accumulating with profits contracts are not covered in the syllabus.

4.3 Premiums payable m times per year

If premiums are payable m times per year, then the expected present value of premiums and level annual expenses must be determined using expressions for m-thly annuities as derived in Chapter 4.

Example: Whole life assurance

The equation of value is:

$$S\,\bar{A}_x + I + e\,\ddot{a}_x^{(m)} + f\,\bar{A}_x = G\,\ddot{a}_x^{(m)}$$

with a given basis and expenses. Using the approximations described the equation can be solved for G.

Note that expenses and premiums can be assumed to be payable at different frequencies.

Example

Sam, aged 40, buys a 20-year term assurance with a sum assured of £150,000 payable immediately on death. Calculate the quarterly premium payable by Sam for this policy. Assume that initial expenses are 60% of total annual premium plus £110, renewal expenses are £30 *pa* from year two onwards.

Basis: AM92 Select, 4% *pa* interest.

Solution

Let P denote Sam's quarterly premium. Then:

$$\text{EPV premiums} = 4P\ddot{a}^{(4)}_{[40]:\overline{20|}} = 4P\left[\ddot{a}_{[40]:\overline{20|}} - \frac{3}{8}\left(1 - \frac{D_{60}}{D_{[40]}}\right)\right]$$

using the formula on Page 36 of the *Tables*. Now:

$$\ddot{a}_{[40]:\overline{20|}} = 13.930$$

and:

$$\frac{D_{60}}{D_{[40]}} = \frac{882.85}{2,052.54} = 0.43013$$

So:

$$\text{EPV premiums} = 4P\ddot{a}^{(4)}_{[40]:\overline{20}|} = 4P\left[13.930 - \frac{3}{8}\left(1 - 0.43013\right)\right] = 54.865P$$

Also:

$$\text{EPV benefits} = 150{,}000\bar{A}^{1}_{[40]:\overline{20}|}$$

$$= 150{,}000 \times 1.04^{½}\left(A_{[40]:\overline{20}|} - \frac{D_{60}}{D_{[40]}}\right)$$

$$= 150{,}000 \times 1.04^{½}\left(0.46423 - 0.43013\right)$$

$$= 5{,}216.30$$

and:

$$\text{EPV expenses} = 0.6 \times 4P + 110 + 30\left(\ddot{a}_{[40]:\overline{20}|} - 1\right)$$

$$= 2.4P + 110 + 30 \times 12.930$$

$$= 2.4P + 497.90$$

Alternatively we can think of the expense terms as 60% of the total annual premium plus £80 initially, and then £30 per year every year including the first. This will give us:

$$0.6 \times 4P + 80 + 30\ddot{a}_{[40]:\overline{20}|}$$

for the expenses. Both give the same numerical value. Use whichever method seems easier to you.

So the equation of value is:

$$54.865P = 5{,}216.30 + 2.4P + 497.90$$

$$\Rightarrow P = £108.91$$

The same approach (and functions) can be used to determine equations of value for the gross premiums of annuity contracts with annuity benefits payable *m* thly.

Example

Calculate the single premium that should be paid by Mrs S, aged 60, for an annuity of £8,500 *pa* payable monthly in advance, allowing for initial expenses of 1.5% of premium and administration expenses payable at the start of each year, including the first. Administration expenses are £120 at the start of Year 1 and increase at the rate of 1.9231% *pa*. Assume AM92 Select mortality and 6% *pa* interest.

Solution

The single premium P is equal to the expected present value of benefits and expenses. We have

$$\begin{aligned}\text{EPV benefits} &= 8{,}500\,\ddot{a}^{(12)}_{[60]} \ @\ 6\% = 8{,}500\left(\ddot{a}_{[60]} - \frac{11}{24}\right)\\ &= 8{,}500\left(11.919 - \frac{11}{24}\right) = 97{,}415.67\end{aligned}$$

and:

$$\begin{aligned}\text{EPV expenses} &= 0.015P + 120\left(1 + 1.019231\,v\,p_{[60]} + 1.019231^2 v^2\,{}_2p_{[60]} + \cdots\right)\\ &= 0.015P + 120\ddot{a}_{[60]} \ @\ 4\%\\ &= 0.015P + 120 \times 14.167\\ &= 0.015P + 1{,}700.04\end{aligned}$$

So:

$$P = 97{,}415.67 + 0.015P + 1{,}700.04 \Rightarrow P = £100{,}625$$

5 Calculating gross premiums using simple criteria other than the equivalence principle

Using the equivalence principle implies that:

E[present value of future loss] = 0

so that on average (provided the assumptions used are true) the contract will "break even". It is usual to load premiums for profit so that:

E[present value of future loss] < 0

If a criterion based on this expected value is chosen to reflect the "loading for profit" required, then a gross premium including a loading for profit can be determined.

Example: Whole life assurance

If the criterion specifies an expected present value of future loss of $-\pi$, then the equation of value becomes:

$$S\,\bar{A}_x + I + e\,\ddot{a}_x^{(m)} + f\,\bar{A}_x + \pi = G\,\ddot{a}_x^{(m)}$$

Here we are assuming that premiums and renewal expenses are payable $m$thly in advance. Note that we are adding a loading π for profit, which means that, in addition to covering benefits and expenses, our premiums must also cover this profit requirement. So the profit element will have the same sign as benefits and expenses, even though we might intuitively think that profit is "nice" and should therefore have the opposite sign.

Thinking about the impact on premium, this will give us the right result: adding an explicit profit requirement will give larger premiums if we treat the profit requirement in the same way as benefits and expenses, and we would expect to need greater premiums if the shareholders are demanding greater profits.

Question 7.12

Calculate the annual premium, payable monthly in advance, for a deferred annuity of £12,400 *pa* to be paid quarterly in advance from age 60 to a male now aged 40. Initial expenses are 80% of the annual premium, renewal expenses are 4% of the annual premium incurred at the start of each year from Year 2 onwards, annuity payment expenses are £15 per payment, and the EPV of profit is 2% of the annual premium. Assume AM92 Ultimate mortality and 4% *pa* interest

6 Gross premium reserves

6.1 Gross premium prospective reserves

The expression for the gross future loss at policy duration t can be used to determine the gross premium prospective reserve at policy duration t. This is done by determining the expected value of the gross future loss random variable.

It should be noted that the calculation of gross premium prospective reserves will be done according to mortality, interest, expense, and (if applicable) future bonus assumptions specifically chosen for the purpose. These assumptions, called the gross premium valuation basis, may well be different from the underlying basis used to calculate the actual gross premiums, which the company charges to policyholders.

Note

Unlike in the net premium valuation, the premium used in the calculation of the gross premium reserve is always the actual gross premium being charged – that is, it is not recalculated on the reserving basis.

The gross reserve is the amount of money that the company should hold to pay for eventual benefits and expenses, allowing for any future premium income. It therefore makes sense to look at the reserve as the expected value of the future "loss", *ie* the shortfall between future premiums and future benefits / expenses.

We calculate prospective reserves as:

EPV of benefits + EPV of expenses – EPV of premiums

Example: Whole life assurance

The expected value of the gross future loss random variable at policy duration t is:

$$S\,\bar{A}_{x+t} + e\,\ddot{a}^{(m)}_{x+t} + f\,\bar{A}_{x+t} - G\,\ddot{a}^{(m)}_{x+t}$$

Again, we are assuming that premiums and renewal expenses are payable $m$thly in advance, and claims are payable immediately on death.

This can be evaluated using an assumed valuation basis to determine the values of the annuity and assurance functions, including an assumption for future expenses. *G* is the gross (*ie* actual) premium payable under the contract .

Note that the gross premium does not *have* to be calculated using the equivalence principle: the key point is that the premium G must always be the *actual* premium that the policyholder is contracted to pay.

We can immediately see two major differences compared with net premium reserves:

- we allow explicitly for expenses, and
- we value future office premiums (the premiums that will actually be received), rather than the net premiums.

In order to calculate a gross reserve for regular premium products we therefore first need to calculate the gross premium. This will not be necessary for single premium policies.

Example

Calculate the gross premium reserve that should be held at the end of Year 10 for a 25-year regular premium endowment assurance policy with sum assured of £75,000 payable on maturity or at the end of the year of earlier death. The policy was taken out by a person aged exactly 45 at entry.

Expenses are 75% of the first premium and 5% of each subsequent premium, plus an initial expense of £250.

Assume AM92 Select mortality and 4% *pa* interest (for the both the premium and the reserve).

Solution

We first need to calculate the gross premium. This is £2,132 from 0.

The prospective reserve at the end of Year 10 is:

$$_{10}V^{pro} = \text{EPV future benefits \& expenses} - \text{EPV future premiums}$$

The components are:

$$\text{EPV future premiums} = P\ddot{a}_{55:\overline{15}|} = 2{,}132 \times 11.0187 = 23{,}492$$

$$\text{EPV future benefits} = 75{,}000 A_{55:\overline{15}|} = 75{,}000 \times 0.57620 = 43{,}215$$

$$\text{EPV future expenses} = 0.05 P\ddot{a}_{55:\overline{15}|} = 0.05 \times 23{,}492 = 1{,}175$$

So we have:

$$_{10}V^{pro} = 43{,}215 + 1{,}175 - 23{,}492 = 20{,}898$$

Question 7.13

Andy, aged 40, purchases a single premium whole life annuity of £8,400 *pa* payable monthly in advance from age 60. Expenses are 2% of premium (initial), £60 *pa* (renewal, Year 2 onwards, and also during payment of the annuity, assumed incurred annually in advance throughout).

Calculate the reserve for Andy's policy at time 10. Assume interest of 4% *pa*, mortality AM92 Ultimate (in deferment), PMA92C20 (from age 60).

Gross premium prospective reserves for conventional with-profits policies

In the case of conventional with-profits policies, a careful definition of how bonuses are to be allowed for, both in the future benefits to be valued and the gross premium, is required.

Note

Unlike in the case of calculating the net premium prospective reserve, there is no set definition of how bonuses must be allowed for in the calculation of a gross premium reserve.

If you have to calculate a gross premium reserve for a with-profit policy, just follow the instructions given in the question.

Normally, the future benefits to be valued will include at least the level of bonuses added to the point of calculation of the reserve. However, normally also the full value of the future gross premium payments will be deducted, and this will effectively discount all the loadings for bonuses contained within the gross premiums that are still to be received. In the context of a reserve calculated at some point in a policy's life, the historical premiums paid to date plus the discounted value of the future premiums will effectively capitalise all the premium loadings for bonuses, both those already added as at the date of the reserve calculation and any which could be added thereafter. It can then be seen that calculating benefits allowing just for bonuses to date, but then deducting the full value of the gross premiums, may produce a rather weak reserve.

For this reason, some level of future bonus is normally also valued as a prospective future benefit. This may or may not include an allowance for terminal bonus, depending mainly on the purpose of the calculation.

6.2 Gross premium retrospective reserve

The idea of net premium retrospective reserves is discussed in Chapter 5. Gross premium retrospective reserves at policy duration t take account of expected expenses incurred between policy duration 0 and policy duration t, as well as the expected premiums and benefits paid. All the expected present values are determined at policy duration t.

What we are doing with a retrospective reserve is determining the "pot of money" represented by the accumulation of premiums and interest, less the cost of cover provided to date. For a gross premium retrospective reserve, we will also knock off the cost of expenses incurred to date.

So a generic definition of a gross premium retrospective reserve would be:

Accumulated value of past premiums – *Accumulated value of past benefits and expenses*

Example: Whole life assurance

The gross premium retrospective reserve at policy duration t is:

$$\frac{D_x}{D_{x+t}}\left\{G\,\ddot{a}^{(m)}_{x:\overline{t}|} - S\,\bar{A}^{1}_{x:\overline{t}|} - I - e\,\ddot{a}^{(m)}_{x:\overline{t}|} - f\,\bar{A}^{1}_{x:\overline{t}|}\right\}$$

Again, we are assuming that the premiums and renewal expenses are payable m thly in advance.

The expression in parentheses gives us "the value that the policy will have at time *t*, but valued at time 0". The *D* factors outside the brackets then translate that into a value at time *t*.

To evaluate this expression we need assumptions to determine the assurance and annuity functions and information or assumptions about expenses incurred.

Similar expressions can be determined using the same approach for other standard contracts.

Example

Calculate the retrospective gross premium reserve at time $t=2$ for a 5-year single premium endowment assurance with sum assured £30,000 payable on maturity or at the end of year of earlier death, issued to a 48-year old. Expenses were £360 (initial), £45 (renewal from Year 2 onwards). Assume AM92 Select mortality and 4% *pa* interest for premiums and reserves.

Solution

We first need to calculate the premium paid at inception. This is given by:

$$P = 30{,}000A_{[48]:\overline{5}|} + 360 + 45\left(\ddot{a}_{[48]:\overline{5}|} - 1\right)$$

Now:

$$A_{[48]:\overline{5}|} = A_{[48]} - v^5\, {}_5p_{[48]}\, A_{53} + v^5\, {}_5p_{[48]}$$

$$= 0.30664 - 1.04^{-5} \times \frac{9{,}630.0522}{9{,}748.8603} \times 0.36448 + 1.04^{-5} \times \frac{9{,}630.0522}{9{,}748.8603}$$

$$= 0.82263$$

and:

$$\ddot{a}_{[48]:\overline{5}|} = \ddot{a}_{[48]} - v^5\, {}_5p_{[48]}\, \ddot{a}_{53}$$

$$= 18.027 - 1.04^{-5} \times \frac{9{,}630.0522}{9{,}748.8603} \times 16.524$$

$$= 4.611$$

So:

$$P = 30{,}000 \times 0.82263 + 360 + 45\left(4.611 - 1\right) = £25{,}201.39$$

The retrospective reserve at time $t = 2$ will be:

$$\left(25{,}201.39 - 30{,}000 A^{\;1}_{[48]:\overline{2}|} - 360 - 45\, a_{[48]:\overline{1}|}\right)\left(1+i\right)^2 \frac{l_{[48]}}{l_{50}}$$
$$= \left(25{,}201.39 - 30{,}000 \times 0.0035519 - 360 - 45 \times 0.95999\right) \times 1.04^2 \times \frac{9{,}748.8603}{9{,}712.0728}$$
$$= £26{,}807.73$$

Note that the assurance function in the reserve is a term assurance function. The only benefit that could have been paid out in the first two years is a death benefit. So a term assurance function gives the appropriate value.

In fact, we can see how this has developed from policy inception. The retrospective reserve at the end of Year 1 will be the premium less initial expenses, with interest and less the cost of cover, divided by the probability of surviving the year:

$$\frac{(25{,}201.39 - 360) \times 1.04 - 30{,}000\, q_{[48]}}{p_{[48]}} = 25{,}828.34$$

The retrospective reserve at the end of Year 2 will be the end-of-Year-1 reserve less renewal expenses, plus interest and less the cost of cover, divided by the probability of surviving the year:

$$\frac{(25{,}828.34 - 45) \times 1.04 - 30{,}000 \times q_{[48]+1}}{p_{[48]+1}} = 26{,}807.75$$

The difference between the two calculations is due to rounding errors.

We shall say more later in the chapter about the process of finding the value of the reserve at any time from the corresponding value of the reserve in a previous year.

Question 7.14

Calculate the gross premium retrospective reserve at time $t = 3$ for the example above. Justify your answer by reconciling it with the result for time $t = 2$.

6.3 Equality of gross premium prospective and retrospective reserves

In order to determine reserves, assumptions must be made about:

- **mortality and interest rates used to determine the values of annuity and assurance functions,**
- **expenses to be valued, and**
- **premium to be valued.**

Conditions for equality

If:

- **the mortality and interest rate basis used is the same as that used to determine the gross premium at the date of issue of the policy, and**
- **the expenses valued are the same as those used to determine the original gross premium, and**
- **the gross premium is that determined on the original basis (mortality, interest, expenses) using the equivalence principle**

then the retrospective and prospective gross premium reserves are equal.

If every element of the bases used to determine the retrospective and prospective reserves is the same, we should intuitively expect that the retrospective and prospective reserves at any point in time will be equal for whatever policy we are considering.

Question 7.15

Calculate the prospective reserve at $t = 2$ for the policy considered in the previous example, and comment on your answer.

Demonstrating the equality of prospective and retrospective gross premium reserves

We will demonstrate that under the conditions given above, the retrospective and prospective gross premium reserves for a whole life assurance are equal. This method can be extended to all standard contracts.

Example: Whole life assurance

The three key expressions are those for prospective reserves, retrospective reserves and the original equation of value used to determine the gross premium.

These are:

(i) $S\,\bar{A}_{x+t} + e\,\ddot{a}^{(m)}_{x+t} + f\,\bar{A}_{x+t} - G\,\ddot{a}^{(m)}_{x+t}$

(ii) $\dfrac{D_x}{D_{x+t}}\left\{G\,\ddot{a}^{(m)}_{x:\overline{t}|} - S\,\bar{A}^{1}_{x:\overline{t}|} - I - e\,\ddot{a}^{(m)}_{x:\overline{t}|} - f\,\bar{A}^{1}_{x:\overline{t}|}\right\}$

(iii) $G\,\ddot{a}^{(m)}_{x} - S\,\bar{A}_{x} - I - e\,\ddot{a}^{(m)}_{x} - f\,\bar{A}_{x} = 0$

Then, if we add:

$$\frac{D_x}{D_{x+t}}\left\{G\,\ddot{a}^{(m)}_{x} - S\,\bar{A}_{x} - I - e\,\ddot{a}^{(m)}_{x} - f\,\bar{A}_{x}\right\}$$

***ie* 0 to the expression for the prospective reserve (i), and rearrange the terms, we will obtain the expression for the retrospective reserve (ii).**

Alternatively, we could split the premium equation up at time t and rearrange as follows:

$$G\,\ddot{a}^{(m)}_{x} = (S+f)\,\bar{A}_{x} + I + e\,\ddot{a}^{(m)}_{x}$$

$$\Rightarrow G\left(\ddot{a}^{(m)}_{x:\overline{t}|} + v^t\;{}_tp_x\,\ddot{a}^{(m)}_{x+t}\right) = (S+f)\left(\bar{A}^{1}_{x:\overline{t}|} + v^t\;{}_tp_x\bar{A}_{x+t}\right)$$
$$+I + e\left(\ddot{a}^{(m)}_{x:\overline{t}|} + v^t\;{}_tp_x\,\ddot{a}^{(m)}_{x+t}\right)$$

$$\Rightarrow G\ddot{a}^{(m)}_{x:\overline{t}|} - (S+f)\,\bar{A}^{1}_{x:\overline{t}|} - I - e\ddot{a}^{(m)}_{x:\overline{t}|} = v^t\;{}_tp_x\left((S+f)\,\bar{A}_{x+t} + e\ddot{a}^{(m)}_{x+t} - G\ddot{a}^{(m)}_{x+t}\right)$$

Dividing both sides by $v^t\;{}_tp_x$ then gives:

$$\left(G\ddot{a}^{(m)}_{x:\overline{t}|} - (S+f)\,\bar{A}^{1}_{x:\overline{t}|} - I - e\ddot{a}^{(m)}_{x:\overline{t}|}\right)\frac{(1+i)^t}{{}_tp_x} = (S+f)\,\bar{A}_{x+t} + e\ddot{a}^{(m)}_{x+t} - G\ddot{a}^{(m)}_{x+t}$$

The expression on the left-hand side of this equation is the gross premium retrospective reserve at time t and the expression on the right is the gross premium prospective reserve at time t.

A similar result can be shown for net premiums P, *ie* those determined assuming all expenses are zero ($I = e = f = 0$).

Question 7.16

Prove that the retrospective and prospective reserves are equal at time t for an immediate annuity (payable annually in arrears) of amount B with initial expenses I and renewal expenses R.

The fact that prospective and retrospective reserves are equal for equality of bases has two immediate uses:

- some policies with complicated and varying future benefit levels may require complex calculations to arrive at the reserve prospectively, but a retrospective calculation may be much easier, and
- you may find that retrospective reserves are a more tangible concept than prospective reserves, in which case thinking about aspects of life insurance involving reserves may be easier if you think in retrospective terms.

On the other hand, under some circumstances the prospective calculation will be easier. For example:

- for policies where there are no further premiums to be paid, the prospective calculation is often simpler, because the term "present value of future premiums" disappears
- for policies that have already undergone a complex alteration, it may be easier to work prospectively, rather than retrospectively.

For equality of prospective and retrospective reserves we require equality of bases. In practice, we shall often find that the bases are not equal. The retrospective basis may be just the past experience – for instance, the mortality experienced by our policyholders was 86% of AM80 – while the prospective basis may be our estimate of future experience, and it might be deliberately prudent (especially in the context of calculating reserves to demonstrate the solvency of the company).

However, we might want to ignore recent experience and "arbitrarily" set the past basis to be equal to the future basis, in order to use the retrospective method to calculate reserves rather than the prospective method.

On the other hand, this will only give a valid answer if the reserving basis is identical to the pricing basis of the gross premium involved. Although in most of our examples so far, and in most of the Question and Answer Bank material, the two bases are identical, it is possible for them to be different. This is a situation that occurs often in real life. However, *in the exam*, it is quite common to have the same basis. So, unless a question specifically states the approach you have to adopt, you would then be free to choose retrospective or prospective calculations. In such cases, however, *the prospective method is usually the easier method to do*.

Very often, companies will price products using "best estimates" of future experience for interest, mortality and expenses, or best estimate with a small margin for prudence. However, the supervisory authority may insist that the reserving basis is chosen to be much more prudent, in order to be more certain that life companies will be capable of honouring their financial commitments in the event of deteriorating future conditions. For instance, if our best estimate of future interest rates was 6½% we might calculate premiums using 5%, but then might need to calculate statutory reserves using 3% interest.

In this case you would *have* to calculate the reserve prospectively, as the reserving basis is different from the pricing basis.

The effect of these bases being different is studied in Chapter 11.

6.4 Recursive relationship between reserves for annual premium contracts

If the expected cashflows (*ie* premiums, benefits and expenses) during the policy year $(t, t+1)$ are evaluated and allowance is made for the time value of money, a recursive relationship linking gross premium reserves in successive years can be developed.

For such a relationship to hold, the reserves at successive durations must be calculated on the same basis.

We illustrate this using a whole life assurance secured by level annual premiums, but the method extends to all standard contracts.

Example: Whole life assurance

Gross premium policy value at duration t	${}_tV'$
Premium less expenses paid at t	$G-e$
Expected claims plus expenses paid at $t+1$	$q_{x+t}(S+f)$
Gross premium policy value at duration $t+1$	${}_{t+1}V'$

Then the equation of value at time $t+1$ for these cashflows is:

$$\left({}_tV' + G - e\right)(1+i) - q_{x+t}(S+f) = \left(1-q_{x+t}\right){}_{t+1}V' \qquad (*)$$

This equation will only be satisfied if all quantities are calculated on mutually consistent bases: *ie* using the same interest, mortality and expense assumptions for the reserves, premium and experience over the year. In this case, the equation then gives a recursive relationship between policy values in successive years.

Question 7.17

(i) Summarise the above relationship in words.

(ii) Write down the relationship for a single premium whole life policy.

We can also rearrange the above to give:

$$\left({}_{t}V' + G - e\right)\left(1+i\right) - q_{x+t}\left(S - {}_{t+1}V' + f\right) = {}_{t+1}V'$$

So the end-year reserve is the start-year reserve, plus premium and less expenses, rolled up with one year's interest, less the claim cost. So the cost of claims depends on the difference between the benefit payable (plus expenses) and the reserves set up at the end of the year.

Example

Consider a 25-year regular premium endowment assurance policy with a sum assured of £75,000 on maturity (or at the end of the year of earlier death), taken out by a 45-year old. Expenses are 75% of the first premium and 5% of each subsequent premium, plus an initial expense of £250.

Given a gross annual premium of £2,132 and a gross premium reserve at the end of Year 10 of £20,898, calculate the gross premium reserves at the end of Year 11.

Assume AM92 Select mortality and 4% *pa* interest.

Solution

The recursive relationship is:

$$\left({}_{10}V' + G - e\right)\left(1+i\right) - Sq_{x+t} = p_{x+t}\ {}_{11}V'$$

So:

$$\left(20{,}898 + 2{,}132 \times 0.95\right) \times 1.04 - 75{,}000 q_{55} = p_{55}\ {}_{11}V'$$

giving:

$${}_{11}V' = £23{,}611$$

Question 7.18

For the policy described in the above example, calculate ${}_{9}V'$ using the recursion relationship and the value ${}_{10}V' = £20{,}898$.

If expenses are assumed to be zero and we use a net premium P rather than the gross premium G, we obtain the recursive relationship between net premium policy values, ${}_tV$, that we saw in Section 8 of Chapter 5.

Question 7.19

Write down the recursive relationship for the net premium reserves of a whole life policy.

Where any of these bases differ, then the equation (*) can be reformulated to represent the profit over the year, *ie*:

$$\boldsymbol{PRO_t = \left({}_tV' + G - e\right)(1+i) - q_{x+t}(S+f) - (1 - q_{x+t})\,{}_{t+1}V'}$$

This profit relates to the year *starting* at policy duration t, that is for policy *year* $t+1$.

In this equation, the elements are now deemed to represent the *actual* experience, rather than the experience expected according to the insurer's assumptions. To make this clearer, we shall re-express the actual experience elements using primed functions. So we have, for policy year $t+1$:

$$PRO_t = \left({}_tV' + G - e'\right)(1+i') - (S+f')\,q'_{x+t} - \left(1 - q'_{x+t}\right)\,{}_{t+1}V'$$

Question 7.20

For the 25-year endowment assurance described in the previous example, calculate the profit earned during year 11 if:

– the start and end of year reserves were as given/calculated in the example
– the insurer earned 3.8% on it's investments during the year
– renewal expenses of £78 were incurred at the start of the year
– claim expenses of £150 per death claim were incurred
– 73% of expected mortality occurred during the year.

This formula can be used to calculate the expected profit in each future year of a policy in a procedure known as *profit testing* – this is covered in detail later in the course.

7 Exam-style questions

Here are some exam-style questions on the material in this chapter.

Question 7.21 (Subject A2, September 1998, Question 15 (part), adapted)

A life aged exactly 40 purchases a special single premium deferred annuity. The annuity payments are to commence at age 60, and are payable monthly in advance for life. The amount of the first monthly payment is to be £1,000, but once in payment the amount is to increase in line with the rate of inflation.

There are no death benefits payable in the event of death during the deferred period.

(i) Show that the single premium is £54,543.

Basis:

mortality:	AM92 Select before age 60
	PMA92C20 after age 60
interest:	6% *pa*
inflation:	1.9231% *pa*
expenses:	initial: £500
	claim: 1% of each annuity payment

[6]

(ii) The life insurance company calculates prospective gross premium reserves on the same basis as above. Calculate the reserve held in respect of the policy at the end of the 10th policy year, assuming that the policyholder is still alive. [3]

[Total 9]

Question 7.22 *(Subject 105, April 2001, Question 13, adapted)*

A life insurance company sells with-profit whole life policies, with the sum assured payable immediately on the death of the life assured and with premiums payable annually in advance ceasing with the policyholder's death or on reaching age 65 if earlier.

The company markets two versions of this policy, one with simple reversionary bonuses and the other with compound reversionary bonuses. In both cases the bonuses are added at the end of the policy year.

The company prices the products using the following basis:

Mortality		AM92 Select
Interest		4% per annum
Expenses	initial	£250
	renewal	2% of second and subsequent premiums
	claim	£150 at termination of contract
Bonuses	simple	6% of basic sum assured per annum
	compound	4% of accumulated sum assured and bonuses per annum

(i) Write down an expression for the gross future loss at the point of sale for each of these policies, assuming they are sold to a life aged x exact ($x < 65$) at outset. Write the expression in terms of functions of the random variables T_x and K_x, which represent the exact future lifetime and the curtate future lifetime of (x), respectively. [5]

(ii) Calculate the gross premium required for each of the two policies for a sum assured of £200,000 and a life aged 40 exact at outset, using the equivalence principle. [8]

(iii) After 10 years, bonuses totalling £90,000 have been declared for the compound reversionary bonus contract. Calculate the net premium reserve for that policy at that time, using AM92 ultimate mortality and interest of 4% per annum. [4]

[Total 17]

Question 7.23 ***(Subject CT5, April 2005, Question 3)***

A life insurance company sells an annual premium whole life assurance policy where the sum assured is payable at the end of the year of death. Expenses are incurred at the start of each policy year, and claim expenses are nil.

(a) Write down a recursive relationship between the gross premium reserves at successive durations, with reserves calculated on the premium basis. Define all the symbols that you use.

(b) Explain in words the meaning of the relationship. [4]

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8 End of Part 2

What next?

1. Briefly **review** the key areas of Part 2 and/or re-read the **summaries** at the end of Chapters 5 to 7.
2. Attempt some of the questions in Part 2 of the **Question and Answer Bank**. If you don't have time to do them all, you could save the remainder for use as part of your revision.
3. Attempt **Assignment X2**.

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Chapter 7 Summary

Expenses

In order to price contracts profitably, life insurance companies need to determine the expenses that will be incurred by the contracts and adjust the premiums accordingly.

To determine the expenses per policy, we conduct an expense investigation. In this investigation, expenses are broken down into overheads and direct expenses. We then apportion overheads to policies on a per-policy basis, and apportion direct expenses to policies according to the relevant drivers (per policy, per £ premium, or per £ sum assured).

These per-policy expenses then need to be broken down into initial expenses, renewal expenses and termination expenses. We can then allow for them when calculating the premium.

Care needs to be taken to allow for inflation of renewal expenses.

Gross premiums

Gross premiums (also called office premiums) are calculated allowing for expenses, using the relation:

EPV of benefits + EPV of expenses = EPV of premiums

This may be modified if we have an explicit profit criterion:

EPV of benefits + EPV of expenses + EPV profit = EPV of premiums

Gross premium reserves

Gross prospective reserves can be calculated as:

EPV of future benefits + EPV of future expenses – EPV of future premiums

Gross retrospective reserves can be calculated as:

Accumulated value of (past premiums – past benefits and expenses)

Premiums and reserves that satisfy probabilities

A premium for a new policy can be found that satisfies a probability defined in terms of the future loss random variable, L. For example, we can define the premium as being the smallest premium for which $P(L>0) \leq \alpha$, where α is some acceptably small probability.

A similar approach can be used to calculate the reserve for an existing policy.

Equality of reserves

Given equality of bases used to calculate the premium and the reserves, the prospective and retrospective reserves of any policy at any given time t will be equal.

Recursive formula for reserves

Reserves at successive values of time t are related by the relation:

$$\left({}_tV' + G - e\right)(1+i) - q_{x+t}\left(S + f\right) = p_{x+t}\ {}_{t+1}V'$$

Profit

The profit for the year between policy durations t and $t+1$ (*ie* for policy year $t+1$) can be calculated using:

$$PRO_{t+1} = \left({}_tV' + G - e'\right)\left(1+i'\right) - q'_{x+t}\left(S + f'\right) - p'_{x+t}\ {}_{t+1}V'$$

where primed functions represent the actual experience over the year.

Chapter 7 Solutions

Solution 7.1

(i) Board director's remuneration is an overhead (unless linked in any way to new business volumes).

(ii) Commission payments to brokers are a direct expense (we pay an extra chunk of commission on every policy sold).

(iii) The production bonus is a direct expense (it varies with production).

(iv) Head office canteen workers' salaries are an overhead.

(v) New business admin department's salary costs: this is a good example of a "grey" area where there is no clear right or wrong. In the very short term, writing one more policy will not change the costs. But a rush of business over a few weeks may require overtime payments, and a rush of business over a few months may require more staff to be taken on.

So some companies will treat it as an overhead, other companies will treat it as a direct expense. Others could do both by splitting down into basic salaries of normal staff (overhead) plus overtime / salaries of temporary staff (direct)!

Solution 7.2

(i) Commission payment: per unit premium.

(ii) Medical exams: per unit of sum insured (because low sum insured policies may not require medicals, very high sum insured policies will require medicals plus extra specialist medical tests).

(iii) New business admin department's salary costs: per policy (the amount of work required to process a policy does not change if we double premium or sum insured, but it doubles if we have to process two policies rather than one).

Solution 7.3

Per policy cost is 320,000 / 7,620 = £42 per policy

Per £ premium cost is 1,020,000 / 4,435,000 = 23p per £1 of premium

Per £ SA cost is 170,000 / 211,700,000 = 0.08p per £1 of sum insured

Solution 7.4

The value of the renewal and claim expenses is:

$$65a_{x:\overline{n-1}|} + 0.06Pa_{x:\overline{n-1}|} + 25A^{1}_{x:\overline{n}|} + 25A_{x:\overline{n}|}$$

but with all functions calculated at a rate of interest $i' = \dfrac{1+i}{1.04} - 1 \approx i - 0.04$.

In practice a large portion of the 6% of premium renewal expense would be represented by renewal commission, which would therefore not increase with inflation.

Solution 7.5

It assumes that they are paid continuously at the constant rate of e per annum.

Solution 7.6

Assume that the expense structure is as in the whole of life example above. Then the gross future loss random variable is:

$$S\,v^{\min(T_x,n)} + I + e\,\bar{a}_{\overline{\min(T_x,n)}|} + f\,v^{\min(T_x,n)} - G\,\bar{a}_{\overline{\min(T_x,n)}|}$$

where I, e, f, S and G are defined as before.

We are assuming here that premiums are payable continuously, and that the sum assured is paid immediately on death (or at the end of n years on survival).

Note: if you had assumed that death benefits were payable at the end of the year of death, then T_x would be replaced by K_x+1 in the multipliers of S and f. If premiums and regular expenses were assumed to be payable annually in advance, then T_x would be replaced by K_x+1 in the multipliers of e and G, and the annuity functions would be $\ddot{a}$ rather than $\bar{a}$.

Solution 7.7

Because premiums and benefits are payable annually, we can use K_x, the curtate future lifetime, rather than T_x.

Assuming that we have initial expenses I, regular expenses during the premium payment term of e, regular benefit payment expenses of e', the gross future loss random variable will be:

$$I + e\,\ddot{a}_{\overline{K_x+1|}} - G\,\ddot{a}_{\overline{K_x+1|}} \qquad \text{for} \qquad K_x < n$$

(*ie* if the life does not survive to the end of the deferred period), and:

$$I + e\,\ddot{a}_{\overline{n|}} - G\,\ddot{a}_{\overline{n|}} + B\,\ddot{a}_{\overline{K_x+1-n|}}\,v^n + e'\ddot{a}_{\overline{K_x+1-n|}}\,v^n \qquad \text{for} \qquad K_x \geq n$$

(*ie* if the life survives to receive payments under the annuity).

Alternatively, we could have written this in one line as:

$$I - (G-e)\,\ddot{a}_{\overline{\min(K_x+1,\,n)|}} + (B+e')\left[\ddot{a}_{\overline{\max(K_x+1,\,n)|}} - \ddot{a}_{\overline{n|}}\right]$$

or as:

$$I - (G-e)\,\ddot{a}_{\overline{\min(K_x+1,\,n)|}} + (B+e')\,\ddot{a}_{\overline{\max(K_x+1-n,\,0)|}}v^n$$

Solution 7.8

The future loss that needs to be covered by the reserve is:

$$L = 75{,}000v^{K_{58}+1} - 1{,}500\ddot{a}_{\overline{K_{58}+1|}}$$

If V is the reserve, then we need the smallest value of V such that:

$$P(L \leq V) \geq 0.975$$

Let V_n be the reserve needed to cover the loss exactly when $K_{58} = n$ $(n = 0, 1, 2, ...)$. So:

$$V_n = 75,000\, v^{n+1} - 1,500\ddot{a}_{\overline{n+1|}}$$

Now:

$$P\left(L \le V \middle| V = V_n\right) = P\left(K_{58} \ge n\right)$$

because the larger the value of K_{58} (*ie* the later that death occurs) the smaller will be the value of L.

However, we also need:

$$P\left(K_{58} \ge n\right) \ge 0.975 \quad (*)$$

Now:

$$P\left(K_{58} \ge n\right) = {}_n p_{58} = \frac{l_{58+n}}{l_{58}}$$

so we require:

$$\frac{l_{58+n}}{l_{58}} \ge 0.975 \quad \Rightarrow \quad l_{58+n} \ge 0.975 \times l_{58} = 0.975 \times 9,413.8004 = 9,178.4554$$

To find the smallest reserve to cover the loss, we need the largest value of n that satisfies (*). From the *Tables*, we find:

$$l_{61} > 9,178.4554 > l_{62}$$

and so the largest value of n that will satisfy (*) is 3. Therefore the smallest reserve we can hold to cover the loss with a probability of at least 97.5% is:

$$V = V_3 = 75,000\, v^4 - 1,500\, \ddot{a}_{\overline{4|}} = 75,000 \times 1.03^{-4} - 1,500 \times \left[\frac{1 - 1.03^{-4}}{0.03 / 1.03}\right] = £60,894$$

Solution 7.9

Option I is correct.

The two losses are independent but identically distributed. If L_1 and L_2 are the present values of the losses on the two policies, then:

$$E\left[L_1 + L_2\right] = 2E\left[L\right]$$

and:

$$\text{var}\left[L_1 + L_2\right] = 2\,\text{var}\left[L\right]$$

This means that the standard deviation of the sum of the losses is:

$$SD\left[L_1 + L_2\right] = \sqrt{2\,\text{var}\left[L\right]} = 1.414\,SD\left[L\right]$$

As the standard deviation has increased by (considerably) less than double, then $P\left(L_1 + L_2 > 2 \times 60{,}894\right)$ will be much smaller than 2.5%. That is, we can hold a total reserve that is considerably smaller than $2 \times 60{,}894$ and still be able to meet the required probability of loss.

This is the "law of large numbers" by which all insurance companies are able to pool their risks by insuring a large number of independent lives or policies (which you will have come across as the central limit theorem in Statistics). The greater the number of independent risks the insurer has on its books, the smaller the reserve required ***per policy*** *that will leave the insurer with an acceptable risk of loss. With enough policies in force, the reserve required per policy will be close in size to the expectation of the present value of the loss – that is, a prospective reserve calculation.*

Solution 7.10

The present value of the loss from the policy is:

$$L = 1.01 X a_{\overline{K_{60}}|} - 200{,}000$$

where X is the annual annuity benefit. If we set $X_{(k)}$ such that:

$$L = 1.01 X_{(k)}\, a_{\overline{k}|} - 200{,}000 = 0$$

$$\Rightarrow X_{(k)} = \frac{200{,}000}{1.01 a_{\overline{k}|}}$$

then the insurer will only produce a loss when $K_{60} > k$ (for annuities, the loss will increase the longer the person lives).

So, an annuity of $X_{(k)}$ implies $P(L>0) = P(K_{60} > k)$. Now, to find the largest value of $X_{(k)}$ that satisfies:

$$P(L>0) \le 0.1$$

we need the smallest value of k that satisfies:

$$P(K_{60} > k) \le 0.1 \qquad ie \qquad P(K_{60} \ge k+1) \le 0.1 \quad \text{or} \quad {}_{k+1}p_{60} \le 0.1$$

That is, the smallest value of k for which:

$$l_{60+k+1} \le 0.1 l_{60} = 0.1 \times 9{,}826.131 = 982.6131$$

From the *Tables* we find:

$$l_{95} = 1{,}020.409 \qquad (k = 34)$$

$$l_{96} = 798.003 \qquad (k = 35)$$

So the required (smallest) value of k is 35. So finally calculate $X_{(35)}$ from:

$$X_{(35)} = \frac{200{,}000}{1.01 a_{\overline{35}|}^{5\%}} = \frac{200{,}000}{1.01 \times 16.3742} = £12{,}093$$

Solution 7.11

The equation of value is:

EPV premiums = EPV benefits + EPV expenses

If the premium is P, then:

$$\text{EPV premiums} = P\ddot{a}_{[45]:\overline{25}|}$$

$$= P\left(\ddot{a}_{[45]} - \frac{D_{70}}{D_{[45]}}\ddot{a}_{70}\right)$$

$$= P\left(18.829 - \frac{517.23}{1,677.42}\times 10.375\right)$$

$$= 15.630P$$

$$\text{EPV benefits} = 75,000A_{[45]:\overline{25}|}$$

$$= 75,000\left(A_{[45]} - \frac{D_{70}}{D_{[45]}}A_{70} + \frac{D_{70}}{D_{[45]}}\right)$$

$$= 75,000\left(0.27583 - \frac{517.23}{1,677.42}\times 0.60097 + \frac{517.23}{1,677.42}\right)$$

$$= 75,000 \times 0.39887 = £29,915$$

$$\text{EPV expenses} = 0.75P + 0.05P(\ddot{a}_{[45]:\overline{25}|} - 1) + 250$$

$$= 0.75P + 0.05 \times 14.630P + 250$$

$$= 1.4815P + 250$$

(Note that $\ddot{a}_{[45]:\overline{25}|} - 1$ is the same as $a_{[45]:\overline{24}|}$. We have done it this way to save work, since $\ddot{a}_{[45]:\overline{25}|}$ is already calculated.)

So the equation of value is:

$$15.630P = 29,915 + 1.4815P + 250 \Rightarrow P = 30,165/14.1485 = 2,132$$

So the premium is £2,132 *pa*.

Solution 7.12

The equation of value is:

EPV premiums = EPV benefits + EPV expenses + EPV profit

For an annual premium P:

$$\text{EPV premiums} = P\ddot{a}^{(12)}_{40:\overline{20}|} = P\left(\ddot{a}_{40:\overline{20}|} - \tfrac{11}{24}\left(1 - v^{20}\frac{l_{60}}{l_{40}}\right)\right) = 13.666P$$

$$\text{EPV benefits} = 12{,}400\ {}_{20|}\ddot{a}^{(4)}_{40}$$

$$= 12{,}400v^{20}\frac{l_{60}}{l_{40}}\ddot{a}^{(4)}_{60}$$

$$= 12{,}400 \times 1.04^{-20} \times \frac{9{,}287.2164}{9{,}856.2863}\left(14.134 - \tfrac{3}{8}\right)$$

$$= 12{,}400 \times 5.91687$$

$$= 73{,}369.24$$

(Note that there was no benefit on death during deferment, so this is rather a theoretical contract!)

$$\text{EPV expenses} = 0.8P + 0.04Pa_{40:\overline{19}|} + 4 \times 15\ {}_{20|}\ddot{a}^{(4)}_{40}$$

$$= 0.8P + 0.04P \times 12.927 + 60 \times 5.91687$$

$$= 1.3171P + 355.01$$

$$\text{EPV profit} = 0.02P$$

So we have:

$$13.666P = 73{,}369.24 + 1.3171P + 355.01 + 0.02P$$

giving:

$$P = \frac{73{,}724.25}{12.3289} = £5{,}980$$

Solution 7.13

It is a single premium contract so we do not need to calculate the premium in order to calculate the prospective reserves. The prospective gross premium reserve will be

EPV of future benefits + EPV of future expenses

The expected present value of the future benefits is:

$$EPV(B) = 8,400 \, {}_{10|}\ddot{a}_{50}^{(12)} = 8,400 \times v^{10} \frac{l_{60}}{l_{50}} \ddot{a}_{60}^{(12)}$$

where l_{50} and l_{60} are taken from the AM92 Table, and $\ddot{a}_{60}^{(12)}$ is taken from the PMA92C20 Table. Therefore:

$$EPV = 8,400 \times 1.04^{-10} \times \frac{9,287.2164}{9,712.0728} \times \left(15.632 - \tfrac{11}{24}\right)$$

$$= 82,339.85$$

The expected present value of the future expenses is:

$$EPV(E) = 60\ddot{a}_{50} = 60\left(\ddot{a}_{50:\overline{10}|} + {}_{10|}\ddot{a}_{50}\right)$$

with $\ddot{a}_{50:\overline{10}|}$ taken from the AM92 Table. So:

$$EPV(E) = 60\left(8.314 + 1.04^{-10} \times \frac{9,287.2164}{9,712.0728} \times 15.632\right)$$

$$= 1,104.75$$

We need to be careful about valuing the pre-age 60 and the post-age 60 elements separately due to the different bases required for deferment and vesting.

Thus the reserve is £83,445.

Solution 7.14

The retrospective reserve at time $t = 3$ will be

$$ {}_3V^{retro} = \left(25,201.39 - 30,000A^{1}_{[48]:\overline{3}|} - 360 - 45a_{[48]:\overline{2}|}\right)(1+i)^3 \frac{l_{[48]}}{l_{51}} $$

The required factors are:

$$ (1+i)^3 \frac{l_{[48]}}{l_{51}} = 1.04^3 \times \frac{9,748.8603}{9,687.7149} = 1.131964 $$

$$ A^{1}_{[48]:\overline{3}|} = A_{[48]} - v^3\, {}_3p_{[48]}\, A_{51} = 0.30664 - \frac{1}{1.131964} \times 0.34058 = 0.00576 $$

$$ \begin{aligned} a_{[48]:\overline{2}|} &= \ddot{a}_{[48]:\overline{3}|} - 1 \\ &= \ddot{a}_{[48]} - v^3\, {}_3p_{[48]}\, \ddot{a}_{51} - 1 \\ &= 18.027 - \frac{1}{1.131964} \times 17.145 - 1 \\ &= 1.881 \end{aligned} $$

So:

$$ \begin{aligned} {}_3V^{retro} &= (25,201.39 - 30,000 \times 0.00576 - 360 - 45 \times 1.881) \times 1.131964 \\ &= £27,828.13 \end{aligned} $$

We can also build this reserve up from the end-of-Year-2 reserve as follows:

$$ \frac{(26,807.75 - 45) \times 1.04 - 30,000q_{50}}{p_{50}} = 27,827.81 $$

The difference is due to rounding error.

Solution 7.15

The prospective reserve at time $t = 2$ will be:

$$ {}_2V^{pro} = 30{,}000 A_{50:\overline{3}|} + 45\ddot{a}_{50:\overline{3}|} $$

Now:

$$
\begin{aligned}
A_{50:\overline{3}|} &= A_{50} - v^3\, {}_3p_{50}\, A_{53} + v^3\, {}_3p_{50} \\
&= 0.32907 - 1.04^{-3} \times \frac{9{,}630.0522}{9{,}712.0728} \times 0.36448 + 1.04^{-3} \times \frac{9{,}630.0522}{9{,}712.0728} \\
&= 0.88927
\end{aligned}
$$

and:

$$
\begin{aligned}
\ddot{a}_{50:\overline{3}|} &= \ddot{a}_{50} - v^3\, {}_3p_{50}\, \ddot{a}_{53} \\
&= 17.444 - 1.04^{-3} \times \frac{9{,}630.0522}{9{,}712.0728} \times 16.524 \\
&= 2.878
\end{aligned}
$$

So:

$$ {}_2V^{pro} = 30{,}000 \times 0.88927 + 45 \times 2.878 = £26{,}807.62 $$

This is equal to the retrospective reserve, because both reserves have been calculated using the assumptions of the gross premium basis. *The slight difference is due to rounding error.*

Solution 7.16

The premium equation is:

$$P = Ba_x + I + Ra_x = (B+R)a_x + I$$

Now split at time t to obtain:

$$P = (B+R)\left(a_{x:\overline{t|}} + v^t \, {}_tp_x \, a_{x+t}\right) + I$$

Rearrange:

$$\Rightarrow P - I - (B+R)a_{x:\overline{t|}} = v^t \, {}_tp_x \, (B+R)a_{x+t}$$

Finally divide through by $v^t \, {}_tp_x$:

$$\Rightarrow \left[P - I - (B+R)a_{x:\overline{t|}}\right]\frac{(1+i)^t}{{}_tp_x} = (B+R)a_{x+t}$$

ie ${}_tV^{retro} = {}_tV^{pro}$.

Solution 7.17

(i) The reserves at the end of the year for those policies still in force are equal to the reserves at the beginning of the year with premiums and investment income, less expenses and the cost of cover.

(ii) $\left({}_tV' - e\right)(1+i) - q_{x+t}(S+f) = p_{x+t} \; {}_{t+1}V'$

Solution 7.18

We have:

$$\left({}_9V' + G - e\right)\left(1+i\right) - Sq_{54} = p_{54}\ {}_{10}V'$$

This gives:

$$\left({}_9V' + 2{,}132 \times 0.95\right) \times 1.04 - 75000q_{54} = p_{54}\ \times 20{,}898$$

So:

$${}_9V' = £18{,}276$$

Solution 7.19

For net premium P we have:

$$\left({}_tV + P\right)\left(1+i\right) - q_{x+t}S = p_{x+t}\ {}_{t+1}V$$

Solution 7.20

Incorporating the additional features of the question, the profit earned is:

$$PRO_{10} = \left({}_{10}V' + G - e'\right)(1+i') - (S + f')\,q'_{55} - \left(1 - q'_{55}\right){}_{11}V'$$

where the primed functions indicate the actual experience. Putting in the numbers:

$$PRO_{11} = \left(20{,}898 + 2{,}132 - 78\right) \times 1.038 - (75{,}000 + 150) \times 0.73 \times q_{55}$$
$$- \left(1 - 0.73 \times q_{55}\right) \times 23{,}611$$

From the *Tables* $q_{55} = 0.004469$, which gives:

$$PRO_{11} = £45.04$$

Solution 7.21

(i) ***Premium***

The expected present value of the annuity at age 60 is:

$$12{,}000\ddot{a}_{60}^{(12)} \;@\, 4\% = 12{,}000\left(\ddot{a}_{60} - \frac{11}{24}\right) = 12{,}000\left(15.632 - \frac{11}{24}\right) = 182{,}084$$

Note that we use 4% interest here since:

$$\frac{1.06}{1.019231} - 1 = 4\%$$

Discounting this back to age 40, allowing for interest and mortality, we get:

$$\begin{aligned}\text{EPV deferred annuity} &= 182{,}084\, v^{20}\ {}_{20}p_{[40]} \\ &= 182{,}084 \times 1.06^{-20} \times \frac{9{,}287.2164}{9{,}854.3036} \\ &= 53{,}507.43\end{aligned}$$

(There are no inflationary increases before age 60, so we use 6% interest here.)

The expected present value of the expenses at age 40 is:

$$\text{EPV expenses} = 500 + 0.01 \times 53{,}507.43 = 1{,}035.07$$

So the single premium is:

$$P = 53{,}507.43 + 1{,}035.07 = £54{,}543$$

(ii) ***Reserve at time 10***

The reserve at time 10 is:

$$\begin{aligned}{}_{10}V &= 12{,}000 \times 1.01 \times 1.06^{-10}\ {}_{10}p_{50}\ a_{60}^{(12)@4\%} \\ &= 12{,}000 \times 1.01 \times 1.06^{-10} \times \frac{9{,}287.2164}{9{,}712.0728} \times \left(15.632 - \frac{11}{24}\right) \\ &= £98{,}199\end{aligned}$$

Solution 7.22

(i) ***Gross future loss***

(To simplify the notation in the following formulae, K will represent K_x and T will represent T_x throughout.)

Simple bonus policy

$$\text{PV premiums} = P\,\ddot{a}_{\overline{\min(K+1,65-x)}|}$$

$$\text{PV expenses} = 250 + 150\,v^T + 0.02 \times P \times \left(\ddot{a}_{\overline{\min(K+1,65-x)}|} - 1\right)$$

$$\text{PV benefits} = S\left(1 + 0.06 \times K\right) v^T$$

So the total future loss is:

$$\left[S\left(1+0.06\times K\right)+150\right]v^T + 250 - P\left(0.98\,\ddot{a}_{\overline{\min(K+1,65-x)}|} + 0.02\right)$$

Compound bonus policy

This is identical to the above except for the present value of the benefits. Now:

$$\text{PV benefits} = S\,(1.04)^K\, v^T$$

So the future loss random variable becomes:

$$\left[S\,(1.04)^K + 150\right]v^T + 250 - P\left(0.98\,\ddot{a}_{\overline{\min(K+1,65-x)}|} + 0.02\right)$$

(ii) ***Calculating the premium***

Simple bonus policy

$$\text{EPV premiums} = P\,\ddot{a}_{[40]:\overline{25}|} = 15.887P$$

$$\text{EPV expenses} = 250 + 150\,\bar{A}_{[40]} + 0.02P\left(\ddot{a}_{[40]:\overline{25}|} - 1\right)$$

$$= 250 + 150 \times 1.04^{½} \times 0.23041 + 0.02P(15.887 - 1)$$

$$= 285.25 + 0.298\,P$$

$$\text{EPV benefits} = 12{,}000(I\bar{A})_{[40]} + 188{,}000\bar{A}_{[40]}$$

$$= 1.04^{½}\left[12{,}000(IA)_{[40]} + 188{,}000A_{[40]}\right]$$

$$= 1.04^{½}\left[12{,}000 \times 7.95835 + 188{,}000 \times 0.23041\right]$$

$$= 141{,}566.40$$

So the equation of value is:

$$15.887P = 141{,}566.40 + 285.25 + 0.298P$$

$$\Rightarrow P = £9{,}099.47$$

Compound bonus policy

$$\text{EPV benefits} = 200{,}000\left[v^{½}\,q_{[40]} + v^{1½}(1.04)\;{}_{1|}q_{[40]} + v^{2½}(1.04)^2\;{}_{2|}q_{[40]} + \ldots\right]$$

$$= \frac{200{,}000}{1.04^{½}}\left[\,q_{[40]} + {}_{1|}q_{[40]} + {}_{2|}q_{[40]} + \ldots\right]$$

$$= \frac{200{,}000}{1.04^{½}}$$

$$= 196{,}116.14$$

The equation of value is then:

$$15.887P = 196{,}116.14 + 285.25 + 0.298P$$

$$\Rightarrow P = £12{,}598.72$$

(iii) ***Net premium reserve***

The equation for the net premium P_{net} is:

$$P_{net}\,\ddot{a}_{40:\overline{25}|} = 200{,}000\bar{A}_{40}$$

So:

$$P_{net} = \frac{200{,}000\times 1.04^{½}\times 0.23056}{15.884} = £2{,}960.54$$

The net premium reserve at time 10 is:

$$\begin{aligned} {}_{10}V &= 290{,}000\,\bar{A}_{50} - P_{net}\,\ddot{a}_{50:\overline{15}|} \\ &= 290{,}000\times 1.04^{½}\times 0.32907 - 2{,}960.54\times 11.253 \\ &= £64{,}005 \end{aligned}$$

Solution 7.23

(a) ***Recursive formula***

The recursive relationship between the gross premium reserves at integer times t and $t+1$ is:

$$\left({}_tV + P - e_t\right)(1+i) = {}_{t+1}V\times p_{x+t} + S\times q_{x+t}$$

where:

x is the age of the policyholder at entry

${}_tV$ is the gross premium reserve at time t

P is the annual gross premium

e_t is the expense incurred at the start of the $(t+1)$th policy year, *ie* at time t

i is the effective annual rate of interest

S is the sum assured under the whole life policy

q_{x+t} is the probability that a policyholder aged $x+t$ dies in the coming year

p_{x+t} is the probability that a policyholder aged $x+t$ survives for at least one year

(b) ***Explanation***

We start with the reserve at time t, *ie* the fund held in respect of the policy at time t. Then we add in the premium income and take off the expenses payable at the time t. Roll this up with interest to time $t+1$, and this is exactly enough to set up the reserve at time $t+1$ for the policyholders who have survived that policy year and also to pay the death benefit of S to the policyholders who have died during that policy year.

Chapter 8

Simple annuities and assurances involving two lives

Syllabus objectives

(vi) Define and use functions involving two lives.

1. Extend the techniques of objectives (i) – (v) to deal with cashflows dependent upon the death or survival of either or both of two lives.

2. Extend the techniques of 1 to deal with functions dependent upon a fixed term as well as age.

0 Introduction

In this chapter we shall extend the concepts developed earlier to deal with situations involving two lives; for instance, "what is the value of an annuity payable until the last of two lives dies?"

This chapter covers the following topics:

- random variables describing basic joint life functions
- determining simple joint life probabilities
- present values of simple joint life policies.

1 Random variables to describe joint life functions

1.1 Single life functions

So far we have described annuity and assurance functions which depend upon the death or survival of a single life aged x. Central to the development of these functions is the random variable measuring the future lifetime of a life now aged x, T_x, or its curtate counterpart K_x.

Recall that K_x (the curtate future lifetime of x) is the integer part of T_x (the complete future lifetime of x).

1.2 Joint life functions

We now consider annuity and assurance functions that depend upon the death or survival of two lives. The random variables of interest are T_x and T_y, the future lifetimes of two lives, one aged x and the other aged y. Throughout the analysis of these problems, we assume that T_x and T_y are independent random variables.

So we assume that the mortality of life x is independent of the mortality of life y. In practice, this is seldom the case since many joint life policies involve husband/wife (and similar) partnerships. The mortality of these partners will not be independent because of the significant possibility of both lives dying as a result of the same accident or illness. However, the assumption of independence makes the theory far more manageable.

The random variable T_{xy} measures the joint lifetime of (x) and (y) *ie* the time while both (x) and (y) remain alive, which is the time until the first death of (x) and (y). We can write:

$$T_{xy} = \min\{T_x, T_y\}$$

The cumulative distribution function of this random variable can be written:

$$F_{T_{xy}}(t) = P\left[T_{xy} \le t\right]$$

which we write as:

$$\begin{aligned} F_{T_{xy}}(t) &= 1 - {}_t p_{xy} \\ &= P\left[\min\left\{T_x, T_y\right\} \le t\right] \\ &= 1 - P\left[T_x > t \text{ and } T_y > t\right] \\ &= 1 - P\left[T_x > t\right] P\left[T_y > t\right] \end{aligned}$$

since the random variables T_x and T_y are independent. So:

$$F_{T_{xy}}(t) = 1 - {}_t p_x \, {}_t p_y$$

using the life table notation.

It is extremely helpful at this point to consider the idea of a *status*, which we also met in Chapter 1. The random variable T_{xy} is characterised by the *joint life* status xy, which is the status of both lives x and y being alive. If either one of x or y dies, the joint life status is said to fail. The random variable T_{xy} represents the future time until the failure of the status, which in this case is the joint life status xy.

We shall see examples of other statuses later on in this chapter.

Question 8.1

So what does ${}_t p_{xy}$ mean?

In the same way, when we consider the temporary annuity $\ddot{a}_{x:\overline{n}|}$, we are looking at an annuity payable while the joint status $x:\overline{n}|$ is still active, that is, the life x is still alive and the n-year period has not yet expired. In this case the joint status is made up of an active life and a time period, but the underlying logic is still the same.

The density function of T_{xy} can be obtained by differentiating the cumulative distribution function:

$$f_{T_{xy}}(t) = \frac{d}{dt}\left[1 - {}_tp_x\,{}_tp_y\right]$$

$$= -{}_tp_x\left(-{}_tp_y\,\mu_{y+t}\right) - {}_tp_y\left(-{}_tp_x\,\mu_{x+t}\right)$$

using the results from Subject CT4, Models.

Question 8.2

Prove the above result.

This rearranges to give:

$$f_{T_{xy}}(t) = {}_tp_x\,{}_tp_y\left(\mu_{x+t} + \mu_{y+t}\right)$$

Question 8.3

By considering what happens over the infinitesimally small time interval $(t, t+\delta t)$ express the above result in words.

1.3 Joint lifetime random variables and joint life table functions

When functions of a single life *eg* T_x are considered, it is helpful to introduce the life table functions l_x, d_x and q_x as an aid to the calculation of the numerical values of expressions which are the solution of actuarial problems.

In an exactly similar way it is helpful to develop the joint life functions l_{xy}, d_{xy} and q_{xy} to help in the numerical evaluation of expressions which are the solution to problems involving more than one life. We define these functions in terms of the single life functions. Recall that:

$${}_tp_{xy} = {}_tp_x\,{}_tp_y$$

Using the independence assumption:

$${}_tp_{xy} = \frac{l_{x+t}}{l_x}\cdot\frac{l_{y+t}}{l_y}$$

So we write:

$$l_{xy} = l_x l_y$$

and:

$${}_tp_{xy} = \frac{l_{x+t:y+t}}{l_{xy}}$$

only separating the subscripts with colons when the exact meaning of the function would be unclear if the colons were omitted.

Then:

$$d_{xy} = l_{xy} - l_{x+1:y+1}$$

$$q_{xy} = \frac{d_{xy}}{l_{xy}}$$

Q

Question 8.4

Explain in words what q_{xy} means.

Similarly, the expression ${}_tq_{xy}$ represents the probability that the joint life status fails within the next t years, *ie* by the end of t years there has been at least one death.

The force of failure of the joint life status can be derived in the usual way:

$$\mu_{x+t:y+t} = -\frac{1}{l_{x+t:y+t}}\frac{d}{dt}l_{x+t:y+t}$$

This can then be related to the forces of mortality in the life tables for the single lives *x* and *y*:

$$\begin{aligned}\mu_{x+t:y+t} &= -\frac{d}{dt}\log_e l_{x+t:y+t} \\ &= -\frac{d}{dt}\log_e l_{x+t}l_{y+t} \\ &= -\frac{d}{dt}\left\{\log_e l_{x+t} + \log_e l_{y+t}\right\} \\ &= \mu_{x+t} + \mu_{y+t}\end{aligned}$$

Notice that:

- **this relationship is additive in contrast to the previous relationships, which were multiplicative, and**
- **that there is no "simple" relationship for d_{xy}.**

So to get the joint life *force of mortality* we *add* the forces for the individual lives. For the joint *probability of survival* we *multiply* the individual probabilities of survival.

So we write:

$$\begin{aligned}f_{T_{xy}}(t) &= {}_tp_x\,{}_tp_y\left(\mu_{x+t} + \mu_{y+t}\right) \\ &= {}_tp_{xy}\;\mu_{x+t:y+t}\end{aligned}$$

We can define a discrete random variable that measures the curtate joint future lifetime of x and y :

$$K_{xy} = \text{integer part of } T_{xy}$$

and develop the probability function of K_{xy} :

$$\begin{aligned} P\left[K_{xy} = k\right] &= P\left[k \le T_{xy} < k+1\right] \\ &= F_{T_{xy}}(k+1) - F_{T_{xy}}(k) \\ &= \left(1 - {}_{k+1}p_{xy}\right) - \left(1 - {}_{k}p_{xy}\right) \\ &= {}_{k}p_{xy} - {}_{k+1}p_{xy} \\ &= {}_{k}p_{xy} - {}_{k}p_{xy}\, p_{x+k:y+k} \\ &= {}_{k}p_{xy}\, q_{x+k:y+k} \\ &= {}_{k|}q_{xy} \end{aligned}$$

> ***Question 8.5***
>
> What does this symbol mean?

The joint life table functions l_{xy}, d_{xy}, q_{xy} and μ_{xy} are not tabulated in the Formulae and Tables for Examinations. However, these functions can be evaluated using the tabulated single life functions l_x, q_x and μ_x.

You should spend a few minutes familiarising yourself with what is in the *Tables* for joint lives. Make sure you have a copy of the 2002 edition of the *Tables*. In particular, you should be aware of the existence and location of the following:

Table	*Interest*	*Functions*
PMA92C20 and PFA92C20	4%	Joint life annuities for males and females of different ages (ultimate mortality) (page 115)

We may sometimes refer to these as "the PA92 Tables", when it is clear from the context whether male or female lives are involved, and that the cohort C20 is indicated.

There are fewer joint life functions tabulated in the yellow tables than there were in the old green ones. You should be aware of this when you get closer to the exam date and start to attempt old past exam questions, as you may find you have a different range of joint life functions available.

Question 8.6

Assuming the mortality of AM92 (Ultimate) for both lives, calculate the following:

(i) ${}_3p_{45:41}$

(ii) $q_{66:65}$

(iii) $\mu_{38:30}$

1.4 *Last survivor lifetime random variables*

Two common types of life policy are:

- an annuity payable to a couple while at least one of them is alive, and
- an assurance payable on the second death of a couple.

These are both examples of last survivor policies, where the payment is contingent on what happens to the second life to die, rather than the first. In order to calculate the value of such benefits, and so be able to calculate suitable premiums, we need to develop the following theory.

The random variable $T_{\overline{xy}}$ measures the time until the last death of (x) and (y), *ie* the time while at least one of (x) and (y) remains alive. We can write:

$$T_{\overline{xy}} = \max\left\{T_x, T_y\right\}$$

So the last survivor status fails on the second death. The last survivor status is indicated by $\overline{xy}$.

The cumulative distribution function of this random variable can be written:

$$F_{T_{\overline{xy}}}(t) = P\left[T_{\overline{xy}} \le t\right] = {}_tq_{\overline{xy}}$$

This is the probability that the last survivor status fails within t years, *ie* the probability that both lives are dead by the end of t years. Now:

$$F_{T_{\overline{xy}}}(t) = P\left[\max\left\{T_x, T_y\right\} \le t\right]$$

$$= P\left[T_x \le t \text{ and } T_y \le t\right]$$

$$= P\left[T_x \le t\right]P\left[T_y \le t\right] = {}_tq_x \times {}_tq_y$$

since the random variables T_x and T_y are independent.

So:

$$F_{T_{\overline{xy}}}(t) = \left(1 - {}_tp_x\right)\left(1 - {}_tp_y\right)$$

$$= 1 - {}_tp_x - {}_tp_y + {}_tp_x\,{}_tp_y$$

$$= \left(1 - {}_tp_x\right) + \left(1 - {}_tp_y\right) - \left(1 - {}_tp_x\,{}_tp_y\right)$$

$$= F_{T_x}(t) + F_{T_y}(t) - F_{T_{xy}}(t)$$

using the life table notation.

This last result is a particular example of a general relationship that will be key to enabling us to calculate multiple life functions easily:

Important result

$$\begin{pmatrix}\text{function of last}\\ \text{survivor status } \overline{xy}\end{pmatrix} = \begin{pmatrix}\text{function of single}\\ \text{life status } x\end{pmatrix} + \begin{pmatrix}\text{function of single}\\ \text{life status } y\end{pmatrix} - \begin{pmatrix}\text{function of joint}\\ \text{life status } xy\end{pmatrix}$$

The function involved must be the same for all statuses. We will subsequently refer to this relationship as just:

$$\text{last survivor (L)} = \text{single (S)} + \text{single (S)} - \text{joint (J)}$$

In the above Core Reading, the function was the cumulative distribution function of the lifetime of the status at time t, *ie* the probability of the status failing by time t. So we have:

$$ {}_tq_{\overline{xy}} = {}_tq_x + {}_tq_y - {}_tq_{xy} $$

The density function of $T_{\overline{xy}}$ can be obtained by differentiating the cumulative distribution function:

$$ f_{T_{\overline{xy}}}(t) = \frac{d}{dt}\left[1 - {}_tp_x - {}_tp_y + {}_tp_x\,{}_tp_y\right] $$

$$ = {}_tp_x\mu_{x+t} + {}_tp_y\mu_{y+t} - {}_tp_x\,{}_tp_y\left(\mu_{x+t} + \mu_{y+t}\right) $$

Using the results from Section 1.3 above, we can write:

$$ f_{T_{\overline{xy}}}(t) = {}_tp_x\mu_{x+t} + {}_tp_y\mu_{y+t} - {}_tp_{xy}\mu_{x+t:y+t} $$

$$ = f_{T_x}(t) + f_{T_y}(t) - f_{T_{xy}}(t) $$

In other words, it is the $L = S + S - J$ relationship again, but this time for the probability density function at time t. We could have alternatively found the above result from:

$$ f_{T_{\overline{xy}}}(t) = \frac{\partial}{\partial t}F_{T_{\overline{xy}}}(t) = \frac{\partial}{\partial t}\left[F_{T_x}(t) + F_{T_y}(t) - F_{T_{xy}}(t)\right] $$

$$ = f_{T_x}(t) + f_{T_y}(t) - f_{T_{xy}}(t) $$

Tip

Notice that for joint life statuses, it is often easier to work with p-type (survival) functions, since ${}_tp_{xy} = {}_tp_x \times {}_tp_y$. For last survivor statuses, it is often easier to work with q-type (mortality) functions, since ${}_tq_{\overline{xy}} = {}_tq_x \times {}_tq_y$.

2 Determining simple probabilities involving two lives

We now apply the random variable theory developed above to see how joint life and last survivor probabilities of death and survival can be evaluated.

2.1 Evaluating probabilities of death or survival of either or both of two lives

We can define a discrete random variable that measures the curtate last survivor lifetime of x and y :

$$K_{\overline{xy}} = \text{ integer part of } T_{\overline{xy}}$$

and develop the probability function of $K_{\overline{xy}}$:

$$\begin{aligned}
P\left[K_{\overline{xy}} = k\right] &= P\left[k \le T_{\overline{xy}} < k+1\right] \\
&= {}_{k|}q_{\overline{xy}} \\
&= F_{T_{\overline{xy}}}(k+1) - F_{T_{\overline{xy}}}(k) \\
&= F_{T_x}(k+1) + F_{T_y}(k+1) - F_{T_{xy}}(k+1) - \left\{F_{T_x}(k) + F_{T_y}(k) - F_{T_{xy}}(k)\right\} \\
&= P\left[K_x = k\right] + P\left[K_y = k\right] - P\left[K_{xy} = k\right] \\
&= {}_{k|}q_x + {}_{k|}q_y - {}_{k|}q_{xy}
\end{aligned}$$

Recall from the last section, **the cumulative distribution function of $T_{\overline{xy}}$ is given by:**

$${}_tq_{\overline{xy}} = 1 - {}_tp_x - {}_tp_y + {}_tp_x\, {}_tp_y$$

So the survival function of $T_{\overline{xy}}$ is:

$$ {}_tp_{\overline{xy}} = P\left[T_{\overline{xy}} > t\right] = 1 - P\left[T_{\overline{xy}} \le t\right] = {}_tp_x + {}_tp_y - {}_tp_x\,{}_tp_y $$

(*ie* $L = S + S - J$)

which can be factorised into:

$$ {}_tp_x\,{}_tp_y + \left(1 - {}_tp_x\right){}_tp_y + \left(1 - {}_tp_y\right){}_tp_x $$

where each of the three terms corresponds to one of the mutually exclusive and exhaustive events which result in the last survivor of (*x*) and (*y*) living for at least *t* years, *ie*:

- **both (x) and (y) alive after t years,**
- **(x) dead, but (y) alive after t years, and**
- **(x) alive, but (y) dead after t years.**

These probabilities can be evaluated directly. The probability of the complementary event that both lives die within t years is:

$$ {}_tq_{\overline{xy}} = (1 - {}_tp_x)(1 - {}_tp_y) = {}_tq_x\,{}_tq_y $$

that we had earlier. In summary, there are three alternatives for calculating ${}_tp_{\overline{xy}}$:

- $1 - {}_tq_{\overline{xy}}$ (1)
- ${}_tp_x + {}_tp_y - {}_tp_{xy}$ (2)
- ${}_tp_x\,{}_tp_y + \left(1 - {}_tp_x\right){}_tp_y + \left(1 - {}_tp_y\right){}_tp_x$ (3)

All three can be useful, but you may as well try using (1) first and (3) last, as obviously more calculations are involved as you move down the list.

Note that we also have $p_{\overline{xy}} = p_x + p_y - p_{xy}$ (for example), where $p_{\overline{xy}}$ is just the probability that the last survivor status remains active for at least one year.

2.2 *Evaluating last survivor functions*

We have already derived:

$$F_{T_{\overline{xy}}}(t) = F_{T_x}(t) + F_{T_y}(t) - F_{T_{xy}}(t) = 1 - {}_t p_x - {}_t p_y + {}_t p_{xy}$$

$$f_{T_{\overline{xy}}}(t) = f_{T_x}(t) + f_{T_y}(t) - f_{T_{xy}}(t) = {}_t p_x \, \mu_{x+t} + {}_t p_y \, \mu_{y+t} - {}_t p_{xy} \, \mu_{x+t:y+t}$$

and so it seems that all last survivor functions can be expressed in terms of single life and joint life functions. This is true and provides a method of evaluating such functions without the need to develop any additional functions to help in computation.

This is the result of the relationship between the joint lifetime and last survivor lifetime random variables:

$$T_{xy} + T_{\overline{xy}} = \min\left\{T_x, T_y\right\} + \max\left\{T_x, T_y\right\} = T_x + T_y$$

Similarly:

$$K_{xy} + K_{\overline{xy}} = \min\left\{K_x, K_y\right\} + \max\left\{K_x, K_y\right\} = K_x + K_y$$

which gives the result:

$$P\left[K_{\overline{xy}} = k\right] = P\left[K_x = k\right] + P\left[K_y = k\right] - P\left[K_{xy} = k\right]$$

So this is the rationale underlying the important relationship that we stated in Section 1.4 above.

So curtate last survivor functions can be evaluated from the corresponding joint life and single life functions.

Question 8.7

Calculate $p_{\overline{62:65}}$ and ${}_3q_{\overline{50:50}}$ using AM92 Ultimate mortality.

3 Determining present values involving two lives

There is no new theory in this section because we are going to use the same assurance and annuity functions that we met in Chapters 1 and 2, but using multiple life statuses instead of the single life status. The concepts that are covered here should therefore be very familiar to you.

3.1 Present values of joint life and last survivor assurances

Consider an assurance under which the benefit (of 1) is paid immediately on the ending (failure) of a status u. This status u could be any joint lifetime or last survivor status, *eg* $xy, \overline{xy}$. Let T_u be a continuous random variable representing the future lifetime of the status u and let $f_{T_u}(t)$ be the probability density function of T_u.

The present value of the assurance can be represented by the random variable:

$$\bar{Z}_u = v_i^{T_u}$$

where i is the valuation rate of interest.

The expected value of $\bar{Z}_u$ is denoted by $\bar{A}_u$ where:

$$E\left[\bar{Z}_u\right] = \bar{A}_u = \int_{t=0}^{t=\infty} v^t f_{T_u}(t)dt$$

and the variance can be written as:

$$\text{var}\left(\bar{Z}_u\right) = E\left[\bar{Z}_u^2\right] - \left(E\left[\bar{Z}_u\right]\right)^2$$

$$= E\left[\left(v^{T_u}\right)^2\right] - \left(\bar{A}_u\right)^2$$

$$= \int_{t=0}^{t=\infty} v^{2t} f_{T_u}(t)dt - \bar{A}_u^2$$

$$= {}^2\bar{A}_u - \left(\bar{A}_u\right)^2$$

where ${}^2\bar{A}_u$ is evaluated at a valuation rate of interest of $i^* = 2i + i^2$.

Question 8.8

Show that:

$$\int_{t=0}^{\infty} v^{2t} f_{T_u}(t)\, dt = {}^2\bar{A}_u$$

where ${}^2\bar{A}_u$ is valued using $i' = 2i + i^2$, or, equivalently, at $\delta' = 2\delta$.

For example, using the results from Section 1.2, if $u = xy$ we can write the mean (*ie* the expected value) and variance of the present value of an assurance payable immediately on the ending of the joint lifetime of (x) and (y) as:

Mean: $$\bar{A}_{xy} = \int_{t=0}^{t=\infty} v^t\, {}_tp_{xy}\, \mu_{x+t:y+t}\, dt$$

$$= \int_{t=0}^{t=\infty} v^t\, {}_tp_{xy} \left(\mu_{x+t} + \mu_{y+t}\right) dt$$

Variance: $${}^2\bar{A}_{xy} - \left(\bar{A}_{xy}\right)^2$$

Again, using the results of Section 1.2, if $u = \overline{xy}$ we can write the mean and variance of the present value of an assurance payable immediately on the death of the last survivor of (x) or (y) as:

Mean: $$\bar{A}_{\overline{xy}} = \int_{t=0}^{t=\infty} v^t \left({}_tp_x\, \mu_{x+t} + {}_tp_y\, \mu_{y+t} - {}_tp_{xy}\, \mu_{x+t:y+t}\right) dt$$

$$= \bar{A}_x + \bar{A}_y - \bar{A}_{xy}$$

Variance: $${}^2\bar{A}_{\overline{xy}} - \left(\bar{A}_{\overline{xy}}\right)^2 = \left({}^2\bar{A}_x + {}^2\bar{A}_y - {}^2\bar{A}_{xy}\right) - \left(\bar{A}_x + \bar{A}_y - \bar{A}_{xy}\right)^2$$

So last survivor functions can be evaluated in terms of single and joint life functions.

If the assurance benefit is paid at the end of the year in which the status ends, then we can use a (discrete) random variable K_u with a present value function

$$Z_u = v_i^{K_u + 1}$$

where i is the valuation rate of interest.

In the above expression we need the extra "1" to take us to the end of the year in which the status fails, which is when the payment is made, because K_u will take us only to the start of that year. For example, if the status fails in the first year, the present value of the benefit is v and not $v^K = v^0 = 1$.

A similar analysis for K_{xy} and $K_{\overline{xy}}$ gives the means and variances of the present values of the joint life and last survivor assurances with sums assured payable at the end of the year of death.

Joint life

Mean: $$A_{xy} = \sum_{t=0}^{t=\infty} v^{t+1} \; {}_{t|}q_{xy}$$

Variance: $${}^2A_{xy} - \left(A_{xy}\right)^2$$

Question 8.9

Prove the above results for the mean and variance.

Last survivor

Mean: $$A_{\overline{xy}} = A_x + A_y - A_{xy}$$

Variance: $${}^2A_{\overline{xy}} - \left(A_{\overline{xy}}\right)^2 = \left({}^2A_x + {}^2A_y - {}^2A_{xy}\right) - \left(A_x + A_y - A_{xy}\right)^2$$

It is important when calculating a reserve for a last survivor assurance to remember that it would be necessary to establish whether both lives remain alive or one has already died. This is because, of course, the contract still remains in force whether one or two lives remain alive. The premium being paid will still be the original premium calculated on a last survivor basis.

Question 8.10

For a policy where the male and female are both aged 63, list the 3 possible states that a last survivor assurance may be in after a few years. For which of these would the reserve be largest?

Thus for example if both x and y are alive a net premium reserve for a whole life last survivor contract would be:

$$_tV_{\overline{x:y}} = A_{\overline{x+t:y+t}} - P_{\overline{x:y}}\,\ddot{a}_{\overline{x+t:y+t}}$$

Whereas if say (y) had previously died the reserve would be:

$$_tV_{\overline{x:y}} = A_{x+t} - P_{\overline{x:y}}\,\ddot{a}_{x+t}$$

Thus on first death a significant increase in reserve will take place.

Question 8.11

For a policy where the male and female are both aged 63, would this increase usually be greater if the first death was the male life or the female life?

3.2 Present values of joint life and last survivor annuities

Consider an annuity under which a benefit of 1 *pa* is paid continuously so long as a status u continues. The present value of these annuity payments can be represented by the random variable:

$$\bar{a}_{\overline{T_u}|}$$

The expected present value of this benefit is denoted by $\bar{a}_u$ where:

$$E\left[\bar{a}_{\overline{T_u}|}\right] = \bar{a}_u = \int_{t=0}^{t=\infty} \bar{a}_{\overline{t}|} f_{T_u}(t)\,dt$$

This is most simply expressed by using assurance functions:

$$E\left[\bar{a}_{\overline{T_u|}}\right] = E\left[\frac{1-v^{T_u}}{\delta}\right] = \frac{1-E\left[v^{T_u}\right]}{\delta} = \frac{1-\bar{A}_u}{\delta}$$

The variance can be expressed in a similar way:

$$\begin{aligned}
\text{var}\left(\bar{a}_{\overline{T_u|}}\right) &= \text{var}\left(\frac{1-v^{T_u}}{\delta}\right) \\
&= \text{var}\left(\frac{1}{\delta}-\frac{1}{\delta}v^{T_u}\right) \\
&= \frac{1}{\delta^2}\text{var}\left(v^{T_u}\right) \\
&= \frac{1}{\delta^2}\left\{{}^2\bar{A}_u - \left(\bar{A}_u\right)^2\right\}
\end{aligned}$$

The results from Section 3.1 can be used to determine the means and variances for $u = xy$ (the joint life annuity) and $u = \overline{xy}$ (the last survivor annuity).

The means and variances of the present values of annuities payable in advance and in arrear can be evaluated using the (discrete) random variables:

- **in advance:** $\ddot{a}_{\overline{K_u+1}|}$
- **in arrear:** $a_{\overline{K_u}|}$

giving the results:

	In advance	*In arrear*
Mean	$\ddot{a}_u = \dfrac{1-A_u}{d}$	$a_u = \dfrac{(1-d)-A_u}{d}$
Variance	$\dfrac{1}{d^2}\left\{{}^2A_u - (A_u)^2\right\}$	$\dfrac{1}{d^2}\left\{{}^2A_u - (A_u)^2\right\}$

which can be applied to joint lifetime and last survivor statuses.

Proof

The mean of the present value of an annuity payable in advance is:

$$E\left[\ddot{a}_{\overline{K_u+1}|}\right] = E\left[\frac{1-v^{K_u+1}}{d}\right] = \frac{1-E\left[v^{K_u+1}\right]}{d} = \frac{1-A_u}{d}$$

The variance of the present value of an annuity payable in advance is:

$$\text{var}\left[\ddot{a}_{\overline{K_u+1}|}\right] = \text{var}\left[\frac{1-v^{K_u+1}}{d}\right] = \frac{1}{d^2}\text{var}\left[v^{K_u+1}\right] = \frac{1}{d^2}\left\{{}^2A_u - \left(A_u\right)^2\right\}$$

Question 8.12

Prove the corresponding results for an annuity payable annually in arrears.

4 Exam-style questions

We now conclude this chapter with a couple of exam-style questions.

Question 8.13

A life insurance company issues a joint-life annuity to a male, aged 68, and a female, aged 65. The annuity of £10,000 *pa* is payable annually in arrears and continues until both lives have died.

The insurance company values this benefit using PA92C20 mortality (males or females as appropriate) and 4% *pa* interest.

(i) (a) Calculate the expected present value of this annuity.

(b) Derive an expression for the variance of the present value of this annuity in terms of appropriate single and joint-life assurance functions.

(ii) If the insurance company charges a premium of £150,000 for this policy, calculate the probability that it makes a profit on the contract.

Question 8.14

The random variable T_{xy} represents the time to failure of the joint-life status (xy). (x) is subject to a constant force of mortality of 0.01, and (y) is subject to a constant force of mortality of 0.02. Calculate the value of $E\left(T_{xy}\right)$ assuming that (x) and (y) are independent with respect to mortality.

Chapter 8 Summary

We have developed the following random variables, with associated simple functions:

Random variable	***Modelling***
T_x	Complete future lifetime of a life aged x
$T_{xy} = \min\{T_x, T_y\}$	Time to failure of the joint life status xy, *ie* the time until the first death of x and y
$T_{\overline{xy}} = \max\{T_x, T_y\}$	Time to failure of the last survivor status $\overline{xy}$, *ie* the time until the last death of x and y
K_x	Curtate future lifetime of a life aged x
$K_{xy} = \min\{K_x, K_y\}$	Curtate time to failure of the joint life status xy, *ie* the curtate time until the first death of x and y
$K_{\overline{xy}} = \max\{K_x, K_y\}$	Curtate time to failure of the last survivor status $\overline{xy}$, *ie* the curtate time until the second death of x and y

We have also developed the following notation, with associated formulae:

Symbol	***Description***	***Formula***
l_{xy}	Life table survival function for two independent lives x and y	$l_x l_y$
μ_{xy}	Force of failure of the joint life status xy	$\mu_x + \mu_y$
${}_tp_{xy}$	Probability that the joint life status xy is still active in t years' time, *ie* the probability that both x and y survive for at least t years	${}_tp_x \ {}_tp_y$
${}_tp_{\overline{xy}}$	Probability that the last survivor status $\overline{xy}$ is still active in t years' time, *ie* the probability that at least one of x and y survive for at least t years	$1 - {}_tq_x \, {}_tq_y$ $= {}_tp_x + {}_tp_y - {}_tp_x \ {}_tp_y$
${}_tq_{xy}$	Probability that the joint life status xy fails within t years, *ie* the probability that at least one of x and y dies in next t years	$1 - {}_tp_x \, {}_tp_y$ $= {}_tq_x + {}_tq_y - {}_tq_x \ {}_tq_y$
${}_tq_{\overline{xy}}$	Probability that the last survivor status $\overline{xy}$ fails within t years, *ie* the probability that both x and y die in next t years	${}_tq_x \ {}_tq_y$

The most common benefits, and their values, are:

Function	***Value (algebraic)***	***Value (stochastic)***
$\bar{a}_{xy}$	$\int_0^\infty v^t \; {}_tp_{xy} \; dt$	$E\left[\bar{a}_{\overline{\min\{T_x,T_y\}}\rceil}\right]$
$\bar{a}_{\overline{xy}}$	$\int_0^\infty v^t \; {}_tp_{\overline{xy}} \; dt$	$E\left[\bar{a}_{\overline{\max\{T_x,T_y\}}\rceil}\right]$
$\bar{A}_{xy}$	$\int_0^\infty v^t \; {}_tp_{xy} (\mu_{x+t} + \mu_{y+t}) \; dt$	$E\left[v^{\min\{T_x,T_y\}}\right]$
$\bar{A}_{\overline{xy}}$	$\int_0^\infty v^t ({}_tq_x \; {}_tp_y \mu_{y+t} + {}_tq_y \; {}_tp_x \mu_{x+t}) \; dt$	$E\left[v^{\max\{T_x,T_y\}}\right]$

Equivalent results hold for discrete functions, *eg*:

$$a_{xy} = \sum_{t=1}^{\infty} v^t \; {}_tp_{xy} = E\left[a_{\overline{\min\{K_x,K_y\}}\rceil}\right]$$

This page has been left blank so that you can keep the chapter summaries together for revision purposes.

Chapter 8 Solutions

Solution 8.1

It is the probability that the joint life status of x and y survives for t years, *ie* both lives survive for at least t years.

Solution 8.2

$$f_{T_{xy}}(t) = \frac{d}{dt}\left[1 - {}_t p_x \, {}_t p_y\right]$$

$$= -\frac{d}{dt} \, {}_t p_x \, {}_t p_y$$

$$= -{}_t p_x \frac{d}{dt} \, {}_t p_y - {}_t p_y \frac{d}{dt} \, {}_t p_x \qquad \text{(using the product rule)}$$

$$= -{}_t p_x \left(-{}_t p_y \mu_{y+t}\right) - {}_t p_y \left(-{}_t p_x \mu_{x+t}\right)$$

Solution 8.3

The probability of the joint life status xy failing over the interval of time δt is approximately

$${}_t p_{xy} \left(\mu_{x+t} + \mu_{y+t}\right) \delta t$$

which is the product of:

- the probability of both x and y surviving to time t, ${}_t p_{xy}$, and
- the probability of x or y dying at time t, $\left(\mu_{x+t} + \mu_{y+t}\right) \delta t$

Solution 8.4

q_{xy} is the probability that the joint life status has failed by the end of a year, *ie* it is the probability that *at least one* of x and y dies within the year.

Solution 8.5

It is the probability that the joint life status fails within a one-year period, starting in k years' time, *ie* it is the probability that both lives survive for k years, and then at least one of the lives dies in the following year.

Solution 8.6

(i) $${}_3p_{45:41} = \frac{l_{48:44}}{l_{45:41}} = \frac{l_{48}l_{44}}{l_{45}l_{41}} = \frac{9,753.4714 \times 9,814.3359}{9,801.3123 \times 9,847.0510} = 0.9918$$

(ii) $$q_{66:65} = 1 - p_{66:65} = 1 - \frac{l_{67:66}}{l_{66:65}} = 1 - \frac{l_{67}}{l_{66}}\frac{l_{66}}{l_{65}} = 1 - \frac{l_{67}}{l_{65}} = 0.030$$

(iii) $$\mu_{38:30} = \mu_{38} + \mu_{30} = 0.000788 + 0.000585 = 0.001373$$

Solution 8.7

(i) $$p_{\overline{62:65}} = 1 - q_{\overline{62:65}} = 1 - q_{62}q_{65} = 1 - 0.010112 \times 0.014243 = 0.99986$$

or:

$$p_{\overline{62:65}} = p_{62} + p_{65} - p_{62:65} = \frac{l_{63}}{l_{62}} + \frac{l_{66}}{l_{65}} - \frac{l_{63:66}}{l_{62:65}} = 0.99986$$

(ii) $${}_3q_{\overline{50:50}} = \left({}_3q_{50}\right)^2 = \left(1 - \frac{l_{53}}{l_{50}}\right)^2 = \left(1 - \frac{9,630.0522}{9,712.0728}\right)^2 = 0.0000713$$

Solution 8.8

We can write:

$$\int_{t=0}^{\infty} v^{2t} f_{T_u}(t)dt = \int_{t=0}^{\infty} \left(v^2\right)^t f_{T_u}(t)dt = \int_{t=0}^{\infty} (v')^t f_{T_u}(t)dt$$

This is equal to ${}^2\bar{A}_u$ where the (^{2}) denotes a rate of interest i' such that:

$$i' = \frac{1}{v'} - 1 = \frac{1}{v^2} - 1 = (1+i)^2 - 1 = 2i + i^2$$

or, equivalently, the (^{2}) denotes a force of interest δ' such that:

$$\delta' = -\ln v' = -\ln v^2 = -2\ln v = 2\delta$$

Solution 8.9

The expected value is:

$$E\left[v^{K_{xy}+1}\right] = \sum_{t=0}^{\infty} v^{t+1}\, {}_t p_{xy} q_{x+t:y+t} = \sum_{t=0}^{\infty} v^{t+1}\, {}_{t|}q_{xy}$$

The variance is:

$$\text{var}\left[v^{K_{xy}+1}\right] = E\left[\left(v^{K_{xy}+1}\right)^2\right] - \left\{E\left[v^{K_{xy}+1}\right]\right\}^2$$

$$= E\left[v^{2\left(K_{xy}+1\right)}\right] - \left(A_{xy}\right)^2$$

$$= E\left[\left(v^2\right)^{K_{xy}+1}\right] - \left(A_{xy}\right)^2$$

$$= {}^2A_{xy} - \left(A_{xy}\right)^2$$

Solution 8.10

The 3 possible states are:

- both lives still alive
- male life still alive
- female life still alive.

We would expect that the sum assured be paid out earliest on the policy where just the male is still alive. This policy would therefore most have the largest reserve.

Solution 8.11

The policy will have a higher reserve if the male life is still alive. So, the increase will be greater if the female life dies.

Solution 8.12

The expected present value of an annuity payable in arrears is:

$$E\left[a_{\overline{K_u}|}\right] = E\left[\frac{1-v^{K_u}}{i}\right] = \frac{1-E\left[v^{K_u}\right]}{i} = \frac{1-\frac{1}{v}E\left[v^{K_u+1}\right]}{i}$$

$$= \frac{v-A_u}{iv} = \frac{(1-d)-A_u}{d}$$

The variance of the present value of an annuity payable in arrears is:

$$\text{var}\left[a_{\overline{K_u}|}\right] = \text{var}\left[\frac{1-v^{K_u}}{i}\right]$$

$$= \frac{1}{i^2}\text{var}\left[v^{K_u}\right]$$

$$= \frac{1}{i^2v^2}\text{var}\left\{v^{K_u+1}\right\}$$

$$= \frac{1}{d^2}\left\{{}^2A_u - \left(A_u\right)^2\right\}$$

Solution 8.13

(i)(a) ***Expected present value***

The expected present value of the annuity is:

$$10,000\,a_{\overline{68:65}} = 10,000\left(a_{68} + a_{65} - a_{68:65}\right)$$
$$= 10,000\left(11.412 + 13.871 - 10.112\right)$$
$$= 151,710$$

(i)(b) ***Variance of the present value***

The present value random variable for the contract is:

$$10,000\,a_{\left.\overline{K_{\overline{68:65}}}\right|}$$

The variance of this random variable is given by:

$$\text{var}\left(10,000\,a_{\left.\overline{K_{\overline{68:65}}}\right|}\right) = 10,000^2\ \text{var}\left(\frac{1 - v^{K_{\overline{68:65}}}}{i}\right)$$
$$= \frac{10,000^2}{i^2}\text{var}\left(v^{K_{\overline{68:65}}}\right)$$
$$= \frac{10,000^2}{d^2}\text{var}\left(v^{K_{\overline{68:65}}+1}\right)$$

Now:

$$\text{var}\left(v^{K_{\overline{68:65}}+1}\right) = E\left[\left(v^{K_{\overline{68:65}}+1}\right)^2\right] - \left[E\left(v^{K_{\overline{68:65}}+1}\right)\right]^2 = {}^2A_{\overline{68:65}} - \left(A_{\overline{68:65}}\right)^2$$

where the "2" on the assurance function indicates that it is evaluated at twice the normal force of interest.

But:

$$A_{\overline{68:65}} = A_{68} + A_{65} - A_{68:65}$$

and:

$${}^2A_{\overline{68:65}} = {}^2A_{68} + {}^2A_{65} - {}^2A_{68:65}$$

So:

$$\begin{aligned}\operatorname{var}\left(10{,}000\, a_{\overline{K_{\overline{68:65}}}|}\right) &= \frac{10{,}000^2}{d^2}\left[{}^2A_{\overline{68:65}} - \left(A_{\overline{68:65}}\right)^2\right] \\ &= \frac{10{,}000^2}{d^2}\left[{}^2A_{68} + {}^2A_{65} - {}^2A_{68:65} - \left(A_{68} + A_{65} - A_{68:65}\right)^2\right]\end{aligned}$$

(ii) ***Probability that the insurance company makes a profit***

The life insurance company charges a premium of £150,000. It will make a profit if the annuity paid is less than £150,000, *ie*:

$$10{,}000\, a_{\overline{K_{\overline{68:65}}}|} < 150{,}000$$

Now:

$$\begin{aligned}10{,}000\, a_{\overline{K_{\overline{68:65}}}|} < 150{,}000 &\Leftrightarrow a_{\overline{K_{\overline{68:65}}}|} < 15 \\ &\Leftrightarrow \frac{1 - v^{K_{\overline{68:65}}}}{i} < 15 \\ &\Leftrightarrow v^{K_{\overline{68:65}}} > 1 - 15i \\ &\Leftrightarrow K_{\overline{68:65}} < \frac{\ln(1 - 15 \times 0.04)}{-\ln 1.04} = 23.4\end{aligned}$$

Since $K_{\overline{68:65}}$ can take only integer values:

$$P\left(K_{\overline{68:65}} < 23.4\right) = P\left(K_{\overline{68:65}} \le 23\right) = P\left(K_{\overline{68:65}} < 24\right)$$

So we want the probability that the last survivor status fails within 24 years. This will happen if both lives die within 24 years. The required probability is therefore:

$$
\begin{aligned}
{}_{24}q_{68} \times {}_{24}q_{65} &= \left(1-\frac{l_{92}}{l_{68}}\right)\left(1-\frac{l_{89}}{l_{65}}\right) \\
&= \left(1-\frac{1{,}911.771}{9{,}440.717}\right)\left(1-\frac{4{,}533.230}{9{,}703.708}\right) \\
&= 0.42493
\end{aligned}
$$

Solution 8.14

The expected value of T_{xy} is:

$$E\left(T_{xy}\right) = \mathring{e}_{xy} = \int_0^\infty {}_t p_{xy}\, dt = \int_0^\infty {}_t p_x \; {}_t p_y \, dt$$

Using the result:

$${}_t p_x = \exp\left(-\int_0^t \mu_{x+s}\, ds\right)$$

we can write:

$$
\begin{aligned}
E\left(T_{xy}\right) &= \int_0^\infty e^{-0.01t}\, e^{-0.02t}\, dt = \int_0^\infty e^{-0.03t}\, dt \\
&= \left[-\frac{1}{0.03}e^{-0.03t}\right]_0^\infty = \frac{1}{0.03} = 33.3 \text{ years}
\end{aligned}
$$

Chapter 9

Contingent and reversionary benefits

Syllabus objectives

(vi) Define and use straightforward functions involving two lives.

1. Extend the techniques of objectives (i) – (v) to deal with cashflows dependent upon the death of survival of either or both of two lives.

2. Extend the techniques of 1 to deal with functions dependent upon a fixed term as well as age.

0 Introduction

In this chapter we will consider the implication of specifying the order in which the two lives die, which leads to the two types of benefit:

- *contingent assurances* – which are payable on the death of one life, contingent upon another life being in a specified state (alive or dead); and
- *reversionary annuities* – which are payable to one life from the moment of death of another life.

We also look at multiple life functions that depend on duration in a variety of ways.

1 Contingent probabilities of death

So far we have examined events which depend on the joint lifetime xy or last survivor lifetime $\overline{xy}$ of two lives. We now look at events that depend on the order in which the deaths occur.

We will study two events:

- **the event that (x) is the first to die of two lives (x) and (y):** $\overset{1}{x}y$
- **the event that (x) is the second to die of two lives (x) and (y):** $\overset{2}{x}y$

Events that depend upon the order in which the lives die are called *contingent* events.

Then we use ${}_nq^{1}_{xy}$ and ${}_nq^{2}_{xy}$ to denote the probabilities that each of these two events occurs in the next n years.

Let's consider the meaning of ${}_nq^{1}_{xy}$. The way to read this is, first, *only* to consider the probability as it would relate to the life *with the number above it* ((x) in this case), *ie* initially ignoring the other life (y).

So we first read this as we would ${}_nq_x$:

- the probability that (x) dies within n years.

We then bring in the second life, and notice that the number superscript over the x is a "1", so this tells you that (x) has to die first. But we have already said something about (x), so we need to express this in terms of what happens to (y); that is, additionally to the above we have:

- (y) must die after (x).

(Note that (y) may or may not die within the n years.)

So the whole thing reads:

$${}_nq^{1}_{xy} = \text{probability that } (x) \text{ dies within } n \text{ years, and } (y) \text{ dies (any time) after } (x).$$

> ***Question 9.1***
>
> Use the same approach to specify the meaning of ${}_nq^{2}_{xy}$.

These probabilities can be evaluated by an appropriate integration of the density functions of the random variables T_x and T_y.

For example, we can write:

$$ {}_n q_{xy}^{1} = \int_{t=0}^{t=n} \int_{s=t}^{s=\infty} {}_t p_x \ \mu_{x+t} \, {}_s p_y \ \mu_{y+s} \, ds \, dt $$

which corresponds to the event $T_x \le T_y$, $T_x \le n$.

If we rewrite this as:

$$ {}_n q_{xy}^{1} = \int_{t=0}^{t=n} {}_t p_x \ \mu_{x+t} \left\{ \int_{s=t}^{s=\infty} {}_s p_y \ \mu_{y+s} \, ds \right\} dt $$

$$ = \int_{t=0}^{t=n} {}_t p_x \mu_{x+t} \ {}_t p_y \, dt $$

then we are calculating the product of:

- the probability that (x) dies at exact moment t $(\simeq {}_t p_x \, \mu_{x+t} \, dt)$
- the probability that (y) is still living at exact moment t $(= {}_t p_y)$, *ie* that (y) dies after time t and therefore after (x) dies

and then summing (= integrating) over all possible times at which (x) could die over the next n years.

Finally we can neaten this up slightly by writing:

$$ {}_n q_{xy}^{1} = \int_{t=0}^{t=n} {}_t p_{xy} \ \mu_{x+t} \, dt $$

The ability to express probabilities of life and death for single and joint lives as integrals is important.

The key considerations are:

- what event do I model as occurring at time t (*eg* should it be the moment of death or just "being alive")?
- who is dying at time t (if anyone)?
- who has to be alive/dead at time t?
- over what range of times t am I evaluating the integral?

So, in expressing ${}_nq^{1}_{x\,y}$ as an integral, we could proceed as follows.

- we want to model x's death as occurring at time t (we choose x rather than y on this occasion because we know that x has got to die within n years, so when we integrate over 0 to n we will have covered all the possible events);
- at time t, x must have survived to that point in order then to die, and then die instantly (so we want the factor ${}_tp_x\,\mu_{x+t}\,dt$);
- at time t, life y needs to be alive (so we want the factor ${}_tp_y$);
- we are evaluating this over the range of times from 0 to n.

So the integral must be: $\displaystyle\int_{t=0}^{n} {}_tp_x\,\mu_{x+t}\;{}_tp_y\,dt$

(Note that, in the probability ${}_nq^{1}_{x\,y}$, y may or may not die within the n year period – y can die as long as x is already dead!)

Q

Question 9.2

Without looking ahead, express as an integral the probability ${}_nq^{2}_{xy}$.

From the solution to the previous question, writing $\int_{s=0}^{s=t} {}_sp_y\,\mu_{y+s}\,ds$ for ${}_tq_y$ we get:

$$ {}_nq_{xy}^{2} = \int_0^n {}_tp_x\,\mu_{x+t}\left(\int_{s=0}^{s=t} {}_sp_y\,\mu_{y+s}\,ds\right)dt \qquad (1)$$

or:

$$\boldsymbol{ {}_nq_{xy}^{2} = \int_{t=0}^{t=n}\int_{s=0}^{s=t} {}_tp_x\,\mu_{x+t}\,{}_sp_y\,\mu_{y+s}\,ds\,dt }$$

This corresponds to the event $T_y < T_x \le n$.

If instead we substitute $\left(1-{}_tp_y\right)$ for ${}_tq_y$ in the solution to Question 9.2, then we can also write:

$$\boldsymbol{ {}_nq_{xy}^{2} = \int_{t=0}^{t=n} (1-{}_tp_y)\,{}_tp_x\;\mu_{x+t}\,dt }$$

$$= \int_0^n {}_tp_x\,\mu_{x+t}\,dt - \int_0^n {}_tp_{xy}\,\mu_{x+t}\,dt$$

$$\boldsymbol{ = {}_nq_x - {}_nq_{xy}^{1} }$$

This result implies that "second death" probabilities can always be expressed in terms of "single death" and "first death" probabilities. This provides a method of evaluating "second death" probabilities. The truth of this expression can be argued by general reasoning if it is rewritten as:

$$\boldsymbol{ {}_nq_x = {}_nq_{xy}^{1} + {}_nq_{xy}^{2} }$$

Question 9.3

Express this result verbally.

By changing the order of integration in expression (1) for $_nq_{xy}^{2}$ we can show that:

$$_nq_{xy}^{2} = {}_nq_{xy}^{1} - {}_np_x\, {}_nq_y$$

> ***Question 9.4***
>
> Prove this result.

This relationship can also be argued by considering the events defined by each of the functions.

So, the events defined by $_nq_{xy}^{1}$ are:

(A) y dies followed by x dying, all within n years, or

(B) y dies within n years and x dies after n years.

On the other hand, only event (A) is covered by $_nq_{xy}^{2}$. So $_nq_{xy}^{1} - {}_nq_{xy}^{2}$ must be the probability of event (B) only, which is $_nq_y\, {}_np_x$, as above.

The expressions become simple and easy to evaluate when $x = y$. We can write

$$_nq_x = {}_nq_{xx}^{1} + {}_nq_{xx}^{2}$$

$$_nq_{xx}^{2} = {}_nq_{xx}^{1} - {}_np_x\, {}_nq_x$$

(by substituting x in place of y in the previous two results). Note that when we use this formula we are assuming not only that the two lives are the same age, but that they are subject to identical mortality, and that mortality operates independently between the two lives.

Recall that:

$$_nq_{xx} = 1 - {}_np_{xx} = 1 - {}_np_x\, {}_np_x$$

So we have:

$$
\begin{aligned}
{}_n q^{\;\;2}_{xx} &= {}_n q^{1}_{xx} - {}_n p_x (1 - {}_n p_x) \\
&= {}_n q^{1}_{xx} - {}_n p_x + {}_n p_{xx} \\
&= {}_n q^{1}_{xx} - (1 - {}_n q_x) + (1 - {}_n q_{xx}) \\
&= {}_n q^{1}_{xx} + {}_n q_x - {}_n q_{xx} \\
&= {}_n q^{1}_{xx} + \left({}_n q^{1}_{xx} + {}_n q^{\;\;2}_{xx} \right) - {}_n q_{xx}
\end{aligned}
$$

Taking ${}_n q^{\;\;2}_{xx}$ from both sides we get:

$$0 = 2\, {}_n q^{1}_{xx} - {}_n q_{xx}$$

to give:

$$\boldsymbol{{}_n q^{1}_{xx} = \tfrac{1}{2}\, {}_n q_{xx}}$$

If $n = \infty$ then ${}_\infty q_{xx} = 1$, leading to:

$$\boldsymbol{{}_\infty q^{1}_{xx} = {}_\infty q^{\;\;2}_{xx} = \tfrac{1}{2}}$$

Note also that:

$${}_n q^{\;\;2}_{xx} = \tfrac{1}{2}\, {}_n q_{\overline{xx}} = \tfrac{1}{2} \left({}_n q_x \right)^2$$

Arguments of symmetry can often lead to simplifications when we have joint life problems with lives of equal ages. However, it is often useful to start off by considering the more general case with unequal ages.

Example

Simplify the sum ${}_nq_{\overset{1}{x}y} + {}_nq_{x\overset{1}{y}}$, and use the result to prove ${}_nq_{\overset{1}{x}x} = \frac{1}{2}\,{}_nq_{xx}$.

Solution

${}_nq_{\overset{1}{x}y} + {}_nq_{x\overset{1}{y}}$ is the probability that either x dies in n years with y alive at the time of x's death, or that y dies in n years with x alive at the time of y's death. So this is the probability that either x or y dies in n years with the other alive at the time of their death, *ie* ${}_nq_{xy}$.

So:

$${}_nq_{\overset{1}{x}y} + {}_nq_{x\overset{1}{y}} = {}_nq_{xy}$$

Now if we have two lives of equal age this becomes:

$${}_nq_{\overset{1}{x}x} + {}_nq_{x\overset{1}{x}} = {}_nq_{xx}$$

and:

$${}_nq_{\overset{1}{x}x} = {}_nq_{x\overset{1}{x}} \quad \text{(we are considering two different lives of the same age)}$$

So we find:

$${}_nq_{\overset{1}{x}x} = \frac{1}{2}\,{}_nq_{xx}$$

Question 9.5

Assuming AM92 Ultimate mortality, calculate ${}_5q_{40:\overset{2}{40}}$.

2 Present values of contingent assurances

In Section 1 we saw that contingent events depending on the future lifetime of two lives (x) and (y) can be written in terms of the random variables T_x and T_y. The random variables representing the present value of contingent assurances can be expressed as functions of these two random variables.

For example, the present value of a sum assured of 1 paid immediately on the death of (x) provided that (y) is still alive can be expressed as:

$$\bar{Z} = \begin{cases} v_i^{T_x} & \text{if } T_x \le T_y \\ 0 & \text{if } T_x > T_y \end{cases}$$

where i is the valuation rate of interest.

It is important to realise that the sum assured is not necessarily paid *to* (y) when (x) dies.

Using similar methods to those used in Section 1 the mean of $\bar{Z}$ is:

$$E\left[\bar{Z}\right] = \bar{A}^{1}_{x\,y} = \int_{t=0}^{t=\infty} v^t \, {}_t p_{xy} \, \mu_{x+t} \, dt$$

The actuarial notation is consistent with the term assurance notation $\bar{A}^{1}_{x:\overline{n}|}$ that we defined in Chapter 1, but here it involves a second life-status y instead of the duration status.

So, $\bar{A}^{1}_{xy}$ means:

- EPV of 1 unit sum assured paid immediately on the specified event
- payable *only* on the death of life (x)
- where (x) must die first out of the two lives (x) and (y).

A point to note is that the *positioning* of the number (over the x) tells us *on whose death* the benefit will be paid. Then the *value* of that number (1 in this case) tells us the required *order* in which the deaths must occur.

Using a stochastic approach we obtain the mean as shown. We can alternatively derive this integral using general reasoning. As before, we consider an integral based on some event happening at time t and what conditions have to hold at time t. In addition we need to discount to the time of the benefit payment. It is therefore normally necessary for the event that we are modelling at time t to be the event that triggers payment. So for this assurance we have:

- a benefit payable on x's death while y is still alive
- so we model x's death at time t
- at time t we require x to have survived to t and then to die, giving ${}_tp_x\mu_{x+t}$
- at time t we require life y to be alive, giving ${}_tp_y$
- we discount to the moment of benefit payment at time t, giving v^t
- and we evaluate the integral from time 0 to ∞ as the term is not limited.

So the integral is:

$$\int_{t=0}^{\infty} v^t \, {}_tp_{xy}\mu_{x+t} \, dt$$

The variance of $\bar{Z}$ is:

$$\text{var}(\bar{Z}) = {}^2\bar{A}^{1}_{x\,y} - \left(\bar{A}^{1}_{x\,y}\right)^2$$

where ${}^2\bar{A}$ is evaluated at a valuation rate of interest $i^2 + 2i$.

Question 9.6

Prove this result for $\text{var}(\bar{Z})$.

These functions are usually evaluated by using numerical methods, *eg* Simpson's Rule to determine the values of the integrals.

Simpson's rule is similar to the trapezium rule except that, instead of joining pairs of consecutive points with straight-line segments, we take groups of three consecutive points and fit quadratics to them.

In some cases joint and single life values obtained from tables can be useful in conjunction with the following and similar relationships:

i) $\bar{A}_{xy} = \bar{A}^{1}_{x\,y} + \bar{A}_{x\,\overset{1}{y}}$

Proof

$$\bar{A}^{1}_{x\,y} + \bar{A}_{x\,\overset{1}{y}} = \int_{t=0}^{\infty} v^t \, {}_t p_{xy} \mu_{x+t} \, dt + \int_{t=0}^{\infty} v^t \, {}_t p_{xy} \mu_{y+t} \, dt$$

$$= \int_{t=0}^{\infty} v^t \, {}_t p_{xy} \left(\mu_{x+t} + \mu_{y+t} \right) dt$$

$$= \int_{t=0}^{\infty} v^t \, {}_t p_{xy} \mu_{x+t:y+t} \, dt$$

$$= \bar{A}_{xy}$$

Question 9.7

Give a general reasoning explanation of this result.

ii) $\bar{A}_x = \bar{A}^{1}_{x\,y} + \bar{A}^{2}_{x\,y}$

Question 9.8

Prove the above Core Reading result, and give a general reasoning explanation.

iii) $\bar{A}^{1}_{x\,x} = ½\bar{A}_{x\,x}$

Proof

We already have $\bar{A}_{xy} = \bar{A}^{1}_{x\,y} + \bar{A}_{x\,\overset{1}{y}}$

So making the two lives both the same age gives:

$$\bar{A}_{xx} = \bar{A}^{1}_{x\,x} + \bar{A}_{x\,\overset{1}{x}}$$

By symmetry:

$$\bar{A}^{1}_{x\,x} = \bar{A}_{x\,x}^{\;\;1}$$

So:

$$\bar{A}^{1}_{x\,y} = \tfrac{1}{2}\bar{A}_{x\,x}$$

iv) $\bar{A}^{2}_{xx} = \tfrac{1}{2}\bar{A}_{\overline{xx}}$

Question 9.9

Prove the above result.

If the benefit is payable at the end of the policy year in which the contingent event occurs then we can show that:

$$A^{1}_{xy} = \sum_{t=0}^{t=\infty} v^{t+1}\,{}_{t}p_{xy}\,q^{\;\;1}_{x+t:y+t}$$

with analogous expressions for the variance to those derived for assurances payable immediately on the occurrence of the contingent event.

Such expressions are usually evaluated by using the approximate relationship:

$$A^{1}_{xy} \approx (1+i)^{-\frac{1}{2}}\,\bar{A}^{1}_{xy}$$

and similar expressions (which you will find in Chapter 1)**.**

3 Present values of reversionary annuities

The simplest form of a reversionary annuity is one that begins on the death of (x), if (y) is then alive, and continues during the lifetime of (y). The life (x) is called the counter or failing life, and the life (y) is called the annuitant. The random variable, $\bar{Z}$ representing the present value of this reversionary annuity if it is payable continuously can be written as a function of the random variables T_x and T_y, where:

$$\bar{Z} = \begin{cases} \bar{a}_{\overline{T_y}|} - \bar{a}_{\overline{T_x}|} & \text{if } T_y > T_x \\ 0 & \text{if } T_y \le T_x \end{cases}$$

So, if $T_y > T_x$, $\bar{Z}$ is the present value of a continuously payable annuity of 1 *pa* beginning in exactly T_x years time (when (x) dies) and ending in exactly T_y years from now (when (y) dies). So we can write, for $T_y > T_x$:

$$\bar{Z} = \int_{T_x}^{T_y} v^t \, dt = \int_0^{T_y} v^t \, dt - \int_0^{T_x} v^t \, dt = \bar{a}_{\overline{T_y}|} - \bar{a}_{\overline{T_x}|}$$

as before.

Also, we can see that, by substituting $r = t - T_x$:

$$\bar{Z} = \int_0^{T_y - T_x} v^{T_x + r} \, dr = v^{T_x} \int_0^{T_y - T_x} v^r \, dr = v^{T_x} \bar{a}_{\overline{T_y - T_x}|}$$

(remembering that this is only for $T_y > T_x$).

So the reversionary annuity is paid for a total of $T_y - T_x$ years, beginning from (and therefore discounted from) T_x years from now.

Question 9.10

Can you find an expression for $\bar{Z}$ that involves just one component (*ie* avoiding the need to condition on $T_y > T_x$ or otherwise)?

Using similar methods to those used for contingent assurances, we can show that:

$$E\left[\bar{Z}\right] = \bar{a}_{x|y} = \bar{a}_y - \bar{a}_{xy} = \frac{\bar{A}_{xy} - \bar{A}_y}{\delta}$$

$$= \int_{t=0}^{t=\infty} v^t \, \bar{a}_{y+t} \; {}_tp_{xy} \, \mu_{x+t} \, dt$$

> ***Question 9.11***
>
> Give a general reasoning explanation for this integral expression.

Note that $\bar{a}_y - \bar{a}_{xy}$ is generally the most useful result for calculating expected present values of reversionary annuities.

The variance can also be expressed as an integral in this way.

> ***Question 9.12***
>
> Prove that:
>
> $$\bar{a}_{x|y} = \bar{a}_y - \bar{a}_{xy} = \frac{\bar{A}_{xy} - \bar{A}_y}{\delta}$$

If the annuity begins at the end of the year of death of (x) and is then paid annually in arrear during the lifetime of (y), the random variable Z representing the present value of the payments can be written as a function of K_x and K_y, where:

$$Z = \begin{cases} a_{\overline{K_y}|} - a_{\overline{K_x}|} & \text{if } K_y > K_x \\ 0 & \text{if } K_y \le K_x \end{cases}$$

$$a_{\overline{K_y}|} - a_{\overline{K_{xy}}|}$$

We can show that:

$$E[Z] = a_{x|y} = a_y - a_{xy} = \frac{A_{xy} - A_y}{d}$$

Example

Calculate $a_{65\,|\,60}$, assuming PA92C20 mortality and 4% *pa* interest

(i) if (65) is male and (60) is female

(ii) if (65) is female and (60) is male

Solution

Using the *Tables*:

(i) $a^{m\ f}_{65|60} = a^{f}_{60} - a^{m\ f}_{65:60} = 15.652 - 11.682 = 3.970$

(ii) $a^{f\ m}_{65|60} = a^{m}_{60} - a^{f\ m}_{65:60} = 14.632 - 12.101 = 2.531$

Note that to find the values of reversionary annuities in arrears, we are taking the tabulated values of the annuities-due in the *Tables*, and subtracting one from both. The two –1 terms actually cancel out, so that we can just say:

$$a_{x|y} = a_y - a_{xy} = (\ddot{a}_y - 1) - (\ddot{a}_{xy} - 1) = \ddot{a}_y - \ddot{a}_{xy} = \ddot{a}_{x|y}$$

4 Present values of functions with specified terms

All of the theory involving assurance and annuity benefits considered above can be easily modified to allow for a specified term.

4.1 Expected present values of joint life assurances and annuities which also depend upon term

Joint life assurances that are dependent on a fixed term of n years can be term assurances or endowment assurances. Their expected present values, if they are paid immediately on death, can be expressed as:

$$\bar{A}^{\;1}_{\overbrace{xy}:\overline{n|}} = \int_{t=0}^{t=n} v^t \, {}_tp_{xy} \, \mu_{x+t:y+t} \, dt$$

$$\bar{A}_{xy:\overline{n|}} = \bar{A}^{\;1}_{\overbrace{xy}:\overline{n|}} + \bar{A}_{xy:\overset{1}{\overline{n|}}}$$

where:

$$\bar{A}_{xy:\overset{1}{\overline{n|}}} = {}_np_{xy} v^n$$

The bracket used in the notation for the term assurance $\bar{A}^{\;1}_{\overbrace{xy}:\overline{n|}}$ indicates that the joint status must end within the fixed term of n years.

Question 9.13

Describe precisely the meaning of each of the three actuarial symbols (the $\bar{A}$ functions) used in the above formulae.

The expected present value of the temporary joint life annuity payable continuously can be written as:

$$\bar{a}_{xy:\overline{n|}} = \int_{t=0}^{t=n} v^t \, {}_tp_{xy} \, dt$$

Similar expressions involving summation operators can be developed if assurances are paid at the end of the year of death or if annuities are payable annually in advance or in arrear.

4.2 *Expected present values of last survivor assurances and annuities that also depend upon term*

Last survivor assurances that are dependent on a fixed term of n years can be term assurances or endowment assurances. Their expected present values can be expressed in terms of single and joint life functions by making use of the results set out in Chapter 8 which can be generalised to any two statuses u and v. So we write:

$$T_{\overline{uv}} = T_u + T_v - T_{uv}$$

$$K_{\overline{uv}} = K_u + K_v - K_{uv}$$

The resulting expressions for assurances payable immediately on death are:

$$\bar{A}_{\overline{xy}:\overline{n}|} = \bar{A}_{x:\overline{n}|} + \bar{A}_{y:\overline{n}|} - \bar{A}_{xy:\overline{n}|}$$

$$\bar{A}^{\,1}_{\overline{xy}:\overline{n}|} = \bar{A}^{\,1}_{x:\overline{n}|} + \bar{A}^{\,1}_{y:\overline{n}|} - \bar{A}^{\,1}_{\overset{\frown}{xy}:\overline{n}|}$$

The expected present value of the temporary last survivor annuity payable continuously can be written as:

$$\bar{a}_{\overline{xy}:\overline{n}|} = \bar{a}_{x:\overline{n}|} + \bar{a}_{y:\overline{n}|} - \bar{a}_{xy:\overline{n}|}$$

Similar expressions involving summation operators can be developed if assurances are paid out at the end of the year of death or if annuities are payable annually in advance or arrear.

Question 9.14

Express $A_{\overline{60:60:\overline{5}|}}$ in terms of single and joint life functions (assuming the two lives have identical, but independent, mortality).

With last survivor assurances and annuities you must be very careful when allowing for duration. For example:

$$\ddot{a}_{\overline{xy}:\overline{n}|} \neq \ddot{a}_{\overline{xy}} - v^n \, {}_n p_{\overline{xy}} \, \ddot{a}_{\overline{x+n:y+n}}$$

Question 9.15

Explain why.

Note also the following (and for similar reasons):

$$A^{\;1}_{\overline{xy}:\overline{n}|} \neq A_{\overline{xy}} - v^n \, {}_np_{\overline{xy}} \, A_{\overline{x+n:y+n}}$$

Question 9.16

Using the PA92C20 tables and 4% *pa* interest, calculate $\ddot{a}_{\overline{50:50}:\overline{20}|}$, assuming that one life is male and the other is female.

4.3 More complex conditions

We frequently employ a technique of describing probabilities and benefit values with appropriately constructed integrals. We can generally use this technique to deal with more complex conditions, and this is often the most efficient way to solve such problems. The technique can be summarised as follows:

(1) identify the critical dates / times in the problem,

(2) express as an integral,

(3) simplify (often by making some appropriate substitution), and

(4) re-express in terms of simpler (non-integral) functions.

For discrete benefits we would use a summation expression rather than an integral.

Example

Find an expression for the probability that Xanthia aged x will die more than 5 years after the death of her sister Yolanda aged y. Express the answer in terms of single life probabilities and probabilities based on the first death.

Solution

When deriving probabilities for two lives, we can usually condition on either death. In this case, if Yolanda dies at time t, we require Xanthia to be alive 5 years later. This can be expressed in integral form as:

$$\int_0^\infty {}_{t+5}p_x \; {}_tp_y \mu_{y+t} \, dt$$

But, ${}_{t+5}p_x = {}_5p_x \; {}_tp_{x+5}$. So the integral can be written:

$$ {}_5p_x \int_0^\infty {}_tp_{x+5} \; {}_tp_y \mu_{y+t} \, dt = {}_5p_x \; {}_\infty q_{x+5:\overset{1}{y}}$$

Example

Xanthia wishes to take out an assurance with sum assured £50,000 payable immediately on her death under the conditions described above. Find an expression in terms of single life assurances and assurances payable on the first death for the present value of the benefit.

Solution

For assurances, it is easier to condition on the assured life. If Xanthia dies at time t, the sum assured will be paid provided Yolanda died before $t-5$, for $t \geq 5$. The present value of the assurance per £1 sum assured is therefore:

$$\int_5^\infty v^t (1 - {}_{t-5}p_y)\, {}_t p_x \mu_{x+t}\, dt$$

Substituting $s = t-5$, the integral becomes:

$$\int_0^\infty v^{s+5} (1 - {}_s p_y)\, {}_{s+5} p_x \mu_{x+s+5}\, ds = v^5\, {}_5 p_x \int_0^\infty v^s (1 - {}_s p_y)\, {}_s p_{x+5} \mu_{x+5+s}\, ds$$

Multiplying out the bracket, we have:

$$v^5\, {}_5 p_x \left[\int_0^\infty v^s\, {}_s p_{x+5} \mu_{x+5+s}\, ds - \int_0^\infty v^s\, {}_s p_y\, {}_s p_{x+5} \mu_{x+5+s}\, ds \right]$$

Expressing this in terms of assurance functions and multiplying by the sum assured of £50,000 gives a present value of:

$$50{,}000 v^5\, {}_5 p_x (\bar{A}_{x+5} - \bar{A}^{\,1}_{x+5:y})$$

You should find this approach very useful in working through the next section.

4.4 *Expected present values of reversionary annuities that depend upon term*

There are several different types of reversionary annuities that depend on term, and some of the possibilities are listed below. You should aim to be able to obtain for yourselves all the various expressions shown for the different types.

Type 1 – an annuity payable after a fixed term has elapsed

A reversionary annuity in which the counter or failing status is a fixed term of n years is exactly equivalent to a deferred life annuity. The expected present value of an annuity that is paid continuously can be written:

$$\bar{a}_{\overline{n}|\,|y} = {}_{n|}\bar{a}_y = \bar{a}_y - \bar{a}_{y:\overline{n}|}$$

However, for calculation purposes it is much quicker to use:

$${}_{n|}\bar{a}_y = v^n \, {}_n p_y \, \bar{a}_{y+n}$$

as we have done before.

Type 2 – an annuity payable to (y) on the death of (x), but ceasing at time n

If a reversionary annuity ceases in any event after n years *ie* is payable to (y) after the death of (x) with no payment being made after n years, the expected present value can be expressed as:

$$\bar{a}_{y:\overline{n}|} - \bar{a}_{xy:\overline{n}|}$$

We can show this using an integral, by considering payments made at time t. So, payment will be made at time t if:

- (y) is alive at time t
- (x) is dead by time t
- $t < n$

So the expected present value must be:

$$\int_0^n v^t \, {}_t p_y \, {}_t q_x \, dt = \int_0^n v^t \, {}_t p_y \left(1 - {}_t p_x\right) dt = \int_0^n v^t \, {}_t p_y \, dt - \int_0^n v^t \, {}_t p_{xy} \, dt = \bar{a}_{y:\overline{n}|} - \bar{a}_{xy:\overline{n}|}$$

Example

Ralph and Betty are both aged 65 exact. Upon Betty's death, Ralph will receive £20,000 *pa* payable annually in advance starting from the end of the year of Betty's death. There will be no payments on or beyond Ralph's 80th birthday in any circumstances.

Ralph's mortality follows PMA92C20, Betty's mortality follows PFA92C20 and the interest rate for all future years is $i = 4\%$ *pa*. Calculate the EPV of this benefit to Ralph.

Solution

We have:

$$\ddot{a}_{65:\overline{15}|} - \ddot{a}_{65:65:\overline{15}|}$$

$$= \ddot{a}_{65} - v^{15}\,{}_{15}p_{65}\,\ddot{a}_{80} - \left[\ddot{a}_{65:65} - v^{15}\,{}_{15}p_{65:65}\,\ddot{a}_{80:80}\right]$$

$$= 13.666 - 1.04^{-15} \times \frac{6,953.536}{9,647.797} \times 7.506$$

$$- \left[11.958 - 1.04^{-15} \times \frac{6,953.536}{9,647.797} \times \frac{7,724.737}{9,703.708} \times 5.857\right]$$

$$= 0.5700$$

So, the EPV to Ralph is:

$$£20,000 \times 0.5700 = £11,401$$

Type 3 – an annuity payable to (y) on the death of (x) provided that (x) dies within n years

If the payment commences on the death of (x) within n years and then continues until the death of (y), the expected present value can be expressed as:

$$\int_{t=0}^{t=n} v^t \, {}_tp_{xy} \, \mu_{x+t} \, \bar{a}_{y+t} \, dt = \bar{a}_y - \bar{a}_{xy} - v^n \, {}_np_{xy} (\bar{a}_{y+n} - \bar{a}_{x+n:y+n})$$

$$= \bar{a}_{x|y} - v^n \, {}_np_{xy} \, \bar{a}_{x+n|y+n} \qquad (2)$$

So this is what we might most accurately describe as a "temporary reversionary annuity".

Question 9.17

Prove this result, using the fact that $\int_0^\infty v^t \, {}_tp_{xy} \, \mu_{x+t} \, \bar{a}_{y+t} \, dt = \bar{a}_{x|y}$ from Section 3.

Question 9.18

Explain in words the rationale for both the left and right hand sides of expression (2).

Example

Ralph and Betty are both aged 65 exact. Upon Betty's death, Ralph will receive £20,000 *pa* payable annually in advance for the rest of his life starting from the end of the year of Betty's death, provided that Betty dies within 10 years.

Ralph's mortality follows PMA92C20, Betty's mortality follows PFA92C20 and the interest rate for future years is $i = 4\%$ *pa*. Calculate the EPV of this benefit to Ralph.

Solution

We have:

$$\ddot{a}^{m}_{65} - \ddot{a}_{65:65} - v^{10}{}_{10}p_{65:65}(\ddot{a}^{m}_{75} - \ddot{a}_{75:75})$$

$$= 13.666 - 11.958 - 1.04^{-10} \times \frac{8,405.160}{9,647.797} \times \frac{8,784.955}{9,703.708} \times (9.456 - 7.679)$$

$$= 0.76117$$

So, the EPV to Ralph is:

$$£20,000 \times 0.76117 = £15,223$$

Type 4 – an annuity payable to (y) on the death of (x) for a maximum of n years

If the conditions of payment say that the payment will:

- **begin on the death of (x) and**
- **cease on the death of (y) or n years after the death of (x) (whichever event occurs first),**

then the expected present value can be expressed as:

$$\int_{t=0}^{t=\infty} v^t \, {}_tp_{xy} \, \mu_{x+t} \, \bar{a}_{y+t:\overline{n}|} \, dt = \bar{a}_{y:\overline{n}|} + v^n \, {}_np_y \, \bar{a}_{x:y+n} - \bar{a}_{xy}$$

$$= \bar{a}_y - v^n \, {}_np_y \, \bar{a}_{y+n} + v^n \, {}_np_y \, \bar{a}_{x:y+n} - \bar{a}_{xy}$$

$$= \bar{a}_y - \bar{a}_{xy} - v^n \, {}_np_y \left(\bar{a}_{y+n} - \bar{a}_{x:y+n} \right)$$

$$= \bar{a}_{x|y} - v^n \, {}_np_y \, \bar{a}_{x|y+n}$$

Question 9.19

Explain in words the integral formula on the left hand side of this expression.

Question 9.20

Can you find a way to derive the formula on the right-hand side of this expression, *ie* $\bar{a}_{x|y} - v^n {}_n p_y \, \bar{a}_{x|y+n}$, by considering payments made at time t (using a similar approach as used for the Type 2 annuity above)?

Example

Ralph and Betty are both aged 70 exact. Upon Betty's death, Ralph will receive £20,000 *pa* payable annually in advance starting from the end of the year of Betty's death and ceasing on Ralph's death. Ralph may receive a maximum of 20 payments.

Ralph's mortality follows PMA92C20, Betty's mortality follows PFA92C20 and the interest rate for all future years is $i = 4\%$ *pa*. Calculate the EPV of this benefit to Ralph.

Solution

We have:

$$\ddot{a}^{m}_{70:\overline{20|}} + v^{20} {}_{20}p^{m}_{70} \ddot{a}^{f\,m}_{70:90} - \ddot{a}_{70:70}$$

$$= 11.562 - 1.04^{-20} \times \frac{2,675.203}{9,238.134} \times 4.527 + 1.04^{-20} \times \frac{2,675.203}{9,238.134} \times 4.339 - 9.766$$

$$= 1.771154$$

So, the EPV to Ralph is:

$$£20,000 \times 1.77115 = £35,423$$

Type 5 – an annuity payable to (y) on the death of (x) and guaranteed for n years

The expected present value of this benefit is:

$$\bar{A}^{1}_{x:y} \, \bar{a}_{\overline{n|}} + v^n \, {}_n p_y \, \bar{a}_{x|y+n}$$

Question 9.21

Prove this result.

The first term in the above expression is the expected present value of the guaranteed benefit, which is paid to (y) following the death of (x). The second term is the expected present value of the benefit paid to (y) once the n-year guarantee period is up.

This second term can also be found by considering payments made at time t that are n or more years after (x)'s death. So, payments will be made at time t if:

- (y) is alive (probability ${}_tp_y$)
- (x) died before time $t-n$ (probability ${}_{t-n}q_x$)

So we have:

$$EPV = \int_n^\infty v^t \, {}_tp_y \, {}_{t-n}q_x \, dt$$

noting that there can be no payment when $t < n$. Substituting $s = t - n$:

$$EPV = \int_0^\infty v^{n+s} \, {}_{n+s}p_y \, {}_sq_x \, ds$$

Taking out constant factors involving n we get:

$$EPV = v^n \, {}_np_y \int_0^\infty v^s \, {}_sp_{y+n} \, {}_sq_x \, ds$$

$$= v^n \, {}_np_y \, \bar{a}_{x|y+n}$$

as before.

Example

Ralph and Ted are both aged 60 exact. Upon Ted's death, Ralph will receive £20,000 *pa* for the rest of his life payable annually in advance starting from the end of the year of Ted's death. The payments to Ralph are guaranteed for 5 years.

Ralph and Ted's mortality both follow PMA92C20 and the interest rate for all future years is $i = 4\%$ *pa*. You are given that $A_{60:60} = 0.47585$ and that $A_{60:65} = 0.51084$ where both lives follow PMA92C20. Calculate the EPV of this benefit to Ralph.

Solution

We have:

$$A^{1}_{60:60}\, \ddot{a}_{\overline{5}|} + v^5\, {}_5p_{60}\, \ddot{a}_{60|65}$$

$$= 0.5 A_{60:60} \ddot{a}_{\overline{5}|} + v^5\, {}_5p_{60} \left(\ddot{a}_{65} - \frac{1 - A_{60:65}}{d} \right)$$

$$= 0.5 \times 0.47585 \times \frac{1 - 1.04^{-5}}{0.04/1.04} + 1.04^{-5} \times \frac{9,647.797}{9,826.131} \times \left(13.666 - \frac{1 - 0.51084}{0.04/1.04} \right)$$

$$= 1.86648$$

So, the EPV to Ralph is:

$$£20,000 \times 1.86648 = £37,330$$

Type 6 – an annuity payable to (y) on the death of (x) and continuing for n years after (y)'s death

The expected present value of this benefit is:

$$\bar{a}_{x|y} + \bar{A}^{\;\;2}_{x:y}\, \bar{a}_{\overline{n}|}$$

The first term is the expected present value of the benefit payable after the death of (x) while (y) is still alive. The second term is the expected present value of the annuity paid for n years following the death of (y), provided that (y) dies after (x).

Example

Ralph and Ted are both aged 60 exact and their mortality follows PMA92C20. Upon Ted's death, Ralph will receive £20,000 *pa* payable annually in advance for the rest of his life, starting from the end of the year of Ted's death. The payments to Ralph will continue for 12 years after Ralph has died. No payments are made if Ralph dies first.

The interest rate for all future years is $i = 4\%$ *pa*. You are given that $A_{60:60} = 0.47585$ where both lives follow PMA92C20. Calculate the EPV of this benefit to Ralph.

Solution

We will need to calculate:

$$A_{\overline{60:60}} = A_{60} + A_{60} - A_{60:60}$$

$$= 2\left(1 - \frac{0.04}{1.04} \times 15.632\right) - 0.47585$$

$$= 0.32169$$

We have:

$$\ddot{a}_{60|60} + A_{60:\overset{2}{60}}\, \ddot{a}_{\overline{12|}}$$

$$= \ddot{a}_{60} - \frac{1 - A_{60:60}}{d} + 0.5 \times A_{\overline{60:60}}\, \ddot{a}_{\overline{12|}}$$

Substituting in the value from above, we get:

$$= 15.632 - \frac{1.04}{0.04} + \frac{1.04 \times 0.47585}{0.04} + 0.5 \times 0.32169 \times 9.7605$$

$$= 3.57402$$

So, the EPV to Ralph is:

$$£20,000 \times 3.57402 = £71,480$$

Tip

If you are attempting a question and you're not sure of the correct formula, try writing down an integral expression to help you figure out what's going on.

Question 9.22

Jack, aged 60, wants to buy a reversionary annuity. If he dies before age 65 and before his wife Vera, who is also now aged 60, she will receive an income of £10,000 *pa*. The income will be paid annually in arrears (from the end of the year of Jack's death) until Vera's 75th birthday or until her earlier death.

Calculate the single premium payable assuming PA92C20 mortality and 4% *pa* interest.

4.5 Expected present values of contingent assurances that depend upon term

Only term assurances are meaningful in this context. The expected present value of an assurance payable immediately on the death of (x) within n years provided (y) is then alive is written:

$$\bar{A}^{\;1}_{xy:\overline{n}|} = \int_{t=0}^{t=n} v^t \, {}_t p_{xy} \, \mu_{x+t} \, dt$$

with a similar expression involving summation operators if the sum assured is payable at the end of the year of death.

If the sum assured is payable at the end of the year of (x)'s death, then the expected present value is:

$$A^{1}_{x\,y:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+1} \, {}_k p_{xy} \, q^{\;1}_{x+k:y+k}$$

5 Expected present value of annuities payable m times a year

In Chapter 4 we determined (for a single life status x) the approximations:

$$\ddot{a}_x^{(m)} \cong \ddot{a}_x - \frac{(m-1)}{2m}$$

and:

$$a_x^{(m)} \cong a_x + \frac{m-1}{2m}$$

It is important to note that the nature of the above approximation means that the single life status x can equally be replaced by any life status, “u” say.

In particular, with $u = xy$, we obtain the joint life annuity approximation:

$$a_{xy}^{(m)} \cong a_{xy} + \frac{m-1}{2m}$$

For a last survivor annuity we write:

$$a_{\overline{xy}}^{(m)} = a_x^{(m)} + a_y^{(m)} - a_{xy}^{(m)}$$

and use the result with statuses x, y and xy to obtain:

$$a_{\overline{xy}}^{(m)} \cong a_x + a_y - a_{xy} + \frac{m-1}{2m}$$

in its simplest form.

For a reversionary annuity we write:

$$a_{x|y}^{(m)} = a_y^{(m)} - a_{xy}^{(m)}$$

and use the result with statuses y and xy to obtain:

$$a_{x|y}^{(m)} \cong a_y - a_{xy}$$

in its simplest form. Notice that there is no “correction term” in this case.

Similar expressions may be developed if the $m$thly annuities are temporary.

We can write, for the generalised status u :

$$a_{u:\overline{n}|}^{(m)} = a_u^{(m)} - {}_{n|}a_u^{(m)}$$

We know from above that $a_u^{(m)} \cong a_u + \dfrac{m-1}{2m}$.

Also:

$$\begin{aligned}
{}_{n|}a_u^{(m)} &= v^n \frac{l_{u+n}}{l_u} a_{u+n}^{(m)} \\
&\cong v^n \frac{l_{u+n}}{l_u}\left(a_{u+n} + \frac{m-1}{2m}\right) \\
&= {}_{n|}a_u + \frac{m-1}{2m} v^n \frac{l_{u+n}}{l_u}
\end{aligned}$$

Hence:

$$a_{u:\overline{n}|}^{(m)} \cong a_u + \frac{m-1}{2m} - \left({}_{n|}a_u + \frac{m-1}{2m} v^n \frac{l_{u+n}}{l_u}\right)$$

or:

$$a_{u:\overline{n}|}^{(m)} \cong a_{u:\overline{n}|} + \frac{m-1}{2m}\left(1 - v^n \frac{l_{u+n}}{l_u}\right)$$

Hence we obtain the following expressions:

$$a_{xy:\overline{n}|}^{(m)} \cong a_{xy:\overline{n}|} + \frac{m-1}{2m}\left(1 - v^n \frac{l_{x+n}\, l_{y+n}}{l_x\, l_y}\right)$$

and:

$$\begin{aligned}
a_{\overline{xy}:\overline{n}|}^{(m)} &= a_{x:\overline{n}|}^{(m)} + a_{y:\overline{n}|}^{(m)} - a_{xy:\overline{n}|}^{(m)} \\
&\cong a_{x:\overline{n}|} + a_{y:\overline{n}|} - a_{xy:\overline{n}|} + \frac{m-1}{2m}\left(1 - v^n \frac{l_{x+n}}{l_x} - v^n \frac{l_{y+n}}{l_y} + v^n \frac{l_{x+n}\, l_{y+n}}{l_x\, l_y}\right)
\end{aligned}$$

For a reversionary annuity which ceases in any event after n years we can write:

$$a_{y:\overline{n}|}^{(m)} - a_{xy:\overline{n}|}^{(m)} \cong a_{y:\overline{n}|} - a_{xy:\overline{n}|} + \frac{m-1}{2m}\left(v^n \frac{l_{x+n}\, l_{y+n}}{l_x\, l_y} - v^n \frac{l_{y+n}}{l_y}\right)$$

Similar expressions can be developed for annuities payable in advance and, letting $m \to \infty$, continuous annuities.

Example

Jim and Dot, both aged 60, buy an annuity payable monthly in advance for at most 20 years, which is payable while at least one of them is alive. Calculate the expected present value of the annuity assuming 4% *pa* interest and PA92C20 mortality.

Solution

The expected present value of the annuity is:

$$\ddot{a}_{\overline{60:60}:\overline{20}|}^{(12)} = \ddot{a}_{60(m):\overline{20}|}^{(12)} + \ddot{a}_{60(f):\overline{20}|}^{(12)} - \ddot{a}_{60:60:\overline{20}|}^{(12)}$$

Now:

$$\ddot{a}_{60(m):\overline{20}|}^{(12)} = \ddot{a}_{60(m)}^{(12)} - v^{20}\ {}_{20}p_{60(m)}\ddot{a}_{80(m)}^{(12)}$$

$$= \ddot{a}_{60(m)} - \frac{11}{24} - v^{20}\ {}_{20}p_{60(m)}\left(\ddot{a}_{80(m)} - \frac{11}{24}\right)$$

$$= 15.632 - \frac{11}{24} - 1.04^{-20} \times \frac{6,953.536}{9,826.131} \times \left(7.506 - \frac{11}{24}\right)$$

$$= 12.898$$

Similarly:

$$\ddot{a}_{60(f):\overline{20}|}^{(12)} = \ddot{a}_{60(f)} - \frac{11}{24} - v^{20}\ {}_{20}p_{60(f)}\left(\ddot{a}_{80(f)} - \frac{11}{24}\right)$$

$$= 16.652 - \frac{11}{24} - 1.04^{-20} \times \frac{7,724.737}{9,848.431} \times \left(8.989 - \frac{11}{24}\right)$$

$$= 13.140$$

Finally, the joint life annuity is given by:

$$\ddot{a}^{(12)}_{60:60:\overline{20|}} = \ddot{a}^{(12)}_{60:60} - v^{20}\ {}_{20}p_{60:60}\ \ddot{a}^{(12)}_{80:80}$$

$$= \ddot{a}_{60:60} - \frac{11}{24} - v^{20}\ {}_{20}p_{60:60}\left(\ddot{a}_{80:80} - \frac{11}{24}\right)$$

$$= 14.090 - \frac{11}{24} - 1.04^{-20} \times \frac{6,953.536}{9,826.131} \times \frac{7,724.737}{9,848.431} \times \left(5.857 - \frac{11}{24}\right)$$

$$= 12.264$$

So:

$$\ddot{a}^{(12)}_{\overline{60:60}:\overline{20|}} = 12.898 + 13.140 - 12.264 = 13.77$$

6 Further aspects

We round off this chapter with consideration of:

- premium conversion relationships,
- the premium payment term.

6.1 Premium conversion relationships

The results seen in Chapter 4 that are very useful here are:

$$A_x = 1 - d\ddot{a}_x$$

and similarly:

$$A_{x:\overline{n}|} = 1 - d\ddot{a}_{x:\overline{n}|}$$

As we have seen, these have analogous equivalents for joint life and last survivor statuses. The relationship is particularly useful for such cases because the *Tables* present only annuity functions; we can therefore work out joint life assurance functions from the tabulated annuity functions.

Question 9.23

Marge and Homer, both aged exactly 55, take out a policy that provides a lump sum of £50,000 payable immediately on the second death. Premiums are payable annually in advance while at least one of Marge and Homer is alive. Calculate the annual premium for the policy assuming PA92C20 mortality, 4% *pa* interest, and no expenses.

6.2 *Miscellaneous harder questions*

Consideration of premium payment term

One of the principal uses of the theory developed in this chapter is to allow the actuary to determine the premium suitable for any given assurance or annuity benefit involving two lives. However, we may then find a complication in that it is possible for the joint life function defining the premium payment annuity to be different from that defining the desired benefit.

For instance, if a monogamous man took out a reversionary annuity contract to provide a pension for his wife (specified by name) after his death, it would be silly to expect him to continue paying premiums if his wife died. Thus premium payment would continue only up to the first death of the pair, *ie* failure of their joint life status.

So normally for contracts involving two lives, premiums will be payable until one of the following events occurs:

- the benefit is paid,
- the term of the contract expires,
- the person paying the premium dies, or
- it becomes impossible for the benefit to be paid at any time in the future.

Example

The following assurance functions represent the factors used for valuing the benefits from annual premium insurance contracts:

(a) A_{xy} (b) $A_{\overline{xy}}$

Specify the appropriate annuity factor to be used in valuing premiums.

Solution

(a) A_{xy} represents a benefit payable on the first death. So premiums should be payable while both lives are still alive. The appropriate annuity factor is $\ddot{a}_{xy}$.

(b) $A_{\overline{xy}}$ represents a benefit payable on the second death. So premiums should be payable while either life is still alive. The appropriate annuity factor is $\ddot{a}_{\overline{xy}}$.

Question 9.24

The following assurance functions represent the factors used for valuing the benefits from annual premium insurance contracts:

(i) $A_{\overset{1}{x}y}$

(ii) $A_{x\overset{2}{y}}$

Specify the appropriate annuity factor to be used in valuing premiums.

7 Exam-style questions

Now try the following exam-style questions yourself.

Question 9.25

Given that $\mu_x = \dfrac{1}{100-x}$ for $0 \le x < 100$, calculate the value of ${}_{30}q^{2}_{50:60}$.

Question 9.26 ***(Subject 105, April 2000, Question 13, updated)***

A pension scheme provides the following benefit to the spouse of a member, following the death of the member in retirement:

A pension of £10,000 *pa* payable during the lifetime of the spouse, but ceasing 15 years after the death of the member if that is earlier. All payments are made on the anniversary of the member's retirement.

Calculate the expected present value of the spouse's benefit in the case of a female member retiring now on her 60th birthday, who has a husband aged exactly 55.

Basis: PA92C20 mortality, 4% *pa* interest [8]

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Chapter 9 Summary

Contingent probabilities

${}_tq^{\ 1}_{x\,y}$ represents the probability that (x) dies within the next t years, with (y) still alive at the time of (x)'s death. It can be expressed in terms of an integral as follows:

$${}_tq^{\ 1}_{x\,y} = \int_0^t {}_sp_x\, \mu_{x+s}\ {}_sp_y\, ds$$

${}_tq^{\ 2}_{x\,y}$ represents the probability that (x) dies within the next t years, with (y) already dead at the time of (x)'s death. It can be expressed in terms of an integral as follows:

$${}_tq^{\ 2}_{x\,y} = \int_0^t {}_sp_x\, \mu_{x+s}\ {}_sq_y\, ds$$

By writing ${}_sq_y = 1 - {}_sp_y$ in the integral above, we see that:

$$\begin{aligned}
{}_tq^{\ 2}_{x\,y} &= \int_0^t {}_sp_x\, \mu_{x+s} \left(1 - {}_sp_y\right) ds \\
&= \int_0^t {}_sp_x\, \mu_{x+s}\, ds - \int_0^t {}_sp_x\, \mu_{x+s}\ {}_sp_y\, ds \\
&= {}_tq_x - {}_tq^{\ 1}_{x\,y}
\end{aligned}$$

When we have two lives of the same age (and we assume that their mortality is identical), we can use a symmetry argument to write:

$${}_tq^{\ 1}_{x\,x} = ½\ {}_tq_{x\,x}$$

and:

$${}_{\infty}q^{\ 1}_{x\,x} = ½$$

Contingent assurances

The present value of a benefit of 1 payable immediately on the death of (x) provided that (y) is still alive is:

$$\bar{Z} = \begin{cases} v^{T_x} & \text{if } T_x \le T_y \\ 0 & \text{if } T_x > T_y \end{cases}$$

The expected present value of this benefit is denoted by $\bar{A}^{1}_{x\,y}$ and can be expressed in integral form as follows:

$$E\left(\bar{Z}\right) = \bar{A}^{1}_{x\,y} = \int_0^\infty v^t \; {}_tp_x \, \mu_{x+t} \; {}_tp_y \, dt$$

The variance of the present value random variable is:

$$\operatorname{var}\left(\bar{Z}\right) = {}^2\bar{A}^{1}_{x\,y} - \left(\bar{A}^{1}_{x\,y}\right)^2$$

Reversionary annuities

The present value of an annuity of 1 *pa* payable continuously throughout life to (y) following the death of (x) is:

$$\bar{Y} = \begin{cases} \bar{a}_{\overline{T_y}|} - \bar{a}_{\overline{T_x}|} & \text{if } T_y > T_x \\ 0 & \text{if } T_y \le T_x \end{cases}$$

The expected present value of this benefit is denoted by $\bar{a}_{x|y}$ and can be expressed in integral form as:

$$E\left(\bar{Y}\right) = \bar{a}_{x|y} = \int_0^\infty v^t \; {}_tp_x \, \mu_{x+t} \; {}_tp_y \, \bar{a}_{y+t} \, dt$$

We can also write:

$$\bar{a}_{x|y} = \bar{a}_y - \bar{a}_{xy}$$

Similar expressions involving summation operators can be developed if the assurance is payable at the end of the year of death or the annuity is payable at discrete intervals.

Premium conversion formulae

$$A_{xy} = 1 - d\ddot{a}_{xy} \qquad A_{xy:\overline{n}|} = 1 - d\ddot{a}_{xy:\overline{n}|}$$

$$\bar{A}_{xy} = 1 - \delta\bar{a}_{xy} \qquad \bar{A}_{xy:\overline{n}|} = 1 - \delta\bar{a}_{xy:\overline{n}|}$$

Similar results hold for last survivor annuities and assurances.

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Chapter 9 Solutions

Solution 9.1

The number is over the x (again), so we can first read this as ${}_nq_x$, *ie* the probability that (x) dies within n years.

But, because the full expression is ${}_nq^{2}_{xy}$, the probability is also conditional on (x) being the second of the two lives to die.

So, altogether:

$${}_nq^{2}_{xy} = \text{probability that } (x) \text{ dies within } n \text{ years, and } (y) \text{ dies before } (x).$$

Solution 9.2

For ${}_nq^{2}_{xy}$, (x) has to die within the n years. So we model (x) as dying at time t (over $0 < t < n$), and so we will need a factor of:

$${}_tp_x\,\mu_{x+t}\,dt$$

(which can be thought of as the probability that (x) dies at exact future time t).

However, we also need to include the probability that (y) has already died by this time, *ie* by the probability that (y) dies before time t. This is:

$${}_tq_y$$

The integral therefore becomes:

$${}_nq^{2}_{xy} = \int_0^n {}_tp_x\,\mu_{x+t}\,{}_tq_y\,dt$$

Solution 9.3

The probability that a life aged x dies in an n-year period either before or after a life aged y, is equal to the probability that the life aged x dies in an n-year period (*ie* totally regardless of what happens to (y)).

Solution 9.4

We have:

$$ {}_nq^{\;2}_{xy} = \int_0^n {}_tp_x\,\mu_{x+t}\left(\int_0^t {}_sp_y\,\mu_{y+s}\,ds\right)dt $$

Changing the order of integration we have:

$$ {}_nq^{\;2}_{xy} = \int_0^n {}_sp_y\,\mu_{y+s}\left(\int_s^n {}_tp_x\,\mu_{x+t}\,dt\right)ds $$

Note that in the original double integration we have $0 \le s \le t \le n$. So when we change the order we have s going from 0 to n and t going from s to n.

Now $\int_s^n {}_tp_x\,\mu_{x+t}\,dt$ is the probability that x dies between time s and time n, and so:

$$ \int_s^n {}_tp_x\,\mu_{x+t}\,dt = {}_sp_x - {}_np_x $$

Substituting this expression into the equation above gives:

$$ \begin{aligned} {}_nq^{\;2}_{xy} &= \int_0^n {}_sp_y\,\mu_{y+s}\left({}_sp_x - {}_np_x\right)ds \\ &= \int_0^n {}_sp_y\,\mu_{y+s}\,{}_sp_x\,ds - {}_np_x\int_0^n {}_sp_y\,\mu_{y+s}\,ds \\ &= {}_nq^{\;1}_{xy} - {}_np_x\,{}_nq_y \end{aligned} $$

as required.

Solution 9.5

We have:

$$ {}_5q^{\;2}_{40:40} = ½\,{}_5q_{\overline{40:40}} = ½\left({}_5q_{40}\right)^2 = ½\left(1-\frac{l_{45}}{l_{40}}\right)^2 = 0.000016 $$

Solution 9.6

We start by writing:

$$\text{var}\left(\overline{Z}\right) = E\left[\overline{Z}^2\right] - \left\{E\left[\overline{Z}\right]\right\}^2$$

Define T = random variable defining time to the event $\overset{1}{x}\,y$. Assume that T takes the value $+\infty$ if (y) dies before (x).

Then:

$$E\left[\overline{Z}^2\right] = E\left[\left(v^T\right)^2\right] = E\left[v^{2T}\right] = \int_{t=0}^{\infty} v^{2t}\,{}_tp_{xy}\mu_{x+t}\;dt = \int_{t=0}^{\infty} \left(v^2\right)^t\,{}_tp_{xy}\mu_{x+t}\;dt$$

and this is equal to ${}^2\overline{A}^{\,1}_{x\,y}$ at interest $i^2 + 2i$ (or force of interest 2δ).

So we have $\text{var}(\overline{Z}) = {}^2\overline{A}^{\,1}_{x\,y} - \left(\overline{A}^{\,1}_{x\,y}\right)^2$ as required.

Solution 9.7

The assurance on the left-hand side is the present value of 1, paid at the moment of the first death. This first death can either be x or y. So the sum of the two assurances on the right hand side must equal the assurance on the left-hand side.

Solution 9.8

We have:

$$\bar{A}^{1}_{xy} + \bar{A}^{2}_{xy} = \int_0^\infty v^t \, {}_tp_{xy} \mu_{x+t} \, dt + \int_0^\infty v^t \, {}_tp_x \mu_{x+t} \, {}_tq_y \, dt$$

$$= \int_0^\infty v^t \, {}_tp_{xy} \mu_{x+t} \, dt + \int_0^\infty v^t \, {}_tp_x \mu_{x+t} \left(1 - {}_tp_y\right) dt$$

$$= \int_0^\infty v^t \, {}_tp_{xy} \mu_{x+t} \, dt + \int_0^\infty v^t \, {}_tp_x \mu_{x+t} \, dt - \int_0^\infty v^t \, {}_tp_{xy} \mu_{x+t} \, dt$$

$$= \int_0^\infty v^t \, {}_tp_x \mu_{x+t} \, dt$$

$$= \bar{A}_x$$

We can also see that this must be true by general reasoning. The present value of 1 paid on x's death must be equal to the sum of the amount paid if x dies first and the amount paid if x dies second. These two possibilities are mutually exclusive and exhaustive.

Solution 9.9

We have:

$$\bar{A}^{2}_{xy} + \bar{A}_{x\overset{2}{y}} = \bar{A}_{\overline{xy}}$$

If $y = x$, this becomes:

$$\bar{A}^{2}_{xx} + \bar{A}_{x\overset{2}{x}} = \bar{A}_{\overline{xx}}$$

But $\bar{A}^{2}_{xx} = \bar{A}_{x\overset{2}{x}}$, so:

$$\bar{A}^{2}_{xx} = \tfrac{1}{2} \bar{A}_{\overline{xx}}$$

Solution 9.10

We need to specify that the term of the annuity should be automatically zero when $T_y < T_x$. So we could write:

$$\bar{Z} = v^{T_x}\, \bar{a}_{\overline{\max\left(0,\, T_y - T_x\right)}|}$$

Alternatively:

$$\bar{Z} = \bar{a}_{\overline{T_y}|} - \bar{a}_{\overline{\min\left(T_x,\, T_y\right)}|}$$

so that when $T_x < T_y$ the value is $\bar{a}_{\overline{T_y}|} - \bar{a}_{\overline{T_x}|}$, and when $T_x > T_y$ the value is $\bar{a}_{\overline{T_y}|} - \bar{a}_{\overline{T_y}|} = 0$, as it should.

Finally, because $\min\left(T_x, T_y\right) = T_{xy}$, then we can conveniently write:

$$\bar{Z} = \bar{a}_{\overline{T_y}|} - \bar{a}_{\overline{T_{xy}}|}$$

This is a very useful form for the present value of a reversionary annuity.

Solution 9.11

An annuity will begin at exact time t if (x) dies at that moment (with probability density ${}_t p_x\, \mu_{x+t}$) and (y) is still alive (probability ${}_t p_y$).

From time t onwards, an annuity of 1 *pa* would be payable continuously until the subsequent death of (y). As (y) is aged $y+t$ at the time the annuity starts, the expected present value of the annuity as at time t is $\bar{a}_{y+t}$, and discounting to time 0 requires further multiplication by v^t.

Finally, integrating over all possible values of t will cover all possible start times for the annuity payments (*ie* over all the moments at which (x) could die with (y) still living).

Solution 9.12

First of all:

$$\bar{a}_{x|y} = \int_0^\infty v^t \, {}_tp_y \, {}_tq_x \, dt$$

The logic here is that payments are made at (and hence discounted from) time t provided (y) is alive at that moment and (x) is dead by that time.

Then:

$$\bar{a}_{x|y} = \int_0^\infty v^t \, {}_tp_y \left(1 - {}_tp_x\right) dt$$

$$= \int_0^\infty v^t \, {}_tp_y \, dt - \int_0^\infty v^t \, {}_tp_{xy} \, dt$$

$$= \bar{a}_y - \bar{a}_{xy}$$

Also:

$$\bar{a}_y = \frac{1 - \bar{A}_y}{\delta}$$

by premium conversion. So:

$$\bar{a}_{x|y} = \bar{a}_y - \bar{a}_{xy} = \frac{1 - \bar{A}_y}{\delta} - \frac{1 - \bar{A}_{xy}}{\delta} = \frac{\bar{A}_{xy} - \bar{A}_y}{\delta}$$

Solution 9.13

$\bar{A}^{\;1}_{\overset{\frown}{xy}:\overline{n}|}$ = EPV of 1 paid immediately on the first death out of (x) and (y), provided that death occurs within n years.

$\bar{A}_{xy:\overline{n}|}$ = EPV of 1 paid in n years time, or immediately on the first death of (x) and (y) if this occurs sooner.

$\bar{A}_{xy:\overset{1}{\overline{n|}}}$ = EPV of 1 paid in n years time, provided neither (x) nor (y) has died by then.

Solution 9.14

$$A_{\overline{60:60}:\overline{5|}} = 2A_{60:\overline{5|}} - A_{60:60:\overline{5|}}$$

Solution 9.15

After n years the status $\overline{xy}$ could have (x), (y) or both living. However, the annuity value $\ddot{a}_{\overline{x+n:y+n}}$ is conditional only on both being living at that time.

Instead, we would have to allow for all three possibilities separately. That is:

$$\ddot{a}_{\overline{xy}:\overline{n|}} = \ddot{a}_{\overline{xy}} - v^n \left[{}_n p_x \ {}_n q_y \ \ddot{a}_{x+n} + {}_n q_x \ {}_n p_y \ \ddot{a}_{y+n} + {}_n p_{xy} \ \ddot{a}_{\overline{x+n:y+n}} \right]$$

The formulae shown in the Core Reading therefore turn out to be simpler.

Solution 9.16

The last survivor temporary annuity can be thought of as the sum of the two single life (temporary) annuities less the temporary joint life annuity:

$$\ddot{a}_{\overline{50:50}:\overline{20|}} = \ddot{a}_{50:\overline{20|}} + \ddot{a}_{50:\overline{20|}} - \ddot{a}_{50:50:\overline{20|}}$$

Then:

$$\ddot{a}_{50:\overline{20|}} = \ddot{a}_{50} - v^{20} \frac{l_{70}}{l_{50}} \ddot{a}_{70}$$

which for males gives:

$$\ddot{a}_{50:\overline{20|}} = 18.843 - 1.04^{-20} \times \frac{9,238.134}{9,941.923} \times 11.562 = 13.940$$

and for females gives:

$$\ddot{a}_{50:\overline{20|}} = 19.539 - 1.04^{-20} \times \frac{9,392.621}{9,952.697} \times 12.934 = 13.968$$

Also:

$$\ddot{a}_{50:50:\overline{20|}} = \ddot{a}_{50:50} - 1.04^{-20} \times \frac{9,238.134}{9,941.923} \times \frac{9,392.621}{9,952.697} \times \ddot{a}_{70:70}$$

$$= 17.688 - 1.04^{-20} \times \frac{9,238.134}{9,941.923} \times \frac{9,392.621}{9,952.697} \times 9.7666$$

$$= 13.780$$

So:

$$\ddot{a}_{\overline{50:50}:\overline{20|}} = 13.940 + 13.968 - 13.780 = 14.128$$

Solution 9.17

We already have, from Section 3, the result

$$\int_{t=0}^{\infty} v^t \, \overline{a}_{y+t} \, {}_tp_{xy} \, \mu_{x+t} \, dt = \overline{a}_y - \overline{a}_{xy}$$

So we can use this as follows:

$$\int_{t=0}^{n} v^t \, \overline{a}_{y+t} \, {}_tp_{xy} \, \mu_{x+t} \, dt = \int_{t=0}^{\infty} v^t \, \overline{a}_{y+t} \, {}_tp_{xy} \, \mu_{x+t} \, dt - \int_{t=n}^{\infty} v^t \, \overline{a}_{y+t} \, {}_tp_{xy} \, \mu_{x+t} \, dt$$

$$= \overline{a}_y - \overline{a}_{xy} - \int_{s=0}^{\infty} v^{s+n} \overline{a}_{y+s+n} \, {}_{s+n}p_{xy} \, \mu_{x+s+n} \, ds$$

on substituting $s = t - n$.

Pulling a factor of $v^n\,{}_np_{xy}$ outside the integral, we get:

$$\int_{t=0}^{n} v^t\,\bar{a}_{y+t}\;{}_tp_{xy}\,\mu_{x+t}\,dt = \bar{a}_y - \bar{a}_{xy} - v^n\,{}_np_{xy}\int_{s=0}^{\infty} v^s\,\bar{a}_{y+s+n}\;{}_sp_{x+n:y+n}\,\mu_{x+s+n}\,ds$$

$$= \bar{a}_y - \bar{a}_{xy} - v^n\,{}_np_{xy}\left(\bar{a}_{y+n} - \bar{a}_{x+n:y+n}\right)$$

Solution 9.18

Left hand side

The only restriction from the normal whole life version is that payments following the death of (x) after n years will not be made. We can therefore calculate the payments we need by simply integrating to n rather than to infinity.

Right hand side

We need to deduct all annuity payments, currently included in $\bar{a}_{x|y}$, that relate to the death of (x) after n years with (y) alive at the time of (x)'s death. (Note that the death of (x) after n years, with (y) dead at the time of (x)'s death, is already excluded from $\bar{a}_{x|y}$). The required deduction therefore equates to a reversionary annuity from time n, with both (x) and (y) alive at that point, appropriately discounted to the start of the contract.

Solution 9.19

The annuity begins at time t, which is the instant of (x)'s death. Hence the factor ${}_tp_x\,\mu_{x+t}\,dt$.

The annuity is payable to life (y) for life thereafter, but for a maximum of n years. Hence:

- (y) has to be alive at time t (probability ${}_tp_y$)
- the expected present value of the temporary annuity from that time is $\bar{a}_{y+t:\overline{n}|}$

and we need to discount this value to time zero (using v^t).

Solution 9.20

First, consider an annuity that starts n years after the death of (x) and is payable to (y). This annuity will be in payment at time $t + n$ if:

- (x) has died before time t and
- (y) is alive at time $t + n$.

So, the expected present value of this annuity is:

$$\int_0^\infty v^{t+n} {}_tq_x \, {}_{t+n}p_y \, dt$$

Now, the Type 4 annuity is equal to a reversionary annuity less this one above. So, the expected present value of the Type 4 annuity is:

$$\int_0^\infty v^t {}_tq_x \, {}_tp_y \, dt - \int_0^\infty v^{t+n} {}_tq_x \, {}_{t+n}p_y \, dt$$

$$= \bar{a}_{x|y} - v^n {}_np_y \int_0^\infty v^t {}_tq_x \, {}_tp_{y+n} \, dt$$

$$= \bar{a}_{x|y} - v^n {}_np_y \ddot{a}_{x|y+n}$$

Solution 9.21

We can write the expected present value of this benefit in terms of an integral as follows:

$$\int_0^\infty v^t \, {}_tp_x \, \mu_{x+t} \, {}_tp_y \left(\bar{a}_{\overline{n}|} + v^n \, {}_np_{y+t} \, \bar{a}_{y+t+n} \right) dt$$

Here we're thinking about (x) dying at time t, with (y) still alive. Then (y) receives an annuity that is guaranteed for n years. Furthermore, if (y) is still alive at time $t+n$, he then receives an annuity throughout his remaining lifetime.

If we multiply out the brackets, then the first term is just:

$$\bar{a}_{\overline{n}|} \int_0^\infty v^t \, {}_tp_x \, \mu_{x+t} \, {}_tp_y \, dt = \bar{a}_{\overline{n}|} \, \bar{A}^1_{x:y}$$

The second term can be written as:

$$\int_0^\infty v^{t+n}\ {}_t p_x\ \mu_{x+t}\ {}_{t+n} p_y\ \bar{a}_{y+t+n}\,dt = v^n\ {}_n p_y \int_0^\infty v^t\ {}_t p_x\ \mu_{x+t}\ {}_t p_{y+n}\ \bar{a}_{y+t+n}\,dt$$

$$= v^n\ {}_n p_y\ \bar{a}_{x|y+n}$$

Solution 9.22

If the annuity were payable until Vera's 65th birthday (or her earlier death), then the expected present value of the benefit would be:

$$10,000\left(a_{60(f):\overline{5}|} - a_{60(m):60(f):\overline{5}|}\right)$$

However, if Jack dies before age 65 and Vera is still alive at age 65, the annuity continues until Vera reaches age 75 or until her earlier death. The expected present value of this part of the benefit is:

$$10,000\,v^5\ {}_5 q_{60(m)}\ {}_5 p_{60(f)}\ a_{65(f):\overline{10}|}$$

So the single premium is given by:

$$P = 10,000\left(a_{60(f):\overline{5}|} - a_{60(m):60(f):\overline{5}|} + v^5\ {}_5 q_{60(m)}\ {}_5 p_{60(f)}\ a_{65(f):\overline{10}|}\right)$$

Now:

$$a_{60(f):\overline{5}|} = a_{60(f)} - v^5\ {}_5 p_{60(f)}\ a_{65(f)}$$

$$= 15.652 - 1.04^{-5} \times \frac{9,703.708}{9,848.431} \times 13.871$$

$$= 4.419$$

$$a_{60(m):60(f):\overline{5}|} = a_{60(m):60(f)} - v^5\ {}_5 p_{60(m)}\ {}_5 p_{60(f)}\ a_{65(m):65(f)}$$

$$= 13.090 - 1.04^{-5} \times \frac{9,647.797}{9,826.131} \times \frac{9,703.708}{9,848.431} \times 10.958$$

$$= 4.377$$

$$a_{65(f):\overline{10}|} = a_{65(f)} - v^{10}\ {}_{10}p_{65(f)}\ a_{75(f)}$$

$$= 13.871 - 1.04^{-10} \times \frac{8,784.955}{9,703.708} \times 9.933$$

$$= 7.796$$

So:

$$P = 10,000\left[4.419 - 4.377 + 1.04^{-5} \times \left(1 - \frac{9,647.797}{9,826.131}\right) \times \frac{9,703.708}{9,848.431} \times 7.796\right]$$

$$= £1,566$$

Solution 9.23

The equation of value is:

$$P\ddot{a}_{\overline{55:55}} = 50,000\bar{A}_{\overline{55:55}}$$

The annuity function is:

$$\ddot{a}_{\overline{55:55}} = \ddot{a}_{55(m)} + \ddot{a}_{55(f)} - \ddot{a}_{55:55} = 17.364 + 18.210 - 16.016 = 19.558$$

By premium conversion, we have:

$$A_{\overline{55:55}} = 1 - d\,\ddot{a}_{\overline{55:55}} = 1 - \frac{0.04}{1.04} \times 19.558 = 0.24777$$

So:

$$\bar{A}_{\overline{55:55}} \approx 1.02 \times 0.24777 = 0.25272$$

and:

$$P = \frac{50,000 \times 0.25272}{19.558} = £646.09$$

Solution 9.24

(i) A^{1}_{xy} represents a benefit payable if (x) dies first. If (x) dies first, the benefit is paid. If (y) dies first, the benefit can never be paid. So premiums should stop if either life dies. The appropriate annuity factor is $\ddot{a}_{xy}$.

(ii) $A_{x\overset{2}{y}}$ represents a benefit payable at the end of the year of (y)'s death if (y) dies second. When (y) dies, either the benefit will be paid (if (x) is already dead), or payment of the benefit will become impossible (if (x) is still alive). So (y)'s death is the deciding factor and premiums should stop when (y) dies. The appropriate annuity factor is $\ddot{a}_{y}$.

Solution 9.25

The probability can be expressed in integral form as follows:

$$ {}_{30}q^{\;\;2}_{50:60} = \int_0^{30} {}_tp_{50}\, \mu_{50+t}\, {}_tq_{60}\, dt $$

Now:

$$ \mu_{50+t} = \frac{1}{50-t} $$

and:

$$ \begin{aligned} {}_tp_x &= \exp\left(-\int_0^t \mu_{x+s}\, ds\right) = \exp\left(-\int_0^t \frac{1}{100-x-s}\, ds\right) \\ &= \exp\Big[\ln(100-x-s)\Big]_0^t = \exp\left[\ln\left(\frac{100-x-t}{100-x}\right)\right] \\ &= \frac{100-x-t}{100-x} \end{aligned} $$

So:

$$ {}_tp_{50} = \frac{50-t}{50} \text{ and } {}_tq_{60} = \frac{t}{40} $$

Substituting these terms into the integral, we get:

$$ {}_{30}q^{\,2}_{50:60} = \int_0^{30} \left(\frac{50-t}{50} \times \frac{1}{50-t} \times \frac{t}{40} \right) dt $$

$$ = \int_0^{30} \frac{t}{2,000} dt $$

$$ = \left[\frac{t^2}{4,000} \right]_0^{30} $$

$$ = 0.225 $$

Solution 9.26

The expected present value of the spouse's benefit is:

$$ 10,000 \left(a_{60|55} - v^{15}\, {}_{15}p_{55}\, a_{60|70} \right) $$

The terms in this expression are:

$$ a_{60|55} = a_{55} - a_{60:55} = 16.364 - 14.121 = 2.243 $$

$$ a_{60|70} = a_{70} - a_{60:70} = 10.562 - 9.978 = 0.584 $$

and:

$$ v^{15}\, {}_{15}p_{55} = 1.04^{-15} \times \frac{9,238.134}{9,904.805} = 0.51789 $$

So the expected present value of the benefit (to the nearest £100) is:

$$ 10,000 (2.243 - 0.51789 \times 0.584) = 19,400 $$

If you have trouble getting started on a question like this, imagine that the annuity benefit is payable continuously. Then write down an integral expression, and simplify it. In this case we could use:

$$\int_0^\infty {}_tp_{60}\,\mu_{60+t}\,{}_tp_{55}\,v^t\,\bar{a}_{55+t:\overline{15|}}\,dt$$

$$=\int_0^\infty {}_tp_{60}\,\mu_{60+t}\,{}_tp_{55}\,v^t\left(\bar{a}_{55+t}-v^{15}\,{}_{15}p_{55+t}\,\bar{a}_{70+t}\right)dt$$

$$=\int_0^\infty {}_tp_{60}\,\mu_{60+t}\,{}_tp_{55}\,v^t\,\bar{a}_{55+t}\,dt-v^{15}\,{}_{15}p_{55}\int_0^\infty {}_tp_{60}\,\mu_{60+t}\,{}_tp_{70}\,v^t\,\bar{a}_{70+t}\,dt$$

$$=\bar{a}_{60|55}-v^{15}\,{}_{15}p_{55}\,\bar{a}_{60|70}$$

Discretising then gives us the required formula:

$$a_{60|55}-v^{15}\,{}_{15}p_{55}\,a_{60|70}$$

Or you could have used the same method as we used in Question 9.20, as this is an example of a "Type 4" reversionary annuity, as described in Section 4.4.

Chapter 10

Competing risks

Syllabus objectives

(vii) Describe and illustrate methods of valuing cashflows that are contingent upon multiple transition events.

1. Define health insurance, and describe simple health insurance premium and benefit structures.

2. Explain how a cashflow, contingent upon more than one risk, may be valued using a multiple state Markov model.

3. Construct formulae for the expected present values of cashflows that are contingent upon multiple transition events, including simple health insurance premiums and benefits, and calculate these in simple cases. Regular premiums and sickness benefits are payable continuously and assurance benefits are payable immediately on transition.

(viii) Describe and use methods of projecting and valuing expected cashflows that are contingent upon multiple decrement events.

1. Define a multiple decrement model as a special case of a multiple-state Markov model.

2. Derive dependent probabilities for a multiple decrement model in terms of given forces of transition, assuming forces of transition are constant over single years of age.

3. Derive forces of transition from given dependent probabilities, assuming forces of transition are constant over single years of age.

4. Describe the construction and use of multiple decrement tables.

Continued...

6. *Use multiple decrement tables to evaluate expected present values of cashflows dependent upon more than one decrement, including those of pension schemes.*

0 Introduction

So far we have considered contingencies where a life is exposed to death only. If we suppose that a life is subject to more than one transition, then the transitions are referred to as a set of *competing risks*. For example, a member of a pension scheme can, in order for the associated pension scheme benefits to be valued, be regarded as exposed to the competing risks of retirement and death.

In a similar way, a person with a health insurance policy who is in good health, can be considered as exposed to the competing risks of becoming sick and dying.

Some of the ideas in this chapter involve concepts you will have seen in Subject CT4. You may wish to refer back to this material as you read.

1 Health insurance contracts

In the same way as insurance contracts exist that pay benefits contingent upon death or survival, so contracts also exist that pay benefits contingent upon the state of health of a person. In this case a policyholder can be considered to be subject to the competing risks of death and of becoming sick.

An *income protection* insurance contract pays an income to the policyholder while that policyholder is deemed as being "sick" (with the definition of sickness being carefully specified in the policy conditions). If the policyholder recovers, the cover under the policy usually continues, so that subsequent bouts of qualifying sickness would merit further benefit payments.

Such policies are usually subject to a *deferred period* (*eg* 3 months) of continuous sickness that has to have elapsed before any benefits start to be paid, and during which no benefit is payable.

Premiums for these policies would normally be regular (*eg* monthly) and would typically be *waived* during periods of qualifying sickness. This means that premiums would not be paid at the same time as benefits are payable.

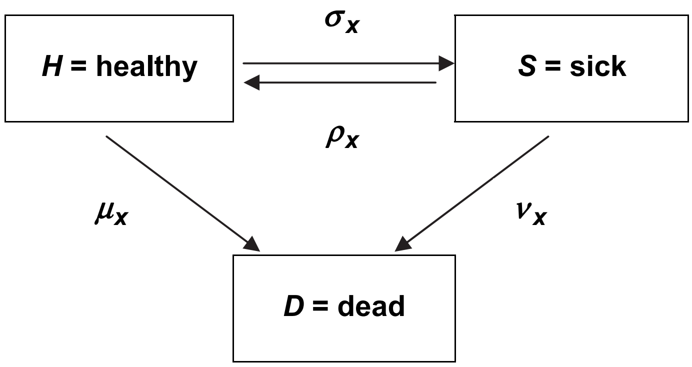
There are many varieties of insurance contracts that can be based on health-dependent contingencies. Other examples (covered in Subject ST1) include:

- ***critical illness* insurance, which normally pays a lump sum on diagnosis of a defined "critical" illness (such as cancer)**
- ***long-term care* insurance, which pays an income contingent upon the policyholder requiring long-term care, and hence supports the costs of receiving that care.**

2 Multiple state models

Multiple state models of the type described in Subject CT4 are well suited to valuing cashflows that are dependent on multiple transitions, such as of health insurance contracts. The model will be chosen to include the relevant states, and transitions between states, that are necessary to replicate the required cashflows for the contract concerned.

For example, for the simple income protection policy described in the previous section (with no deferred period), the general three-state healthy-sick-dead model would be suitable.



Recall that the transition intensity from *sick* to *dead* is represented by ν (nu, the 13th letter of the Greek alphabet), not v (the 22nd letter of the English alphabet).

2.1 Notation

Let i and j denote any two different states. Define μ_x^{ij} be the transition intensity from state i to state j at age x (so, for example, $\mu_x^{HS} = \sigma_x$ in the above model). Also define the related transition probability:

$${}_tp_x^{ij} = P[\text{In state } j \text{ at age } x+t \mid \text{In state } i \text{ at age } x]$$

where now i and j need not be different.

For example, for the model shown above:

${}_tp_x^{HD}$ represents the probability that a life in the *healthy* state at age x will be in the *dead* state at age $x+t$. This probability encompasses all possible routes from *healthy* to *dead*, which may or may not include one or more visits to the *sick* state.

${}_t p_x^{SS}$ represents the probability that a life in the *sick* state at age x will still be in the *sick* state at age $x+t$. This includes the probability that the life remained in the *sick* state throughout the period from x to $x+t$ and the probability that the life made one or more visits to the *healthy* state, returning on each occasion to the *sick* state.

The event whose probability is defined by the expression:

$$ {}_t p_x^{ij} = P\left[\text{In state } j \text{ at age } x+t \mid \text{ In state } i \text{ at age } x\right] $$

does not specify what must happen between age x and $x+t$, however. In particular, if $i=j$, it does not require that the life remains in state i between these ages. So for any state i, also define the related transition probability:

$$ {}_t p_x^{\overline{ii}} = P\left[\text{In state } i \text{ from age } x \text{ to } x+t \mid \text{ In state } i \text{ at age } x\right] $$

If return to state i is impossible, then ${}_t p_x^{\overline{ii}} = {}_t p_x^{ii}$, but this is not true (for example) in the case of states H and S in the healthy-sick-dead model above.

This is sometimes referred to as the *occupancy probability*.

Question 10.1

A life insurance company uses the three-state healthy-sick-dead model as described above to calculate premiums for a 3-year sickness policy issued to healthy policyholders aged 60.

S_t denotes the state occupied by the policyholder at age $60+t$, so that $S_0 = H$ and $S_t = H, S$ or D for $t = 1, 2, 3$.

The transition probabilities used by the insurer are defined in the following way:

$$p_{60+t}^{jk} = P\left(S_{t+1} = k \mid S_t = j\right)$$

For $t = 0, 1, 2$, it is assumed that:

$$p_{60+t}^{HH} = 0.9 \qquad p_{60+t}^{HS} = 0.08$$

$$p_{60+t}^{SH} = 0.7 \qquad p_{60+t}^{SS} = 0.25$$

What is the probability that a new policyholder is:

(a) sick at exact age 62?

(b) dead at exact age 62?

2.2 *Kolmogorov differential equations*

In Subject CT5, we assume that the above probabilities, as well as the transition intensities, are available. The procedures by which the transition probabilities ${}_t p_x^{ij}$ and ${}_t p_x^{\overline{ii}}$ are obtained from the transition intensities, involving the solution of the relevant differential equations, are examined in Subject CT4 and will not be examined again here.

For example, the Kolmogorov forward differential equations are:

$$\frac{\partial}{\partial t}\, {}_t p_x^{ij} = \sum_{k \neq j} \left({}_t p_x^{ik}\, \mu_{x+t}^{kj} - {}_t p_x^{ij}\, \mu_{x+t}^{jk} \right)$$

and:

$$\frac{\partial}{\partial t}\,{}_t p_x^{\overline{ii}} = -\,{}_t p_x^{\overline{ii}} \sum_{j \neq i} \mu_{x+t}^{ij}$$

(These equations were derived in Subject CT4 and you will not be required to derive them in Subject CT5.)

The calculation of probabilities from the estimated intensities requires the solution of these equations, which is a straightforward numerical exercise. (Again, you won't be required to solve these equations in Subject CT5.)

However, importantly for Subject CT5, the above **differential equation for ${}_t p_x^{\overline{ii}}$ has the closed form solution:**

$$ {}_t p_x^{\overline{ii}} = \exp\left(-\int_0^t \sum_{j \neq i} \mu_{x+s}^{ij}\, ds \right)$$

Question 10.2

Prove this result (*Subject CT4 revision!*)

This result is particularly important in the construction of multiple decrement models, which are described in Section 3.

We will first consider how we can value cashflows for the more general, multiple state scenarios that we have described so far.

2.3 Valuing continuous cashflows using multiple state models

Consider, for example, a healthy life who is subject to the competing risks of sickness and death. A multiple state model can be used to construct integral expressions for the EPVs of the following types of cashflows:

- **a lump sum paid immediately on transition from one state to another (Type 1)**
- **an income payable while occupying a particular state (Type 2).**

Examples

The healthy-sickness-dead model described in the previous section will be used.

1. The EPV of a lump sum of 1 payable on death (whether directly from healthy or from having first become sick) of a healthy life currently aged x is

$$\int_0^\infty e^{-\delta t}\left({}_tp_x^{HH}\mu_{x+t} + {}_tp_x^{HS}\nu_{x+t}\right)dt$$

(assuming a constant force of interest δ).

This is a Type 1 cashflow. You can think about the integral as being built up as follows:

- The benefit of 1 is payable at time t if the policyholder dies at time t. This may be from the healthy state (in which case the policyholder is healthy at time t, then dies from this state at age $x+t$) or from the sick state (in which case the policyholder is sick at time t and dies from the sick state at age $x+t$).
- The benefit is then discounted back to time 0.
- The expression is then integrated over all possible points in time when death might occur.

2. The EPV of an annuity of 1 per annum payable continuously during sickness of a healthy life currently aged x is

$$\int_0^\infty e^{-\delta t}\,{}_tp_x^{HS}\,dt$$

This is a Type 2 cashflow. Here we can think about a benefit being paid at time t provided that the life is sick at time t. The benefit amount is then discounted back to time 0, and we integrate over all points in time at which a benefit could be paid.

3. The EPV of a premium of 1 per annum payable continuously, but waived during periods of sickness, by a healthy life currently aged x is

$$\int_0^\infty e^{-\delta t}\,{}_tp_x^{HH}\,dt$$

This is also a Type 2 cashflow. The premium is payable at time t only by someone who is healthy at time t. So we need the probability that the currently healthy life is also healthy at time t, discounted back to time 0, and then integrated over all points in time during which the premium could be paid.

Question 10.3 ***(Tricky)***

You are using the three-state healthy-sick-dead model to price various sickness policies. Using the notation defined in Section 2.1, write down an expression for the expected present value of each of the following sickness benefits for a healthy life aged 30.

(i) £3,000 *pa* payable continuously while sick, but ceasing at age 60.

(ii) £3,000 *pa* payable continuously throughout the first period of sickness only, but ceasing at age 60.

(iii) £3,000 *pa* payable continuously while sick provided that the life has been sick for at least one year. Again, any benefit ceases to be paid at age 60.

The actual evaluation of the above integrals may, in general, be done numerically.

For example, we could use a numerical technique such as the trapezium rule to approximate the value of an integral. The trapezium rule says that if we want to integrate a function $f(x)$ between the limits of $x = a$ and $x = b$, we divide the interval $[a,b]$ into n strips of equal length $h = \dfrac{b-a}{n}$ and use the formula:

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f(a) + 2f(a+h) + 2f(a+2h) + \cdots + 2f(b-h) + f(b)\right]$$

Example

We consider a simple income protection insurance contract that pays at a rate of 20,000 per annum continuously while a policyholder is sick. The policy is issued to a healthy life aged 50 exact for a term of three years. Calculate an approximate EPV of the sickness benefit at outset on the following assumptions:

- $_1p_{50}^{HS} = 0.02$, $_2p_{50}^{HS} = 0.04$, $_3p_{50}^{HS} = 0.07$
- **Interest of 3% per annum**

(with symbols as defined in Section 2.1).

Solution

As an integral the EPV is:

$$\text{EPV} = \int_0^3 20{,}000\, e^{-\delta t}\, {}_tp_{50}^{HS}\, dt = 20{,}000 \int_0^3 v^t\, {}_tp_{50}^{HS}\, dt$$

If we assume the function $v^t\, {}_tp_{50}^{HS}$ varies linearly over each policy year, we can approximate the integral using the trapezium rule:

$$\text{EPV} \simeq 20{,}000\left(\tfrac{1}{2}v^0\, {}_0p_{50}^{HS} + v^1\, {}_1p_{50}^{HS} + v^2\, {}_2p_{50}^{HS} + \tfrac{1}{2}v^3\, {}_3p_{50}^{HS}\right)$$

$$= 20{,}000\left(0 + \frac{0.02}{1.03} + \frac{0.04}{1.03^2} + \tfrac{1}{2} \times \frac{0.07}{1.03^3}\right)$$

$$= 1{,}783$$

(Other suitable approximations for the integral could be used).

3 Multiple decrement models

A multiple decrement model is a multiple state model which has:

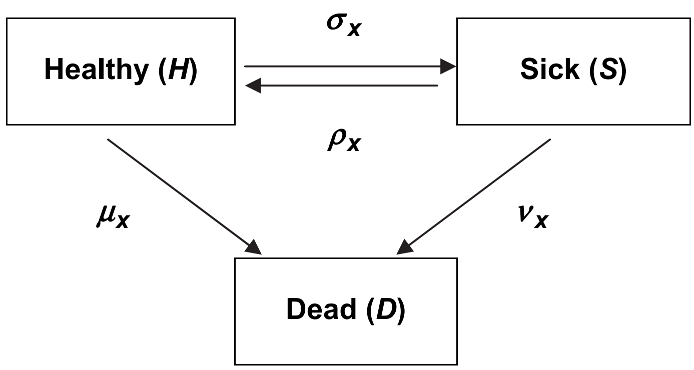
- **one active state, and**
- **one or more absorbing exit states**

Many practical situations involving competing risks can be modelled adequately using this simplified model structure.

3.1 A simple example

We will use a simple example to illustrate the operation of multiple decrement models.

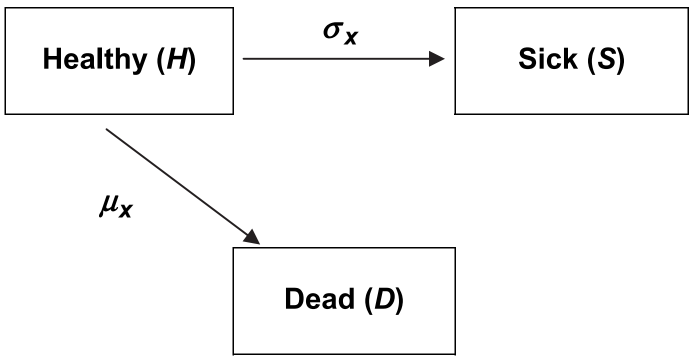
We begin with the general healthy-sick-dead multiple state model described in Section 2:



Suppose we have an insurance contract which only pays:

- **a once-only payment of X on transition from healthy to sick, or**
- **a once-only payment of Y on transition from healthy to dead.**

In this case, the following multiple decrement model should be sufficient to enable us to value these cashflows adequately:



In this multiple decrement model, H is the active state, and there are two absorbing exit states, S and D. The life H is subject to the competing risks of S and D.

3.2 *Multiple decrement probabilities*

In a multiple decrement model, we only need to define two types of probability.

The first is ${}_t(aq)_x^r$, which is defined as the *dependent* probability that an individual aged x in the active state will be removed from that state between ages x and $x+t$ by the decrement r. (By "dependent", we mean *in the presence of all other risks of decrement in the population*). When $t=1$ this is written as $(aq)_x^r$.

The second is ${}_t(ap)_x$, which is defined as the dependent probability that an individual aged x in the active state will still be in the active state at age $x+t$. When $t=1$ this is written as $(ap)_x$.

For comparison, using multiple state notation the dependent probabilities for our example would be written:

$$(aq)_x^s = {}_1p_x^{HS}$$

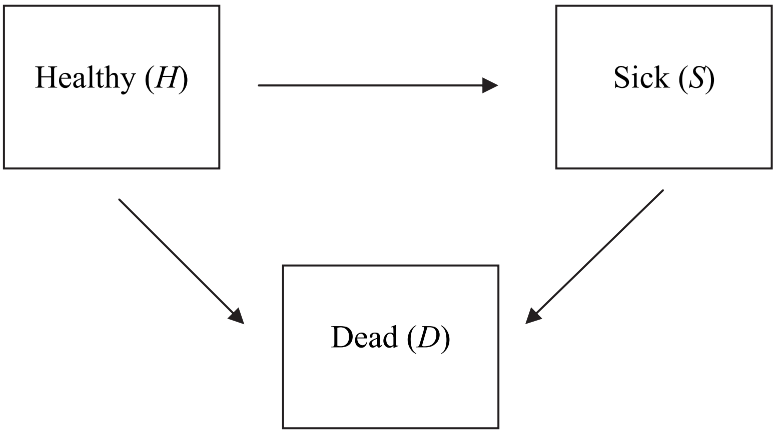
$$(aq)_x^d = {}_1p_x^{HD}$$

$$(ap)_x = {}_1p_x^{HH}$$

These equalities are *only* true in the case of the revised healthy-sick-dead model above. They are *not* true for the general healthy-sick-dead multiple state model described at the start of Section 3.1.

Question 10.4

Consider the following 3-state model:



Explain whether or not each of the following is true, and if not, state with reasons which of the two probabilities is the larger.

(a) $(aq)_x^s = {}_1p_x^{HS}$

(b) $(aq)_x^d = {}_1p_x^{HD}$

(c) $(ap)_x = {}_1p_x^{HH}$

Question 10.5

Explain whether $(ap)_x = {}_1p_x^{HH}$ for the general healthy-sick-dead model defined at the start of Section 3.1.

We also note that

$$\boldsymbol{(ap)_x + (aq)_x^s + (aq)_x^d = 1}$$

and that we also write

$$\boldsymbol{(aq)_x = (aq)_x^s + (aq)_x^d}$$

so that

$$\boldsymbol{(ap)_x + (aq)_x = 1}$$

Example

Active members of a pension scheme are subject to the following probabilities of decrement at the given ages (where r and d stand for retirement and death respectively).

Age x	$(aq)_x^r$	$(aq)_x^d$
60	0.1	0.03
61	0.2	0.04

Calculate the following probabilities, all relating to an active member who is currently exactly aged 60.

(i) The probability of retiring during the year of age 61 to 62.

(ii) The probability of dying as an active member before age 62 (*ie* without retiring first).

(iii) The probability of still being an active member at age 62.

Solution

(i) This is $(ap)_{60} \times (aq)^r_{61}$, where:

$$(ap)_{60} = 1-(aq)^r_{60}-(aq)^d_{60} = 1-0.1-0.03 = 0.87$$

So the required probability is $0.87 \times 0.2 = 0.174$

Note that this probability can also be written as ${}_{1|}(aq)^r_{60}$.

(ii) This probability is:

$$\begin{aligned} {}_2(aq)^d_{60} &= (aq)^d_{60} + (ap)_{60} \times (aq)^d_{61} \\ &= 0.03 + 0.87 \times 0.04 \\ &= 0.0648 \end{aligned}$$

(iii) This is:

$${}_2(ap)_{60} = (ap)_{60} \times (ap)_{61} = 0.87 \times \left[1-0.2-0.04\right] = 0.6612$$

It is also useful to consider the special case of a single decrement model, which only has one cause of decrement. For this we define ${}_tq^r_x$ to be the *independent* probability that an individual aged x in the active state will be removed from that state between ages x and $x+t$ by the decrement r. (By "independent", we mean *when r is the only risk of decrement acting on the population*). When $t=1$ this is written as q^r_x.

Independent probabilities of decrement assume that there are no other decrements operating on the population of interest. Dependent probabilities of decrement take into account the competing forces of decrement that operate on the population.

Q

Question 10.6

A population is subject to two causes of decrement, death (d) and withdrawal (w). Explain whether the value of q^d_x would be larger or smaller than the value of $(aq)^d_x$.

3.3 *Deriving probabilities from transition intensities*

We can use the Kolmogorov forward differential equations to derive transition probabilities, as in the case of multiple state models. We note from Section 2.2 that, in the multiple state model, this produces the following general result:

$$ {}_t p_x^{\overline{ii}} = \exp\left(-\int_0^t \sum_{j \neq i} \mu_{x+s}^{ij} ds \right) $$

Question 10.7

Describe this formula carefully.

In the case of the multiple decrement model, in which return to the active state is not possible, we have:

$$ {}_t(ap)_x = {}_t p_x^{HH} = {}_t p_x^{\overline{HH}} $$

For our double decrement model with decrements of sickness and death we have:

$$ {}_t(ap)_x = {}_t p_x^{\overline{HH}} = \exp\left[-\int_{s=0}^{t} \left(\mu_{x+s}^{HS} + \mu_{x+s}^{HD} \right) ds \right] = \exp\left[-\int_{s=0}^{t} \left(\sigma_{x+s} + \mu_{x+s} \right) ds \right] $$

Therefore, assuming constant transition intensities:

$$ {}_t(ap)_x = e^{-(\mu+\sigma)t} \qquad \textbf{(1)} $$

For the other probabilities, the differential equations (again assuming constant transition intensities) are:

$$ \frac{\partial}{\partial t} {}_t(aq)_x^s = \sigma . {}_t(ap)_x = \sigma e^{-(\mu+\sigma)t} $$

and

$$ \frac{\partial}{\partial t} {}_t(aq)_x^d = \mu \, {}_t(ap)_x = \mu e^{-(\mu+\sigma)t} $$

These differential equations have the closed form solutions (with $t = 1$):

$$(aq)_x^s = \frac{\sigma}{\mu+\sigma}\left(1 - e^{-(\mu+\sigma)}\right) \qquad \textbf{(2)}$$

and

$$(aq)_x^d = \frac{\mu}{\mu+\sigma}\left(1 - e^{-(\mu+\sigma)}\right) \qquad \textbf{(3)}$$

These solutions are obtained by integrating the differential equations with respect to t between the limits of $t = 0$ and $t = 1$. For example, integrating the first equation gives:

$$\left[{}_t(aq)_x^s\right]_0^1 = \int_0^1 \sigma e^{-(\mu+\sigma)t}\, dt$$

$$\Rightarrow (aq)_x^s - {}_0(aq)_x^s = \frac{-\sigma}{\mu+\sigma}\left[e^{-(\mu+\sigma)t}\right]_0^1$$

$$\Rightarrow (aq)_x^s = \frac{\sigma}{\mu+\sigma}\left[1 - e^{-(\mu+\sigma)}\right]$$

since ${}_0(aq)_x^s = 0$.

Note that $(aq)_x^s$ is the product of:

- $1 - e^{-(\mu+\sigma)} = 1 - (ap)_x$, *ie* the probability that a life who is in the active state at the start of the year, is not in the active state at the end of the year, and
- $\dfrac{\sigma}{\mu+\sigma}$, *ie* the conditional probability that the transition from the active state takes the life into the sick state, given that there is a transition out of the active state.

You will have seen these ideas before in Subject CT4.

We now have formulae for the dependent probabilities in terms of transition intensities. So, if we can estimate the transition intensities, it is very easy to estimate the dependent probabilities using these formulae.

Question 10.8 ***(Revision)***

How do we estimate σ and μ?

For the independent (single decrement) probabilities we obtain:

$$q_x^s = 1 - e^{-\sigma} \qquad (4)$$

and:

$$q_x^d = 1 - e^{-\mu}$$

When $t \neq 1$, the differential equations have the following closed form solution (for example):

$$_t(aq)_x^s = \frac{\sigma}{\mu + \sigma}\left(1 - e^{-t(\mu+\sigma)}\right)$$

provided the transition intensities are constant over $[x, x+t]$ (and similar for other cases).

Question 10.9

A population of healthy people over the year of age 50 to 51 is subject to a constant force of decrement due to sickness of 0.08 per annum, and a constant force of mortality of 0.002 per annum.

(i) Assuming that a double decrement model is used, calculate:

(a) the probability that a healthy person aged exactly 50 will still be healthy at exact age 51

(b) the probability that a healthy person aged exactly 50 will die due to any cause other than from sickness before exact age 51

(c) the independent probability of a life aged exactly 50 dying before exact age 51.

(ii) Explain why the probability in (i)(c) is unlikely to be a realistic estimate of the probability of a healthy life aged exactly 50 dying before exact age 51.

4 Multiple decrement tables

A multiple decrement table is a computational tool for dealing with a population subject to multiple decrements.

We introduce the following notation as an extension of the (single decrement) life table approach:

$$(al)_x = \text{active population at age } x$$

and $\alpha, \beta, \gamma, \ldots$ the labels for the types of independent decrements to which the population is subject.

Then the multiple decrement table is a numerical representation of the development of the population, such that

$$\begin{aligned}(al)_{x+1} = (al)_x &- \text{number of lives removed between ages } x \text{ and } x+1 \\ &\quad \text{due to decrement } \alpha \\ &- \text{number of lives removed between ages } x \text{ and } x+1 \\ &\quad \text{due to decrement } \beta \\ &- \text{number of lives removed between ages } x \text{ and } x+1 \\ &\quad \text{due to decrement } \gamma \\ &- \cdots\end{aligned}$$

In general, as the decrements are assumed to operate independently, the number of lives removed due to decrement "k" will depend on the preceding population $(al)_x$ *as well as* the numbers removed by every other decrement other than k.

We need to be clear about what the Core Reading means by "independently" here. It means that the presence of multiple causes of decrement in a population does not affect the *forces* of decrement by each cause – *ie* that the forces of decrement are independent of each other.

However, the *numbers* of decrements by any cause will certainly be affected by how many decrements from other causes occur.

We define the number of lives removed over the year of age due to decrement k as $(ad)_x^k$. Hence we have

$$(aq)_x^k = \frac{(ad)_x^k}{(al)_x}$$

$$_n(aq)_x^k = \frac{(ad)_x^k + (ad)_{x+1}^k + \ldots + (ad)_{x+n-1}^k}{(al)_x}$$

$$(ap)_x = \frac{(al)_{x+1}}{(al)_x}$$

$$_n(ap)_x = \frac{(al)_{x+n}}{(al)_x}$$

for $n = 0, 1, \ldots$

Question 10.10

You are given the following extract from a double decrement table:

Age x	$(al)_x$	$(ad)_x^d$	$(ad)_x^w$
50	100,000	175	2,490
51	97,335	180	2,160
52	94,995		

where d and w refer to death and withdrawal respectively.

Calculate:

(a) $(ap)_{51}$

(b) $(aq)_{51}^d$

(c) $_2(aq)_{50}^w$

(d) $_2(ap)_{50}$

We can also use the table to calculate *deferred* dependent probabilities of the form:

$$_{n|}(aq)_x^k = \frac{(ad)_{x+n}^k}{(al)_x}$$

This is the probability that an active life, currently aged x, leaves the population by cause k in the year of age $x+n$ to $x+n+1$.

The expression can be constructed from:

$$_{n|}(aq)_x^k = {}_n(ap)_x \times (aq)_{x+n}^k = \frac{(al)_{x+n}}{(al)_x} \times \frac{(ad)_{x+n}^k}{(al)_{x+n}} = \frac{(ad)_{x+n}^k}{(al)_x}$$

Question 10.11

Using the multiple decrement table in Question 10.10, calculate $_{1|}(aq)_{50}^d$, and state what this probability means in words.

It is also conventional to write the transition intensity (or *force of decrement*) due to cause k at age x in the multiple decrement model as $(a\mu)_x^k$.

4.1 Associated single decrement tables

For each of the causes in a multiple decrement table, it is possible to define a single decrement table which only involves a particular cause of decrement. In effect the other modes of decrement are assumed not to operate.

A life table, such as AM92 or ELT15, is an example of a single decrement table, where the only decrement is death. We can similarly construct other single decrement tables (at least in theory) where the decrement is just withdrawal, for example.

We define functions as in any single decrement table except that each symbol has a superscript indicating the mode of decrement that is being modelled.

The notation used is $l_x^j, d_x^j, q_x^j, p_x^j, \mu_x^j$ *etc* for mode of decrement j.

We've already met the independent probability q_x^j.

Q

Question 10.12

Suppose we have a single decrement table for withdrawals, so q_x^w is the probability of a life aged x withdrawing over the year of age when no other decrements (including mortality!) are possible.

What would you understand by the symbol p_x^w?

The importance of single decrement tables in practice is that they are often used, at least as a starting point, in the construction of a multiple decrement table.

For example, we might wish to construct a double decrement table for endowment assurance policyholders incorporating decrements of death and withdrawal, in which the underlying (or "independent") mortality basis is represented by a standard mortality table, such as AM92 Select. The probabilities q_x^d and p_x^d can be read off from the table, and then used to help construct the relevant independent probabilities, as described in Sections 4.3 and 4.4 below.

4.2 Relationships between single and multiple decrement tables

The linking assumption between the tables is

$$(a\mu)_x^j = \mu_x^j \text{ for all } j \text{ and all } x$$

Normally we would expect the independent decrement probabilities and the dependent decrement probabilities not to be equal. For instance, $(aq)_x^d$ and q_x^d might not be equal because the dependent probability will be affected by the fact that some people will withdraw, or retire (if these are the other decrements operating) "before they could die" in our population. So the fact that a lot of things could happen over the course of a year causes a problem.

But if we look at the transition intensities (forces of decrement) μ then we are in effect looking at some infinitesimally small time interval. In that very small time interval there is only time for one decrement – *eg* death – and so the number of deaths occurring in that time interval will not be reduced by people withdrawing. Thus it is reasonable to assume that the independent intensity of death μ_x^d will be equal to the dependent intensity of death $(a\mu)_x^d$. The same reasoning will hold for any decrement.

The assumption is often called the "independence of decrements". One of its consequences is that if we observe the forces of decrement from cause j in two groups, one which includes all those who have not left the population as a result of decrement from cause i and the other which includes all those who have left the population as a result of decrement from cause i, then the observed forces of decrement from cause j in the two groups are the same. So the force of decrement from cause j is independent of the force from cause i.

For instance, if we have a population of pension scheme members where cause j is death and cause i is withdrawal, then "independence of decrements" implies that:

"the mortality of active pension scheme members is equal to that of pension scheme members who have withdrawn."

This means that any change in $(a\mu)_x^w$ would not affect $(a\mu)_x^d$.

So a consequence of the independence of decrements is that the decrements are non-selective – they do not alter the decrement experience of those "left behind". In practice this may not be true, especially if we are considering populations of life assurance policyholders, for example, where it is very likely that people who lapse or surrender their policies would have lower than average mortality. However, the independence assumption is required to make the theory more tractable.

So, **there are a number of theoretical shortcomings of these assumptions but they are beyond the scope of this course and application of these assumptions provides us with a viable working model for actuaries in practice.**

The sum of all the (dependent) forces of mortality is denoted by:

$$(a\mu)_x = \sum_{all\ j} (a\mu)_x^j$$

4.3 Constructing a multiple decrement table

To construct a multiple decrement table, we need to obtain the relevant dependent probabilities $(aq)_x^k$ at each age and for each cause of decrement k (as described in Section 4.4 below).

Once these are obtained we then choose a suitable starting age α for the table, and a suitable value for the radix $(al)_\alpha$.

Then construct:

$$(ad)_x^k = (al)_x \, (aq)_x^k$$

for all k, and

$$(al)_{x+1} = (al)_x - \sum_k (ad)_x^k$$

recursively for all $x = \alpha, \alpha + 1, \ldots$

4.4 *Obtaining dependent probabilities*

The formulae in this section all require the independence of decrements assumption, defined in Section 4.3, to hold.

From the forces of decrement

The most logical starting point is to begin with the relevant forces of decrement, and assume these are constant over single years of age. Formulae such as (1), (2) and (3) of Section 3.3 can then be used to calculate the dependent probabilities directly.

Example

In a certain population, forces of decrement are assumed to be constant over individual years of age.

The following independent forces of decrement will be assumed for this population between the exact ages of 50 and 52:

	Force of decrement for year of age commencing from exact age x	
	due to mortality	due to sickness
Age x		
50	0.011	0.075
51	0.012	0.081

Construct a double decrement table including the two decrements of mortality and sickness, for this population between exact ages 50 and 52, assuming a radix of $(al)_{50} = 100,000$.

Solution

First we need the dependent probabilities of decrement by each cause at each age. We will assume decrements are independent, so that the given forces can be assumed to apply when the two decrements occur together in the same population. We will use $\mu_{\bar{x}}^{j}$ to be the constant force of decrement due to cause j operating over the year of age x to $x+1$, where $j = d$ (death), s (sickness).

From Section 3.3 we have:

$$(aq)_x^j = \frac{\mu_{\bar{x}}^j}{\mu_{\bar{x}}^d + \mu_{\bar{x}}^s}\left(1 - e^{-(\mu_{\bar{x}}^d + \mu_{\bar{x}}^s)}\right)$$

We then obtain:

$$(aq)_{50}^d = \frac{0.011}{0.086}\left(1 - e^{-0.086}\right) = 0.010540$$

$$(aq)_{50}^s = \frac{0.075}{0.086}\left(1 - e^{-0.086}\right) = 0.071865$$

$$(aq)_{51}^d = \frac{0.012}{0.093}\left(1 - e^{-0.093}\right) = 0.011459$$

$$(aq)_{51}^s = \frac{0.081}{0.093}\left(1 - e^{-0.093}\right) = 0.077348$$

To construct the table, we use the radix of $(al)_{50} = 100,000$ and the formulae in Section 4.3, which give us:

Age x	$(al)_x$	$(ad)_x^d$	$(ad)_x^s$
50	100,000	1,054	7,186.5
51	91,759.5	1,051.47	7,097.41
52	83,610.62		

From an existing multiple decrement table

Here we will need to calculate the implied (constant) forces of decrement underlying the existing table. Taking formula (2) from Section 3.3 as an example (where we have decrements of sickness and mortality)**:**

$$\boldsymbol{(aq)_x^s = \frac{\sigma}{\mu+\sigma}\left(1-e^{-(\mu+\sigma)}\right)}$$

$$= \frac{\sigma}{\mu+\sigma}\left(1-(ap)_x\right)$$

$$\boldsymbol{= \frac{\sigma}{\mu+\sigma}(aq)_x}$$

Rearranging:

$$\boldsymbol{\Rightarrow \sigma = \frac{(aq)_x^s}{(aq)_x}(\mu+\sigma)}$$

$$\boldsymbol{= \frac{(aq)_x^s}{(aq)_x}\left(-\ln(ap)_x\right)}$$

where:

$$(aq)_x = (aq)_x^s + (aq)_x^d$$

$$= \sum_{\text{all } j} (aq)_x^j$$

Example

We now wish to extend the multiple decrement table we constructed in the previous example to include decrements of withdrawal (w). It is believed that the independent forces of withdrawal will conform to those underlying the withdrawal decrement in the multiple decrement table given in Question 10.10. Calculate the first line of the triple decrement table (*ie* between ages 50 and 51) incorporating the three decrements d , s and w . You should assume that the forces of sickness and mortality are unchanged.

Solution

We will first need the independent force of withdrawal from the table in Question 10.10. We shall use:

$$\mu_x^w = \frac{(aq)_x^w}{(aq)_x}\left[-\ln(ap)_x\right] = \frac{(ad)_x^w}{(ad)_x^d + (ad)_x^w}\left[-\ln\left(\frac{(al)_{x+1}}{(al)_x}\right)\right]$$

So, for $x = 50$:

$$\mu_{50}^w = \frac{2,490}{175+2,490}\left[-\ln\left(\frac{97,335}{100,000}\right)\right] = 0.025238$$

Indicating the new triple-decrement functions with "*b*" rather than "*a*" prefixes, the new dependent probabilities for age 50 now become:

$$(bq)_{50}^j = \frac{\mu_{50}^j}{\mu_{50}^d + \mu_{50}^s + \mu_{50}^w}\left(1 - e^{-(\mu_{50}^d + \mu_{50}^s + \mu_{50}^w)}\right) \qquad j = d, s, w$$

So we have:

$$\mu_{50}^d + \mu_{50}^s + \mu_{50}^w = 0.011 + 0.075 + 0.025238 = 0.111238$$

$$1 - e^{-0.111238} = 0.105274$$

Then:

$$(bq)_{50}^d = \frac{0.011}{0.111238} \times 0.105274 = 0.010410$$

$$(bq)_{50}^s = \frac{0.075}{0.111238} \times 0.105274 = 0.070979$$

$$(bq)_{50}^w = \frac{0.025238}{0.111238} \times 0.105274 = 0.023885$$

And so the first line of the new triple decrement table is:

Age x	$(bl)_x$	$(bd)_x^d$	$(bd)_x^s$	$(bd)_x^w$
50	100,000	1,041	7,097.9	2,388.5
51	89,472.6			

Question 10.13

Calculate the second line of the new triple decrement table.

From existing single decrement tables

Here we will need to calculate the implied (constant) forces of decrement underlying the existing single decrement tables. Taking formula (4) from Section 3.3 as an example:

$$q_x^s = 1 - e^{-\sigma}$$

$$\Rightarrow \sigma = -\ln\left(1 - q_x^s\right)$$

In all cases, the forces of decrement obtained may need to be adjusted before being applied to construct the dependent probabilities for any particular application.

Example

Following on from the previous example, it is now desired to construct a double decrement table for mortality and sickness only, where the forces of sickness are unchanged from before but the independent forces of mortality will be 80% of the forces of mortality according to the ELT15-Males mortality table. As before, all forces of decrement are assumed to be constant over individual years of age. Calculate the first line of the revised double decrement table.

Solution

The required force of mortality is found from:

$$\mu_{50}^{d} = 0.8\left[-\ln\left(1-q_{50}^{d}\right)\right]$$

where q_{50}^{d} is the relevant probability of mortality in the ELT15-Males table. We find that $q_{50}^{d} = 0.00464$, and so:

$$\mu_{50}^{d} = 0.8\left[-\ln\left(1-0.00464\right)\right] = 0.003721$$

Using the prefix "*c*" to indicate the new multiple decrement model functions, we have:

$$(cq)_{50}^{d} = \frac{0.003721}{0.003721+0.075}\left(1-e^{-(0.003721+0.075)}\right) = 0.003578$$

$$(cq)_{50}^{s} = \frac{0.075}{0.078721}\times\left(1-e^{-0.078721}\right) = 0.072124$$

So the first line of the revised double decrement table is:

Age x	$(cl)_x$	$(cd)_x^d$	$(cd)_x^s$
50	100,000	357.8	7,212.4
51	92,429.8		

These formulae can also be used to construct single decrement tables from existing multiple decrement tables, if desired. The key is always to define any existing tables in terms of their underlying forces of decrement, and then apply these forces to calculate the dependent or independent probabilities as required.

The second exam-style question in Section 6 is an example of this.

There is a direct relationship between the independent and dependent probabilities, which is sometimes useful.

It works as follows. If we have n decrements, labelled $j = 1, 2, ...n$:

$$
\begin{aligned}
{}_t(ap)_x &= \exp\left[-\int_{s=0}^{t} (a\mu)_{x+s} \, ds \right] \\
&= \exp\left[-\int_{s=0}^{t} \sum_{j=1}^{n} (a\mu)_{x+s}^{j} \, ds \right] \\
&= \prod_{j=1}^{n} \exp\left[-\int_{s=0}^{t} (a\mu)_{x+s}^{j} \, ds \right]
\end{aligned}
$$

Assuming $(a\mu)_{x+s}^{j} = \mu_{x+s}^{j}$ for all j and s, as usual, then:

$$
\begin{aligned}
{}_t(ap)_x &= \prod_{j=1}^{n} \exp\left[-\int_{s=0}^{t} \mu_{x+s}^{j} \, ds \right] \\
&= \prod_{j=1}^{n} {}_t p_x^{j}
\end{aligned}
$$

So we can obtain the overall dependent probability of remaining in the active state, by multiplying all the independent "not-leaving" probabilities for all causes together.

4.5 *Integral formulae for multiple decrement probabilities*

We can also obtain expressions for multiple decrement probabilities without making the assumption that forces of decrement are constant over each year of age.

For example, the forward differential equations of Section 3.3 for ${}_t(aq)_x^s$ would become:

$$
\frac{\partial}{\partial t} {}_t(aq)_x^s = {}_t(ap)_x \, \sigma_{x+t} = {}_t(ap)_x \, (a\mu)_{x+t}^s
$$

Integrating over $t = 0$ to $t = 1$ we obtain:

$$
{}_1(aq)_x^s - {}_0(aq)_x^s = \int_{t=0}^{1} {}_t(ap)_x \, (a\mu)_{x+t}^s \, dt
$$

As ${}_0(aq)_x^s = 0$:

$$(aq)_x^s = \int_{t=0}^{1} {}_t(ap)_x \, (a\mu)_{x+t}^s \, dt$$

Question 10.14

Write down the equivalent integral expression for $(aq)_x^d$.

5 Using multiple decrement tables to evaluate expected present values of cashflows

The approach to calculating EPVs of cashflows that are contingent on multiple decrements is similar to the single decrement case described in earlier units.

5.1 Continuous approach

Integrals can be used to express the expected present value of cashflows when these are continuously contingent on multiple decrements.

Example

An employee benefits package pays 50,000 immediately on the death of an employee while in employment. The employee population is assumed to be subject to multiple decrements, of which death in employment is one. All benefits (including death benefits) under the package cease on the 70th birthday.

Write down an integral expression to represent the expected present value (EPV) of the death benefits to an employee who is currently aged exactly 48.

Solution

The payment of 50,000 needs to be discounted back from the moment of death. Writing time t to be the moment of death, we have:

$$EPV = \int_{t=0}^{22} 50{,}000\, v^t \, {}_t(ap)_{48} \, (a\mu)^d_{48+t} \, dt$$

We integrate over $[0 < t < 22]$ because no benefit is payable after 22 years (which is when the member attains age 70).

5.2 Discrete approach

As usual, the basic approach is summarised (non-rigorously) as:

$$EPV = \sum \{\text{amount}\} \times \{\text{discount}\} \times \{\text{probability}\}$$

where the summation is over all future payment periods. In the case of multiple decrements, the probability will be the relevant dependent probability that the cashflow will occur.

Example

An endowment assurance pays 10,000 in three years' time or immediately on earlier death of a life aged 50 at entry. On surrender at any time, a surrender value equal to 75% of premiums paid by the date of surrender (without interest) would be payable. Surrender payments are assumed to occur immediately at the time of surrender. A level annual premium of 3,000 is paid at the start of each year.

Calculate the expected present value of the death, maturity and surrender benefits for a single policy at outset, using the following assumptions:

- **mortality: AM92 Ultimate**
- **annual force of decrement due to surrender: 0.1 in year 1, and 0.05 in each of years 2 and 3**
- **interest: 3% per annum compound**

State any further assumptions you make.

Solution

We will require a multiple decrement model with decrements of surrender and death (represented by s and d respectively).

Assuming that decrements occur half way through each year of age, on average, the EPV of the death and maturity benefits is:

$$EPV_{D,M} = 10{,}000\left\{v^{½}\,{}_{0|}(aq)_{50}^{d} + v^{1½}\,{}_{1|}(aq)_{50}^{d} + v^{2½}\,{}_{2|}(aq)_{50}^{d} + v^{3}\,{}_{3}(ap)_{50}\right\} \qquad (*)$$

$$= 10{,}000\left\{\frac{v^{½}(ad)_{50}^{d} + v^{1½}(ad)_{51}^{d} + v^{2½}(ad)_{52}^{d} + v^{3}(al)_{53}}{(al)_{50}}\right\} \qquad (**)$$

The EPV of the surrender benefits is:

$$EPV_S = 3{,}000\left\{v^{\frac{1}{2}}\ {}_{0|}(aq)_{50}^{s} + 2v^{1\frac{1}{2}}\ {}_{1|}(aq)_{50}^{s} + 3v^{2\frac{1}{2}}\ {}_{2|}(aq)_{50}^{s}\right\}\times 0.75 \qquad (*)$$

$$= 3{,}000\left\{\frac{v^{\frac{1}{2}}(ad)_{50}^{s} + 2v^{1\frac{1}{2}}(ad)_{51}^{s} + 3v^{2\frac{1}{2}}(ad)_{52}^{s}}{(al)_{50}}\right\}\times 0.75 \qquad (**)$$

We can either use the dependent probabilities directly, using (*), or we can construct a multiple decrement table and then use (**).

Calculating EPVs using probabilities directly

The required values are shown in the following table:

Age x Year t	50 1	51 2	52 3
μ^d	0.002511	0.002813	0.003157
μ^s	0.1	0.05	0.05
$(aq)_x$	0.097432	0.051443	0.051769
$(aq)_x^d$	0.002387	0.002740	0.003075
$(aq)_x^s$	0.095045	0.048703	0.048694
$(ap)_x$	0.902568	0.948557	0.948231
${}_t(ap)_x$	0.902568	0.856137	0.811816

where:

$$\mu^d = -\ln\left(1 - q_x^d\right)$$

$$(aq)_x = 1 - e^{-(\mu^d + \mu^s)}$$

$$(aq)_x^d = \frac{\mu^d}{\mu^d + \mu^s}(aq)_x$$

$$(aq)_x^s = (aq)_x - (aq)_x^d$$

$$(ap)_x = 1-(aq)_x$$

$$_t(ap)_x = \prod_{r=0}^{t-1}(ap)_{x+r}$$

Using (*):

$$EPV_{D,M} = 10{,}000\left\{\frac{0.002387}{1.03^{½}}+\frac{0.902568\times 0.002740}{1.03^{1½}}+\frac{0.856137\times 0.003075}{1.03^{2½}}+\frac{0.811816}{1.03^{3}}\right\}$$

$$= 7{,}500.89$$

$$EPV_S = 3{,}000\left\{\frac{0.095045}{1.03^{½}}+\frac{2\times 0.902568\times 0.048703}{1.03^{1½}}+\frac{3\times 0.856137\times 0.048694}{1.03^{2½}}\right\}\times 0.75$$

$$= 661.30$$

Calculating EPVs using a multiple decrement table

Using (**) we first need to construct the multiple decrement table. We choose an arbitrary radix of $(al)_{50} = 100{,}000$:

Age x	$(al)_x$	$(ad)_x^d$	$(ad)_x^s$
50	100,000	238.7	9,504.5
51	90,256.8	247.3	4,395.8
52	85,613.7	263.3	4,168.9
53	81,181.5		

using:

$$(ad)_x^d = (al)_x \times (aq)_x^d$$
$$(ad)_x^s = (al)_x \times (aq)_x^s$$
$$(al)_{x+1} = (al)_x - (ad)_x^d - (ad)_x^s$$

Then:

$$EPV_{D,M} = \frac{10,000}{100,000}\Big\{1.03^{-\frac{1}{2}} \times 238.7 + 1.03^{-1\frac{1}{2}} \times 247.3$$

$$+1.03^{-2\frac{1}{2}} \times 263.3 + 1.03^{-3} \times 81,181.5\Big\}$$

$$= 7,500.89$$

$$EPV_S = \frac{3,000}{100,000}\Big\{1.03^{-\frac{1}{2}} \times 9,504.5 + 2 \times 1.03^{-1\frac{1}{2}} \times 4,395.8$$

$$+3 \times 1.03^{-2\frac{1}{2}} \times 4,168.9\Big\} \times 0.75$$

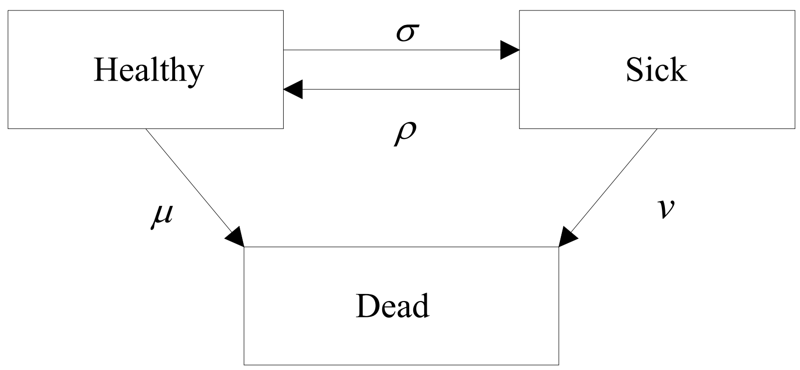
$$= 661.30$$

6 Exam-style questions

Here are two past exam questions. They are typical of questions on this topic.

Question 10.15 ***(Subject CT5, September 2005, Question 7)***

A life insurance company prices its long-term sickness policies using the following three-state continuous-time Markov model, in which the forces of transition σ, ρ, μ and ν are assumed to be constant:



The company issues a particular long-term sickness policy with a benefit of £10,000 per annum payable continuously while sick, provided that the life has been sick continuously for at least one year. Benefit payments under this policy cease at age 65 exact.

Write down an expression for the expected present value of the sickness benefit for a healthy life aged 20 exact. Define the symbols that you use. [5]

Question 10.16 ***(Subject CT5, April 2010, Question 10, adapted)***

The decrement table extract below is based on the historical experience of a very large multinational company's workforce:

Age x	Number of employees $(al)_x$	Deaths $(ad)_x^d$	Withdrawals $(ad)_x^w$
40	10,000	25	120
41	9,855	27	144
42	9,684		

Recent changes in working conditions have resulted in an estimate that the annual independent force of withdrawal is now 75% of that previously used.

Calculate a revised table assuming no changes to the independent forces of mortality, stating your results to one decimal place. [7]

Chapter 10 Summary

Multiple-state models

Definitions

$\mu_x^{ij} =$ force of transition (transition intensity) from state i to state j at exact age x

${}_t p_x^{ij} =$ probability of being in state j at age $x+t$ given in state i at age x

${}_t p_x^{\overline{ii}} =$ probability of staying continuously in state i between ages x and $x+t$ given in state i at age x

$$= \exp\left(-\int_0^t \sum_{j \neq i} \mu_{x+s}^{ij} ds \right)$$

Valuing cashflows

A lump sum benefit of S payable immediately on transition from state i to state j, for a life currently in state a, has expected present value:

$$S \int_0^\infty v^t \, {}_t p_x^{ai} \, \mu_{x+t}^{ij} \, dt$$

An income benefit of B *pa* payable continuously while in state j, for a life currently in state a, has expected present value:

$$B \int_0^\infty v^t \, {}_t p_x^{aj} \, dt \qquad \text{or alternatively:} \qquad B \int_0^\infty v^t \sum_i {}_t p_x^{ai} \, \mu_{x+t}^{ij} \, \bar{a}_{x+t}^{\overline{jj}} \, dt$$

This last form can be adjusted if necessary to allow for certain conditions, *eg* no payments during the first year of being continuously in state j.

Integral expressions can be evaluated using numerical techniques, such as the trapezium rule.

Multiple-decrement models

A multiple-decrement model is appropriate if there is one active state, out of which transitions occur into one or more absorbing exit states only.

Dependent and independent probabilities

The dependent probability $(aq)_x^{\alpha}$ is the probability that a life aged x in a particular state will be removed from that state within a year by the decrement α, in the presence of all other decrements in the population.

The independent probability q_x^{α} is the probability that a life aged x in a particular state will be removed from that state within a year by the decrement α, where α is the only decrement acting on the population.

Results for the sickness-death (2 decrement) model

If the active state A is subject to decrements to exit states D and S, with constant forces of transition μ and σ over integer ages, then the independent probabilities are given by:

$$q_x^d = 1 - e^{-\mu} \quad \text{and} \quad q_x^s = 1 - e^{-\sigma}$$

The dependent probabilities are given by:

$$(aq)_x^d = p_x^{AD} = \frac{\mu}{\mu+\sigma}\left(1 - e^{-(\mu+\sigma)}\right) \quad \text{and} \quad (aq)_x^s = p_x^{AS} = \frac{\sigma}{\mu+\sigma}\left(1 - e^{-(\mu+\sigma)}\right)$$

The probability of remaining in the active state is:

$$(ap)_x = 1 - (aq)_x^d - (aq)_x^s = e^{-(\mu+\sigma)}$$

Given dependent probabilities, the forces of transition can be obtained as:

$$\mu = \frac{(aq)_x^d}{(aq)_x}\left(-\ln(ap)_x\right) \quad \text{and} \quad \sigma = \frac{(aq)_x^s}{(aq)_x}\left(-\ln(ap)_x\right)$$

where $(aq)_x = (aq)_x^d + (aq)_x^s$.

Independence of decrements

The "independence of decrements" equation is $(a\mu)_x^j = \mu_x^j$ from which it follows that:

$$(ap)_x = p_x^1\, p_x^2\, p_x^3 \ldots p_x^m$$

where there are *m* causes of decrement $j = 1, 2, \ldots m$, and $p_x^j = 1 - q_x^j$.

The dependent probability expressed in integral form is:

$$(aq)_x^j = \int_0^1 {}_t(ap)_x\, (a\mu)_{x+t}^j\, dt$$

$$= \int_0^1 {}_t(ap)_x\, \mu_{x+t}^j\, dt$$

(the second expression is true only if the independence of decrements assumption holds).

Multiple-decrement tables

Definitions

$(al)_x =$ expected number of active lives at exact age x

$(ad)_x^j =$ expected number of decrements due to cause j over the year of age x to $x+1$, given a radix of $(al)_\alpha$ lives active at age α.

Calculating probabilities

$$(aq)_x^j = \frac{(ad)_x^j}{(al)_x} \qquad {}_n(ap)_x = \frac{(al)_{x+n}}{(al)_x}$$

$$ {}_n(aq)_x^j = \frac{\sum_{r=0}^{n-1} (ad)_{x+r}^j}{(al)_x} \qquad {}_{n|}(aq)_x^j = \frac{(ad)_{x+n}^j}{(al)_x}$$

Valuing cashflows

Cashflows can be evaluated either using integrals (as for multiple state models), or using a discrete, annual summation approach:

$$EPV = \sum \{\text{amount}\} \times \{\text{discount}\} \times \{\text{probability}\}$$

Construction of multiple decrement tables

$$(ad)_x^j = (al)_x \, (aq)_x^j$$

$$(al)_{x+1} = (al)_x - \sum_{j=1}^{m} (ad)_x^j$$

where there are *m* causes of decrement $j = 1, 2, \ldots m$.

Chapter 10 Solutions

Solution 10.1

Since the policyholder is healthy at age 60, the probability that he is sick at age 62 is:

$${}_2p_{60}^{HS} = \left(p_{60}^{HH} \times p_{61}^{HS}\right) + \left(p_{60}^{HS} \times p_{61}^{SS}\right) = (0.9 \times 0.08) + (0.08 \times 0.25) = 0.092$$

and the probability that he is dead at age 62 is:

$$\begin{aligned} {}_2p_{60}^{HD} &= p_{60}^{HD} + \left(p_{60}^{HH} \times p_{61}^{HD}\right) + \left(p_{60}^{HS} \times p_{61}^{SD}\right) \\ &= 0.02 + (0.9 \times 0.02) + (0.08 \times 0.05) \\ &= 0.042 \end{aligned}$$

Note that $p_{60}^{HH} + p_{60}^{HS} + p_{60}^{HD} = 1$*, and a similar equation holds for lives starting from the sick state.*

Solution 10.2

Beginning with the differential equation:

$$\frac{\partial}{\partial t}\, {}_tp_x^{\overline{ii}} = -\,{}_tp_x^{\overline{ii}} \sum_{j \neq i} \mu_{x+t}^{ij}$$

we can write:

$$\frac{\partial}{\partial s} \ln {}_sp_x^{\overline{ii}} = \left(\frac{\partial}{\partial s}\, {}_sp_x^{\overline{ii}}\right) \Big/ {}_sp_x^{\overline{ii}} = -\sum_{j \neq i} \mu_{x+s}^{ij}$$

Integrating both sides with respect to s between the limits of $s = 0$ and $s = t$ gives:

$$\left[\ln {}_sp_x^{\overline{ii}}\right]_0^t = -\int_0^t \sum_{j \neq i} \mu_{x+s}^{ij}\, ds$$

Finally, since $\ln {}_0p_x^{\overline{ii}} = \ln 1 = 0$, we have the result:

$$ {}_tp_x^{\overline{ii}} = \exp\left(-\int_0^t \sum_{j \neq i} \mu_{x+s}^{ij}\, ds\right) $$

Solution 10.3

(i) $3{,}000\int_0^{30} v^t\, {}_tp_{30}^{HS}\, dt$

Note that in all of these solutions v^t could be replaced by $e^{-\delta t}$.

(ii) $3{,}000\int_0^{30} v^t\, {}_tp_{30}^{\overline{HH}}\, \sigma_{30+t}\left(\int_0^{30-t} v^s\, {}_sp_{30+t}^{\overline{SS}}\, ds\right)dt$

This can be reasoned as follows. The probability that the policyholder falls sick for the first time at age $30+t$ is ${}_tp_{30}^{\overline{HH}}\, \sigma_{30+t}$. Given that this has occurred, the policyholder will receive sickness benefit until he recovers or reaches age 60, whichever is earlier. So the expected value at time t of this benefit is $3{,}000\int_0^{30-t} v^s\, {}_sp_{30+t}^{\overline{SS}}\, ds$. Discounting back to time 0 and integrating over all possible values of t gives the required result.

(iii) $3{,}000\int_0^{29} v^t\, {}_tp_{30}^{HH}\, \sigma_{30+t}\left(\int_1^{30-t} v^s\, {}_sp_{30+t}^{\overline{SS}}\, ds\right)dt$

This can be reasoned in a similar way. The probability that the policyholder falls sick at time t is ${}_tp_{30}^{HH}\, \sigma_{30+t}$. Given that this occurs, the policyholder will receive benefit if he is still sick at age $30+t+s$, where $1 \le s < 30-t$. The expected value at time t of the benefit payable in respect of this bout of sickness is $3{,}000\int_1^{30-t} v^s\, {}_sp_{30+t}^{\overline{SS}}\, ds$. Discounting back to time 0 and integrating over all possible values of t gives the required result. Note that if the policyholder falls sick after age 59, there will be no benefit payable in respect of this bout of sickness.

You might like to think about how these formulae should be adapted in slightly different situations, eg a different age at which benefits cease or a different duration of sickness at which benefits start.

Solution 10.4

(a) $(aq)_x^s = {}_1p_x^{HS}$

$(aq)_x^s$ is the probability that a healthy life aged x will leave the healthy state due to sickness during the next year. ${}_1p_x^{HS}$ is the probability that a life who is healthy at age x, will be sick at age $x+1$.

${}_1p_x^{HS}$ will be smaller than $(aq)_x^s$, because some of the lives who become sick during the year go on to die during the same year, and so are not present in State S at age $x+1$.

(b) $(aq)_x^d = {}_1p_x^{HD}$

$(aq)_x^d$ is the probability that a healthy life aged x will leave the healthy state due to death during the next year. ${}_1p_x^{HD}$ is the probability that a life who is healthy at age x, will be dead by age $x+1$.

${}_1p_x^{HD}$ will be larger than $(aq)_x^d$, because some of those who start the year healthy and who end the year dead, will have become sick and then died from sick during the same year. So this will include some lives who leave the healthy state through sickness, as well as all those who leave the healthy state directly through death.

(c) $(ap)_x = {}_1p_x^{HH}$

$(ap)_x$ is the probability that a healthy life aged x stays healthy until at least age $x+1$. ${}_1p_x^{HH}$ is the probability that a life who is healthy at age x, is also healthy at age $x+1$.

Because (in *this* model) it is impossible to return to State H once the state has been left, then ${}_1p_x^{HH}$ also implies that the life remains in H for at least one year, and so these two probabilities are the same.

Solution 10.5

As in Question 10.4, $(ap)_x$ means the probability of staying (continuously) healthy for at least one year, and ${}_1p_x^{HH}$ means the probability of a life, who is healthy at age x, being also healthy one year later. However, these are now no longer the same: in the general healthy-sick-dead model the lives are able to leave healthy and go back again during the same year, which means that:

$$(ap)_x < {}_1p_x^{HH}$$

Actually, $(ap)_x$ is equal to the occupancy probability, ${}_1p_x^{\overline{HH}}$, in this case.

Solution 10.6

q_x^d is the probability of a life aged x dying between ages x and $x+1$, where death is the only cause of decrement occurring.

$(aq)_x^d$ is the probability of a life aged x leaving the population directly through death, when withdrawals are also taking place. This probability is lower than it would be if death were the only decrement, because lives may withdraw before they die during the same year.

Solution 10.7

${}_tp_x^{\overline{ii}}$ is the probability that a life stays continuously in state i from age x to age $x+t$, given it was in state i at age x.

$\sum_{j \neq i} \mu_{x+s}^{ij}$ is the sum of all the forces of transition *out* of state i, at exact age $x+s$. This total force of leaving state i is then integrated over the age range x to $x+t$; the exponential of its negative value then gives the required probability.

Solution 10.8

We carry out an investigation involving lives between the ages of x and $x+1$ and record:

$n_{AS} =$ the number of transitions from active to sick

$n_{AD} =$ the number of transitions from active to dead

$T_A =$ the total time spent by the lives in the active state.

Then σ is estimated by $\dfrac{n_{AS}}{T_A}$ and μ is estimated by $\dfrac{n_{AD}}{T_A}$.

Note: this knowledge is examined in Subject CT4 and is unlikely to be examined again in Subject CT5.

Solution 10.9

(i) ***Probabilities***

(a) The probability of staying healthy is:

$$(ap)_{50} = e^{-(0.08+0.002)} = 0.921272$$

(b) The dependent probability of leaving the population through death is:

$$(aq)_{50}^{d} = \frac{0.002}{0.082}\left(1-e^{-0.082}\right) = 0.001920$$

(c) The independent probability of dying is:

$$q_x^d = 1-e^{-0.002} = 0.001998$$

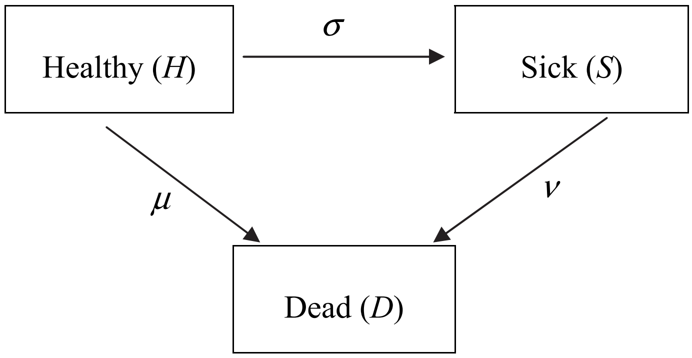
(ii) ***Why (i)(c) is not realistic***

The calculation in (i)(c) is unlikely to equal the probability of a healthy life dying over the year, because it makes the implicit assumption that the force of mortality is the same for a sick life as for a healthy life.

A person who becomes sick during the year is likely to have a considerably higher force of mortality for the rest of that year, after becoming sick, and so the actual probability of a healthy life dying by the end of the year should be higher.

The double decrement model we have used is therefore adequate for calculating probabilities such as (i)(a) and (b), but not for a probability such as (i)(c).

A more suitable model for the latter would be the following multiple state model:



Where mortality is assumed to be different for healthy and sick lives.

Solution 10.10

(a) $$(ap)_{51} = \frac{(al)_{52}}{(al)_{51}} = \frac{94,995}{97,335} = 0.975959$$

(b) $$(aq)_{51}^{d} = \frac{(ad)_{51}^{d}}{(al)_{51}} = \frac{180}{97,335} = 0.001849$$

(c) $${}_2(aq)_{50}^{w} = \frac{(ad)_{50}^{w} + (ad)_{51}^{w}}{(al)_{50}} = \frac{2,490 + 2,160}{100,000} = 0.0465$$

(d) $${}_2(ap)_{50} = \frac{(al)_{52}}{(al)_{50}} = \frac{94,995}{100,000} = 0.94995$$

Solution 10.11

The probability is:

$$_{1|}(aq)_{50}^{d} = \frac{(ad)_{51}^{d}}{(al)_{50}} = \frac{180}{100,000} = 0.0018$$

This is the probability that a person aged exactly 50 leaves the population through death between exact ages 51 and 52.

Solution 10.12

This is the probability of a life aged exactly x remaining in a population over at least one year, when withdrawal is the only way in which people can exit from the population.

Solution 10.13

From the multiple decrement table in Question 10.10 we can calculate:

$$\mu_{51}^{w} = \frac{(ad)_{51}^{w}}{(ad)_{51}^{d} + (ad)_{51}^{w}}\left[-\ln\left(\frac{(al)_{52}}{(al)_{51}}\right)\right]$$

$$= \frac{2,160}{180+2,160}\left[-\ln\left(\frac{94,995}{97,335}\right)\right] = 0.022463$$

For the new table:

$$\mu_{51}^{d} + \mu_{51}^{s} + \mu_{51}^{w} = 0.012 + 0.081 + 0.022463 = 0.115463$$

$$1 - e^{-0.115463} = 0.109046$$

Then:

$$(bq)_{51}^{d} = \frac{0.012}{0.115463} \times 0.109046 = 0.011333$$

$$(bq)_{51}^{s} = \frac{0.081}{0.115463} \times 0.109046 = 0.076499$$

$$(bq)_{51}^{w} = \frac{0.022463}{0.115463} \times 0.109046 = 0.021214$$

The first two years of the table are now as follows (with the new entries indicated in bold):

Age x	$(bl)_x$	$(bd)_x^d$	$(bd)_x^s$	$(bd)_x^w$
50	100,000	1,041	7,097.9	2,388.5
51	89,472.6	**1,013.99**	**6,844.56**	**1,898.07**
52	**79,715.98**			

Solution 10.14

This is:

$$(aq)_x^d = \int_{t=0}^{1} {}_t(ap)_x \, (a\mu)_{x+t}^d \, dt$$

Solution 10.15

Suppose the policyholder, who is now aged 20 *and is healthy, falls sick at age* $20+t$. *The "probability" of being healthy just before age* $20+t$ *and falling sick at age* $20+t$ *is:*

$${}_t p_{20}^{HH} \, \sigma_{20+t}$$

We're not saying that the policyholder is continuously healthy between the ages of 20 *and* $20+t$. *If we restricted ourselves to that case (and used* ${}_t p_{20}^{\overline{HH}}$ *instead of* ${}_t p_{20}^{HH}$*), then we would end up valuing a benefit that is payable only during the first bout of sickness.*

Now we consider the expected present value at age $20+t$ *of the benefit payable in respect of this bout of sickness. It is:*

$$10{,}000 \int_1^{45-t} v^s \, {}_s p_{20+t}^{\overline{SS}} \, ds$$

This integral says that a benefit of 10,000 pa is payable continuously while the policyholder stays sick, and has been sick for between 1 *and* $45-t$ *years, ie between the ages of* $20+t+1$ *and 65.*

Multiplying this by ${}_tp_{20}^{HH}\,\sigma_{20+t}$*, discounting back to time 0 and integrating over all possible times when the life could fall sick and subsequently be entitled to benefit (ie times 0 to 44) gives the expected present value of the benefit. Note that the upper limit on the outer integral is 44 since no benefit is payable in respect of any bout of sickness that starts after age 64.*

The expected present value of the sickness benefit is:

$$10{,}000\int_0^{44} v^t\ {}_tp_{20}^{HH}\ \sigma_{20+t}\left(\int_1^{45-t} v^s\ {}_sp_{20+t}^{\overline{SS}}\,ds\right)dt$$

We are using the following notation here:

$${}_tp_x^{ab} = P\left(\text{in state } b \text{ at age } x+t \mid \text{ in state } a \text{ at age } x\right)$$

$${}_tp_x^{\overline{aa}} = P\left(\text{in state } a \text{ from age } x \text{ to age } x+t \mid \text{ in state } a \text{ at age } x\right)$$

H represents the healthy state

S represents the sick state

$$v = \frac{1}{1+i}$$

i is the valuation rate of interest

Furthermore, since the forces of transition are assumed to be constant, this integral simplifies to:

$$10{,}000\int_0^{44} v^t\ {}_tp_{20}^{HH}\ \sigma\left(\int_1^{45-t} v^s\ {}_sp_{20+t}^{\overline{SS}}\,ds\right)dt$$

Solution 10.16

We first need to calculate the (dependent) forces of decrement from the existing multiple decrement table. Assuming forces of decrement are constant over individual years of age, and that independent and dependent forces are equal, we can use:

$$\mu_{\bar{x}}^j = (a\mu)_{\bar{x}}^j = \frac{(ad)_x^j}{(ad)_x^d + (ad)_x^w}\left[-\ln\left(\frac{(al)_{x+1}}{(al)_x}\right)\right]$$

Using this formula gives the following values:

Age x	$\mu_{\bar{x}}^d$	$\mu_{\bar{x}}^w$
40	0.002518	0.012088
41	0.002764	0.014740

Reducing the forces of withdrawal by 25% gives:

Age x	$*\mu_{\bar{x}}^d$	$*\mu_{\bar{x}}^w$
40	0.002518	0.009066
41	0.002764	0.011055

We now need to obtain the revised dependent probabilities, using:

$$*(aq)_x^j = \frac{*\mu_{\bar{x}}^j}{*\mu_{\bar{x}}^d + *\mu_{\bar{x}}^w}\left(1 - e^{-(*\mu_{\bar{x}}^d + *\mu_{\bar{x}}^w)}\right)$$

This gives:

Age x	$*(aq)_x^d$	$*(aq)_x^w$
40	0.002504	0.009014
41	0.002745	0.010979

Finally using:

$$*(al)_{40} = 10,000$$

$$*(ad)_x^j = *(al)_x \times *(aq)_x^j$$

$$*(al)_{x+1} = *(al)_x - *(ad)_x^d - *(ad)_x^w$$

we obtain the new table as:

Age x	$*(al)_x$	$*(ad)_x^d$	$*(ad)_x^w$
40	10,000	25.0	90.1
41	9,884.8	27.1	108.5
42	9,749.2		

Chapter 11

Pension funds

Syllabus objectives:

(viii) Describe and use methods of projecting and valuing expected cashflows that are contingent upon multiple decrement events.

5. Describe the typical benefit and contribution structures of pension schemes, including:

- *defined contribution schemes*
- *defined benefit (final salary) schemes*

6. Use multiple decrement tables to evaluate expected present values of cashflows dependent upon more than one decrement, including those of pension schemes.

0 Introduction

Many companies provide some form of pension benefit for their employees. This acts as an attractive form of remuneration for the employee. It may also be fiscally advantageous for the company to give some of its reward to workers in the form of pension benefits. The exact form that the benefits take will vary greatly from company to company and from country to country. The design of the benefits is heavily influenced by local regulations, especially fiscal constraints (*ie* government spending and taxation policies).

In this chapter we see how the benefits of a pension scheme can be valued. We mainly consider pension benefits that are related to salary. Alternatively pension benefits can be defined in cash terms (*eg* the company builds up a fund in respect of each employee of 10% of salary, and uses that at retirement to purchase an immediate annuity). This alternative is briefly described, but not covered in great detail in Subject CT5.

1 *Salary-related pensions benefits and contributions*

1.1 *Age retirement*

Pension schemes usually have a fixed age, *eg* 65, at which members normally retire on grounds of age. Sometimes this fixed age is termed normal pension age (NPA). In many schemes members can opt to retire at a range of ages before NPA *eg* 60 to 65. Such retirements are termed early retirements.

The pension entitlement is usually related to the length of pensionable service. Pensionable service is as defined in the scheme rules. Examples are:

- **the number of years of scheme membership at the date of retirement or earlier leaving**
- **the curtate number of years of membership, *ie* only complete years count.**

The pension entitlement for each year of pensionable service is usually related to the level of pensionable salary, *eg* 1/80th of the pensionable salary at date of retirement for each year of service. The increase in the fraction of the pensionable salary that is made for each year of pensionable service is termed the accrual rate.

Pensionable salary is usually defined (in the scheme rules) in one of three ways:

(i) annual rate of salary at retirement, termed final salary

(ii) average annual salary in the last few years (usually 3 to 5 years) before retirement, termed final average salary

(iii) average annual salary during scheme membership, termed career-average salary or lifetime earnings

To mitigate the effects of inflationary increases, salaries are sometimes revalued to the date of retirement using an index of prices or earnings before being averaged.

In addition to, or in place of, part of the pension entitlement some pension schemes provide a lump sum on retirement which is related to the pension entitlement (and thus to salary), *eg* $3\times$ amount of pension.

Example

Typical pension schemes in the UK might offer benefits structured as follows (using *PS* to denote pensionable salary, and *n* to denote years of service):

Company A

A level pension on retirement at age 60 equal to:

$$PS \times \frac{n}{60}$$

where pensionable salary *PS* is defined as total earnings over the year before retirement and years of service n includes months, but is restricted to a maximum of 40 years.

Company B

A pension on retirement at age 60 which increases in line with the national retail prices index (but no more than 5% *pa*) and starts at retirement at an amount of:

$$PS \times \frac{n}{80}$$

where pensionable salary *PS* is defined as the average total earnings over the last five years before retirement and n is the number of years of service completed.

1.2 Ill-health retirement

Schemes usually allow members to retire on grounds of ill-health and receive a pension benefit after a minimum length of scheme service. To prevent selection against other scheme members, entitlement usually depends on evidence of ill-health being provided. Benefits are usually related to salary at the date of ill-health retirement in similar ways to age retirement benefits.

However, pensionable service is usually more generous than under age retirement with years beyond those served in the scheme being credited to the member *eg* actual pensionable service subject to a minimum of 20 years, or pensionable service that would have been completed by normal retirement age.

1.3 Death in service benefit

The dependants of the active members of the scheme who die before NPA usually receive a lump sum at the date of the member's death. This lump sum is usually a multiple of salary (rather than pensionable salary) at the date of death *eg* $\times 2$ or $\times 4$.

Death in service benefits may also include a pension to the widow or widower, perhaps set at 50% of what the member would have accrued on survival to normal pension age, but based on pensionable salary at death.

1.4 Benefits on withdrawal

Members who leave service before NPA may be entitled to:

- **a refund of their own contributions, or**
- **a deferred pension payable from NPA based on service to the date of leaving**

The valuation of deferred benefits is outside the scope of this course.

Example

So the overall picture for the pension scheme benefits offered by company A in the previous example might be:

Normal age retirement

A level pension on retirement at age 60 equal to:

$$PS \times \frac{n}{60}$$

where PS is defined as total earnings over the year before retirement and years of service n includes months, but is restricted to a maximum of 40 years.

On death in retirement, the pension continues for any surviving spouse (contingent on being the spouse at retirement) but at a reduced 50% level.

Ill-health retirement

A level pension equal to $PS \times \dfrac{m}{60}$ where m is the sum of years of service n plus half of the time remaining to serve to age 60. On death in retirement, conditions as above apply.

Death in service

A cash lump sum equal to four times PS (where PS in this case is the pensionable salary as at the moment of death).

Withdrawal from service

On withdrawal with less than two full years of service, a refund of the member's total contributions paid into the scheme.

On withdrawal with two or more full years of service, a preserved annual pension payable from age 60 equal to:

$$PS \times \frac{n}{60}$$

where PS is now defined as total earnings over the year before withdrawal, and n is the number of years (including fractions of a year) of service completed by the date of withdrawal. The amount of pension is re-valued in deferment at the lower of a specified published inflation index and 3% *pa*.

1.5 Scheme contributions

Members usually pay a fixed percentage of their salary each month *ie* effectively continuously during membership, *eg* 5%, 0% (a non-contributory scheme). The employer pays a regular contribution equal to a percentage of the total salaries of members. This contribution is designed to meet the balance of the cost of the scheme. It may be varied from time to time in order to satisfy this objective. The employer may also make special additional payments over a limited period to fund one-off shortfalls in a scheme.

As we mentioned before, everything else being equal, companies provide pension benefits as a way of remunerating employees. The lower the members' contribution rate, the more generous the scheme and so (possibly) the better will be the quality of recruitment, worker morale *etc*; but the company will then find the scheme much more expensive to maintain.

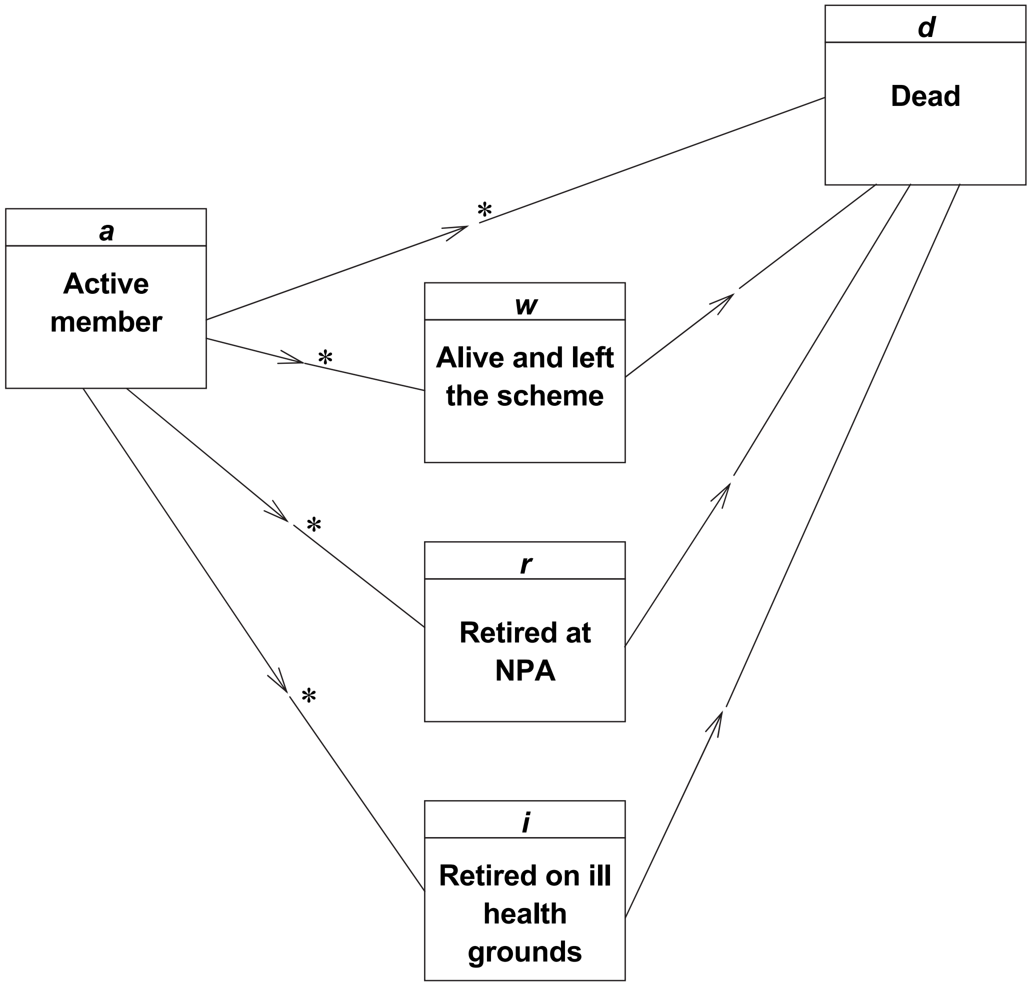
It is common for the employees' rate to be fixed leaving the company's rate variable. This could end up very expensive for the company, especially in the event of falling interest rates.

Question 11.1

So why do we not have schemes with fixed company contribution rates, leaving the members' rates variable?

2 Multiple decrement service table for pensions calculations

Consider the following multiple state model.



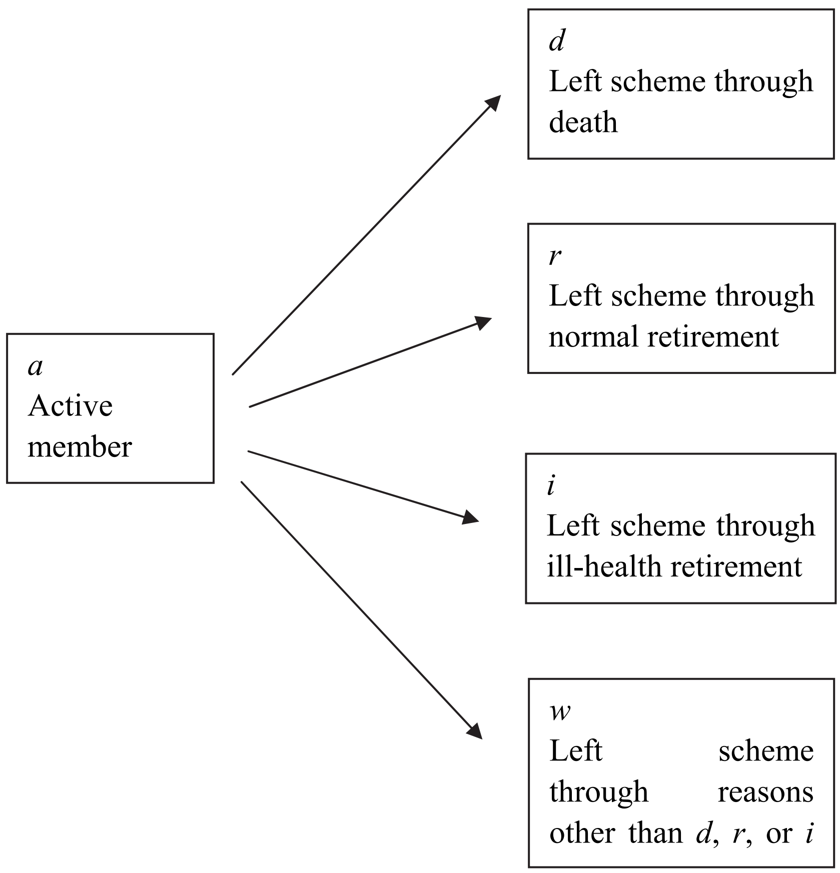
Question 11.2

Why will we need to distinguish ill-health retirements from purely age-related retirements?

This model represents the states and transitions for the members of a typical pension scheme, where the benefit provided on withdrawal is a refund of contributions. Contributions are paid while members are in the "active" state, benefits are paid when members move between states, *eg* return of contributions on leaving, death in service benefit, or while members remain in a particular state, *eg* age retirement and ill-health retirement pensions.

To value contributions and benefits it is necessary to be able to calculate the probabilities of transfer between particular states. For the active members of a scheme these transition probabilities can be presented in the form of a multiple decrement table, with decrements of death, withdrawal, normal retirement and ill-health retirement. Such a table for the active members of a pension scheme is usually termed a *service table*.

The multiple decrement model, showing the four decrements, is:



The required decrement probabilities are:

$$(aq)_x^d,\ (aq)_x^r,\ (aq)_x^i,\ (aq)_x^w$$

ie the dependent probabilities of decrement as we defined in the previous chapter.

The service table (*ie* the corresponding multiple decrement table) **for the active lives is calculated in the usual way, starting from an assumed radix (for example at age 20) based on the underlying assumed forces of decrement.**

For ease of notation, it is conventional to abbreviate the symbols used for the elements of the service table to the following:

$$l_x = (al)_x\,,\;\; d_x = (ad)_x^d\,,\;\; w_x = (ad)_x^w\,,\;\; r_x = (ad)_x^r\,,\;\; i_x = (ad)_x^i$$

The service table would then be displayed as:

$$x \qquad l_x \qquad w_x \qquad d_x \qquad i_x \qquad r_x$$

These are calculated from the probabilities in the usual way, as follows:

$$d_x = l_x\,(aq)_x^d\,,\quad r_x = l_x\,(aq)_x^r\,,\quad i_x = l_x\,(aq)_x^i\,,\quad w_x = l_x\,(aq)_x^w$$

$$l_{x+1} = l_x - d_x - r_x - i_x - w_x$$

Question 11.3

Explain how r_x and d_x, calculated using the multiple-decrement model, will be different from the r_x and d_x values calculated using the multiple state model defined at the start of this section.

It is worth clarifying exactly what we mean by each of the symbols we use (in the context of a pension scheme). The precise definitions are:

Function	*Description*
l_x	Number of active members aged exactly x
r_x	Number of age retirements between exact ages x and $x+1$ ($x <$ NPA) Number of age retirements at exact age x ($x =$ NPA)
i_x	Number of ill health retirements between exact ages x and $x+1$
d_x	Number of deaths in service between exact ages x and $x+1$
w_x	Number of withdrawals between exact ages x and $x+1$

An example of a service table for a hypothetical pension scheme is given in the "*Formulae and Tables for Examinations*".

You will see this service table on Page 142. This table also contains the salary index s_x, which is explained later.

Question 11.4

Using the service table in the *Tables:*

(i) At what ages is normal retirement possible?

(ii) What are the possible decrements for a 40-year old active member?

(iii) How many new entrants do we expect at age 30?

Question 11.5

Calculate the probabilities of the following events using the service table in the *Tables*:

(i) A member aged 40 exactly will die from service during the next year.

(ii) A member aged 45 exactly will withdraw from service by age 50.

(iii) A member aged 38 exactly will retire through ill health in the year ending on his 60th birthday.

(iv) A member aged 42 exactly will retire in normal health on his 65th birthday.

The valuations connected with a pension scheme will also need separate life tables for age retirement pensioners, ill-health retirement pensioners and deferred pensioners.

These are needed so that we can value the income streams (pensions) payable after these members have retired.

3 Expected present values of fixed benefit payments

Calculating expected present values (EPVs) of pension scheme benefits and contributions allows us to check on whether the scheme has enough assets to cover its liabilities, and also to determine a suitable level of future contributions. This is therefore a very important aspect of a pension actuary's day-to-day work.

We will start by considering how to calculate expected present values of simple (not salary-related) benefits, before including the complexities of the benefit structures found in practice (which we have just described in Section 1 above).

As with any type of contingent cashflow, we can calculate any EPV using the general construction:

$$EPV = \sum \{\text{amount of payment}\} \times \{\text{discount function}\} \times \{\text{probability of payment}\}$$

In the case of a pension scheme, the amount of payment is determined in accordance with the rules of the scheme, and the probabilities we need are the dependent probabilities of death, retirement, *etc*, that we described in Section 2. That is, we need a service table, typically defined by the functions l_x, w_x, d_x, i_x, and r_x.

3.1 Valuing a lump sum retirement benefit

Consider a lump sum of £10,000 paid immediately on the event of normal retirement of a scheme member, who is currently aged exactly 55. For the sake of this example, we will assume that normal retirement can occur at any age up to 65 inclusive, and that all members who are still active on reaching their 65th birthday retire on that day.

Now:

$$EPV = \sum \{\text{amount of payment}\} \times \{\text{discount function}\} \times \{\text{probability of payment}\}$$

where each element of the sum relates to each possible year of future retirement following the current age of 55. So, the first element relates to the value of the benefit paid if the member retired between exact ages 55 and 56 (*ie* between those two birthdays). We assume that normal retirements occur uniformly over the year of age, so that on average we will be discounting the £10,000 payment by half a year, *ie* its present value will be $10,000\,v^{½}$. We then multiply by the probability of retirement occurring in that year of age.

So we can write down the first element of the EPV sum as:

$$10,000\, v^{\frac{1}{2}} \frac{r_{55}}{l_{55}}$$

Question 11.6

Without looking ahead, write down the second element of this EPV summation formula.

Continuing in this way, we obtain:

$$EPV = 10,000\left[v^{\frac{1}{2}} \frac{r_{55}}{l_{55}} + v^{1\frac{1}{2}} \frac{r_{56}}{l_{55}} + \cdots + v^{9\frac{1}{2}} \frac{r_{64}}{l_{55}} + v^{10} \frac{r_{65}}{l_{55}} \right]$$

noting that r_{65} is the expected number of retirements occurring on the exact 65th birthday.

Question 11.7

Explain why the discount function for age 65 is v^{10} rather than $v^{10\frac{1}{2}}$.

Using commutation functions

The calculation of pension EPVs can be aided by the use of commutation functions. These are not included in the standard international actuarial notation, so any symbols we use must be defined as we go along. The idea will become clearer from some examples.

So, we will take the EPV of the lump sum we have just described, and define some suitable commutation functions that we can use. The main step in this process is to *multiply the numerator and denominator of the EPV formula by* v^x, where x is the *current age of the member* (the "valuation age"). So, in our case, we multiply and divide through by v^{55} to obtain:

$$EPV = \frac{10,000}{v^{55}\, l_{55}} \left[v^{55\frac{1}{2}} r_{55} + v^{56\frac{1}{2}} r_{56} + \cdots + v^{64\frac{1}{2}} r_{64} + v^{65} r_{65} \right]$$

We then define symbols to represent all the elements of the expression.

While we could use any symbols for these (provided we define them!), we will adopt the same notation as used in the Pension Scheme Tables. So for the denominator we define:

$$D_x = v^x l_x$$

and for the numerator define:

$$C_x^r = \begin{cases} v^{x+½}\, r_x & x < 65 \\ v^{65}\, r_{65} & x = 65 \end{cases}$$

Note that we have to define the function differently for the two age ranges to accommodate the retirements occurring on the exact 65th birthday.

So the EPV now looks like this:

$$EPV = \frac{10,000}{D_{55}} \sum_{t=0}^{10} C_{55+t}^r$$

We now tidy this up further by defining:

$$M_x^r = \sum_{t=0}^{65-x} C_{x+t}^r$$

and so:

$$EPV = 10,000 \frac{M_{55}^r}{D_{55}}$$

Question 11.8

Calculate the expected present value of a lump sum benefit of £50,000 paid on normal age retirement, for a scheme member aged exactly 52, assuming that this benefit is paid:

(a) regardless of the actual age at retirement

(b) only if the member retires after the 64th birthday.

Use the assumptions underlying the Pension Scheme Tables, and 4% *pa* interest.

From this question note that, in general:

- an M_x^j function will include all decrements by cause j occurring at exact age x and in all future years after age x
- a C_x^j function will include decrements by cause j occurring at age x and in the single year of age beginning at age x only.

Dividing by D_y (where y is the valuation age) will ensure that the correct discounting and survival probabilities are included.

3.2 Lump sums paid on other types of decrement

The other possible decrements are withdrawal, death and ill-health retirement. Assuming, as before, that all remaining active members retire "normally" on the 65th birthday, then we know that:

$$w_{65} = d_{65} = i_{65} = 0$$

Other than this, the formulae used are of identical construction to before. So, for example, the EPV of a lump sum of £100,000 payable immediately on death in service of a member currently aged 44 would be:

$$EPV = 100,000 \frac{M_{44}^d}{D_{44}}$$

where:

$$D_x = v^x \, l_x$$

$$M_x^d = \sum_{t=0}^{64-x} C_{x+t}^d$$

$$C_x^d = v^{x+½} \, d_x$$

noting that the M_x^d function sums to $t = 64 - x$ (not $65 - x$), because there is no C_{65}^d function to include.

Question 11.9

Write down an expression, using commutation functions, for the EPV of a lump sum of £150,000 paid immediately on the event of ill-health retirement, for an active pension scheme member aged exactly 48. Assume that ill-health retirement is only permitted before the member reaches his 60th birthday.

Define all the symbols that you use.

3.3 Valuing a pension

Suppose now our 55 year-old member is entitled to a pension of £10,000 *a year*, payable *from* normal retirement (which again we will assume can occur at any time up to and including the member's 65th birthday).

Consider the first possible year of retirement (between ages 55 and 56), where we assume retirement occurs (on average) at age 55½, as usual. So, if retirement occurs *in* that year, the expected cost of the pension benefits have an equivalent *present* value (as at the average retirement date) of:

$$10{,}000\,\bar{a}^{r}_{55\frac{1}{2}}$$

where $\bar{a}^{r}_{x+\frac{1}{2}}$ is the EPV of an annuity of $1pa$ payable to a person who has just retired at age $x+\frac{1}{2}$ (in normal health) according to the rules of the scheme. (The bar over the a symbol has no special significance other than meaning "according to the rules of the scheme", in particular it does not necessarily imply continuous payments!)

Question 11.10

Describe what you would expect $\bar{a}^{i}_{x+\frac{1}{2}}$ to represent, and explain how you would expect it to differ in value from $\bar{a}^{r}_{x+\frac{1}{2}}$.

We can now construct our total EPV as follows.

$$EPV = \frac{10{,}000}{l_{55}}\left[\bar{a}^{r}_{55\frac{1}{2}}v^{\frac{1}{2}}r_{55} + \bar{a}^{r}_{56\frac{1}{2}}v^{1\frac{1}{2}}r_{56} + \cdots + \bar{a}^{r}_{64\frac{1}{2}}v^{9\frac{1}{2}}r_{64} + \bar{a}^{r}_{65}v^{10}r_{65}\right]$$

Multiplying and dividing by v^{55}:

$$EPV = \frac{10,000}{v^{55} l_{55}} \left[\bar{a}^{r}_{55½} v^{55½} r_{55} + \bar{a}^{r}_{56½} v^{56½} r_{56} + \cdots + \bar{a}^{r}_{64½} v^{64½} r_{64} + \bar{a}^{r}_{65} v^{65} r_{65} \right]$$

Defining:

$$D_x = v^x l_x$$

$$C_x^{ra} = \begin{cases} \bar{a}^{r}_{x+½} \, v^{x+½} \, r_x & x < 65 \\ \bar{a}^{r}_{65} \, v^{65} \, r_{65} & x = 65 \end{cases}$$

then:

$$EPV = \frac{10,000}{D_{55}} \sum_{t=0}^{10} C_{55+t}^{ra} = 10,000 \frac{M_{55}^{ra}}{D_{55}}$$

where:

$$M_x^{ra} = \sum_{t=0}^{65-x} C_{x+t}^{ra}$$

Question 11.11

On retirement due to any reason (*ie* normally or through ill-health) before age 65, a pension scheme provides a pension of £20,000 *pa* paid from the date of retirement for the remainder of life. However, should the member retire *on* the 65th birthday, this annual pension is increased to £25,000 *pa*, also payable for life.

Calculate the EPV of this benefit, for an active member currently aged exactly 57, using the assumptions of the Pension Scheme Tables and 4% *pa* interest.

4 Expected present values of salary-related pensions benefits and contributions

4.1 Classification of benefits

For the purposes of evaluating expected cashflows and determining the present value of liabilities arising from the cashflows it is useful to divide benefits into three types.

An *accrued benefit* (sometimes referred to as past service benefit) is a benefit that has been earned as a result of pensionable service (or credited service) prior to the valuation date *eg* a pension of $n/80$ ths of final average salary where n is the number of years of past pensionable service at the valuation date.

A *future service benefit* is a benefit that is expected to be earned as a result of pensionable service after the valuation date *eg* a pension of $(65-x)/80$ *ths* of final average salary on age retirement at NPA of 65 for a member aged x at the valuation date.

A *prospective service benefit* does not depend on either past or future service, although it may depend on total expected pensionable service, *eg* a pension of $m/80$ ths of final average salary at date of ill-health retirement where m is the total expected pensionable service before NPA. An example of a benefit which is prospective but which does not depend on amount of service would be **a death in service benefit of $4\times$ salary at the date of death.**

In determining expected cashflows and the present values of benefits (and contributions) it is usually most straightforward to determine the components that arise as a result of past service and future service separately, and then combine these to obtain the total expected cashflow or the total present value.

4.2 Salary scales

In pension schemes where future benefits and contributions are directly related to future salary levels the valuation of such benefits and contributions will require the estimation of future salaries. This is achieved using a salary scale.

Salaries are typically assumed to increase with age as a result of "merit" (promotional) increases and with time as a result of "inflationary" increases.

At any time an employer will tend to pay its older staff more than its younger staff to reflect the greater experience, expertise and/or responsibility they usually have. These increases in salary that occur with age (even in a zero-inflation world) are the "merit" increases.

In addition, during calendar year t a person then aged x would expect to earn more than someone the same age would have earned in any previous calendar year, due to inflation (assuming this is positive).

Overall, a person aged x in year t will expect salary increases over the next n years both through promotion (merit increases between ages x and $x+n$), and inflation (inflationary increases between calendar years t and $t+n$).

Using data about past merit increases (obtained by removing inflation from past salary trends) we can build up a salary scale unadjusted for inflation s_x^* such that:

$$\frac{s_{x+t}^*}{s_x^*} = \frac{\text{salary expected to be received in age interval } (x+t,\, x+t+1) \text{ before inflation}}{\text{salary expected to be received in age interval } (x,\, x+1) \text{ before inflation}}$$

Inflationary increases can then be valued by adjusting for an assumed rate of inflation of f per annum, defining an inflation adjusted scale $s_x = (1+f)^x s_x^*$. This inflation adjusted salary scale is used throughout the remainder of this chapter.

Care needs to be taken in using pension scheme service tables to understand what, if any, inflation adjustment has been applied to the salary scale. The table (for s_x) **given in "Formulae and Tables for Examinations" contains both merit and inflation increases.**

Example

Workers at a company have pay reviews on their birthdays and retire on their 60th birthdays. Using the salary scale (including inflation increases) given in the *Tables*, estimate the total earnings in the year before retirement of a worker now aged exactly 30 who was paid a total of £15,000 last year.

Solution

s_{29} is an index representing the worker's pay last year, *ie* in the year starting at exact age 29. Similarly, s_{59} represents the worker's pay in the year before retirement, *ie* in the year starting at exact age 59. So, the required estimate of the worker's pay is:

$$15,000\frac{s_{59}}{s_{29}} = 15,000 \times \frac{10.510}{4.991} = £31,587$$

Example

Workers at a company have a pay review on 1 January each year and retire on their 60th birthdays. Using the salary scale (including inflation increases) given in the *Tables*, estimate the total earnings in the year before retirement of a worker aged exactly 30 on 1 October who is paid an annual rate of salary of £15,000 on that date.

Solution

The salary of £15,000 came into force on 1 January when the worker was aged 29¼. s_{59} again represents the worker's pay in the year before retirement *ie* in the year starting at exact age 59. So, the required estimate of the worker's pay, approximating $s_{29\frac{1}{4}}$ by linear interpolation, is:

$$15,000\frac{s_{59}}{s_{29\frac{1}{4}}} = 15,000 \times \frac{10.510}{0.75 \times 4.991 + 0.25 \times 5.278} = £31,139$$

Unless otherwise stated, we will be referring to the salary scale including inflation increases, throughout this course.

Question 11.12

On 1 October 2003 a pension scheme member was aged exactly 45, and had earned £40,000 over the previous year.

The salary scale s_x is defined such that for a life aged exactly x and for any integer $t > 0$:

$$\frac{s_{x+t}}{s_x} = \frac{\text{expected earnings between ages } x+t \text{ and } x+t+1}{\text{expected earnings between ages } x \text{ and } x+1}$$

Final salary is defined as the earnings received in the year immediately prior to retirement. Salaries are increased on 1 January each year.

Write down an expression for the expected final salary for this member, given that he intends to retire on 31 December following his 60th birthday.

5 Using commutation functions to value salary-related benefits and contributions

We can now turn to constructing formulae for calculating EPVs for the kinds of salary-related benefits and contributions we described in Section 1. This has featured frequently in the Subject CT5 exam.

The determination of the expected present value of benefits and contributions involves the discounting of all expected cashflows to the valuation date and their summation. This summation and its evaluation is aided by the development and use of "commutation functions".

We have already seen some examples of commutation functions in Section 3. We will now build on these functions in order to value the more complex benefits covered here. We will generally be using the same functions as used in the Pension Scheme Tables, though in practice differences in benefit types or definitions may require us to modify these as we see fit. Ultimately, it does not matter what we do as long as we *define each new factor* very carefully. (A hard idea to get used to in this part of the course is the concept of inventing our own commutation functions.)

When the events leading to a cashflow are compound, *eg* age retirement at age $x+u$ and survival from age $x+u$ to receive a pension payment in $(x+t, x+t+1)$, the expected present values of the cashflows at the time of the first event are usually explicitly expressed as an annuity function.

In other words, we will be discounting all pension annuity values to the date of retirement, using annuity functions of the types $\bar{a}^{i}_{x+½}$ and $\bar{a}^{r}_{x+½}$ as we described in Section 3.

We also have to be careful about what assumptions we are making – for instance, do we assume retirements occur half-way through each year on average? If so, we need to make it clear that we assume this.

Formulae are usually developed separately for entitlement arising from past and future service and then summed to give the total expected present value.

We illustrate the development of commutation functions by using examples.

5.1 Death benefit

Death benefit of $4\times$ salary at date of death for a member aged x at valuation date:

The EPV of the benefit is:

$$\sum_{t=0}^{t=NPA-x-1} 4S\frac{s_{x+t}}{s_{x-1}}\frac{d_{x+t}}{l_x}\cdot\frac{v^{x+t+½}}{v^x}$$

where S is the salary received in the year immediately before the valuation date, salaries are reviewed continuously and deaths in $(x+t, x+t+1)$ are assumed to occur uniformly over the year of age.

Let's see where this comes from.

The amount of benefit on death will be four times the annual rate of salary at the time of death, which will be on average at age $x+t+½$. So this annual rate of salary will be:

the expected annual rate of salary at age $x+t+½$

$\approx$ the expected average annual rate of salary received over the year of age $[x+t,\ x+t+1]$

$=$ expected total salary received over the year of age $[x+t,\ x+t+1]$

$$= S\frac{s_{x+t}}{s_{x-1}}$$

We are assuming here that a total salary of S was earned over the year of age $[x-1,\ x]$.

So the expected amount of cashflow from death during this year of age will be four times this rate, multiplied by the probability of death occurring during the year, *ie* d_{x+t}/l_x.

Then, assuming that the amount is paid immediately on death, it needs to be discounted back from age $x+t+½$ to age x. We can do this using the term $\dfrac{v^{x+t+½}}{v^x}$.

We then sum over all possible years in which death could occur. This gives us the expression above.

We define the following commutation functions:

$$^{s}C^{d}_{x+t} = s_{x+t}\, d_{x+t}\, v^{x+t+½}$$

$$^{s}D_{x} = s_{x-1}\, l_{x}\, v^{x}$$

$$^{s}M^{d}_{x} = \sum_{t=0}^{NPA-x-1} {}^{s}C^{d}_{x+t}$$

In the *Tables*, $^{s}D_x$ is defined to be $s_x\, D_x = s_x\, v^x\, l_x$. The Core Reading uses a different definition here. This is OK because there is no standard notation for pensions commutation functions. As we said at the start of this section, when you are deriving formulae for the expected present value of pensions benefits, you must make sure that you define all your notation carefully.

Then the expected present value of the death benefit can be expressed as follows:

EPV of death benefit of 4 times salary at the date of death

The expected present value for a member aged x at the valuation date is:

$$4S\,\frac{^{s}M^{d}_{x}}{^{s}D_{x}}$$

Note that values of the function $^{s}M^{d}_{x}$ are not listed in the *Tables*.

Question 11.13

Explain how the result would differ if S is the annual rate of income at the valuation date, and salaries are reviewed continuously.

5.2 Age retirement benefit

Consider an age retirement benefit of $n/80$ ths of average salary over the five years immediately preceding retirement, where n is the total number of years service at date of retirement. Let the member's salary in the year preceding the valuation date be S, and let the member aged x at the valuation date have exactly m years past service.

Consider first the pension that would be payable if the member retired during the first year of age, x to $x+1$ (*ie* at age $x+½$ on average). The *total* annual amount of pension would be:

$$\frac{(m+½)}{80} PS_{x+½}$$

where $PS_{x+½}$ is the pensionable salary as at age $x+½$. Now $PS_{x+½}$ is defined as the average salary over the 5 years preceding exact age $x+½$, so the *projected* amount of this would be:

$$PS_{x+½} = \frac{1}{5} S \left[\frac{s_{x-½} + s_{x-1½} + s_{x-2½} + s_{x-3½} + s_{x-4½}}{s_{x-1}} \right]$$

We use the symbol z_x to make this less cumbersome. In this case we could choose:

$$z_x = \frac{1}{5}\left[s_{x-1} + s_{x-2} + s_{x-3} + s_{x-4} + s_{x-5} \right]$$

so that:

$$PS_{x+½} = S \frac{z_{x+½}}{s_{x-1}}$$

and the total pension per annum from age $x+½$ would be:

$$\frac{(m+½)}{80} S \frac{z_{x+½}}{s_{x-1}}$$

Note that we have s_{x-1} in the denominator here since the salary S is to be received during the year of age $x-1$ to x.

Question 11.14

How much of this annual pension is accrued from (and therefore relates to) the member's past service, and how much is accrued from future service?

Question 11.15

A common definition of final pensionable salary is "the average of the last three years' earnings." What is the corresponding definition of z_x ?

Past service liability

In practice, it is often necessary to calculate the value of past service benefits separately from future service benefits. So we will deal with past service only for the time being.

Still considering the year of retirement $[x, x+1]$, the expected cost of the annual past service pension discounted to age $x+½$ is:

$$\frac{m}{80} S \frac{z_{x+½}}{s_{x-1}} \bar{a}^r_{x+½}$$

In the same way as for our simple pension benefits in Section 3.1, we now need to discount this to age x (*ie* by half a year) and multiply by the probability of retiring in this year, so we obtain:

$$\frac{m}{80} S \frac{z_{x+½}}{s_{x-1}} \bar{a}^r_{x+½} \frac{v^{x+½}}{v^x} \frac{r_x}{l_x}$$

Now, to obtain the total EPV, we calculate similar terms for all future pension ages, remembering to allow for the fact that retirements at the oldest possible age (NPA) retire at *exact* age NPA, and then sum all terms.

Question 11.16

Write down the term in the past service EPV formula that relates to retirement:

(a) in the year beginning on the 62nd birthday

(b) at NPA.

Assume that $x < 62 < NPA$.

So, in total, **the expected present value of the benefits arising from past service is:**

$$\sum_{t=0}^{t=NPA-x-1} \frac{m}{80} \cdot \frac{z_{x+t+½}}{s_{x-1}} \cdot S \cdot \frac{r_{x+t}}{l_x} \cdot \frac{v^{x+t+½}}{v^x} \cdot \bar{a}^r_{x+t+½}$$

$$+ \frac{m}{80} \cdot \frac{z_{NPA}}{s_{x-1}} \cdot S \cdot \frac{r_{NPA}}{l_x} \cdot \frac{v^{NPA}}{v^x} \cdot \bar{a}^r_{NPA}$$

where $\bar{a}^r_x$ is the present value of a pension of 1 *pa* payable according to the rules of the scheme to a member who retires on age grounds at age x, and z_y is defined as

$$z_y = \frac{1}{5}\left(s_{y-1} + s_{y-2} + s_{y-3} + s_{y-4} + s_{y-5}\right)$$

Retirements in $(x+t, x+t+1)$ are assumed to occur uniformly over the year of age. However, retirements in $(NPA, NPA+1)$ are assumed to occur exactly at *NPA*.

If we define the following commutation functions:

$$ {}^{z}C^{ra}_{x+t} = z_{x+t+\frac{1}{2}}\, r_{x+t}\, v^{x+t+\frac{1}{2}}\, \bar{a}^{r}_{x+t+\frac{1}{2}} \qquad t = 0, 1, 2, \ldots, NPA - x - 1 $$

$$ {}^{z}C^{ra}_{NPA} = z_{NPA}\, r_{NPA}\, v^{NPA}\, \bar{a}^{r}_{NPA} $$

$$ {}^{s}D_{x} = s_{x-1}\, l_{x}\, v^{x} $$

$$ {}^{z}M^{ra}_{x} = \sum_{t=0}^{t=NPA-x} {}^{z}C^{ra}_{x+t} $$

then the expected present value of the age retirement benefit as a result of past service can be written as follows:

EPV of past service liability of age retirement benefit

The expected present value of the benefit in respect of past service is:

$$ \frac{m}{80}\, S\, \frac{{}^{z}M^{ra}_{x}}{{}^{s}D_{x}} $$

Future service liability

We now need to value the future pension payments that will be accrued from future service. Going back to Question 11.14, we saw that the future service annual pension on retirement in the year x to $x+1$ was:

$$ \frac{\frac{1}{2}}{80}\, S\, \frac{z_{x+\frac{1}{2}}}{s_{x-1}} $$

Similarly, if retirement occurred in the year $x+1$ to $x+2$, the future service pension from age $x+1\frac{1}{2}$ (on average) would be:

$$ \frac{1\frac{1}{2}}{80}\, S\, \frac{z_{x+1\frac{1}{2}}}{s_{x-1}} $$

Consider the 1½ factor in the numerator here. It is the *total* future service between now (the valuation age of x) and the retirement age of $x+1\frac{1}{2}$. One year of this relates to the year of future service x to $x+1$, while the "½" relates to the year of service $x+1$ to $x+2$.

Similarly, on retiring during the next year, at $x+2\frac{1}{2}$ (on average), we can split the 2½ years of future service into:

Year of future service	Number of years accrued
x to $x+1$	1 year
$x+1$ to $x+2$	1 year
$x+2$ to $x+3$	½ year
Total	2½ years

To value these benefits conveniently, using commutation functions, we consider the pension earned by *each year of future service separately*.

For example, the pension benefits accrued *from the year of future service* x *to* $x+1$ are:

Retirement year	Average retirement age	Pension *pa* from retirement age
x to $x+1$	$x+\frac{1}{2}$	$\frac{\frac{1}{2}}{80} S \frac{z_{x+\frac{1}{2}}}{s_{x-1}}$
$x+1$ to $x+2$	$x+1\frac{1}{2}$	$\frac{1}{80} S \frac{z_{x+1\frac{1}{2}}}{s_{x-1}}$
$x+2$ to $x+3$	$x+2\frac{1}{2}$	$\frac{1}{80} S \frac{z_{x+2\frac{1}{2}}}{s_{x-1}}$
⋮	⋮	⋮
$NPA-1$ to NPA	$NPA-\frac{1}{2}$	$\frac{1}{80} S \frac{z_{NPA-\frac{1}{2}}}{s_{x-1}}$
NPA	NPA	$\frac{1}{80} S \frac{z_{NPA}}{s_{x-1}}$

Question 11.17

Construct an equivalent table of the pension benefits accrued from the future year of service $x+1$ to $x+2$.

In general, the table of pension benefits accrued from future year of service relating to the year of age $x+t$ to $x+t+1$ is:

Retirement year	Average retirement age	Pension *pa* from retirement age
$x+t$ to $x+t+1$	$x+t+½$	$\frac{½}{80} S \frac{z_{x+t+½}}{s_{x-1}}$
$x+t+1$ to $x+t+2$	$x+t+1½$	$\frac{1}{80} S \frac{z_{x+t+1½}}{s_{x-1}}$
$\vdots$	$\vdots$	$\vdots$
$NPA-1$ to NPA	$NPA-½$	$\frac{1}{80} S \frac{z_{NPA-½}}{s_{x-1}}$
NPA	NPA	$\frac{1}{80} S \frac{z_{NPA}}{s_{x-1}}$

Now, we multiply each of these values by the annuity, discount and probability factors, as usual, to obtain the expected present value of these benefits.

The expected present value at the valuation date (age x) of benefits arising from future service in the year of age $(x+t, x+t+1)$ is:

$$\frac{1}{2} \cdot \frac{1}{80} \cdot \frac{z_{x+t+½}}{s_{x-1}} \cdot S \frac{r_{x+t}}{l_x} \frac{v^{x+t+½}}{v^x} \bar{a}^r_{x+t+½}$$

$$+ \frac{1}{80} \frac{z_{x+t+1½}}{s_{x-1}} S \frac{r_{x+t+1}}{l_x} \frac{v^{x+t+1½}}{v^x} \bar{a}^r_{x+t+1½}$$

$$+ \ldots$$

$$+ \frac{1}{80} \frac{z_{NPA-1+½}}{s_{x-1}} S \frac{r_{NPA-1}}{l_x} \frac{v^{NPA-1+½}}{v^x} \bar{a}^r_{NPA-1+½}$$

$$+ \frac{1}{80} \frac{z_{NPA}}{s_{x-1}} S \frac{r_{NPA}}{l_x} \frac{v^{NPA}}{v^x} \bar{a}^r_{NPA}$$

Question 11.18

Write the above expression in terms of the ${}^{z}C^{ra}_{x+t}$ symbol.

We define the following additional commutation function:

$${}^{z}\bar{M}_x^{ra} = \sum_{t=0}^{t=NPA-x} {}^{z}C_{x+t}^{ra} - \frac{1}{2}\,{}^{z}C_x^{ra}$$

and the expected present value can be written as:

$$\frac{1}{80} S \frac{{}^{z}\bar{M}_{x+t}^{ra}}{{}^{s}D_x}$$

So this is the present value of the benefit in respect of the future *year of service* from $x+t$ to $x+t+1$. We now need to consider all of the future years of service.

Now we add over all possible years of future service *ie* $(x, x+1)$, $(x+1, x+2)$, ..., $(NPA-1, NPA)$ to give the total expected present value of the age retirement entitlement arising from all future service:

$$\sum_{t=0}^{t=NPA-x-1} \frac{1}{80} S \frac{{}^{z}\bar{M}_{x+t}^{ra}}{{}^{s}D_x}$$

Note how the upper limit of this summation goes only to age $NPA-1$ because we are now considering years of service, and so the highest possible term here must be in respect of the year of service from $NPA-1$ to NPA.

If we define a commutation function:

$${}^{z}\bar{R}_x^{ra} = \sum_{t=0}^{t=NPA-x-1} {}^{z}\bar{M}_{x+t}^{ra}$$

then the future liability can be expressed as follows:

EPV of future service liability of age retirement benefit

The expected present value of the benefit in respect of future service is:

$$\frac{1}{80} S \frac{{}^{z}\bar{R}_x^{ra}}{{}^{s}D_x}$$

Adding the past service liability and the future service liability, we obtain the following result:

EPV of total service liability for age retirement benefit

The total expected present value is:

$$\frac{1}{80}\frac{S}{{}^{s}D_x}\left\{m\ {}^{z}M_x^{ra} + {}^{z}\bar{R}_x^{ra}\right\}$$

Question 11.19

Johnny "Hot Spats" Monachesi has been a member of the Al Capone Gangsters Ltd pension scheme since starting work with the firm at the age of 25. He is currently aged 40. The scheme gives a pension on retirement equal to *n*/60ths of final pensionable salary, where n is the years of service to retirement (including fractions) and final pensionable salary is the average of the prior three years' earnings.

Calculate the value of his past and future service pension liabilities in the pension scheme, assuming that scheme experience follows that of the Pension Scheme Tables, if his earnings in the previous year totalled $45,000.

Assume interest of 4%.

Accruing ill-health benefits

The expected present values of the pensions benefits on ill-health retirement and the benefits on leaving service can be evaluated using similar methods.

Instead of the service table function r_x, we use i_x to denote the number of ill-health retirements between the ages of x and $x+1$, $x < NPA$. If the benefits described in this section were payable on ill-health retirement before *NPA*, then the expected present value of the past service benefit would be:

$$\sum_{t=0}^{NPA-x-1} \frac{m}{80} \cdot \frac{z_{x+t+½}}{s_{x-1}} \cdot S \cdot \frac{i_{x+t}}{l_x} \cdot \frac{v^{x+t+½}}{v^x} \cdot \bar{a}^i_{x+t+½}$$

where $\bar{a}^i_x$ is the expected present value of a pension of 1 *pa* payable according to the scheme rules to a member who retires on ill-health grounds at age x.

If we define:

$$^{z}C^{ia}_{x+t} = z_{x+t+½}\, i_{x+t}\, v^{x+t+½}\, \bar{a}^{i}_{x+t+½} \qquad t = 0, 1, 2, ..., NPA - x - 1$$

$$^{s}D_{x} = s_{x-1}\, l_{x}\, v^{x}$$

and:

$$^{z}M^{ia}_{x} = \sum_{t=0}^{NPA-x-1} {}^{z}C^{ia}_{x+t}$$

then the expected present value of the past service benefits can be written as:

$$\frac{m}{80} S \frac{^{z}M^{ia}_{x}}{^{s}D_{x}}$$

Adjusting the derivation for the future service benefit for age retirement in a similar way, we find that the expected present value of the future service benefit payable on ill-health retirement before *NPA* is:

$$\frac{1}{80} S \frac{^{z}\bar{R}^{ia}_{x}}{^{s}D_{x}}$$

where:

$$^{z}\bar{R}^{ia}_{x} = \sum_{t=0}^{NPA-x-1} {}^{z}\bar{M}^{ia}_{x+t}$$

and:

$$^{z}\bar{M}^{ia}_{x} = \sum_{t=0}^{NPA-x-1} {}^{z}C^{ia}_{x+t} - \frac{1}{2}\, {}^{z}C^{ia}_{x}$$

5.3 Modifications to age retirement benefits

We can apply the above methodology to value almost any benefit structure by suitable modification of the commutation functions seen so far. We now describe how we can deal with some of the common variants of the "standard" accruing final salary benefits scheme studied above.

Restricted age range

Sometimes, members are only eligible for a particular benefit over a specified age range. At other ages the benefit amount can be considered to be zero.

Example

Write a formula for calculating the total service liability for an ill health pension of 2/3rds of final average salary (as defined in the *Tables*) for a member aged exactly x $(x < 55)$ based on the commutation functions in the *Tables* if the ill health pension is payable only after age 55. Members who leave the scheme before age 55 through ill health, are not entitled to any ill-health retirement benefits.

Solution

Since no benefit is payable on ill health before age 55, the total service liability is:

$$\tfrac{2}{3} S \left[\frac{{}^{z}C_{55}^{ia}}{{}^{s}D_x} + \frac{{}^{z}C_{56}^{ia}}{{}^{s}D_x} + \cdots + \frac{{}^{z}C_{64}^{ia}}{{}^{s}D_x} \right] = \tfrac{2}{3} S \frac{{}^{z}M_{55}^{ia}}{{}^{s}D_x}$$

where

S is the salary which will be earned over the year x to $x+1$

$${}^{s}D_x = s_x v^x l_x$$

$$z_x = \tfrac{1}{3}\left(s_{x-3} + s_{x-2} + s_{x-1}\right)$$

$${}^{z}C_y^{ia} = z_{y+½} v^{y+½} i_y \bar{a}^{i}_{y+½}$$

$${}^{z}M_x^{ia} = \sum_{y=x}^{64} {}^{z}C_y^{ia}$$

Limits on service

In some situations, an upper limit ("ceiling") or a lower limit ("floor") may be specified in the definition of the benefit.

Example

A pension scheme provides a pension of $n/60$ths of final average salary payable on normal age retirement. However, n is limited to a maximum of 40 years. Write down a commutation function formula for calculating the total service liability for a member currently aged x, with a whole number of years' past service who joined the scheme:

(a) after age 25,

(b) before age 25, with less than 40 years of past service as at current age x, and

(c) before age 25, with more than 40 years of past service as at current age x.

Solution

Let m the number of years of past service. Define S, z_x and ${}^sD_x$ as in the previous example.

(a) ***Joining after age 25***

A member who joined after age 25 (*ie* $x-m>25$) won't be affected by the 40-year restriction. We define:

$$ {}^zC_y^{ra} = z_{y+½} v^{y+½}\, r_y\, \bar{a}^r_{y+½} \qquad (y \le 64) $$

$$ {}^zC_{65}^{ra} = z_{65} v^{65}\, r_{65}\, \bar{a}^r_{65} $$

$$ {}^zM_x^{ra} = \sum_{y=x}^{65} {}^zC_y^{ra} $$

$$ {}^z\bar{M}_x^{ra} = {}^zM_x^{ra} - ½\, {}^zC_x^{ra} \qquad (x \le 64) $$

and $$ {}^z\bar{R}_x^{ra} = \sum_{y=x}^{64} {}^z\bar{M}_y^{ra} \qquad (x \le 64) $$

Then the total service liability is as derived above, *ie*:

$$ \frac{m}{60} S \frac{{}^zM_x^{ra}}{{}^sD_x} + \frac{1}{60} S \frac{{}^z\bar{R}_x^{ra}}{{}^sD_x} \qquad (x-m>25) $$

(b) ***Joining before age 25, less than 40 years of current past service***

For a member who joined before age 25 (*ie* $x-m<25$), benefits will stop accruing once he hits the 40-year maximum (at age $x-m+40$). If we consider each possible year in the future when retirement might take place, we see that the total service liability is:

$$\frac{1}{60}S\left[(m+½)\frac{{}^zC_x^{ra}}{{}^sD_x}+(m+1½)\frac{{}^zC_{x+1}^{ra}}{{}^sD_x}+\cdots+39½\frac{{}^zC_{x-m+39}^{ra}}{{}^sD_x}\right.$$
$$\left.+40\frac{{}^zC_{x-m+40}^{ra}}{{}^sD_x}+\cdots+40\frac{{}^zC_{65}^{ra}}{{}^sD_x}\right]$$

Here, the top row corresponds to the period when benefits are still accruing, while the bottom row corresponds to the period after they have hit the 40-year maximum. If we now sum the C's, we can write this as:

$$\frac{m}{60}S\frac{{}^zM_x^{ra}}{{}^sD_x}+\frac{1}{60}S\left[\frac{{}^z\overline{R}_x^{ra}-{}^z\overline{R}_{x-m+40}^{ra}}{{}^sD_x}\right] \qquad (x-m<25)$$

(c) ***Joining before age 25, more than 40 years of current past service***

For this situation, service has already stopped accruing by the current age x. So all we need to calculate is the past service liability, which is:

$$\frac{40}{60}S\frac{{}^zM_x^{ra}}{{}^sD_x}$$

Question 11.20 ***(Tricky)***

Justify the last step in part (b) of the preceding example.

Complete years only

Sometimes pension scheme rules specify that the calculation of benefit amounts should be based on complete years of service only *ie* fractions of a year are ignored in the calculations.

The formulae we have developed so far assume that credit is given for part years. These formulae can be modified for valuing benefits where only complete years count.

Example

Derive formulae for calculating the past and future service liabilities for a pension of a fixed amount P per annum for each complete year of service payable to an individual member now aged exactly 30 with exactly 5 years' past service.

Solution

The pension relating to past service will be $5P$. So the past service liability is:

$$5P\frac{M_{30}^{ra}}{D_{30}}$$

If the member leaves between ages 30 and 31, there will be no extra benefit in respect of future service since the part year will be ignored. If the member leaves between ages 31 and 32, there will be one extra year's benefit *etc*. This leads to the following expression for the future service liability:

$$P\left(\frac{C_{31}^{ra}}{D_{30}}+\frac{2C_{32}^{ra}}{D_{30}}+\cdots+\frac{34C_{64}^{ra}}{D_{30}}+\frac{35C_{65}^{ra}}{D_{30}}\right)=P\frac{R_{31}^{ra}}{D_{30}}$$

where $R_x^{ra}=\sum_{y=x}^{65}M_y^{ra}$.

Question 11.21

Justify the last step in this example.

Question 11.22

A pension fund offers the following benefit to its members: upon age retirement at any age, a member will receive an annual pension equal to 1/60th of final pensionable salary for each year of service, up to a maximum of two thirds of final salary. The pension fund is valued on every 1 July.

Salaries are increased each year on 1 July, final pensionable salary is defined to be the average salary earned over the three years preceding retirement, and normal pension age is 65.

At a particular valuation date, one member is aged 40 exactly, has 18 years of past service and earned £32,000 over the last year. Calculate the expected present value of the past and future service benefits for this member assuming that mortality, retirement, interest and salary scale are as given in the *Tables.*

5.4 Members contributions

Members' contributions are k % of salary assumed to be paid continuously *ie* on average at mid-year. The expected present value of future contributions can be expressed as:

$$\sum_{t=0}^{t=NPA-x-1} \frac{k}{100} S \frac{s_{x+t}}{s_{x-1}} \frac{v^{x+t+½}}{v^x} \frac{l_{x+t+½}}{l_x}$$

We define the following commutation functions:

$${}^sD_x = s_{x-1} v^x l_x \qquad {}^s\bar{D}_x = s_x v^{x+½} l_{x+½} \qquad {}^s\bar{N}_x = \sum_{t=0}^{t=NPA-x-1} {}^s\bar{D}_{x+t}$$

and the total expected present value can be written as follows:

Expected present value of members' contributions

The expected present value of a member's future contributions is given by:

$$\frac{k}{100} S \frac{{}^s\bar{N}_x}{{}^sD_x}$$

Of course, the value of member's contributions can be calculated in different ways, and different assumptions may be used. You will need to read the question very carefully when answering questions in this area, since there is a variety of different situations that could be examined.

Question 11.23

Calculate the annual contribution, as a percentage of salary, which should be made by the company in respect of Johnny "Hot Spats" Monachesi (see Question 11.19) over his remaining term to retirement. Assume that he will contribute 4% of salary *pa* and that the scheme has already built up $37,000 in respect of his pension.

5.5 An important point about notation

Unlike the mainstream annuity and assurance terminology (usefully summarised for the interested reader in the "International Actuarial Notation" section of the "Formulae and Tables for Examinations") it is important to note that the commutation functions used above, to value pensions benefits and contributions, have no universally accepted definitions.

Therefore, in any particular case, care needs to be taken to define carefully all symbols used. Similarly, the use of tables – such as that reproduced in the "Formulae and Tables for Examinations" – should pay due regard to the precise definition of the functions contained therein.

Exam tip

Exam questions on pension funds come in two varieties:

- calculating the expected present value of a benefit
- deriving a formula for evaluating the expected present value of a benefit.

When you are attempting derivation-type questions, it is important that you define all of the symbols that you use. Start with i for the valuation rate of interest and $v = \dfrac{1}{1+i}$ for the discount factor (assuming that you have used both i and v in your commutation function formulae).

You cannot assume anything when you write down the solution to this type of question.

6 Defined contribution schemes

The other, and increasingly popular, type of pension scheme is the defined contribution scheme which, as the name suggests, is contribution-based rather than benefit-based.

Essentially, members (and/or employers on behalf of their members) pay contributions into separately identifiable investment vehicles (such as a unit-linked fund, described in a later chapter). The accumulating value of each member's fund may be used to pay for the scheme benefits required for the member from time to time (*eg* death benefits), or these benefits may be paid for using specific additional contributions or premiums. The residual accumulated fund available at the member's retirement date is then available to fund their post-retirement income, for example by purchasing an immediate annuity at that point.

Question 11.24

If a defined contribution pension fund grows at 3% *pa* compound at all times (net of any charges), calculate the annual contribution that would have to be paid into the fund in order to secure a pension of £2,000 per month from the retirement date.

Assume that:

- contributions are paid annually in advance for 40 years
- retirement occurs at the end of the 40-year period
- retirement occurs exactly on the 67th birthday
- mortality after retirement follows the PMA92C20 table
- interest after retirement is 4% *pa*
- the pension is payable monthly in advance, guaranteed for a minimum of 5 years.

These schemes therefore essentially operate as pooled savings vehicles for their members, and as there are few, if any, contingent payments involved, the need for actuarial involvement in performing valuations is minimal.

There can be plenty of scope for actuaries to add value to these schemes, however, for example in the setting of contribution levels and in scheme design, and this is covered in later subjects.

7 Exam-style questions

Here are two exam-style questions on pensions. There are few marks available in both, so you've no time to derive the formulae. Make sure that you memorise the pensions formulae (or are able to work them out very quickly) before the exam.

Question 11.25 ***(Subject 105, April 2000, Question 7)***

A pension scheme provides a pension of $\frac{1}{45}$ of final pensionable salary for each year of service, with a maximum of $\frac{2}{3}$ of final pensionable salary, upon retirement at age 65.

Final pensionable salary is defined as average annual salary over the 3 years immediately preceding retirement.

A member is now aged exactly 47 and has 14 years of past service. He earned £40,000 in the previous 12 months.

Calculate the expected present value now of this member's total pension on retirement, using the symbols defined in, and assumptions underlying, the Formulae and Tables for Actuarial Examinations. [3]

Question 11.26

A pension scheme provides a pension on age retirement of 1/60th of final pensionable salary for each year of service, with part years counting proportionately. Final pensionable salary is defined to be the average salary over the 3 years prior to retirement. Members contribute 6% of their salaries to the pension fund.

One member aged exactly 50 has 18 years of past service and earned £45,000 in the last year. Using the Pension Scheme Tables from the Actuarial Formulae and Tables, calculate:

(i) the expected present value of this member's past service benefit [2]

(ii) the expected present value of this member's future service benefit [2]

(iii) the expected present value of this member's future contributions. [2]

[Total 6]

8 End of Part 3

What next?

1. Briefly **review** the key areas of Part 3 and/or re-read the **summaries** at the end of Chapters 8 to 11.
2. Attempt some of the questions in Part 3 of the **Question and Answer Bank**. If you don't have time to do them all, you could save the remainder for use as part of your revision.
3. Attempt **Assignment X3**.

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Chapter 11 Summary

Pension scheme service table

$l_x =$ number of active members at exact age x

$i_x, d_x, r_x, w_x =$ numbers expected to leave service through ill-health retirement, death, normal retirement, and for any other reason, between ages x and $x+1$.

$\frac{r_x}{l_x} =$ probability of a currently active member aged x exact, retiring normally during the year of age $[x, x+1]$

and similarly for other causes.

Salary scale

A salary scale (including merit and inflation increases) is tabulated as a function of age, s_x, such that:

$$\frac{s_{x+t}}{s_x} = \frac{\text{salary expected to be received in age interval } (x+t, x+t+1)}{\text{salary expected to be received in age interval } (x, x+1)}$$

Pension scheme benefits

A pension scheme may provide several types of benefits. Consider a member aged x on the valuation date, who received a salary of S in the year preceding the valuation date. Let i denote the valuation rate of interest, and $v=(1+i)^{-1}$.

Death benefit

The expected present value at age x of a benefit of 4 times salary at the date of death for this member is:

$$\sum_{t=0}^{NPA-x-1} 4S\,\frac{s_{x+t}}{s_{x-1}}\,\frac{d_{x+t}}{l_x}\,\frac{v^{x+t+½}}{v^x}$$

where:

$d_y =$ the number of deaths between the ages of y and $y+1$

$l_y =$ the number of active lives at age y

according to some suitable service table, and s_y is a salary scale as defined above. If we define the following commutation functions:

$$^sC^d_{x+t} = s_{x+t}\,d_{x+t}\,v^{x+t+½} \qquad ^sD_x = s_{x-1}\,l_x\,v^x \qquad ^sM^d_x = \sum_{t=0}^{NPA-x-1} {}^sC^d_{x+t}$$

then the expected present value of the death benefit can be expressed as:

$$4S\,\frac{^sM^d_x}{^sD_x}$$

Age retirement benefit

Suppose that the member has m years of past service at the valuation date, and is entitled to a pension of $1/K$th of final pensionable salary in respect of each year of service, with part years counting proportionately. Final pensionable salary is defined to be the average salary in the n years immediately preceding retirement.

We further define:

$$z_x = \frac{s_{x-1} + s_{x-2} + \cdots + s_{x-n}}{n}$$

$r_y =$ number of age retirements between the ages of y and $y+1$, $y < NPA$

$r_{NPA} =$ number of age retirements at NPA

$\bar{a}_y^r =$ EPV at age y of a pension of 1 *pa* payable on age retirement at age y

We assume that age retirements occur halfway between birthdays, for retirements before NPA, or at exact NPA.

Past service benefits

The expected present value of the benefits arising from past service is:

$$\sum_{t=0}^{NPA-x-1} \frac{m}{K} \cdot \frac{z_{x+t+½}}{s_{x-1}} . S. \frac{r_{x+t}}{l_x} \cdot \frac{v^{x+t+½}}{v^x} . \bar{a}_{x+t+½}^r$$

$$+ \frac{m}{K} \cdot \frac{z_{NPA}}{s_{x-1}} . S . \frac{r_{NPA}}{l_x} \cdot \frac{v^{NPA}}{v^x} . \bar{a}_{NPA}^r$$

If we also define the following commutation functions:

$${}^zC_{x+t}^{ra} = z_{x+t+½}\, r_{x+t}\, v^{x+t+½}\, \bar{a}_{x+t+½}^r \qquad \text{for } t \neq NPA - x$$

$${}^zC_{NPA}^{ra} = z_{NPA}\, r_{NPA}\, v^{NPA}\, \bar{a}_{NPA}^r$$

$${}^sD_x = s_{x-1}\, l_x\, v^x$$

$${}^zM_x^{ra} = \sum_{t=0}^{NPA-x} {}^zC_{x+t}^{ra}$$

then the value of the past service liability is $\frac{m}{K} S \frac{{}^zM_x^{ra}}{{}^sD_x}$.

Future service benefits

The expected present value of the future service benefits in the year of age $x+t$ to $x+t+1$ is:

$$\frac{1}{2}\frac{1}{K}\frac{z_{x+t+½}}{s_{x-1}}S\frac{r_{x+t}}{l_x}\frac{v^{x+t+½}}{v^x}\bar{a}^r_{x+t+½}$$
$$+\frac{1}{K}\frac{z_{x+t+1½}}{s_{x-1}}S\frac{r_{x+t+1}}{l_x}\frac{v^{x+t+1½}}{v^x}\bar{a}^r_{x+t+1½}$$
$$+\cdots$$
$$+\frac{1}{K}\frac{z_{NPA-1+½}}{s_{x-1}}S\frac{r_{NPA-1}}{l_x}\frac{v^{NPA-1+½}}{v^x}\bar{a}^r_{NPA-1+½}$$
$$+\frac{1}{K}\frac{z_{NPA}}{s_{x-1}}S\frac{r_{NPA}}{l_x}\frac{v^{NPA}}{v^x}\bar{a}^r_{NPA}$$

If we define the additional commutation function:

$${}^z\bar{M}^{ra}_x = {}^zM^{ra}_x - \tfrac{1}{2}{}^zC^{ra}_x$$

then the expression above can be written as:

$$\frac{1}{K}S\frac{{}^z\bar{M}^{ra}_{x+t}}{{}^sD_x}$$

Adding over all possible years of future service gives the total expected present value of the future service benefits:

$$\sum_{t=0}^{NPA-x-1}\frac{1}{K}S\frac{{}^z\bar{M}^{ra}_{x+t}}{{}^sD_x}$$

We finally define:

$${}^z\bar{R}^{ra}_x = \sum_{t=0}^{NPA-x-1}{}^z\bar{M}^{ra}_{x+t}$$

So the future service liability can be expressed as:

$$\frac{1}{K} S \frac{{}^{z}\overline{R}^{ra}_{x+t}}{{}^{s}D_x}$$

Similar formulae to the above can be developed for valuing ill-health retirement benefits.

Members' contributions

If the member contributes $k\%$ of salary, and contributions are assumed to be made continuously, then the expected present value of the members' future contributions is:

$$\sum_{t=0}^{NPA-x-1} \frac{k}{100} S \frac{s_{x+t}}{s_{x-1}} \frac{v^{x+t+½}}{v^x} \frac{l_{x+t+½}}{l_x}$$

If we define:

$${}^{s}D_x = s_{x-1}\, l_x\, v^x$$

$${}^{s}\overline{D}_x = s_x v^{x+½} l_{x+½}$$

$${}^{s}\overline{N}_x = \sum_{t=0}^{NPA-x-1} {}^{s}\overline{D}_{x+t}$$

then the expected present value can be expressed as follows:

$$\frac{k}{100} S \frac{{}^{s}\overline{N}_x}{{}^{s}D_x}$$

Defined contribution schemes

Contributions accumulate in a fund, which is then available to purchase benefits for the individual members.

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Chapter 11 Solutions

Solution 11.1

The total contribution rate (company plus employee) may need to be increased if investment returns fall, or legislation changes significantly (*eg* introduction of new minimum benefits).

If the company says that the company's rate is fixed and that members' rates may need to be increased at some time in the future, the members would be taking all of the risk involved in the scheme. An increase in members' contributions would, in effect, be a pay cut. This would make the scheme very unattractive, and thwart the original intention of offering a scheme which will be welcomed by employees and keep the company competitive in the recruitment marketplace. There may also be legislative constraints.

Solution 11.2

The benefits on ill-health retirement and normal age retirement may be different. In addition, we will probably need to value the cost of procuring an annuity for the members at retirement, and this cost will depend on the mortality of those who retire. We would expect people retiring due to ill health to have much higher mortality than normal age retirements. So we need to distinguish between the two types of retirement.

Solution 11.3

Multiple state model

r_x will include only those lives who retired normally during the year of age x to $x+1$, and who are still living at age $x+1$.

d_x will include all people, originally active at age x, who died between ages x to $x+1$. This will therefore include those who retired or otherwise left the scheme over the year, but who then also died before the end of the year.

Multiple decrement model

r_x will include all the people who retired normally during the year of age, including those who subsequently died before age $x+1$.

d_x will only include those who died while active members of the scheme, over the year of age.

This means that:

$$r_x(ms) < r_x(md) \qquad \text{and} \qquad d_x(ms) > d_x(md)$$

(ms = multiple state; md = multiple decrement)

Solution 11.4

(i) Age retirement is possible from ages 60 to 65.

(ii) The possible decrements at age 40 are withdrawal, death and ill-health retirement.

(iii) New entrants do not come into the population as described by the table, which deals only with an in-force population that gradually "declines and falls".

Solution 11.5

The calculations are:

(i) $$\frac{d_{40}}{l_{40}} = \frac{14}{15,397} = 0.00091$$

(ii) $$\frac{w_{45} + w_{46} + \ldots + w_{49}}{l_{45}} = \frac{162+120+79+52+26}{13,602} = 0.03227$$

(iii) $$\frac{i_{59}}{l_{38}} = \frac{278}{16,506} = 0.01684$$

(iv) $$\frac{r_{65}}{l_{42}} = \frac{3,757}{14,536} = 0.25846$$

Solution 11.6

This is the value of £10,000 discounted from half-way through the year of age $[56,\ 57]$, multiplied by the probability of retiring in that year. Hence we need:

$$10,000\, v^{1½} \frac{r_{56}}{l_{55}}$$

Solution 11.7

At age 65, normal retirements can only occur on the birthday, which is exactly 10 years from the present age of 55.

For all other retirement years, retirement can occur at any point during the year, so we assume that they retire half way through the year on average.

Solution 11.8

(a) ***Lump sum paid at any retirement age***

This is the same as the previous example, except it relates to a different valuation age and amount. So:

$$EPV = 50,000\frac{M_{52}^r}{D_{52}} = 50,000 \times \frac{782}{1,629} = 24,002$$

(b) ***Lump sum paid only if retire after 64th birthday***

From first principles this is:

$$50,000\left[\frac{v^{12½}\, r_{64}}{l_{52}} + \frac{v^{13}\, r_{65}}{l_{52}}\right] = \frac{50,000}{v^{52}\, l_{52}}\left[v^{64½}\, r_{64} + v^{65}\, r_{65}\right]$$

$$= \frac{50,000}{D_{52}}\left[C_{64}^r + C_{65}^r\right] = 50,000\frac{M_{64}^r}{D_{52}}$$

$$= 50,000 \times \frac{321}{1,629} = 9,853$$

Solution 11.9

The value is:

$$EPV = 150{,}000 \frac{M_{48}^i}{D_{48}}$$

where:

$$M_x^i = \sum_{t=0}^{59-x} C_{x+t}^i$$

$$C_x^i = v^{x+½}\, i_x$$

$$D_x = v^x\, l_x$$

i_x = expected number of ill-health retirements in the year beginning at exact age x

l_x = expected number of active members in the scheme at exact age x

$v = (1+i)^{-1}$ where i is the valuation rate of interest.

*Note that **all** symbols used have to be defined, not just the commutation functions!*

Solution 11.10

$\bar{a}_{x+½}^i$ would be the EPV of an annuity of $1pa$ payable to a person who has just retired through ill health at age $x+½$.

The annuity value would be based on the expected mortality of ill-health retirees, which would be higher than that of normal-age retirees. So we would expect $\bar{a}_{x+½}^i < \bar{a}_{x+½}^r$, all else being equal, as we expect fewer payments to be made to those retiring in ill health.

Solution 11.11

The EPV is:

$$EPV = \frac{20,000}{D_{57}}\left[M_{57}^{ia} + M_{57}^{ra}\right] + 5,000\frac{C_{65}^{ra}}{D_{57}}$$

$$= \frac{20,000}{1,235} \times \left[2,644 + 11,915\right] + 5,000 \times \frac{4,075}{1,235} = £252,300$$

Solution 11.12

The income of £40,000 is earned between 1 October 2002 (when aged 44) and 1 October 2003 (when aged 45). Since salaries are reviewed on 1 January each year, we should break this income into two different salary year components.

Earnings from 1 October 2002 to 31 December 2002 relate to the salary year starting on 1 January 2002, when the member was aged 43.25. Earnings from 1 January 2003 to 1 October 2003 relate to the salary year starting on 1 January 2003, when the member was aged 44.25. So the required denominator is:

$$0.25 s_{43.25} + 0.75 s_{44.25}$$

For the numerator, we need the age at the beginning of the year of earnings related to final salary on 31 December after the member's 60th birthday. This is 59.25.

So the member's expected final salary is:

$$40,000 \frac{s_{59.25}}{0.25 s_{43.25} + 0.75 s_{44.25}}$$

Alternatively, we can simply use s_{44} *for the denominator, as the person was aged exactly 44 at the start of the year in which the 40,000 was earned. The formula for the expected final salary is then:*

$$40,000 \frac{s_{59.25}}{s_{44}}$$

The two approaches give almost identical numerical answers and both are acceptable in the exam.

Solution 11.13

If S is the annual rate of income at the valuation date and salaries are reviewed continuously, then we need to modify the definition of ${}^{s}D_x$ to $s_{x-½}\, l_x v^x$. See Section 4.

Solution 11.14

The expected annual amount of pension, starting at age $x+½$, that has accrued from *past* service is:

$$\frac{m}{80} S \frac{z_{x+½}}{s_{x-1}} \quad (\textit{per annum})$$

The expected annual amount of pension, starting at age $x+½$, that has accrued from *future* service is:

$$\frac{½}{80} S \frac{z_{x+½}}{s_{x-1}} \quad (\textit{per annum})$$

Solution 11.15

The corresponding definition is:

$$z_x = \tfrac{1}{3}\left(s_{x-3} + s_{x-2} + s_{x-1}\right)$$

Note that this is the definition assumed in the Pension Scheme section of the Tables.

Solution 11.16

(a) ***Retiring in the year beginning at age 62***

The EPV of the benefits for a member aged x is:

$$\frac{m}{80} S \frac{z_{62½}}{s_{x-1}} \bar{a}^{r}_{62½} \frac{v^{62½}}{v^x} \frac{r_{62}}{l_x}$$

(b) ***Retiring at exact age NPA***

The EPV of the benefits for a member aged x is:

$$\frac{m}{80} S \frac{z_{NPA}}{s_{x-1}} \bar{a}^r_{NPA} \frac{v^{NPA}}{v^x} \frac{r_{NPA}}{l_x}$$

Solution 11.17

The table is:

Retirement year	Average retirement age	Pension *pa* from retirement age
x to $x+1$	$x+½$	0
$x+1$ to $x+2$	$x+1½$	$\frac{½}{80} S \frac{z_{x+1½}}{s_{x-1}}$
$x+2$ to $x+3$	$x+2½$	$\frac{1}{80} S \frac{z_{x+2½}}{s_{x-1}}$
$x+3$ to $x+4$	$x+3½$	$\frac{1}{80} S \frac{z_{x+3½}}{s_{x-1}}$
$\vdots$	$\vdots$	$\vdots$
$NPA-1$ to NPA	$NPA-½$	$\frac{1}{80} S \frac{z_{NPA-½}}{s_{x-1}}$
NPA	NPA	$\frac{1}{80} S \frac{z_{NPA}}{s_{x-1}}$

Solution 11.18

The whole expression can be rewritten as:

$$\frac{S}{80} \frac{1}{{}^sD_x} \left\{ \tfrac{1}{2}\, {}^zC^{ra}_{x+t} + {}^zC^{ra}_{x+t+1} + \cdots + {}^zC^{ra}_{NPA} \right\}$$

Solution 11.19

The past service liability will be of value $\dfrac{15S}{60}\dfrac{{}^{z}M_{40}^{ra}}{s_{39}D_{40}}=\dfrac{15S}{60}\dfrac{{}^{z}M_{40}^{ra}}{{}^{s}D_{40}}\dfrac{s_{40}}{s_{39}}$

where ${}^{s}D_{x}=s_{x}D_{x}$ to be consistent with the *Tables*.

This is:

$$0.25\times 45,000\times\frac{128,026}{25,059}\times\frac{7.814}{7.623}=58,916$$

Likewise the future service liability will be of value:

$$\frac{1}{60}S\frac{{}^{z}\bar{R}_{40}^{ra}}{{}^{s}D_{40}}\frac{s_{40}}{s_{39}}=88,487$$

giving a total of \$147,403.

Solution 11.20

The value of the future service liability is:

$$\frac{S}{60}\left[\tfrac{1}{2}\frac{{}^{z}C_{x}^{ra}}{{}^{s}D_{x}}+1\tfrac{1}{2}\frac{{}^{z}C_{x+1}^{ra}}{{}^{s}D_{x}}+\cdots+\left(39\tfrac{1}{2}-m\right)\frac{{}^{z}C_{x+39-m}^{ra}}{{}^{s}D_{x}}\right.$$

$$\left.+(40-m)\left\{\frac{{}^{z}C_{x+40-m}^{ra}}{{}^{s}D_{x}}+\frac{{}^{z}C_{x+41-m}^{ra}}{{}^{s}D_{x}}+\cdots+\frac{{}^{z}C_{65}^{ra}}{{}^{s}D_{x}}\right\}\right]$$

which is:

$$\frac{S}{60}\left[\left\{\tfrac{1}{2}\frac{{}^{z}C_{x}^{ra}}{{}^{s}D_{x}}+1\tfrac{1}{2}\frac{{}^{z}C_{x+1}^{ra}}{{}^{s}D_{x}}+\cdots\right\}-\left\{\tfrac{1}{2}\frac{{}^{z}C_{x+40-m}^{ra}}{{}^{s}D_{x}}+1\tfrac{1}{2}\frac{{}^{z}C_{x+41-m}^{ra}}{{}^{s}D_{x}}+\cdots\right\}\right]$$

Each of the functions in the curly brackets are $\overline{R}$ functions at the appropriate starting age, and so the value can be written:

$$\frac{S}{60}\left\{\frac{{}^{z}\overline{R}_{x}^{ra}-{}^{z}\overline{R}_{x-m+40}^{ra}}{{}^{s}D_{x}}\right\}$$

giving the required result.

Solution 11.21

$$R_{31}^{ra}=\sum_{t=0}^{34}M_{31+t}^{ra}=\sum_{t=0}^{34}\sum_{s=0}^{65-31-t}C_{31+t+s}^{ra}=C_{31}^{ra}+2C_{32}^{ra}+\cdots+35C_{65}^{ra}$$

Solution 11.22

The salary of £32,000 relates to the year of age 39 to 40. So the expected present value of the past service liability is:

$$32,000\times\frac{18}{60}\times\frac{{}^{z}M_{40}^{ra}}{s_{39}D_{40}}=9,600\times\frac{128,026}{7.623\times 3,207}=£50,274$$

Note that you would get the same answer if you used ${}^{z}M_{60}^{ra}$ *in the numerator, as the service table has* $r_x=0$ *for* $x<60$.

The pension is restricted to 2/3rds of final pensionable salary. So the maximum number of 60ths that can earned in respect of future service is 22, *ie* future accrual stops at age 62. So the expected present of the future service liability is:

$$32,000\times\frac{1}{60}\times\frac{{}^{z}\overline{R}_{40}^{ra}-{}^{z}\overline{R}_{62}^{ra}}{s_{39}D_{40}}=32,000\times\frac{1}{60}\times\frac{2,884,260-159,030}{7.623\times 3,207}=£59,453$$

Solution 11.23

The value of Hot Spats's contributions at a rate of $k\%$ *pa* is:

$$\frac{k}{100}\, S\, \frac{{}^s\bar{N}_{40}}{{}^sD_{40}} \frac{s_{40}}{s_{39}} = 6,692k$$

The shortfall between the value of the liabilities and the fund already established is:

$$147,403 - 37,000 = 110,403$$

so setting $6,692k = 110,403$ gives a total required contribution rate of 16.5%, which is to be split 4% Johnny, 12.5% the Firm.

Solution 11.24

If X is the annual amount of contribution, then X satisfies:

$$X\, \ddot{s}_{\overline{40|}}^{@3\%} = 24,000\, \ddot{a}_{\overline{67:5|}}^{(12)}$$

where:

$$\ddot{s}_{\overline{40|}}^{@3\%} = \frac{1.03^{40} - 1}{0.03/1.03} = 77.66330$$

$$\ddot{a}_{\overline{67:5|}}^{(12)} = \ddot{a}_{\overline{5|}}^{(12)} + v^5 \frac{l_{72}}{l_{67}} \ddot{a}_{72}^{(12)}$$

$$= \frac{1 - 1.04^{-5}}{d^{(12)}} + 1.04^{-5} \times \frac{8,968.099}{9,521.065} \times \left(\ddot{a}_{72} - \frac{11}{24} \right)$$

$$= 12.485$$

using $d^{(12)} = 0.039157$ and $\ddot{a}_{72} = 10.711$.

So:

$$X = \frac{24,000 \times 12.485}{77.66330} = £3,858$$

Solution 11.25

The question says that retirement takes place at age 65 (only). By this stage, the member will have accrued the maximum number of years' service.

The salary figure given in the data relates to the year starting at exact age 46.

So the value of the member's total pension is:

$$\frac{2}{3} \times \frac{40,000}{s_{46}} \times \frac{{}^{z}C_{65}^{ra}}{D_{47}} = \frac{2}{3} \times \frac{40,000}{8.628} \times \frac{45,467}{2,087} = £67,334$$

Solution 11.26

(i) ***Past service benefit***

The expected present value of the member's past service benefit is:

$$\frac{18}{60} \times 45,000 \times \frac{{}^{z}M_{50}^{ra}}{s_{49}\, D_{50}} = \frac{18}{60} \times 45,000 \times \frac{128,026}{9.031 \times 1,796} = £106,559$$

(ii) ***Future service benefit***

The expected present value of the member's future service benefit is:

$$\frac{1}{60} \times 45,000 \times \frac{{}^{z}\bar{R}_{50}^{ra}}{s_{49}\, D_{50}} = \frac{1}{60} \times 45,000 \times \frac{1,604,000}{9.031 \times 1,796} = £74,169$$

(iii) ***Future contributions***

The expected present value of the member's future contributions is:

$$0.06 \times 45,000 \times \frac{{}^{s}\bar{N}_{50}}{s_{49}\, D_{50}} = 0.06 \times 45,000 \times \frac{163,638}{9.031 \times 1,796} = £27,240$$

Chapter 12

Profit testing

Syllabus objectives:

(ix) Describe and use cashflow techniques, where and as appropriate for use in pricing, reserving, and assessing profitability.

1. Define a unit-linked contract.

2. Project expected future cashflows for whole life, endowment and term assurances, annuities, unit-linked contracts, and unitised with-profits contracts, incorporating multiple decrement models as appropriate.

3. Profit test simple annual premium contracts of the types listed in 2 and determine the profit vector, the profit signature, the net present value, and the profit margin.

4. Show how a profit test may be used to price a product, and use a profit test to calculate a premium for a conventional (without profits) life insurance contract.

0 Introduction

In this chapter we introduce the technique of profit testing. This is the process of projecting the income and outgo emerging from a policy, and discounting the results. The results can then be used for various different purposes, such as setting the premium for a life policy that will give us our required level of profitability.

We shall see in the next chapter how we can also use profit tests to set reserves, and various other applications.

Before discussing profit tests in detail, we describe the structure of unit-linked contracts, and how profit emerges from them.

1 Unit-linked contracts

We begin by introducing a further type of contract that we will discuss in this chapter.

Unit-linked assurances **(typically whole life or endowment) have benefits that are directly linked to the value of the underlying investments. Each policyholder receives the value of the units allocated to the policy. There is no pooling of investments or allocation of the pooled surplus. As each premium is paid, a specified proportion (the "allocation percentage") is invested in an investment fund chosen by the policyholder. The investment fund is divided into units which are priced continuously.**

When each investment allocation is made, the number of units purchased by the policyholder is recorded. The value at the date of death or survival (*ie* at the time of the claim) **of the cumulative number of units purchased is the sum assured under the policy. Sometimes a minimum guaranteed sum assured is specified in the terms of the contract to ensure that the policyholder avoids any difficulties arising from a particularly poor investment performance.**

In addition, a minimum guaranteed sum assured will give the policyholder (or his estate) some benefit in the event of an early death. Without it the policy becomes a pure savings policy, with no element of life insurance involved.

Unit-linked policies can have a minimum guaranteed sum assured payable on survival to a specified date, on death, or both. Death benefit guarantees are generally more common than survival (or *maturity*) guarantees. In fact, policies with investment guarantees have proved very costly in the UK in the past.

In order to price and value unit-linked contracts, details of allocation percentages (usually specified in the policy) and an assumption about the future growth in the price of the units purchased are needed.

Question 12.1

Describe the type of benefit provided by a unit-linked contract.

Important terminology that you will come across includes:

- Unit account – the total value of the units in respect of the policy at any time.
- Bid and offer price – most life companies use what is called the bid/offer spread to help cover expenses and contribute to profit. The policyholder buys units with his premium at the *offer* price, and at maturity the company buys those units back from the policyholder at the *bid* price. The bid price will typically be 5% lower than the offer price.
- Charges – the company will deduct money from the unit account on a periodic basis, for instance monthly. The company normally reserves the right to vary charges in the light of experience.

The following example illustrates how these all work.

Example

A unit-linked endowment assurance policy could look like this:

Premiums of £1,500 are paid annually for ten years. 97% of every premium paid is used to buy units. The policyholder can choose between five different funds, *eg* the company's European Equities fund. There is a 5% bid/offer spread.

There is a guaranteed sum insured on death of £40,000 but no guaranteed benefit on maturity.

Every year the company deducts 1% from the bid value of the fund. This charge is made monthly in arrears (*ie* at the end of each month $1.01^{1/12}-1$ of the bid value of the units is deducted). This charge is called a fund management charge and is to pay for administration expenses. The company also deducts £50 *pa* from the policy's unit account. Again, this charge is made monthly in arrears. These charges may be varied in line with the company's experience.

In the first year the policyholder pays the premium of £1,500. Of this 97% = £1,455 is used to buy units, but the company also takes off the bid/offer spread now, so we really buy just 95% of £1,455, *ie* £1,382 worth of units.

The policyholder chooses to buy units entirely in the company's Mixed Fund, which is a mix of local and international bonds and equities.

Due to the investment manager's fortunate decision to invest heavily in a company that specialises in publishing historical romances about actuaries at the time of the French Revolution, the fund grows by 12.3% over the year.

By the end of the year, the unit account has (approximately) become:

$$1{,}382 \times 1.123 \times 0.99 - 50$$

(Actually a calculation of this type would have been done at least monthly, and possibly daily, so the above is slightly inaccurate.)

Note that contracts that are non-unit-linked are normally called *conventional*.

The most important thing to bear in mind with unit-linked contracts is that we have two "worlds" to keep track of: the *unit* world, and a *cash* (or *non-unit*) world. The policyholder pays premiums to acquire units, and the eventual benefit is normally denominated in these units, so we will need to keep track of the number of units bought by a policyholder, how they are growing, and what charges we are deducting from them.

However, the policyholder pays the life insurance company in real money. So we need to keep track of the cash not used to buy units, because that cash is a source of profit to the life insurance company. Conversely, if the policyholder dies there might be a cash denominated sum insured, and so we need to keep track of the cash outgo on claims. Another very significant cash outgo to consider is comprised of the company's expenses. These will include expenses incurred in underwriting and maintaining the policy, as well as commission payments to whoever sold it.

Question 12.2

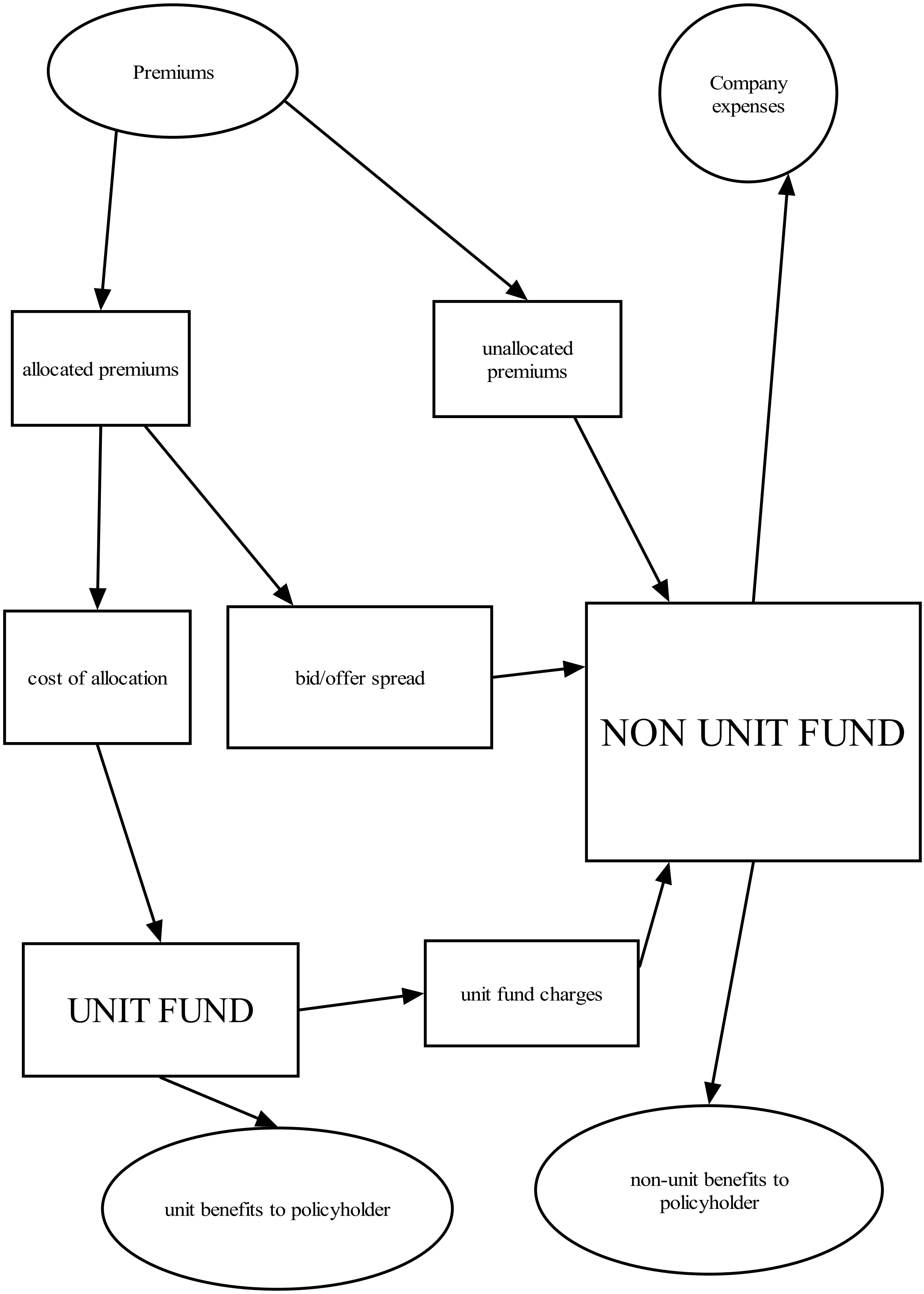
Before glancing at the diagram on Page 6, write down as many of the following as you can think of:

- different charges in the product, and
- different cash outgoes which the company must pay.

To clarify the situation, it is useful to consider things diagrammatically. Note that here, and at most other times in the course, we do not worry about splitting the policyholder's units into different funds, *eg* American Equities, Peruvian Goldmines, *etc*, but consider a generic unit fund.

It is also common to think of the cash world as a specific cash fund. We will refer to this as the *non-unit* fund.

The inter-relationship between the policyholder, the company, units and cash can be encapsulated in the diagram overleaf.


Premiums
Company expenses
allocated premiums
unallocated premiums
cost of allocation
bid/offer spread
NON UNIT FUND
UNIT FUND
unit fund charges
unit benefits to policyholder
non-unit benefits to policyholder

Unit benefits are given to the policyholder on surrender, claim or maturity.

Non-unit benefits will include, for instance, any sum insured payable on death in excess of the value of the units, or any guaranteed maturity value in excess of the value of the units.

Unit fund charges comprise:

- a fund management charge, for instance 0.5% *pa* of the unit fund in respect of fund management expenses,
- a policy fee (monetary deduction from units) to cover other administration expenses, and
- a charge to cover the cost of providing any additional non-unit benefits, *eg* for any extra money paid out (over and above the unit fund value) when there is a guaranteed minimum death benefit.

Question 12.3

One very important influence on the eventual experience of the policy is not shown in the diagram, because it occurs entirely "within" the boxes. What is it?

The important things that come out of this diagram are:

- The unit fund is worth only the *bid* value of the *allocated* premium – everything else in the premium goes to the non-unit fund.
- The unit fund charges are made to pay for the expense of fund management but this is not the same thing as these actual fund management expenses.
- The profit or loss to the life company in each year will be the balance in the non-unit fund between all sources of income (charges, unallocated premium, bid/offer spread) and all sources of outgo (expenses, non-unit benefits).
- The unit fund is what the policyholder sees – for instance, unit growth and all charges will be communicated to the policyholder. The non-unit fund is what goes on within the company, and the policyholder does not see anything at this level.

Question 12.4

For the example given on Page 3, calculate the (accurate) unit fund value at the end of the first month of the policy (deducting all charges at the end of the month) for a 40-year old policyholder, and calculate the total charges arising for the same period.

Question 12.5

Suppose that the company's expenses in respect of this policy in the first month were 55% of annual premium plus £178, and that on average the mortality experience of all such policyholders was 58% of AM92 Ultimate.

Death payments are made at the end of the month, after all charges have been deducted.

Calculate the profit or loss to the company for that first month (ignoring the impact of interest on the non-unit fund).

1.1 Unitised with-profits contracts

These contracts are described in Chapter 6. They operate in a similar way to unit-linked contracts, in that the benefits are expressed as an accumulating (unit) fund of premiums and explicit charges are deducted to cover expenses and other costs. Non-unit cashflows can therefore be projected in the same way as for unit-linked contracts, and used in profit testing as described in the next sections.

2 Projecting expected cashflows for various contract types

The first step is to project the expected cashflows of a hypothetical policy. We first need to decide what units of time to use – for instance, are we going to consider every month of the policy's lifetime, or only every year?

The standard approach is to divide the total duration of a contract into a series of non-overlapping time periods. The length of each time period is chosen so that it is reasonable to make simple assumptions about the cashflows within each period, *eg* funds earn a constant rate of interest during the period, a particular cashflow accrues uniformly during the period. These assumptions allow the expected cashflows during the period to be evaluated.

We consider in great detail below exactly what these assumptions are.

The arithmetic of these calculations is usually most straightforward when the expected cashflows per contract in force at the start of the time period are calculated… as opposed to the expected cashflows per contract in force at policy inception.

In practice, time periods will be short where there are many rapidly changing cashflows, *eg* the start of a contract, and longer where there are fewer cashflows.

So for instance we might project monthly for the first year of a policy's lifetime, and then yearly thereafter. In fact, given the extent of computational power now easily available, it may be simpler to just carry on with a monthly breakdown for the duration of the policy.

Question 12.6

Why are cashflows often changing rapidly in the first year of a contract, to the extent that a yearly breakdown could be very inaccurate?

The expected cashflows are used to construct a projected revenue account (per contract in force at the start of the period) for each time period. For some contracts, *eg* life assurance, there is only one way in which a contract remains in force and so only one projected revenue account is needed.

The revenue account for a life company is:

(+) premiums

(+) investment income

(–) expenses

(–) benefit payouts (claims, maturity, surrender values)

(–) increase in reserves

= *profit gross of tax*

(–) tax

= *profit net of tax*

(Here the investment income is the income on reserves, plus fractional interest on the balance of the cashflow itself; we exclude the investment income on the life company's other assets unconnected with the policies under consideration.)

So all that we are trying to do is to construct such a revenue account in respect of an individual (and probably hypothetical) policy for some given contract type. If we are doing this for a unit-linked contract then our projections get more complicated; as you might guess from Section 1, we shall need to project the unit fund and the non-unit fund separately. We shall consider an example of this later.

For other contracts, *eg* disability insurance or long term care insurance, there is more than one way in which the contract remains in force, *eg* policyholder alive and not receiving disability benefits, policyholder alive and receiving disability benefits. Projecting cashflows in such cases will require separate projected revenue accounts for each in-force status, but the details of this will not be examined in this subject.

The balancing item in the projected revenue account is the profit emerging at the end of the time period.

How do we go about calculating the individual elements of the projected revenue account for a policy? To do this, we require a *basis* – meaning simply a set of assumptions. We have already come across assumptions used to calculate premiums (*eg* AM92 Select, 6% interest, expenses of … *etc*).

We refer to the set of assumptions used for any particular task, such as calculating premiums, as a basis. Very often, the basis for these projected cashflows will be a realistic estimate of expected future experience. This is as opposed to the more prudent estimates we might use for reserving. We consider the question of different bases in more detail in the next chapter.

In order to calculate the expected cashflows the following information is needed:

- **premiums paid and their times of payment**
- **expected expenses (from the basis) and their times of payment**
- **contingent benefits payable under the contract, *eg* death benefit, annuity payment, survival benefit for endowment, difference between guaranteed sum assured and value of unit fund for unit-linked endowment**
- **other benefits payable under the contract, *eg* surrender values**
- **other expected cash payments, *eg* taxes**
- **other expected cash receipts, *eg* management charges levied on a unit fund**
- **the reserves required for a contract, usually at the beginning and end of the time period, calculated using the valuation basis**

together with the different probabilities of the various events leading to the payment of particular cash amounts. Any balance on the expected revenue account during the time period will be invested, and an assumption about the rate of return on these funds is needed. This allows the expected investment income during the period to be calculated and credited at the end of the period.

2.1 Example 1: Conventional whole life assurance

The contract is issued to a select life aged x and has a sum assured of S secured by level annual premiums of P. The premium basis assumes initial expenses of I and renewal expenses of e. The reserving basis requires reserves of $S\,{}_tV$ for an in force policy with sum assured S at policy duration t. The basis assumes that invested funds earn an effective rate i. The surrender value basis determines that a surrender value $(SV)_t$ will be paid to policies surrendered at policy duration t. The probabilities of events are determined from a multiple decrement table with decrements of death, d, and surrender, w, having dependent probabilities at age, x, of $(aq)_x^d$ and $(aq)_x^w$.

<u>Income</u>

Premiums	P **(from data)**
Interest on reserves	$iS\,{}_tV$
Interest on balances	$(P-e)i$

<u>Expenditure</u>

Expenses	e **(from data)**
Expected surrender value	$(aq)_{[x]+t}^w\,(SV)_{t+1}$
Expected death claims	$(aq)_{[x]+t}^d\,S$
Transfer to reserves	$(ap)_{[x]+t}\,S\,{}_{t+1}V - S\,{}_tV$
Profit	**Balancing item**

This assumes that expenses occur at the start of each time period, and death claims and surrender values are paid at the end of each time period.

Temporary insurances and endowment assurances follow similarly.

Question 12.7

Explain in words what the item "transfer to reserves" represents.

Example

A life insurance company sells a 5-year regular-premium endowment assurance policy to a 55-year old male. The sum insured is £10,000 (payable at the end of year of death). Initial expenses are 50% of annual premium, renewal expenses are 5% of subsequent premiums. Premiums are payable annually in advance.

There is a surrender benefit payable equal to return of premiums paid, with no interest. This is paid at the end of the year of withdrawal.

The company is required to hold net premium reserves, calculated ignoring surrenders.

Calculate the projected yearly cashflows per policy in force at the start of each year, using the following bases.

For pricing: AM92 Ultimate mortality, 4% *pa* interest, expenses as above and ignoring surrenders.

For valuation: Interest and mortality as per pricing.

For future cashflow projection: Interest and expenses as per pricing, dependent surrender and mortality probabilities as in the table below.

Age x	$(aq)_x^d$	$(aq)_x^w$
55	0.005	0.1
56	0.006	0.05
57	0.007	0.05
58	0.008	0.01
59	0.009	0

Solution

We first need to calculate the annual premium payable, *P*. This will satisfy the equation

$$P\ddot{a}_{55:\overline{5}|} = 10,000A_{55:\overline{5}|} + 0.5P + 0.05Pa_{55:\overline{4}|}$$

So:

$$P = \frac{10,000A_{55:\overline{5}|}}{0.95\ddot{a}_{55:\overline{5}|} - 0.45} = £2,108.81$$

We now need to work out the reserves that the company must hold over the term of the policy. The reserve at time t will be:

$${}_tV = 10,000\left(1 - \frac{\ddot{a}_{55+t:\overline{5-t}|}}{\ddot{a}_{55:\overline{5}|}}\right)$$

So we have:

$${}_0V = 0$$

$${}_1V = 10,000\left(1 - \frac{\ddot{a}_{56:\overline{4}|}}{\ddot{a}_{55:\overline{5}|}}\right) = 1,832.06$$

$${}_2V = 10,000\left(1 - \frac{\ddot{a}_{57:\overline{3}|}}{\ddot{a}_{55:\overline{5}|}}\right) = 3,740.46$$

and so on.

A useful check here is that we expect the reserves to be vaguely similar to total net premiums notionally paid to date. Thinking about retrospective reserve calculations, reserves will be equal to net premiums rolled up with interest and less cost of cover. The roll-up increases the reserves, the cost of cover decreases them, and they will, very roughly, cancel out.

(Here the net premium is $10,000\,A_{55:\overline{5}|}\big/\ddot{a}_{55:\overline{5}|} = £1,796.40$, which is close to ${}_1V$; two net premiums is about £3,600, which is close to ${}_2V$, and so on.)

This gives the following development of reserves.

Year t	Reserve at start of year, ${}_{t-1}V$
1	0
2	1,832.06
3	3,740.46
4	5,736.10
5	7,818.97

We now need to start calculating the cashflows for a policy in force at the start of each year. For instance, for the first year we will have:

Premium paid in of	2,108.81	
Expenses	–1,054.41	from $0.5 \times 2{,}108.81$
Interest on other cashflow	42.18	from $(P-e)i$
Expected death claims	–50	from $10{,}000(aq)_{55}^{d}$
Expected surrenders	–210.88	from $P(aq)_{55}^{w}$
Expected maturities	0	

Increase in reserves

At the start of the year we have no reserve. So there is no interest earned on the reserve.

At the end of the year we need 1,832.06 per policy still in force. The probability that a policy in force at the start of Year 1 is still in force at the end of Year 1 is $(ap)_{55} = 1-(aq)_{55}^{d}-(aq)_{55}^{w} = 0.895$.

So the expected cost of increasing the reserve is $1{,}832.06 \times 0.895 = 1{,}639.69$

This gives an expected profit of –803.99 at the end of Year 1.

We can carry on and do the same over the remaining years of the policy's term:

Year	Prem	Exps	Interest	Claim cost	Cost of surrs and maturity	Cost of increase in reserves	Profit
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1	2108.81	–1054.41	42.18	–50	–210.88	–1639.69	–803.99
2	2108.81	–105.44	80.13	–60	–210.88	–1625.65	186.97
3	2108.81	–105.44	80.13	–70	–316.32	–1519.06	178.12
4	2108.81	–105.44	80.13	–80	–84.35	–1712.68	206.47
5	2108.81	–105.44	80.13	–90	–9910	8131.73	215.23

where the cost of surrenders column (6) includes the cost of maturing policies.

Question 12.8

Explain why there is a big loss in the first year, despite the fact that the expenses "experienced" were the same as those used to price the policy.

Question 12.9

Verify the entries for Year 3 of this policy.

Question 12.10

For Year 5, verify the entries for cost of maturities and cost of increase in reserves.

2.2 Example 2: Unit-linked endowment assurance

The contract is issued to a life aged x and has a sum assured equal to the bid value, at the time of death, of the units purchased, subject to a minimum guaranteed sum assured of S. It is secured by level annual premiums of P, of which a_t% is allocated to the unit fund at the start of policy year t at the offer price.

(Note that the $a_t\%$ is an arbitrary number – it should not be confused with an annuity value.)

The premium basis assumes initial expenses of I and renewal expenses of e per annum incurred at the beginning of each of the second and subsequent policy years. Unit reserves are held in the unit fund, and no allowance is made for reserves in the non-unit (cash) fund.

By unit reserve, here we mean the bid value of units, *ie* we reserve for what the policyholder would get on withdrawal. We shall see in the next chapter that it is sometimes necessary to hold not only unit reserves but also non-unit reserves, *ie* a cash-denominated reserve as a contingency against future negative (non-unit) cashflows. In this example no such reserves are required.

Investments in the non-unit fund are assumed to grow at i% *pa*. Suppose the unit fund projections show a fund value of F_t at policy duration t after management charges at a rate k_t% *pa* have been paid to the non-unit fund. The bid price (sale price) of units is $(1-\lambda)\times$offer price, *ie* there is a bid-offer spread of $100\lambda\%$. F_t is evaluated at the bid price.

The annual management charge paid from the unit fund to the non-unit fund together with the bid-offer spread (*ie* units are sold to policyholders at a price greater than their underlying value) are analogous to charges covering the level annual expenses for a traditional assurance policy.

They are analogous in that the main way the company covers level annual expenses with unit-linked contracts is via these two charges. However, the fund management charge has a very different "shape" to what we see with conventional contracts because, as the fund grows, the fund management charge will grow. This growth will be very significant for regular premium contracts.

Note that the charge of $(1-a_t\%)$, which results from the allocation percentage imposed by the company, can be used to cover both initial expenses (with a very low a_1) and renewal expenses (with a sufficiently low a_t for $t>1$).

On death, the bid value of the unit fund, after management charges, is paid at the end of the year of death, subject to the minimum guaranteed sum assured of S.

The expected cashflows to and from the non-unit fund in policy year t can be evaluated, where $t=1$ denotes the first policy year.

Income

Premiums not allocated to unit fund $\left(1-\frac{a_t}{100}\right)P$

Bid-Offer Spread $\lambda\frac{a_t}{100}P$

Management charge on the unit fund (taken at the year-end)

$$\frac{F_{t+1}}{\left(1-\frac{k_t}{100}\right)}\left(\frac{k_t}{100}\right) \quad \text{(from unit fund projections)}$$

Investment Income on Balances $\left\{\left(1-\frac{a_t}{100}+\frac{\lambda a_t}{100}\right)P-I\right\}i \quad t=1$

$\left\{\left(1-\frac{a_t}{100}+\frac{\lambda a_t}{100}\right)P-e\right\}i \quad t>1$

Expenditure

Expenses e **(from data) (or I when $t=1$)**

Expected cost of death claims $(S-F_{t+1})q_{x+t}$ **if positive**

Profit **Balancing item**

The fact that no non-unit reserves were deemed necessary means that there is no "income on reserves" element in the income section.

It is important to realise that the above gives us only the cashflows in the non-unit fund. It will normally be necessary first to calculate the projected unit fund, so as to calculate the F_{t+1} values needed for several of the above items.

We shall see how this works in the following numerical example.

Example

A life insurance company is studying the profitability of a 5-year unit-linked endowment assurance contract. Details are as follows:

Age at inception	50	
Annual premium	£2,000	
Benefit	The greater of the bid value of units and £5,000 (paid at the end of the policy year)	
Allocation	First year:	60%
	Other years:	98%
Bid/offer spread	5%	
Management charge	1% (deducted at end of year)	
Unit growth	6%	
Interest for non-unit fund	6%	
Mortality	AM92 Ultimate	
Expenses	Initial	£1,150
	Renewal	£75 at the start of the second year, subsequently inflating at 4% *pa*

Calculate the expected profit or loss on the non-unit fund in each year, per policy in force at the start of each year.

Solution

We first need to project the size of the unit fund over the term of the policy.

Question 12.11

Using the diagram from Page 6 as a guide, calculate the unit fund at the end of the first year.

Solution (continued)

We can repeat the logic seen in the first year's development of the unit fund and project the fund to the end of the five-year term:

Year	Prem rec'd	Prem all'd	Cost of all'n	Fund after all'n	Fund before mgt charge	Mgt charge	Fund at year end
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1	2,000	1,200	1,140	1,140.00	1,208.40	–12.08	1,196.32
2	2,000	1,960	1,862	3,058.32	3,241.81	–32.42	3,209.40
3	2,000	1,960	1,862	5,071.40	5,375.68	–53.76	5,321.92
4	2,000	1,960	1,862	7,183.92	7,614.96	–76.15	7,538.81
5	2,000	1,960	1,862	9,400.81	9,964.86	–99.65	9,865.21

So we have the columns developing as follows:

(3) premium allocated = premium received × allocation percentage

(4) cost of allocation = premium allocated (3) × (1 – bid/offer spread)

(5) fund after allocation = fund at end of previous year (8) + cost of allocation (4)

(6) fund on day 364 = fund after allocation (5) × (1 + unit growth rate)

(7) fund management charge = fund on day 364 (6) × management charge

(8) fund at end year = fund on day 364 (6) – management charge (7)

Question 12.12

Verify the entries for Year 3.

Solution (continued)

Having done this, we can start to calculate the non-unit movements. For instance, in the first year we will have the following elements of outgo

Expenses 1,150

Death cost $q_{50}\left(sum\ assured - unit\ \ fund\right) = 0.002508 \times \left(5{,}000 - 1{,}196.32\right)$

$= 9.54$

Question 12.13

Calculate the income to the non-unit fund in this first year, remembering to allow for interest and the fund management charge.

Question 12.14

Hence calculate the non-unit profit or loss in the first year.

Solution (continued)

We can do the same for every year of the contract, to give the profit in each year per policy in force at the start of that year.

One area where we need to be slightly careful is in calculating the cost of death cover, because if the value of the units goes above the guaranteed sum insured then the death cost will be zero.

We find the following development of non-unit cashflows. It helps to think chronologically through the company's cashflows when constructing tables like this.

Year t	Premium less cost of all'n	Expenses	Interest	Extra death cost	Mgt charge	Profit
(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	860	–1,150.00	–17.40	–9.54	12.08	–304.86
2	138	–75.00	3.78	–5.03	32.42	94.17
3	138	–78.00	3.60	0.00	53.76	117.36
4	138	–81.12	3.41	0.00	76.15	136.44
5	138	–84.36	3.22	0.00	99.65	156.51

We have the following derivations:

(2) Premium less cost of allocation from the unit fund table (2) – (4)

(3) Expenses from the set of assumptions

(4) Interest as $\big((2)-(3)\big)i$

(5) Extra death cost as $q_{50+t-1}\big(5{,}000 - unit\ \ fund\ \ end\ \ year, (8)\ in\ unit\ \ fund\ \ table\big)$

(6) Management charge from unit fund table (7)

(7) Profit = sum of all of these

Be careful about signs here. From the point of view of the company (*ie* the non-unit fund's point of view), the management charge is a positive because it is something deducted from the unit fund (the "policyholder's fund") – although it was a negative when we were considering it in the context of the unit fund.

One way of thinking about all of these items is "if it's good for profits, it must be positive" – so a management charge must be positive, because if the company increases it then profits go up. Similarly the death cost must be negative: if we get more deaths, company profits go down.

2.3 Example 3: Single premium unitised with-profits contract

For this example we assume the contract has a five-year term, paid for by a single premium. The allocation rate is 100% and there is no bid-offer spread. The only charges under the contract are an admin fee equal to 0.5% of the bid value of the unit fund, deducted at the end of each year by cancelling units, and a penalty on surrender calculated as a percentage of the unit fund value depending on policy duration as follows:

After 1 year: 2.4%

After 2 years: 1.8%

After 3 years: 1.2%

After 4 years: 0.6%

The unit price increases each year according to the insurer's declared bonus interest rate. The full value of the unit fund, plus any discretionary terminal bonus, is paid out on maturity or at the end of the year of earlier death. Surrender is permitted only at the end of each year, when the fund value less the above surrender penalty would be paid out.

You can assume that, over the long term, all profits earned from investment returns are fully distributed to policyholders through the bonus payments (we explain the significance of this assumption after the solution, below).

Calculate the projected profit for each policy year based on the following assumptions:

Age at entry:	45 exact
Single premium:	£20,000
Mortality:	$q_x = 0.002$ for $x = 45, 46, ..., 49$
Surrender rate:	5% of all policies in force at the end of each of Years 1-4
Bonus interest rate:	3.5% *pa*
Non-unit interest:	2% *pa*
Initial expenses:	£300
Renewal expenses:	£25 at the start of Year 2, and thereafter at the start of each subsequent year, inflating at 2.5% *pa*
Claim expenses:	£100 per death, surrender, or maturity
Non-unit reserves:	£50 per policy in force at the start of Year 5

Solution

The first thing we need to do is to project the unit fund values and the policy fees each year. At the end of the first year, the fund value (before deduction of the policy fee) is the single premium plus one year's bonus:

$$20,000 \times 1.035 = 20,700$$

The policy fee at the end of the year is:

$$20,700 \times 0.005 = 103.50$$

which means that the fund remaining at the end of the year is:

$$20,700 - 103.50 = 20,596.50$$

Working through the other years in a similar way, we obtain the following table of projected values:

Year t	Fund at end of year before deduction of policy fee (A)	Policy fee (B)	Fund at end of year after deduction of policy fee (C)
1	20,700	103.50	20,596.50
2	21,317.38	106.59	21,210.79
3	21,953.17	109.77	21,843.40
4	22,607.92	113.04	22,494.88
5	23,282.20	116.41	23,165.79

where, for years $t = 2, 3, 4, 5$:

$$(A)_t = (C)_{t-1} \times 1.035$$

$$(B) = (A) \times 0.005$$

$$(C) = (A) - (B)$$

Next we need to work out the surrender penalties. These are equal to the end-year fund values (after charges) multiplied by the appropriate percentage rates:

Year t	Fund at end of year	Surrender penalty
1	20,596.50	494.32
2	21,210.79	381.79
3	21,843.40	262.12
4	22,494.88	134.97
5	23,165.79	0

We will also need the dependent probabilities of decrement by death and surrender. Because surrenders take place at the end of each year, the dependent probability of dying is the same as the independent probability, *ie* equal to 0.002 each year. The dependent surrender probability is found as:

$$(1-0.002)\times 0.05 = 0.0499$$

as we are told that 5% of the *end* of year in-force policies surrender each year.

The dependent probability of a policy staying in force over any particular year is then:

$$1-0.002-0.0499 = 0.9481$$

(*or alternatively this can be calculated as* $(1-0.002)\times(1-0.05) = 0.9481$).

The non-unit cashflows can now be calculated, which for the first four years of the contract are as follows:

Year t	Initial and renewal Expenses (1)	Interest (2)	Policy fee (3)	Expected surrender profit (4)	Expected claim expenses (5)	Expected non-unit Cashflow (6)
1	– 300	– 6	103.5	24.67	– 5.19	– 183.02
2	– 25	– 0.5	106.59	19.05	– 5.19	94.95
3	– 25.62	– 0.51	109.77	13.08	– 5.19	91.53
4	– 26.27	– 0.53	113.04	6.73	– 5.19	87.78
5						

where:

$$(1)_t = (1)_{t-1} \times 1.025 \quad (3 \le t \le 5)$$

$$(2) = (1) \times 0.02$$

$$(3) = \{\text{fund value}\} \times 0.005$$

$$(4) = \{\text{surrender penalty}\} \times 0.0499$$

$$(5) = -100 \times (0.002 + 0.0499) \quad (1 \le t \le 4)$$

$$(6) = (1) + (2) + (3) + (4) + (5)$$

Question 12.15

Calculate the table entries for Year 5.

Having worked out the expected cashflows, we now need to calculate the expected profit for each year, taking into account the effect of the non-unit reserves.

For this policy, we are told that a non-unit reserve of 50 is required per policy in force at the start of Year 5 (and also therefore per policy in force at the *end* of Year 4); at all other times no non-unit reserve is to be held.

How will this affect the expected profit in Year 4? For a policy in force at the start of Year 4, the insurer expects a cashflow of 87.78 by the end of the year. However, the insurer now has to set aside reserves of 50 for each policy that's remained in force through to the end of the year. The probability of doing this is 0.9481 (from above).

So the expected profit for Year 4 (per policy in force at the start of the year) reduces to:

$$87.78 - 50 \times 0.9481 = 40.38$$

Question 12.16

Calculate the expected profit that will now be earned from a policy in force at the start of Year 5, including the reserve of 50 held at the start of the year.

As there are no reserves held at any time in Years 1-3, the expected profits for these years are all equal to the expected cashflows shown in the table above.

Non-unit reserves, such as in this example, are required for unit-linked and unitised with-profits policies wherever a future negative cashflow is expected.

Question 12.17

Given that there was an expected negative cashflow of 11.06, it might have seemed logical to hold the smallest reserve possible to cover this expected cost, *ie* 11.06 rather than the 50 actually held. Suggest a reason why the insurer might have used the higher figure.

What's the point of profit-testing for unitised with-profits?

As explained in an earlier chapter, the idea of a with-profits contract is to return to the policyholder the profits earned by the insurer on the policy over the policy term. One approach for unitised with-profits is to earmark the investment profits for policyholders (via the unit fund), and deduct explicit charges to cover the insurer's non-unit expenses and other costs, as in the example we have just looked at. By projecting the future non-unit profits, the insurer can check that its charges are on track to cover these outgoes.

This was why in the above example we assumed that all the investment profits were distributed to policyholders. An exam question may or may not state this assumption explicitly, but it generally would be implied, as the main point of profit testing these contracts in this way would be to "test" the adequacy of the charges in covering the non-unit liabilities.

3 Profit tests for annual premium contracts

Having now considered how to project the revenue accounts for a policy, we see how to use that projection.

The first step in the profit testing of a contract is the construction of the projected revenue accounts for the non-unit (cash) fund for each policy year.

For some contracts other funds, *eg* unit fund for unit-linked assurances, reserve fund for traditional assurances, provide cashflows to the non-unit fund.

For conventional products, the Core Reading uses the phrase "reserve fund" to refer to the policy reserves. For conventional profit tests, the whole profit test is really a "non-unit fund projection", where one of the elements will be the interest on reserves.

In such cases these funds will need to be projected so that the expected cashflows to the non-unit fund can be determined. These calculations will require data items about the contract, *eg* proportion of premium allocated to purchase of units, bid-offer spread in unit prices; and assumptions which form a basis for the calculations, *eg* growth rate of unit fund, mortality and interest rate basis used to calculate required reserves.

These expected cashflows, together with the direct expected cashflows into and out of the non-unit fund, are the components of the projected revenue account. The calculation of the direct expected cashflows will also require data items about the contract, *eg* initial and renewal expenses; and assumptions to form a basis, *eg* mortality of policyholders, rate of return earned on non-unit fund.

Profit vector

The vector of balancing items in the projected revenue accounts for each policy year is called the profit vector; $(PRO)_t$, $t = 1, 2, 3, \ldots$. The profit vector gives the expected profit at the end of each policy year per policy in force at the beginning of that policy year.

Example

For instance the endowment assurance policy studied in Section 2.1 gave the following profit vector:

$$(-803.99,\ 186.97,\ 178.12,\ 206.47,\ 215.23)$$

For some contracts the expected profit will depend upon the policyholder's status at the beginning of the policy year, *eg* receiving or not receiving sickness benefit (although examples of this should not appear in the exam).

Profit signature

The vector of expected profits per policy issued is called the profit signature. This is obtained by using transition probabilities from policy duration 0 to policy duration $t-1$.

***Eg* (life assurance)** $(PS)_t = {}_{t-1}p_x \, (PRO)_t$

> ***Note***
>
> Summarising the above, we have the very important distinction:
>
> - profit vector = profits per policy in force at the start of each year
> - profit signature = profits per policy in force at inception

So all of the work we did in Section 2 went to determining the profit vector of contracts, that is, the profits for each year per policy in force at the start of the relevant *year*. For the profit *signature*, we wish to calculate (for each year) the expected amount of profit emerging in that year, per policy in force at *inception*.

For example, for the endowment assurance policy studied in Section 2.1, we found the second element of the profit vector to be £186.97.

> ***Question 12.18***
>
> For which policies in force do we expect a profit of £186.97?

Now consider how we would obtain the expected profit for the same year (year 2), but per policy in force at inception (and hence calculate the second element of the profit signature). First consider the random variable:

P_2 = profit emerging in year 2 from a policy that has just incepted

This can take two possible (conditional expected) values:

$$P_2 = \begin{cases} 186.97 & \text{If policy remains in force to at least time 1} \\ 0 & \text{Otherwise} \end{cases}$$

So, using:

$$E(P_2) = \sum_r r \times P(P_2 = r)$$

we find:

$$E(P_2) = 186.97 \times P(\text{policy remains in force to at least time 1})$$

Question 12.19

Use the relevant table of dependent decrement probabilities from Section 2.1 (reproduced below) to calculate the required probability, and hence calculate the value of $E(P_2)$.

Age x	$(aq)_x^d$	$(aq)_x^w$
55	0.005	0.1
56	0.006	0.05
57	0.007	0.05
58	0.008	0.01
59	0.009	0

This value will therefore form the second element of the profit signature for this policy, for which we use the notation $(PS)_2$.

The other elements of the profit signature can be obtained in the same way: by multiplying the profit vector element for a given year by the *probability of a policy remaining in force from policy inception to the start of the year in question*.

Question 12.20

Calculate the other elements of the profit signature for the endowment insurance policy studied in Section 2.1.

The vector representing the profit signature $(PS)_t$; $t = 1, 2, 3, \ldots$ can be displayed graphically to illustrate the way in which profits are expected to emerge over the lifetime of the policy. However, it is difficult to compare this information for different policies when there is a need to evaluate alternative designs for a product (policy) or to decide which of several different possible policies is the most profitable. Decisions like this are usually made easier by summarising each profit signature as a single figure. We describe two ways in which such summary measures can be determined.

3.2 Summary measures of profit

Summary measures usually involve determining the present values of the expected cashflows. This requires an assumption about the discount rate. This rate is chosen to equal the cost of capital (rate at which funds can be borrowed if this is necessary or rate which funds would otherwise earn if funds are to be diverted from alternative investment opportunities) plus a premium to reflect the risks and uncertainties surrounding the cashflows to and from the policy, and is called the risk discount rate, i_d. So:

Risk discount rate = risk-free rate + margin for risk

Writing a policy can be thought of as an investment by the shareholders of the company, because they supply the capital to make good the shortfall between the premium income and the outgo of expenses and the setting up of reserves. If the shareholders are providing capital, then they expect a return on that capital appropriate to the riskiness of their investment. It is much riskier investing by writing life insurance business than by buying government bonds, because more things could go wrong.

So to quantify the extra risk, we add a margin to the investment returns on relatively risk-free assets such as government bonds, and then price the product using the resultant risk discount rate. This will then give us premiums that contain an adequate allowance for the risk.

How do we price the product using the resultant risk discount rate? This is done by projecting cashflows, and then varying premiums until we meet the profit criterion (as described below) which has been calculated using our risk discount rate.

Net present value (NPV)

This is the present value of the profit signature determined using the risk discount rate.

$$\text{NPV} = \sum_{t=1}^{t=\infty} (1+i_d)^{-t}\,(PS)_t$$

The NPV can be interpreted as the EPV of the future profits from the policy, for a single policy as at the start date of the contract.

Question 12.21

Calculate the net present value of the endowment assurance policy studied in Section 2.1, using a risk discount rate of 7%.

Question 12.22

Give one reason why the net present value is negative.

Profit margin

This is the NPV expressed as a percentage of the EPV of the premium income. If the premium paid at the beginning of the t th policy year is P_t, this is

$$\frac{\sum_{t=1}^{t=\infty}(1+i_d)^{-t}\,(PS)_t}{\sum_{t=1}^{t=\infty}(1+i_d)^{-(t-1)}\,{}_{t-1}p_x\,P_t}$$

In other words, the profit margin is:

$$\frac{\text{NPV}}{\text{EPV premiums}}$$

where the risk discount rate is used to do the discounting in both the numerator and the denominator.

Question 12.23

Calculate the profit margin of the endowment assurance policy studied in Section 2.1, using a risk discount rate of 7%.

3.3 Choosing the risk discount rate

The choice of risk discount rate should lead to a market-consistent assumption as far as possible, using the financial economics idea of risk neutral valuations.

For example, one approach to calculating the NPV would be to take the future cashflows, undiscounted but allowing for the risk attaching to the cashflows, and value them using spot yields on a series of (risk free) zero coupon government bonds of matching duration.

Then the discounted value of the expected cashflows found by this calculation should agree with the expected cashflows, free of risk margins, discounted at the risk discount rate.

4 Determining premiums using a profit test

If the premium for a contract together with all the other data items about the contract are known, then given a basis on which the projected revenue accounts can be calculated, the expected profitability of the contract can be evaluated using the criteria in Section 3.2. The actual profitability is an unknown quantity until each respective contract terminates and the actual experience becomes known.

However, in developing products the expected level of profit will usually be specified as an objective and we ask how the features of the product can be set to achieve this objective. Usually, the benefits and terms and conditions for the payment of these benefits are specified in advance with the result that only the level and pattern of premium payments can be varied to meet the profit objective. In the case of unit-linked contracts there is the additional possibility of varying the charges and typically this would be the approach taken in pricing such contracts to achieve the profit objective.

4.1 Profit criterion

The objective specified for expected level of profit is termed the "profit criterion".

Careful choice of a profit criterion is central to the actuarial management of the company selling the products. It is common for those marketing and selling the products to receive part of their salary in the form of a "productivity" bonus *eg* commission which is a percentage of the total premiums for the policies sold. If the profit criterion chosen is directly related to this "productivity" bonus, then the company's profits will be maximised if the salesforce maximises its income. Such considerations are important in choosing the profit criterion to be used.

Examples of the profit criterion are:

NPV = 40% of Initial Sales Commission

Profit Margin = 3% of EPV of premium income

For conventional products, the profit test is completed using a spreadsheet or similar software, and the premiums are varied until the required "target" *ie* NPV, level of profit margin, is achieved. The premium or price of the product has been determined using a profit test.

For unit-linked products, the management charges are varied to try to achieve an acceptable charging structure (in comparison with other products in the market) which satisfies the profit criterion.

The sensitivity of this profit to variation in the key features of the product design, *eg* benefits offered, and the assumptions made in determining the expected cashflows, *eg* mortality rates, rate of return on investments are usually investigated. This is done by keeping the premium or charging structure fixed and determining the change in the profit criterion for realistic variation in the characteristics of the product and the assumptions made in the basis. (See "sensitivity test assumptions" in Section 1 of Chapter 13.) The objective is to design a product which is robust (*ie* profit changes as little as possible) to possible changes in the data and the assumptions used in the profit test.

So the steps in profit testing could be summarised as follows:

- decide on the structure of the product, *eg* single premium unit-linked deferred annuity
- build a model to project cashflows for the product
- choose some specimen policies – what age, sex, level of cover do we expect?
- decide on a risk discount rate and profit criterion
- choose a basis – probably our best estimate – of all important parameters required for the profit test, *eg* unit growth, levels of withdrawals, *etc*
- decide on some "first draft" premiums (conventional product) / charges (unit-linked product)
- profit test our specimen policies using these premiums
- vary the premiums / charges until our profit criterion is met
- remembering that premiums need to be acceptable in the market
- sensitivity test by doing the cashflow projection varying key parameters, *eg* investment return, mortality, expense inflation, to check that our product design is sufficiently resilient to adverse changes in future experience
- keep varying premiums and if necessary features of the product design until we have a product that:
  - meets the profitability criterion,
  - is marketable, and
  - is resilient to adverse future experience.

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Chapter 12 Summary

Unit-linked contracts

With unit-linked contracts the policyholder's basic entitlement is expressed in terms of units, which represent a portion of a fund or funds run by the life insurer. The value of these units moves in line with the performance of the fund. Given this basic concept, the unit-linked idea can be used to provide many different types of product, for instance:

- a regular premium product offering a guaranteed sum assured on death, to give an endowment assurance;
- a single premium product offering a return of premium on death to give a pure savings bond;
- a regular premium contract offering an annuity payment during periods of disability or unemployment; or,
- a single premium product offering an annuity payment until death.

Unit fund

For the unit fund, we will have the following elements to consider:

- premium received
- premium allocated
- cost of allocation
- fund after allocation
- fund at end year pre-management charge
- management charge

to give:

- fund at year end

Non-unit fund

We project the non-unit fund as:

(+) premium less cost of allocation

(–) expenses

(+) interest on non-unit fund

(–) non-unit death cost

(+) management charge transferred from unit fund

= profit

Conventional products

We can project cashflows for a conventional policy according to certain assumptions about future experience by calculating the following elements:

(+) premiums

(+) investment income

(–) expenses

(–) benefit payouts (claims, maturity, surrender values)

(–) increase in reserves

= *profit gross of tax*

(–) tax

= *profit net of tax*

For a unit-linked or unitised with-profits policy, we can do something similar for the non-unit fund after having first projected the unit fund.

Profit criteria

We define:

- profit vector = profits per policy in force at the start of each year
- profit signature = profits per policy in force at inception

We determine a risk discount rate as:

- risk discount rate = risk-free rate + margin for risk

where the margin for risk reflects the degree of riskiness associated with the product, and use the profit signature to calculate:

- net present value (NPV), which is the present value of the profit signature, discounted using the risk discount rate, or
- profit margin, which is the NPV of the profit signature divided by the expected present value of the premium income.

Profit testing

We can set the premiums for a product to give a desired level of profitability by projecting cashflows under a certain basis of assumptions, deciding on a risk discount rate and profit criterion, and then varying premiums until we achieve that profit criterion.

Unitised with-profits contracts

Profit tests can be used to test the adequacy of the charges for covering the non-unit liabilities. The investment profits are usually assumed to be distributed to policyholders through the unit fund and any additional bonuses.

The profit-testing methods are similar to those used for unit-linked contracts.

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Chapter 12 Solutions

Solution 12.1

With unit-linked contracts the policyholder's basic entitlement is expressed in terms of units, which represent a portion of a fund or funds run by the life insurer. The value of these units moves in line with the performance of the fund.

Given this basic concept, the unit-linked idea can be used to provide many different types of product, for instance:

- *a regular premium product offering a guaranteed sum assured on death, to give an endowment assurance;*
- *a single premium product offering a return of premium on death to give a pure savings bond;*
- *a regular premium contract offering an annuity payment during periods of disability or unemployment; or,*
- *a single premium product offering an annuity payment until death.*

Solution 12.2

Charges in the product:

- margin due to allocation rate less than 100%
- bid/offer spread
- fund management charge
- expense charge (policy fee)

Cash outgoes to be paid by the company:

- expenses
- maturity benefit (excess of any cash-denominated guaranteed maturity value over value of units)
- claim benefit (excess of any cash-denominated guaranteed sum insured over value of units)

Solution 12.3

The fund growth experienced by the units, *eg* if our units are all invested in equities, the combination of any capital appreciation of the equities, plus dividends received.

The insurance company is also likely to earn interest on any money it *holds* in the non-unit fund from time to time.

Solution 12.4

Premiums allocated are $97\% \times £1{,}500 = £1{,}455$.

We then deduct the bid/offer spread from this, so the value of units allocated is $(1-5\%) \times £1{,}455 = £1{,}382.25$.

Growth in the month takes this up to $£1{,}382.25 \times 1.0097 = £1{,}395.68$.

(1.0097 is the growth in units over one month at an assumed growth rate of 12.3% per year, because $1.123^{1/12} = 1.0097$.)

We have the following charges:

- cash policy fee of: $\dfrac{£50}{12} = £4.17$
- management charge of $£1{,}395.68 \times \left(1.01^{\frac{1}{12}} - 1\right) = £1.16$.

So charges total £5.33 and the fund at the end of the month reduces to £1,390.35.

Solution 12.5

The difference between the premium and the cost of the allocated premium is:

$$1,500 - 1,382.25 = 117.75$$

Expenses are $0.55 \times 1,500 + 178 = 1,003$.

Interest on the non-unit fund is to be ignored.

The cost of death benefits was $\frac{1}{12} \times (40,000 - 1,390.35) \times 0.58 \times 0.000937 = £1.75$.

From Solution 12.4, the charges from the unit fund at the end of the month total £5.33.

So the profit is $117.75 - 1,003 - 1.75 + 5.33 = -£881.67$.

Solution 12.6

We would expect the pattern of expenses to be very uneven over the course of the first year: big initial expenses on the first day, then small admin expenses spread over the year.

Solution 12.7

The "transfer to reserves" corresponds to reserves ${}_tV$ at the start of the year being considered a source of income for all policyholders in force at that time, and our needing to pay for the establishment of end-year reserves ${}_{t+1}V$ for those policyholders who survive to the end of that year. So an increase in reserves will cost money.

Solution 12.8

The incidence of expenses is very uneven over the course of the policy: big initial expenses, small renewal expenses. Although we have loaded these into the premium, we have ended up with premiums that are equal in every year: in other words our loading for the initial expenses has ended up spread over all years' premiums. So in the first year the initial expenses are not taken care of by the first premium on its own, and we see a big loss.

Solution 12.9

The premium is £2,108.81.

Expenses are 5% of premium = –105.44.

Interest is $4\% \times (2{,}108.81 - 105.44) = 80.13$.

Claim cost is $-10{,}000(aq)^d_{57} = -10{,}000 \times 0.007 = -70$.

Cost of surrenders is $-3P(aq)^w_{57} = -3 \times 2{,}108.81 \times 0.05 = -316.32$.

The reserve at the start of Year 3 is 3,740.46. This earns interest at the rate of 4% *pa* to become 3,890.08 by the end of Year 3. At the end of Year 3, we need a reserve of:

$$_3V = 10{,}000\left(1 - \frac{\ddot{a}_{58:\overline{2}|}}{\ddot{a}_{55:\overline{5}|}}\right) = 5{,}736.10$$

per policy in force. The probability that a policy in force at the start of Year 3 is still in force at the end of Year 3 is $(ap)_{57} = 1 - (aq)^d_{57} - (aq)^w_{57} = 0.943$. So the cost of the increase in reserves is $5{,}736.10 \times 0.943 - 3{,}890.08 = 1{,}519.06$. (This is a cost to the company so it is shown as a negative entry in the table of cashflows.)

The profit is then:

$$2{,}108.81 - 105.44 + 80.13 - 70 - 316.32 - 1{,}519.06 = £178.12$$

Solution 12.10

The cost of maturities will be the maturity amount multiplied by the probability of having survived to the end of that year, *ie* $-10{,}000 \times (ap)_{59} = -9{,}910$.

At the start of Year 5 we have reserves of 7,818.97 per policy in force. This earns interest at the rate of 4% *pa* to become 8,131.73 by the end of Year 5. We don't need to hold any reserve at time 5 because the policies are finished. So there is a release of reserves of 8,131,73.

Solution 12.11

The bid value of premiums allocated will be $2,000 \times 0.6 \times 0.95 = 1,140$.

The fund at end of year, before management charge, is $1,140 \times 1.06 = 1,208.40$.

The management charge is $1\% \times 1,208.40 = 12.08$.

The fund after deduction of management charge will be $1,208.40 - 12.08 = 1,196.32$ (or do as 99% of fund value).

Solution 12.12

The bid value of premiums allocated will be $2,000 \times 0.98 \times 0.95 = 1,862$.

The fund after allocation is $1,862 + 3,209.40 = 5,071.40$.

The fund at end of year, before management charge, is $5,071.40 \times 1.06 = 5,375.68$.

The management charge is 1% of this = 53.76.

The fund after deduction of this management charge will be:

$$5,375.68 - 53.76 = 5,321.92$$

Solution 12.13

Income to the non-unit fund comes from:

- at the start of the year, the margin between premium and cost of premiums allocated, ie $2,000 - 1,140 = 860$, and
- at the end of the year the fund management charge of 12.08 (from the unit fund calculations).

Note that interest on the non-unit fund is negative in the first year due to the effect of expenses:

$$(860 - 1,150) \times 6\% = -17.40$$

The total of all these elements is 854.68.

Solution 12.14

Non-unit profit/loss is income less outgo, *ie*:

$$854.68 - 1{,}150 - 9.54 = -304.86$$

Solution 12.15

The expenses in Year 5 are the Year 4 expenses increased by inflation at 2.5%. The amount is therefore:

$$26.27 \times 1.025 = 26.93$$

The interest is equal to the interest lost over the year due to the expenses incurred at the start of the year. Its amount is:

$$-26.93 \times 0.02 = -0.54$$

The policy fee income is read from the table, and for Year 5 this is 116.41.

As there is no surrender penalty (and no-one surrenders!) then the expected surrender profit from the surrender penalty is zero.

The claim expenses of 100 are paid out on all policies that become claims during the year. All policies in force at the start of year 5 will ultimately claim at the end of the year – either by surviving and receiving the maturity benefit, or by dying during the year and receiving the death benefit. The expected amount of claim expenses is then equal to:

$$\left(q_{49} + p_{49}\right) \times 100 = 100$$

The expected cashflow for year 5 is then calculated as:

$$\{-\text{ expenses}\} + \{\text{interest}\} + \{\text{policy fee}\} + \{\text{expected surrender profit}\}$$
$$-\{\text{expected claim expenses}\}$$
$$= -26.93 - 0.54 + 116.41 - 100 = -11.06$$

The completed cashflow table now reads as follows:

Year t	Initial and renewal Expenses (1)	Interest (2)	Policy fee (3)	Expected surrender profit (4)	Expected claim expenses (5)	Expected non-unit Cashflow (6)
1	– 300	– 6	103.5	24.67	– 5.19	– 183.02
2	– 25	– 0.5	106.59	19.05	– 5.19	94.95
3	– 25.62	– 0.51	109.77	13.08	– 5.19	91.53
4	– 26.27	– 0.53	113.04	6.73	– 5.19	87.78
5	– 26.93	– 0.54	116.41	0	– 100	– 11.06

Solution 12.16

The expected profit for Year 5 will equal:

$$\{\text{expected cashflow}\} + \{\text{reserve at start of year}\} + \{\text{interest on reserve}\}$$
$$-\{\text{expected cost of reserve at end of year}\}$$

Putting in the values (and noting that the reserve at the end of the year is zero) we obtain:

$$-11.06 + 50 + 0.02 \times 50 - 0 = 39.94$$

Solution 12.17

All reserves need to be prudent, so that there will be a high probability that the liabilities (future outgo) will be covered. The insurer would therefore have projected its cashflows on more cautious assumptions, leading to a somewhat higher negative cashflow than 11.06 in Year 5, and decided that a reserve of 50 was necessary to cover this.

Solution 12.18

£186.97 will be the profit expected by the end of the coming year (Year 2), from a policy that was in force at exact time 1.

Solution 12.19

From the definition given in Section 2.1, the required probability is:

$$_1(ap)_{55} = 1-(aq)_{55}^d - (aq)_{55}^w = 1-0.005-0.1 = 0.895$$

So:

$$E\left(P_2\right) = 186.97 \times {}_1(ap)_{55} = 186.97 \times 0.895 = 167.34$$

Solution 12.20

The first-year element of the profit signature is unchanged from the profit vector, because the amount (-803.99) is already the amount per policy in force at the beginning of year 1, *ie* at inception.

For year 3 we need:

$$(PS)_3 = (PRO)_3 \times {}_2(ap)_{55}$$

and in general we will need:

$$(PS)_t = (PRO)_t \times {}_{t-1}(ap)_{55}$$

We calculate $(ap)_{55}$, $(ap)_{56}$, *etc* from:

$$(ap)_x = 1-(aq)_x^d - (aq)_x^w$$

(using the dependent probabilities given in Section 2.1); and $_2(ap)_{55}$ (for example) from the cumulative probability:

$$_2(ap)_{55} = (ap)_{55}\,(ap)_{56}$$

These give:

Year t	$(ap)_{54+t}$	$_{t-1}(ap)_{55}$
1	0.895	1
2	0.944	0.895
3	0.943	0.8449
4	0.982	0.7967
5	not required	0.7824

(For example, $0.7967 = 0.895 \times 0.944 \times 0.943$.)`

The profit signature is then the product of the profit vector and the cumulative in-force probabilities:

Year t	Profit in year t	$_{t-1}(ap)_{55}$	Profit signature
1	–803.99	1	–803.99
2	186.97	0.895	167.34
3	178.12	0.8449	150.49
4	206.47	0.7967	164.49
5	215.23	0.7824	168.40

So the profit signature is (–803.99, 167.34, 150.49, 164.49, 168.40).

Solution 12.21

Notice that the expected profits for each year include all interest earned (on cashflows and reserves) during the year: so the profits are essentially already accumulated with interest ***to*** *the end of each year, so all we have to do is to discount* ***from*** *the end of each year.*

So, discounting the profit signature at 7%, we get:

Year t	Profit signature	Discount factor v^t	Discounted profit
1	–803.99	0.9346	–751.41
2	167.34	0.8734	146.16
3	150.49	0.8163	122.84
4	164.49	0.7629	125.49
5	168.40	0.7130	120.07
Total			–236.85

So the net present value of profits is –£236.85.

Solution 12.22

One reason is that the pricing of the contract did not allow for withdrawals, and on early withdrawal the company will lose out due to not having recouped the initial expenses of the policy.

A second reason (which makes the answer even more negative) is that we have valued profits using a risk discount rate much higher than the 4% interest rate used in calculating the reserves for the product. The effect of such differences is discussed in more detail in the next chapter.

Solution 12.23

The discounted present value of the premiums is £8,059.11, calculated as in the table below. Note that, because premiums are payable in advance, the discount factors are based on the duration at the *start* of each year.

Year t	Premium	In force probability ${}_{t-1}(ap)_{55}$	Discount factor v^{t-1}	Discounted premium
1	2,108.81	1	1	2,108.81
2	2,108.81	0.895	0.93458	1,763.91
3	2,108.81	0.84488	0.87344	1,556.20
4	2,108.81	0.79672	0.81630	1,371.49
5	2,108.81	0.78238	0.76290	1,258.70
Total				8,059.11

So the profit margin is $\dfrac{-236.85}{8,059.11} = -2.9\%$.

Chapter 13

Profit testing and reserves

Syllabus objectives:

(ix) Describe the technique of discounted emerging costs, for use in pricing, reserving, and assessing profitability.

5. Show how the profit test may be used to determine reserves.

0 Introduction

In Chapter 12 we saw how to project the cashflows for policies. In this chapter, we describe how such techniques can be used to set reserves for both unit-linked and conventional contracts, and how changes in the reserving and pricing assumptions affect profit.

1 Pricing and reserving bases

In this section we consider the different bases that may be used in the financial management of a life insurance contract.

The profit test, as described in Chapter 12, needs many assumptions to be made in order to calculate the expected future profits of a contract for comparison with a stated profit criterion.

The assumptions will, in the first instance, be the insurer's best estimate of expected future experience. This can be termed the *experience basis*.

The experience basis is the best estimate of the future expected experience of the contract. If we set premiums using that basis then there is a 50% probability that experience will be worse than that assumed. Many life insurance companies may see this as acceptable because of the implicit allowance for risk in the pricing basis. In fact, competition may force a company into marketing a "loss leader".

Assuming that the chosen risk discount rate reflects fully the uncertainties in the assumptions, no further margins would be taken.

Thus, we arrive at the *pricing basis* for the contract as the insurer's realistic expected outlook that it chooses to assume in setting premiums.

In practice the actual profits which will arise in the future will depend on the actual future experience values of the items for which assumptions have been made. The insurer must consider two implications of this:

(i) How will profits be affected by the actual values for assumptions turning out to be different from the pricing basis?; and

(ii) Might the actual experience give rise to a need for additional finance?

The answer to both questions lies in re-running the profit test with a different set of assumptions from the pricing basis.

To answer question (i), typically the insurer will choose a variety of different assumptions in order to determine how quickly the expected future profit changes on varying any particular assumption. Such alternative bases represent *sensitivity test assumptions*. Such "sensitivity tests" can give the insurer an understanding of how profits might be increased as well as how they might be endangered. The results of these tests may indicate ways in which a product might be re-designed to minimise changes in expected profits. Any redesign would need to be profit tested itself, so this process can be iterative.

A similar approach could be taken to answer question (ii), but in this case the alternative assumptions would concentrate just on situations where profits would be reduced, and reduced to the extent that external finance would be required. The external capital requirements so assessed must then be supported by *reserves* held by the insurer (see Sections 2.2 and 3 below). The profit test can be re-run for a range of scenarios which give rise to varying levels of reserves. The more the assumptions diverge from the pricing basis, the greater the required reserves will become, and it becomes necessary to choose a single set of assumptions for which the reserves would provide adequate protection to policyholders without being beyond the means of reasonably strong insurers' finances.

This is one of the most important, and difficult, aspects of life insurance company management. Consider the following two extremes:

- A company whose reserving basis is too optimistic runs a significant risk of paying out too much profit to its capital providers. The result could be insufficient assets in future to meet liabilities, and the company would become insolvent.
- A company whose reserving basis is too pessimistic will be holding extremely large reserves. Large reserves require large amounts of capital from the shareholders (or with profit policyholders). The profits from the business then have to be that much higher in order to provide the required rate of return on that capital, which will mean greater cost to the customer through having to pay higher premiums or charges.

Now policyholders will be prepared to pay higher premiums up to a point, in return for the increased security of the fund (and therefore obtaining greater certainty that the company will meet its future obligations to them). On the other hand, if the cost of capital is too great, the company will lose customers and could go out of business.

The actuary therefore has to determine the level of reserve that leaves the company with an acceptably low probability (or risk) of insolvency occurring in the future, whilst at the same time imposing a cost of capital on the company that the policyholders are willing to pay for. The result is that the reserving basis will be prudent: a significantly more pessimistic basis than "best estimate" will be assumed, but it will not be beyond the realms of reasonable possibility (for example, we would not assume that all the company's life assurance policyholders will die on the day after the valuation date!).

This single *valuation (or reserving) basis* is set by an insurer's actuary to ensure that an adequate assessment of the reserves is made. In practice, the valuation basis chosen will have to satisfy any local legislation and professional guidance which exists to protect the interests of policyholders.

Once the actuary has decided on an appropriate level of reserves to be held by the office, the profit test would be finally re-run, still on the original pricing basis, but with the reserves as additional cashflows. This may mean a reassessment of premiums, benefits, and charges with consequential reassessment of valuation bases. The approach of pricing a contract can therefore be iterative.

2 Determining reserves for a unit-linked policy using cashflow techniques

In this section we see how cashflow projections can be used to set reserves. This is particularly important for unit-linked products, where it is the only way of determining appropriate reserves, but we shall also see that we can apply the same methodology to conventional products. We first take another look at reserves in order to put things in perspective.

2.1 Reserves revisited

What is a reserve? You have already come across the idea of a reserve as being a sum of money that the life insurance company puts aside to meet future liabilities. Now that we have discussed cashflow projections, we can look at reserving in a different light.

What would happen if we projected the cashflows for a policy, where no reserves are established: *ie*, if we just looked at the cash income (premiums and investment return) less cash outgo (expenses and claims) for each year? For instance, for a regular premium endowment assurance of term n years we would expect something like the following pattern of cashflows:

- in year 1 — low, perhaps negative, due to initial expenses
- in years 2 to $n-1$ — positive due to premiums exceeding outgo
- in year n — very negative due to maturity payout.

For instance the endowment assurance we looked at in Chapter 12, Section 2 has the following pattern of cashflows per policy in force at the start of each year (Chapter 12, Page 16: column (8) – column (7)):

(836, 1813, 1698, 1919, –7917)

So we need to put aside money early on to provide for the large negative cashflow in the last year.

We have already seen how to calculate the required amount by valuing the stream of future benefit payments and future expenses and deducting future premiums: in other words we take each item of the cashflow and sum (with discounting) over all future years. This will give the same answer as summing each year of the cashflow (with discounting) and multiplying by –1.

Question 13.1

Why is there a factor of -1?

What we are doing is summing all elements of the following box, with suitable discounting for interest and survival:

Year	**Discounted Premium** (positive)	**…**	**Discounted Expenses** (negative)	**Discounted Claims** (negative)	**etc …**	**Sum = Discounted Cashflow**
1	$+P_1$		$-e_1$	$-C_1$		PV at time 0 of premiums, expenses *etc* in year 1
…						
t	$+P_t$		$-e_t$	$-C_t$		PV at time 0 of premiums, expenses *etc* in year t
…						
Sum	Sum of discounted premiums		Sum of discounted expenses			**Sum of sums $\times$ –1 = reserve**

We can sum either the rows first and then sum the row totals, or sum the columns and then sum the column totals. We then multiply by -1 so as to convert from:

"EPV of future income less outgo"

to:

"EPV of future outgo less income"

as required in order to produce a (prospective) reserve value. So we can use our cashflow projections – these are our row totals above – and use these to establish suitable reserves.

We illustrate how this can be done for conventional contracts in Section 3. In the rest of this section we look at the situation for unit-linked contracts. This is slightly different from the simple summation implied above.

2.2 Zeroising negative cashflows

In this section we introduce a method of reserving for unit-linked products. We have already come across the idea that the reserve for a unit-linked contract should be the bid value of the units. However, this will clearly not be adequate if we expect future negative cashflows. In that case we need to set up a cash, or non-unit, reserve now to fund for those future negative cashflows. To do this we need to use our cashflow projections.

It is a principle of prudent financial management that once sold and funded at the outset a product should be self-supporting. This implies that the profit signature has a single negative value (funds are provided by the insurance company) at policy duration zero. This is often termed "a single financing phase at the outset".

Many products "naturally" produce profit signatures which usually have a single financing phase. However some products, particularly those with substantial expected outgo at later policy durations, can give profit signatures which have more than one financing phase. In such cases these later negative cashflows (financing phases) should be reduced to zero by establishing reserves in the non-unit fund at earlier durations. These reserves are funded by reducing earlier positive cashflows. Good financial management dictates that these reserves should be established as late as possible during the term of the contract.

So we are going to find the latest negative cashflow, set up a reserve the year before to fund for that negative, and if necessary carry on working back until we have no negative cashflows.

Example

On planet Ciccio in the star system Pasticcio, interest and mortality are both zero. (This makes the actuarial job market rather small, so remember not to go there when you qualify.)

Suppose we have a four-year unit-linked policy with expected cashflows of:

(10, 15, –5, 8)

What reserves are required in the non-unit fund, and what will be the new cashflow with those reserves?

Solution

We need to set up a reserve to deal with the negative cashflow in the third year. Our start point is the third year because it is the last negative.

We want the –5 in the third year to become 0. How?

If we set up a reserve of 5 at the end of the second year, then the new cashflow in the third year will be 5 + (–5) = 0. (On planet Ciccio all mathematics is done in base 16 but we're all right here.)

But setting up the reserve of 5 in the second year costs 5 in that second year. So the second year's cashflow is reduced by 5, from 15 to 10.

So our pattern of cashflows is now (10, 10, 0, 8).

Question 13.2

"Surely we don't need a reserve at the start of year 3, because we are expecting more than enough money in year 4 to make up the loss? That is:

$$\{\text{Reserve at time 2}\} = 5 - 8 = -3$$

and so we'd assume a reserve of zero for prudence."

Explain the flaws in this argument.

So, when working out non-unit reserves for unit-linked contracts, it is necessary to *ignore all positive cashflows that occur after the last negative cashflow* in our calculations.

A policy has a non-unit cashflow vector (profit vector without non-unit reserves) of $(\text{NUCF})_t$; $t = 1, 2, 3, \ldots$ determined using the methods described in Chapter 12. Non-unit reserves (reserves in the cash fund) are to be set up so that there is only a single financing phase. The reserves to be established at policy duration t are ${}_tV$.

The reserves will be established using a basis of an interest rate of i_s and a mortality basis, where p_x is the one-year survival probability for a life aged x . After establishing non-unit reserves the profit vector is $(\text{PRO})_t$, $t = 1, 2, 3, \ldots$.

So, in the example above, (10, 15, –5, 8) was the cashflow vector, and (10, 10, 0, 8) was the profit vector.

The equation of value at the end of policy year t for cashflows in policy year t , per policy in force at time $t-1$, is

$$(\text{NUCF})_t + {}_{t-1}V(1+i_s) - (ap)_{x+t-1}\,{}_tV = (\text{PRO})_t$$

In this equation, the entry in the profit vector corresponding to the end of year t is equal to:

- the non-unit cashflow at time t
- plus the accumulated amount of the reserve that was set up at time $t-1$
- minus the reserve that has to be set up at time t for each policy that's remained in force over the policy year.

The process of establishing reserves begins at the greatest duration t for which $(\text{NUCF})_t$ is negative. Let this be duration $t = m$. Non-unit reserves will not be required at durations $t \geq m$ because during these policy years the product is expected to be self-financing. Hence we know that ${}_tV = 0$ for $t \geq m$.

For policy year m we can write:

$$(\text{NUCF})_m + {}_{m-1}V(1+i_s) - (ap)_{x+m-1} \times 0 = (\text{PRO})_m$$

where $(\text{NUCF})_m < 0$ and we wish to choose ${}_{m-1}V$ so that $(\text{PRO})_m = 0$.

Thus $_{m-1}V$ should be chosen to be:

$$_{m-1}V=-\frac{(\text{NUCF})_m}{(1+i_s)}$$

$(\text{NUCF})_m<0$ has been "turned into" $(\text{PRO})_m=0$; the expected cashflows have been zeroised.

So we set up a reserve at time $m-1$ (the beginning of the year m). This reserve, when accumulated at the rate of interest i_s assumed in the basis, is exactly enough to cancel out the negative cashflow at the end of year m. So the profit at the end of year m is now 0.

However, setting up a reserve at time $m-1$ will now have an impact on the non-unit profit at time $m-1$.

The reserve $_{m-1}V$ will be established at policy duration $m-1$ out of the funds available at duration $m-1$. A reserve of $_{m-1}V$ is required for every policyholder alive at the start of the mth policy year. The non-unit cashflow $(\text{NUCF})_{m-1}$ at time $m-1$ is $(\text{NUCF})_{m-1}$ for every policyholder alive at the start of the $(m-1)$th policy year. If we take the required reserve $_{m-1}V$ out of this cashflow, then the adjusted cashflow $(\text{NUCF})'_{m-1}$ is given by:

$$(\text{NUCF})'_{m-1}=(\text{NUCF})_{m-1}-(ap)_{x+m-2}\ {}_{m-1}V$$

We use the probability $(ap)_{x+m-2}$ because we are interested in the probability of the policy staying in force to age $x+m-1$ from a year before that age, *ie* from age $x+m-2$ to age $x+m-1$.

If $(\text{NUCF})_{m-1}<0$, then $(\text{NUCF})'_{m-1}$ will be negative. However if $(\text{NUCF})_{m-1}>0$, then $(\text{NUCF})'_{m-1}$ may be positive or negative. If $(\text{NUCF})'_{m-1}>0$, then:

$$(PRO)_{m-1}=(\text{NUCF})'_{m-1}$$

So if the non-unit cashflow is positive after setting up the reserve, then the entry in the profit vector is equal to this cashflow. However, if the non-unit cashflow is now negative, we have to zeroise this as well.

If $(\text{NUCF})'_{m-1}<0$, then we repeat the process establishing non-unit reserves $_{m-2}V$ at policy duration $m-2$.

We write:

$$(\text{NUCF})_{m-1} + {}_{m-2}V\,(1+i_s) - (ap)_{x+m-2} \times {}_{m-1}V = (\text{PRO})_{m-1}$$

and choose ${}_{m-2}V$ **so that** $(PRO)_{m-1} = 0$**, *ie*:**

$${}_{m-2}V = -\frac{(\text{NUCF})_{m-1} - (ap)_{x+m-2}\ {}_{m-1}V}{(1+i_s)} = -\frac{(\text{NUCF})'_{m-1}}{(1+i_s)}$$

Then the adjusted cashflow at time $m-2$ **is:**

$$(\text{NUCF})'_{m-2} = (\text{NUCF})_{m-2} - (ap)_{x+m-3}\ {}_{m-2}V$$

If $(\text{NUCF})'_{m-2} > 0$**, then:**

$$(\text{PRO})_{m-2} = (\text{NUCF})'_{m-2}$$

otherwise the process is repeated.

Example

The in force expected cashflows for a five-year unit-linked policy under which no non-unit reserves are held is:

$$(-60.20,\quad -20.50,\quad -17.00,\quad 50.13,\quad 85.75)$$

Calculate the reserves required if negative cashflows other than in Year 1 are to be eliminated, and give the revised profit vector allowing for the reserves.

Assume that reserves earn interest at a rate of 5% per annum. Ignore mortality.

Solution

We have negative cashflows in Years 2 and 3 that need to be dealt with. If the company can release money held in reserves as follows:

- £20.50 at the end of year 2
- £17.00 at the end of year 3,

then the negative cashflows in the profit vector will be matched exactly by a positive cashflow from reserves and the profit vector will show a zero entry for these two years.

No reserves are required after year three since there are no losses after Year 3.

So, we require a reserve at the start of Year 3, ${}_2V$, such that:

$${}_2V = \frac{17.00}{1.05} = 16.19$$

And we require a reserve at the start of Year 2 that will both fund the negative cashflow in the second year, and give us the end-year reserves of £16.19.

This means we need a total reserve at the start of Year 2, ${}_1V$, such that:

$${}_1V \times 1.05 - {}_2V = 20.50$$

ie $\quad {}_1V = (20.50 + 16.19)/1.05 = 34.94$

This corresponds to the formulae in the above Core Reading because we are ignoring mortality, *ie* treating p_{x+m-2} as if equal to one.

No other reserves are required, since negative cashflows except in Year 1 have been eliminated. So, the reserves required are:

$${}_1V = 34.94, \ {}_2V = 16.19$$

and $\quad {}_0V = {}_3V = {}_4V = {}_5V = 0$

The net effect is that the loss in year one is increased by the amount of the reserve set up, and the losses in Years 2 and 3 have been zeroised, so the profit vector is:

$$(-95.14, \ 0, \ 0, \ 50.13, \ 85.75)$$

Question 13.3

A five-year unit-linked policy has the following in force expected cashflows (prior to setting up non-unit reserves):

$$(-10,\ -20,\ 5,\ -15,\ 40)$$

Calculate the non-unit reserves that should be set up to zeroise the negative cashflows, and give the revised profit vector. Assume 6% *pa* interest, and ignore mortality.

In the above example we have ignored mortality. However, we normally need to take mortality into account in determining the reserves to be established. The next example illustrates how this is achieved.

Example

For the cashflow of $(-60.20,\ -20.50,\ -17.00,\ 50.13,\ 85.75)$ from the previous example, calculate the revised reserves required and the revised profit vector, if the policy is issued to lives aged 55 and mortality is assumed to be such that: $q_{55+t} = 0.01 + 0.001t$ for $t = 0, 1, \ldots, 4$.

Solution

The reserve required at the start of Year 3 (time 2) does not change, *ie* we still have:

$$_2V = 16.19$$

since we still need to reduce the expected loss of 17.00 at the end of year 3 to zero for each policy that was in force at the start of that year (*ie* at time 2).

Now let's consider the reserve we need at time 1, *ie* $_1V$. This is the reserve that needs to be held *per policy in force at time 1*. This reserve has to cover:

(1) the expected cash outgo from this in-force policy, occurring at the end of year 2

(2) the required reserves that need to be carried over to the third policy year, which will be needed *only for those policies that are still in force at the end of year 2*, *ie* for the proportion surviving from time 1 to time 2.

To assess this, we work out the adjusted cashflow at time 2, per policy in force at time 1.

This is:

$$(NUCF)_2' = -20.50 - {}_2V \times p_{56}$$
$$= -20.50 - 16.19 \times (1 - q_{56})$$
$$= 16.19 \times 0.989$$
$$= -36.51$$

In other words, the total expected cash *outgo* at the end of year 2, per policy in force at the *start* of year 2, is $+36.51$, and it is this amount that the reserve for each policy that starts the year needs to cover. So, to eliminate this loss, we need:

$${}_1V = \frac{36.51}{1.05} = 34.77$$

Holding this reserve (and then using it to cover the year's adjusted cashflow) will now result in a zero expected profit for year 2.

The other reserves are zero as before.

Finally turn to year 1. The ${}_1V$ reserves now become part of the cash outgo for this year. However, the profit vector element for this year is defined as the expected profit per policy in force at the start of *that* year, *ie* as at time 0. As the ${}_1V$ reserves are only required for the survivors of year 1, the expected adjusted cashflow at the end of Year 1 is:

$$(NUCF)_1' = -60.20 - {}_1V \times p_{55} = -60.20 - 34.77 \times 0.99 = -94.62$$

which is also the profit for that year.

So, the revised profit vector is:

$$(-94.62,\ 0,\ 0,\ 50.13,\ 85.75)$$

Question 13.4

Calculate the non-unit reserves required to zeroise negative cashflows for the in-force expected cashflows $(-10,\ -20,\ 5,\ -15,\ 40)$ for a five-year policy taken out by a 50 year old. Assume AM92 Ultimate mortality and 6% *pa* interest.

In some cases, the profit vector may show several runs of negative entries. In this case, the above method is repeated as many times as necessary until all the negative entries have been eliminated. The following example illustrates this.

Example

The in force expected cashflows for a 6-year policy issued to lives aged x is:

$$(-131.53, -70.11, 25.00, -20.15, 55.74, 157.91)$$

Calculate the reserves required and the revised profit vector if negative cashflows after year one are to be zeroised.

Assume non-unit reserves earn interest at 6% *pa* and that the probability of death during any year is 0.01.

Solution

No reserves will be required after Year 4. To match the negative cashflow in Year 4, we require a reduction in reserves of 20.15 at the end of the year. So:

$${}_3V \times 1.06 = 20.15 \Rightarrow {}_3V = 19.01$$

The cashflow in the previous year is +25.00. This is sufficient to set up the reserve. Allowing for this, the profit in Year 3 becomes:

$$25.00 - 0.99 \times 19.01 = 6.18$$

We now need to zeroise the negative cashflow in Year 2. This will require a reserve at the start of Year 2 such that:

$${}_1V = 70.11 / 1.06 = 66.14$$

The expected profit for Year 1 becomes:

$$-131.53 - 0.99 \times 66.14 = -197.01$$

The revised profit vector is:

$$(-197.01\,,\ 0\,,\ 6.18\,,\ 0\,,\ 55.74\,,\ 157.91)$$

Question 13.5

Calculate the reserves required for the policy in the above example if the cashflow in Year 3 is 15.00 instead of 25.00.

Summary of the method

To recap all of the above, a summary of the steps required to zeroise negative cashflows is:

Step 1

Starting from the last negative entry in the profit vector (in year *m* say), calculate ${}_{m-1}V$:

$${}_{m-1}V \;=\; \frac{\textit{Loss in year m}}{1 \,+\, i}$$

Step 2

If there is a loss in the previous year, find ${}_{m-2}V$ from:

$${}_{m-2}V \times (1+i) - (ap)_{x+m-2} \times {}_{m-1}V = \textit{Loss in year } m-1$$

using the value of ${}_{m-1}V$ found from Step 1.

Step 3

Carry on working backwards through the profit vector using:

$$_{t-1}V \times (1+i) - (ap)_{x+t-1} \times {}_tV = Loss\ in\ year\ t$$

until either:

- Year 1 is reached, or
- a positive entry in the profit vector is reached.

If Year 1 is reached, the profit for Year 1 is reduced by an amount $(ap)_x \times {}_1V$.

If a positive entry is reached (in year *k* say, $k > 1$), check whether the profit for that year is large enough to set up the required reserve,

ie check if $\quad$ *Profit in year k* $> (ap)_{x+k-1} \times {}_kV$

If the profit for Year k *is* large enough, the process stops for this run of negative entries and the profit for Year k is reduced by $(ap)_{x+k-1} \times {}_kV$.

If the profit in Year k *is not* large enough to set up the reserve, then the reserve at the start of Year k is calculated using the general formula above. Carry on working backwards through the profit vector until a large enough positive entry or Year 1 is reached.

Repeat steps 1 to 3 for any other runs of negative entries.

3 Determining reserves for a conventional policy using cashflow techniques

In the last section we saw how to determine non-unit reserves for unit-linked products by working iteratively back from the last negative cashflow. We can also apply exactly the same methodology to conventional contracts.

A profit test can also be used to determine the reserves for a conventional life assurance (*ie* non unit-linked) policy. We illustrate the procedures by using a without-profit endowment assurance with a term n years and a sum assured of S payable at the end of the year of death which is secured by a level annual premium of P. The experience basis (see Section 1) assumes that funds earn an effective rate i and that mortality is represented by survival probabilities ${}_tp_x$. The expenses incurred at time t are e_t.

In this section we are ignoring any cause of decrement other than mortality, hence the use of ${}_tp_x$ rather than ${}_t(ap)_x$ throughout. If the reserves are needed to cover outgoes due to other decrements as well, then the appropriate multiple decrement probabilities will be required in the formulae that follow.

Then the expected cashflow at time t per policy in force at time $t-1$, $(\text{CF})_t$ ignoring reserves is:

$$(\text{CF})_t = \left(P - e_{t-1}\right)(1+i) - Sq_{x+t-1} \qquad t = 1, 2, 3, \ldots, n-1$$

$$(\text{CF})_n = \left(P - e_{n-1}\right)(1+i) - S$$

These cashflows will usually be positive for earlier years of a contract and negative during the later years. For example a five-year endowment assurance with a sum assured of 1,000 might have cashflows:

t	1	2	3	4	5
$(\text{CF})_t$	156.39	187.41	186.33	185.14	–803.17

If the contract is to be self-funding, then reserves must be established using the earlier positive cashflows. These reserves should be sufficient to pay the later expected negative cashflows. This requirement is exactly analogous to the need to establish reserves in the non-unit fund for a unit-linked endowment assurance. Reserves can be established for conventional assurances using the same procedures as those used to establish non-unit reserves.

Let $(CF)'_t$ denote the cashflows $(CF)_t$ adjusted to allow for reserves.

A basis (the valuation or reserving basis – see Section 1) consisting of an interest rate i_r and a mortality basis ${}_tp'_x$ is chosen.

The valuation basis will probably be more prudent than the experience basis, as we discussed in Section 1.

Calculations begin at the longest policy duration, m, at which there is a negative cashflow.

For $t > m$ $\quad (CF)'_t = (CF)_t$

For policy year m we write:

$$(CF)_m + {}_{m-1}V\left(1+i_r\right) - p'_{x+m-1} \times 0 = (CF)'_m$$

where $(CF)_m < 0$ and we wish to choose ${}_{m-1}V$ so that $(CF)'_m = 0$. This requires that ${}_{m-1}V$ is chosen to be:

$${}_{m-1}V = \frac{-(CF)_m}{(1+i_r)}$$

We set up this reserve from $(CF)_{m-1}$ and determine the adjusted cashflow:

$$(CF)'_{m-1} = (CF)_{m-1} - p'_{x+m-2}\ {}_{m-1}V$$

If $(CF)'_{m-1} > 0$, then ${}_{m-2}V = 0$. If $(CF)'_{m-1} < 0$, then the process is repeated for the $(m-1)$th policy year.

For conventional assurances it is usually the case that reserves are needed at all policy durations. So the calculation begins with $(CF)_n$ and concludes with $(CF)_1$.

For the example cashflows shown above, if a basis of 3% *pa* interest and AM92 Ultimate Mortality is assumed for a policy taken out at age 55 exact, we obtain:

t	1	2	3	4	5
$(CF)_t$	156.39	187.41	186.33	185.14	–803.17
${}_tV$	177.20	371.80	572.52	779.78	0

First we need a reserve at time 4 to cover the negative cashflow at time 5. We simply discount the negative cashflow at time 5 over one year, and multiply by -1. So:

$$ {}_4V = \frac{803.17}{1.03} = 779.78 $$

Next we need to work out the cashflow at time 4 allowing for the reserve. This is:

$$ (CF)'_4 = 185.14 - 779.78\, p_{58} = -589.70 $$

Note that we only need reserves at the end of Year 4 for those policyholders that have *survived* over Year 4, so we multiply the required reserve by p_{58} (the probability of surviving over Year 4). Remember that all of these end-year cashflows are expressed per policy in force at the *start* of each year.

As $(CF)'_4$ is negative, then we need a reserve at the start of Year 4 of:

$$ {}_3V = \frac{589.70}{1.03} = 572.52 $$

This process is continued until $(CF)'_t$ becomes positive or, as in this case, we get to the start date of the policy. You should verify for yourself the other figures in this table.

Note that when a policy has a cashflow pattern like this one, where only the final cashflow is negative, calculating reserves using:

1. the EPV future outgo – EPV future income formula
2. discounted future cashflows
3. zeroisation of future negative cashflows

all give the same result.

This is also true when the cashflows become increasingly negative over time, as in the case of a term assurance. However, as we discussed in Section 2.1, when a policy produces some negative cashflows but the last cashflow is positive, zeroisation of future negative cashflows gives a different answer to the other 2 methods and is preferable on the grounds of prudence.

Usually, the reserves are calculated using a (conservative) valuation basis. Then these reserves are used in a profit test using a pricing basis in order to determine the premium required to satisfy the required profit criterion.

So we:

- calculate reserves on a prudent basis,
- calculate cashflows on a realistic basis, then allow for the prudent reserves so as to give the projected profits for the profit test,
- play around with premiums and product design until we achieve our profitability criterion using these "realistic experience but prudent reserves" cashflows.

Clearly we cannot derive prudent reserves from realistic (*ie* best estimate) cashflows. We go into more detail on the interaction between a realistic experience basis and a prudent reserving basis in the next section of this chapter.

4 *Effect of pricing and reserving bases on a profit test*

The writing of each contract represents an investment of capital of an insurer, and it has been seen from above that the expected profit from a contract is the present value of the projected profit test cashflows at the risk discount rate.

The chosen pricing and reserving assumptions will each have an impact on the expected profit.

The greater the extent that pricing assumptions reduce the expected future profit flows in the profit test, the greater will be the reduction in present value of profits. The assumed investment return in the pricing basis would normally, by definition of the risk discount rate, be less than that risk discount rate and lower assumptions lead to lower present values of profits.

A valuation basis which gives rise to positive reserves will normally reduce the present value of profits from a contract. This is because the reserves reduce otherwise positive cashflows and are then invested in a relatively safe form of assets whose rate of investment return, by definition of the risk discount rate, is expected to be less than the risk discount rate. Over the whole term of a contract, and assuming that experience turns out to be in line with the assumed pricing basis, stronger reserves will not reduce the overall aggregate size of the cashflows but they will delay the cashflows so that they emerge later than they would have done using a weaker reserving basis, hence reducing the present value of profits.

In the above description of the profit test calculations we have assumed that reserves are set up at the end of policy years, and this is conventionally how profit tests are carried out. However, it is more prudent to assume that reserves are set up at the beginning of policy years (as soon as a premium is paid). In this case, making the usual assumption that reserves will earn less than the risk discount rate, the profit would be slightly smaller. In practice, insurance companies would tend to calculate cashflows monthly rather than annually, and therefore the convention of year-end (or, in this case, month-end) **reserves does not lead to a material overstatement of profit.**

This is an example of a more general approach which can be used to discount cashflows, given sufficient computing power. That is, to discount cashflows as soon as they arise, rather than first accumulating them to the end of policy years at an assumed rate of interest. As explained above, this approach would be more prudent in the case of reserve cashflows and indeed is actually more prudent for any other negative cashflows, expenses for example.

For example, if we had an initial expense of 100, accumulating this negative cashflow to the end of the year at 8% and then discounting back at 12% gives:

$$100 \times \frac{1.08}{1.12} = 96.43$$

So it is more prudent to account for the negative cashflow of 100 as soon as it arises rather than accumulate to the end of the policy year and discount.

Example

We can illustrate the above concepts by considering a very simple single premium 5-year pure endowment in a zero-mortality zero-expense world. The sum assured is 1,000.

Suppose we have the following bases for interest:

Pricing	6%	
Experience	7%	(on cash and reserves)
Reserving	5%	
Risk discount rate	9%	

Expenses, mortality and withdrawals are zero for all bases (pricing, reserving and expected experience).

So the premium is $1,000v^5$ at 6% giving 747.26.

The projected cashflows, showing them both before and after reserving, are shown in the tables below.

Year	Premium	Interest	Benefit outgo	Cashflow (before reserves)
1	747.26	52.31	0	799.57
2	–	0	0	0
3	–	0	0	0
4	–	0	0	0
5	–	0	–1,000	–1,000

Year	End of year reserve	Interest earned on reserves	Cost of increasing reserves (ignoring interest on reserves)	Profit vector
1	822.70	0	–822.70	–23.14
2	863.84	57.59	–41.14	16.45
3	907.03	60.47	–43.19	17.28
4	952.38	63.49	–45.35	18.14
5	0.00	66.67	952.38	19.05
Total			0.00	47.78

Note that since there is no mortality, the cost of increasing the reserve at the end of Year 1 is calculated as ${}_1V - {}_0V = {}_1V$. Since this is a cost to the company, it is shown as a negative entry in the table above. The other entries in this column are calculated in a similar way.

Question 13.6

Verify the entries above for the second year.

Question 13.7

Explain why, in a zero-expense world, we get new business strain with this contract.

Question 13.8

What is the one feature in the basis that tells us that the total non-discounted profits will be positive?

Question 13.9

Calculate the net present value of profits.

What is the effect of strengthening the reserving basis, *ie* making it more pessimistic? If we were to reserve at 3%, then the projected cashflows and profits would become:

Year	Premium	Interest	Benefit outgo	Cashflow (before reserves)
1	747.26	52.31	0	799.57
2	–	0	0	0
3	–	0	0	0
4	–	0	0	0
5	–	0	–1,000	–1,000

Year	End of year reserve	Interest on reserves	Cost of increasing reserves	Profit
1	888.49	0	–888.49	–88.92
2	915.14	62.19	–26.65	35.54
3	942.60	64.06	–27.45	36.61
4	970.87	65.98	–28.28	37.70
5	0.00	67.96	970.87	38.83
Total			0.00	59.76

The net present value of these profits is now 28.55. Note that the total amount of profits is slightly higher, because we have set up bigger reserves and those reserves bring in interest – but that interest is lower than the risk discount rate, so the effect is a reduction in the *value* of profits.

We see that the profitability of the contract has gone down due to the increase in the gap between the investment return earned on cash and reserves, and the required rate of return.

We could also have increased that gap by increasing the risk discount rate – this would have reduced the net present value of profits.

We can see also that strengthening the reserving basis has increased the new business strain.

5 Exam-style questions

The following exam-style questions are based on the material from Chapters 12 and 13.

Question 13.10 ***(Subject 105, April 2002, Question 14, adapted)***

A life insurance company issues a number of 3-year term assurance contracts to lives aged exactly 60. The sum assured under each contract is £200,000, payable at the end of the year of death. Premiums are payable annually in advance for the term of the policy, ceasing on earlier death.

The company carries out profit tests for these contracts using the following assumptions:

Initial expenses:	£200 plus 35% of the first year's premium
Renewal expenses:	£25 plus 3% of the annual premium, incurred at the beginning of the second and subsequent years
Mortality:	AM92 Ultimate
Investment return:	7% per annum
Risk discount rate:	15% per annum
Reserves:	One year's office premium

(i) Show that the office premium, to the nearest pound, is £2,527, if the net present value of the profit is 25% of the office premium. [10]

(ii) Calculate the cash flows if the company held zero reserves throughout the contract, using the premium calculated in part (i). [2]

(iii) Explain why the company might not hold reserves for this contract and the impact on profit if they did not hold any reserves. [3]

[Total 15]

Question 13.11

A 5-year unit-linked endowment assurance is issued to a male aged exactly 55. The expected year-end cashflows in the non-unit fund, $(NUCF)_t$ ($t = 1, \dots, 5$) per policy in force at the start of Year t are:

Year t	1	2	3	4	5
$(NUCF)_t$	–200	+20	+45	–60	+480

You are given:

Independent probabilities of mortality: AM92 Select

Independent probabilities of withdrawal: 0.1 for Years 1 and 2
0.05 for Years 3 and 4
0 for Year 5

Withdrawals can occur at any time over the policy year.

(i) Calculate the net present value of profit at a risk discount rate of 10% *pa* assuming that the company holds no non-unit reserves. [4]

The rate of interest earned on non-unit reserves is assumed to be 8% *pa*.

(ii) (a) Calculate the reserves that are required at times $t = 1, \dots, 4$ in order to zeroise future negative cashflows.

(b) Calculate the net present value of the policy assuming that the company holds the non-unit reserves calculated in (i)(a). [6]

(iii) Without carrying out any more calculations, explain the effect on the net present value if non-unit reserves earned interest at the rate of 10% *pa*. [2]

[Total 12]

Chapter 13 Summary

Calculating reserves using cashflow projections

We can use cashflow projections to calculate reserves by:

- identifying negative cashflows (starting from the end of the projection) in year t,
- calculating the reserves needed at the end of the previous year $t-1$ to fund such negative cashflows,
- paying for such a reserve from the cashflow of year $t-1$,
- and if the cashflow of year $t-1$ is now negative, repeat the above process.

We need to do this with unit-linked contracts to ensure that policies in force will not require further financing by the life company.

We can use the same technique with conventional contracts, although usually other approaches will be more convenient.

Different bases

A basis is a set of assumptions which defines what we are saying about the future (*eg* investment return, mortality, expenses).

Assumptions that are our best estimates of the future give an *experience* basis.

The basis we use to set premiums is called the *pricing* basis and can be the same as the experience basis, or slightly more prudent, or more risky!

The basis we use to determine reserves – the *valuation* or *reserving* basis – is always much more prudent than the experience basis. This will defer the emergence of profits from the contract. If the required rate of return is greater than the investment return on reserves, this will reduce the profitability of the contract.

This page has been left blank so that you can keep the chapter summaries together for revision purposes.

Chapter 13 Solutions

Solution 13.1

The reserve is "expected present value of benefits and expenses less expected present value of premiums". This is the negative of the "premiums less benefits and expenses" we see with cashflows.

Solution 13.2

One problem is that the positive cashflow comes after the negative cashflow, so if the company does not have any reserves available in year 3 it would find it physically impossible to pay out all its claims and expenses at the required time.

A second and ultimately more serious problem is that the fourth policy year may never happen (because the policy may lapse at the start of year 4). The insurer is therefore permanently short of money (and, in the scenario we have just described, completely bankrupt).

Solution 13.3

We ignore the +40 in year 5.

We want to zeroise the –15 in year 4. So we need a reserve in place at the start of that year, *ie* at the end of year 3, equal to

$$_3V = \frac{15}{1.06} = 14.15$$

The year 3 cashflow is now going to be $5 - 14.15 = -9.15$ so we have a negative cashflow that we need to sort out.

We shall need an end of year 2 reserve of

$$_2V = \frac{9.15}{1.06} = 8.63$$

This gives a new, year 2 cashflow of $-20-8.63=-28.63$ so we need a reserve at the end of year 1 of

$$_1V = \frac{28.63}{1.06} = 27.01$$

This will change the year 1 cashflow to $-10-27.01=-37.01$.

The revised cashflow (*ie* profit vector) will now be (–37.01, 0, 0, 0, 40).

Solution 13.4

We ignore the +40 in Year 5.

We want to zeroise the –15 in Year 4. So we need a reserve in place at the start of that year, *ie* at the end of Year 3, equal to:

$$_3V = \frac{15}{1.06} = 14.15$$

Note that we do not need a reserve at the end of Year 4 so we have not needed to consider people surviving Year 4.

We now need a reserve at the end of Year 2, which will satisfy:

$$_2V \times 1.06 + 5 = {}_3V \times p_{52}$$

giving:

$$_2V = \frac{14.15 \times 0.996848 - 5}{1.06} = 8.59$$

So we need a reserve at the end of Year 1, which will satisfy:

$$_1V \times 1.06 - 20 = {}_2V \times p_{51}$$

giving:

$$_1V = \frac{8.59 \times 0.997191 + 20}{1.06} = 26.95$$

Solution 13.5

As before, we have a reserve at the end of Year 3 of 19.01. So the profit in Year 3 becomes $15-0.99\times 19.01=-3.82$. We will need to set up a reserve at the end of Year 2 in respect of this negative result.

The end of Year 2 reserve must be $\dfrac{3.82}{1.06}=3.60$.

Then the Year 2 result becomes:

$$-70.11-{}_2V\times p=-70.11-3.60\times 0.99=-73.68$$

So we need a reserve at the end of Year 1 of:

$${}_1V=\frac{73.68}{1.06}=69.51$$

Solution 13.6

There is no premium, so no interest on the premium. There is no benefit either. So the only cashflows come from the reserves.

At the start of the year, the company holds a reserve of 822.70 per policy in force. Interest on reserves is 7%, which gives interest of 57.59.

The reserve required at the end of the year is $1,000v^3$. Valuing at 5%, this is 863.84. Since there is no mortality, the cost of increasing the reserve is:

863.84 – 822.70 = 41.14

So profit is 57.59 – 41.14 = 16.45.

Solution 13.7

New business strain has arisen because the reserving basis is stronger than the pricing basis. The premium contains 747.26 in respect of the value of future benefits, while the reserves we require to set up after receiving the premium have a present value of $\dfrac{822.70}{1.05}=783.53$ at the start of the year.

Solution 13.8

The experience investment income assumption is greater than the pricing assumption.

Solution 13.9

Net present value of profits is

$$-23.14v + 16.45v^2 + \cdots + 19.05v^5 = 31.19$$

discounting at 9%.

Solution 13.10

(i) ***Office premium***

The table for the profit test is as follows:

Year	Premium	Expenses	Interest	Expected death cost
1	P	$-200-0.35P$	$0.07(0.65P-200)$	$-1,604.4$
2	P	$-25-0.03P$	$0.07(0.97P-25)$	$-1,801.8$
3	P	$-25-0.03P$	$0.07(0.97P-25)$	$-2,022.4$

Year	End of year cashflow	Cost of increase in reserves	Profit vector
1	$0.6955P-1,818.4$	$-0.9920P$	$-0.2965P-1,818.4$
2	$1.0379P-1,828.6$	$0.0790P$	$1.1169P-1,828.6$
3	$1.0379P-2,049.2$	$1.07P$	$2.1079P-2,049.2$

Year	Probability in force	Profit signature
1	1	$-0.2965P-1,818.4$
2	0.991978	$1.1079P-1,813.9$
3	0.983041	$2.0722P-2,014.4$

So the expected net present value of the profit is:

$$\frac{-0.2965P - 1{,}818.4}{1.15} + \frac{1.1079P - 1{,}813.9}{1.15^2} + \frac{2.0722P - 2{,}014.4}{1.15^3}$$
$$= 1.9424P - 4{,}277.29$$

Setting this equal to $0.25P$ and solving for P gives $P = £2{,}527$, to the nearest £1.

(ii) ***Cashflows, ignoring reserves***

The cashflows are given in the table below:

Year	Premium	Expenses	Interest	Expected death cost	End of year cashflow
1	2,527	−1,084.45	100.98	−1,604.4	−60.9
2	2,527	−100.81	169.83	−1,801.8	794.2
3	2,527	−100.81	169.83	−2,022.4	573.6

(iii) ***Why reserves are not needed and impact of holding reserves***

As the table above shows, the policy is self-funding after the first year. So there is no need to hold reserves to cover future outgo. If the company didn't hold reserves, this would accelerate the emergence of profit. Since the risk discount rate is higher than the investment return, not holding reserves would increase the net present value of the profit.

Solution 13.11

(i) ***Net present value assuming no non-unit reserves***

The independent probabilities of mortality are:

$$q_{55}^{d} = q_{[55]} = 0.003358$$

$$q_{56}^{d} = q_{[55]+1} = 0.004903$$

$$q_{57}^{d} = q_{57} = 0.005650$$

$$q_{58}^{d} = q_{58} = 0.006352$$

So we have:

$$p_{55}^{d} = 0.996642 \qquad p_{55}^{w} = 0.9$$

$$p_{56}^{d} = 0.995097 \qquad p_{56}^{w} = 0.9$$

$$p_{57}^{d} = 0.994350 \qquad p_{57}^{w} = 0.95$$

$$p_{58}^{d} = 0.993648 \qquad p_{58}^{w} = 0.95$$

and:

$$(ap)_{55} = p_{55}^{d} \times p_{55}^{w} = 0.896978$$

$$(ap)_{56} = p_{56}^{d} \times p_{56}^{w} = 0.895587$$

$$(ap)_{57} = p_{57}^{d} \times p_{57}^{w} = 0.944633$$

$$(ap)_{58} = p_{58}^{d} \times p_{58}^{w} = 0.943966$$

It follows that:

$${}_2(ap)_{55} = (ap)_{55} \times (ap)_{56} = 0.803322$$

$${}_3(ap)_{55} = {}_2(ap)_{55} \times (ap)_{57} = 0.758844$$

$${}_4(ap)_{55} = {}_3(ap)_{55} \times (ap)_{58} = 0.716323$$

So the net present value of the contract is:

$$-\frac{200}{1.1}+\frac{20\times 0.896978}{1.1^2}+\frac{45\times 0.803322}{1.1^3}-\frac{60\times 0.758844}{1.1^4}+\frac{480\times 0.716323}{1.1^5}$$

$$= 42.56$$

(ii)(a) ***Reserves required to zeroise negative cashflows***

We do not need a reserve at time 4 since the cashflow at time 5 is positive. So:

$${}_4V = 0$$

We do need a reserve at time 3 since the cashflow at time 4 is negative. We require:

$${}_3V\left(1+i\right)=60 \Rightarrow {}_3V=\frac{60}{1.08}=55.56$$

The non-unit cashflow at time 3 then becomes:

$$(NUCF)_3' = 45 - {}_3V\,(ap)_{57} = 45-55.56\times 0.944633=-7.48$$

We now need a reserve at time 2 to zeroise this negative:

$${}_2V\left(1+i\right)=7.48 \Rightarrow {}_3V=\frac{7.48}{1.08}=6.93$$

The non-unit cashflow at time 2 then becomes:

$$(NUCF)_2' = 20 - {}_2V\,(ap)_{56} = 20-6.93\times 0.895587=13.80$$

Since this is positive, we do not need a reserve at time 1, *ie*:

$${}_1V = 0$$

(ii)(b) ***Net present value assuming non-unit reserves are set up***

The profit vector is the vector of non-unit cashflows after the reserves have been set up. So for this policy the profit vector is:

$$\left(-200, \quad 13.80, \quad 0, \quad 0, \quad 480\right)$$

The net present value is then:

$$-\frac{200}{1.1}+\frac{13.80\times 0.896978}{1.1^2}+\frac{480\times 0.716323}{1.1^5}=41.91$$

(iii) ***If non-unit reserves earned 10% pa interest***

Holding reserves delays the emergence of profit. If the rate of interest earned on the reserves is 10%, then we are accumulating and discounting at the same rate. In this case, holding reserves will have no effect on the net present value of the contract. So the net present value would be 42.56 as in (i).

Chapter 14

Mortality, selection and standardisation

Syllabus objectives

(x) Describe the principal forms of heterogeneity within a population and the ways in which selection can occur.

1. *Explain why it is necessary to have different mortality tables for different classes of lives.*

2. *Explain the theoretical basis of the use of risk classification in life insurance.*

3. *State the factors which contribute to the variation in mortality and morbidity by region and according to the social and economic environment, specifically:*
   - *occupation*
   - *nutrition*
   - *housing*
   - *climate/geography*
   - *education*
   - *genetics*

4. *Define and give examples of the main forms of selection:*
   - *temporary initial selection*
   - *class selection*
   - *time selection*
   - *spurious selection*
   - *adverse selection*

Continued…

5. *Explain how selection can be expected to occur amongst individuals or groups taking out each of the main types of life insurance contracts, or amongst members of large pension schemes.*

6. *Explain the concept of mortality convergence.*

7. *Explain how decrements can have a selective effect.*

8. *Explain the concept of a single figure index and its advantages and disadvantages for summarising and comparing actual experience.*

9. *Define the terms crude rate, directly standardised and indirectly standardised mortality rate, and standardised mortality ratio, and illustrate their use.*

0 Introduction

Apart from the variation in mortality rates between the sexes and between different ages, mortality rates vary both within a population and between different populations as a result of a number of other *risk factors*, *ie* influences that affect the mortality of individuals. Morbidity rates will also vary and you should note that the syllabus objectives include consideration of morbidity rates.

In this chapter we consider the various risk factors which lead to heterogeneity within a population and the various types of selection that this can give rise to. We also consider ways in which population mortality and morbidity rates can be summarised and compared.

1 The general pattern of mortality

1.1 Mortality characteristics

The first thing we shall consider in this chapter is the pattern of mortality over the lifespan. We shall do this by showing how the various life table functions (*eg* l_x and d_x) vary with age.

Before looking at graphs of the various functions, we will make some general comments about the mortality experienced in the different stages of life. Since the Core Reading uses a life table based on UK experience, we will consider some of the key features of mortality experience from a UK viewpoint for consistency.

Age range	Stage of life	Mortality characteristics	Reasons	Main causes of death
0 – 1	Infancy	Heavy initially	Babies born with severe medical problems die soon after birth	Congenital disorders
2 – 4	Early childhood	Light	Protected environment (constantly supervised)	Accidents, cot deaths
5 – 12	Childhood	Light	Protected environment	Accidents
13 – 16	Adolescence	Slight increase	Active lifestyle	Accidents, drug-related, suicide
17 – 20	Late teenage	Temporary increase ("accident hump")	Active lifestyle	Motor accidents, drug-related, suicide
21 – 40	Early adulthood	Light	Settling down *eg* job, marriage	Various (incl. AIDS)
41 – 70	Middle age	Increasing steadily	Onset of various diseases	Heart disease, cancer
70+	Senescence	Heavy, increasing steadily	Body systems lose robustness	Wide range of diseases

Question 14.1

How do you think the incidence of AIDS-related deaths at different ages will change in industrialised countries such as the UK?

In this section we illustrate the general features of life table functions using ELT15 (Males).

1.2 *The shape of q_x*

Figure 1 shows q_x, the ultimate rates of mortality. The main feature is the rapid, in fact nearly exponential, increase in mortality beyond middle age.

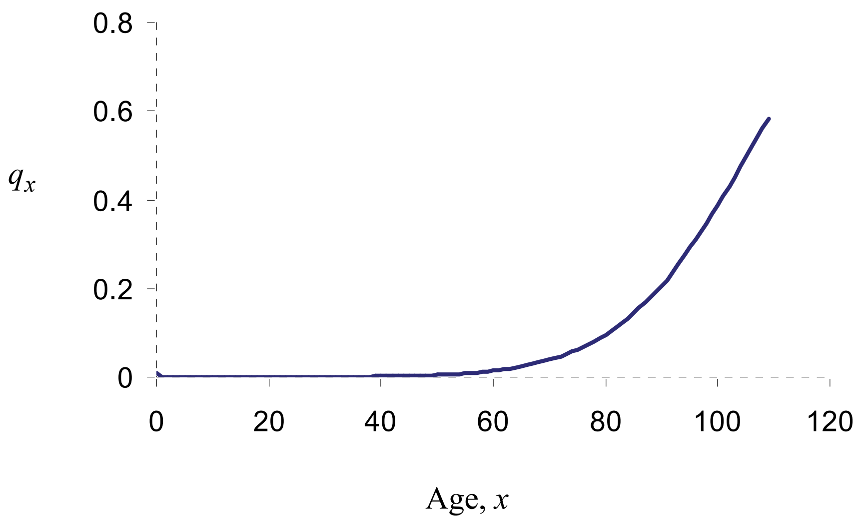

Figure 1

q_x (ELT15 (Males) Mortality Table)

The rates of mortality at older ages are so much larger than those at younger ages that Figure 1 fails to show any detail at younger ages.

In order to see the shape of q_x in more detail, we have to rescale the y-axis by taking logs.

The mortality table is tabulated to age 109. In practice, the value of q_x will rise towards 1 by around age 120.

Figure 2 shows q_x on a vertical base 10 logarithmic scale.

This means we are using a log scale on the y-axis.

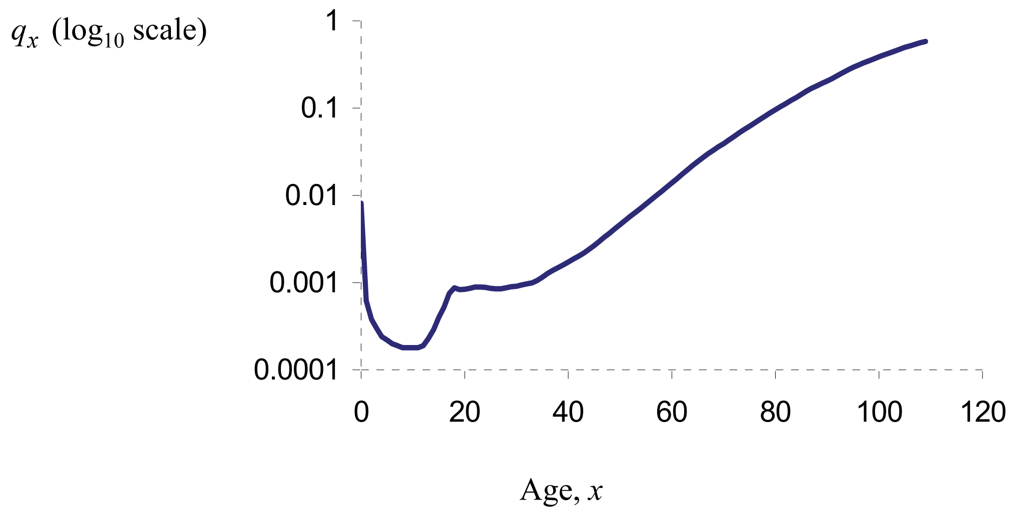

Figure 2

q_x on $\log_{10}$ scale (ELT15 (Males) Mortality Table)

The main features are:

- **high infant mortality,**
- **an "accident hump" at ages around 20, and**
- **the nearly exponential increase at older ages.**

In practice, the graph would effectively reach 1 by age 120.

Question 14.2

A student comments that the increase at older ages in Figure 2 looks linear, not exponential. Explain the student's probable misunderstanding.

1.3 The shape of l_x

Figure 3 shows l_x. The main feature is the very slight fall until late middle age, followed by a steep plunge.

This shape is very much as we would expect from our knowledge of q_x, *ie* that mortality is light at young ages and increases almost exponentially in later life.

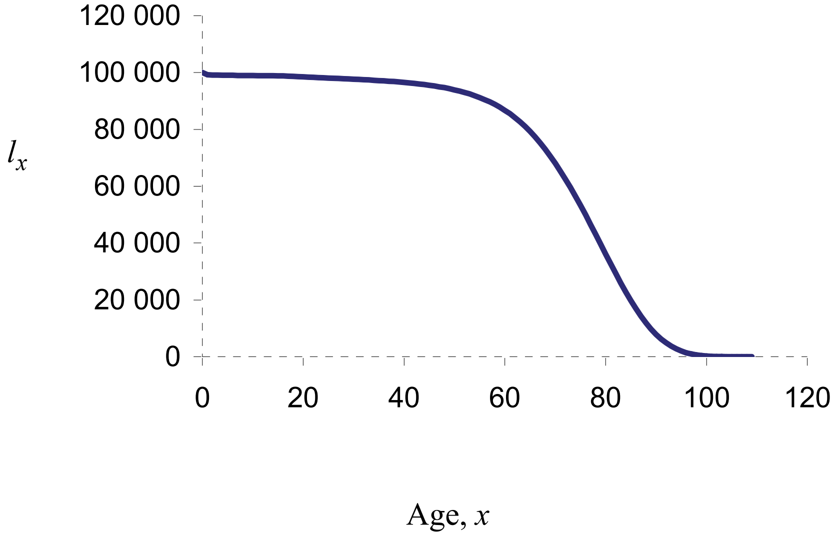

Figure 3

l_x (ELT15 (Males) Mortality Table)

1.4 The shape of d_x

Figure 4 shows d_x. Although the scale is different, this looks the same as the graph of $f_0(t) = {}_tp_0\,\mu_t$ shown in Section 1.6 of Chapter 3. The similarity is explained by the relationship:

$$q_x = \frac{d_x}{l_x} = \int_0^1 {}_tp_x\,\mu_{x+t}\,dt$$

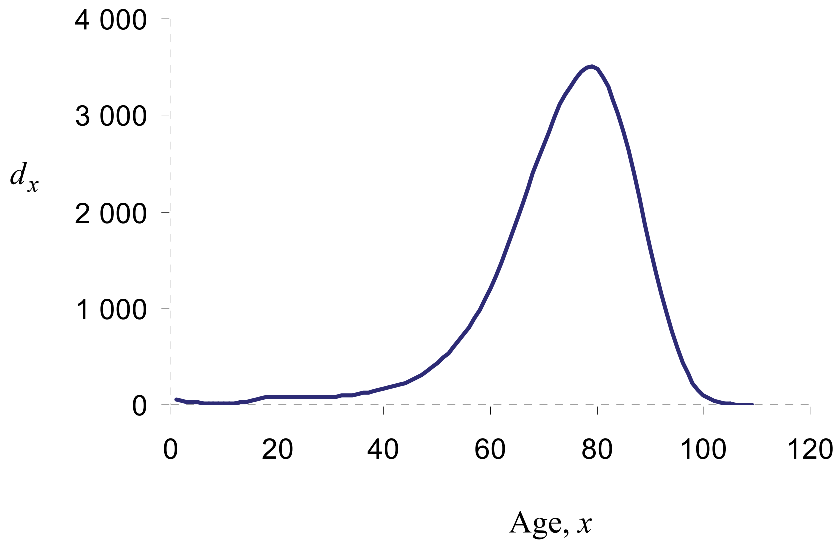

Figure 4

d_x (ELT15 (Males) Mortality Table)

The graph of d_x has a mode at approximately age 80. At birth, male lives are expected to live for about 80 years, with the vast majority of deaths occurring between 60 and 100.

2 Principal factors contributing to variation in mortality and morbidity

In addition to variation by age and sex, mortality and morbidity rates are observed to vary:

- **between geographical areas, *eg* countries, regions of a country, urban and rural areas,**
- **by social class, *eg* manual and non-manual workers,**
- **over time, *eg* mortality rates usually decrease over time.**

None of these categories (*eg* geographical location and time) provide a direct (causal) explanation of the observed differences. Rather they are proxies for the real factors that cause the observed differences. Such factors are:

- **occupation**
- **nutrition**
- **housing**
- **climate**
- **education**
- **genetics.**

It is rare that observed differences in mortality can all be ascribed to a single factor. It is difficult to disentangle the effects of different factors as a result of confounding. For example, mortality rates of those living in sub-standard housing are (usually) higher than those of people living in good quality housing. However, those living in sub-standard housing usually have less well paid occupations and lower educational attainment than those living in good quality housing. Part or all of the observed difference may be due to these other differences and not to housing differences.

It is important for governments to be able to identify risk factors in order to bring about improvements in public health, *eg* by means of appropriately targeted public health campaigns. They are very important in insurance, since identifying specific risk factors for an individual enables insurers to classify that individual's mortality or morbidity risk more precisely, thereby allowing more accurate calculation of premiums and reserves.

Confounding means that the effects of the different factors are so intermingled as to be indistinguishable. In other words, most of these factors could be grouped under the general heading "standard of living". This makes it difficult to isolate the individual effects since they are correlated, with factors tending to operate together. However, it is possible by means of statistical studies to identify the dominant risk factors. So we have two problems. First, we want to identify the *real* factors and not be confused by *apparent* factors. Secondly, we then need to quantify the relative importance of these factors.

Question 14.3

Why would you expect mortality rates to decrease over time and when would this not be the case?

2.1 Occupation

Occupation can have several direct and indirect effects on mortality and morbidity. Occupation determines a person's environment for 40 or more hours each week. The environment may be rural or urban, the occupation may involve exposure to harmful substances, *eg* chemicals, or to potentially dangerous situations, *eg* working at heights. Much of this is moderated by health and safety at work regulations.

Some occupations are more healthy by their very nature, *eg* bus drivers have a sedentary and stressful occupation while bus conductors are more active and less stressed. Some work environments, *eg* publicans, give exposure to a less healthy lifestyle.

The nature of the work activity is clearly important. A sedentary job such as actuarial work is less healthy than being a fitness instructor! Publicans are typically quoted as an unhealthy lot, and they were observed to have unusually high mortality in UK occupational mortality investigations.

Some occupations by their very nature attract more healthy workers. This may be accentuated by health checks made on appointment or by the need to pass regular health checks, *eg* airline pilots. However, this effect can be produced without formal checks, *eg* former miners who have left the mining industry as a result of ill health and then chosen to sell newspapers. This will inflate the mortality rates of newspaper sellers.

Selection of employees with respect to health both at entry to and exit from the occupation may lead to lighter mortality among workers: *ie* a "healthy worker" effect.

Former miners who sell newspapers are observed to have very poor mortality. No checks are made on the health of newspaper sellers and typically it will be the sick former miners who choose to take up this occupation.

People's occupations largely determine their income, and this permits them to adopt a particular lifestyle, *eg* content and pattern of diet, quality of housing. This effect can be positive and negative, *eg* over-indulgence.

Unemployment has a particularly negative effect since it can lead to increased levels of stress and a lower income.

It is not always easy to obtain a reliable estimate of mortality rates for a particular occupation. The rates estimated will be unreliable if deaths are not recorded under the same category of occupation as are the lives in the exposed to risk. This can occur for the following reasons:

- Entries on the census returns, which are used to determine the exposed to risk, may not be specific enough. This may result in the wrong occupation being recorded.
- The families of individuals who die may unintentionally "elevate" the occupation. For example, an electrician may be described as an "electrical engineer".
- Estimates may also be unreliable as a result of the factors considered below.

Previous occupations

Information from censuses and death certificates usually relates to the most recent occupation. A selective effect can occur when workers who become ill and cannot continue their usual job switch to "light duties" (*eg* a factory worker taking up a "desk job"). This can lead to artificially high rates of mortality rates being associated with certain jobs. Recording complete employment histories to eliminate this problem is not a practical solution.

Classification

In the past, wives were categorised under the same occupational group as their husbands. Of course this is no longer appropriate, now that most women follow their own occupations. There is, however, still a correlation between the occupation of husband and wife.

Lack of statistics

For occupations which have only a small number of participants (*eg* actuaries) there may not be sufficient data to provide meaningful statistics. Conversely, however, select occupations are more likely to maintain detailed records of members that can be used to provide accurate statistics.

> ***Question 14.4***
>
> How would you expect the mortality rates for judges and divers to compare?

2.2 *Nutrition*

Good nutrition involves consuming the appropriate amounts of the right types of food.

Nutrition has an important influence on morbidity and in the longer term on mortality.

Poor quality nutrition can increase the risk of contracting many diseases and hinder recovery from sickness. In the longer term the burden of increased sickness can influence mortality. In other words sick people are more likely to die prematurely.

Excessive or inappropriate (*eg* too much fat) eating can lead to obesity and an increased risk of associated diseases (*eg* heart disease, hypertension *ie* high blood pressure**) leading to increased morbidity and mortality.**

Inappropriate nutrition may be the result of economic factors *eg* lack of income to buy appropriate foods, or the result of a lack of health and personal education resulting in poor nutritional choices. Also social and cultural factors encourage or discourage the eating of certain foods, *eg* alcohol consumption.

Lack of sufficient food (*subnutrition*) can lead to a general weakening of the body and a reduction in resistance to disease.

Lack of essential vitamins and minerals can lead to *malnutrition*, which can induce certain medical conditions that can increase mortality rates.

Social factors have their influence on the nature of a society's diet; consider the prevalence of junk food and ready-processed meals with high sugar content in developed countries.

Question 14.5

Fat people in the UK tend to come from a "rich" or a "poor" background, rather than an "average wealth" background. Suggest possible reasons for this.

2.3 Housing

The standard of housing encompasses not only all aspects of the physical quality of housing (*eg* state of repair, type of construction, heating, sanitation) but also the way in which the housing is used (*eg* overcrowding and shared cooking). These factors have an important influence on morbidity, particularly that related to infectious diseases (*eg* from tuberculosis and cholera to colds and coughs) and thus on mortality in the longer term.

The effect of poor housing is often confounded with the general effects of poverty.

Question 14.6

List four factors that could adversely affect the mortality of a homeless person in a developed country.

2.4 Climate and geographical location

Climate and geographical location are closely linked. Levels and patterns of rainfall and temperature lead to an environment that is amicable to certain kinds of diseases, *eg* those associated with tropical regions.

Effects can also be observed within these broad categories, *eg* the differences between rural and urban areas in a geographical region. Some effects may be accentuated or mitigated depending upon the development of an area, *eg* industry leading to better roads and communications.

Natural disasters (such as tidal waves and famines) will also affect mortality and morbidity rates, and may be correlated to particular climates and geographical locations.

The following will also vary according to geographical location.

Access to medical care and transport

The availability of readily accessible, modern medical facilities nearby can reduce the delay in receiving effective medical treatment. Preventative screening can identify some conditions at an early stage. Immunisation programmes can control epidemics.

Road accidents

Individuals living in cities are more likely to be involved in motor accidents, although the traffic speed may be lower, so that they are less likely to be fatally injured.

Natural disasters

Certain countries are known to be susceptible to natural disasters, such as tidal waves, earthquakes, hurricanes, floods, drought and famine. A topical issue is whether the apparent increase in the number of natural disasters in recent years can be attributed to global warming caused by increased levels of carbon dioxide in the atmosphere.

Political unrest

Mortality rates in countries at war or where there is a high level of violence and social unrest will be higher because:

- individuals may be required to take part in direct combat,
- there will be an increased risk of injury to civilians,
- food, clean water and medical facilities may be restricted in a war zone.

2.5 Education

Education influences the awareness of the components of a healthy lifestyle which reduces morbidity and lowers mortality rates. It encompasses both formal education and more general awareness resulting from public health and associated campaigns. This effect manifests itself through many proximate determinants:

- **increased income**
- **choice of a better diet**
- **the taking of exercise**
- **personal health care**
- **moderation in alcohol consumption, smoking**
- **awareness of dangers of drug abuse**
- **awareness of a safe sexual lifestyle.**

Some of these are direct causes of increased morbidity *eg* smoking and excessive alcohol consumption, leading to diseases such as lung and other forms of cancer, and strokes. A healthy lifestyle with improved fitness can lead to an enhanced ability to resist diseases.

Recent studies have emphasised the harmful effects of smoking. As a result, life offices have increased the differentials between premium rates for smokers and non-smokers. The degree of harm from passive smoking (inhaling other people's smoke) is not yet precisely understood.

Although education is believed to affect mortality in its own right, it is highly correlated with other risk factors *eg* occupation, standard of living and social class.

Other aspects of lifestyle which can influence mortality and morbidity rates are listed below.

Dangerous activities

Individuals who take part in dangerous sports (*eg* motor racing, hang gliding) are more likely to be involved in serious and possibly fatal accidents.

Travel

Individuals who travel frequently are more likely to be involved in an accident and will be exposed to a wider range of infectious diseases.

Religious attitudes

Some religions do not permit the use of blood transfusions (increasing mortality very slightly). Others forbid the use of alcohol (potentially reducing mortality).

Marital status

We shall see later in this chapter that mortality rates are dependent on marital status.

Question 14.7

What problem must you be careful to avoid before you can make the claim that a small intake of alcohol is good for you?

2.6 Genetics

Genetics may give information about the likelihood of a person contracting particular diseases, and therefore may provide improved information about the chances of sickness or death. Such information may be used in isolation for the particular life in question or, more usefully, by combining it with the life histories of the current and past generations of the family. However, genetics is a relatively new area of study for the medical profession, and only in the case of a few specific diseases is there as yet any indication that genetic information provides firm predictive evidence as to the chances of sickness or death relative to a person of average health.

2.7 Mortality convergence

The variations in mortality described in this section are noted most strongly at working ages. These variations can be large and material for insurance companies.

The variation has been seen to continue after retirement but reduces at the very highest ages, although the evidence is disputed. This convergence of mortality between subgroups at higher ages is referred to as *mortality convergence*.

Detailed analysis of mortality convergence is complicated by the low volumes of data at the highest ages.

3 Selection

Selection is the process by which lives are divided into separate groups so that the mortality (or morbidity) within each group is homogeneous. That is, the experience of all lives within a particular group can be satisfactorily modelled by the same stochastic model of mortality (or morbidity).

In actuarial practice, however, it has become customary to refer to the source of heterogeneity *itself* as "selection".

"Selection" can therefore refer either to the source of heterogeneity itself, or to the subdivision of heterogeneous data into homogeneous classes. Hence the following descriptions of the various *types* of selection also, by definition, describe the various causes of heterogeneity that may exist in any population.

Sometimes the process of division is one that is imposed on the lives, *eg* the division of proposals for life assurance into classes. Lives in different classes will be charged according to different premium scales, which reflect the mortality differences between the classes. Sometimes the process of division is one chosen by the lives themselves, *eg* deciding whether or not to purchase an immediate annuity. Those who purchase an annuity usually experience lighter mortality than those who decide not to purchase an annuity. This phenomenon is known as *self-selection.*

The difference in mortality levels between groups is called the size of the select effect.

Commonly occurring kinds of selection are classified into categories.

We shall consider in detail the following types of selection:

- temporary initial selection
- class selection
- time selection
- adverse selection
- spurious selection.

3.1 Temporary initial selection

Each group is defined by a specified event (the select event) happening to all the members of the group at a particular age, *eg* buying a life assurance policy at age x, retiring on ill-health grounds at age x.

A select mortality table (representing the stochastic model of mortality) is estimated for each group (see Chapter 3). The mortality patterns in each group are observed to differ only for the first s years after the select event. The length of select period is s years. The differences are temporary, producing the phenomenon called *temporary initial selection* (see Chapter 3).

Temporary initial selection occurs when heterogeneity is present in a group that was selected on the basis of a criterion whose effects wear off over time. The relative numbers at each duration since selection in the select group will affect the risk levels within the select group. We have already seen in Chapter 3 that getting through the medical underwriting hurdle of a life company gives policyholders particularly good mortality, but that this effect wears off as time since the medical increases.

Example of temporary initial selection

Lives who have been infected with the HIV virus. Mortality rates will be a function of duration since infection. For example, individuals infected five years ago will experience higher mortality rates than others of the same age who were infected only one year ago.

3.2 Class selection

Each group is specified by a category or class of a particular characteristic of the population, *eg* sex with categories of male and female, occupation with categories of manual and non-manual employment. The stochastic models (life tables) are different for each class. There are no common features to the models, they are different for all ages. This is termed *class selection*.

Class selection refers to a factor which is *permanent* in its effect with respect to mortality *ie* the source of the heterogeneity, in this case, is due to a permanent attribute of the individuals concerned.

Age is one such factor. It can be described as "permanent" because lives subdivided by age will always be expected to show different mortality. Similarly lives (which are otherwise homogeneous) subdivided by sex will also always be expected to show different mortality.

Examples of class selection

- Different races have different susceptibilities to disease.
- Individuals who have lived abroad may have been exposed to tropical diseases.
- More highly paid individuals have a higher standard of living and experience lower mortality rates.

3.3 Time selection

Within a population mortality (or morbidity) varies with calendar time. This effect is usually observed at all ages. The usual pattern is for mortality rates to become lighter (improve) over time, although there can be exceptions, due, for example, to the increasing effect of AIDS in some countries.

A separate model or table will be produced for different calendar periods, *eg* English Life Table No 15 1990–92 and English Life Table No 16 2000–02. The difference between the tables is termed *time selection*.

A mortality investigation carried out over a number of years involves grouping together lives who attain the same age in different time periods. Where time selection is occurring (*ie* mortality varies between time periods) then the combined sample of data taken at different times will be heterogeneous with respect to the lives' true underlying mortality rates. Hence the average rate will not reflect the true underlying rates for each life over the investigation period.

Examples of time selection

- Individuals living 20 years ago experienced higher mortality rates than individuals of the same age living today.
- Individuals with life assurance policies where the sum assured exceeds £100,000 are more likely to have taken out the policy recently.

3.4 Adverse selection

***Adverse selection* is characterised by the way in which the select groups are formed rather than by the characteristics of those groups. So any of the previous forms of selection may also exhibit adverse selection. Adverse selection usually involves an element of self-selection, which acts to disrupt (act against) a controlled selection process which is being imposed on the lives. This adverse selection tends to reduce the effectiveness of the controlled selection.**

Underwriting is the process by which life insurance companies divide lives into homogeneous risk groups by using the values of certain factors (rating factors) recorded for each life. If prospective policyholders know that a company does not use a particular rating factor, *eg* smoking status, then lives who smoke will opt to buy a policy from this company rather than a company that uses smoking status as a rating factor. The outcome will be to lessen the effect of the controlled selection being used by the company as part of the underwriting process. The effect of self-selection by smokers is adverse to the company's selection process. It is an example of adverse selection.

Note that the decision whether to join the select group need not always be a deliberate conscious decision on the part of the individual. It may be just a statistical effect that people with certain characteristics are more likely to join the group or a "double negative" effect where people who do *not* have certain characteristics are *less* likely to join the group.

Example of adverse selection

Individuals who purchase an annuity at retirement are more likely to be in good health than the general population. If these individuals thought that they were likely to die in the near future they would not convert a capital lump sum into a lifetime annuity as this would represent a poor investment.

3.5 Spurious selection

When homogeneous groups are formed we usually tacitly infer that the factors used to define each group are the cause of the differences in mortality observed between the groups. However, there may be other differences in composition between the groups, and it is these differences rather than the differences in the factors used to form the groups that are the true causes of the observed mortality differences.

In other words, the groups are unlikely to be homogeneous. Within each group people will be affected to differing extents by a factor which has not been used in defining the group. Groups with different proportions of affected members will exhibit different mortality.

Ascribing mortality differences to groups formed by factors which are not the true causes of these differences is termed s*purious selection*. For example, when the population of England and Wales is divided by region of residence, some striking mortality differences are observed.

However, a large part of these differences can be explained by the different mix of occupations in each region. The class selection ascribed to regions is spurious and is in part the effect of compositional differences in occupation between the regions.

In statistical terminology the occupational differences in mortality are confounded (mixed up) with the regional differences.

Spurious selection can (theoretically) be removed by identifying all possible sources of heterogeneity within the parent population *ie* by risk classification.

Examples of spurious selection

- Increasing the strictness of underwriting for life insurance products will lead to a lighter mortality experience. This will give the false impression that mortality is improving at a quicker rate than it really is.
- The mortality of individuals of the same age tends to be higher in the North of England than in the South. This gives the impression that class selection is present in respect of regions. However, when the comparison is restricted to individuals in the same occupation, the apparent difference diminishes. This is explained by differences in the relative numbers in high and low risk occupations.

Example

Classify the possible sources of selection that might influence risk levels in the following select groups in the contexts indicated:

(a) an option on a term assurance policy to extend the period of cover without requiring a medical examination (life insurance),

(b) a mortality study based on people earning over £50,000 per annum (life insurance),

(c) a mortality study based on doctors (population mortality),

(d) people being interviewed in the street (market research).

Solution

(a) *Adverse* selection: Individuals who consider they are not in perfect health (and may have difficulty obtaining cover elsewhere) are more likely to take up the option. *Temporary initial selection*: The effects of differences in state of health at the time of selection will wear off over time.

(b) *Class selection*: People earning high salaries will have a different standard of living and lifestyle from other policyholders. *Time selection*: If the study covers several years, this selection criterion will include a greater proportion of more recent policies due to the effect of inflation.

(c) *Class selection*: Doctors form a distinct category with different mortality characteristics from the general population.

(d) *Adverse selection*: The probability that an individual will take part in the survey will depend on the characteristics of the individual *eg* outgoing people with time on their hands are more likely to take part. *Class selection*: The interviewer may have specific instructions only to interview a certain category (*eg* women with children). Also, the interviewer may unconsciously introduce a bias when selecting interviewees *eg* a non-smoking interviewer may prefer not to interview people who are smoking.

Question 14.8

An actuarial student is preparing a mortality table based on data obtained by recording the ages at death on graves in a local cemetery. Suggest how each of the forms of selection described above might be present.

Question 14.9

Criticise the following statements from the viewpoint of how selection might affect the assumed reality:

(a) "There were 1 million reported crimes in 1980 and 2 million in 1990. So the numbers of minor and serious crimes committed have doubled over the last decade."

(b) "1% of the HIV tests carried out in a London clinic proved positive. So there are 500,000 HIV positive people in this country."

(c) "People who take out life assurance policies experience higher mortality rates than the general population because they believe they are likely to die sooner than other people."

4 Selection in life assurance and pensions business

4.1 Life assurance

Selection can arise as a result of decisions by the insurance company (underwriting) or as a result of decisions made by the policyholder (self-selection). The aim of underwriting is to form homogeneous risk classes on the basis of sufficient rating factors, so that the possibility of adverse selection by prospective policyholders is very small. Subsequent decisions made by those who take out policies, *eg* a decision to lapse the policy, can act as selective decrements in respect of the mortality risk.

The initial underwriting process is based on a proposal form and sometimes a medical questionnaire and a medical examination selects out lives in better health. This is a form of class selection. The differences between those underwritten at different ages will usually only depend on policy duration for a limited period of time *ie* there will only be a significant difference between mortality at ages $[x]+t$ and $x+t$ for a limited range of t (up to the end of the select period). **Within the class of assured lives there is temporary initial selection. Each individual rating factor, *eg* occupation and alcohol consumption, will produce mortality differences that are an example of class selection.**

When a restricted number of rating factors are used, *eg* endowment assurances with limited evidence of health, there is more opportunity for adverse selection by prospective policyholders.

The mortality levels amongst the class of assured lives tend to change over calendar time. This is an example of time selection. This effect may be the result of general improvements in mortality over time, or a result of changing underwriting standards over time. If the prime cause is the latter effect, and the observed changes are ascribed to improvements over time, this is an example of spurious selection.

Thus an investigation may reveal a considerable improvement in mortality compared with an earlier period. This might appear to suggest general improvements in mortality rates. However, we need to look more deeply to ensure that the current population is comparable with the earlier investigation. Suppose that the company no longer provides cover for smokers. The company is no longer operating the same underwriting standards and some of the mortality improvement should be attributed to the exclusion of smokers.

Withdrawal often acts as a selective decrement in respect of mortality. Those withdrawing tend to have lighter mortality than those who keep their policies in force. This selective effect results in mortality rates that increase markedly with policy duration and resembles temporary initial selection. Because the cause is not the effect of initial selection, but the effect of selective withdrawal, this is an example of spurious selection.

In other words there are two reasons for the relatively heavy mortality of those who have had a policy for a while compared with new entrants. First, initial selection at the underwriting stage should ensure that the new entrant will be relatively healthy. Secondly, the healthier lives are more likely to lapse their policies and so there will be an increasing concentration of impaired lives as time goes by. It is difficult to separate these two effects.

Homogeneous lapse rates often result from subdivision by the method of selling the policy. This is an example of class selection. Lapse rates tend to be higher at shorter policy durations. This is an example of temporary initial selection.

Question 14.10

Explain why it might be necessary to subdivide policies by method of selling in order to obtain homogeneous lapse rates.

4.2 *Pension funds*

A pension fund includes several different classes of lives, each of which is subject to causes of decrement either from the scheme or from another class. The most important classes are:

(i) active members of the scheme, *ie* those not in receipt of any benefits and for whom contributions are being paid,

(ii) deferred pensioners, *ie* those for whom no contributions are currently being paid and who are entitled to a pension at some future date, and

(iii) pensioners who retired at the normal retirement age and are now receiving a pension,

(iv) pensioners who retired under the ill-health retirement rules of the scheme and are now receiving a pension benefit,

(v) pensioners who retired under the early (before normal pension age) retirement rules of the scheme and are now receiving a pension benefit.

For those currently paying contributions the decrements of interest are death, withdrawal and retirement (in ill health or in good health either early, at or after the normal retirement date)**. For those receiving benefit or entitled to a deferred benefit the only decrement of interest is death.**

The mortality of those who retired early (but in good health) or at normal retirement age is likely to be lower than that of ill-health retirement pensioners. This is an example of class selection. The mortality of ill-health retirement pensioners is likely to depend on duration since retirement for a few years following the date of retirement, and subsequently only on age attained. This is an example of temporary initial selection.

Underwriting at the date of joining a scheme tends to be very limited, *eg* actively at work, and so there tends to be only very slight temporary initial selection. Different sections of a large scheme, *eg* works and staff, may exhibit different levels of mortality. This is an example of class selection.

Among the active members of the scheme ill-health retirement acts as a selective decrement, resulting in lighter mortality among the remaining active members. This is sometimes termed the "healthy worker" or the "active lives mortality" effect.

Withdrawal from a scheme is associated with voluntary or compulsory termination of employment (changing jobs or redundancy). If voluntary resignation is the cause this tends to select those with lighter mortality (and ill-health retirement) rates. If redundancy is the cause withdrawal rates tend to vary markedly over time as economic conditions vary. This is an example of time selection.

Q

Question 14.11

Why will voluntary resignation tend to select those with lighter mortality and ill health-retirement rates?

5 How decrements can have a selective effect

One way in which lives in a population can be grouped is by the operation of a decrement (other than death) *eg* retiring on ill-health grounds, getting married, migrating to a new country. Those who do and do not experience this selective decrement will experience different levels of the primary decrement of interest, often mortality or morbidity.

Those getting married usually experience lighter mortality and morbidity than those of the same age, who do not get married. Marriage is said to have a selective effect in respect of mortality and morbidity.

Part of the reason for the lighter mortality of married persons is that sick and disabled single people at any age are much less likely to marry than healthy individuals. So marriage is operating as a selective decrement, in effect, from the population of single lives (comprising bachelors and spinsters). However, the fact that divorced and widowed lives also show higher mortality than their married counterparts does indicate that there may be survival benefits to be obtained from marriage. The responsibilities of bringing up children may encourage less venturesome activities by parents than might be pursued by unmarried people, and it is possible that the relative stability of a family environment may reduce stress with beneficial consequences for mortality.

6 Why it is necessary to have different mortality tables for different classes of lives

When a life table is constructed it is assumed to reflect the mortality experience of a homogeneous group of lives *ie* all the lives to whom the table applies follow the same stochastic model of mortality represented by the rates in the table. This means that the table can be used to model the mortality experience of a homogeneous group of lives which is suspected to have a similar experience.

If a life table is constructed for a heterogeneous group then the mortality experience will depend on the exact mixture of lives with different experiences that has been used to construct the table. Such a table could only be used to model mortality in a group with the same mixture. It would have very restricted uses.

For this reason separate mortality tables are usually constructed for groups which are expected to be heterogeneous, *eg* separate tables for males and females.

Sometimes only parts of the mortality experience are heterogeneous, *eg* the experience during the initial select period for life assurance policyholders, and the remainder are homogeneous, *eg* the experience after the end of the select period for life assurance policyholders. In such cases the tables are separate (different) during the select period, but combined after the end of the select period. In fact there are separate (homogeneous) mortality tables for each age at selection, but they are tabulated in an efficient (space saving) way.

Question 14.12

What is the risk to a life company of using one mortality table for all classes of lives together?

Question 14.13

What is the problem with producing tables for different classes of lives?

7 Risk classification in life insurance

Life insurance companies provide a service of pooling independent homogeneous risks. If a company is able to do this then as a result of the Central Limit Theorem the profit per policy will be a random variable that follows the normal distribution with a known mean and standard deviation. The Central Limit Theorem was covered in Subject CT3, Probability and Mathematical Statistics.

The company can use this result to set premium rates that ensure that the probability of a loss on a portfolio of policies is at an acceptable level.

The process by which potential insured lives are separated into different homogeneous groups for premium rating purposes, according to the risk they present, is called *risk classification*. It involves trying to identify any risk factors specific to the individual that might influence the likely risk of that individual.

The independence of risks usually follows naturally from the way in which life insurance policies are sold. Rarely does the death of one policyholder influence the mortality of another policyholder.

Careful underwriting is the mechanism by which the company ensures that its risk groups are homogeneous. The risk groups are defined by the use of rating factors, *eg* age, sex, medical history, height, weight, lifestyle. In theory, a company should continue to add rating factors to its underwriting system until the differences in mortality between the different categories of the next rating factor are indistinguishable from the random variation between lives that remains after using the current list of rating factors. In reality the ability of prospective policyholders to provide accurate responses to questions and the cost of collecting information also limit the extent to which rating factors can be used. For example, a proposal form should not ask for information that requires a specialist knowledge. Height is acceptable, but blood pressure is not. For example, the cost of undertaking extensive blood tests has to be weighed against the expected cost of claims that will be "saved" as a result of having this information.

From a marketing point of view proposers are anxious that the process of underwriting should be straightforward and speedy.

In practice, rating factors will be included if they avoid any possibility of selection against the company, and satisfy the time and cost constraints of marketing. This decision is often driven by competitive pressures. If several companies introduce a new rating factor, which in fact influences mortality levels significantly, then other companies will need to follow this lead or risk adverse selection against them.

Question 14.14 ***(Subject CT5, September 2005, Question 1)***

Describe what is meant by adverse selection in the context of a life insurance company's underwriting process and give an example. [2]

Example

In the 1970s the first UK offices started to introduce smoker and non-smoker rates for assurance business. This is now the normal practice except for savings contracts.

Question 14.15

Explain what would be the likely effect on a company's mortality experience if it issued assurance policies for the same rates of premium to smokers and non-smokers, when most of the other companies in the market place charged different rates for the two groups.

8 Use of single figure indices to summarise and compare mortality levels

In the remainder of this chapter we will consider how summary (single figure) mortality indices can be used to enable an observer to assimilate, quantify and compare the mortality experience of different populations. For example, we might want a single number that allows us to compare the mortality of cooks *vs* butlers, or to monitor the progress over time of a country's mortality.

Within any homogeneous population mortality experience will be summarised by age-specific mortality rates and sometimes by graduating and displaying these results in a life table. If we wish to summarise the overall level of mortality in the population we are faced with the problem of evaluating the set of age-specific rates. This is most simply done by calculating a single figure index that is a true reflection of the general level of mortality in the population. Such indices can also be used to facilitate the comparison between the mortality levels in two or more different populations, or within the same population at different points of time.

All summary measures are weighted averages of the age specific rates or some function of the age specific rates. Averaging estimates of age-specific rates, all of which are subject to sampling error, will give summary measures with much reduced sampling error. This will result in more precise comparisons.

For instance if we want to compare the mortality of Brighton residents against Blackpool residents and compare q_{40} for each region, the potential sampling error in each rate makes the comparison quite risky. But if we quantify "Brighton mortality" as a single number and "Blackpool mortality" as a single number, comparison of *these* should not be so adversely affected by sampling errors.

The calculation of some indices requires extensive data. This may limit the situations in which they can be used.

Some indices involve the use of weights that vary between populations. In making comparisons using such indices we must be aware that observed differences in the index may be the result of different weights rather than the result of differing age specific rates in the populations being compared.

The main advantage of the use of single figure indices is their simplicity for summary and comparison compared to the use of a set of age specific rates. Some indices are particularly designed for comparison with the mortality in a standard population, *eg* mortality rates used for premium calculation. This makes their use particularly relevant in an actuarial context.

The main disadvantage of the use of single figure indices is the loss of information as a result of summarising the set of age specific rates, and any distortions that may be introduced by the choice of weights for the averaging process.

8.1 *Single figure indices*

Rates will be presented in the context of mortality but they can be used for any other decremental rates, *eg* retirement, disability inception, lapse.

The following notation will be used:

$E^c_{x,t}$ **Central exposed to risk in population being studied between ages x and $x+t$.**

$m_{x,t}$ **Central rate of mortality either observed or from a life table in population being studied for ages x to $x+t$.**

${}^sE^c_{x,t}$ **Central exposed to risk for a standard population between x and $x+t$.**

${}^sm_{x,t}$ **Central rate of mortality either observed or from a life table in standard population for ages x to $x+t$.**

Remember that m_x is the weighted average force of mortality over the year of age x to $x+1$. Usually summary measures will not require the calculation of the m_x at each age but will instead work with age grouped data *eg* in 5-year age bands. The "t" in the expression $m_{x,t}$ allows for this grouping.

Crude mortality rate

This summary measure is defined as:

$$\frac{\sum_x E^c_{x,t}\, m_{x,t}}{\sum_x E^c_{x,t}} = \frac{\textbf{Actual deaths}}{\textbf{Total exposed to risk}}$$

It is a weighted average of $m_{x,t}$ using $E^c_{x,t}$ as weights. Its main advantage is that a knowledge of $E^c_{x,t}$ and $m_{x,t}$ separately for each age x is not needed (because we just use *total* actual deaths and *total* exposed to risk). **Its main disadvantage is that differences in age structure *ie* $E^c_{x,t}$ between populations will be confounded with differences between mortality rates in using this index to make comparisons of mortality levels between populations.**

Consider the following example based on two towns, Youngsville and Oldsville, whose hypothetical populations consist of 100,000 people all aged 20, 40 or 60:

	YOUNGSVILLE			**OLDSVILLE**		
Age	**Population**	**Deaths**	**Death rate**	**Population**	**Deaths**	**Death rate**
20	70,000	210	0.30%	10,000	15	0.15%
40	20,000	90	0.45%	20,000	60	0.30%
60	10,000	100	1.00%	70,000	525	0.75%
Total	100,000	400	0.40%	100,000	600	0.60%

Looking at the death rates for each age, we see that the mortality rates are higher in Youngsville than in Oldsville at all ages, *ie* any individual in Youngsville is more likely to die at any given age than a corresponding individual in Oldsville. However, the crude death rate is *higher* for Oldsville, which suggests the opposite.

The explanation of this apparent paradox is that, as the names suggest, the age profiles of the two populations are very different. The crude death rate takes no account of the fact that there are far more old people in Oldsville, so that we would expect the total number of deaths to be higher there anyway, irrespective of the influence of other risk factors. So, the crude death rate is primarily reflecting the average age of the population, rather than indicating how "healthy" each town is to live in (although it is affected by the actual mortality rates, as well).

This problem can be dealt with by *standardising* the mortality rates. This is done by applying weighting factors to each age group to compensate for the different population structures. Populations may need to be standardised by age, sex or some other risk factor, *eg* occupation.

Question 14.16 ***(Subject CT5, September 2005, Question 2)***

Describe how occupation affects morbidity and mortality. [3]

An index that is not standardised is called a *crude* index. Standardisation can be done using *direct* or *indirect* standardisation. These concepts are discussed further in the next sections.

In our examples, we will base our calculations on a standard population with the following characteristics:

	STANDARD POPULATION		
Age	**Population**	**Deaths**	**Death rate**
20	40,000	80	0.20%
40	35,000	140	0.40%
60	25,000	200	0.80%
Total	100,000	420	0.42%

Directly standardised mortality rate

This summary measure is defined as:

$$\frac{\sum_x {}^sE^c_{x,t}\, m_{x,t}}{\sum_x {}^sE^c_{x,t}}$$

It is a weighted average of $m_{x,t}$ using ${}^sE^c_{x,t}$ as weights. Its main advantage is that it only reflects differences in $m_{x,t}$ and not differences in age structures between populations. Its main disadvantage is that it requires age specific mortality rates $m_{x,t}$ for its calculation.

We standardise the death rates by working out how many deaths we would have had if the proportions at each age had been the same as in a standard population.

Example

Calculate the directly standardised mortality rates for Youngsville and Oldsville.

Solution

For the group aged 20 in Youngsville:

- There would have been $40\% \times 100{,}000 = 40{,}000$ people.
- By proportioning, $210 \times \dfrac{40{,}000}{70{,}000} = 120$ of these "would have" died.

Carrying out similar calculations for the other age groups gives the following results:

	YOUNGSVILLE			OLDSVILLE		
Age	**Standardised population**	**Deaths**	**Death rate**	**Standardised population**	**Deaths**	**Death rate**
20	40,000	120	0.30%	40,000	60	0.15%
40	35,000	157.5	0.45%	35,000	105	0.30%
60	25,000	250	1.00%	25,000	187.5	0.75%
Total	100,000	527.5	0.53%	100,000	352.5	0.35%

The directly standardised mortality rates are therefore:

Youngsville: $\dfrac{527.5}{100{,}000} = 0.005275 \quad \textit{ie } 0.5275\%$

Oldsville: $\dfrac{352.5}{100{,}000} = 0.003525 \quad \textit{ie } 0.3525\%$

Alternatively, we could have calculated these figures more simply as follows:

Youngsville: $40\% \times 0.30 + 35\% \times 0.45 + 25\% \times 1.00 = 0.5275\%$

Oldsville: $40\% \times 0.15 + 35\% \times 0.30 + 25\% \times 0.75 = 0.3525\%$

Question 14.17

You are given the following data from two populations:

		MADEUPTOWN		**STANDARD POPULATION**	
Sex	**Occupation**	**Population**	**Deaths**	**Population**	**Deaths**
Male	**Office worker**	20,000	100	10,000	50
	Manual worker	60,000	500	20,000	300
	Other	20,000	250	20,000	500
Female	**Office worker**	20,000	50	10,000	50
	Manual worker	30,000	200	10,000	100
	Other	50,000	900	30,000	500
Total		200,000	2,000	100,000	1,500

Calculate the directly standardised mortality rate for Madeuptown, standardising by,

(a) occupation,

(b) sex, and

(c) occupation *and* sex.

Question 14.18

"A directly standardised mortality rate is most heavily influenced by the older ages." True or false?

Indirectly standardised mortality rate

Indirect standardisation is an approximation to direct standardisation, and removes the need for knowledge of the age-specific mortality rates, $m_{x,t}$. These rates are often not available or, because of the small age specific exposed to risk in the population being studied, are estimated with large standard errors.

The crude mortality rate for the population being studied is easily calculated (see above). We attempt to find a factor F^*, such that:

$$\text{directly standardised mortality rate} = F^* \times \text{crude mortality rate}$$

Simple substitution shows that:

$$F^* = \frac{\sum_x {}^sE^c_{x,t}\, m_{x,t}}{\sum_x {}^sE^c_{x,t}} \Bigg/ \frac{\sum_x E^c_{x,t}\, m_{x,t}}{\sum_x E^c_{x,t}}$$

If we approximate $m_{x,t}$ in this ratio by ${}^sm_{x,t}$, then we obtain a similar factor

$$F = \frac{\sum_x {}^sE^c_{x,t}\, {}^sm_{x,t}}{\sum_x {}^sE^c_{x,t}} \Bigg/ \frac{\sum_x E^c_{x,t}\, {}^sm_{x,t}}{\sum_x E^c_{x,t}}$$

F is called the area comparability factor (a term resulting from its initial use in making comparisons between the different regions of England and Wales).

Note also that:

$$F = \frac{\text{crude mortality rate for standard population}}{\text{crude mortality rate for region, using standard mortality}}$$

If this approximation to F^* is used then we obtain an indirectly standardised mortality rate:

$$\frac{\sum_x {}^sE^c_{x,t}\, {}^sm_{x,t}}{\sum_x {}^sE^c_{x,t}} \Bigg/ \frac{\sum_x E^c_{x,t}\, {}^sm_{x,t}}{\sum_x E^c_{x,t}\, m_{x,t}}$$

$$= \begin{array}{c}\textbf{Crude mortality rate}\\ \textbf{for standard population}\end{array} \Bigg/ \frac{\textbf{Expected deaths in population}}{\textbf{Actual deaths in population}}$$

The main advantages of the indirectly standardised rate are the ease of availability of the data to calculate it, and the fact that it is a good approximation to the directly standardised rate and so removes almost all of the effect of differing age structures between populations when any comparisons are made.

Question 14.19

Prove that approximating $m_{x,t}$ by ${}^sm_{x,t}$ in the definition of F gives the above formula for the indirectly standardised mortality rate.

Question 14.20

(a) Is F a measure of mortality?

(b) What does a value of F less than/greater than 1 indicate?

Example

Calculate the area comparability factor for Youngsville using the data for the standard population given earlier.

Solution

We have already seen that the mortality rate based on the standard population and standard population mortality is 0.42%.

The mortality rate based on the regional population and standard population mortality is:

$$70\% \times 0.20 + 20\% \times 0.40 + 10\% \times 0.80 = 0.30\%$$

So the approximate area comparability factor is: $0.42\%/0.30\% = 1.400$

Question 14.21

Calculate the area comparability factor and indirectly standardised mortality rate for Oldsville.

Question 14.22

Calculate a standardised mortality rate for Madeuptown using indirect standardisation by both occupation and sex.

Standardised mortality ratio

If we compare the indirectly standardised mortality rate with the crude mortality rate in the standard population *ie* a ratio, then we obtain:

$$\frac{\sum_x E^c_{x,t}\, m_{x,t}}{\sum_x E^c_{x,t}\ {}^s m_{x,t}} = \frac{\text{Actual deaths in population}}{\text{Expected deaths in population}}$$

Note that we do not need a standard population structure for this.

This is effectively an indirectly standardised rate expressed on a scale where values less than 1 indicate populations with mortality lighter than that in the standard population and values greater than 1 indicate populations with mortality heavier than that in the standard population.

The ratio can be written in the form:

$$\frac{\sum_x E^c_{x,t}\ {}^s m_{x,t}\, \dfrac{m_{x,t}}{{}^s m_{x,t}}}{\sum_x E^c_{x,t}\ {}^s m_{x,t}}$$

which shows it to be the weighted average of the age specific mortality differentials between the standard population and the population being studied

ie $\dfrac{m_{x,t}}{{}^s m_{x,t}}$

weighted by the age specific numbers of deaths in the standard population

ie $E^c_{x,t}\ {}^s m_{x,t}$

Question 14.23

Calculate the standardised mortality ratio for Oldsville.

9 Exam-style questions

Here are some exam-style questions on single figure indices:

Question 14.24 ***(Subject 105, April 2000, Question 14)***

(i) Discuss the suitability of the crude death rate, the standardised mortality rate and the standardised mortality ratio for comparing:

(a) the mortality, at different times, of the population of a given country

(b) the mortality, at a certain time, of two different occupational groups in the same population [6]

(ii) The following table gives a summary of mortality for one of the occupational groups and for the country as a whole.

	Occupation A		Whole country	
Age group	Exposed to risk	Deaths	Exposed to risk	Deaths
20 – 34	15,000	52	960,000	3,100
35 – 49	12,000	74	1,400,000	7,500
50 –64	10,000	109	740,000	7,100
Total	37,000	235	3,100,000	17,700

Calculate the crude death rate, the standardised mortality rate and the standardised mortality ratio for Occupation A. [4]

[Total 10]

Question 14.25

(i) Define in words the following single figure indices:

- the crude death rate
- the directly standardised mortality rate
- the standardised mortality ratio. [4]

(ii) Two states, A and B, of a particular country have produced the following mortality data for a given time period:

State A

Age last birthday	Deaths	Central exposed to risk
0 – 19	75	40,000
20 – 59	1,175	80,000
60 – 100	2,600	60,000
Total	3,850	180,000

State B

Age last birthday	Deaths	Central exposed to risk
0 – 19	150	30,000
20 – 39	300	20,000
40 – 59	375	15,000
60 – 100	600	10,000
Total	1,425	75,000

(a) Calculate the crude death rates and the standardised mortality ratios for the two areas, using ELT15 (Males) as the standard mortality basis, where appropriate. (For this purpose, you can assume that all lives in a particular age band are subject to the force of mortality applicable to the average age of that band.)

(b) Comment on your results. [7]
[Total 11]

10 End of Part 4

What next?

1. Briefly **review** the key areas of Part 4 and/or re-read the **summaries** at the end of Chapters 12 to 14.
2. Attempt some of the questions in Part 4 of the **Question and Answer Bank**. If you don't have time to do them all, you could save the remainder for use as part of your revision.
3. Attempt **Assignment X4**.

And finally ...

Good luck!

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Chapter 14 Summary

Selection

Mortality and morbidity rates vary by age and sex but also vary because of the influence of other risk factors. These include occupation, nutrition, housing, climate/geography, education and genetics. The risk factors give rise to heterogeneity in a population.

The process by which lives in a population are divided into separate homogenous groups is called selection.

There are five main forms of selection:

- *Temporary initial selection*, where the level of risk diminishes or increases since the occurrence of a selection process (or a discriminating event).
- *Class selection*, where a select group is taken from a population consisting of a mixture of different types ("classes") of individual with different characteristics.
- *Time selection*, where a select group is taken from a population of individuals from different calendar years.
- *Adverse selection*, where the individual's own choice influences the composition of a select group.
- *Spurious selection*, where the distorting effect of a confounding factor gives the false impression that one of the other forms of selection is present.

Examples of such selection can be found in both life assurance and pensions business.

Unless insurers classify risks accurately they cannot charge an appropriate premium which reflects the underlying risk. They will therefore be exposed to adverse selection because of differences in premiums charged by competitors. The existence of different mortality tables for different classes of lives enables the insurer to deal with heterogeneity.

Decrements may be found to have a selective effect. A selective decrement will "select" from the population lives whose rate of decrement from another cause differs from that of the whole population.

Standardisation

The mortality of a population can be summarised using a single figure index. This makes comparison easy, since a single figure index is easy to assimilate, but specific features of the underlying mortality rates may be hidden.

The simplest single figure index is the crude mortality rate, but this is heavily influenced by the age and sex structure of the population.

$$\text{Crude mortality rate} = \frac{\text{actual number of deaths (all ages)}}{\text{total central exposed to risk (all ages)}}$$

Standardisation is used to remove the effects of demographic structure. Standardised indices include:

- directly standardised mortality rates,
- indirectly standardised mortality rates.

The indirectly standardised mortality rate uses the area comparability factor F, which adjusts the crude mortality rate to compensate for the composition of the underlying population.

Indirectly standardised mortality rate = $F \times$ crude regional mortality rate

The SMR (which is a ratio, not a rate) can be employed to compare relative mortality levels. It is the ratio of the actual number of deaths in region divided by the expected number of deaths in that region had the standard mortality applied.

The SMR is a useful index, since it only requires the age/sex specific rates for the standard population, not the age/sex specific rates for the region/occupation *etc*.

Even after standardisation, these indices are still heavily dependent on mortality at the older ages *ie* they are much more sensitive to changes in mortality in this age range.

Formulae

Crude mortality rate = $\dfrac{\text{actual number of deaths (all ages)}}{\text{total central exposed to risk (all ages)}} = \dfrac{\sum E^c_{x,t} m_{x,t}}{\sum E^c_{x,t}}$

Directly standardised mortality rate =

$$\frac{\text{expected deaths in standard population (regional mortality)}}{\text{total standard population}} = \frac{\sum {}^sE^c_{x,t} m_{x,t}}{\sum {}^sE^c_{x,t}}$$

Area comparability factor

$$F = \frac{\text{crude mortality rate for standard population}}{\text{expected crude regional mortality rate (standard mortality)}}$$

$$= \frac{\sum {}^sE^c_{x,t}\ {}^sm_{x,t}}{\sum {}^sE^c_{x,t}} \Bigg/ \frac{\sum E^c_{x,t}\ {}^sm_{x,t}}{\sum E^c_{x,t}}$$

Indirectly standardised mortality rate =

$$F \times \text{crude regional mortality rate} = \frac{\sum {}^sE^c_{x,t}\ {}^sm_{x,t}}{\sum {}^sE^c_{x,t}} \Bigg/ \frac{\sum E^c_{x,t}\ {}^sm_{x,t}}{\sum E^c_{x,t} m_{x,t}}$$

Standardised mortality ratio =

$$\frac{\text{actual deaths in region}}{\text{expected deaths in region (standard mortality)}} = \frac{\sum E^c_{x,t} m_{x,t}}{\sum E^c_{x,t}\ {}^sm_{x,t}}$$

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Chapter 14 Solutions

Solution 14.1

AIDS was identified in the early 1980s and advice for preventing infection was publicised some years later. The incidence of AIDS-related deaths at different ages, which is potentially very important in life assurance, can therefore be expected to change over time.

If education has genuinely changed people's behaviour, then we may expect to see fewer AIDS-related deaths amongst the young. We may see an increase in AIDS-related deaths at higher ages as the lives who were most at risk during the mid-80s gradually age.

Recently, concerns have been raised that, without ongoing AIDS education, today's young generation may have become less aware of the dangers of AIDS.

Solution 14.2

The graph shows $\log(q_x)$. So if q_x increases exponentially with age, the graph will show a linear relationship between $\log(q_x)$ and age.

Solution 14.3

Mortality rates tend to improve over time due to improvements in living standards and, in particular, as a result of improvements in medical care. Exceptions to this trend would occur at times of natural or man-made disasters such as epidemics or wars.

Solution 14.4

The crude mortality rate, *ie* number of deaths divided by the whole population, is likely to be higher for judges, because their average age is higher, even though it is inherently a less dangerous job than diving.

Solution 14.5

Affluent individuals can afford to buy rich foods *eg* steak and spirits. Some may overindulge and become overweight through eating too much.

Individuals with limited means may be forced to eat cheaper foods *eg* chips and beer. This can lead to obesity through eating the "wrong" foods.

Solution 14.6

Homeless people who are forced to live outdoors will be subject to the effects of the weather *eg* rain and cold nights.

Many homeless people do not have a reliable source of income and may not be able to afford to eat well.

Homeless people may have to wear the same clothes for prolonged periods, which will increase the likelihood of disease.

Some homeless people have psychiatric disorders or are drug addicts. So their mortality risk may be higher because of a selection effect. We will study selection in Section 3 of this chapter.

Solution 14.7

One of the problems is that some people are teetotal *because* they are in poor health *eg* they have had a heart attack and their doctor has told them to stop smoking and drinking. So it would hardly be surprising if people who didn't drink had higher mortality rates. A study that did not take this into account would simply be showing that people who are in poor health tend to drink less, which does not mean that drinking is good for you.

(In fact, the studies carried out recently have excluded people who do not drink "for medical reasons", and they still appear to show the same effect *ie* that a small intake of alcohol is beneficial. One possible explanation, in addition to the direct physical effects of alcohol consumption, is that a moderate consumption of alcohol reduces stress levels.)

Solution 14.8

Here are some possibilities:

Temporary initial selection: (not applicable, since we're not analysing by duration)

Class selection: Only people living in that area who had connections with that church and could afford a gravestone would be buried there. Only those whose precise age at death was known (*"Here lies X, lost at sea"*) would be recorded.

Self-selection: Only those individuals (or their families) who indicated a wish to have a grave with a gravestone will be there. There may have been special collections for people who died young in tragic circumstances (so such deaths might be over represented).

Time selection: Graves may cover several centuries, during which different general mortality rates and specific causes of death (epidemics *etc*) were in effect. Some of the older graves may have been ignored because it was difficult to read the writing or because they were overgrown by weeds.

Spurious selection: Spurious selection may be present by chance, since there are a lot of possible risk factors that have not been taken into account. For example, the men may have all worked at the local mine, exaggerating the difference between the mortality rates for males and females.

Solution 14.9

(a) Crimes reported may not reflect crimes committed. There are influences from self-selection (*eg* a lot of crimes are never reported), time selection (*eg* changes in procedures for recording and classifying crimes) and class selection (*eg* the relative proportions of minor and serious crimes may have changed over time).

(b) The individuals tested are not representative of the general population. There are influences from self-selection (*eg* people are more likely to have a test if they consider they are at greater risk than the general population or if a sexual partner has been diagnosed as HIV positive) and class selection (*eg* only those who are prepared to undergo a medical test and can afford to pay for the cost of the test, if applicable). Extrapolating the experience in London to the rest of the country may also not be valid. London is likely to have a higher incidence of HIV than say Bournemouth where the population is older *ie* there are influences from class selection.

(c) This statement is not true because the temporary initial selection effect of underwriting outweighs any self-selection effect on the part of the proposer. Risk averse individuals who acknowledge the fact that they *might* die at an early age do not necessarily believe that they *will*.

Solution 14.10

The rate at which people withdraw from a policy differs depending on the way in which the policy was sold. For example, people who are coerced into taking out policies (*eg* by "pushy" sales staff) are more likely to withdraw than people who perceive that their policies meet an important financial need. Hence withdrawal rates vary with sales practices and by sales distribution channel.

Solution 14.11

People who are in ill health will not tend to move to a new job. This is partly because their health may mean that they would be screened out by any initial health checks carried out by a prospective employer and partly because they may not feel up to the stresses of a new job. In addition, pension benefits tend to improve with length of service. Hence people in ill health would be reluctant to move jobs and risk a lower pension benefit.

Solution 14.12

If the life insurer is using the same mortality table for all classes of lives together it will be charging the same premium to lives which present different risks. The premium will be based on the average risk. This practice leaves the company in a risky position because it could easily lose the low risk lives to a competitor who charges differential premium rates. High risks will be attracted to the company and it will be selected against.

Solution 14.13

The problem with producing tables for different classes of lives is that whilst we would wish to subdivide the data into homogeneous groups as far as possible we cannot reduce the size of each group below the level at which observations may be statistically significant. It is also administratively inconvenient to use too many different tables.

Solution 14.14

Adverse selection in a life insurance company's underwriting process is selection that leads to an adverse effect on the company. It usually involves an element of self-selection, which acts to disrupt the underwriting process (*ie* the process by which a life insurance company divides lives into homogeneous risk groups through the use of rating factors).
Examples include:

- People who smoke will tend to seek life assurance from companies that charge identical premiums for smokers and non-smokers, whereas non-smokers will apply to companies that differentiate between them and therefore charge cheaper premiums to non-smokers. The first company will suffer from adverse selection, as the ratio of smoker to non-smoker lives that it takes on will increase.
- Selective withdrawal (of healthy lives) worsens the company's average mortality experience from those policies that remain.

Only one example would be required to answer this question in the exam.

Solution 14.15

It is likely that the company's mortality experience would worsen substantially.

Assume that smoker mortality is higher than non-smoker mortality, and so premium rates for smokers would be higher than for non-smokers. An insurer that does not distinguish between the two groups will charge the same rates for both, and these rates will be in between the market rates of other companies, *ie* its rates for smokers will be cheap, and for non smokers will be expensive.

The company will therefore attract large numbers of smokers. However, non-smokers will find its rates too expensive, and will therefore buy from other companies in the market place. Before long the company may find that its portfolio consists almost entirely of smokers, and its mortality experience will be heavier as a result.

Solution 14.16

People's occupations determine their environment for around 40 hours a week (if they work full time). This environment may be rural or urban. Some work environments give exposure to a less healthy lifestyle (*eg* bar workers). Some may involve exposure to harmful substances (*eg* chemicals) or to infection (*eg* hospital workers), and some may involve potentially dangerous situations (*eg* working at heights, working on an oil rig). Health and safety regulations help to reduce risks in the workplace.

Some occupations are, by nature, more healthy than others (*eg* office workers are less active than fitness instructors). Some require health checks (*eg* armed forces, airline pilots) and, as a result, attract more healthy people. On the other hand, some occupations can attract less healthy workers, *eg* lives who have retired from a job involving manual labour because of ill health, and now have a more sedentary occupation.

A person's occupation largely determines his/her income, and this affects lifestyle and standard of living, *eg* diet, quality of housing. The effect of this can be either positive or negative, *eg* a higher income is sometimes associated with over-indulgence.

Solution 14.17

The crude mortality rate is: $2{,}000/200{,}000 = 0.01$ *ie* 1%.

(a) ***Standardising by occupation***

Standardising by occupation (by calculating how many people and deaths there would have been if the population had conformed to the same proportions as the standard population):

		STANDARDISED POPULATION	
Sex	**Occupation**	**Population**	**Deaths**
Both	**Office worker**	40,000	150.0
	Manual worker	60,000	466.7
	Other	100,000	1,642.9
Total		200,000	2,259.5

For example, the percentage of manual workers in the standard population:

$$(20,000 + 10,000) / 100,000 = 30\%$$

Therefore the number of manual workers in Madeuptown if it had the same percentage of manual workers:

$$30\% \times 200,000 = 60,000$$

Number of deaths amongst manual workers in Madeuptown based on 60,000 rather than 90,000 lives is:

$$\frac{60{,}000}{90{,}000} \times (500 + 200) = 466.7$$

Other entries in the table are derived similarly.
So the death rate standardised by occupation is: $2{,}259.5/200{,}000 = 0.0113$

ie 11.3 per 1,000.

(b) ***Standardising by sex***

Since the proportions of each sex are the same as in the standard population, standardisation by sex makes no difference to the crude mortality rate.

(c) ***Standardising by occupation and sex***

Standardising by occupation and sex gives:

		STANDARDISED POPULATION	
Sex	**Occupation**	**Population**	**Deaths**
Male	**Office worker**	20,000	100.0
	Manual worker	40,000	333.3
	Other	40,000	500.0
Female	**Office worker**	20,000	50.0
	Manual worker	20,000	133.3
	Other	60,000	1,080.0
Total		200,000	2,196.7

So the death rate standardised by sex and occupation is: $2{,}196.7/200{,}000 = 0.0110$ *ie* 11.0 per 1,000.

The crude death rate is lower than both the death rate standardised by occupation alone and the death rate standardised by occupation and sex. Hence Madeuptown has a lower prevalence of the high mortality occupations. Standardising by sex as well as occupation reduces the death rate, which suggests that Madeuptown has a higher prevalence of the weaker sex in the high mortality occupations.

Solution 14.18

True. The weightings used are based on mortality rates, which will be greatest at the older ages.

Solution 14.19

The indirectly standardised mortality rate for a region is:

$F \times$ crude mortality rate for population being studied

$$= \frac{\sum {}^{s}E^{c}_{x,t}\, {}^{s}m_{x,t}}{\sum {}^{s}E^{c}_{x,t}} \Bigg/ \frac{\sum E^{c}_{x,t}\, {}^{s}m_{x,t}}{\sum E^{c}_{x,t}} \times \frac{\sum E^{c}_{x,t} m_{x,t}}{\sum E^{c}_{x,t}} = \frac{\sum {}^{s}E^{c}_{x,t}\, {}^{s}m_{x,t}}{\sum {}^{s}E^{c}_{x,t}} \Bigg/ \frac{\sum E^{c}_{x,t}\, {}^{s}m_{x,t}}{\sum E^{c}_{x,t} m_{x,t}}$$

$$= \text{Crude mortality rate for standard population} \Bigg/ \frac{\text{Expected deaths in population}}{\text{Actual deaths in population}}$$

Solution 14.20

(a) Not directly. F provides information only about the structure of the population being studied, relative to the standard population.

(b) A value of F less than 1 indicates that the regional population structure (*ie* the structure of the population being studied) is more heavily weighted towards individuals who experience heavier mortality *eg* older ages or males. A value of F greater than 1 indicates that the regional population structure is more heavily weighted towards individuals who experience lighter mortality *eg* younger ages or females.

Solution 14.21

The mortality rate based on the standard population and standard population mortality is 0.42%.

The mortality rate based on the regional population and standard population mortality is:

$$10\% \times 0.20 + 20\% \times 0.40 + 70\% \times 0.80 = 0.66\%$$

So the approximate area comparability factor is: $0.42\%/0.66\% = 0.636$
The indirectly standardised mortality rate is: $0.636 \times 0.6\% = 0.38\%$

Solution 14.22

The mortality rate based on the standard population and standard population mortality is:

$$1{,}500/100{,}000 = 1.50\%$$

The mortality rate based on the population of Madeuptown and standard population mortality is:

$$\frac{20{,}000}{200{,}000} \times \frac{50}{10{,}000} + \frac{60{,}000}{200{,}000} \times \frac{300}{20{,}000} + \ldots + \frac{50{,}000}{200{,}000} \times \frac{500}{30{,}000} = 1.37\%$$

So the approximate area comparability factor is: $1.50\%/1.37\% = 1.098$

The indirectly standardised mortality rate is calculated by applying this to the crude mortality rate: $1.098 \times \dfrac{2{,}000}{200{,}000} = 1.10\%$

Solution 14.23

The actual number of deaths in Oldsville is 600.

The expected number of deaths is:

$$0.20\% \times 10{,}000 + 0.40\% \times 20{,}000 + 0.80\% \times 70{,}000 = 660$$

So the indirectly standardised mortality ratio (SMR) is $600/660 = 0.909$

Solution 14.24

(i)(a) ***Comparison of national mortality at different times***

Crude death rate

The crude death rate is easy to calculate but does not take into account the age/sex structure of the population. So it could give misleading results if the structure has changed significantly between the two times.

Standardised mortality rate

The standardised mortality rate takes account of the population structure. So it should provide a good indicator, provided the same standard population is used in both calculations. However, it is more complicated to calculate since it requires a breakdown of the occupational rates for each age/sex group.

Standardised mortality ratio

Again, this takes into account the population structure and should be a good indicator. This requires data for the occupational population numbers in each age/sex group.

(i)(b) ***Comparison of mortality for different occupational groups***

Crude death rate

Since the crude death rate does not take into account the age/sex structure of the population and different occupations may have very different population structures, this would not be a reliable indicator for comparing the effect of different occupations on mortality.

Standardised mortality rate

Because the standardised mortality rate takes account of the population structure, it should provide a good indicator, provided the same standard population is used in both calculations. Again, it requires age/sex-specific rates.

Standardised mortality ratio

Again, this takes into account the population structure and should be a good indicator.

All three indices are heavily influenced by the mortality rates at the older ages, and so may not be helpful if it is the rates at younger ages that are of interest.

(ii) ***Calculations for Occupation A***

Crude death rate

The crude death rate is obtained by dividing the total number of deaths by the total exposed to risk, which gives:

$$CDR = \frac{235}{37,000} = 0.00635 \quad \text{or } 6.35 \text{ per thousand}$$

Standardised mortality rate

The standardised mortality rate is obtained by first calculating the number of deaths that would be expected in the whole country based on the mortality rates for Occupation A, which gives:

$$\begin{aligned} Exp.deaths &= \frac{52}{15,000} \times 960,000 + \frac{74}{12,000} \times 1,400,000 + \frac{109}{10,000} \times 740,000 \\ &= 20,027 \end{aligned}$$

This is then divided by the exposed to risk for the whole country:

$$Std.Rate = \frac{20,027}{3,100,000} = 0.00646 \quad \text{or } 6.46 \text{ per thousand}$$

Standardised mortality ratio

For the standardised mortality ratio, we first need to find the number of deaths that would be expected in Occupation A if national mortality rates applied, which gives:

$$\begin{aligned} Exp.deaths &= \frac{3,100}{960,000} \times 15,000 + \frac{7,500}{1,400,000} \times 12,000 + \frac{7,100}{740,000} \times 10,000 \\ &= 208.7 \end{aligned}$$

The standardised mortality ratio is then found by dividing the actual number of deaths by this figure:

$$SMR = \frac{235}{208.7} = 1.126 \quad \text{or } 112.6\%$$

Solution 14.25

(i) ***Definitions***

Crude death rate

The crude death rate for a particular population is the total number of deaths observed during the period divided by the total central exposed to risk for the same period.

Directly standardised mortality rate

The DSMR is the ratio of:

{the number of deaths that would have occurred in a standard population, had the age-specific mortality of the particular population applied}

to:

{the total central exposed to risk of the standard population}

Alternatively it could be described as the weighted average mortality rate for the study group, where the weights are the central exposed-to-risk values for the standard population.

Standardised mortality ratio

The SMR is the ratio of:

{the number of deaths observed in the particular population}

to:

{the number of deaths that would have occurred in the particular population had the age-specific mortality of the standard population applied}

(ii)(a) ***Calculations***

Crude death rate (CDR)

$$CDR(A) = \frac{3,850}{180,000} = 0.02139$$

$$CDR(B) = \frac{1,425}{75,000} = 0.019$$

Standardised mortality ratio (SMR)

$$SMR(A) = \frac{3.85}{40 \times 0.00018 + 80 \times 0.00166 + 60 \times 0.09675} = 0.648$$

where, for example, 0.00018 is μ_{10} from ELT15 (Males).

$$SMR(B) = \frac{1.425}{30 \times 0.00018 + 20 \times 0.00090 + 15 \times 0.00440 + 10 \times 0.09675} = 1.348$$

(ii)(b) ***Comment***

The CDRs of the two states are very similar.

However, the SMRs are very different, with A having much lower mortality than the standard, and B having somewhat higher mortality than the standard.

The following tables show the age-specific mortality rates (obtained by dividing the deaths by the exposed-to-risk in each row of the two tables).

State A

Age last birthday	Mortality rate
0 – 19	0.0019
20 – 59	0.0147
60 – 100	0.0433

State B

Age last birthday	Mortality rate
0 – 19	0.0050
20 – 39	0.0150
40 – 59	0.0250
60 – 100	0.0600

We can see that State B actually has higher mortality at all age ranges.

A comparison of the CDRs therefore seems misleading. The explanation is that State A has an age structure that is much more heavily weighted towards the older ages than State B. This greatly increases the number of deaths per life year of exposure for A compared to B – *ie* the CDR for A will be elevated compared to B.

The SMR gives a more representative comparison because numerator and denominator are subject to the same age-structure weights, so the distortions cancel out.

Part 1 – Questions

Introduction

The Question and Answer Bank is divided into five parts. The first four parts contain a mixture of development and exam-style questions (see below), primarily related to the material studied in that part, whilst the last part contains a set of exam-style questions covering the whole course. However, for each part the questions may require knowledge from earlier parts of the course.

Each of the first four parts of the Question and Answer Bank is split into two sections:

- Section 1 – Development Questions. The aim of these questions is to build on your understanding, test key Core Reading and bring your knowledge and skills to the level required to tackle Exam-style Questions.

- Section 2 – Exam-style Questions. These questions are of the level of difficulty you are likely to face in the examination. It is very important that you focus on these questions as preparation for the exam.

We recommend that you do not use the questions as a set of material to *learn* but attempt the questions for yourself before looking at the solutions provided.

This last point highlights the difference between active studying and passive studying. Given that the examiners will be aiming to set questions to make you think (and in doing so they will be devising questions you have not seen before) it is much better if you practise the skills that they will be testing.

It may also be useful to you if you group a number of the questions together to attempt under exam time conditions. Ideally three hours would be set aside, but anything from one hour (*ie* 35 marks) upwards will help your time management.

Note that the split between Development Questions and Exam-style Questions is somewhat subjective. For example, there may be some development questions that are as difficult as any exam question, but may be more repetitive than typically found in the exam; equally some of the shorter and more straightforward development questions are similar to the short questions regularly found in the CT5 exam. The Exam-style Questions generally involve more application and greater scope and are typical of the more challenging questions you will meet in the exam.

1 Development Questions

Question 1.1

If T_x and K_x are random variables measuring the complete and curtate future lifetimes, respectively, for a life aged x, write down expressions for the following symbols in terms of expected values.

(i) A_x

(ii) $\bar{A}^1_{x:\overline{n}|}$

(iii) $A_{x:\overline{n}|}^{\ \ 1}$

(iv) $\bar{a}_x$

(v) $\ddot{a}_{x:\overline{n}|}$

(vi) $\bar{a}_{\overline{x:\overline{n}|}}$ [6]

Question 1.2

A whole life annuity is payable continuously to a life now aged 60. The rate of payment at time t is:

$$\rho(t) = 10,000(1.02)^t \quad (t > 0)$$

(i) Write down an expression for the present value of the annuity in terms of annuities-certain. [2]

(ii) Write down expressions for the expected present value and variance of the present value of the annuity. [2]

(iii) Calculate the expected value and the variance of the annuity assuming AM92 Ultimate mortality and 6.08% *pa* interest. [4]

(iv) Simplify your expressions for the present value and its expectation assuming that $i = 0.02$. [2]

(v) Calculate the expected present value of the annuity assuming AM92 Ultimate mortality and 2% *pa* interest. [1]

[Total 11]

Question 1.3

Consider the following assertions relating to a mortality model that assumes uniform decrement rates between consecutive integer ages, where x is an integer and $0 \le t < 1$:

I $\quad {}_tq_x = tq_x$

II $\quad l_{x+t} = l_x - td_x$

III $\quad l_{x+t} = tl_x + (1-t)l_{x+1}$

State whether each of these is true or false. [3]

Question 1.4

State whether each of the following functions could be used as a function to represent the number of lives aged x (over the specified range of ages) in a hypothetical population (not necessarily human)?

I $\quad l_x = (1 + x/2)^{-3} \qquad (x \geq 0)$

II $\quad l_x = -\log(x/100) \qquad (0 \leq x \leq 100)$

III $\quad l_x = x^{-½} \qquad (x \geq 20)$ [3]

Question 1.5

You are given that $p_{80} = 0.988$. Estimate ${}_{0.5}p_{80}$ assuming:

(i) a uniform distribution of deaths between integer ages

(ii) a constant force of mortality between integer ages. [2]

Question 1.6

A student has written down the following equations:

I $\quad {}_{n|}A_x = v^n A_{x+n}$

II $\quad \ddot{a}_x = \ddot{a}_{\overline{n}|} + {}_{n|}\ddot{a}_x$

III $\quad A_{x:\overline{n}|} = A_{x:\overline{t}|} + \dfrac{D_{x+t}}{D_x} A_{x+t:\overline{n-t}|}$

State whether each equation is correct or not, and, where applicable, suggest a correction. [3]

Question 1.7

Let K be a random variable representing the curtate future lifetime of a life aged 40, and let $g(K)$ be the function defined by:

$$g(K) = \begin{cases} 0 & \text{if } 0 \le K < 10 \\ v^{10}\ddot{a}_{\overline{K-9|}} & \text{if } 10 \le K < 35 \\ v^{10}\ddot{a}_{\overline{25|}} & \text{if } 35 \le K \end{cases}$$

Calculate the value of $E[g(K)]$, assuming mortality follows AM92 Ultimate and interest is 4% *pa*. [4]

Question 1.8

A life aged 65 is assumed to be subject to an annual initial rate (probability) of mortality equal to twice that of the AM92 Ultimate tables for the next two years. Calculate the probability that the life will die before age 67. [2]

Question 1.9

Without calculating the functions, judge whether each of the following statements is true or false, based on AM92 mortality and 10% *pa* interest.

(i) $A_{17:\overline{2|}} < A_{27:\overline{2|}}$

(ii) $A_{25:\overline{10|}} > 0.3$

(iii) $\ddot{a}_{20} > 12$

(iv) $A^{1}_{30:\overline{25|}} < 0.3$ [4]

Question 1.10

Evaluate the following functions, assuming the given basis:

(i) $\ddot{a}_{65:\overline{20|}}$ AM92 Ultimate mortality and interest at 4% *pa* [2]

(ii) $A_{68:\overline{2|}}$ AM92 Ultimate mortality and interest at 6% *pa* [2]

(iii) $a_{\overline{62:\overline{5|}}}$ PFA92C20 mortality and interest at 4% *pa* [2]

[Total 6]

Question 1.11

Calculate values for the following functions, assuming AM92 mortality:

(i) $A_{[60]+1}$ at 4% [1]

(ii) $A^{1}_{[40]:\overline{16|}}$ at 4% [1]

(iii) $A^{1}_{[40]:\overline{10|}}$ at 6% [1]

[Total 3]

Question 1.12

(i) Z is a random variable representing the present value of the benefits payable under an immediate life annuity which pays 1 per year, issued to a life aged x.

Show that $\text{var}(Z) = \dfrac{1}{d^2}\left({}^2A_x - (A_x)^2\right)$, where ${}^2A_x$ is an assurance calculated at a rate of interest which you should specify. [5]

(ii) A life office issues such a policy to a life aged exactly 60. The benefit is £100 per annum. Calculate the standard deviation of the annuity.

Basis: Mortality: AM92 Ultimate
Interest: 4% per annum throughout [3]

[Total 8]

Question 1.13

The table below is part of a mortality table used by a life insurance company to calculate survival probabilities for a special type of life insurance policy.

x	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	$l_{[x]+3}$	l_{x+4}
51	1,537	1,517	1,502	1,492	1,483
52	1,532	1,512	1,497	1,487	1,477
53	1,525	1,505	1,490	1,480	1,470
54	1,517	1,499	1,484	1,474	1,462
55	1,512	1,492	1,477	1,467	1,453

(i) Calculate the probability that a policyholder who was accepted for insurance exactly 2 years ago and is now aged exactly 55 will die at age 57 next birthday. [3]

(ii) Calculate the corresponding probability for an individual of the same age who has been a policyholder for many years. [2]

(iii) Comment on your answers to (i) and (ii). [2]

[Total 7]

Question 1.14

Which of the following functions has the highest numerical value when calculated assuming AM92 Select mortality and 4% *pa* interest? (An asterisk denotes a function evaluated at twice the stated force of interest.)

(i) $A_{[50]}$

(ii) A_{50}

(iii) $A_{[31]:\overline{29|}}$

(iv) $A^{*}_{[60]}$ [3]

Question 1.15

Calculate values for the following functions, assuming AM92 mortality:

(i) $\dfrac{D_{50}}{D_{40}} a_{50}$ at 6%

(ii) $A^{1}_{40:\overline{16}|}$ at 4%

(iii) $\ddot{a}_{23:\overline{18}|}$ at 6% [6]

Question 1.16

Calculate the values of:

(a) $\ddot{a}^{(12)}_{50:\overline{15}|}$

(b) $\ddot{a}^{(12)}_{\overline{50:\overline{15}|}}$

using AM92 Ultimate mortality and 4% *pa* interest. [4]

Question 1.17

Sometimes when no standard mortality table has been produced for female lives, actuaries use the corresponding male table, but apply an "age rating" of 4 years (say), *ie* they consider a female aged x to be equivalent to a male aged $x-4$. Explain the rationale underlying this approach. [2]

Question 1.18

Calculate the present value of a payment of £2,000 payable 6 months after the death of a life now aged exactly 60, assuming AM92 Select mortality and 6% *pa* interest. [2]

Question 1.19

(i) Calculate $\bar{A}_{30:\overline{25|}}$ and $\bar{a}_{30:\overline{25|}}$ independently, assuming AM92 Ultimate mortality and 6% *pa* interest. [5]

(ii) State the assumptions underlying the approximations on which your values calculated in part (i) are based. [2]

(iii) Verify that the usual premium conversion relationship holds approximately between these two functions. [2]

[Total 9]

Question 1.20

The mortality of a certain population is governed by the life table function $l_x = 100 - x$, $0 \le x \le 100$. Calculate the values of the following expressions:

(i) ${}_{10}p_{30}$ [1]

(ii) μ_{30} [2]

(iii) $P(T_{30} < 20)$ [2]

(iv) $P(K_{30} = 20)$ [2]

(v) $\mathring{e}_{30}$ [2]

[Total 9]

Question 1.21

A whole of life assurance policy pays £10,000 immediately on death of a policyholder currently aged 50 exact, but only if death occurs after the age of 60.

(i) Write down an expression for the present value random variable of the benefit payable under this policy. [2]

(ii) Derive an expression, in the form of an integral, for the expected present value of the benefit payment, and express your answer using standard actuarial notation. [2]

(iii) Derive an expression for the variance of the present value of the benefit payment, expressing your answer using standard actuarial notation, defining any other symbols you may use. [3]

(iv) Calculate the standard deviation of the present value of the policy benefit, using the following assumptions:

Mortality:	AM92 Ultimate
Interest:	4% per annum

[4]

[Total 11]

2 Exam-style Questions

Question 1.22

A 10-year temporary annuity is payable continuously to a life now aged x. The rate of payment is 100 *pa* for the first 5 years and 150 *pa* for the next 5 years.

(i) Write down an expression in terms of T_x for the present value of this annuity. [3]

(ii) Calculate the expected present value of the annuity assuming a constant force of interest of 0.04 *pa* and a constant force of mortality of 0.01 *pa*. [8]
[Total 11]

Question 1.23

A special 25-year life insurance policy is issued to a life aged x and provides the following benefits:

- a lump sum of £75,000 (payable at the end of the policy year) if death occurs during the first 10 years
- a dependants' pension (payable in the form of an annuity certain) of £5,000 *pa* payable on each remaining policy anniversary during the term (including the 25th anniversary) if death occurs after 10 years but before the end of the term of the policy
- a pension of £7,500 *pa* commencing on the day after the term of the policy expires and with payments on each subsequent policy anniversary while the policyholder is still alive.

Write down an expression for the present value random variable of the benefits under this policy. [3]

Question 1.24

A select table is to be constructed with select period two years added to the ELT15 table (Males), which is to be treated as the ultimate table. Select rates are to be derived by applying the same ratios select:ultimate seen in the AM92 table, thus

$$q_{[x]} = \frac{q'_{[x]}}{q'_x} q_x \text{ and } q_{[x]+1} = \frac{q'_{[x]+1}}{q'_{x+1}} q_{x+1}$$

where the dash notation q'_x refers to AM92 mortality.

Find the value of $l_{[60]}$. [3]

Question 1.25

If $l_{40} = 1{,}000$ and $l_{40+t} = l_{40} - 5t$ for $t = 1, 2, \ldots, 10$, calculate the value of $A_{40:\overline{10}|}$ at 6% *pa* interest. [3]

Question 1.26

In a mortality table with a one-year select period $q_{[x]} = a.q_x$ for all $x \geq 0$ and for a certain constant $0 < a < 1$.

(i) Let:

$K_{[x]+t}$ = curtate future lifetime of a person aged $x+t$ who became select t years ago

K_x = curtate future lifetime of a person aged x whose mortality is reflected by the ultimate part of the mortality table.

Explain whether the expected value of $K_{[x]+1}$ is less than, greater than, or equal to the expected value of K_{x+1}. [2]

(ii) Calculate $\bar{A}_{[45]:\overline{20}|}$ according to the following assumptions:

$a = 0.9$

Interest: 4% *pa*

Ultimate mortality: ELT15 (Males)

Note that financial functions using ELT15 (Males) mortality are to be found on page 136 of the Tables. [8]

[Total 10]

Question 1.27

You are given the following select and ultimate mortality table:

x	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	$l_{[x]+3}$	l_{x+4}
51	1,537	1,517	1,502	1,492	1,483
52	1,532	1,512	1,497	1,487	1,477
53	1,525	1,505	1,490	1,480	1,470
54	1,517	1,499	1,484	1,474	1,462
55	1,512	1,492	1,477	1,467	1,453

(i) Calculate the probability that a life selected at age 51 and still alive at age 53 will survive to age 55. [1]

(ii) Given that $\ddot{a}_{55:\overline{10}|} = 8.078$ when $i = 0.04$, calculate the corresponding select mortality annuity, $\ddot{a}_{[55]:\overline{10}|}$. Comment on your answer. [5]

[Total 6]

Question 1.28

(i) An annuity is payable continuously throughout the lifetime of a person now aged exactly 60, but for at most 10 years. The rate of payment at all times t during the first 5 years is £10,000 *pa*, and thereafter it is £12,000 *pa*. The force of mortality of this life is 0.03 *pa* between the ages of 60 and 65, and 0.04 *pa* between the ages of 65 and 70. Calculate the expected present value of this annuity assuming a force of interest of 0.05 *pa*. [4]

(ii) Assuming that mortality and interest are as in (i) above, calculate the expected present value of:

(a) a 10-year term assurance issued to the life in (i), which pays £50,000 immediately on death

(b) a 10-year endowment assurance issued to the life in (i), which pays £50,000 on maturity or immediately on earlier death. [4]

[Total 8]

Question 1.29

The following is an expression for the present value of a special assurance policy:

$$W = v^{\max\{K_{x+1},\, n\}}$$

where K_x is the curtate future lifetime of a person currently aged exactly x.

(i) Describe the benefit paid under this policy. [1]

(ii) Obtain an expression for $E[W]$ in terms of standard actuarial notation. [2]

(iii) Obtain an expression for the variance of W, in terms of standard actuarial notation, defining any other symbols that you use. [2]

An annuity pays 1 *pa* in advance for *n* years, or until the death of a life currently aged *x* if this occurs after *n* years.

(iv) Using your answer to (iii), obtain an expression for the variance of the present value of these annuity payments. [3]

[Total 8]

Part 1 – Solutions

Solution 1.1

(i) $A_x = E\left(v^{K_x+1}\right)$ [1]

(ii) $\bar{A}^{1}_{x:\overline{n}|} = E\left[g\left(T_x\right)\right]$ where $g\left(T_x\right) = \begin{cases} v^{T_x} & \text{if } T_x < n \\ 0 & \text{if } T_x \geq n \end{cases}$ [1]

(iii) $A_{x:\overset{\,1}{\overline{n}|}} = E\left[h\left(K_x\right)\right]$ where $h\left(K_x\right) = \begin{cases} 0 & \text{if } K_x < n \\ v^n & \text{if } K_x \geq n \end{cases}$ [1]

Alternatively, you could have replaced K_x with T_x here.

(iv) $\bar{a}_x = E\left[\bar{a}_{\overline{T_x}|}\right]$ [1]

(v) $\ddot{a}_{x:\overline{n}|} = E\left[l\left(K_x\right)\right]$ where $l\left(K_x\right) = \begin{cases} \ddot{a}_{\overline{K_x+1}|} & \text{if } K_x < n \\ \ddot{a}_{\overline{n}|} & \text{if } K_x \geq n \end{cases}$ [1]

Alternatively, you could write $\ddot{a}_{x:\overline{n}|} = E\left[\ddot{a}_{\overline{\min\{K_x+1, n\}}|}\right]$.

(vi) $\bar{a}_{\overline{x:\overline{n}|}} = E\left[l\left(T_x\right)\right]$ where $l\left(T_x\right) = \begin{cases} \bar{a}_{\overline{n}|} & \text{if } T_x < n \\ \bar{a}_{\overline{T_x}|} & \text{if } T_x \geq n \end{cases}$ [1]

Alternatively, you could write $\bar{a}_{\overline{x:\overline{n}|}} = E\left[\bar{a}_{\overline{\max\{T_x,\, n\}}|}\right]$.

[Total 6]

Solution 1.2

(i) The present value of the annuity is:

$$\int_0^{T_{60}} 10{,}000(1.02)^t\, v^t\, dt \qquad [1]$$

If we let $v' = 1.02v$, then this can also be written as:

$$\int_0^{T_{60}} 10{,}000(v')^t\, dt = 10{,}000\,\bar{a}_{\overline{T_{60}}|} \,@\, i' \qquad [½]$$

where $i' = \dfrac{1+i}{1.02} - 1 = \dfrac{i - 0.02}{1.02}$. [½]

[Total 2]

(ii) The expected present value is $10{,}000\,\bar{a}_{60}$. [1]

The variance of the present value is $10{,}000^2 \dfrac{\left({}^2\bar{A}_{60} - \left(\bar{A}_{60}\right)^2\right)}{\delta^2}$. [1]

Both of these should be evaluated using the interest rate i'. [Total 2]

(iii) If $i = 0.0608$, then $i' = \dfrac{0.0608 - 0.02}{1.02} = 0.04$. [½]

So:

$$\begin{aligned} EPV &= 10{,}000\,\bar{a}_{60} \,@\, 4\% \\ &\approx 10{,}000\left(\ddot{a}_{60} - ½\right) \\ &= 10{,}000\left(14.134 - ½\right) \\ &= 136{,}340 \end{aligned} \qquad [1]$$

and:

$$\text{var}(PV) = 10{,}000^2 \frac{\left({}^2\bar{A}_{60} - \left(\bar{A}_{60}\right)^2\right)}{\delta^2}$$

We can estimate $\bar{A}_{60}$ as follows:

$$\bar{A}_{60} \approx (1.04)^{½} A_{60} = (1.04)^{½} \times 0.45640 = 0.46544 \qquad [½]$$

Similarly:

$$^2\bar{A}_{60} \approx 1.04 \times {}^2A_{60} = 1.04 \times 0.23723 = 0.24672 \qquad [1]$$

So:

$$\text{var}(PV) = 10,000^2 \frac{\left(0.24672 - (0.46544)^2\right)}{(\ln 1.04)^2}$$

$$= 1,955,815,110$$

$$= (44,225)^2 \qquad [1]$$

[Total 4]

(iv) When $i = 0.02$ the present value of the annuity is:

$$10,000\,\bar{a}_{\overline{T_{60}}|} \text{ @ } 0\% = 10,000\,T_{60} \qquad [1]$$

and the expected present value is:

$$10,000E(T_{60}) = 10,000\,\mathring{e}_{60} \qquad [1]$$

[Total 2]

(v) Evaluating using AM92 ultimate mortality gives:

$$10,000E(T_{60}) \approx 10,000(e_{60} + ½)$$

$$= 10,000(20.670 + ½)$$

$$= 211,700 \qquad \text{[Total 1]}$$

Solution 1.3

I is true. This is one of the definitions of uniform decrement rates. [1]

II is true. This is a standard result that follows from a uniform decrement assumption. [1]

III is false. The multipliers t and $t-1$ are transposed. For example, putting $t=0$ gives $l_x = l_{x+1}$, instead of $l_x = l_x$. [1]

[Total 3]

Solution 1.4

A survival function l_x must satisfy the following criteria:

1. $0 \le l_x < \infty$ over the relevant range of ages

2. l_x is a non-increasing function of x

In practice, survival functions describing human mortality would also have to satisfy other conditions such as $\lim_{x\to\infty} l_x \to 0$*, and* $e_x < \infty$*, but these are not logically required.*

I and III are decreasing functions, taking positive finite values over the specified ranges.

II is not defined when $x = 0$. So it cannot be used over a range that includes age 0. [1]

Hence I and III are correct. [2]

[Total 3]

Solution 1.5

(i) ${}_{0.5}p_{80} = 1 - {}_{0.5}q_{80} \overset{UDD}{=} 1 - 0.5q_{80} = 0.99400$ [1]

(ii) ${}_{0.5}p_{80} = \left(p_{80}\right)^{0.5} = 0.99398$ [1]

[Total 2]

Solution 1.6

I is not correct. The RHS should also include an ${}_np_x$ factor, since the life will only receive cover from the deferred assurance on the LHS if he or she is still alive at age $x+n$. [1]

II is not correct. The $\ddot{a}_{\overline{n}|}$ term on the RHS will make payments even if the life has died – *ie* this would be correct for a whole life annuity that was guaranteed for a minimum of n years. So instead we could either write:

$$\ddot{a}_{\overline{\overline{x:n}|}} = \ddot{a}_{\overline{n}|} + {}_{n|}\ddot{a}_x$$

or:

$$\ddot{a}_x = \ddot{a}_{x:\overline{n}|} + {}_{n|}\ddot{a}_x\,. \qquad [1]$$

III is not correct. The $A_{x:\overline{t}|}$ on the RHS is an endowment assurance factor, which provides a payment on survival to age t, as well as life cover up to that point. This factor should be replaced with $A^1_{x:\overline{t}|}$. [1]

[Total 3]

Solution 1.7

The payments correspond to a benefit that is deferred for 10 years, then makes payments annually in advance during the remaining lifetime, up to a maximum of 25 payments. [1]

In terms of actuarial symbols, the expected present value is:

$$E[g(K)] = \frac{D_{50}}{D_{40}}\ddot{a}_{50:\overline{25}|}$$

$$= \frac{D_{50}}{D_{40}}\left(\ddot{a}_{50} - \frac{D_{75}}{D_{50}}\ddot{a}_{75}\right)$$

$$= \frac{1{,}366.61 \times 17.444 - 363.11 \times 8.524}{2{,}052.96}$$

$$= 10.104 \qquad [3]$$

[Total 4]

Solution 1.8

The adjusted annual rates of mortality will be:

$$q'_{65} = 2q_{65} = 2 \times 0.014243 = 0.028486 \quad [½]$$

and: $$q'_{66} = 2q_{66} = 2 \times 0.015940 = 0.031880 \quad [½]$$

The probability of surviving for 2 years can then be calculated as:

$$_2p'_{65} = p'_{65} \times p'_{66} = (1 - 0.028486)(1 - 0.031880) = 0.94054 \quad [½]$$

So the probability of dying within 2 years is:

$$_2q'_{65} = 1 - 0.94054 = 0.05946 \quad [½]$$

[Total 2]

Solution 1.9

(i) is false. Mortality is higher at ages 17 and 18 on the AM92 tables than at ages 27 and 28 because of the accident hump. So the benefit on the LHS will, on average, be paid more often on death. [1]

(ii) is true. Since mortality is very light over the age range 25 to 35, $A_{25:\overline{10}|}$ will be very slightly more than $v^{10} = 0.386$. [1]

(iii) is false. The value of a whole life annuity cannot be greater than the corresponding perpetuity, which would be $\ddot{a}_{\overline{\infty}|} = 1/d = 1.1/0.1 = 11$. [1]

(iv) is true. The probability that a 30-year old will die before age 55 is much less than 30%. So the value of this function (even before discounting) could not be greater than 0.3. [1]

[Total 4]

Solution 1.10

(i) We can write:

$$\ddot{a}_{65:\overline{20}|} = \ddot{a}_{65} - \frac{D_{85}}{D_{65}}\ddot{a}_{85} = 12.276 - \frac{120.71}{689.23} \times 5.333 = 11.342 \qquad [2]$$

(ii) If death occurs during the first year, the payment will be made at the end of the first year. Otherwise, it will be made at the end of the second year. So the value is:

$$A_{68:\overline{2}|} = vq_{68} + v^2 p_{68}$$

$$= 1.06^{-1} \times 0.019913 + 1.06^{-2} \times 0.980087 = 0.89106 \qquad [2]$$

(iii) This is the EPV of a guaranteed annuity, payable annually in arrears for a minimum of 5 years and for the remaining lifetime of a person currently aged 62 exact. We can write:

$$a_{\overline{62:\overline{5}|}} = a_{\overline{5}|} + {}_{5|}a_{62}$$

$$= \frac{1-v^5}{i} + v^5 \frac{l_{67}}{l_{62}} a_{67}$$

$$= \frac{1-1.04^{-5}}{0.04} + 1.04^{-5} \frac{9,605.483}{9,804.173} \times (14.111-1)$$

$$= 15.010$$

[2]
[Total 6]

Solution 1.11

(i) $$A_{[60]+1} = v\,q_{[60]+1} + v\,p_{[60]+1}\,A_{62}$$

$$= 1.04^{-1}\left(0.008680 + 0.99132 \times 0.48458\right)$$

$$= 0.4702$$

[1]

(ii) $$A^{\ \ 1}_{[40]:\overline{16}|} = A_{[40]} - \frac{D_{56}}{D_{[40]}} A_{56} = 0.23041 - \frac{1,058.15}{2,052.54} \times 0.40240 = 0.0230 \qquad [1]$$

(iii) $A^{\;\;1}_{[40]:\overline{10|}} = A_{[40]} - v^{10}\dfrac{l_{50}}{l_{[40]}}A_{50}$

$$= 0.12296 - \frac{1}{1.06^{10}} \times \frac{9{,}712.0728}{9{,}854.3036} \times 0.20508$$

$$= 0.0101$$

[1]
[Total 3]

Solution 1.12

This question is Subject 104 September 2004 Question 4.

(i) ***Proof***

The random variable Z is defined by the equation:

$$Z = a_{\overline{K_x|}}$$ [½]

So:

$$\text{var}(Z) = \text{var}\left(a_{\overline{K_x|}}\right) = \text{var}\left(\frac{1-v^{K_x}}{i}\right) = \frac{1}{i^2}\text{var}\left(v^{K_x}\right)$$ [1]

v^{K_x} is the present value of a benefit of 1 at the beginning of the year of death of (x). To make the benefit payable at the end of the year of death, we have to increase the power of v by 1. So:

$$\text{var}(Z) = \frac{1}{i^2}\text{var}\left(\frac{v^{K_x+1}}{v}\right)$$

$$= \frac{1}{i^2 v^2}\text{var}\left(v^{K_x+1}\right)$$

$$= \frac{1}{d^2}\left[E\left(v^{2(K_x+1)}\right) - \left(E\left(v^{K_x+1}\right)\right)^2\right]$$ [1½]

By definition:

$$A_x = E\left(v^{K_x+1}\right)$$ [½]

and:

$$^2A_x = E\left(v^{2(K_x+1)}\right) \qquad [½]$$

So:

$$\text{var}(Z) = \frac{1}{d^2}\left[{}^2A_x - (A_x)^2\right] \qquad [½]$$

and ^2A_x is evaluated at the rate of interest $i' = (1+i)^2 - 1$. [½]

[Total 5]

(ii) ***Standard deviation of the annuity***

The variance of the present value of this annuity is:

$$\frac{100^2}{d^2}\left[{}^2A_{60} - (A_{60})^2\right] = \frac{100^2 \times 1.04^2}{0.04^2}\left[0.23723 - 0.45640^2\right]$$

$$= (442.22)^2 \qquad [2½]$$

So the standard deviation of the present value is £442.22. [½]

[Total 3]

Solution 1.13

(i) *Note that the table in the question is not laid out in the same way as AM92 in the tables.*

The policyholder is currently age $[53]+2$. [1]

So the probability of dying between ages 56 and 57 is:

$$\frac{l_{[53]+3} - l_{57}}{l_{[53]+2}} = \frac{1,480 - 1,470}{1,490} = 0.00671 \qquad [2]$$

[Total 3]

(ii) The corresponding probability for an ultimate policyholder is:

$$\frac{l_{56}-l_{57}}{l_{55}}=\frac{1,477-1,470}{1,483}=0.00472$$ [Total 2]

(iii) For the usual types of policies (life assurance and annuities), policyholders experience lighter mortality during the select period. [1]

But here, the mortality rate is higher in (i) than in (ii). [1]
[Total 2]

This is called reverse selection. It could occur, for example, if the policy was an annuity sold to individuals who had recently had a particular form of medical treatment that increased their mortality rates during the first few years, or if they had recently arrived/returned from a third world country.

Solution 1.14

The values of the functions are:

(i) $A_{[50]}=0.32868$ [½]

(ii) $A_{50}=0.32907$ [½]

(iii) $A_{[31]:\overline{29}|}=0.32926$ [½]

(iv) $A_{[60]}^{@8.16\%}=0.23547$ [½]

So function (iii) has the largest numerical value. [1]
[Total 3]

Solution 1.15

(i) $$\frac{D_{50}}{D_{40}}a_{50}=v^{10}\frac{l_{50}}{l_{40}}(\ddot{a}_{50}-1)=\frac{1}{1.06^{10}}\times\frac{9,712.0728}{9,856.2863}\times(14.044-1)=7.1771$$ [2]

(ii) $$A_{40:\overline{16}|}^{1}=A_{40}-\frac{D_{56}}{D_{40}}A_{56}=0.23056-\frac{1,058.15}{2,052.96}\times0.40240=0.02315$$ [2]

(iii) $\ddot{a}_{23:\overline{18|}} = \ddot{a}_{23} - v^{18}\frac{l_{41}}{l_{23}}\ddot{a}_{41} = 16.753 - \frac{1}{1.06^{18}} \times \frac{9{,}847.0510}{9{,}964.9313} \times 15.375 = 11.430$ [2]

[Total 6]

Solution 1.16

(a) This function equals:

$$\ddot{a}^{(12)}_{50:\overline{15|}} = \ddot{a}_{50:\overline{15|}} - \frac{11}{24}\left(1 - \frac{D_{65}}{D_{50}}\right) = 11.253 - \frac{11}{24}(1 - 0.5043) = 11.0258$$ [2]

(b) This function equals:

$$\ddot{a}^{(12)}_{50:\overline{15|}} = \ddot{a}^{(12)}_{\overline{15|}} + \frac{D_{65}}{D_{50}}\ddot{a}^{(12)}_{65}$$

$$= \frac{1 - v^{15}}{d^{(12)}} + \frac{D_{65}}{D_{50}}\left(\ddot{a}_{65} - \frac{11}{24}\right)$$

$$= \frac{1 - 1.04^{-15}}{0.039157} + \frac{689.23}{1{,}366.61} \times \left(12.276 - \frac{11}{24}\right)$$

$$= 17.318$$ [2]

[Total 4]

Solution 1.17

The rationale is that females experience lower mortality than males. If the pattern of mortality is similar for the two sexes, but the average age at death of females exceeds that of males by 4 years, then we can approximate the mortality of a female by that of a male 4 years younger. [2]

Solution 1.18

$$2{,}000\ \bar{A}_{[60]}\, v^{1/2} \approx 2{,}000\ A_{[60]} = 2{,}000 \times 0.32533 = 650.66$$ [2]

Solution 1.19

(i) ***Calculations***

The endowment assurance function can be calculated from:

$$\bar{A}_{30:\overline{25|}} = \bar{A}^{1}_{30:\overline{25|}} + \bar{A}_{30:\overline{25|}}^{\;\;\;\;\;\;\;\;1} \approx 1.06^{½} A^{1}_{30:\overline{25|}} + \frac{D_{55}}{D_{30}}$$

Remember that only the death benefit is accelerated.

The functions on the RHS can be calculated as follows:

$$\frac{D_{55}}{D_{30}} = v^{25} \frac{l_{55}}{l_{30}} = 0.22437$$

$$A^{1}_{30:\overline{25|}} = A_{30} - \frac{D_{55}}{D_{30}} A_{55} = 0.07328 - 0.22437 \times 0.26092 = 0.01474$$

So:

$$\bar{A}_{30:\overline{25|}} \approx 1.06^{½} \times 0.01474 + 0.22437 = 0.2395 \qquad [3]$$

Applying claims acceleration factors of $1+i/2$ *or* i/δ *is also acceptable.*

The temporary annuity function can be calculated from:

$$\bar{a}_{30:\overline{25|}} = \bar{a}_{30} - \frac{D_{55}}{D_{30}} \bar{a}_{55} \approx (16.372 - ½) - 0.22437 \times (13.057 - ½) = 13.055 \qquad [2]$$

[Total 5]

(ii) ***Assumptions***

The assurance factor has been approximated by assuming that deaths between the ages of 30 and 55 occur in the middle of the year of age on average. [1]

The $1+i/2$ *adjustment is based on mid-year deaths with simple interest. The* i/δ *adjustment assumes uniform deaths over each life year.*

The annuity function has been approximated by assuming the value lies midway between the two annual functions $a_{30:\overline{25|}}$ and $\ddot{a}_{30:\overline{25|}}$. [1]

[Total 2]

(iii) ***Verification of premium conversion formula***

We would expect the following premium conversion relationship to hold:

$$\bar{A}_{30:\overline{25|}} = 1 - \delta \bar{a}_{30:\overline{25|}} \qquad [1]$$

We have calculated the LHS to be 0.2395. The RHS is:

$$1 - \ln 1.06 \times 13.055 = 0.2393$$

The answers agree to 3 significant figures (an error of only 0.08%). The slight discrepancy is due to the different assumptions underlying the two approximations used.

[1]

[Total 2]

Solution 1.20

(i) $${}_{10}p_{30} = \frac{l_{40}}{l_{30}} = \frac{60}{70} = \frac{6}{7} \qquad [1]$$

(ii) We can write:

$$\begin{aligned}\mu_x &= -\frac{\partial}{\partial x}\ln l_x \\ &= -\frac{\partial}{\partial x}\ln(100-x) \\ &= \frac{1}{100-x}\end{aligned}$$

So:

$$\mu_{30} = \frac{1}{70} \qquad [2]$$

(iii) $$P(T_{30} < 20) = {}_{20}q_{30} = 1 - \frac{l_{50}}{l_{30}} = 1 - \frac{50}{70} = \frac{2}{7} \qquad [2]$$

(iv) $$P(K_{30} = 20) = {}_{20}p_{30} \times q_{50} = \frac{l_{50} - l_{51}}{l_{30}} = \frac{50-49}{70} = \frac{1}{70} \qquad [2]$$

(v) $\overset{\circ}{e}_{30} = \int_0^{70} {}_t p_{30}\, dt = \int_0^{70} \frac{l_{30+t}}{l_{30}}\, dt = \frac{1}{70}\int_0^{70} (70-t)\, dt = \left[\frac{70t - \frac{1}{2}t^2}{70}\right]_0^{70} = 35$ [2]

Alternatively, you could argue that for $0 \le k < 70$*:*

$$P(K_{30} = k) = {}_k p_{30} - {}_{k+1} p_{30} = \frac{l_{30+k} - l_{30+k+1}}{l_{30}}$$

$$= \frac{(70-k)-(70-k-1)}{70} = \frac{1}{70}$$

So if (30) is equally likely to die in each future year up to age 100, then his expected future lifetime is $\frac{100-30}{2} = 35$ *years.*

Solution 1.21

(i) ***Present value random variable***

We can write this as:

$$X = \begin{cases} 0 & \text{if } T_{50} \le 10 \\ 10{,}000 v^{T_{50}} & \text{if } T_{50} > 10 \end{cases}$$ [2]

(ii) ***Expected present value***

Now:

$$E[X] = \int_0^{10} 0 \times f_{T_{50}}(t)\, dt + \int_{10}^{\infty} 10{,}000\, v^t f_{T_{50}}(t)\, dt$$

$$= 0 + 10{,}000 \int_{10}^{\infty} v^t\, {}_t p_{50}\, \mu_{50+t}\, dt$$

$$= 10{,}000 \times {}_{10|}\bar{A}_{50}$$ [2]

(iii) ***Variance***

We need:

$$\text{var}\left[X\right]=E\left[X^2\right]-\left(E\left[X\right]\right)^2$$

$$=\int_{10}^{\infty}10{,}000^2\,(v^t)^2\,{}_tp_{50}\,\mu_{50+t}\,dt-\left(10{,}000\;{}_{10|}\bar{A}_{50}\right)^2$$ [1½]

Now:

$$\int_{10}^{\infty}10{,}000^2\,(v^t)^2\,{}_tp_{50}\,\mu_{50+t}\,dt=10{,}000^2\int_{10}^{\infty}(v^2)^t\,{}_tp_{50}\,\mu_{50+t}\,dt$$

$$=10{,}000^2\;{}_{10|}{}^2\bar{A}_{50}$$ [½]

where ${}^2\bar{A}$ is calculated at rate of interest $(1+i)^2-1$. [½]

So:

$$\text{var}\left[X\right]=10{,}000^2\left[{}_{10|}{}^2\bar{A}_{50}-\left({}_{10|}\bar{A}_{50}\right)^2\right]$$ [½]

[Total 3]

(iv) ***Standard deviation***

We will use the formula for the variance from (iii). We need:

$${}_{10|}\bar{A}_{50}=\frac{D_{60}}{D_{50}}\bar{A}_{60}\simeq\frac{882.85}{1{,}366.61}\times1.04^{\frac{1}{2}}\times0.45640=0.30068$$ [1]

$${}_{10|}{}^2\bar{A}_{50}=(v^2)^{10}\times\frac{l_{60}}{l_{50}}\times{}^2\bar{A}_{60}$$

$$=1.04^{-20}\times\frac{l_{60}}{l_{50}}\times\left[1.04^2\right]^{\frac{1}{2}}\times{}^2A_{60}$$

$$=1.04^{-20}\times\frac{9{,}287.2164}{9{,}712.0728}\times1.04\times0.23723$$

$$=0.10767$$ [2]

The standard deviation is the square root of the variance:

$$SD[X] = 10,000\left[{}_{10|}{}^2\bar{A}_{50} - \left({}_{10|}\bar{A}_{50}\right)^2\right]^{1/2}$$

$$= 10,000 \times \left(0.10767 - 0.30068^2\right)^{1/2}$$

$$= 1,313.97$$ [1]

[Total 4]

Solution 1.22

(i) The present value of the annuity is $g(T_x)$, where:

$$g(T_x) = \begin{cases} 100\,\bar{a}_{\overline{T_x}|} & \text{if } T_x < 5 \\ 150\,\bar{a}_{\overline{T_x}|} - 50\,\bar{a}_{\overline{5}|} & \text{if } 5 \le T_x < 10 \\ 150\,\bar{a}_{\overline{10}|} - 50\,\bar{a}_{\overline{5}|} & \text{if } T_x \ge 10 \end{cases}$$

[Total 3]

This could also be written as:

$$g(T_x) = \begin{cases} 100\,\bar{a}_{\overline{T_x}|} & \text{if } T_x < 5 \\ 100\,\bar{a}_{\overline{5}|} + 150v^5\,\bar{a}_{\overline{T_x-5}|} & \text{if } 5 \le T_x < 10 \\ 100\,\bar{a}_{\overline{5}|} + 150v^5\,\bar{a}_{\overline{5}|} & \text{if } T_x \ge 10 \end{cases}$$

or as:

$$g(T_x) = \int_0^{\min\{T_x,10\}} \rho(s)\,v^s\,ds \quad \text{where} \quad \rho(s) = \begin{cases} 100 & \text{for } 0 \le s < 5 \\ 150 & \text{for } 5 \le s < 10 \end{cases}$$

(ii) The expected present value of the annuity is:

$$\int_0^5 100\,\bar{a}_{\overline{t}|}\ {}_tp_x\,\mu_{x+t}\,dt + \int_5^{10}\left(150\,\bar{a}_{\overline{t}|} - 50\bar{a}_{\overline{5}|}\right){}_tp_x\,\mu_{x+t}\,dt$$

$$+\int_{10}^{\infty}\left(150\,\bar{a}_{\overline{10}|} - 50\bar{a}_{\overline{5}|}\right){}_tp_x\,\mu_{x+t}\,dt \qquad [1½]$$

The first integral is:

$$\int_0^5 100\,\bar{a}_{\overline{t}|}\ {}_tp_x\,\mu_{x+t}\,dt = 100\int_0^5\left(\frac{1-e^{-0.04t}}{0.04}\right)0.01e^{-0.01t}\,dt$$

$$=\frac{100\times 0.01}{0.04}\int_0^5\left(e^{-0.01t}-e^{-0.05t}\right)dt$$

$$=25\left[\frac{1-e^{-0.05}}{0.01}-\frac{1-e^{-0.25}}{0.05}\right]$$

$$=11.32683 \qquad [2]$$

The second integral is:

$$\int_5^{10}\left(150\,\bar{a}_{\overline{t}|} - 50\bar{a}_{\overline{5}|}\right){}_tp_x\,\mu_{x+t}\,dt$$

$$=150\int_5^{10}\left(\frac{1-e^{-0.04t}}{0.04}\right)0.01e^{-0.01t}\,dt - 50\bar{a}_{\overline{5}|}\left({}_5p_x - {}_{10}p_x\right)$$

$$=\frac{150\times 0.01}{0.04}\int_5^{10}\left(e^{-0.01t}-e^{-0.05t}\right)dt$$

$$-50\left(\frac{1-e^{-0.04\times 5}}{0.04}\right)\left(e^{-0.05}-e^{-0.1}\right)$$

$$=37.5\left[\frac{e^{-0.05}-e^{-0.1}}{0.01}-\frac{e^{-0.25}-e^{-0.5}}{0.05}\right]-10.51181$$

$$=34.25563 \qquad [2]$$

The third integral is:

$$\int_{10}^{\infty}\left(150\,\bar{a}_{\overline{10|}} - 50\bar{a}_{\overline{5|}}\right)\,{}_tp_x\,\mu_{x+t}\,dt$$

$$=\left(150\,\bar{a}_{\overline{10|}} - 50\bar{a}_{\overline{5|}}\right)\,{}_{10}p_x$$

$$=\left[150\left(\frac{1-e^{-0.4}}{0.04}\right)-50\left(\frac{1-e^{-0.2}}{0.04}\right)\right]\times e^{-0.1}$$

$$=913.62635$$ [2]

So the expected present value of the annuity is:

$$11.32683+34.25563+913.62635=959.2088$$ [½]
[Total 8]

Solution 1.23

The present value random variable is:

$$\begin{cases} 75,000v^{K_x+1} & \text{if } 0\le K_x\le 9 \\ 5,000v^{K_x+1}\,\ddot{a}_{\overline{25-K_x|}} & \text{if } 10\le K_x\le 24 \\ 7,500v^{25}\,\ddot{a}_{\overline{K_x-24|}} & \text{if } K_x\ge 25 \end{cases}$$

A possible alternative solution is:

$$\begin{cases} 75,000v^{K_x+1} & \text{if } 0\le K_x\le 9 \\ 5,000\left(a_{\overline{25|}}-a_{\overline{K_x|}}\right) & \text{if } 10\le K_x\le 24 \\ 7,500\left(a_{\overline{K_x|}}-a_{\overline{24|}}\right) & \text{if } K_x\ge 25 \end{cases}$$ [3]

Solution 1.24

We need to derive $l_{[60]}$ from:

$$l_{[60]}\left(1-q_{[60]}\right)=l_{[60]+1} \qquad [½]$$

and:

$$l_{[60]+1}\left(1-q_{[60]+1}\right)=l_{62} \qquad [½]$$

Now:

$$q_{[60]+1}=\frac{q'_{[60]+1}}{q'_{61}}\times q_{61}=\frac{0.00868}{0.009009}\times 0.0156=0.01503 \qquad [½]$$

and:

$$q_{[60]}=\frac{q'_{[60]}}{q'_{60}}\times q_{60}=\frac{0.005774}{0.008022}\times 0.01392=0.01002 \qquad [½]$$

So we have:

$$l_{[60]+1}=\frac{l_{62}}{1-q_{[60]+1}}=\frac{84173}{1-0.01503}=85,457.4 \qquad [½]$$

and then:

$$l_{[60]}=\frac{l_{[60]+1}}{1-q_{[60]}}=\frac{85,457.4}{1-0.01002}=86,322.4 \qquad [½]$$

[Total 3]

Solution 1.25

The definition of l_x tells us that for every 1,000 lives age 40, we would expect 5 to die at each age 40, 41, *etc*. [1]

So the value of the endowment assurance is:

$$A_{40:\overline{10}|} = \frac{5v + 5v^2 + 5v^2 + \ldots + 5v^{10} + 950v^{10}}{1,000}$$

$$= \frac{5a_{\overline{10}|} + 950v^{10}}{1,000} @ 6\%$$

$$= \frac{5 \times 7.3601 + 950 \times 0.55839}{1,000} = 0.5673 \qquad [2]$$

[Total 3]

Solution 1.26

(i) ***Longevity comparison***

The mortality of the life aged $[x]+1$ is equal to that of a life aged $x+1$ because we are told that the table has a one-year select period. Thus:

$$E\left[K_{[x]+1}\right] = E\left[K_{x+1}\right] \qquad \text{[Total 2]}$$

(ii) ***Calculation***

Since the select period is one year, we can write:

$$\bar{A}_{[45]:\overline{20}|} = \bar{A}^{1}_{[45]:\overline{1}|} + {}_{1|}\bar{A}_{[45]:\overline{19}|} = \bar{A}^{1}_{[45]:\overline{1}|} + \frac{D_{46}}{D_{[45]}} \bar{A}_{46:\overline{19}|} \qquad [2]$$

where:

$$\bar{A}^{1}_{[45]:\overline{1}|} = v^{½} q_{[45]}$$

$$= v^{½} \times 0.9 \times q_{45}$$

$$= 0.980581 \times 0.9 \times 0.00266$$

$$= 0.0023475 \qquad [2]$$

$$\frac{D_{46}}{D_{[45]}} = \frac{l_{46}}{l_{[45]}} v$$

$$= p_{[45]} v$$

$$= \left(1 - q_{[45]}\right) v$$

$$= \left(1 - 0.9 \times 0.00266\right) \times 0.961538$$

$$= 0.959236 \qquad [2]$$

$$\bar{A}_{46:\overline{19}|} = 1 - \delta \bar{a}_{46:\overline{19}|} = 1 - 0.0392207 \times 12.74 = 0.500328 \text{ (\textit{Tables}, Page 136)} \qquad [1]$$

giving:

$$\bar{A}_{[45]:\overline{20}|} = 0.0023475 + 0.959236 \times 0.500328 = 0.48228 \qquad [1]$$

[Total 8]

Solution 1.27

(i) ***Probability***

The probability that a life selected at age 51 and still alive at age 53 survives to age 55 is:

$$\frac{l_{55}}{l_{[51]+2}} = \frac{1,483}{1,502} = 0.98735 \qquad [1]$$

(ii) ***Annuity***

First consider the ultimate annuity. This can be written as:

$$\ddot{a}_{55:\overline{10}|} = 1 + \frac{l_{56}}{l_{55}} v + \frac{l_{57}}{l_{55}} v^2 + \frac{l_{58}}{l_{55}} v^3 + \cdots + \frac{l_{64}}{l_{55}} v^9 \qquad [1]$$

Multiplying both sides by l_{55}, we get:

$$l_{55} \times \ddot{a}_{55:\overline{10}|} = l_{55} + l_{56} v + l_{57} v^2 + l_{58} v^3 + \cdots + l_{64} v^9 \qquad \ldots(1)$$

The select annuity can be written:

$$\ddot{a}_{[55]:\overline{10}|} = 1 + \frac{l_{[55]+1}}{l_{[55]}}v + \frac{l_{[55]+2}}{l_{[55]}}v^2 + \frac{l_{[55]+3}}{l_{[55]}}v^3 + \cdots + \frac{l_{64}}{l_{[55]}}v^9 \qquad [1]$$

Similarly, multiplying both sides by $l_{[55]}$, we get:

$$l_{[55]} \times \ddot{a}_{[55]:\overline{10}|} = l_{[55]} + l_{[55]+1}v + l_{[55]+2}v^2 + l_{[55]+3}v^3 + \cdots + l_{64}v^9 \qquad \ldots(2)$$

Now in the RHS of equations (1) and (2), the terms in powers higher than v^3 are the same in both expressions, since these are beyond the end of the select period.

So, subtracting one from the other we get:

$$\begin{aligned}&l_{[55]}\ddot{a}_{[55]:\overline{10}|} - l_{55}\ddot{a}_{55:\overline{10}|}\\ &= l_{[55]} - l_{55} + (l_{[55]+1} - l_{56})v + (l_{[55]+2} - l_{57})v^2 + (l_{[55]+3} - l_{58})v^3 + 0 + \ldots + 0\end{aligned} \qquad [1]$$

Using the numbers given in the question, we can calculate $\ddot{a}_{[55]:\overline{10}|}$:

$$\begin{aligned}&1512\ddot{a}_{[55]:\overline{10}|} - 1483 \times 8.078\\ &= 1512 - 1483 + (1492 - 1477)v + (1477 - 1470)v^2 + (1467 - 1462)v^3\end{aligned}$$

So:

$$\ddot{a}_{[55]:\overline{10}|} = 7.959 \qquad [1]$$

We see here that the value of the select annuity is actually lower than the value of the ultimate annuity. We would normally expect the reverse since the (normally lighter) select mortality would increase the expected present value of the annuity payments. Here the opposite is true because the select mortality in the table is actually heavier than the corresponding ultimate mortality. This is known as reverse selection. [1]

[Total 5]

Solution 1.28

(i) ***EPV of annuity***

The expected present value of this annuity is:

$$10,000\,\bar{a}_{60:\overline{5}|} + 12,000\,v^5\ {}_5p_{60}\ \bar{a}_{65:\overline{5}|} \qquad [½]$$

Since the force of mortality is constant between age 60 and age 65:

$$v^5\ {}_5p_{60} = e^{-5\delta}e^{-5\mu} = e^{-5(0.05+0.03)} = e^{-0.4} = 0.67032 \qquad [1]$$

Also:

$$\bar{a}_{60:\overline{5}|} = \int_0^5 v^t\ {}_tp_x\,dt$$

$$= \int_0^5 e^{-(\delta+\mu)t}\ dt$$

$$= \frac{1}{\delta+\mu}\left[1-e^{-5(\delta+\mu)}\right]$$

$$= \frac{1}{0.08}\left(1-e^{-0.4}\right) = 4.12100 \qquad [1]$$

and similarly:

$$\bar{a}_{65:\overline{5}|} = \frac{1}{0.05+0.04}\left(1-e^{-0.45}\right) = 4.02635 \qquad [½]$$

So the expected present value of the annuity is:

$$(10,000\times 4.12100) + (12,000\times 0.67032\times 4.02635) = £73,597 \qquad [1]$$

[Total 4]

(ii)(a) ***EPV of term assurance***

The expected present value of this term assurance is:

$$50,000\bar{A}^{1}_{60:\overline{10|}} = 50,000\left(\bar{A}^{1}_{60:\overline{5|}} + v^5\ {}_5p_{60}\ \bar{A}^{1}_{65:\overline{5|}}\right) \qquad [½]$$

Since the force of mortality is constant between age 60 and age 65:

$$\bar{A}^{1}_{60:\overline{5|}} = \int_0^5 v^t\ {}_tp_x\ \mu_{x+t}\,dt$$

$$= \mu\,\bar{a}_{60:\overline{5|}}$$

$$= 0.03 \times 4.12100$$

$$= 0.12363 \qquad [1]$$

and similarly:

$$\bar{A}^{1}_{65:\overline{5|}} = 0.04 \times 4.02635 = 0.16105 \qquad [½]$$

So the expected present value is:

$$50,000\left(0.12363 + 0.67032 \times 0.16105\right) = £11,579 \qquad [½]$$

[Total 2½]

(ii)(b) ***EPV of endowment assurance***

The EPV of the maturity benefit is $50,000A_{60:\overset{\;1}{\overline{10|}}}$, where:

$$A_{60:\overset{\;1}{\overline{10|}}} = v^{10}\ {}_5p_{60}\ {}_5p_{65} = e^{-10\times0.05} \times e^{-5\times0.03} \times e^{-5\times0.04} = e^{-0.85} = 0.42741 \qquad [1]$$

So the EPV of the endowment assurance is:

$$50,000\left(0.12363 + 0.67032 \times 0.16105 + 0.42741\right) = £32,950 \qquad [½]$$

[Total 1½]

Solution 1.29

(i) ***Description of benefit***

This assurance policy pays 1 in n years time, or at the end of the year of death, if later. [1]

(ii) $\boldsymbol{E[W]}$

We can write:

$$W = \begin{cases} v^n & \text{if } K_x < n \\ v^{K_x+1} & \text{if } K_x \geq n \end{cases}$$

So:

$$E[W] = v^n\,{}_nq_x + v^n\,{}_np_x\,A_{x+n} \qquad [2]$$

(iii) $\mathbf{var}[\boldsymbol{W}]$

We need:

$$\begin{aligned} \operatorname{var}[W] &= E\left[W^2\right] - \left(E[W]\right)^2 \\ &= v^{2n}\,{}_nq_x + v^{2n}\,{}_np_x\,{}^2A_{x+n} - \left(v^n\,{}_nq_x + v^n\,{}_np_x\,A_{x+n}\right)^2 \end{aligned} \qquad [2]$$

(iv) ***Variance of the present value of the annuity payments***

The present value of the annuity payments is:

$$Y = \ddot{a}_{\overline{\max[K_x+1,\, n]|}} \qquad [1]$$

We need:

$$\begin{aligned}
\text{var}[Y] &= \text{var}\left[\ddot{a}_{\overline{\max\{K_x+1,\, n\}}|}\right] \\
&= \text{var}\left[\frac{1-v^{\max\{K_x+1,\, n\}}}{d}\right] \\
&= \text{var}\left[\frac{1-W}{d}\right] \\
&= \frac{1}{d^2}\,\text{var}[W] \\
&= \frac{1}{d^2}\left[v^{2n}\,{}_nq_x + v^{2n}\,{}_np_x\;{}^2A_{x+n} - \left(v^n\,{}_nq_x + v^n\,{}_np_x\,A_{x+n}\right)^2\right]
\end{aligned}$$

[2]

[Total 3]

Part 2 – Questions

Note that the split between Development Questions and Exam-style Questions is somewhat subjective. For example, there may be some development questions that are as difficult as any exam question, but may be more repetitive than typically found in the exam; equally some of the shorter and more straightforward development questions are similar to the short questions regularly found in the CT5 exam. The Exam-style Questions generally involve more application and greater scope and are typical of the more challenging questions you will meet in the exam.

1 *Development Questions*

Question 2.1

Calculate the annual premium payable in advance by a life now aged exactly 32 in respect of a deferred annuity payable from age 60 for 5 years certain and for life thereafter. The amount of the annuity is £400 *pa*, payable annually in arrear, but the insurer incurs an additional administration cost of £2 when each annuity payment is made.

Basis: AM92 Ultimate mortality, 6% *pa* interest [4]

Question 2.2

Calculate the annual premium payable by a man aged 32 for a temporary assurance with a sum assured of £5,000 and a term of 12 years. Assume AM92 Ultimate mortality and 4% *pa* interest. [4]

Question 2.3

On 30 June 1999 an annual premium 25-year endowment assurance policy was issued to a life aged 45 who was assumed to be subject to AM92 Select mortality rated down 5 years. Write down an expression in terms of annuities (in which all functions are based on AM92 mortality with no age adjustment) giving the net premium reserve per unit sum assured on 1 July 2004 for this policy, assuming that all premiums due by that date have been paid. [3]

Question 2.4

An annual premium conventional with-profits endowment assurance policy is issued to a life aged 35. The initial sum assured is £50,000 and the term of the policy is 25 years. The death benefit of the sum assured and attaching bonuses is payable at the end of the year of death. The office declares compound reversionary bonuses. Given that bonuses of 3% *pa* have been declared annually in advance for each year of the contract, find the net premium reserve at the end of the fifth policy year.

Basis: AM92 Select mortality, 4% *pa* interest [3]

Question 2.5

A 10-year regular-premium term assurance policy is issued to a group of lives aged 40. The sum assured is £20,000 and is payable at the end of the year of death. Write down an expression for the retrospective net premium reserve immediately before the 6th premium is due, assuming that reserves are calculated on the same basis as the premium P. [3]

Question 2.6

If K is a random variable representing the curtate future lifetime of a life aged x, state whether a_x is greater than or smaller than $a_{\overline{E(K)}|}$ for a positive rate of interest. Justify your answer. [5]

Question 2.7

A temporary annuity of £3,000 *pa* payable annually in arrears for a term of 10 years was purchased one year ago by a life then aged exactly 60 by payment of a single premium. Show algebraically that the current retrospective and prospective net premium reserves are equal, assuming that the pricing and reserving bases are the same. [4]

Question 2.8

The payments under a special deferred annuity are payable continuously from age 60 and increase continuously at the rate of 5% *pa* compound. The payment stream starts at the rate of £200 *pa*. Assuming AM92 select mortality before age 60 and PFA92C20 mortality after age 60, calculate the value of the annuity for a female life now aged 40, if interest is 5%. [5]

Question 2.9

Graham, aged 40, purchases a conventional with-profits whole life assurance with sum assured £2,000 plus attaching bonuses, payable at the end of the year of death. Assuming allowance for simple bonuses of 3% *pa*, vesting at the start of each policy year, what is Graham's level annual premium payable throughout life?

Basis: AM92 Ultimate mortality, 4% *pa* interest, no expenses [3]

Question 2.10

A special ten-year increasing endowment assurance policy provides a sum assured of £10,000 during the first year, which increases by £1,000 in each subsequent year. The sum payable on maturity at age 60 is £25,000. Write down an expression for the prospective net premium reserve immediately before the 5th premium is paid. [4]

Question 2.11

A with-profit whole life insurance policy was issued to a life aged 50 with level premiums payable annually in advance. The original sum assured was £20,000 and bonuses declared in the first 5 years total £1,500. The sum assured is payable at the end of the year of death.

Calculate the net premium reserve for the contract just before the 6th premium is due, assuming AM92 Ultimate mortality and 4% *pa* interest. [3]

Question 2.12

Chris, aged 32, buys a 12-year temporary assurance with a sum assured of £5,000 payable at the end of year of death. Calculate the annual premium for this policy. Assume AM92 Select mortality, 4% *pa* interest and that premiums are payable annually in advance. Allow for initial expenses of £100 and renewal expenses of 3% of each premium, excluding the first. [5]

Question 2.13

Wendy, aged 32, buys a deferred annuity that is payable from age 60 for 5 years certain and for life thereafter. The amount of the annuity is £400 *pa*, payable annually in arrears. Calculate the annual premium payable in advance for this annuity.

Basis: AM92 Ultimate mortality, 6% *pa* interest, expenses of £2 per annuity payment.
[5]

Question 2.14

An annual premium whole life assurance policy provides a sum assured of £30,000 payable immediately on death. Write down an expression for the retrospective reserve after 20 years in respect of a life aged 30 at entry. Expenses are £100 payable initially, with renewal expenses of 5% of each premium after the first. [4]

Question 2.15

Show how you would modify the premium equation below to allow for the expenses indicated:

$$P\ddot{a}_{x:\overline{n}|} = SA_{x:\overline{n}|}$$

(i) initial expenses of 2% of the sum assured [1]

(ii) renewal expenses of 2% of each premium, including the first [1]

(iii) claim expenses of 2% of the sum assured [1]

(iv) initial expenses of 50% of first premium plus renewal expenses of 3% of each premium excluding the first. [1]
[Total 4]

Question 2.16

A temporary annuity of £3,000 *pa* payable quarterly in arrears for a term of 10 years was purchased one year ago by Jim, on his 60th birthday, by payment of a single premium. Show algebraically that the current retrospective and prospective net reserves are equal assuming that the premium and reserving bases are the same. Ignore expenses. [4]

Question 2.17

The premiums under a whole life assurance with sum assured S issued to a life aged x are payable annually in advance throughout life. The premium is calculated assuming that the following expenses will be incurred:

Initial expenses:	I
Renewal expenses:	$100k$ % of each premium after the first
Claim related expenses:	$100c$ % of the sum assured

Write down equations linking the gross premium reserves at the end of successive policy years. [4]

Question 2.18

Find the value of £200 paid by a person aged exactly 25, accumulated with the benefit of survivorship and interest at 7½% *pa*, assuming ELT15 (Females) mortality, at the end of:

(i) 5 years [2]

(ii) 20 years [2]

[Total 4]

2 Exam-style Questions

Question 2.19

On 1 January 2009, a life insurance company sold a number of 10-year pure endowment policies, each with a benefit amount of £40,000, to lives then aged 30. Level premiums are payable annually in advance.

(i) Calculate the annual premium. [3]

(ii) On 1 January 2010, there were 50 of these policies still in force. During 2010, one policyholder died. Calculate the company's mortality profit for 2010. [5]

Assume AM92 Select mortality and 4% *pa* interest. Ignore expenses. [Total 8]

Question 2.20

An insurer issues a combined term assurance and annuity contract to a life aged 35. Level premiums are payable monthly in advance for a maximum of 30 years.

On death before age 65 a benefit is paid immediately. The benefit is £200,000 on death in the first year of the contract, £195,000 on death in the second year, £190,000 on death in the third year, *etc*, with the benefit decreasing by £5,000 each year until age 65. No benefit is payable on death after age 65.

On attaining age 65 the life receives a whole of life annuity of £10,000 *pa* payable monthly in arrears.

Calculate the monthly premium on the basis of:

Mortality:	up to age 65:	AM92 Select
	over age 65:	PFA92C20
Interest:	4% *pa*	
Expenses:	none	

[8]

Question 2.21

A life office sold a portfolio of 10,000 term assurance policies on 1/1/2003. The policies had a term of 2 years, with premiums paid annually in advance and were sold to a group of males aged 60 exactly at that date. Each policy had a sum assured of £50,000, which is payable at the end of the year of death. The company prices the product assuming AM92 Ultimate mortality.

(i) The same premium was charged for each year. The premium was calculated from an equation of value, in which the expected present value of the premiums is set to equal the expected present value of the benefit payments plus 10% of the standard deviation of the present value of the benefit payments. Calculate the premium assuming 4% *pa* interest. [7]

(ii) During the first policy year 75 policyholders died. Calculate the net premium reserve at the end of 2003 and hence the mortality profit for the portfolio for calendar year 2003. [5]

(iii) A director of the company has calculated the profit of the business as premiums received less sum assured paid on death less the net premium reserve. He calculates the profit as "just under £3.5*m*" and writes to ask why this conflicts with the mortality profit set out above. Show that the director's figures are numerically correct and then explain why the two figures differ. [6]

[Total 18]

Question 2.22

On 1 January 2010 a pension scheme had 100 members aged 75 exact, each eligible for a pension of £10,000 *pa*, payable annually in advance. In addition, the members were entitled to a death benefit of £20,000 payable at the end of the year of death. No premiums were being paid in respect of these contracts after January 2010. Given that 4 of the lives died during 2010, calculate the mortality profit for these contracts for calendar year 2010 using the following basis:

Mortality: PFA92C20
Interest: 4% *pa*
Expenses: none [5]

Question 2.23

(i) Express in the form of symbols, and also explain in words, the expressions "death strain at risk", "expected death strain" and "actual death strain". [3]

(ii) On 1 January 2001 an office issued a number of annual premium policies to a group of lives, each of whom was then aged exactly 45. All policies were for a term of 20 years and were of the following types:

- endowment assurances under which the sum assured was payable on survival to the end of the term or at the end of the year of earlier death

- temporary assurances under which the sum assured was payable only at the end of the year of death within the policy term

- pure endowments under which the only benefit payable is the sum assured on survival to the end of the policy term

Assuming that there is no source of decrement other than death, calculate the profit or loss from mortality for the calendar year 2010 in respect of the policies issued to this group of lives, given the following information:

Type of policy	*Sums assured in force on 1 January 2010*	*Sums assured discontinued by death during 2010*
Endowment assurance	£600,000	£4,000
Temporary assurance	£200,000	£2,000
Pure endowment	£80,000	£500

Reserving basis: AM92 Ultimate mortality, 4% *pa* interest. Ignore expenses.

[12]

[Total 15]

Question 2.24

A life office sells 100 independent 5-year term assurance policies to lives aged 60. Each policy has a sum assured of £10,000 payable at the end of the year of death. Premiums of £200 are payable annually in advance throughout the 5-year term or until earlier death.

Let L denote the present value of the insurer's loss on these policies. Assuming AM92 Ultimate mortality and 5½% *pa* interest, calculate the expected value and standard deviation of L. [7]

Question 2.25

A woman now aged exactly 64 has paid £20,000 a year into an accumulating with-profits contract at the start of each of the last four years.

The policy has incurred the following charges:

- £1,000 deducted at the start of year 1
- £100 deducted at the start of each subsequent year.

The following rates of regular bonus interest have been applied:

Year t	1	2	3	4
Bonus interest b_t	2.9%	3.1%	3.2%	3.4%

Additionally, there is a terminal bonus on contractual claim, currently payable at the rate of $0.015\times(t-1)$ of the fund value, where t is the number of years the policy has been in force at the time of claim.

The policy is now maturing, and the woman is using all of the maturity proceeds to buy a level annuity from the insurance company. The annuity will be payable monthly in advance for a minimum of 5 years and for the whole of life thereafter.

Calculate the monthly amount of annuity that the woman will receive, if the insurance company uses the following basis in its annuity pricing:

Mortality: PFA92C20 with a 3-year age deduction
Interest: 4% *pa*
Expenses: £400 initial plus 0.35% of each annuity payment. [7]

Question 2.26

An insurance company issues a special single premium annuity contract, which pays £10,000 *pa* in arrears for 10 years. If the policyholder dies within the 10-year term the annuity payments cease, and a lump sum benefit is paid out immediately on death. The amount of the death benefit is calculated as:

$$100,000 - 10,000k$$

where k is the curtate duration of the policy at the time of death.

The policy cannot be terminated for any reason other than through death.

1,500 of these policies were issued during a particular year to lives who were all aged exactly 55 when they took out the policy. It is now more than four years since the most recent of these policies was issued.

The mortality experience to date of this group of policyholders is given as follows:

- number of policyholders receiving exactly 2 or fewer annuity payments = 8
- number of policyholders receiving exactly 3 annuity payments = 4
- number of policyholders receiving exactly 4 or more annuity payments = 1,488

Calculate the mortality profit earned for the insurance company in the fourth policy year of this block of business, on the following basis:

Mortality: AM92 Ultimate
Interest: 4% *pa*
Expenses: None. [10]

Question 2.27

An annual premium conventional with-profits 20-year endowment assurance policy, issued to a life aged exactly 40 has a basic sum assured of £10,000 payable at the end of the year of death. Premiums are calculated assuming AM92 Select mortality, 4% *pa* interest, initial expenses of £150 and claim related expenses of 3% of the base sum assured (payable on death or maturity).

(i) Calculate the premium if the policy is assumed to provide simple bonuses of 2% of the sum assured vesting at the end of each policy year (*ie* the basic benefit amount will be increased by £200 at the end of each policy year for future claims). [6]

(ii) Calculate the premium if the policy is assumed to provide compound bonuses of 4% *pa* of the sum assured vesting at the end of each policy year (*ie* the basic benefit amount will be increased by a factor of 1.04 at the end of each policy year for future claims). [5]

[Total 11]

Question 2.28

A whole life assurance policy pays a benefit of £50,000 at the end of the year of death. The policyholder is currently aged 30 and is paying an annual premium of £700 at the start of each year. A premium has just been paid.

Use the following basis to calculate the reserve the company needs to hold at the present time so that the probability of covering the liability in full will be 99%.

Mortality: AM92 Select
Interest: 3% *pa*
Expenses: 5% of each future premium [5]

Question 2.29

A life insurance company issues a 20-year conventional with-profits endowment assurance policy to Russell, aged 40. The sum assured of £20,000 plus declared reversionary bonuses is payable immediately on death, or on survival to the end of the term.

(i) Calculate the quarterly premium payable by Russell throughout the term of the policy if the office assumes that future reversionary bonuses will be declared at the rate of 1.92308% of the sum assured, compounded and vesting at the end of each policy year on the following basis:

Mortality:	AM92 Select
Interest:	6% *pa*
Initial expenses:	114% of the first premium and 2.5% of the basic sum assured
Renewal expenses:	4% of each quarterly premium, excluding the first [8]

(ii) The life office values its with-profit business using the net premium method, assuming an interest rate of 4% *pa* and AM92 Ultimate mortality.

Calculate the prospective reserve for Russell's contract described above just before the 13th quarterly premium is payable, given that the total reversionary bonus declared up to that time is £600. [5]

[Total 13]

Question 2.30

The premiums payable under a deferred annuity contract issued to women aged exactly 60 are limited to 5 years. The annuity commences at age 65, provided the policyholder is still alive at that age. The annuity provides payments of £3,500 payable annually in advance for 5 years certain (*ie* it continues to be paid for 5 years even if the annuitant dies before age 69) and for life thereafter. There is no benefit if the policyholder dies before age 65.

(i) Calculate the annual premium. [6]

(ii) Calculate the retrospective and prospective reserves after the policy has been in force for each of 5 and 10 years. [10]

Basis: PFA92C20 mortality, 4% *pa* interest, no expenses. [Total 16]

Question 2.31

A 10-year "double endowment" assurance policy issued to a group of lives aged 50, a sum assured of £10,000 is payable at the end of the year of death and £20,000 is paid if the life survives to reach the maturity date. Premiums are payable annually in advance.

You are given the following information:

Reserve at the start of the 8th year (per policy in force):	£12,940
Number of policies in force at the start of the 8th year:	200
Number of deaths during the 8th year:	3
Annual net premium (per policy)	£1,591

(i) Assuming that the reserving basis uses ELT15 (Males) mortality and 4% *pa* interest, calculate the profit or loss arising from mortality in the 8th year. [5]

(ii) Comment on your results. [1]
[Total 6]

Question 2.32

On 1 January 1998 a life insurance company issued a number of 20-year pure endowment policies to a group of lives aged 40 exact. In each case, the sum assured was £75,000 and premiums were payable annually in advance.

On 1 January 2012, 500 policies were still in force. During 2012, 3 policyholders died, and no policy lapsed for any other reason.

The office calculates net premiums and net premium reserves on the following basis:

Interest: 4% per annum
Mortality: AM92 Select

(i) Calculate the profit or loss from mortality for this group for the year ending 31 December 2012. [6]

(ii) Explain why the mortality profit or loss has arisen. [2]

(iii) Calculate the probability of there being a mortality loss (as opposed to a mortality profit) in 2013. [3]
[Total 11]

Part 2 – Solutions

Solution 2.1

The premium equation is:

$$P\ddot{a}_{32:\overline{28|}} = (400+2)\frac{D_{60}}{D_{32}}\left(a_{\overline{5|}} + \frac{D_{65}}{D_{60}}a_{65}\right) \qquad [1]$$

The factors are:

$\ddot{a}_{32:\overline{28|}} = 14.053$ (tabulated on Page 106 of the *Tables*) [½]

$$\frac{D_{60}}{D_{32}} = \frac{1}{1.06^{28}}\frac{l_{60}}{l_{32}} = 0.18327 \qquad [½]$$

$a_{\overline{5|}} = 4.2124$ [½]

$$\frac{D_{65}}{D_{60}} = 0.70977 \qquad [½]$$

$a_{65} = (\ddot{a}_{65} - 1) = 9.569$ [½]

So the premium equation becomes:

$$14.053P = 402 \times 0.18327 \times 11.004 = 810.74$$

$\Rightarrow P = 810.74/14.053 = £57.69$ [½]

[Total 4]

Solution 2.2

The premium equation is:

$$P\ddot{a}_{32:\overline{12}|} = 5,000 A^{1}_{32:\overline{12}|} \quad [1]$$

The factors are:

$$\ddot{a}_{32:\overline{12}|} = \ddot{a}_{32} - \frac{D_{44}}{D_{32}}\ddot{a}_{44} = 21.520 - \frac{1,747.41}{2,825.89} \times 19.075 = 9.725 \quad [1]$$

$$A^{1}_{32:\overline{12}|} = A_{32} - \frac{D_{44}}{D_{32}} A_{44} = 0.17230 - \frac{1,747.41}{2,825.89} \times 0.26636 = 0.007594 \quad [1]$$

So the premium equation becomes:

$$9.725P = 5,000 \times 0.007594 \quad \Rightarrow P = \frac{37.97}{9.725} = £3.90 \quad [1]$$

[Total 4]

Solution 2.3

We need to find the reserve at the end of the 5th year, then add on the premium just received. The formula for the net premium reserve for an *n*-year endowment assurance policy is:

$$ {}_tV_{x:\overline{n}|} = 1 - \frac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x:\overline{n}|}} \quad [½]$$

So the reserve at the end of the 5th year is:

$$ {}_5V_{[45-5]:\overline{25}|} = 1 - \frac{\ddot{a}_{[45-5]+5:\overline{25-5}|}}{\ddot{a}_{[45-5]:\overline{25}|}} = 1 - \ddot{a}_{45:\overline{20}|} / \ddot{a}_{[40]:\overline{25}|} \quad [½]$$

The annual net premium is:

$$\frac{A_{[40]:\overline{25}|}}{\ddot{a}_{[40]:\overline{25}|}} = \frac{1 - d\,\ddot{a}_{[40]:\overline{25}|}}{\ddot{a}_{[40]:\overline{25}|}} = 1/\ddot{a}_{[40]:\overline{25}|} - d \quad [1]$$

So the correct answer is:

$$1-\ddot{a}_{45:\overline{20|}}/\ddot{a}_{[40]:\overline{25|}}+1/\ddot{a}_{[40]:\overline{25|}}-d \quad = \quad 1-d-(\ddot{a}_{45:\overline{20|}}-1)/\ddot{a}_{[40]:\overline{25|}} \qquad [1]$$

[Total 3]

Solution 2.4

The reserve is calculated based on the net premium (ignoring bonuses) and allowing for declared bonuses only.

The net premium is found from the premium equation:

$$P\ddot{a}_{[35]:\overline{25|}} = 50,000A_{[35]:\overline{25|}}$$

ie $\quad 16.029P = 50,000 \times 0.3835$

$$\Rightarrow P = £1,196.27 \qquad [1]$$

The reserve at the end of the fifth year (by which time, five bonuses will have been declared) is:

$$\begin{aligned} {}_5V^{pro} &= 50,000 \times 1.03^5\, A_{40:\overline{20|}} - P\ddot{a}_{40:\overline{20|}} \\ &= 50,000 \times 1.03^5 \times 0.46433 - 13.927 \times 1,196.27 \\ &= £10,254 \end{aligned} \qquad [2]$$

[Total 3]

Solution 2.5

The retrospective net premium reserve can be calculated as the expected present value at the outset of (net premiums – benefits), accumulated with interest and allowing for survival to age 45. This gives:

$$\frac{D_{40}}{D_{45}}(P\ddot{a}_{40:\overline{5|}} - 20,000A^1_{40:\overline{5|}}) \qquad [3]$$

Solution 2.6

The annuity certain is:

$$a_{\overline{E(K)}|} = a_{\overline{e_x}|} = v + v^2 + \ldots + v^{e_x} \qquad [½]$$

The life annuity is:

$$a_x = v\frac{l_{x+1}}{l_x} + v^2\frac{l_{x+2}}{l_x} + \ldots \qquad [½]$$

Each of these functions can be expressed in the form of a "perpetuity certain" (with varying payment amounts):

$$a_{\overline{e_x}|} = 1 \times v + 1 \times v^2 + \ldots + 1 \times v^{e_x} + 0 \times v^{e_x+1} + 0 \times v^{e_x+2} + \ldots$$

$$a_x = \frac{l_{x+1}}{l_x}v + \frac{l_{x+2}}{l_x}v^2 + \ldots + \frac{l_{x+e_x}}{l_x}v^{e_x} + \frac{l_{x+e_x+1}}{l_x}v^{e_x+1} + \frac{l_{x+e_x+2}}{l_x}v^{e_x+2} + \ldots$$

In the first equation, a payment of 1 is made for e_x years. So the total amount paid is e_x.
[½]

For simplicity, we have assumed here that e_x is a whole number. In fact, the result also holds when e_x takes a fractional value (which it usually does).

In the second equation, the payment in year t is $\frac{l_{x+t}}{l_x}$. So the total amount paid is:

$$\frac{l_{x+1}}{l_x} + \frac{l_{x+2}}{l_x} + \ldots = e_x \qquad [½]$$

So the total amount paid for both these perpetuities is the same. The only difference between them is the timing of the payments. [1]

Since $\frac{l_{x+t}}{l_x} < 1$, the "payments" (*ie* the coefficients) in each of the first e_x years are smaller for the second annuity than for the first. This shortfall in the payments in the early years has been redistributed over the years from $e_x + 1$ onwards in the second annuity. So the average time until payment is longer for the second annuity. Its present value must therefore be smaller.

So $\quad a_x < a_{\overline{E(K)}|}$ [2]

[Total 5]

In practice, the values of these two functions are quite similar, and an annuity certain for the expected lifetime can be used as a reasonableness check on the value of a life annuity.

Solution 2.7

The prospective reserve is:

$${}_1V^{pro} = 3,000 \times a_{61:\overline{9}|} \qquad [1]$$

The retrospective reserve is the accumulated value of the premium less benefits:

$${}_1V^{retro} = \frac{D_{60}}{D_{61}} \times P - \frac{D_{60}}{D_{61}} \times 3,000 a_{60:\overline{1}|} \qquad [½]$$

where P is the single premium. Now:

$$P = 3,000 a_{60:\overline{10}|} \qquad [½]$$

so:

$${}_1V^{retro} = \frac{D_{60}}{D_{61}} \times 3,000 a_{60:\overline{10}|} - \frac{D_{60}}{D_{61}} \times 3,000 a_{60:\overline{1}|}$$

$$= 3,000 \frac{D_{60}}{D_{61}} (a_{60:\overline{10}|} - a_{60:\overline{1}|}) \qquad [1]$$

Since the difference between the annuities in the brackets represents the value at age 60 of annual payments made in arrears for the 9 years between ages 61 and 70, this can be written as:

$${}_1V^{retro} = 3,000 \frac{D_{60}}{D_{61}} \times \left(\frac{D_{61}}{D_{60}} \times a_{61:\overline{9}|}\right) = 3,000 a_{61:\overline{9}|} = {}_1V^{pro} \qquad [1]$$

[Total 4]

Solution 2.8

The expected present value of the deferred annuity is:

$$v^{20}\frac{l_{60}}{l_{[40]}}\int_0^\infty 200\times 1.05^t\, v^t\; {}_tp_{60}\,dt = 200v^{20}\frac{l_{60}}{l_{[40]}}\bar{a}_{60}^{@0\%} \qquad [2]$$

We have:

$$v^{20}\frac{l_{60}}{l_{[40]}} = 1.05^{-20}\times\frac{9,287.2164}{9,854.3036} = 0.3552 \qquad [1]$$

The annuity factor at 0% is the same as the complete expectation of life $\mathring{e}_{60}$. [1]

Putting these factors together gives:

$$EPV = 200\times 0.3552\times 27.41 = £1,947 \qquad [1]$$

[Total 5]

Solution 2.9

The premium equation is:

$$P\ddot{a}_{40} = 2,000A_{40} + 0.03\times 2,000(IA)_{40} \qquad [2]$$

ie $\quad 20.005P = 2,000\times 0.23056 + 60\times 7.95699$

$$\Rightarrow \; P = £46.92 \qquad [1]$$

[Total 3]

Solution 2.10

The policy starts when the policyholder is age 50. The 5th premium is paid on the policyholder's 54th birthday, when the remaining term will be 6 years.

If death occurs at age 54 last birthday, the benefit amount will be £14,000, which will increase by £1,000 each year. The maturity value is £25,000. So the prospective net reserve is:

$$ {}_4V^{pro} = 13{,}000 A^1_{54:\overline{6}|} + 1{,}000 (IA)^1_{54:\overline{6}|} + 25{,}000 \frac{D_{60}}{D_{54}} - P\ddot{a}_{54:\overline{6}|} \qquad [2] $$

where the premium is given by:

$$ P\ddot{a}_{50:\overline{10}|} = 9{,}000 A^1_{50:\overline{10}|} + 1{,}000 (IA)^1_{50:\overline{10}|} + 25{,}000 \frac{D_{60}}{D_{50}} \qquad [2] $$

[Total 4]

Note that, in order to specify the calculation of the reserve precisely, it is necessary to state how the premium is calculated.

Solution 2.11

The net premium is found from the equation:

$$ P\ddot{a}_{50} = 20{,}000 A_{50} $$

ie $17.444P = 20{,}000 \times 0.32907$

$$ \Rightarrow P = £377.29 \qquad [1] $$

The reserve at the end of the fifth year (*ie* just before the 6th premium) is:

$$ \begin{aligned} {}_5V &= (20{,}000 + 1{,}500) A_{55} - P\ddot{a}_{55} \\ &= 21{,}500 \times 0.3895 - 377.29 \times 15.873 \\ &= £2{,}386 \qquad [2] \end{aligned} $$

[Total 3]

Solution 2.12

The premium equation is:

$$P\ddot{a}_{[32]:\overline{12}|} = 5{,}000A^{1}_{[32]:\overline{12}|} + 100 + 0.03P(\ddot{a}_{[32]:\overline{12}|} - 1) \qquad [2]$$

The factors are:

$$\ddot{a}_{[32]:\overline{12}|} = \ddot{a}_{[32]} - \frac{D_{44}}{D_{[32]}}\ddot{a}_{44} = 21.523 - \frac{1{,}747.41}{2{,}825.48} \times 19.075 = 9.726 \qquad [1]$$

$$A^{1}_{[32]:\overline{12}|} = A_{[32]} - \frac{D_{44}}{D_{[32]}}A_{44} = 0.17218 - \frac{1{,}747.41}{2{,}825.48} \times 0.26636 = 0.00745 \qquad [1]$$

So the premium equation becomes:

$$9.726P = 5{,}000 \times 0.00745 + 100 + 0.03(9.726 - 1)P$$

So:

$$P = £14.50 \qquad [1]$$

[Total 5]

Solution 2.13

The premium equation is:

$$P\ddot{a}_{32:\overline{28}|} = 402\frac{D_{60}}{D_{32}}\left(a_{\overline{5}|} + \frac{D_{65}}{D_{60}}a_{65}\right) \qquad [1]$$

The factors are:

$$\ddot{a}_{32:\overline{28}|} = 14.053 \text{ (from the } \textit{Tables}\text{)} \qquad [½]$$

$$\frac{D_{60}}{D_{32}} = \frac{1}{1.06^{28}}\frac{l_{60}}{l_{32}} = 0.18327 \qquad [1]$$

$$a_{\overline{5}|} = 4.2124 \qquad [½]$$

$$\frac{D_{65}}{D_{60}} = 0.70977 \qquad [½]$$

$$a_{65} = 9.569 \qquad [½]$$

So the premium equation becomes:

$$14.053P = 402 \times 0.18327 \times \left(4.2124 + 0.70977 \times 9.569\right) = 810.729$$

and: $P = 810.729/14.053 = £57.69$ [1]

[Total 5]

Solution 2.14

The retrospective reserve at the end of year 20 is:

$${}_{20}V^{retro} = \frac{D_{30}}{D_{50}}\left[P\ddot{a}_{30:\overline{20}|} - 30,000\bar{A}^{1}_{30:\overline{20}|} - 100 - 0.05P(\ddot{a}_{30:\overline{20}|} - 1)\right] \qquad [2]$$

where the premium is given by:

$$P\ddot{a}_{30} = 30,000\bar{A}_{30} + 100 + 0.05P(\ddot{a}_{30} - 1) \qquad [2]$$

[Total 4]

Solution 2.15

(i) $P\ddot{a}_{x:\overline{n}|} = SA_{x:\overline{n}|} + 0.02S$ [1]

(ii) $P\ddot{a}_{x:\overline{n}|} = SA_{x:\overline{n}|} + 0.02P\ddot{a}_{x:\overline{n}|}$ which simplifies to $0.98P\ddot{a}_{x:\overline{n}|} = SA_{x:\overline{n}|}$ [1]

(iii) $P\ddot{a}_{x:\overline{n}|} = SA_{x:\overline{n}|} + 0.02SA_{x:\overline{n}|}$ which simplifies to $P\ddot{a}_{x:\overline{n}|} = 1.02SA_{x:\overline{n}|}$ [1]

(iv) $P\ddot{a}_{x:\overline{n}|} = SA_{x:\overline{n}|} + 0.5P + 0.03P(\ddot{a}_{x:\overline{n}|} - 1)$ which simplifies to:

$$0.97P\ddot{a}_{x:\overline{n}|} = SA_{x:\overline{n}|} + 0.47P$$ [1]

[Total 4]

Solution 2.16

The prospective reserve is:

$${}_1V^{pro} = 3{,}000a^{(4)}_{61:\overline{9}|}$$ [1]

The retrospective reserve is:

$${}_1V^{retro} = \text{Acc.value past premiums - Acc.value past benefits}$$

$$= \frac{D_{60}}{D_{61}} \times 3{,}000a^{(4)}_{60:\overline{10}|} - \frac{D_{60}}{D_{61}} \times 3{,}000a^{(4)}_{60:\overline{1}|}$$

$$= 3{,}000\frac{D_{60}}{D_{61}}(a^{(4)}_{60:\overline{10}|} - a^{(4)}_{60:\overline{1}|})$$ [2]

Since the "difference of annuities" in the brackets represents the value at age 60 of quarterly payments made in arrears for the 9 years between ages 61 and 70, this can be written as:

$${}_1V^{retro} = 3{,}000\frac{D_{60}}{D_{61}} \times \frac{D_{61}}{D_{60}}a^{(4)}_{61:\overline{9}|} = 3{,}000a^{(4)}_{61:\overline{9}|} = {}_1V^{pro}$$ [1]

[Total 4]

Solution 2.17

For the first year, the equation of equilibrium is:

$$(P-I)(1+i)=(1+c)Sq_x+{}_1V^{gross}\,p_x \qquad [2]$$

For subsequent years:

$$({}_tV^{gross}+P-kP)(1+i)=(1+c)Sq_{x+t}+{}_{t+1}V^{gross}\,p_{x+t} \qquad [2]$$

[Total 4]

Solution 2.18

(i) The accumulated value at the end of 5 years is:

$$200\frac{D_{25}}{D_{30}}=200(1.075)^5\frac{l_{25}}{l_{30}}=£287.65 \qquad [2]$$

(ii) The accumulated value at the end of 20 years is:

$$200\frac{D_{25}}{D_{45}}=200(1.075)^{20}\frac{l_{25}}{l_{45}}=£862.51 \qquad [2]$$

[Total 4]

Solution 2.19

(i) ***Annual premium***

Let P denote the annual premium. Then:

$$\text{EPV premiums} = P\,\ddot{a}_{[30]:\overline{10}|} = P\left(\ddot{a}_{[30]} - \frac{D_{40}}{D_{[30]}}\ddot{a}_{40}\right) \quad [½]$$

From the *Tables*:

$$\frac{D_{40}}{D_{[30]}} = \frac{2,052.96}{3,059.68} = 0.67097 \quad [½]$$

$$\ddot{a}_{[30]} = 21.837 \text{ and } \ddot{a}_{40} = 20.005 \quad [½]$$

So:

$$\text{EPV premiums} = P\left(21.837 - 0.67097 \times 20.005\right) = 8.414P \quad [½]$$

Also:

$$\text{EPV benefits} = 40,000\frac{D_{40}}{D_{[30]}} = 26,838.89 \quad [½]$$

So the annual premium is:

$$P = \frac{26,838.89}{8.414} = £3,189.71 \quad [½]$$

[Total 3]

(ii) ***Mortality profit***

The reserve per policy in force at the end of 2010 is:

$$ {}_2V = 40{,}000\frac{D_{40}}{D_{32}} - 3{,}189.71\,\ddot{a}_{32:\overline{8}|} $$ [½]

From the *Tables*:

$$ \frac{D_{40}}{D_{32}} = \frac{2{,}052.96}{2{,}825.89} = 0.72648 $$ [½]

and:

$$ \ddot{a}_{32:\overline{8}|} = \ddot{a}_{32} - \frac{D_{40}}{D_{32}}\ddot{a}_{40} = 21.520 - 0.72648 \times 20.005 = 6.987 $$ [1]

So:

$$ {}_2V = 40{,}000 \times 0.72648 - 3{,}189.71 \times 6.987 = 6{,}773.71 $$ [½]

There is no death benefit, so the death strain at risk (DSAR) for calendar year 2010 is:

$$ DSAR = S - {}_2V = 0 - 6{,}773.71 = -6{,}773.71 $$ [½]

The expected death strain (EDS) is:

$$ EDS = 50q_{[30]+1} \times DSAR = 50 \times 0.000569 \times (-6{,}773.71) = -192.71 $$ [1]

and the actual death strain (ADS) is:

$$ ADS = 1 \times DSAR = -6{,}773.71 $$ [½]

So the company's mortality profit for calendar year 2010 is:

$$ EDS - ADS = -192.71 - (-6{,}773.71) = £6{,}581 $$ [½]

[Total 5]

Solution 2.20

If the monthly premium is P, the premium equation is:

$$12P\ddot{a}^{(12)}_{[35]:\overline{30|}} = 205,000\bar{A}^{1}_{[35]:\overline{30|}} - 5,000(I\bar{A})^{\ 1}_{[35]:\overline{30|}} + 10,000\frac{D_{65}}{D_{[35]}}a^{(12)}_{65} \qquad [2]$$

The factors (calculated using the appropriate mortality and interest rates) are:

$$\ddot{a}^{(12)}_{[35]:\overline{30|}} = \ddot{a}_{[35]:\overline{30|}} - \frac{11}{24}\left(1 - \frac{D_{65}}{D_{[35]}}\right) = 17.631 - \frac{11}{24}\times 0.72508 = 17.299 \qquad [1]$$

$$\bar{A}^{\ 1}_{[35]:\overline{30|}} = 1.02\left(A_{[35]:\overline{30|}} - A_{[35]:\overline{30|}}^{\ \ \ \ \ \ \ 1}\right) = 1.02\left(0.32187 - \frac{689.23}{2,507.02}\right) = 0.04789 \qquad [1]$$

$$(I\bar{A})^{\ 1}_{[35]:\overline{30|}} = 1.02\left((IA)_{[35]} - \frac{D_{65}}{D_{[35]}}\left((IA)_{65} + 30A_{65}\right)\right)$$

$$= 1.02\left(7.47005 - \frac{689.23}{2,507.02}\left(7.89442 + 30\times 0.52786\right)\right)$$

$$= 0.96506 \qquad [2]$$

$$a^{(12)}_{65} = a_{65} + \tfrac{11}{24} = 13.871 + \tfrac{11}{24} = 14.329 \qquad [1]$$

So the premium equation becomes:

$$12P\times 17.299 = 205,000\times 0.04789 - 5,000\times 0.96506$$
$$+10,000\times 0.27492\times 14.329$$

So:

$$P = \frac{44,385.4}{12\times 17.299} = £213.82 \text{ per month} \qquad [1]$$

[Total 8]

Solution 2.21

(i) ***Premium calculation***

The expected benefit outgo per policy is:

$$50,000 \times A^1_{60:\overline{2}|} = 50,000\left(A_{60} - \frac{D_{62}}{D_{60}} A_{62} \right)$$

$$= 50,000\left(0.45640 - \frac{802.40}{882.85} \times 0.48458 \right)$$

$$= 50,000 \times 0.0159775$$

$$= £798.88$$ [1]

The variance per unit sum assured is equal to ${}^2A^1_{60:\overline{2}|} - (A^1_{60:\overline{2}|})^2$.

From first principles, the variance is:

$$vq_{60} + v^2 p_{60} q_{61} - (0.0159775)^2$$ [1]

where v is calculated at $1.04^2 - 1 = 8.16\%$. So the variance is:

$$\left(1.0816^{-1} \times 0.008022 + 1.0816^{-2} \times 0.991978 \times 0.009009\right) - (0.0159775)^2$$

$$= 0.014801$$ [2]

Therefore, assuming that the lives are independent, then the variance is:

$$50,000^2 \times 0.014801 = 37,002,178$$

and the standard deviation is $6,082.9$. [1]

The expected present value of the premiums is:

$$P\ddot{a}_{60:\overline{2}|} = P\left(1 + v\, p_{60}\right) = P\left(1 + \frac{0.991987}{1.04}\right) = 1.953825P$$ [1]

Hence the premium is:

$$P = \frac{798.88 + 608.29}{\ddot{a}_{60:\overline{2}|}} = \frac{1,407.17}{1.953825} = 720.21 \qquad [1]$$

[Total 7]

(ii) ***Net premium reserve and mortality profit***

The net premium reserve is equal to the value of future benefits less the value of net premiums.

The net premium is equal to:

$$\frac{798.88}{1.953825} = 408.88 \qquad [1]$$

So the reserve for each policy is:

$$\begin{aligned} 50,000 \times q_{61} \times 1.04^{-1} - 408.88 &= 50,000 \times 0.009009 \times 1.04^{-1} - 408.88 \\ &= 24.25 \qquad [2] \end{aligned}$$

Therefore the mortality profit is equal to:

$$(10,000q_{60} - 75) \times (50,000 - 24.25) = 260,873$$

ie a profit of around £261,000. [2]

[Total 5]

(iii) ***Why the figures differ***

The director's figures are:

- premiums with interest of $10,000 \times 720.21 \times 1.04 = 7,490,000$
- death payments of $50,000 \times 75 = 3,750,000$
- reserves of $(10,000 - 75) \times 24.25 = 240,700$
- profit £3,499,000, *ie* "just under £3.5m" as calculated by the director. [3]

The figures conflict since the "mortality profit" in part (ii) ignores the contingency loading in the premium rates. This is the source of the profit. [1]

If the director recalculated his figures with the premium ignoring the loading, *ie* premiums of $408.88 \times 10,000 = 4,088,800$, before interest, the profit would now be a profit of £262K. So the numbers would be consistent. [2]

[Total 6]

Solution 2.22

The death strain at risk (DSAR) for a single contract for calendar year 2010 is:

$$20,000 - {}_{31.12.10}V \qquad [½]$$

where:

$$\begin{aligned} {}_{31.12.10}V &= 10,000\ddot{a}_{76} + 20,000A_{76} \\ &= 10,000\ddot{a}_{76} + 20,000\left(1 - d\ddot{a}_{76}\right) \\ &= 10,000 \times 10.536 + 20,000\left(1 - \frac{0.04}{1.04} \times 10.536\right) \\ &= 117,255.38 \end{aligned} \qquad [2]$$

So the DSAR is:

$$20,000 - 117,255.38 = -97,255.38 \qquad [½]$$

The expected number of deaths during 2010 is:

$$100\,q_{75} = 100 \times 0.019478 = 1.9478 \qquad [½]$$

So the expected death strain is:

$$EDS = 100\,q_{75} \times DSAR = -189,434.04 \qquad [½]$$

The actual number of deaths during 2010 is 4. So the actual death strain is:

$$ADS = 4 \times DSAR = -389,021.54 \qquad [½]$$

Hence the mortality profit for the group of policies for 2010 is:

$$EDS - ADS = £199,587 \qquad [½]$$

[Total 5]

Solution 2.23

(i) ***Definitions***

The "death strain at risk" for a policy for year $t+1$ ($t = 0,1,2,\ldots$) is the excess of the sum assured (*ie* the present value at time $t+1$ of all benefits payable on death during year $t+1$) over the end of year reserve. [1]

The "expected death strain" for year $t+1$ is the total death strain that would be incurred in respect of all policies in force at the start of year $t+1$ if deaths conformed to the numbers expected.

$$\text{EDS for year } t+1 = \sum_{\substack{\text{policies in force} \\ \text{at start of year}}} q(S - {}_{t+1}V) \qquad [1]$$

The "actual death strain" for year $t+1$ is the total death strain incurred in respect of all claims actually arising during year $t+1$.

$$\text{ADS for year } t+1 = \sum_{\text{claims during year}} (S - {}_{t+1}V) \qquad [1]$$

[Total 3]

(ii) ***Mortality profit***

The net premiums per unit sum assured for the three types of policies can be found as follows:

$$P_a \ddot{a}_{45:\overline{20|}} = A_{45:\overline{20|}} \Rightarrow P_a = 0.46998/13.780 = 0.03411 \qquad [1]$$

$$P_b \ddot{a}_{45:\overline{20|}} = A^{1}_{45:\overline{20|}} \Rightarrow P_b = 0.05923/13.780 = 0.00430 \qquad [1]$$

$$P_c = P_a - P_b = 0.02981 \qquad [1]$$

The net premium reserves at the end of the year per unit sum assured are:

$$ {}_{10}V_a = A_{55:\overline{10}|} - P_a \ddot{a}_{55:\overline{10}|} = 0.68388 - 0.03411 \times 8.219 = 0.4036 \qquad [1]$$

$$ {}_{10}V_b = A^{1}_{55:\overline{10}|} - P_b \ddot{a}_{55:\overline{10}|} = 0.06037 - 0.00430 \times 8.219 = 0.02504 \qquad [1]$$

$$ {}_{10}V_c = \frac{D_{65}}{D_{55}} - P_c \ddot{a}_{55:\overline{10}|} = 0.62351 - 0.02981 \times 8.219 = 0.3785 \qquad [1]$$

The total expected death strain is:

$$\begin{aligned} EDS &= EDS_a + EDS_b + EDS_c \\ &= q_{54}[600,000(1 - {}_{10}V_a) + 200,000(1 - {}_{10}V_b) + 80,000(0 - {}_{10}V_c)] \\ &= 0.003976[600,000(1 - 0.4036) + 200,000(1 - 0.02504) + 80,000(-0.3785)] \\ &= 2,080 \end{aligned} \qquad [3]$$

The total actual death strain is:

$$\begin{aligned} ADS &= ADS_a + ADS_b + ADS_c \\ &= 4,000(1 - 0.4036) + 2,000(1 - 0.02504) + 500(-0.3785) \\ &= 4,150 \end{aligned} \qquad [2]$$

So there is a profit of $2,080 - 4,150 = -2,070$, *ie* a loss of £2,070. [1]

[Total 12]

Solution 2.24

The loss on the i th policy is:

$$L_i = \begin{cases} 10,000 v^{K_i+1} - 200 \ddot{a}_{\overline{K_i+1}|} & \text{if } K_i = 0,1,2,3,4 \\ -200 \ddot{a}_{\overline{5}|} & \text{if } K_i \geq 5 \end{cases}$$

where K_i denotes the curtate future lifetime of the i th policyholder.

Since we don't have any tabulated functions for 5½%, we will need to do the calculations for the expected value numerically (by considering the possible cases for the time of death), rather than attempting an algebraic method using A functions etc.

The table below shows the possible values of L_i and their associated probabilities:

Curtate future lifetime, K_i	Loss, L_i	Probability
0	9,278.67	$q_{60} = 0.0080220$
1	8,594.95	$p_{60}\, q_{61} = 0.0089367$
2	7,946.87	${}_2p_{60}\, q_{62} = 0.0099405$
3	7,332.58	${}_3p_{60}\, q_{63} = 0.011039$
4	6,750.31	${}_4p_{60}\, q_{64} = 0.012234$
≥ 4	–901.03	${}_5p_{60} = 0.94983$

[2]

The probabilities can alternatively be calculated as $\frac{d_{60}}{l_{60}}, \frac{d_{61}}{l_{60}}, \frac{d_{62}}{l_{60}}, \frac{d_{63}}{l_{60}}, \frac{d_{64}}{l_{60}}$ *and* $\frac{l_{65}}{l_{60}}$.

The expected present value of the loss on a single policy is:

$$E(L_i) = (9,278.67 \times 0.0080220) + (8,594.95 \times 0.0089367) + \cdots$$

$$+ (-901.03 \times 0.94983)$$

$$= -462.06$$

So the expected present value of the loss on the group of policies is:

$$E(L) = -462.06 \times 100 = -£46,206$$

ie a profit of £46,206. [2]

Since the policies are independent, the variance of L is given by:

$$\text{var}(L) = \text{var}\left(L_1 + \cdots + L_{100}\right) = 100\,\text{var}\left(L_i\right) = 100\left[E\left(L_i^2\right) - \left[E\left(L_i\right)\right]^2\right]$$

Now:

$$E\left(L_i^2\right) = \left(9,278.67^2 \times 0.0080220\right) + \cdots + \left(901.03^2 \times 0.94983\right)$$

$$= 3,900,700 \text{ (5 s.f.)}$$

So:

$$\text{var}(L) = 100\left[3,900,700 - (-462.06)^2\right] = (£19,202)^2$$

ie the standard deviation of the present value of the loss is £19,202. [3]

[Total 7]

Solution 2.25

First we need to calculate the maturity benefit of the accumulating with-profits policy. The fund at maturity can be found recursively using:

$$F_0 = 0$$

$$F_t = \left(F_{t-1} + P - E_t\right)\left(1 + b_t\right) \qquad t = 1, 2, 3, 4$$

where:

F_t = fund value at time t

P = premium

E_t = expense charge at start of year t

b_t = bonus interest for year t

This produces the fund values shown in the following table:

Year	Premium	Expense charge	Bonus interest rate	Fund at end of year
1	20,000	1,000	2.9%	19,551.00
2	20,000	100	3.1%	40,673.98
3	20,000	100	3.2%	62,512.35
4	20,000	100	3.4%	85,214.37

[2]

The terminal bonus rate at maturity is $0.015 \times 3 = 0.045$, so the total maturity value is:

$$85,214.37 \times 1.045 = 89,049.01 \qquad [1]$$

Using the equivalence principle, the monthly annuity payment (of *X*) can be obtained from the equation:

$$89,049.01 = 400 + (12X) \times 1.0035 \times \ddot{a}^{(12)}_{\overline{61:5|}}$$

$$\Rightarrow \quad X = \frac{89,049.01 - 400}{12 \times 1.0035 \times \ddot{a}^{(12)}_{\overline{61:5|}}} \qquad [1½]$$

Now:

$$\ddot{a}^{(12)}_{\overline{61:5|}} = \ddot{a}^{(12)}_{\overline{5|}} + v^5 \, {}_5p_{61} \, \ddot{a}^{(12)}_{66}$$

$$= \frac{1 - v^5}{d^{(12)}} + v^5 \frac{l_{66}}{l_{61}} \left(\ddot{a}_{66} - \frac{11}{24} \right)$$

$$= \frac{1 - 1.04^{-5}}{0.039157} + 1.04^{-5} \times \frac{9,658.285}{9,828.163} \times \left(14.494 - \frac{11}{24} \right)$$

$$= 15.88456 \qquad [2]$$

And so the monthly amount of the annuity is:

$$X = \frac{89,049.01 - 400}{12 \times 1.0035 \times 15.88456} = £463.45 \qquad [½]$$

[Total 7]

Solution 2.26

If the policyholder dies in the fourth policy year, the curtate duration will equal 3. So the death strain at risk in the fourth policy year is:

$$DSAR = (100{,}000 - 3 \times 10{,}000) \times (1+i)^{½} - 10{,}000 - {}_4V \qquad [2]$$

where ${}_4V$ is the reserve at the end of year 4. Now:

$${}_4V = 70{,}000 \bar{A}^{\,1}_{59:\overline{6}|} - 10{,}000 (I\bar{A})^{1}_{59:\overline{6}|} + 10{,}000\, a_{59:\overline{6}|} \qquad [1]$$

where:

$$\begin{aligned}\bar{A}^{\,1}_{59:\overline{6}|} &= (1+i)^{½} \left(A_{59} - \frac{D_{65}}{D_{59}} A_{65} \right) \\ &= 1.04^{½} \times \left(0.44258 - \frac{689.23}{924.76} \times 0.52786 \right) \\ &= 0.050136 \end{aligned} \qquad [1]$$

$$\begin{aligned}(I\bar{A})^{1}_{59:\overline{6}|} &= (1+i)^{½} \left\{ (IA)_{59} - \frac{D_{65}}{D_{59}} \left[(IA)_{65} + 6A_{65} \right] \right\} \\ &= 1.04^{½} \times \left(8.42588 - \frac{689.23}{924.76} \times [7.89442 + 6 \times 0.52786] \right) \\ &= 0.185205 \end{aligned} \qquad [1½]$$

$$a_{59:\overline{6}|} = (\ddot{a}_{59} - 1) - \frac{D_{65}}{D_{59}} (\ddot{a}_{65} - 1) = 13.493 - \frac{689.23}{924.76} \times 11.276 = 5.089 \qquad [1]$$

So:

$${}_4V = 70{,}000 \times 0.050136 - 10{,}000 \times 0.185205 + 10{,}000 \times 5.089 = 52{,}546.66 \qquad [½]$$

Hence:

$$DSAR = 70{,}000 \times 1.04^{½} - 10{,}000 - 52{,}546.66 = 8{,}839.61 \qquad [½]$$

There were 1,488 policies in force at the start of the fourth policy year, all then aged 58. [½]

So the expected death strain was:

$$EDS = 1,488 \times q_{58} \times DSAR$$

$$= 1,488 \times 0.006352 \times 8,839.61 = 83,550 \qquad [1]$$

4 policyholders died during the year, so the actual death strain was:

$$ADS = 4 \times 8,839.61 = 35,358 \qquad [½]$$

And so the mortality profit was:

$$EDS - ADS = 83,550 - 35,358 = £48,192 \qquad [½]$$

[Total 10]

Solution 2.27

(i) ***Simple bonuses***

The benefit amount on death will be £10,000 for the first year, £10,200 for the second year, £10,400 for the third year, *etc*, and the benefit amount on maturity will be £14,000.

So the premium equation can be expressed as:

$$P\ddot{a}_{[40]:\overline{20}|} = 9,800 A_{[40]:\overline{20}|} + 200(IA)^{\,1}_{[40]:\overline{20}|} + 4,200\frac{D_{60}}{D_{[40]}} + 150 + 300 A_{[40]:\overline{20}|} \qquad [2]$$

The increasing assurance function can be calculated as:

$$(IA)^{\,1}_{[40]:\overline{20}|} = (IA)_{[40]} - \frac{D_{60}}{D_{[40]}}\left[(IA)_{60} + 20A_{60}\right]$$

$$= 7.95835 - \frac{882.85}{2,052.54}\left[8.36234 + 20 \times 0.45640\right]$$

$$= 0.43531 \qquad [2]$$

So the premium equation becomes:

$$13.930P = 9{,}800 \times 0.46423 + 200 \times 0.43531 + 4{,}200 \times 0.43013$$
$$+150 + 300 \times 0.46423$$
$$= 6{,}732.33 \qquad [1]$$

Hence the premium is:

$$P = \frac{6{,}732.33}{13.930} = £483 \qquad [1]$$

[Total 6]

(ii) ***Compound bonuses***

The benefit amount on death will be £10,000 for the first year, $£10{,}000 \times 1.04$ for the second year, $£10{,}000 \times 1.04^2$ for the third year, *etc*, and the benefit amount on maturity will be $£10{,}000 \times 1.04^{20}$.

So the premium equation can be expressed as:

$$P\ddot{a}_{[40]:\overline{20}|} = \frac{10{,}000}{1.04} A^{1\ @0\%}_{[40]:\overline{20}|} + 10{,}000 \frac{D_{60}}{D_{[40]}}^{@0\%} + 150 + 300 A^{@4\%}_{[40]:\overline{20}|} \qquad [1]$$

The functions evaluated at 0% are:

$$A^{1\ @0\%}_{[40]:\overline{20}|} = \frac{l_{[40]} - l_{60}}{l_{[40]}} = 0.05755 \qquad [1]$$

and:

$$\frac{D_{60}}{D_{[40]}}^{@0\%} = \frac{l_{60}}{l_{[40]}} = 0.94245 \qquad [1]$$

So the premium equation becomes:

$$13.930P = \frac{10{,}000}{1.04} \times 0.05755 + 10{,}000 \times 0.94245 + 150 + 300 \times 0.46423$$
$$= 10{,}267 \qquad [1]$$

and the premium is:

$$P = \frac{10,267}{13.930} = £737$$ [1]

[Total 5]

Solution 2.28

The reserve V should be such that the probability of making a positive future loss should be less than 1%, *ie* such that:

$$P\left(50,000v^{K_{[30]}+1} - 0.95 \times 700 a_{\overline{K_{[30]}}|} - V > 0\right) < 0.01$$ [1]

noting that the next premium is due in one year's time, hence we use the annuity function for payments in arrears.

We need to calculate the reserve as:

$$V(r) = 50,000v^{r+1} - 0.95 \times 700 a_{\overline{r}|}$$ [½]

for a value of r such that:

$$P\left(K_{[30]} < r\right) < 0.01 \quad \text{and} \quad P\left(K_{[30]} < r+1\right) \geq 0.01$$ [1½]

Rearranging the above we require:

$$P\left(K_{[30]} \geq r\right) > 0.99 \quad \text{and} \quad P\left(K_{[30]} \geq r+1\right) \leq 0.99$$

or $\quad {}_r p_{[30]} > 0.99 \quad$ and $\quad {}_{r+1}p_{[30]} \leq 0.99$

or $\quad l_{[30]+r} > 0.99 l_{[30]} \quad$ and $\quad l_{[30]+r+1} \leq 0.99 l_{[30]}$

From the *Tables* we find $l_{[30]} = 9,923.7497$ which makes $0.99 l_{[30]} = 9,824.5122$. We then find that $l_{[30]+13} = l_{43} = 9,826.2060$ and $l_{[30]+14} = l_{44} = 9,814.3359$, which means that we take $r = 13$. [1½]

So the required reserve is:

$$V(13) = 50{,}000v^{14} - 0.95 \times 700\left(\frac{1-v^{13}}{0.03}\right) = £25{,}984 \qquad [½]$$

[Total 5]

Solution 2.29

(i) ***Premium calculation***

If P denotes the quarterly premium, the present value of the premiums is:

$$\begin{aligned}
\text{EPV Premiums} &= 4P\ddot{a}^{(4)\ @6\%}_{[40]:\overline{20}|} \\
&= 4P\left[\ddot{a}_{[40]:\overline{20}|} - \frac{3}{8}\left(1 - \frac{v^{20}l_{60}}{l_{[40]}}\right)\right] \\
&= 4P \times 11.7352 \\
&= 46.9408P
\end{aligned} \qquad [2]$$

Since $1.06/1.0192308 = 1.04$, the present value of the benefits (including the reversionary bonuses) uses 4% in the discount factor. The EPV of the benefits is:

$$\begin{aligned}
\text{EPV Benefits} &= 20{,}000\left[\frac{1.06^{½}}{1.0192308}A^{\ 1}_{[40]:\overline{20}|} + A_{[40]:\overline{20}|}^{\quad\ \ 1}\right] \\
&= 20{,}000\left[\frac{1.06^{½}}{1.0192308}\left(A_{[40]} - \frac{D_{60}}{D_{[40]}}A_{60}\right) + \frac{D_{60}}{D_{[40]}}\right] \\
&= 20{,}000\left[\frac{1.06^{½}}{1.0192308}\left(0.23041 - \frac{882.84}{2{,}052.54} \times 0.45640\right)\right. \\
&\qquad\qquad \left. + \frac{882.84}{2{,}052.54}\right] \\
&= 9{,}291.39
\end{aligned} \qquad [3]$$

Note that the claims acceleration factor $1.06^{1/2}$ *is based on the true interest rate of 6%, not the adjusted rate of 4% (which is just a trick for dealing with the bonuses). The 1.0192308 factor is required because bonuses vest at the <u>end</u> of the year, but we only adjust the death benefit part of the formula (if the policyholder survives to maturity, he will receive twenty years' bonuses).*

The present value of the expenses is:

$$\text{EPV Expenses} = 1.14P + 0.025 \times 20{,}000 + 0.04P(4\ddot{a}^{(4)}_{[40]:\overline{20|}} - 1)$$

$$= 1.14P + 500 + 0.04P \times 45.9408$$

$$= 2.9776P + 500 \qquad [2]$$

So the premium equation becomes:

$$46.9408P = 9{,}291.39 + 2.9776P + 500$$

So:

$$P = \frac{9{,}791.39}{43.9632} = £222.72 \qquad [1]$$

[Total 8]

(ii) ***Net premium reserve***

The net quarterly premium P_{net} is found from the equation of value (with functions calculated at 4% and with ultimate mortality):

$$4P_{net}\ddot{a}^{(4)}_{40:\overline{20|}} = 20{,}000\bar{A}_{40:\overline{20|}} = 20{,}000\left\{(1.02)A^{1}_{40:\overline{20|}} + \frac{D_{60}}{D_{40}}\right\} \qquad [2]$$

ie $13.713 \times 4P_{net} = 20{,}000 \times 0.46502$

$$\Rightarrow P_{net} = £169.55 \qquad [1]$$

The reserve at the end of the third policy year can then be calculated prospectively as:

$${}_3V = 20{,}600\bar{A}_{43:\overline{17|}} - 4P_{net}\ddot{a}^{(4)}_{43:\overline{17|}}$$

$$= 20{,}600 \times 0.52144 - 4 \times 169.55 \times 12.268 = £2{,}422 \qquad [2]$$

[Total 5]

Note that here we have calculated the net premium based on the original sum insured, but calculate the reserves allowing for bonuses accrued to date.

Solution 2.30

(i) ***Annual premium***

If the annual premium is P, then:

$$\begin{aligned}\text{EPV premiums} &= P\,\ddot{a}_{60:\overline{5}|} \\ &= P\left(\ddot{a}_{60} - v^5 \frac{l_{65}}{l_{60}}\ddot{a}_{65}\right) \\ &= \left(16.652 - 1.04^{-5} \times \frac{9,703.708}{9,848.431} \times 14.871\right) \\ &= 4.6087P\end{aligned}$$ [2]

$$\begin{aligned}\text{EPV benefits} &= 3,500\frac{D_{65}}{D_{60}}\left(\ddot{a}_{\overline{5}|}^{4\%} + \frac{D_{70}}{D_{65}}\ddot{a}_{70}\right) \\ &= 3,500 \times 1.04^{-5} \times \frac{9,703.708}{9,848.431} \\ &\qquad \times \left(4.6299 + 1.04^{-5} \times \frac{9,392.621}{9,703.708} \times 12.934\right) \\ &= 42,290\end{aligned}$$ [3]

So the premium equation is:

$$4.6087P = 42,290$$

So $P = £9,176$. [1]

[Total 6]

(ii) ***Prospective and retrospective reserves***

Prospective reserve after 5 years

This is equal to the present value of future benefits, *ie*:

$$3,500 \times \left(\frac{D_{70}}{D_{65}} \times \ddot{a}_{70} + \ddot{a}_{\overline{5}|}^{4\%}\right) = 3,500 \times (0.79558 \times 12.934 + 4.6299) = 52,220$$ [2]

Retrospective reserve after 5 years

This is equal to the accumulated premiums allowing for mortality, *ie*:

$$9{,}176 \times \ddot{s}_{60:\overline{5|}} = 9{,}176 \times \frac{D_{60}}{D_{65}} \ddot{a}_{60:\overline{5|}} = 9{,}176 \times 1.04^5 \frac{l_{60}}{l_{65}} \ddot{a}_{60:\overline{5|}}$$
$$= 9{,}176 \times 1.2348 \times 4.6087 = 52{,}219 \qquad [2]$$

Prospective reserve after 10 years

Prospective reserve is equal to the value of future benefits.

Hence the reserve is equal to $3{,}500 \times \ddot{a}_{70} = 3{,}500 \times 12.934 = 45{,}269$ [2]

Retrospective reserve after 10 years

The retrospective reserve is equal to the accumulated reserve less the accumulated payments, adjusted to allow for deaths between 65 and 70.

The accumulated payments equal:

$$3{,}500 \times \ddot{s}_{\overline{5|}}^{4\%} = 3{,}500 \times 5.6330 = 19{,}715 \qquad [1]$$

Therefore the reserve per surviving policy is:

$$52{,}219 \times 1.04^5 - 19{,}715 = 43{,}817 \qquad [1]$$

Adjusting by the survival function gives the reserve as:

$$43{,}817 \times \frac{l_{65}}{l_{70}} = 43{,}817 \times \frac{9{,}703.708}{9{,}392.621} = 45{,}269 \qquad [2]$$

[Total 10]

Solution 2.31

(i) ***Mortality profit***

The reserve required (per policy) at the end of the 8th year can be found from the equation of equilibrium:

$$1.04({}_7V + P) = q_{57} \times 10,000 + p_{57} \times {}_8V \qquad [1]$$

Inserting the values gives:

$$1.04(12,940 + 1,591) = 0.00995 \times 10,000 + 0.99005 \times {}_8V$$

So:

$${}_8V = 15,012.74 / 0.99005 = 15,163.62 \qquad [1]$$

The expected death strain is:

$$200q_{57}(10,000 - {}_8V) = 1.99(10,000 - 15,163.62) = -10,275.60 \qquad [1]$$

The actual death strain is:

$$3(10,000 - {}_8V) = 3(10,000 - 15,163.62) = -15,490.85 \qquad [1]$$

So the mortality profit for the year is:

$$Profit = EDS - ADS = -10,275.60 - (-15,490.85) = 5,215 \qquad [1]$$

[Total 5]

(ii) ***Comment***

In this case the reserve exceeds the death benefit, so the company makes a profit when people die. More people than expected died, so the result is a mortality profit.

[Total 1]

Solution 2.32

This question is CT5 September 2007 Question 12 (extended and with the dates changed).

(i) ***Mortality profit***

2012 is the 15th policy year so we need to calculate the death strain at risk (DSAR) in this year. This will require the reserve at time 15, which in turn requires that we first calculate the net premium from policy commencement.

The net premium can be found from the equation:

$$£75,000 A_{[40]:\overline{20|}}^{\;\;\;\;\;\;\;1} = P\ddot{a}_{[40]:\overline{20|}}$$

$$P = \frac{£75,000 v^{20} l_{60}}{\ddot{a}_{[40]:\overline{20|}}\, l_{[40]}}$$

$$= \frac{£75,000 \times 1.04^{-20} \times 9,287.2164}{13.930 \times 9,854.3036}$$

$$\Leftrightarrow \quad = £2,315.81 \qquad [1½]$$

Or, you could use $\dfrac{D_{60}}{D_{[40]}}$ *to calculate* $A_{[40]:\overline{20|}}^{\;\;\;\;\;\;\;1}$*, leading to the same answer.*

So, the reserve per policy at time 15 is:

$$_{15}V = £75,000 A_{55:\overline{5|}}^{\;\;\;\;\;1} - £2,315.81\ddot{a}_{55:\overline{5|}}$$

$$= £75,000 \times 1.04^{-5} \times \frac{9,287.2164}{9,557.8179} - £2,315.81 \times 4.585$$

$$= £49,281.26 \qquad [1]$$

So, the DSAR per policy in the 15th year is:

$$DSAR = 0 - 49,281.26$$

$$= -£49,281.26 \qquad [½]$$

The expected death strain (EDS) across the 500 policies is:

$$\begin{aligned} EDS &= 500q_{54} \times DSAR \\ &= 500 \times 0.003976 \times (-£49,281.26) \\ &= -£97,971.15 \end{aligned}$$ [1]

The actual death strain (ADS) is:

$$\begin{aligned} ADS &= 3 \times DSAR \\ &= 3 \times (-£49,281.26) \\ &= -£147,843.78 \end{aligned}$$ [1]

So, the mortality profit (MP) is:

$$\begin{aligned} MP &= EDS - ADS \\ &= -97,971.15 + 147,843.78 \\ &= £49,873 \end{aligned}$$ [1]

[Total 6]

(ii) ***Why the mortality profit has arisen***

When asked to explain why mortality profit or loss has arisen you should think about the type of policy, in this case a pure endowment. Ask yourself whether, to maximise profits for the life office, you would want more or fewer deaths than expected.

In this case, the expected number of deaths is:

$$500q_{54} = 500 \times 0.003976 = 1.988 < 3$$ [1]

and so we have more actual deaths than expected. However, this is a pure endowment so no benefit is paid upon death. This means that if more lives die than expected, the life office will make a mortality profit. [1]

[Total 2]

(iii) ***Probability of there being a mortality loss in 2013***

This will occur if we have less deaths than expected. The expected number of deaths in 2013 is:

$$497q_{55} = 497 \times 0.004469 = 2.2211$$

So, there will be a mortality loss if there are 2 or fewer deaths. [1]

The number of deaths has a binomial distribution:

$$D \sim Bin(497, 0.004469)$$ [½]

So, the probability is:

$$(1-0.004469)^{497} + 497 \times 0.004469 \times (1-0.004469)^{496}$$
$$+ \binom{497}{2} \times 0.004469^2 \times (1-0.004469)^{495}$$
$$= 0.6169$$ [1½]

[Total 3]

Part 3 – Questions

Note that the split between Development Questions and Exam-style Questions is somewhat subjective. For example, there may be some development questions that are as difficult as any exam question, but may be more repetitive than typically found in the exam; equally some of the shorter and more straightforward development questions are similar to the short questions regularly found in the CT5 exam. The Exam-style Questions generally involve more application and greater scope and are typical of the more challenging questions you will meet in the exam.

1 Development Questions

Question 3.1

The entries for age 60 in a multiple decrement table with decrements for death and retirement are as follows:

Age	$(al)_x$	$(ad)_x^d$	$(ad)_x^r$
60	10,000	200	400

If $(al)_{62} = 9{,}000$, $q_{61}^d = q_{60}^d$ and the forces of decrement are constant over each year of age, calculate the value of q_{61}^r. [6]

Question 3.2

Consider the following equations relating to a multiple decrement table with three decrements α, β and γ:

I $\quad (ap)_x = (1 - q_x^{\alpha})(1 - q_x^{\beta})(1 - q_x^{\gamma})$

II $\quad (aq)_x^{\alpha} = \int_0^1 {}_t(ap)_x^{\alpha} \; \mu_{x+t}^{\alpha} \, dt$

III $\quad \mu_x^{\alpha} = -\dfrac{d}{dx} \log(ad)_x^{\alpha}$

Which of these equations are necessarily correct, assuming that all the functions are differentiable, and that $(a\mu)_x^{\alpha} = \mu_x^{\alpha}$? [3]

Question 3.3

Consider the following assertions relating to a multiple decrement table:

I The dependent probabilities of decrement can never exceed the corresponding independent probabilities.

II Forces of decrement can never exceed 100%.

III The total of all the decrement numbers summed over all ages equals the initial radix of the table.

Which of these assertions are true? [3]

Question 3.4

The active male membership of a large pension scheme enjoys the experience of the multiple decrement table below.

Age x	Active lives $(al)_x$	Age retirements $(ad)_x^r$	Ill-health retirements $(ad)_x^i$	Withdrawals $(ad)_x^w$	Deaths $(ad)_x^d$
16	10,000	0	0	750	5
17	9,245	0	0	600	5
18	8,640	0	0	500	5
…	…	…	…	…	…
39	3,150	0	2	30	7
40	3,111	0	3	20	8
…	…	…	…	…	…
63	2,200	100	25	0	20
64	2,055	200	30	0	25
65	1,800	1,800	-	-	-

You may assume that decrements occur continuously, except for age retirements at age 65 which all occur on the 65th birthday.

Stating any other assumptions that you make, calculate the following:

(a) The probability that a man who joins the scheme on his 18th birthday, will retire after his 63rd birthday. [2]

(b) The independent probability of withdrawal between the ages of 17 and 18. [3]

(c) The expected present value of a lump sum retirement benefit of £10,000 payable on retirement at age 65, for a member now aged exactly 18, calculated using 4% *pa* interest. [1]

(d) The expected present value of a lump sum death benefit of £20,000 payable immediately on death in service after a member's 63rd birthday, for a member now aged exactly 40, using 4% *pa* interest. [2]

[Total 8]

Question 3.5

Consider the following assertions relating to the liabilities in respect of a benefit payable from a defined benefit pension scheme:

I The past service liability for the benefit is the present value of the benefit entitlement of a member who leaves on the valuation date.

II The future service liability can be calculated by subtracting the past service liability from the total service liability.

III For a new entrant at age 16, the total service liability equals the future service liability.

Which of these assertions are true? [3]

Question 3.6

A pension scheme provides pensions on normal age retirement that are a function of final salary and the number of years service accrued by the member at that date. The scheme also provides pensions on ill health retirement, equal to 50% of the pensions the members would have received if they had remained in service until normal retirement age, based on the salary they were earning at the time they became ill.

Consider the following assertions about the ill-health retirement benefit:

I This is a career average benefit.

II This is an accruing benefit.

III This is a final salary benefit.

Which of these assertions are true? [3]

Question 3.7

The index of expected earnings during the year of age x to $x+1$ used in the valuation of a pension scheme is defined by the formula

$$s_x = (100+x)^{1/2} \qquad \text{for } x = 16,17,\ldots,64$$

The corresponding index of average earnings is denoted by z_x. If the definition of final average salary is "total earnings during the 12 months immediately prior to retirement" and pay levels are reviewed on members' birthdays, find an approximate value for z_{60}.
[2]

Question 3.8

In order to value the benefits in a final salary pension scheme as at 1 January 2005, a salary scale s_x has been defined so that s_{x+t}/s_x is the ratio of a member's total earnings between ages $x+t$ and $x+t+1$ to the member's total earnings between ages x and $x+1$. Salary increases take place on 1 July every year.

Ross, whose date of birth is 1 October 1973, has an annual salary rate of £30,000 on the valuation date. Write down an expression for Ross's expected earnings during 2005. [2]

Question 3.9

On age retirement between the ages of 60 and 65 a pension fund provides a pension of one eightieth of final pensionable salary for each year of scheme service. Final pensionable salary is defined as the average annual salary earned over the 36 months prior to retirement.

Bob is now aged 40 exact and earned £35,000 over the last year.

Assuming an interest rate of 4% *pa* and that all decrements and the salary scale follow the Pension Fund Tables of the Formulae and Tables for Examinations, what is the expected present value (to the nearest £100) of Bob's future pension? [2]

Question 3.10

If retirement is not permitted before age 55, write down the relationship between the pension fund commutation functions ${}^zM_{50}^{ra}$ and ${}^zM_{55}^{ra}$. [2]

Question 3.11

An actuary has calculated a column of commutation functions for valuing a spouse's pension payable on death in service using the formula:

$$ {}^{z}C_x^{dwa} = v^{x+½} d_x z_{x+½} h_{x+½} \bar{a}'_{y+½} $$

The benefit is based on pensionable salary, which is defined to be the average salary earned during the 24 months immediately prior to death. The pension is payable for life to the spouses of men and women who have been married for at least 6 months. The pension incorporates automatic annual increases of 5% *pa* (compound).

Define all the symbols involved in this definition and state any assumptions made. [8]

Question 3.12

If ${}_nq_x = 0.3$ and ${}_nq_y = 0.5$, calculate ${}_nq_{xy}$ and ${}_nq_{\overline{xy}}$. [2]

Question 3.13

Write down a random variable that is a function of the complete future lifetime, whose variance is given by the expression $\dfrac{1}{\delta^2}\left[{}^2\bar{A}_{xy} - (\bar{A}_{xy})^2\right]$. [2]

Question 3.14

Consider the following stochastic mortality expressions:

I $E\left(\bar{a}_{\overline{T_y}|} - \bar{a}_{\overline{T_x}|}\right)$

II $E\left[\max(\bar{a}_{\overline{T_y}|} - \bar{a}_{\overline{T_x}|}, 0)\right]$

III $E\left(\bar{a}_{\overline{T_y}|} - \bar{a}_{\overline{T_{xy}}|}\right)$

State whether each of these expressions is correct for $\bar{a}_{x|y}$. [3]

Question 3.15

An actuarial student wishes to calculate $\ddot{a}^{(12)}_{68|60}$ on the following basis:

Mortality:	First life:	PMA92C20
	Second life	PFA92C20
Interest:	4% *pa*	

Calculate the value of this annuity. [2]

Question 3.16

Consider the following relationships:

I $\quad \bar{A}^{\;2}_{xy} + \bar{A}_{x\overset{2}{y}} = \bar{A}_{xy}$

II $\quad \bar{A}_{xy} - \bar{A}_{y} = \bar{A}^{\;1}_{xy}$

III $\quad \bar{A}^{\;1}_{xy} + \bar{A}^{\;2}_{xy} + \bar{A}_{x\overset{1}{y}} + \bar{A}_{x\overset{2}{y}} = \bar{A}_{\overline{xy}}$

For each of these relationships, explain whether it is correct or not. [3]

Question 3.17

The symbol $a_{\overline{60:60}}^{[1]}$ represents the present value of an annuity of 1 *pa* payable annually in arrears while exactly one of two lives aged 60 is alive. Consider the following assertions about the actuarial function $a_{\overline{60:60}}^{[1]}$, which is evaluated at a positive rate of interest and relates to lives subject to the same mortality rates:

I $\quad a_{\overline{60:60}}^{[1]}$ equals $a_{60|60}$

II $\quad a_{\overline{60:60}}^{[1]}$ is greater than $a_{\overline{60:60}}$

III $\quad a_{\overline{60:60}}^{[1]}$ represents an immediate annuity

Which of these assertions are correct? [3]

Question 3.18

(i) Explain what is meant by a dependent probability of decrement and by an independent probability of decrement. [2]

(ii) A multiple decrement table is subject to 3 modes of decrement: α, β and γ. You are given the following extract from the table:

Age, x	$(al)_x$	$(ad)_x^\alpha$	$(ad)_x^\beta$	$(ad)_x^\gamma$
50	5,000	86	52	14
51	4,848	80	56	20

(a) Calculate the probability that a 50-year old leaves the population through decrement γ between the ages of 51 and 52.

(b) Assuming that forces of decrement are constant between integer ages, calculate the independent probabilities q_{50}^α and ${}_{1|}q_{50}^\alpha$. [5]

[Total 7]

Question 3.19

Determine the values of $\ddot{a}_{65|60}$ and $\ddot{a}_{60|65}$ based on PMA92C20, PFA92C20 mortality and 4% *pa* interest, if (65) represents a male life and (60) represents a female life. [4]

Question 3.20

Arrange the following actuarial functions into ascending order by value. The functions relate to two healthy lives of the same sex subject to typical UK mortality rates and are evaluated at the same positive interest rate.

(a) $a_{60|60}$ (b) a_{60} (c) $a_{\overline{60:60}}$ (d) $a_{\overline{60:60}}^{[1]}$ (e) $a_{60:60}$

The symbol used in (d) is as defined in Question 3.17 above. [4]

Question 3.21

Describe in words what the symbol $\bar{A}_{xy}^{\;2}$ represents, and give both a stochastic mortality definition and an integral definition for it. [5]

Question 3.22

Describe in words what the symbol ${}_{\infty}q_{xy}^{1}$ represents and write down an integral expression defining it. [2]

Question 3.23

Express $\bar{A}_{xy}^{1}$ in the form $E\left[g(T_x, T_y)\right]$. [2]

Question 3.24

Two lives, each aged *x*, are subject to the same mortality table. According to the mortality table, and at a certain rate of interest, $A_x = 0.4$ and $A_{xx} = 0.6$. What is the value of $A_{xx}^{\;2}$ according to the same mortality table and interest rate? [2]

Question 3.25

Consider each of the symbols listed below:

(a) ${}_{10}q^{\;2}_{xy}$

(b) $p_{\overline{xy}}$

(c) $\bar{A}_{xy}$

(d) $\bar{A}_{\overline{xy}}$

(e) $\bar{a}_{y|x}$

Explain carefully the meaning of each of these symbols and calculate the value of each, assuming that:

- (x) is subject to a constant force of mortality of 0.01 *pa*
- (y) is subject to a constant force of mortality of 0.02 *pa*
- the lives are independent with respect to mortality
- the force of interest is 0.04 *pa*. [13]

2 Exam-style Questions

Question 3.26

A life office uses a multiple state sickness model to calculate premiums for a three-year combined sickness and endowment assurance policy issued to a healthy policyholder aged 57 at inception. Premiums are payable annually in advance for 3 years and are waived if the policyholder is sick at the time when any premium is due. At the end of each of the first 2 years a benefit of £10,000 is payable if the life is then sick.

A sum assured of £15,000 is payable at the end of the year of death if this occurs during the term of the policy or at the end of the three years if the life is alive and has never claimed any sickness benefit. The benefit payable at the end of year 3 is £10,000 if the life has previously claimed sickness benefit.

The transition probabilities are as follows (for $t = 0, 1, 2$):

$$P\left(\text{dead at } t+1 \mid \text{healthy at } t\right) = 0.02$$

$$P\left(\text{dead at } t+1 \mid \text{sick at } t\right) = 0.05$$

$$P\left(\text{sick at } t+1 \mid \text{healthy at } t\right) = 0.10$$

$$P\left(\text{sick at } t+1 \mid \text{sick at } t\right) = 0.09$$

(i) Calculate the probabilities that the policyholder aged 57 is in each state at times $t = 0, 1, 2, 3$. [3]

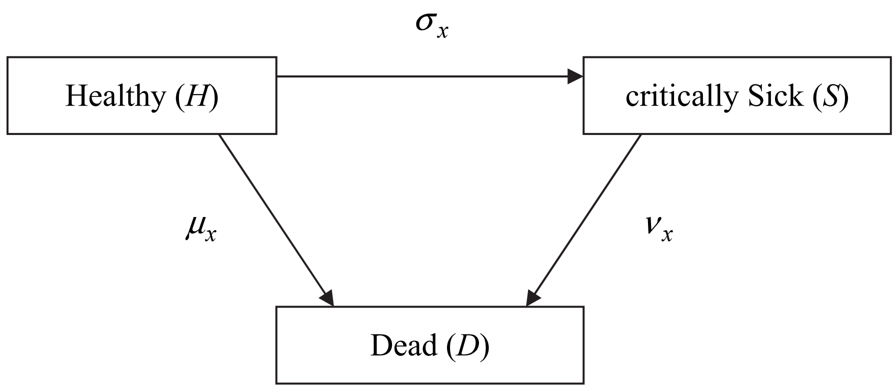
(ii) Calculate the annual premium for this policy using the equivalence principle, based on the above transition probabilities and the following additional assumptions:

Interest:	3% *pa*
Initial expenses:	£200 incurred on payment of the first premium
Renewal expenses:	£40 at times $t = 1, 2$ whether healthy or sick
Claim expense:	£30 at the date of payment of any benefit

[13]
[Total 16]

Question 3.27

An insurance company is considering the sale of a "critical illness extra" term assurance policy. The critical illness benefit is £25,000, payable immediately on diagnosis of a critical illness within the 25-year term. The death benefit is £75,000, payable immediately on death. Only one benefit is payable under any one policy and once the benefit has been paid, both the premiums and the cover cease. Annual premiums of $\pounds P\ pa$ are payable continuously. The company assesses the profitability of the policy using the following multiple state model:



$p_{x,t}^{ab}$ is defined as the probability that a life who is in state a at age x ($a = H, S, D$) is in state b at age $x + t$ ($t \geq 0$ and $b = H, S, D$).

(i) Suggest with reasons, one group of customers the insurance company may wish to target in their marketing of this policy. [1]

(ii) Express in integral form, using the probabilities and the various forces of transition, the expected present value of one such policy with an annual premium of £1,200, that has just been sold to a life aged exactly 50. [2]

After careful consideration, the company modifies the policy by changing both the death benefit and the critical illness benefit to be £50,000.

(iii) Explain how the modification could considerably reduce the cost of assessing claims. [2]

(iv) You are given $\mu_x = 0.0006$, $\nu_x = 0.03x$, $\sigma_x = 0.0014$ for all $45 < x < 70$ and the force of interest is $\delta = 4\%$. Calculate the expected present value of the benefits for the modified policy sold to a life aged exactly 45. [4]

[Total 9]

Question 3.28

On death in service before retirement, a pension scheme provides a lump sum equal to 4 times the annual rate of salary at the time of death. Salaries are revised annually on 1 July each year. The maximum normal retirement date from the scheme is the member's 65th birthday.

Develop formulae, including appropriate commutation functions, to value these benefits as at 1 July of the current year, for an existing member of the fund aged x nearest birthday ($x < 65$). Assume that salaries increase by 3% per annum, in addition to promotional increases.

State clearly any assumptions you make and define carefully all the symbols you use.

[7]

Question 3.29

Two lives aged x and y take out a policy that will pay out £15,000 on the death of (x) provided that (y) has died at least 5 years earlier and no more than 15 years earlier.

(i) Express the present value of this benefit in terms of the random variables denoting the future lifetimes of (x) and (y). [2]

(ii) Give an integral expression (in terms of single integrals only) for the expected present value of the benefit. [3]

(iii) Prove that the expected present value is equal to:

$$15{,}000\left[v^5\ {}_5p_x\ \bar{A}^{\,2}_{x+5:y} - v^{15}\ {}_{15}p_x\ \bar{A}^{\,2}_{x+15:y}\right]$$ [3]

(iv) State the appropriate premium payment term for this policy, assuming premiums are to be paid annually in advance. [2]

[Total 10]

Question 3.30

A level assurance is payable immediately on the death of a man aged 40 provided

(1) he dies after his 50th birthday, and

(2) his death occurs after, but within 15 years of, the death of another life now aged exactly 45.

Derive an expression in terms of assurance functions payable on first death for valuing this benefit, assuming that both lives are subject to the same mortality table. [7]

Question 3.31

A life insurance company issues 1,000 last survivor annuities of £5,000 *pa* payable continuously to pairs of lives aged 60. Each pair comprises one male and one female. The single premium charged is £80,000.

Calculate the expected present value of the profit to the life office and the standard deviation of the profit.

Basis: PA92C20, 4% *pa* interest. (You are given that $\bar{A}_{\overline{60:60}} = 0.20021$ at 8.16%.) [9]

Question 3.32

A multiple decrement model involves three decrements a, b and c. Decrements a and b occur continuously over $(x, x+1)$, but decrement c occurs only at age $x + ¼$. Also:

- the forces of decrement due to causes a and b are constant over $(x, x+1)$ and equal to 0.03 and 0.01 per annum respectively
- the probability of decrement by cause c at exact age $x + ¼$ is 0.06

Calculate the value of $(aq)_x^a$. [6]

Question 3.33

William, aged 75, and Laura aged 80, are the guardians of a child. They take out an annual premium life assurance policy that provides a payment of £25,000 payable immediately when the last guardian dies.

(i) State the conditions under which premiums would normally be payable. [1]

(ii) State with reasons which, if any, of the guardians' lives should be examined medically. [1]

(iii) Calculate the premium, assuming that premium payments are limited to a maximum of 10 years, the medical examinations were satisfactory and the lives are subject to PA92C20 mortality. Assume 4% *pa* interest and allow for expenses of 5% of each premium and an initial expense of £250. [9]
[Total 11]

Question 3.34

Ken and Barbie, aged 60 and 64 respectively, take out a policy under which the benefits are:

- A lump sum of £50,000 payable at the end of the year of the first death provided this occurs within 10 years.

- An annuity payable annually in advance with the first payment due to be made 10 years from the date of issue. The annuity will be of £10,000 *pa* for so long as both Ken and Barbie are still alive or £5,000 while only one of them is alive.

Level premiums are payable annually in advance for at most 10 years and will cease on the first death if this occurs earlier.

Calculate the amount of the annual premium on the following basis:

Interest: 4% *pa*

Mortality: Ken: PMA92C20
Barbie: PFA92C20

Expenses: None [8]

Question 3.35

A 65-year old male and a 62-year old female take out a joint whole life policy with sum insured £10,000 and premium payment term of 5 years. The death benefit is payable immediately on the first death. Premiums are payable monthly in advance.

(i) Show that the net monthly premium is £100. [7]

(ii) Calculate the net premium reserve after three years. [4]
[Total 11]

Basis: PA92C20 mortality, 4% *pa* interest

Question 3.36

An annuity policy provides the following benefits:

- an annuity of £20,000 *pa* payable monthly in advance. It is guaranteed to last for 10 years and continuing thereafter until the first death out of the two policyholders (a woman aged 63 exact and her husband aged 66 exact);
- a reversionary annuity of £15,000 *pa* payable monthly in advance to the surviving spouse, commencing on the next payment date following the death of the first of the lives to die, or from the 10th policy anniversary if this is later.

Calculate the net single premium for this policy assuming mortality follows that of the PA92C20 mortality tables (males or females as appropriate) rated down 3 years and using an interest rate of 4% *pa*. [11]

Question 3.37

The rules of a pension scheme require members to contribute 5% of earnings each month for a maximum of 40 years. Assuming that the interest rate, salary scales and decrements are the same as in the *Tables*, calculate:

(i) the expected present value of future contributions for a member now aged exactly 25 with 3 years of past service whose earnings during the last 12 months were £15,000. [3]

(ii) the expected present value of an ill-health retirement benefit of £1,000 *pa* payable continuously to the member in (i) above. [2]
[Total 5]

Question 3.38

A large pension scheme only permits retirement, by any cause, between the ages of 60 and 65 exact inclusive. Normal age retirement occurs on any birthday (only) over this age range, and retirement through ill health occurs at any intermediate age. Mortality is the only other cause of decrement.

(i) During a year, 750 members of the scheme pass their 59th birthdays. Calculate using the basis below:

(a) the expected number of these members who will retire through ill health between the ages of 60 and 61, and between 61 and 62 (separately)

(b) the expected number of these members retiring on their 60th and 61st birthdays.

Probability of retirement at age 60 exact:	0.3	
Probability of retirement at age 61 exact:	0.1	
Force of mortality (at all ages):	0.01	
Force of ill-health retirement (for all ages over 60):	0.05	[6]

(ii) The pension scheme provides one-eightieth of final pensionable salary for each year of service, with part years counting proportionately, for normal age retirement. For ill-health retirement the benefit is the same, except the number of years of service is taken to be the total service the member would have achieved if she had retired at her 65th birthday. All pensions are payable monthly in advance, ceasing immediately on death. Final pensionable salary is defined to be the total salary received over the year preceding the retirement date.

A member of the scheme is exactly 59 at 31 December 2005, has exactly 35 years of past service, and currently earns an annual salary of £37,000.

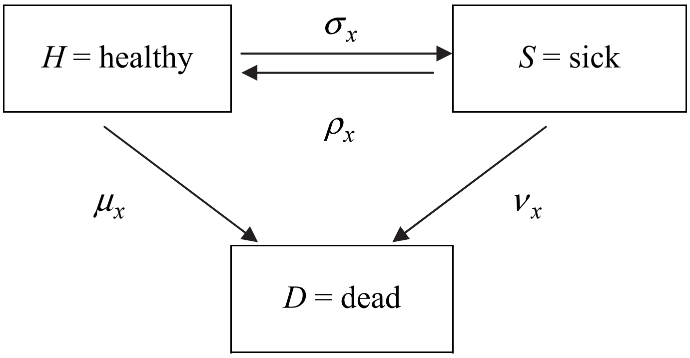
Calculate the expected present value as at 31 December 2005 of this member's ill-health and normal age retirement benefits payable as a result of all such retirements occurring up to, but excluding, retirement on the 62nd birthday. Identify any approximations you make. Use the following assumptions:

Mortality and retirement rates as in part (i)
Salaries increase continuously at 4% *pa*
Mortality post retirement:
  normal age retirement: PFA92C20
  ill-health retirement: PFA92C20 plus 8 years to the actual age
Interest: 4% *pa* [9]

[Total 15]

Question 3.39

A life insurance company uses a three-state illness-death model as shown below to calculate premiums for a 2-year sickness policy issued to healthy policyholders aged 58.



S_t denotes the state occupied by the policyholder at age $58+t$, so that $S_0 = H$ (healthy) and $S_t = H, S$ or D (healthy, sick or dead) for $t = 1, 2$.

The transition probabilities used by the insurer are defined in the following way:

$$p_{58+t}^{jk} = P\left(S_{t+1} = k \mid S_t = j\right), \qquad t = 0, 1$$

For $t = 0, 1$, it is assumed that:

$$p_{58+t}^{HH} = 0.92 \qquad p_{58+t}^{HS} = 0.05$$

$$p_{58+t}^{SH} = 0.65 \qquad p_{58+t}^{SS} = 0.25$$

The policy provides a benefit of £5,000 at the end of each year if the policyholder is then sick, and a benefit of £20,000 at the end of the year of death. Calculate the expected present value of the benefits under this policy, assuming an interest rate of 5% *pa*. [5]

Question 3.40

An employer provides the following benefits for his employees:

- immediately on death in service, a lump sum of £20,000
- immediately on withdrawal from service (other than on death or in ill health), a lump sum equal to £1,000 for each completed year of service
- immediately on leaving due to ill health, a benefit of £5,000 *pa* payable monthly in advance for 5 years certain and then ceasing, and
- on survival in service to age 65, a pension of £2,000 *pa* for each complete year of service, payable monthly in advance from age 65 for 5 years certain and life thereafter.

The forces of decrement for the employees at each age, assumed to be constant over each year of age, are as follows:

Age x	$\mu_{\bar{x}}^d$	$\mu_{\bar{x}}^i$	$\mu_{\bar{x}}^w$
62	0.018	0.10	0.020
63	0.020	0.15	0.015
64	0.023	0.20	0.010

Where $\mu_{\bar{x}}^j$ is the (assumed constant) force of decrement by cause j over $(x, x+1)$, d represents death, i ill-health retirement and w withdrawal.

(i) Construct a multiple decrement table with radix $(al)_{62} = 100,000$ to show the numbers of deaths, ill-health retirements and withdrawals at ages 62, 63 and 64, and the number remaining in employment until age 65. [5]

(ii) Calculate the expected present value of each of the above benefits for a new entrant aged exactly 62. Assume that interest is 6% *pa* before retirement and 4% *pa* thereafter, and that mortality after retirement follows the PMA92C20 table. [10]

[Total 15]

Question 3.41

Two independent lives aged x and y have complete future lifetime random variables T_x and T_y respectively, and their joint complete future lifetime random variable is T_{xy}.

(i) An annuity of 1 *pa* is continuously payable until the second death of the two lives.

Write down an expression for the present value random variable for this annuity, in terms of the random variables defined above only. [1]

(ii) A man aged 60 exact and a woman aged 65 exact wish to purchase an annuity of £25,000 *pa* payable while they are both alive. If the man dies first, the annuity will reduce to £20,000 *pa* payable for the remainder of the woman's life. If the woman dies first, the annuity will reduce to £15,000 payable for the remainder of the man's life.

(a) Write down an expression for the present value random variable of this benefit, assuming all annuities are payable continuously.

(b) Hence or otherwise, calculate the expected present value of this annuity benefit, assuming payment is actually made in monthly instalments in arrears, on the following basis:

Mortality: male life – PMA92C20
female life – PFA92C20
Interest: 4% *pa* [5]

[Total 6]

Question 3.42

A special life and health insurance policy pays the following benefits:

- £200,000 paid after diagnosis of one of a specified list of critical illnesses, provided diagnosis occurs within the policy term
- £100,000 paid on death within the policy term (provided the policyholder has not already been accepted for a critical illness claim).

On average, critical illness claims are paid 6 months after the date of diagnosis, while death claims are paid 1½ months after the date of death.

Premiums cease immediately from the date of death or critical illness diagnosis.

If no claim is made under the policy, 50% of all premiums paid are returned to the policyholder, without interest, at the expiry of the policy term. On surrender, 25% of all premiums paid are returned to the policyholder, again without interest. Surrender payments are paid 6 months after the date of the first unpaid premium.

Only one payment (critical illness claim, death claim, surrender payment or expiry payment) will be made under a single policy.

Level premiums are paid annually in advance while the policy is in force. The policy term is three years, and is to be taken out by a man aged exactly 57.

Using the equivalence principle, calculate the gross annual premium for this policy using the following basis:

- Independent probabilities of death: 60% of AM92 Select mortality rates
- Forces of decrement due to critical illness diagnosis:
  - year 1: 0.0020
  - year 2: 0.0035
  - year 3: 0.0040
- Probabilities of policy renewal:
  - 90% for policies in force at the start of year 2
  - 95% for policies in force at the start of year 3
- Interest: 3% *pa*
- Expenses:
  - initial: 20% of one annual premium plus £150
  - renewal: 1% of each premium paid at the start of years 2 and 3
  - claim: £500 per critical illness claim, £200 per death claim, and £100 per surrender and policy expiry payment, all assumed to be incurred at the time of claim payment. [20]

Question 3.43

A special 3-year temporary assurance will pay £100,000 on the first death of a man aged 60 exact and a woman aged 58 exact. The benefit is paid at the end of the year of death.

A level annual gross premium of £1,299 is paid in advance for as long as the policy remains in force.

(i) Calculate the reserve that would be held for a policy in force at the end of the first year, immediately before the second premium is paid, according to each of the following two bases:

Basis A

Reserve type:	Prospective net premium reserve
Mortality:	Male life – PMA92C20 plus five years to the actual age
	Female life – PFA92C20 plus five years to the actual age
Interest:	4% *pa*

Basis B

Reserve type:	Prospective gross premium reserve
Expenses:	£75 incurred at the start of each year

Mortality and interest as for Basis A [8]

(ii) Comment on your answers to part (i). [2]

[Total 10]

Part 3 – Solutions

Solution 3.1

With constant forces of decrement over the year of age, we can calculate q_{61}^r using:

$$q_{61}^r = 1 - e^{-\mu_{61}^r}$$

where μ_x^j is defined to be the constant force of decrement due to cause *j* over the year of age $[x, x+1]$. So we need to find the value of $\overline{\mu}_{61}^r$. We can do this using:

$$(ap)_{61} = \exp\left[-\left(\mu_{61}^r + \mu_{61}^d\right)\right]$$

$$\Rightarrow \mu_{61}^r = -\ln\left[(ap)_{61}\right] - \mu_{61}^d \qquad [1]$$

Now:

$$(ap)_{61} = \frac{(al)_{62}}{(al)_{61}} \qquad [½]$$

The value of $(al)_{62}$ is given, and we can work out $(al)_{61}$ using:

$$(al)_{61} = (al)_{60} - (ad)_{60}^d - (ad)_{60}^r = 10{,}000 - 200 - 400 = 9{,}400 \qquad [½]$$

So:

$$(ap)_{61} = \frac{9{,}000}{9{,}400} = 0.957447$$

$$\Rightarrow -\ln\left[(ap)_{61}\right] = 0.043485 \qquad [½]$$

In order to obtain μ_{61}^d we use the fact that:

$$q_{61}^d = q_{60}^d \;\Rightarrow\; \mu_{61}^d = \mu_{60}^d \qquad [½]$$

We can therefore obtain:

$$\mu_{61}^{d} = \mu_{60}^{d} = \frac{(aq)_{60}^{d}}{(aq)_{60}}\left(-\ln\left[(ap)_{60}\right]\right)$$

$$= \frac{(ad)_{60}^{d}}{(ad)_{60}}\left(-\ln\left[\frac{(al)_{61}}{(al)_{60}}\right]\right)$$

$$= \frac{200}{600}\left(-\ln\left[\frac{9,400}{10,000}\right]\right)$$

$$= 0.020625$$ [1½]

So we have:

$$\mu_{61}^{r} = 0.043485 - 0.020625 = 0.022860$$ [½]

And so finally:

$$q_{61}^{r} = 1 - e^{-0.022860} = 0.022601$$ [1]

[Total 6]

Solution 3.2

Equation I is correct (whatever the incidence of decrements occur during the year). [1]

Equation II is incorrect. The ${}_{t}(ap)_{x}^{\alpha}$ in the integral should be replaced by ${}_{t}(ap)_{x}$. [1]

Equation III is incorrect. The RHS is $\dfrac{-1}{(ad)_{x}^{\alpha}}\dfrac{d}{dx}(ad)_{x}^{\alpha}$, which cannot represent a force of decrement, since it would give a negative answer where $(ad)_{x}^{\alpha}$ increases with age, and it uses the number of exits, not the number of lives exposed, as a denominator. [1]

[Total 3]

Solution 3.3

I is true. For the dependent probability, the other decrements operate to reduce the exposure, so that a smaller proportion of lives are expected to leave by a particular cause than if the decrement was operating on its own. [1]

II is false. Forces of decrement are rates of transition per unit time, and there is no restriction on their size (other than they cannot be negative). [1]

III is true. All the active lives at any age will eventually become a decrement by one cause or another at some future age. [1]
[Total 3]

Solution 3.4

(a) The probability that an 18-year old will retire (voluntarily or through ill health) after attaining age 63 is:

$$\frac{[(ad)^r_{63}+(ad)^r_{64}+(ad)^r_{65}]+[(ad)^i_{63}+(ad)^i_{64}]}{(al)_{18}}$$

$$=\frac{(100+200+1{,}800)+(25+30)}{8{,}640}=0.249 \qquad [2]$$

(b) Assuming the force of decrement due to withdrawal is constant over the year of age 17 to 18, the independent probability of withdrawal at age 17 can be calculated from:

$$q^w_{17}=1-e^{-\mu^w_{17}}$$

where μ^j_x is defined to be the (assumed constant) force of decrement due to cause j over the year of age $[x, x+1]$.

At age 17, there are two decrements operating (withdrawal and death). Again assuming constant forces of decrement over the year of age, we can find the value of $\bar{\mu}_{17}^{w}$ from:

$$\mu_{17}^{w} = \frac{(aq)_{17}^{w}}{(aq)_{17}}\left(-\ln\left[(ap)_{17}\right]\right)$$

$$= \frac{(ad)_{17}^{w}}{(ad)_{17}}\left(-\ln\left[\frac{(al)_{18}}{(al)_{17}}\right]\right)$$

$$= \frac{600}{605}\left(-\ln\left[\frac{8,640}{9,245}\right]\right)$$

$$= 0.067121 \qquad [2]$$

Therefore:

$$q_{17}^{w} = 1 - e^{-0.067121} = 0.064918 \qquad [1]$$

(c) The expected present value is:

$$EPV = 10,000 \times \frac{1,800}{8,640} \times v^{65-18} = £329.76 \qquad [1]$$

(d) The expected present value (assuming deaths occur mid-year) is:

$$EPV = 20,000 \times \left(\frac{20}{3,111} v^{63½-40} + \frac{25}{3,111} v^{64½-40}\right) = £112.64 \qquad [2]$$

[Total 8]

Solution 3.5

I is false. The past service liability is the present value of the benefits for an *active member* based on past service. The amounts of the benefits an active member will receive will not usually be the same as for a leaver (who may get nothing). For example, the retirement pension for an active member's benefit will be based on salary at retirement, whereas the deferred pension for a leaver's benefit will be based on the member's salary at the time of leaving. Also, the probabilities of receiving the benefits will be different for active members and deferred pensioners. [1]

II is true. The total service liability must be the sum of the past and future service liabilities. [1]

III is true. A new entrant has no past service. So the past service liability must be zero, which means that the total service liability equals the future service liability. [1]
[Total 3]

Solution 3.6

I is false. The benefit amount is linked to the members' pay at the time they become ill, not on the average salary over their whole period of employment. [1]

II is false. Since the benefit is based on potential service, the period of service used in the calculation is constant. It does not increase in line with members' actual service. [1]

III is true. The benefit amount is linked to the members' pay at the time they become ill. [1]
[Total 3]

Solution 3.7

z_{60} is the value of the index of average salary for a member aged exactly 60. With the definition of final average salary given here, this corresponds to the member's total pay earned between the member's 59th and 60th birthdays, which corresponds to s_{59}.

So:

$$z_{60} = s_{59} = (100+59)^{\frac{1}{2}} = 12.61$$ [2]

Note that, in this question, the timing of pay reviews did not affect the answer.

Solution 3.8

Salaries change only on the 1 July of each year, so a rate of £30,000 at 1 January 2005 must mean that the person is paid an amount of £30,000 from 1 July 2004 to 1 July 2005. This corresponds to $s_{30.75}$. Earnings from 1 July 2005 to 1 July 2006 correspond to $s_{31.75}$. So the answer is:

$$\frac{30,000}{2}\left[1+\frac{s_{31.75}}{s_{30.75}}\right] \qquad [2]$$

Alternative solution

We are given a salary rate, which is no use: we need to convert it into a salary amount. Salaries change only on the 1 July of each year, so a rate of £30,000 at 1 January 2005 must mean that the person is paid an amount of £30,000 from 1 July 2004 to 1 July 2005. This corresponds to $s_{30.75}$. *Earnings over the year 2005 correspond to* $s_{31.25}$. *So the answer is* $30,000\,\dfrac{s_{31.25}}{s_{30.75}}$. *[2]*

Solution 3.9

The normal formula for valuing future service liability for such a pension arrangement would be:

$$\frac{1}{80}35,000\,\frac{{}^{z}\bar{R}^{ra}_{40}}{{}^{s}D_{40}}$$

However this formula presupposes that the earnings of £35,000 correspond to s_{40}, *ie* earnings from age 40 to age 41. Here, we are given earnings from age 39 to age 40, so we require a salary index of s_{39} in the denominator. So the expression becomes:

$$\frac{1}{80}35,000\,\frac{{}^{z}\bar{R}^{ra}_{40}}{s_{39}\,D_{40}} \qquad [1]$$

$$=\frac{35,000}{80}\times\frac{2,884,260}{7.623\times 3,207}=£51,600 \qquad [1]$$

[Total 2]

Solution 3.10

Since retirement is not permitted before age 55:

$$^{z}C_{50}^{ra} = {}^{z}C_{51}^{ra} = {}^{z}C_{52}^{ra} = {}^{z}C_{53}^{ra} = {}^{z}C_{54}^{ra} = 0 \quad [1]$$

So $^{z}M_{50}^{ra}$ and $^{z}M_{55}^{ra}$ are equal. [1]

[Total 2]

Solution 3.11

$^{z}C_{x}^{dwa}$ is the commutation function used to value the spouse's death in service pension for a member aged x nearest birthday on the valuation date. [1]

x represents a member's exact age. [½]

v is calculated as $v = 1/(1+i)$ where i is the valuation interest rate. [½]

d_x is the number of deaths, according to the service table, between ages x and $x+1$. [1]

z_x is an index corresponding to members' pensionable salary at exact age x. It is calculated as $z_x = ½(s_{x-2} + s_{x-1})$, where s_x is an index of members' salary earned between ages x and $x+1$. [1]

h_x is the proportion of members with a spouse to whom they have been married for at least 6 months. [1]

y is the average age of the spouses of members aged exactly x. [1]

$\bar{a}'_{y+½}$ is an annuity factor representing the present value of a pension payable to a spouse at the rate of 1 unit *pa*, calculated at the valuation interest rate, using spouses' mortality rates and incorporating automatic annual increases of 5% *pa* (compound). [1]

It is assumed that deaths take place on average in the middle of the year of age. [½]

It is assumed that the average age of members aged x nearest birthday is x. [½]

[Total 8]

Solution 3.12

$${}_nq_{xy} = 1 - {}_np_{xy} = 1 - {}_np_x\,{}_np_y = 1 - \left(1 - {}_nq_x\right)\left(1 - {}_nq_y\right) = 1 - 0.7 \times 0.5 = 0.65 \qquad [1]$$

$${}_nq_{\overline{xy}} = {}_nq_x\;{}_nq_y = 0.15 \qquad [1]$$

[Total 2]

Solution 3.13

The formula given is the variance of the present value of an annuity payable continuously during the joint lives of (x) and (y). So the required random variable is $\bar{a}_{\overline{\min(T_x,T_y)}|}$. [2]

Solution 3.14

I is incorrect. If (y) is subject to heavier mortality than (x), it will give a negative answer. [1]

II is correct. $\bar{a}_{x|y}$ is the expected present value of annuity payable during the future lifetime of (y), but not during the future lifetime of (x), provided this is greater than zero. [1]

III is correct. Using the joint life in the second annuity ensures that the expression inside the expectation cannot be negative. [1]

So **II** and **III** are correct. [Total 3]

Solution 3.15

We can write:

$$\ddot{a}^{(12)}_{68|60} = \ddot{a}^{(12)}_{60} - \ddot{a}^{(12)}_{68:60} \approx \ddot{a}_{60} - \ddot{a}_{68:60} \qquad [1]$$

where the 60-year old is female and the 68-year old is male.

From the *Tables* and interpolating, we get:

$$\ddot{a}_{60} - \ddot{a}_{68:60} = 16.652 - \left\{\tfrac{2}{5}\times 11.372 + \tfrac{3}{5}\times 11.849\right\} = 4.994 \qquad [1]$$

[Total 2]

Solution 3.16

I is incorrect. The LHS is the value of a payment payable on the second death, whichever life that is. The RHS is the value of a payment payable on first death. [1]

Note that the "2" superscript in $\bar{A}^{2}_{xy}$ *indicates that payment is made at the time of the second death, provided it is* (x) *who dies second.*

II is incorrect. If (y) dies second, the first term on the LHS would make no payment and the second term would make a payment of 1, making a total of –1 for the LHS. However, the term on the RHS would not make a payment. So the two sides do not match in this case. [1]

III is incorrect. The LHS is the value of a payment payable on the first death and the second death. The RHS is the value of a payment payable on second death only. [1]

Solution 3.17

I is incorrect. In fact, $a_{\overset{[1]}{\overline{60:60}}}$ equals $2\,a_{60|60}$, since the order of the lives in the first function is not specified. [1]

II is incorrect. The conditions for payment of $a_{\overset{[1]}{\overline{60:60}}}$ are more restrictive. So it has a smaller value. [1]

III is incorrect. Payments do not start until one of the lives has died. So it is a deferred annuity. [1]

So none of the assertions are correct. [Total 3]

Solution 3.18

(i) ***Dependent and independent probabilities***

A dependent probability of decrement takes into account the action of other decrements operating on the population. For example, the dependent probability $(aq)_x^\alpha$ is the probability that a life aged x will leave the active population through decrement α before age $x+1$, while all other decrements are operating. [1]

An independent probability of decrement is a purely theoretical quantity that assumes there are no other decrements operating. For example, q_x^α is the probability that a life aged x will leave the active population through decrement α before age $x+1$, where α is the only decrement operating. [1]

[Total 2]

(ii)(a) ***Probability***

The probability that a 50-year old member of the population leaves through decrement γ between the ages of 51 and 52 is:

$$\frac{(ad)_{51}^\gamma}{(al)_{50}} = \frac{20}{5{,}000} = 0.004 \qquad [1]$$

(ii)(b) ***Calculation of independent probabilities***

Since the forces of decrement are constant over each year of age, we have:

$$q_x^\alpha = 1 - e^{-\mu_{\bar{x}}^\alpha}$$

where $\mu_{\bar{x}}^j$ is defined to be the (assumed constant) force of decrement due to cause j over the year of age $[x, x+1]$.

We can find the value of $\mu_{\bar{x}}^\alpha$ from:

$$\mu_{\bar{x}}^\alpha = \frac{(aq)_x^\alpha}{(aq)_x}\left(-\ln\left[(ap)_x\right]\right)$$

$$= \frac{(ad)_x^\alpha}{(ad)_x}\left(-\ln\left[\frac{(al)_{x+1}}{(al)_x}\right]\right) \qquad [1]$$

So:

$$\mu_{50}^{\alpha} = \frac{86}{86+52+14}\left(-\ln\left[\frac{4,848}{5,000}\right]\right)$$

$$= 0.017467$$

$$\Rightarrow q_{50}^{\alpha} = 1 - e^{-0.017467}$$

$$= 0.017315 \qquad [1]$$

and:

$$\mu_{51}^{\alpha} = \frac{80}{80+56+20}\left(-\ln\left[\frac{4,848-(80+56+20)}{4,848}\right]\right)$$

$$= 0.016773$$

$$\Rightarrow q_{51}^{\alpha} = 1 - e^{-0.016773}$$

$$= 0.016633 \qquad [1]$$

From these we can then calculate:

$$_{1|}q_{50}^{\alpha} = \left(1 - q_{50}^{\alpha}\right) \times q_{51}^{\alpha} = (1 - 0.017315) \times 0.016633 = 0.016345 \qquad [1]$$

[Total 5]

Solution 3.19

$$\ddot{a}_{65|60} = \ddot{a}_{60}^{f} - \ddot{a}_{65:60}^{m\ f} = 16.652 - 12.682 = 3.97 \qquad [2]$$

$$\ddot{a}_{60|65} = \ddot{a}_{65}^{m} - \ddot{a}_{65:60}^{m\ f} = 13.666 - 12.682 = 0.984 \qquad [2]$$

[Total 4]

Solution 3.20

Assuming a positive interest rate, we can make the following observations:

1. (c) has the highest value since it starts immediately and is paid until the second life dies. [1]

2. Including more lives in a joint life annuity reduces the value (but not by very much at age 60). So (e) < (b). [1]

3. The value of (d) is twice the value of (a) since $a_{\overline{xy}}^{[1]} = a_{x|y} + a_{y|x}$. The value of these functions is quite small since they represent annuities that are deferred typically for 15 or 20 years and they are paid during the lifetime of an older life. [1]

So the order is:

$$(a)\ a_{60|60} < (d)\ a_{\overline{60:60}}^{[1]} < (e)\ a_{60:60} < (b)\ a_{60} < (c)\ a_{\overline{60:60}} \quad [1]$$

[Total 4]

Solution 3.21

This symbol represents the expected present value of a payment of 1 unit made immediately on the death of (y), provided (y) dies after (x). [1]

A stochastic mortality definition would be:

$$\bar{A}_{x\,y}^{\ 2} = E\left[g(T_x, T_y)\right] \text{ where } \quad g(T_x, T_y) = \begin{cases} v^{T_y} & \text{if } T_y > T_x \\ 0 & \text{if } T_y \le T_x \end{cases} \quad [2]$$

An integral definition would be:

$$\bar{A}_{x\,y}^{\ 2} = \int_0^\infty v^t\, {}_tq_x\, {}_tp_y \mu_{y+t}\, dt \quad [2]$$

[Total 5]

Solution 3.22

${}_\infty q_{xy}^1$ represents the probability that (x) dies before (y). [1]

It can be defined by the integral:

$${}_\infty q_{xy}^1 = \int_0^\infty {}_tp_{xy}\, \mu_{x+t}\, dt \quad [1]$$

[Total 2]

Solution 3.23

$\overline{A}^{1}_{xy}$ can be expressed in the form $E\left[g(T_x,T_y)\right]$ where the function $g(T_x,T_y)$ is defined by:

$$g(T_x,T_y) \quad = \quad \begin{cases} v^{T_x} & \text{if } T_x < T_y \\ 0 & \text{if } T_x \geq T_y \end{cases} \qquad [2]$$

Solution 3.24

$$A_{x\overset{2}{x}} + A^{2}_{xx} = A_{\overline{xx}} = A_x + A_x - A_{xx} = 2A_x - A_{xx} = 0.2 \qquad [1]$$

But $A_{x\overset{2}{x}} = A^{2}_{xx}$ by symmetry, so we must have

$$A_{x\overset{2}{x}} = \tfrac{1}{2} A_{\overline{xx}} = 0.1 \qquad [1]$$

[Total 2]

Alternatively:

$$A_{x\overset{2}{x}} = \int_0^\infty {}_t p_x \, \mu_{x+t} \, (1 - {}_t p_x) \, dt$$

$$= \int_0^\infty {}_t p_x \, \mu_{x+t} \, dt - \int_0^\infty {}_t p_{xx} \, \mu_{x+t} \, dt$$

$$= A_x - \tfrac{1}{2} \int_0^\infty {}_t p_{xx} \, \mu_{x+t:x+t} \, dt$$

$$= A_x - \tfrac{1}{2} A_{xx} = 0.4 - 0.3 = 0.1$$

Solution 3.25

(a) ${}_{10}q^{\;2}_{x\,y}$ is the probability that (x) dies after (y) but within 10 years. [1]

$$
\begin{aligned}
{}_{10}q^{\;2}_{x\,y} &= \int_0^{10} {}_tp_x\,\mu_{x+t}\;{}_tq_y\,dt \\
&= \int_0^{10} 0.01e^{-0.01t}\left(1-e^{-0.02t}\right)dt \\
&= 0.01\int_0^{10}\left(e^{-0.01t}-e^{-0.03t}\right)dt \\
&= \left[-e^{-0.01t}+\frac{1}{3}e^{-0.03t}\right]_0^{10} \\
&= -e^{-0.1}+\frac{1}{3}e^{-0.3}+1-\frac{1}{3} \\
&= 0.008769
\end{aligned}
$$

[2]

[Total 3]

(b) $p_{\overline{xy}}$ is the probability that at least one of (x) and (y) is alive in one year's time. [1]

$$p_{\overline{xy}} = 1-q_{\overline{xy}} = 1-q_x\,q_y = 1-\left(1-e^{-0.01}\right)\left(1-e^{-0.02}\right) = 0.999803$$ [1]

[Total 2]

(c) $\bar{A}_{xy}$ is the expected present value of a benefit of 1 payable immediately on the failure of the joint life status xy. So the benefit is paid immediately upon the first death. [1]

$$
\begin{aligned}
\bar{A}_{xy} &= \int_0^{\infty} v^t\,{}_tp_{xy}\,\mu_{x+t:y+t}\,dt \\
&= \int_0^{\infty} e^{-0.04t}e^{-0.03t}\,0.03\,dt \\
&= 0.03\int_0^{\infty} e^{-0.07t}\,dt \\
&= 0.03\left[\frac{-e^{-0.07t}}{0.07}\right]_0^{\infty} = \frac{3}{7}
\end{aligned}
$$

[1]

[Total 2]

(d) $\bar{A}_{\overline{xy}}$ is the expected present value of a benefit of 1 payable immediately on the failure of the last survivor status $\overline{xy}$. So the benefit is paid immediately upon the second death. [1]

We can write:

$$\bar{A}_{\overline{xy}} = \bar{A}_x + \bar{A}_y - \bar{A}_{xy}$$ [½]

where:

$$\bar{A}_x = \int_0^\infty v^t \, {}_tp_x \, \mu_{x+t} \, dt = 0.01\int_0^\infty e^{-0.05t} \, dt = 0.01\left[\frac{-e^{-0.05t}}{0.05}\right]_0^\infty = \frac{1}{5}$$ [½]

$$\bar{A}_y = \int_0^\infty v^t \, {}_tp_y \, \mu_{y+t} \, dt = 0.02\int_0^\infty e^{-0.06t} \, dt = 0.02\left[\frac{-e^{-0.06t}}{0.06}\right]_0^\infty = \frac{1}{3}$$ [½]

So:

$$\bar{A}_{\overline{xy}} = \frac{1}{5} + \frac{1}{3} - \frac{3}{7} = \frac{11}{105} = 0.104762$$ [½]

[Total 3]

(e) $\bar{a}_{y|x}$ is the expected present value of a reversionary annuity, payable continuously at the rate of 1 *pa*, following the death of (y). [1]

$$\bar{a}_{y|x} = \bar{a}_x - \bar{a}_{xy} = \frac{1-\bar{A}_x}{\delta} - \frac{1-\bar{A}_{xy}}{\delta} = \frac{\bar{A}_{xy} - \bar{A}_x}{\delta} = \frac{3/7 - 1/5}{0.04} = 5.7143$$ [2]

[Total 3]

Solution 3.26

(i) ***Probabilities***

We can represent the transition probabilities by the matrix:

$$\begin{array}{c} \\ \text{H} \\ \text{S} \\ \text{D} \end{array}\begin{array}{c} \begin{array}{ccc} \text{H} & \text{S} & \text{D} \end{array} \\ \begin{pmatrix} 0.88 & 0.10 & 0.02 \\ 0.86 & 0.09 & 0.05 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

The vector of probabilities of being in the states (H, S, D) at time 0 is (1, 0, 0). [½]

We can multiply the probability vector at time t by the transition matrix to get the probability vector at time $t+1$.

This gives probabilities as follows:

$t=1$ (0.8800, 0.1000, 0.0200) [½]

$t=2$ (0.8604, 0.0970, 0.0426) [1]

$t=3$ (0.8406, 0.0948, 0.0647) [1]

[Total 3]

(ii) ***Premium***

Expected present value of death benefit and death claim expenses

This is:

$$EPV[D] = 15,030 \times \left[{}_1p_{57}^{HD} v + \left({}_2p_{57}^{HD} - {}_1p_{57}^{HD} \right) v^2 + \left({}_3p_{57}^{HD} - {}_2p_{57}^{HD} \right) v^3 \right]$$

$$= 15,030 \times \left[\frac{0.02}{1.03} + \frac{(0.0426 - 0.02)}{1.03^2} + \frac{(0.0647 - 0.0426)}{1.03^3} \right]$$

$$= 916.00$$ [3]

Expected present value of sickness benefit and sickness claim expenses

This is:

$$EPV[S] = 10,030 \times \left[{}_1p_{57}^{HS} v + {}_2p_{57}^{HS} v^2 \right]$$

$$= 10,030 \times \left[\frac{0.1}{1.03} + \frac{0.097}{1.03^2} \right]$$

$$= 1,890.85 \qquad [2]$$

Expected present value of maturity benefit and maturity claim expenses

A benefit of (at least) 10,000 is paid, and the expense of 30 incurred, if the policyholder is alive at time 3 (*ie* healthy or sick). A further 5,000 is paid if the policyholder is alive at that point *and* was healthy at each of times 1 and 2. So:

$$EPV[M] = 10,030\left(1 - {}_3p_{57}^{HD}\right)v^3 + 5,000\,{}_1p_{57}^{HH}\,{}_1p_{58}^{HH}\left(1 - {}_1p_{59}^{HD}\right)v^3$$

$$= \frac{10,030 \times (1 - 0.0647) + 5,000 \times 0.88^2 \times (1 - 0.02)}{1.03^3}$$

$$= 12,057.56 \qquad [3]$$

Expected present value of other expenses

This is:

$$EPV[E] = 200 + 40 \times \left[\left(1 - {}_1p_{57}^{HD}\right)v + \left(1 - {}_2p_{57}^{HD}\right)v^2 \right]$$

$$= 200 + 40 \times \left[\frac{(1 - 0.02)}{1.03} + \frac{(1 - 0.0426)}{1.03^2} \right]$$

$$= 274.16 \qquad [2]$$

Expected present value of premiums

Using P for the annual premium, this is:

$$EPV[P] = P \times \left[1 + {}_1p_{57}^{HH} v + {}_2p_{57}^{HH} v^2\right]$$

$$= P \times \left[1 + \frac{0.88}{1.03} + \frac{0.8604}{1.03^2}\right]$$

$$= P \times 2.66538 \qquad [2]$$

Equating the EPV of the premiums with the EPV of the benefits and expenses, and solving for P, we obtain:

$$P = \frac{916.00 + 1{,}890.85 + 12{,}057.56 + 274.16}{2.66538}$$

$$= 5{,}679.70 \qquad [1]$$

[Total 13]

Solution 3.27

(i) ***Possible customers***

This policy would be suitable for anybody with a family who has recently taken out a 25-year mortgage. If the policyholder dies then their family would have some money to reduce the outstanding balance on the mortgage. If the policyholder becomes critically ill then they would receive some money to either pay or considerably reduce the outstanding balance on the mortgage at a time of financial distress. [1]

(ii) ***Expected present value of the policy***

The expected present value is:

$$1200\int_0^{25} e^{-\delta t}\, p_{50,t}^{HH}\, dt - 75{,}000\int_0^{25} e^{-\delta t}\, p_{50,t}^{HH}\, \mu_{x+t}\, dt - 25{,}000\int_0^{25} e^{-\delta t}\, p_{50,t}^{HH}\, \sigma_{x+t}\, dt$$

$$= \int_0^{25} e^{-\delta t}\, p_{50,t}^{HH} \left(1200 - 75{,}000\mu_{x+t} - 25{,}000\sigma_{x+t}\right) dt \qquad [2]$$

(iii) ***Reducing the cost of underwriting claims***

The main problem with this policy is that the death benefit is three times as large as the critical illness benefit. Families of policyholders who become critically ill and then die may try (possibly fraudulently) to claim the death benefit instead of the critical illness benefit. The company would need to ensure that death claims are genuine in that the policyholder was not already critically ill when they died. This would involve running certain checks as part of the claims underwriting procedure and would cost money. If the death benefit is equal to the critical illness benefit then there is no need for these extra checks and hence costs. [2]

(iv) ***Expected present value of the benefits***

The rate of death, from being critically ill, ν_x is irrelevant since, once a policyholder has become critically ill, the period of cover has expired.

In fact, the transition from state S to state D is an unnecessary complication of the model for this whole question: a double decrement model with decrements of S and D would have been perfectly adequate.

The probability that a 45-year old remains alive for t years is:

$$p_{45,t}^{HH} = \exp\left\{-\int_{45}^{45+t} \left(\mu_s + \sigma_s\right) ds\right\} = \exp\left\{-\int_{45}^{45+t} 0.002 ds\right\}$$

$$= \exp\left\{-\left[0.002s\right]_{45}^{45+t}\right\}$$

$$= \exp\left(-0.002t\right) \qquad [2]$$

The expected present value of the benefits is:

$$50,000\int_0^{25} e^{-\delta t} p_{45,t}^{HH}(\mu_{x+t} + \sigma_{x+t})dt = 50,000\int_0^{25} e^{-0.04t}e^{-0.002t}(0.0006 + 0.0014)dt$$

$$= 50,000\int_0^{25} 0.002e^{-0.042t} dt$$

$$= \frac{100}{0.042} \times \left[-e^{-0.042t}\right]_0^{25}$$

$$= £1,548 \qquad [2]$$

[Total 4]

Solution 3.28

We need an appropriate service table for active members that includes the following entries:

l_x represents the number of lives in the service table who are at exact age x.

d_x represents the number of lives in the service table who are assumed to die between exact ages x and $x+1$. [½]

Deaths are assumed to take place on average in the middle of the year of age. [½]

We need an appropriate salary scale in which:

s_x is an index of the salary earned by a member between exact ages x and $x+1$. [½]

In this case since we need to allow for salary increases of 3% *pa* as well as promotional increases, it is therefore convenient to build this into our s_x function by taking:

$s_x = 1.03^x \times s'_x$, where s'_x is an index relating to promotional increases only [1]

We also need the following:

$v = \dfrac{1}{1+i}$, where i is the valuation interest rate

S is the expected earnings for a member during the year starting on the valuation date. [½]

We assume that the member is aged exactly x on the valuation date. [½]

If the member dies between ages y and $y+1$, the benefit amount will be:

$$4S\frac{s_y}{s_x}$$ [½]

because it is based on the current annual rate of salary.

The value of this benefit payment is:

$$4S\frac{s_y}{s_x}v^{y-x+½}\frac{d_y}{l_x} = 4S\frac{s_y v^{y+½} d_y}{s_x v^x l_x} = 4S\frac{{}^sC_y^d}{{}^sD_x} \qquad [1]$$

where ${}^sC_y^d = s_y v^{y+½} d_y$ [½]

Because death can occur during any of the years starting at ages x to 64 inclusive, the value of the benefit for death at any age is:

$$4S\left[\frac{{}^sC_x^d}{{}^sD_x} + \frac{{}^sC_{x+1}^d}{{}^sD_x} + \ldots + \frac{{}^sC_{64}^d}{{}^sD_x}\right] = 4S\frac{{}^sM_x^d}{{}^sD_x} \qquad [1]$$

where ${}^sM_x^d = \sum_{y=x}^{64} {}^sC_y^d$ [½]

[Total 7]

Solution 3.29

(i) ***Present value random variable***

Using the random variables T_x and T_y to denote the future lifetimes of (x) and (y), the present value of the benefit is some random variable $Z_{x,y}$ defined as:

$$Z_{x,y} = \begin{cases} 15,000v^{T_x} & \text{if } T_y + 5 < T_x < T_y + 15 \\ 0 & \text{otherwise} \end{cases}$$

[Total 2]

(ii) ***Integral expression for expected present value***

The expected present value of the benefit is:

$$E\left[Z_{x,y}\right] = 15,000\left\{\int_5^{15} v^t\ {}_tp_x\ \mu_{x+t}\ {}_{t-5}q_y\,dt + \int_{15}^{\infty} v^t\ {}_tp_x\mu_{x+t}\left({}_{t-5}q_y - {}_{t-15}q_y\right)dt\right\}$$

[Total 3]

This can also be written as:

$$E\left[Z_{x,y}\right]=15,000\left\{\int_{5}^{\infty} v^{t}\,{}_{t}p_{x}\,\mu_{x+t}\;{}_{t-5}q_{y}\,dt-\int_{15}^{\infty} v^{t}\,{}_{t}p_{x}\,\mu_{x+t}\;{}_{t-15}q_{y}\,dt\right\}$$

(iii) ***Proof***

Consider the first of the integrals in the expression above. Making the substitution $t=s+5$, we see that:

$$\int_{5}^{\infty} v^{t}\,{}_{t}p_{x}\,\mu_{x+t}\;{}_{t-5}q_{y}\,dt=\int_{0}^{\infty} v^{s+5}\;{}_{s+5}p_{x}\,\mu_{x+s+5}\;{}_{s}q_{y}\,ds$$

$$=v^{5}\;{}_{5}p_{x}\int_{0}^{\infty} v^{s}\;{}_{s}p_{x+5}\,\mu_{x+s+5}\;{}_{s}q_{y}\,ds$$

$$=v^{5}\;{}_{5}p_{x}\,\bar{A}^{2}_{x+5:y}$$ [2]

Similarly:

$$\int_{15}^{\infty} v^{t}\,{}_{t}p_{x}\,\mu_{x+t}\;{}_{t-15}q_{y}\,dt=v^{15}\;{}_{15}p_{x}\,\bar{A}^{2}_{x+15:y}$$

Hence:

$$E\left[Z_{x,y}\right]=15,000\left[v^{5}\;{}_{5}p_{x}\,\bar{A}^{2}_{x+5:y}-v^{15}\;{}_{15}p_{x}\,\bar{A}^{2}_{x+15:y}\right]$$ [1]

[Total 3]

(iv) ***Premium payment term***

For a premium payment that makes any sense we require (x) to be alive (since the benefit is paid on (x)'s death) and either (y) to be alive, or (y) to have died no more than 15 years ago. [Total 2]

Solution 3.30

The expected present value of the benefit per unit sum assured is:

$$EPV = \int_{10}^{15} v^t \; {}_tp_{40}\mu_{40+t}(1-{}_tp_{45})\,dt + \int_{15}^{\infty} v^t \; {}_tp_{40}\mu_{40+t}({}_{t-15}p_{45} - {}_tp_{45})\,dt$$

$$= \int_{10}^{15} v^t \; {}_tp_{40}\mu_{40+t}\,dt - \int_{10}^{\infty} v^t \; {}_tp_{40}\mu_{40+t}\;{}_tp_{45}\,dt + \int_{15}^{\infty} v^t \; {}_tp_{40}\mu_{40+t}\;{}_{t-15}p_{45}\,dt$$

[3]

Using the substitutions $s = t-10$ in the first two integrals and $s = t-15$ in the third gives:

$$EPV = v^{10}\;{}_{10}p_{40}\int_0^5 v^s\;{}_sp_{50}\;\mu_{50+s}\,ds - v^{10}\;{}_{10}p_{40:45}\int_0^{\infty} v^s\;{}_sp_{50}\;\mu_{50+s}\;{}_sp_{55}\,ds$$

$$+v^{15}\;{}_{15}p_{40}\int_0^{\infty} v^s\;{}_sp_{55}\;\mu_{55+s}\;{}_sp_{45}\,ds \qquad [2]$$

This can be written in terms of assurance functions as:

$$EPV = v^{10}\;{}_{10}p_{40}\;\bar{A}^{\,1}_{50:\overline{5}|} - v^{10}\;{}_{10}p_{40:45}\;\bar{A}^{\,1}_{50:55} + v^{15}\;{}_{15}p_{40}\;\bar{A}^{\,1}_{55:45} \qquad [2]$$

[Total 7]

Solution 3.31

If $T_{\overline{xy}}$ is a random variable representing the time to failure of the last survivor status $\overline{xy}$ then the present value of the profit, X, on a single policy is:

$$X = 80,000 - 5,000\,\bar{a}_{\overline{T_{\overline{xy}}}|} \qquad [1]$$

The expected value of X is:

$$E[X] = E\left[80,000 - 5,000\,\bar{a}_{\overline{T_{\overline{xy}}}|}\right]$$

$$= 80,000 - 5,000\,E\left[\bar{a}_{\overline{T_{\overline{xy}}}|}\right]$$

$$= 80,000 - 5,000\,\bar{a}_{\overline{xy}} \qquad [1]$$

Here x and y are both equal to 60, and we can calculate the last survivor annuity from:

$$\bar{a}_{\overline{60:60}} = \bar{a}^m_{60} + \bar{a}^f_{60} - \bar{a}_{60:60} = 15.132 + 16.152 - 13.590 = 17.694 \qquad [1]$$

So, the expected value of the present value of the profit on a single policy is:

$$80,000 - 5,000 \times 17.694 = -£8,470 \qquad [½]$$

Assuming that the lives are independent with respect to mortality, the expected value of the present value of the profit on the block of 1,000 policies is –£8,470,000. [½]

We can calculate the variance of X from:

$$\begin{aligned}
\text{var}[X] &= \text{var}\left[80,000 - 5,000\,\bar{a}_{\overline{T_{\overline{xy}}}|}\right] \\
&= 5,000^2\ \text{var}\left[\frac{1 - v^{T_{\overline{xy}}}}{\delta}\right] \\
&= \left(\frac{5,000}{\delta}\right)^2 \text{var}\left[v^{T_{\overline{xy}}}\right] \qquad [1]
\end{aligned}$$

and $\text{var}\left[v^{T_{\overline{xy}}}\right] = {}^2\bar{A}_{\overline{xy}} - (\bar{A}_{\overline{xy}})^2$. [1]

Using the premium conversion relationship $\bar{A}_{\overline{xy}} = 1 - \delta\,\bar{a}_{\overline{xy}}$ we obtain:

$$\bar{A}_{\overline{60:60}} = 1 - \ln 1.04 \times 17.694 = 0.30603 \qquad [1]$$

We are given that ${}^2\bar{A}_{\overline{60:60}} = 0.20021$, so the variance of the present value of the profit for a single policy is:

$$\frac{5,000^2}{\delta^2}(0.20021 - 0.30603^2) = 1,731,764,026 \qquad [1]$$

The variance of the present value of the profit on the block of 1,000 policies is then:

$$1,000 \times 1,731,764,026$$

assuming independent lives.

So the standard deviation of the total profit is:

$$\sqrt{1,000\times 1,731,764,026} = £1,315,965 \qquad [1]$$

[Total 9]

Solution 3.32

Functions such as ${}_t(ap)_x{}'$ and ${}_t(aq)_x{}'$ will be used to represent dependent probabilities that *ignore* decrement c. We can then write:

$$(aq)_x^a = {}_{1/4}(aq)_x^{a\,\prime} + {}_{1/4}(ap)_x{}'\left(1-q_{x+1/4}^c\right){}_{3/4}(aq)_{x+1/4}^{a}{}' \qquad [2]$$

Now for $0 \le r < 1$ and $0 < t \le 1-r$:

$${}_t(ap)_{x+r}{}' = \exp\left[-\left(\mu^a+\mu^b\right)t\right]$$

$$= e^{-0.04t} \qquad [1]$$

and:

$${}_t(aq)_{x+r}^a{}' = \frac{\mu^a}{\mu^a+\mu^b}\left(1-{}_t(ap)_{x+r}{}'\right)$$

$$= \frac{0.03}{0.04}\times\left(1-e^{-0.04t}\right) \qquad [1]$$

So:

$${}_{1/4}(aq)_x^{a\,\prime} = 0.75\left(1-e^{-0.04\times 0.25}\right)$$

$${}_{1/4}(ap)_x{}' = e^{-0.04\times 0.25}$$

$${}_{3/4}(aq)_{x+1/4}^a{}' = 0.75\left(1-e^{-0.04\times 0.75}\right) \qquad [1½]$$

Therefore:

$$(aq)_x^a = 0.75\times\left(1-e^{-0.01}\right)+e^{-0.01}\left(1-0.06\right)\times 0.75\times\left(1-e^{-0.03}\right) = 0.028091 \qquad [½]$$

[Total 6]

Solution 3.33

(i) ***Conditions under which premiums would normally be payable***

Premiums would normally be payable while either of the guardians is still alive. (In practice, the premium payment period for older lives may be limited, *eg* to a maximum of 10 years.) [Total 1]

(ii) ***Which of the lives should be medically examined***

Both lives should be examined medically, since premature death of either life can only advance the timing of the benefit payment. [Total 1]

(iii) ***Premium***

The premium equation is:

$$P\,\ddot{a}_{\overline{75:80}:\overline{10|}} = 25,000\,\bar{A}_{\overline{75:80}} + 0.05P\,\ddot{a}_{\overline{75:80}:\overline{10|}} + 250 \qquad [1]$$

We can calculate $\ddot{a}_{\overline{75:80}:\overline{10|}}$ using:

$$\ddot{a}_{\overline{75:80}:\overline{10|}} = \ddot{a}_{75:\overline{10|}} + \ddot{a}_{80:\overline{10|}} - \ddot{a}_{75:80:\overline{10|}}$$

From the *Tables*:

$$\begin{aligned}\ddot{a}_{75:80:\overline{10|}} &= \ddot{a}_{75:80} - v^{10}\,\frac{l_{85}}{l_{75}}(male)\,\frac{l_{90}}{l_{80}}(female)\,\ddot{a}_{85:90}\\ &= 6.822 - \frac{1}{1.04^{10}} \times \frac{4,892.878}{8,405.16} \times \frac{4,118.693}{7,724.737} \times 3.87\\ &= 6.011\end{aligned} \qquad [2]$$

Also:

$$\ddot{a}_{75:\overline{10|}} = \ddot{a}_{75} - v^{10}\,\frac{l_{85}}{l_{75}}(male)\ddot{a}_{85} = 9.456 - 0.39326 \times 5.842 = 7.159 \qquad [1]$$

and:

$$\ddot{a}_{80:\overline{10|}} = \ddot{a}_{80} - v^{10}\,\frac{l_{90}}{l_{80}}(female)\ddot{a}_{90} = 8.989 - 0.360199 \times 5.713 = 6.9312 \qquad [1]$$

So:

$$\ddot{a}_{\overline{75:80}:\overline{10|}} = 7.159 + 6.9312 - 6.011 = 8.079 \qquad [1]$$

The assurance factor can be calculated by premium conversion:

$$\begin{aligned}\bar{A}_{\overline{75:80}} &\approx 1.02 A_{\overline{75:80}} \\ &= 1.02(1 - d\,\ddot{a}_{\overline{75:80}}) \\ &= 1.02\left[1 - \frac{0.04}{1.04} \times (9.456 + 8.989 - 6.822)\right] \\ &= 0.5640 \qquad [2]\end{aligned}$$

The premium equation becomes:

$$8.079P = 25,000 \times 0.5640 + 0.05 \times 8.079P + 250$$

So the premium is:

$$P = \frac{14,350}{7.6751} = £1,870 \qquad [1]$$

[Total 9]

Reasonableness check: Mortality is very high at these ages. The annuity factor of 8.079 corresponds to the annuity certain $\ddot{a}_{\overline{9.5|}} = 8.087$*, which suggests that the second life will typically die in around 9.5 years' time. If the second life definitely died at this time, the premium would be* $(25,000v^{9.5} + 250) / 0.95\ddot{a}_{\overline{9.5|}} = £2,274$.

Solution 3.34

The expected present value of future premiums is:

$$P\ddot{a}_{60:64:\overline{10|}} = P\left(\ddot{a}_{60:64} - {}_{10}p_{64}\,\frac{v^{10} l_{70}}{l_{60}}\,\ddot{a}_{70:74}\right) \qquad [1]$$

where the 60-year old is subject to male mortality and the 64-year old is subject to female mortality.

So, using the *Tables*:

$$P\ddot{a}_{60:64:\overline{10}|} = \left(13.325 - \frac{8937.791}{9742.64} \times \frac{1.04^{-10} \times 9238.134}{9826.131} \times 9.005\right) P = 8.078P \qquad [1]$$

The expected present value of the benefit payable on death is $50,000 A^{\ \ 1}_{\overline{60:64}:\overline{10}|}$.

Using premium conversion we can evaluate the assurance function as:

$$1 - d\,\ddot{a}_{60:64:\overline{10}|} - {}_{10}p_{64}\frac{D_{70}}{D_{60}} = 1 - 0.03846 \times 8.078 - 0.5827 = 0.1066 \qquad [1]$$

So the expected present value of the death benefit is £5,330. [1]

The deferred annuity is £10,000 *pa* while both lives are alive or £5,000 *pa* if only one life is alive. This is equivalent to giving each life a single life annuity of £5,000 (since £10,000 will be paid in total if both are alive). [1]

So the expected present value of this benefit is:

$$5{,}000\left(\frac{v^{10} l_{70}}{l_{60}}\ddot{a}_{70} + \frac{v^{10} l_{74}}{l_{64}}\ddot{a}_{74}\right) = 5{,}000(0.63514 \times 11.562 + 0.61976 \times 11.333)$$

$$= 71,836 \qquad [2]$$

Setting the EPV of premiums equal to the EPV of the benefits gives:

$$8.078P = 5,330 + 71,836$$

$$\Rightarrow P = £9,553 \qquad [1]$$

[Total 8]

Solution 3.35

(i) *Monthly premium*

Let P denote the monthly premium. Then the equation of value is:

$$12P\ddot{a}^{(12)}_{65:62:\overline{5}|} = 10,000\,\bar{A}_{65:62} \qquad [½]$$

Now:

$$\ddot{a}^{(12)}_{65:62:\overline{5}|} = \ddot{a}^{(12)}_{65:62} - v^5 \times \frac{l_{70}}{l_{65}} \times \frac{l_{67}}{l_{62}} \times \ddot{a}^{(12)}_{70:67} \qquad [1]$$

$$\ddot{a}^{(12)}_{65:62} = \ddot{a}_{65:62} - \frac{11}{24} = 12.427 - \frac{11}{24} = 11.969 \qquad [1]$$

$$\ddot{a}^{(12)}_{70:67} = \ddot{a}_{70:67} - \frac{11}{24} = 10.233 - \frac{11}{24} = 9.775 \qquad [1]$$

and:

$$v^5 \times \frac{l_{70}}{l_{65}} \times \frac{l_{67}}{l_{62}} = 1.04^{-5} \times \frac{9,238.134}{9,647.797} \times \frac{9,605.483}{9,804.173} = 0.77108 \qquad [1]$$

So:

$$\ddot{a}^{(12)}_{65:62:\overline{5}|} = 11.969 - 0.77108 \times 9.775 = 4.432 \qquad [1]$$

We need to use the premium conversion relationship to determine the value of the benefit:

$$\bar{A}_{65:62} = (1+i)^{½} A_{65:62} = 1.04^{½} \left(1 - d\, \ddot{a}_{65:62}\right) = 0.53238 \qquad [1]$$

So the monthly premium is:

$$P = \frac{10,000\, \bar{A}_{65:62}}{12\, \ddot{a}^{(12)}_{65:62:\overline{5}|}} = \frac{10,000 \times 0.53238}{12 \times 4.432} = £100 \qquad [½]$$

[Total 7]

(ii) ***Net premium reserve after 3 years***

The reserve after three years will be:

$$ {}_3V = 10,000\bar{A}_{68:65} - 12P\ddot{a}^{(12)}_{68:65:\overline{2}|} \qquad [1]$$

where:

$$\ddot{a}^{(12)}_{68:65:\overline{2}|} = \ddot{a}^{(12)}_{68:65} - v^2 \times \frac{l_{70}}{l_{68}} \times \frac{l_{67}}{l_{65}} \times \ddot{a}^{(12)}_{70:67} \qquad [½]$$

Now:

$$\ddot{a}^{(12)}_{68:65} = \ddot{a}_{68:65} - \frac{11}{24} = 11.112 - \frac{11}{24} = 10.654 \qquad [½]$$

$$\ddot{a}^{(12)}_{70:67} = 9.775 \text{ from (i)}$$

and:

$$v^2 \times \frac{l_{70}}{l_{68}} \times \frac{l_{67}}{l_{65}} = 1.04^{-2} \times \frac{9,238.134}{9,440.717} \times \frac{9,605.483}{9,703.708} = 0.89556 \qquad [½]$$

So:

$$\ddot{a}^{(12)}_{68:65:\overline{2}|} = 10.654 - 0.89556 \times 9.775 = 1.900 \qquad [½]$$

Also:

$$\bar{A}_{68:65} = (1+i)^{½} A_{68:65} = (1+i)^{½}\left(1 - d\,\ddot{a}_{68:65}\right) = 0.58396 \qquad [½]$$

So the reserve is:

$$10,000 \times 0.58396 - 1,200 \times 1.900 = £3,560 \qquad [½]$$

[Total 4]

Note that this is the right order of magnitude (three years of premiums with interest less cost of cover).

Solution 3.36

The value of the first part of the benefit described, (of £20,000 *pa*), is a guaranteed annuity for ten years, plus a joint life benefit deferred ten years. Noting that a joint life benefit will only begin at time 10 if both survive the period, we have:

$$20,000\left(\ddot{a}_{\overline{10}|}^{(12)} + {}_{10}p_{63}^{m}\ {}_{10}p_{60}^{f} v^{10}(\ddot{a}_{73:70} - \tfrac{11}{24})\right) \qquad [2]$$

where we use the *m* and *f* superscripts to denote male and female mortality respectively. This helps to clarify which lives we are talking about.

We have:

$$\ddot{a}_{\overline{10}|}^{(12)} = \frac{i}{d^{(12)}} a_{\overline{10}|} = 8.2856 \qquad [1]$$

$${}_{10}p_{63}^{m} = 0.90398 \qquad [½]$$

$${}_{10}p_{60}^{f} = 0.95372 \qquad [½]$$

$$\ddot{a}_{73:70} = \frac{2}{5}\times 10.233 + \frac{3}{5}\times 8.110 = 8.9592 \qquad [1]$$

(interpolating because the *Tables* do not contain exactly what we need)

This gives a total value of £264,736. [1]

The value of the reversionary annuity of £15,000 is slightly messier. There are three events to consider:

- x dies in next 10 years, y survives these 10 years
- x survives next 10 years, y dies in these 10 years
- x and y both survive the next 10 years

For the first two events, we just have simple single-life annuities to consider. For the third event, we have a reversionary annuity to consider – but note that we actually have two reversionary annuities, $a_{x+10|y+10}$ *and* $a_{y+10|x+10}$ *(ignoring monthly payment etc).*

Whenever we get a reversionary annuity we should immediately simplify down. For the third event above, we have, at the end of the ten-year period:

$$15,000\left(\ddot{a}^{(12)}_{x+10|y+10}+\ddot{a}^{(12)}_{y+10|x+10}\right)$$

$$=15,000\left\{\left(\ddot{a}^{(12)}_{y+10}-\ddot{a}^{(12)}_{x+10:y+10}\right)+\left(\ddot{a}^{(12)}_{x+10}-\ddot{a}^{(12)}_{x+10:y+10}\right)\right\}$$

$$=15,000\left(\ddot{a}^{(12)}_{x+10}+\ddot{a}^{(12)}_{y+10}-2\ddot{a}^{(12)}_{x+10:y+10}\right) \qquad [2]$$

Note that we do the above thinking using generic ages x and y – in general, manipulate and simplify as much as possible with generic ages before putting in the hard-coded figures. This will also reduce the chance of any major arithmetic slips.

So putting the three events together we get:

$$15,000v^{10}\Big[{}_{10}p^{m}_{63}\left(1-{}_{10}p^{f}_{60}\right)\ddot{a}^{(12)}_{73^m}+\left(1-{}_{10}p^{m}_{63}\right){}_{10}p^{f}_{60}\,\ddot{a}^{(12)}_{70^f}$$

$$+{}_{10}p_{63^m:60^f}\left(\ddot{a}_{73}+\ddot{a}_{70}-2\ddot{a}_{73:70}\right)\Big]$$

$$=15,000\times 0.67556\times\Big[0.90398\times(1-0.95372)\times\left(10.288-\tfrac{11}{24}\right)$$

$$+(1-0.90398)\times 0.95372\times\left(12.934-\tfrac{11}{24}\right)$$

$$+0.90398\times 0.95372\times\left(10.288+12.934-2\times 8.9592\right)\Big]$$

$$=£62,079 \qquad [2]$$

This gives a total benefit value of $264,736+62,079=£326,815$.

So the appropriate single premium is £326,815. [1]

[Total 11]

Solution 3.37

(i) ***Future contributions***

This member joined at age 22 and the salary figure given relates to the year of age 24 to 25.

So the EPV of future contributions is:

$$0.05 \times 15,000 \left(\frac{{}^s\bar{N}_{25} - {}^s\bar{N}_{62}}{s_{24} D_{25}} \right) = 750 \times \frac{915,673 - 13,152}{3.605 \times 14,550} = £12,905 \qquad [3]$$

(ii) ***Ill-health retirement benefit***

The expected present value of the ill-health retirement benefit is:

$$1,000 \times \frac{M_{25}^{ia}}{D_{25}} = 1,000 \times \frac{7,023}{14,550} = £483 \qquad [2]$$

Solution 3.38

(i)(a) ***Expected number retiring through ill health***

Functions such as ${}_t(ap)_x{}'$ and ${}_t(aq)_x{}'$ will be used to represent dependent probabilities that *ignore* normal age retirement.

The expected number of members retiring through ill health between 60 and 61 is:

$$750\,(ap)_{59}\,(aq)_{60}^i = 750\,(ap)_{59}{}'\left(1 - q_{60}^r\right)(aq)_{60}^i{}' \qquad [1]$$

where:

$$(ap)_{59}{}' = e^{-0.01} \qquad [½]$$

$$(aq)_{60}^i{}' = \frac{\mu^i}{\mu^i + \mu^d}\left(1 - e^{-\left(\mu^i + \mu^d\right)}\right) = \frac{0.05}{0.06}\left(1 - e^{-0.06}\right) = 0.048530 \qquad [1½]$$

Therefore the expected number of ill-health retirements during this year:

$$750 \times e^{-0.01} \times (1 - 0.3) \times 0.048530 = 25.22 \qquad [½]$$

The expected number of members retiring through ill-health between 61 and 62 is:

$$750\,{}_2(ap)_{59}\,(aq)_{61}^i = 750 \times e^{-0.01} \times 0.7 \times (ap)_{60}{}'\left(1 - q_{61}^r\right)(aq)_{61}^i{}' \qquad [½]$$

Now:

$$(ap)_{60}' = e^{-0.06}$$

and:

$$(aq)_{61}^{i}{}' = (aq)_{60}^{i}{}' = 0.048530$$

So, the expected number of retirements during this year is:

$$750 \times e^{-0.01} \times 0.7 \times e^{-0.06} \times (1-0.1) \times 0.048530 = 21.38 \qquad [1]$$

(i)(b) ***Expected number retiring at normal ages***

The expected number of members retiring at age 60 exact is:

$$750\,(ap)_{59}' \times q_{60}^{r} = 750 \times e^{-0.01} \times 0.3 = 222.76 \qquad [½]$$

The expected number of members retiring at age 61 exact is:

$$750\,(ap)_{59}\,(ap)_{60}'\,q_{61}^{r} = 750 \times e^{-0.01} \times 0.7 \times e^{-0.06} \times 0.1 = 48.95 \qquad [½]$$

[Total 6]

(ii) ***Value of the retirement benefits***

Ill-health retirement between 60 and 61

Assuming retirement occurs on average at 60½ (an approximation), we will need the expected salary earned over [59½, 60½]. This will be:

$$37{,}000\int_{½}^{1½} 1.04^{t}\,dt = 37{,}000\int_{½}^{1½} e^{t\ln 1.04}\,dt = \frac{37{,}000}{\ln 1.04}\Big[1.04^{t}\Big]_{½}^{1½} = 38{,}482.47 \qquad [1]$$

Alternatively, you could further approximate as $37{,}000 \times 1.04 = 38{,}480$.

The annuity factor for age 60½ (including the 8-year age rating) is:

$$\ddot{a}_{60½+8}^{(12)} \approx \tfrac{1}{2}\big[\ddot{a}_{68} + \ddot{a}_{69}\big] - \frac{11}{24} = \frac{13.723 + 13.330}{2} - \frac{11}{24} = 13.068 \qquad [1]$$

The value of this benefit at age 59 is then:

$$\frac{(65-59)+35}{80}\times 38,482.47\times\frac{13.068}{1.04^{1½}}\times\frac{25.22}{750}=8,173.02 \qquad [1½]$$

Note that the last fraction is the probability of a member aged 59 retiring through ill-health between 60 and 61.

Using the approximation of 38,480 for the expected salary gives 8,170.94. This would gain full credit.

Ill-health retirement between 61 and 62

Similar calculations produce the following:

Salary in the year of age [60½, 61½] will be $38,482.47\times 1.04=40,021.77$. [½]

Alternatively, this could be approximated as $38,480\times 1.04=40,019.20$.

The annuity factor is $\ddot{a}^{(12)}_{61½+8}\approx\frac{1}{2}\left[\ddot{a}_{69}+\ddot{a}_{70}\right]-\frac{11}{24}=12.674$. [½]

The value at age 59 is $\frac{41}{80}\times 40,021.77\times\frac{12.674}{1.04^{2½}}\times\frac{21.38}{750}=6,718.24$. [½]

Using the approximation of 40,019.20 for expected salary gives 6,717.99.

The total EPV of the ill-health retirement benefits as at age 59 is then:

$$8,173.02+6,718.24=£14,891 \qquad [½]$$

Using the approximation gives £14,889. This would gain full credit.

Normal age retirement at exact age 60

The salary in the year [59, 60] will be:

$$37,000\int_0^1 1.04^t\,dt=\frac{37,000}{\ln 1.04}\left[1.04^t\right]_0^1=37,735.16 \qquad [½]$$

This could be approximated as $37,000\times 1.04^{½}=37,732.74$.

The annotity factor will be $\ddot{a}_{60}^{(12)} \simeq \ddot{a}_{60} - \dfrac{11}{24} = 16.652 - \dfrac{11}{24} = 16.194$ [½]

The value of the pension as at age 59 is:

$$\frac{36}{80} \times 37,735.16 \times \frac{16.194}{1.04} \times \frac{222.76}{750} = 78,533.59$$ [½]

Using the approximation of 37,732.74 gives 78,528.55. This would gain full credit.

Normal age retirement at exact age 61

The salary in the year [60, 61] will be $37,735.16 \times 1.04 = 39,244.57$ [½]

Alternatively, we could use the approximation $37,732.74 \times 1.04 = 39,242.05$.

The annuity factor will be $\ddot{a}_{61}^{(12)} = \ddot{a}_{61} - \dfrac{11}{24} = 16.311 - 0.458 = 15.853$ [½]

So the value at age 59 will be $\dfrac{37}{80} \times 39,244.57 \times \dfrac{15.853}{1.04^2} \times \dfrac{48.95}{750} = 17,363.11$ [½]

Using the approximation of 39,242.05 gives 17,362.00.

So the total EPV of the normal age retirement benefits as at age 59 is:

$$78,533.59 + 17,363.11 = £95,897$$ [½]

Using the above approximations, this EPV is 95,890. This would gain full credit.

[Total 9]

Solution 3.39

The expected present value of the sickness benefit is:

$$5,000\left(v\, p_{58}^{HS} + v^2\, {}_2p_{58}^{HS}\right)$$ [½]

Now:

$${}_2p_{58}^{HS} = p_{58}^{HH}\, p_{59}^{HS} + p_{58}^{HS}\, p_{59}^{SS} = 0.92 \times 0.05 + 0.05 \times 0.25 = 0.0585$$ [1]

So the expected present value of the sickness benefit is:

$$5{,}000\left(\frac{0.05}{1.05}+\frac{0.0585}{1.05^2}\right)=£503.40$$ [½]

The expected present value of the death benefit is:

$$20{,}000\left[vp_{58}^{HD}+v^2\left(p_{58}^{HH}\,p_{59}^{HD}+p_{58}^{HS}\,p_{59}^{SD}\right)\right]$$

$$=20{,}000\left[\frac{0.03}{1.05}+\frac{0.92\times 0.03+0.05\times 0.10}{1.05^2}\right]$$

$$=£1{,}162.81$$ [2½]

So the total expected present value of the benefits is:

$$503.40+1{,}162.81=£1{,}666.21$$ [½]

[Total 5]

Solution 3.40

(i) ***Multiple decrement table***

Since the forces of decrement are constant over each year of age, we calculate the dependent probabilities of decrement using formulae of the form:

$$(aq)_x^d \approx \frac{\mu_{\bar{x}}^d}{\mu_{\bar{x}}^d+\mu_{\bar{x}}^i+\mu_{\bar{x}}^w}\left[1-e^{-(\mu_{\bar{x}}^d+\mu_{\bar{x}}^i+\mu_{\bar{x}}^w)}\right]$$

The dependent probabilities of decrement are then:

x	$(aq)_x^d$	$(aq)_x^i$	$(aq)_x^w$
62	0.01681	0.09341	0.01868
63	0.01826	0.13694	0.01369
64	0.02052	0.17841	0.00892

[3]

A multiple decrement table with radix $(al)_{62} = 100,000$ can then be created:

x	$(al)_x$	$(ad)_x^d$	$(ad)_x^i$	$(ad)_x^w$
62	100,000	1,681	9,341	1,868
63	87,110	1,591	11,929	1,193
64	72,397	1,486	12,916	646
65	57,349			

[2]
[Total 5]

(ii) ***Expected present value of benefits***

Death benefit

Assuming that death occurs, on average, halfway through the year, the expected present value of the death benefit is:

$$\frac{20,000}{(al)_{62}}\left(v^{½}(ad)_{62}^d + v^{1½}(ad)_{63}^d + v^{2½}(ad)_{64}^d\right) \quad [1½]$$

$$= \frac{20,000}{100,000}\left(\frac{1,681}{(1.06)^{½}} + \frac{1,591}{(1.06)^{1½}} + \frac{1,486}{(1.06)^{2½}}\right) = £875 \quad [½]$$

Withdrawal benefit

Assuming that withdrawal occurs, on average, halfway through the year, the expected present value of the withdrawal benefit is:

$$1,000v^{1½}\frac{(ad)_{63}^w}{(al)_{62}} + 2,000v^{2½}\frac{(ad)_{64}^w}{(al)_{62}} \quad [1½]$$

$$= \frac{1,000}{100,000}\left(\frac{1,193}{(1.06)^{1½}} + \frac{2\times 646}{(1.06)^{2½}}\right) = £22 \quad [½]$$

Ill-health benefit

Assuming that ill-health retirement occurs, on average, halfway through the year, the expected present value of the ill-health retirement benefit is:

$$\frac{5{,}000}{(al)_{62}}\ddot{a}^{(12)}_{\overline{5}|}\left(v^{½}(ad)^{i}_{62}+v^{1½}(ad)^{i}_{63}+v^{2½}(ad)^{i}_{64}\right) \qquad [1½]$$

$$=\frac{5{,}000\times 4.4518\times 1.021537}{100{,}000}\left(\frac{9{,}341}{(1.06)^{½}}+\frac{11{,}929}{(1.06)^{1½}}+\frac{12{,}916}{(1.06)^{2½}}\right)$$

$$=£7{,}087 \qquad [1]$$

Normal retirement benefit

The expected present value of the normal retirement benefit is:

$$3\times 2{,}000\times v^{3}\times\frac{(al)_{65}}{(al)_{62}}\times\left(\ddot{a}^{(12)}_{\overline{5}|}+v^{5}\frac{l_{70}}{l_{65}}\ddot{a}^{(12)}_{70}\right) \qquad [2]$$

$$=\frac{6{,}000}{1.06^{3}}\times\frac{57{,}349}{100{,}000}\times\Big[4.4518\times 1.021537 + \frac{1}{1.04^{5}}\times\frac{9{,}238.134}{9{,}647.797}\times\left(11.562-\frac{11}{24}\right)\Big]$$

$$=£38{,}386 \qquad [1½]$$

noting that $\ddot{a}^{(12)}_{\overline{5}|}=a_{\overline{5}|}\times\dfrac{i}{d^{(12)}}$. [Total 10]

Solution 3.41

(i) ***Present value random variable***

This is:

$$\bar{a}_{\overline{T_{\overline{xy}}}|}=\bar{a}_{\overline{T_x}|}+\bar{a}_{\overline{T_y}|}-\bar{a}_{\overline{T_{xy}}|}$$

where $T_{\overline{xy}}=\max\{T_x,T_y\}$. [Total 1] (ii)(a) ***Present value random variable***

This is:

$$25{,}000\,\bar{a}_{\overline{T_{60:65}}|} + 20{,}000\left[\bar{a}_{\overline{T_{65}}|} - \bar{a}_{\overline{T_{60:65}}|}\right] + 15{,}000\left[\bar{a}_{\overline{T_{60}}|} - \bar{a}_{\overline{T_{60:65}}|}\right]$$

$$= 20{,}000\,\bar{a}_{\overline{T_{65}}|} + 15{,}000\,\bar{a}_{\overline{T_{60}}|} - 10{,}000\,\bar{a}_{\overline{T_{60:65}}|}$$ [2]

(ii)(b) ***Expected present value***

This is:

$$EPV = 20{,}000\,a_{65(f)}^{(12)} + 15{,}000\,a_{60(m)}^{(12)} - 10{,}000\,a_{60(m):65(f)}^{(12)}$$ [½]

Now:

$$a_x^{(12)} \approx a_x + \frac{11}{24} = \ddot{a}_x - \frac{13}{24}$$ (from Page 36 of the *Tables*) [1]

So:

$$EPV = 20{,}000\,\ddot{a}_{65(f)} + 15{,}000\,\ddot{a}_{60(m)} - 10{,}000\,\ddot{a}_{60(m):65(f)} - \frac{13}{24} \times 25{,}000$$

$$= 20{,}000 \times 14.871 + 15{,}000 \times 15.632 - 10{,}000 \times 13.101 - 13{,}541.667$$

$$= £387{,}348$$ [1½]

[Total 5]

Solution 3.42

It will be easiest to construct a multiple decrement table for the process. For this we will need to calculate the dependent probabilities of critical illness and death claims.

Dependent probabilities

Assuming forces of decrement are constant over each year of age, we need, for $x = 57,\ 58,\ 59$:

$$(aq)_x^i \approx \frac{\mu_{\bar{x}}^i}{\mu_{\bar{x}}^d + \mu_{\bar{x}}^i}\left[1 - e^{-(\mu_{\bar{x}}^d + \mu_{\bar{x}}^i)}\right]$$

$$(aq)_x^d \approx \frac{\mu_{\bar{x}}^d}{\mu_{\bar{x}}^d + \mu_{\bar{x}}^i}\left[1 - e^{-(\mu_{\bar{x}}^d + \mu_{\bar{x}}^i)}\right]$$

where $\mu_{\bar{x}}^j$ is defined to be the (assumed constant) force of decrement due to cause j over the year of age $[x, x+1]$, i and d are critical illness and death respectively.

We obtain $\mu_{\bar{x}}^d$ using:

$$\mu_{\overline{57+r}}^d = -\ln\left(1 - 0.6\, q_{[57]+r}^d\right) \quad \text{for } r = 0, 1, 2$$

where $q_{[57]+r}^d$ is the corresponding mortality rate from the AM92 Select table. These give the following values:

x	$\mu_{\bar{x}}^d$	$(aq)_x^d$	$(aq)_x^i$
57	0.0025057	0.0025001	0.0019955
58	0.0037149	0.0037015	0.0034874
59	0.0042932	0.0042754	0.0039835

[4]

Multiple decrement table

Using a radix of $(al)_{57} = 100,000$ (say), this results in the following multiple decrement table:

x	$(al)_x$	$(ad)_x^d$	$(ad)_x^i$	$(ad)_x^s$
57	100,000.00	250.01	199.55	9,955.04
58	89,595.40	331.64	312.46	4,447.57
59	84,503.74	361.29	336.62	0
60	83,805.83			

[2]

where:

$$(ad)_x^j = (al)_x \, (aq)_x^j \qquad\qquad j = i, d$$

$$(ad)_x^s = q_{x+1}^s \left[(al)_x - (ad)_x^i - (ad)_x^d \right] \qquad\qquad *$$

$q_{x+1}^s =$ independent probability of not renewing the policy on reaching age $x+1$

and:

$$(al)_{x+1} = (al)_x - (ad)_x^i - (ad)_x^d - (ad)_x^s$$

* So, for example:

$$(ad)_{57}^s = 0.1\left[100,000 - 199.55 - 250.01\right] = 9,955.04$$

EPV critical illness claims and claim expenses

Assuming diagnosis occurs half way through each year on average:

$$EPV(CI) = \frac{200,500}{(al)_{57}} \left[v\,(ad)_{57}^i + v^2\,(ad)_{58}^i + v^3\,(ad)_{59}^i \right]$$

(which allows for the average delay of 6 months between the dates of diagnosis and claim payment).

So:

$$EPV(CI) = \frac{200,500}{100,000}\left[\frac{199.55}{1.03} + \frac{312.46}{1.03^2} + \frac{336.62}{1.03^3}\right] = 1,596.60 \qquad [2]$$

EPV death claims and claim expenses

Assuming death occurs half way through each year on average:

$$EPV(D) = \frac{100,200}{(al)_{57}}\left[v^{0.625}\,(ad)_{57}^{d} + v^{1.625}\,(ad)_{58}^{d} + v^{2.625}\,(ad)_{59}^{d}\right]$$

(which allows for the average delay of 1½ months between the dates of death and claim payment).

So:

$$EPV(D) = \frac{100,200}{100,000}\left[\frac{250.01}{1.03^{0.625}} + \frac{331.64}{1.03^{1.625}} + \frac{361.29}{1.03^{2.625}}\right] = 897.63 \qquad [2]$$

EPV surrender payments and payment expenses

These occur half a year after the date of the first unpaid premium. If P is the annual premium:

$$EPV(S) = \frac{1}{(al)_{57}}\left[v^{1.5}\,(0.25P+100)(ad)_{57}^{s} + v^{2.5}\,(0.25\times 2P+100)(ad)_{58}^{s}\right]$$

$$= \frac{1}{100,000}\left[\frac{(0.25P+100)\times 9,955.04}{1.03^{1.5}} + \frac{(0.25\times 2P+100)\times 4,447.57}{1.03^{2.5}}\right]$$

$$= 0.044462P + 13.65 \qquad [2]$$

EPV of expiry payment and payment expenses

This is:

$$EPV(E) = \left(0.5\times 3P+100\right)\frac{(al)_{60}}{(al)_{57}}v^{3}$$

$$= \left(1.5P+100\right)\times\frac{83,805.83}{100,000}\times 1.03^{-3}$$

$$= 1.150413P + 76.69 \qquad [2]$$

EPV of initial and renewal expenses

This is:

$$EPV(X) = P\left[0.2 + 0.01 \times \frac{(al)_{58} v + (al)_{59} v^2}{(al)_{57}}\right] + 150$$

$$= P\left[0.2 + 0.01 \times \left(\frac{0.8959540}{1.03} + \frac{0.8450374}{1.03^2}\right)\right] + 150$$

$$= 0.216664P + 150$$ [2]

EPV premiums

This is:

$$EPV(P) = \frac{P}{(al)_{57}}\left[(al)_{57} + (al)_{58} v + (al)_{59} v^2\right]$$

$$= P\left[1 + \frac{0.8959540}{1.03} + \frac{0.8450374}{1.03^2}\right]$$

$$= 2.666387P$$ [2]

Equation of value

Finally equate:

$$EPV(P) = EPV(CI) + EPV(D) + EPV(S) + EPV(E) + EPV(X)$$

$$\Rightarrow \quad 2.666387P = 1{,}596.60 + 897.63 + 13.65 + 76.69 + 150$$

$$+\left(0.044462 + 1.150413 + 0.216664\right)P$$

$$\Rightarrow \quad P = \frac{2{,}734.57}{2.666387 - 1.411539} = £2{,}179$$ [2]

[Total 20]

Solution 3.43

(i) ***Reserve***

Basis A: net premium reserve

The net premium reserve uses an artificial net premium, calculated to cover the policy benefits (only) according to the reserving basis assumptions. The net premium is as follows, where the ages shown include the additional 5 years age rating:

$$NP = 100{,}000 \frac{A_{\overset{1}{65:63}:\overline{3|}}}{\ddot{a}_{65:63:\overline{3|}}} \qquad [½]$$

Now:

$$\ddot{a}_{65:63:\overline{3|}} = \ddot{a}_{65:63} - v^3 \frac{l_{68(m)}}{l_{65(m)}} \times \frac{l_{66(f)}}{l_{63(f)}} \times \ddot{a}_{68:66}$$

$$= 12.282 - \frac{1}{1.04^3} \times \frac{9{,}440.717}{9{,}647.797} \times \frac{9{,}658.285}{9{,}775.888} \times 10.966$$

$$= 12.282 - 0.859450 \times 10.966$$

$$= 2.85727 \qquad [1½]$$

$$A_{\overset{1}{65:63}:\overline{3|}} = A_{65:63:\overline{3|}} - A_{65:63:\overset{1}{\overline{3|}}}$$

$$= 1 - d\ddot{a}_{65:63:\overline{3|}} - v^3 \frac{l_{68(m)}}{l_{65(m)}} \times \frac{l_{66(f)}}{l_{63(f)}}$$

$$= 1 - \frac{0.04}{1.04} \times 2.85727 - 0.859450$$

$$= 0.030655 \qquad [1]$$

Therefore:

$$NP = 100{,}000 \times \frac{0.030655}{2.85727} = £1{,}072.88 \qquad [½]$$

The net premium reserve at time 1 is:

$${}_1V^{net} = 100,000 A_{\overset{1}{66}:64:\overline{2}|} - 1,072.88 \ddot{a}_{66:64:\overline{2}|}$$ [½]

where:

$$\ddot{a}_{66:64:\overline{2}|} = 1 + v\, p_{66(m)}\, p_{64(f)}$$

$$= 1 + \frac{1}{1.04} \times \frac{l_{67(m)}}{l_{66(m)}} \times \frac{l_{65(f)}}{l_{64(f)}}$$

$$= 1 + \frac{1}{1.04} \times \frac{9,521.065}{9,589.602} \times \frac{9,703.708}{9,742.640}$$

$$= 1.95085$$ [1]

Alternatively, you could have used $(1-q)$ *instead of* l/l *for* p *in the above.*

$$A_{\overset{1}{66}:64:\overline{2}|} = 1 - d\ddot{a}_{66:64:\overline{2}|} - A_{66:64:\overset{1}{\overline{2}|}}$$

$$= 1 - \frac{0.04}{1.04} \times 1.95085 - \frac{1}{1.04^2} \times \frac{l_{68(m)}}{l_{66(m)}} \times \frac{l_{66(f)}}{l_{64(f)}}$$

$$= 1 - \frac{0.04}{1.04} \times 1.95085 - \frac{1}{1.04^2} \times \frac{9,440.717}{9,589.602} \times \frac{9,658.285}{9,742.640}$$

$$= 0.0226463$$ [1½]

Therefore:

$${}_1V^{net} = 100,000 \times 0.0226463 - 1,072.88 \times 1.95085$$

$$= £171.61$$ [½]

If you had used the above alternative you would have got £171.59. This shows how your answers can be quite sensitive to rounding.

Basis B: gross premium reserve

Using the actual gross premium of £1,299, the gross premium reserve is:

$$
\begin{aligned}
{}_1V^{gross} &= 100,000 A_{\overset{1}{66:64}:\overline{2}|} - (1,299 - 75)\,\ddot{a}_{66:64:\overline{2}|} \\
&= 100,000 \times 0.0226463 - 1,224 \times 1.95085 \\
&= -£123.21
\end{aligned}
$$

[1]

[Total 8]

(ii) ***Comment***

The gross premium reserve is lower than the net premium reserve, and is also negative. [½]

This is because the gross premium reserve takes credit for the present value of the full future gross premiums, including all margins for expenses and profits. [½]

The net premium, on the other hand, only takes credit for a calculated net premium, which in this case is considerably smaller than the actual gross premium. [½]

In effect, the net premium reserve is implying a future regular expense equal to the difference between the actual premium and the net premium (*ie* £226 *pa*), which is much higher than either the gross premium reserve assumption (£75 *pa*) or the pricing basis assumption (£50 *pa*). [1]

It would be imprudent for the company to use the gross premium reserve as its actual reserve, because the future profits (from the future premiums) may not actually materialise (*eg* because the policy could lapse). [½]

[Maximum 2]

Part 4 – Questions

Note that the split between Development Questions and Exam-style Questions is somewhat subjective. For example, there may be some development questions that are as difficult as any exam question, but may be more repetitive than typically found in the exam; equally some of the shorter and more straightforward development questions are similar to the short questions regularly found in the CT5 exam. The Exam-style Questions generally involve more application and greater scope and are typical of the more challenging questions you will meet in the exam.

1 *Development Questions*

Question 4.1

Describe the main features of a unit-linked policy. [7]

Question 4.2

An actuary is profit testing a 15-year endowment assurance policy. The sum assured is £25,000 payable on survival or at the end of the year of earlier death. If the policyholder surrenders then she will receive a return of premiums without interest at the end of the year of surrender.

A level premium of £1,500 *pa* is payable annually in advance.

For a policy in force at the start of the eighth year the remaining details are as follows:

	(£)
Renewal expenses	35
Claim expenses on death or surrender	75
Reserve at the start of year, ${}_7V$	8,000
Reserve at end of year per survivor, ${}_8V$	9,300
Rate of interest	8% *pa*
Dependent probability of death	0.02
Dependent probability of surrender	0.05

Calculate the profit expected to emerge at the end of the eighth year, per policy in force at the start of that year. [3]

Question 4.3

A life insurance company issues five-year without profit endowment assurances for an annual premium of £3,600 and a sum assured of £20,000 payable on maturity or at the end of the year of death if earlier.

The company uses the following assumptions for profit testing:

Year	Mortality probability	Surrender probability	Expenses at start year per policy	Reserves at end of year per policy	Surrender value at end of year per policy
1	0.01	0.05	£750	£3,100	£2,800
2	0.01	0.05	£15	£6,800	£6,250
3	0.01	0.05	£15	£10,900	£10,000
4	0.01	0.05	£15	£15,300	£14,500
5	0.01	0	£15	–	–

Surrenders occur only at the end of a year immediately before a premium is paid. The surrender probabilities shown in the table above are applied to the number of policies in force at each year-end.

(i) Set out the column headings and the formulae you would use to calculate the profit arising each year per policy in force at the beginning of the year. [8]

(ii) Calculate (to the nearest 1%) the internal rate of return obtained by the company.

Basis: Interest 6% *pa* [13]

[Total 21]

Question 4.4

A unit-linked policy issued to lives aged 50 has a minimum death benefit of £3,000 (payable at the end of the year). Write down an expression for the extra death cost in the non-unit fund for year 2 expressed in terms of F_i, the size of the unit fund at the end of year i. [2]

Question 4.5

A profit test for policies issued to lives aged 60 who are subject to AM92 Ultimate mortality has been carried out using an accumulation rate of 6% and a risk discount rate of 8%. The profit vector is as follows:

Year	*In force expected profit*
1	–30
2	–12
3	–6
4	20
5	30

Write down an expression for the expected loss at the end of Year 1 after zeroisation. [2]

Question 4.6

(i) Explain the terms "unit fund" and "non-unit fund" in the context of a unit-linked life assurance contract, listing the various items that make up the non-unit fund. [6]

(ii) Explain why a life insurance company might need to set up non-unit reserves in respect of a unit-linked life assurance contract. [2]
[Total 8]

Question 4.7

Explain how a change in the stringency of underwriting may give rise to spurious selection. [3]

Question 4.8

What are the main difficulties associated with the measurement of social and economic factors in mortality? [2]

Question 4.9

Discuss the factors that affect the stated rate of mortality for different occupations. You should consider the factors that affect both the real and apparent levels of mortality. [15]

Question 4.10

The following statements have been made concerning the standardised mortality ratio:

I It is an example of indirect standardisation.

II It is heavily biased towards the relative mortality at the older ages.

III It requires knowledge of the age-sex specific mortality rates for the population under consideration.

Explain for each of these statements whether it is correct or not. [3]

Question 4.11

Explain the difference between direct standardisation and indirect standardisation and give two disadvantages of direct standardisation. [3]

Question 4.12

A statistic for a regional population is calculated as the ratio of the average standard population mortality rate weighted by the exposed to risk for the standard population to the average standard population mortality rate weighted by the exposed to risk for the regional population. Name the statistic. [1]

2 Exam-style Questions

Question 4.13

Maxine, aged 60, buys an endowment assurance from Company ABC with a term of five years. The sum assured is £25,000, payable at the end of the five years or at the end of the year of death if earlier. Premiums are payable annually in advance throughout the term of the policy.

Company ABC assumes that initial expenses will be £200, and renewal expenses, which are incurred at the beginning of the second and subsequent years of the policy, will be £20 plus 2.5% of the premium. The funds invested for the policy are expected to earn 7% *pa*, and mortality is expected to follow the AM92 Select life table. Company ABC holds net premium reserves, calculated using AM92 Ultimate mortality and interest of 4% *pa*.

Company ABC sets premiums so that the net present value of the profit on the contract is 10% of the annual premium, using a risk discount rate of 12% *pa*.

(i) Calculate Maxine's annual premium. [15]

(ii) Without carrying out any further calculations, state with brief reasons what the effect on the premium would be in each of the following cases:

(a) The reserves are calculated using a lower rate of interest.

(b) The office uses a risk discount rate of 15%.

(c) Mortality is assumed to be AM92 Ultimate. [6]

[Total 21]

Question 4.14

Amit, aged 60, invests £100 at the beginning of each month in an account earning interest at 1% per month. Amit requires a guaranteed amount of £3,000 at the end of the month of his death. To provide this guarantee, he buys a decreasing term assurance with a sum assured payable at the end of the month following death equal to the difference between the balance in the account and £3,000. The office premium for the assurance is £10 per month. The office incurs initial expenses of £25 and renewal expenses of £5 per month. The mortality basis for premium calculations is AM92 Ultimate and a uniform distribution of deaths over each year of age is assumed.

Determine the expected net outgo for the 18th month of the assurance contract. (Ignore interest earned by the life office.) [5]

Question 4.15

Craig, aged 40, buys a four-year unit-linked endowment policy under which level annual premiums of £1,000 are payable. 75% of the first premium and 105% of each subsequent premium is invested in units. There is a bid/offer spread in unit values, the bid price being 95% of the offer price.

A fund management charge of 0.75% of the value of the policyholder's fund is deducted at the end of each policy year.

The death benefit, which is payable at the end of the year of death, is £3,000 or the bid value of the units if greater. The maturity value is equal to the bid value of the units.

The insurance company incurs expenses of £150 at the start of the first year, £75 at the start of the second year, and £25 at the start of each of the third and fourth years.

The mortality probability (q_x) is assumed to be 0.01 at each age and withdrawals may be ignored.

(i) Assuming that the growth in the unit value is 5% *pa* and that the insurance company holds unit reserves equal to the value of units and zero non-unit reserves, calculate the expected profit emerging in each policy year. [10]

(ii) Calculate the revised profit emerging each year assuming that the office sets up non-unit reserves to ensure that the expected profit emerging in the second and subsequent policy years is non-negative. Non-unit reserves are assumed to earn interest at 5% *pa*. [9]

[Total 19]

Question 4.16

A special endowment policy pays a sum assured of £20,000 to a life who is currently aged exactly 57 after three years or at the end of the year of earlier death.

Annual reversionary bonuses are declared at the end of each policy year, and a terminal bonus is payable at maturity only.

Policies may be surrendered only at the end of each policy year. On surrender, the policyholder receives a return of premiums with interest calculated at the rate of 3% per annum.

A level premium of £8,000 is paid at the start of each year.

The premium basis is as follows:

•	Interest:		7% per annum
•	Mortality:		AM92 Select
•	Surrender rates:		15% of all policies in force at the end of year 1
			5% of all policies in force at the end of year 2
•	Reversionary bonuses:		6% per annum compound
•	Terminal bonus:		10% of all other benefits payable at maturity
•	Expenses:	Initial	£500
		Renewal	£30 at start of year 2
			£35 at start of year 3
		Termination	£100 per termination (death, surrender or maturity)
•	Reserves:		Net premium reserves, using AM92 Ultimate mortality and 4% per annum interest.

(i) Calculate the profit signature for this policy according to the premium basis. [14]

(ii) Explain briefly whether you think the company expects to declare the bonus rates it has assumed in its premium basis, assuming all the other assumptions in the basis are realistic. [2]

[Total 16]

Question 4.17

The following table shows (in £'s) a profit testing calculation with some of the entries missing for a three-year endowment assurance contract issued to a group of lives aged exactly 57 with a sum assured of £5,000 payable at the end of the year of death. Outgo terms are shown as negative entries.

Year	Premium	Expenses	Interest	Expected cost of claims	Expected cost of increasing reserves (*)	Profit vector
1	1,530	–50	?	?	?	–51
2	1,530	?	?	?	?	21
3	1,530	?	?	?	?	45

The mortality probability at each age is 1%. The rate of accumulation used is 6%. Reserves are calculated using an interest rate of 4%. The reserves are zero at the start and end of the contract. The interest earned on the reserve in the third year is £195.

(i) Complete the table. [7]

(ii) Calculate the internal rate of return. [2]

(iii) Explain the effect that changing to a weaker reserving basis would have on the internal rate of return. [2]

(iv) Calculate the net present value using a risk discount rate of 7%. [2]

(v) Explain the effect that changing to a weaker reserving basis would have on the net present value. [2]
[Total 15]

(*) *Net of interest earned on reserves.*

Question 4.18

Discuss how time selection and class selection can affect the results of a mortality investigation and how you consider these factors should be allowed for in developing mortality tables for practical use. [6]

Question 4.19

The following data have been extracted from the mortality records of an overseas country.

	Region A		Region B		Country	
Age group	Population at 30 June 2000 (000s)	Deaths in 2000	Population at 30 June 2000 (000s)	Deaths in 2000	Population at 30 June 2000 (000s)	Deaths in 2000
0 – 25	600	282	3,300	1,106	10,200	3,402
26 – 55	1,284	3,129	1,965	2,214	16,800	18,602
56 +	920	9,617	1,100	8,473	12,900	98,440

Calculate the standardised mortality ratio for each region by reference to the country as a whole. Comment on your results. [5]

Question 4.20

(i) Explain the following terms and give an example of each:

(a) Time selection

(b) Spurious selection [5]

(ii) A life office conducts quinquennial reviews of the mortality experience of male lives accepted at normal rates under its group life assurance business. The data are collected separately:

(1) for “works” and “staff” employees;

(2) according to duration since a life entered into assurance, into durations of less than one year and durations of one year and over.

All lives are underwritten before being accepted for assurance.

The following data relates to the last three investigations:

Year	Curtate Duration 0		Curtate Duration 1^+	
	E_{50}	θ_{50}	E_{50}	θ_{50}
		Works		
1998	10,000	20	20,000	50
2003	8,421	16	20,000	48
2008	5,000	9	16,959	39
		Staff		
1998	10,000	16	5,000	10
2003	7,333	11	14,211	27
2008	2,143	3	33,889	61

(a) Identify all evidence of class selection and temporary initial selection in the above data. Give a justification in each case.

(b) Aggregate the above data in three different ways and in each case determine whether spurious selection is present.

(c) The same annual premium rate is charged for all members of the same age, and when the rates were last reviewed in 1999 it was decided to base them on the aggregate mortality rates at each age which emerged from the quinquennial investigation in 1998. The rates are now being reviewed, and it has been suggested that they should be based on the aggregate mortality rates at each age that emerged from the investigation in 2008.

Comment on the mortality basis proposed for the new premium rates, in the case of the lives aged 50. [15]

[Total 20]

Question 4.21

(i) Explain what is meant by the following terms and give an example of each:

(a) temporary initial selection

(b) time selection

(c) spurious selection [7]

(ii) In North America the select period for published mortality tables for assured lives often extends up to 15 years. Discuss whether or not you consider that a select period of this length can be justified. [4]

[Total 11]

Question 4.22

The following table is extracted from the 1970-72 investigation into the mortality of occupational groups in England and Wales:

	Standard Mortality Ratio (ages 15-64, both sexes)
Electrical engineers (so described)	317
Publicans, innkeepers	155
Physiotherapists	55
Electrical engineers (requiring training of university standard)	42

(i) Suggest possible explanations for each of the statistics above. [8]

(ii) What factors, other than occupation, will influence the mortality differentials? [6]

[Total 14]

Question 4.23

Describe how selection can arise in pension schemes. [4]

Question 4.24

(i) Describe briefly the advantages and disadvantages of using a single figure index for comparing the mortality of two regions of a country. [4]

(ii) The following data has been extracted from the 2001 census for the whole of a country with a developed economy and for two of its administrative regions.

	Region X		Region Y		Country	
Age group	Population at June 30 (000s)	Deaths in 2001	Population at June 30 (000s)	Deaths in 2001	Population at June 30 (000s)	Deaths in 2001
0-14	590	136	408	108	10,200	2,550
15-39	980	820	510	441	16,800	13,950
40-59	1,050	5,690	520	2,816	12,900	70,950
60-79	870	42,630	260	11,980	8,900	418,300
80+	110	18,920	36	6,077	1,200	204,000
Total	3,600	68,196	1,734	21,422	50,000	709,750

(a) Calculate the crude death rate for each region and for the whole country.

(b) Calculate the standardised mortality rate and the standardised mortality ratio for each region by reference to the country as a whole.

(c) Comment on your results. [13]

[Total 17]

Question 4.25

A life insurance company sells 5-year-term, single-premium, unit-linked bonds each for a premium of £10,000. There is no bid/offer spread and the allocation percentage is 100%.

(i) Assuming that the only charge is a 2% annual management charge and assuming unit growth of 9% *pa*, calculate the unit reserve at the start and end of each year and the management charge each year. [3]

(ii) Calculate the net present value of the contract assuming:

- Commission of 5% of the premium
- Initial expenses of £50
- Annual renewal expenses of £20 in the 1st year, inflating at 5% pa.
- Independent probability of mortality is 0.5% at each age
- Independent probability of surrender is 5% at each age
- Non-unit fund interest rate is 9% *pa*
- Risk discount rate 12% *pa*

The company holds unit reserves equal to the full value of the units and zero non-unit reserves.

You may assume that expenses are incurred at the start of the year and that death and surrender payments are made at the end of the year. [4]

[Total 7]

Part 4 – Solutions

Solution 4.1

Features of unit-linked policies:

- Benefits are directly linked to the value of the underlying investment. [1]
- There is no pooling of investments – each policyholder receives the value of the units allocated to their own policy. [1]
- Every time the policyholder pays a premium, part of it (the allocated premium) is invested on the policyholder's behalf in a fund chosen by the policyholder. The remainder goes into the company's non-unit fund. [1]
- The investment fund is divided into units, which are priced continuously. [½]
- Most companies use a bid/offer spread to help cover expenses and contribute to profit. The policyholder buys units at the offer price and sells them back to the company at the bid price. The bid price is usually about 5% lower than the offer price. [1]
- Every time the policyholder pays a premium, the number of units purchased is recorded. When the policy matures or a claim is made, the policyholder receives the value of the cumulative number of units purchased. [1]
- The company will deduct money from the unit account on a periodic basis, *eg* monthly. This is to cover expenses and the cost of providing insurance. The charges are usually variable, and can be modified in the light of the company's experience. [1]
- There may be a minimum guaranteed sum assured to protect the policyholder against poor investment performance, or to provide some benefit in the event of an early death. [1]
- In order to price and value unit-linked contracts, the allocation percentage must be specified and an assumption about the future growth in the price of units must be made. [1]
- The most common types of unit-linked assurance are whole life and endowment assurances. [½]

[Maximum 7]

Solution 4.2

The expected profit is:

$$\begin{aligned}&\left(8,000+1,500-35\right)\times 1.08\\&-\left(8\times 1,500+75\right)\times 0.05-\left(25,000+75\right)\times 0.02\\&-9,300\times 0.93\\&=£468\end{aligned}$$ [3]

Solution 4.3

(i) ***Column headings***

The column headings and formulae required are:

(t)	Year	$t=1,2,\ldots,5$	
(1)	Premium received	£3,600	[½]
(2)	Expenses	£750 (Year 1)	
		£15 (Years 2-5)	[½]
(3)	Interest earned	$0.06\times[(1)-(2)]$	[½]
(4)	Death claims	$0.01\times 20,000$	[1]
(5)	Maturity claims	$0.99\times 20,000$ (Year 5 only)	[1]
(6)	Surrender claims	$0.05\times 0.99\times SV$ (Years 1-4 only)	[1]
	(The surrender value SV is a data item.)		
(7)	In force cashflow	$(1)-(2)+(3)-(4)-(5)-(6)$	[1]
(8)	Increase in reserves	$0.95\times 0.99\times {}_tV-{}_{t-1}V$	[1]
	(The reserve at the end of Year 5 is zero.)		
(9)	Interest on reserves	$0.06\times {}_{t-1}V$	[1]
(10)	Profit	$(7)-(8)+(9)$	[½]

[Total 8]

We have adopted the convention here that the figures in the expenses and claims columns are shown as positive entries. If you have shown these as negative numbers and adjusted the signs in the other column definitions accordingly, that is an equally valid approach.

(ii) ***Internal rate of return***

The calculations required to determine the profit signature are set out in the tables below, which include two extra columns:

(11) Probability in force $0.95^{t-1} \times 0.99^{t-1}$

(12) Profit signature $(10) \times (11)$

We then have:

Year	Premiums	Expenses	Interest	Death claims	Maturity claims	Surrenders
t	(1)	(2)	(3)	(4)	(5)	(6)
1	3,600	750	171	200	–	138.60
2	3,600	15	215.10	200	–	309.38
3	3,600	15	215.10	200	–	495.00
4	3,600	15	215.10	200	–	717.75
5	3,600	15	215.10	200	19,800	–

[4]

Year	In force cashflow	Increase in reserves	Interest on reserves	Profit	Prob in force	Profit signature
t	(7)	(8)	(9)	(10)	(11)	(12)
1	2,682.40	2,915.55	0.00	–233.15	1.0000	–233.15
2	3,290.73	3,295.40	186.00	181.33	0.9405	170.54
3	3,105.10	3,451.45	408.00	61.65	0.8845	54.53
4	2,882.35	3,489.65	654.00	46.70	0.8319	38.85
5	–16,199.90	–15,300	918.00	18.10	0.7824	14.16

[5]

The internal rate of return is the interest rate that satisfies the equation:

$$-233.15v + 170.54v^2 + 54.53v^3 + 38.85v^4 + 14.16v^5 = 0 \qquad [1]$$

By trial and error:

$$i = 11\% \qquad LHS = 2.24 \qquad [1]$$

$$i = 12\% \qquad LHS = -0.68 \qquad [1]$$

So the internal rate of return is approximately 11.8%, or 12% to the nearest percent. [1]
[Total 13]

Solution 4.4

The comparison is made with the fund at the end of year 2 and relates to lives who died at age 51 last birthday. So the expression is:

$$\max\{0,\ 3{,}000 - F_2\} \times q_{51} \qquad [2]$$

Solution 4.5

Equating the present value (calculated at the outset using the accumulation rate) of the expected loss at the end of year 1 (L) and the present value of the expected losses at the end of the first 3 years gives:

$$Lv \quad = \quad 30v + 12v^2 p_{60} + 6v^3 {}_2p_{60} \quad @6\%$$

Dividing through by v gives the relationship:

$$L \quad = \quad 30 + 12vp_{60} + 6v^2 {}_2p_{60} \quad @6\% \qquad [2]$$

Solution 4.6

(i) ***Unit and non-unit funds***

The unit fund is the amount held in units on behalf of the policyholder at any time. [1]

It may not necessarily be the amount that the policyholder is entitled to at that time. For example, if the policy is surrendered, the policyholder may receive only a proportion of the full bid value of the units. [1]

On death, maturity or surrender, the units held will be used to pay the benefit. Any excess/shortfall in the unit fund will give rise to a positive/negative cashflow in the non-unit fund. [1]

The non-unit fund is the net result of the life office's cashflows. [1]

These will arise from the following sources:

- premium less cost of allocation, *ie* the difference between the premium paid by the policyholder and the amount invested in the unit fund on the policyholder's behalf [½]
- expenses incurred by the life office [½]
- interest earned/charged on the non-unit fund [½]
- management charges taken from the unit fund [½]
- extra death or maturity costs (if the benefit payable on death or maturity is greater than the value of the units held at the time of death or maturity) [½]
- profit on surrender (if the benefit payable on surrender is less than the value of the units held at the time of surrender) [½]

[Maximum 6]

(ii) ***Need for reserves in non-unit fund***

The life office may set up reserves in the non-unit fund if the overall cashflow in any year other than year 1 would otherwise be negative. It is desirable that the policy be self-funding after year 1. [1]

Reserves are set up early in the contract so that money can be released from the reserves as required to eliminate the negative cashflow. The regulatory authorities, *eg* FSA in UK, may insist on this. Setting up reserves will result in larger losses (or smaller profits) in the early years of the contract, but the life office will not expect to have to find extra capital to support the policies later on. [1]
[Total 2]

Solution 4.7

A change in the level of underwriting may alter the effect of temporary initial selection, *eg* more rigorous underwriting often leads to increasing the intensity and duration of temporary initial selection. [1]

This may give the impression that mortality rates are improving more quickly than they really are. In other words, the effect of time selection is being distorted, hence spurious selection is occurring. [1]

Conversely, a relaxation of underwriting standards is likely to lessen the effect of temporary initial selection. This may give the impression that mortality rates are worsening or improving at a slower rate than they really are. This again distorts the effect of time selection. [1]
[Total 3]

Solution 4.8

The main difficulty associated with the measurement of social and economic factors in mortality studies is the measurement of the independent variables affecting mortality.
[1]

A person experiences many effects during their lifetime, and it is not merely the values of the factors at the date of death that are the important causes. The retrospective measurement of values during a person's lifetime is difficult. Many of the factors influencing mortality are interrelated, and their individual effects may be extremely difficult (if not impossible) to determine. [1]
[Total 2]

Solution 4.9

It is necessary to consider:

The difficulty of recording the *independent* effect of the occupation on mortality … [1]

…level of intelligence largely dictates type of job and thus it is often the intelligence rather than the occupation that affects the mortality. [1]

Effect of actual occupation

Some occupations require you to work in an unhealthy environment: [1]

eg coal miner. [½]

Others increase risk of accidental death: [1]

eg oil rig worker. [½]

Others are very stressful: [1]

eg football club manager [½]

Non-correspondence (misstatement of occupation) [1]

Where the deaths and exposed to risk do not correspond. [½]

Vagueness in census returns may result in the wrong occupation being recorded. [½]

A widow or widower may elevate the occupation of their spouse so that it sounded as if they had achieved higher office than they actually had. [1]

(For example an electrician may be flatteringly described as an electrical engineer.) [½]

Antecedent occupation – information from censuses and at death usually relates to the most immediate past or present occupation. [½]

An occupation may be selected against: [1]

eg higher mortality might appear to be associated with certain light occupations (*eg* newspaper selling) because they are often taken up following chronic illness or disability emerging during a former occupation. [1]

Lack of statistics – for a small occupation (*eg* actuary) there may not be sufficient data to provide a meaningful statistic. [½]

Conversely, a small occupation is more likely to maintain detailed records of members and thus be able to provide accurate statistics. [½]

Standardised mortality rates should be compared… [1]

…otherwise age and sex distribution will affect the stated rate. [½]

Example of occupation with light mortality [½]

eg actuary

[Bonus marks, ½ each for any other relevant points up to maximum of 1½] [1½]
[Maximum 15]

Solution 4.10

I is correct. The SMR is classed as indirect standardisation because it uses a local population structure rather than a standard structure. [1]

II is correct. The formula for the SMR can be rewritten to show that it is a weighted (by the expected deaths in each age group) average of the relative mortality at different ages. [1]

III is not correct. The SMR does not require the age-sex specific mortality rates of the population under question. This is one of the reasons why it is commonly used, *ie* because it is easy to calculate. (It does however require the age-sex specific mortality rates of a standard population.) [1]
[Total 3]

Solution 4.11

The difference between direct standardisation and indirect standardisation is that the direct method uses the age-sex specific mortality rates of the groups being compared whereas the indirect method merely adjusts the crude death rate of each group. [1]

Direct standardisation thus uses the structure of the standard population, whereas indirect standardisation uses the structure of the local population. [1]

Two disadvantages of direct standardisation are:

- that it requires the age specific rates for each group to be known, and [½]
- the figures are heavily influenced by the mortality rates at older ages. [½]

[Total 3]

Solution 4.12

This is the definition of the area comparability factor. [1]

Solution 4.13

(i) ***Calculating the premium***

To do this, we will need to construct an expression for the net present value. We begin with the in-force expected cashflows. Let P denote the annual premium.

Year t	Premium – expenses + interest	Mortality rate $= q_{[60]+t-1}$	Expected claim cost $= q \times S$ for $(t < 5)$ $= S$ for $(t = 5)$
1	$(P - 200) \times 1.07$	0.005774	144.35
2	$(0.975P - 20) \times 1.07$	0.008680	217.00
3	$(0.975P - 20) \times 1.07$	0.010112	252.80
4	$(0.975P - 20) \times 1.07$	0.011344	283.60
5	$(0.975P - 20) \times 1.07$	0.012716	25,000

[3]

The cashflow each year will then be the premiums less expenses plus interest less expected claim cost. We can then allow for the reserves by calculating the profit vector (the in-force expected profit) in year *t* from:

$$PRO_t = CF_t + {}_{t-1}V \times 1.07 - p_{[60]+t-1} \times {}_tV$$

where ${}_tV$ is the reserve at the beginning of the year (and the other symbols take their obvious meanings).

We calculate the reserves most easily using:

$$_tV = 25{,}000\left(1 - \frac{\ddot{a}_{60+t:\overline{5-t}|}}{\ddot{a}_{60:\overline{5}|}}\right)$$

We obtain:

Year t	$\ddot{a}_{60+t-1:\overline{5-(t-1)}|}$	$_{t-1}V$	$p_{[60]+t-1}$	$_{t-1}V \times 1.07 - p_{[60]+t-1}\ {}_tV$
1	4.550	0	0.994226	–4,523.18
2	3.722	4,549.45	0.991320	–4,353.55
3	2.857	9,302.20	0.989888	–4,182.46
4	1.951	14,280.22	0.988656	–4,004.38
5	1.000	19,505.49	0.987284	20,870.87

[4]

Next we need to multiply the profit vector by the probability of surviving to the start of the year, and by the discount factor calculated at 12% *pa*. So we calculate:

$$NPV_t = PRO_t \times {}_{t-1}p_{[60]} \times v^t$$

For this we need:

Year t	$_{t-1}p_{[60]}$	v^t	$_{t-1}p_{[60]} \times v^t$
1	1	0.892857	0.892857
2	0.994226	0.797194	0.792591
3	0.985596	0.711780	0.701528
4	0.975630	0.635518	0.620031
5	0.964562	0.567427	0.547318

[3]

The profit signature is then given by:

Year t	$PS_t = PRO_t \times {}_{t-1}p_{[60]} \times v^t$
1	$\left[(P-200)\times 1.07 - 144.35 - 4,523.18\right]\times 0.892857$
2	$\left[(0.975P-20)\times 1.07 - 217.00 - 4,353.55\right]\times 0.792591$
3	$\left[(0.975P-20)\times 1.07 - 252.80 - 4,182.46\right]\times 0.701528$
4	$\left[(0.975P-20)\times 1.07 - 283.60 - 4,004.38\right]\times 0.620031$
5	$\left[(0.975P-20)\times 1.07 - 25,000 + 20,870.87\right]\times 0.547318$

[3]

Summing for like terms and setting equal to the profit criterion we get:

$$P(1.07\times 0.892857 + 0.975\times 1.07\times 2.661468)$$

$$-200\times 1.07\times 0.892857 - 20\times 1.07\times 2.661468$$

$$-4,667.53\times 0.892857 - 4,570.55\times 0.792591 - 4,435.26\times 0.701528$$

$$-4,287.98\times 0.620031 - 4,129.13\times 0.547318$$

$$= 0.1\times P$$

So:

$$P\times(3.73193 - 0.1) = 248.03 + 10,901.47 + 4,918.63$$

$$\therefore \quad P = \frac{16,068.13}{3.63193} = £4,424.13$$ [2]

[Total 15]

(ii)(a) ***Reserves calculated using lower interest***

This will increase the reserves at each duration. This will cause the profit to emerge later in the policy term. As the reserves earn a rate of return that is less than the rate at which profits are discounted, then this will reduce the net present value of the profits. In order, therefore, to meet the same profit criterion, the premium will have to be increased. [2]

(ii)(b) ***Risk discount rate increased to 15%***

By discounting profits at a higher rate of interest, the present value of the later emerging profits will be reduced by more than the present value of the profits emerging earlier in the policy term. As the profits become more positive as duration increases (and the profit in the first year is clearly negative), then the positive profits will be reduced by more than the negative ones. As a result, the net present value will decrease, and the premium will have to be increased in order to meet the profit criterion. [2]

(ii)(c) ***Assume ultimate instead of select mortality***

This will increase the expected claim costs during the first two years, reducing the cashflow and profit in those years directly, and will reduce the expected proportion of policyholders surviving each year. This will reduce the expected present value of these later positive profits. Both these will again reduce the overall net present value of the profit, requiring the premium to be increased. [2]

[Total 6]

Solution 4.14

The cashflows occurring at the end of the 18th month are:

(1) Premium of £10 paid if Amit is alive at the beginning of the month

(2) Renewal expenses of £5 paid if Amit is alive at the beginning of the month

(3) A sum assured of $3000 - 100\ddot{s}_{\overline{18}|}^{@1\%}$ paid if Amit dies in the 18th month

The probability that Amit survives to the beginning of the 18th month is:

$$\frac{l_{61\frac{5}{12}}}{l_{60}} \approx \frac{(\frac{7}{12}l_{61} + \frac{5}{12}l_{62})}{l_{60}} = 0.98825 \qquad [1\tfrac{1}{2}]$$

The probability that Amit dies in the 18th month is:

$$\frac{d_{61}}{12l_{60}} = \frac{82.9973}{111,446.6} = 0.0007447 \qquad [1\tfrac{1}{2}]$$

So the expected net outgo is:

$$(3,000 - 1,981.09) \times 0.0007447 - (10 - 5) \times 0.98825 = -£4.18 \qquad [2]$$

[Total 5]

Solution 4.15

(i) ***Emerging profit***

The tables required for the calculations are shown below:

Policy year	Premium allocated	Cost of allocation	Plus fund b/f	Fund before charge	Annual charge	Fund c/f
1	750.00	712.50	712.50	748.13	5.61	742.52
2	1,050.00	997.50	1740.02	1827.02	13.70	1813.32
3	1,050.00	997.50	2810.82	2951.36	22.14	2929.22
4	1,050.00	997.50	3926.72	4123.06	30.92	4092.14

[5]

Policy year	Profit on allocation	Expenses	Non unit interest	Annual charge	Non unit death cost	Profit in each year
1	287.50	150.00	6.88	5.61	22.57	127.42
2	2.50	75.00	–3.63	13.70	11.87	–74.30
3	2.50	25.00	–1.13	22.14	0.71	–2.20
4	2.50	25.00	–1.13	30.92	0.00	7.29

[5]
[Total 10]

(ii) ***Revised profit***

The non-unit reserves required are:

Start of year 3: $2.20/1.05 = 2.10$ [1]

End of year 2: $2.10 \times 0.99 + 74.3 = 76.38$ [1]

Start of year 2: $76.38/1.05 = 72.74$ [1]

End of year 1: $72.74 \times 0.99 = 72.01$ [1]

The calculations of the revised profit figures are shown in the table below:

Policy year	Non unit reserve b/f	Interest on reserves	Increase in reserves	Profit ignoring reserves	Profit allowing for reserves
1	0.00	0.00	72.01	127.42	55.41
2	72.74	3.64	–70.66	–74.30	0.00
3	2.10	0.10	–2.10	–2.20	0.00
4	0.00	0.00	0.00	7.29	7.29

[5]
[Total 9]

Solution 4.16

(i) ***Profit signature***

The calculations are shown in the following tables.

Year t	Premium (1)	Expenses (2)	Interest (3)	Mortality probability (4)	Death benefit + termination expenses (5)	Expected death cost (6)
1	8,000	500	525.00	0.004171	20,100	83.84
2	8,000	30	557.90	0.006180	21,300	131.63
3	8,000	35	557.55	0.007140	22,572	161.16

[3]

Year t	Dependent surrender probability (7)	Surrender value + terminat'n expenses (8)	Expected surrender cost (9)	Maturity value + terminat'n expenses (10)	Survival probability (11)	Expected maturity cost (12)
1	0.149374	8,340.00	1,245.78	0	0.846455	0
2	0.049691	16,827.20	836.16	0	0.944129	0
3	0	0	0	26,302.35	0.992860	26,114.55

[6]

Year t	Cash flow (13)	Reserve at start of year (14)	Interest on reserve (15)	Reserve required at end year (16)	Profit vector (17)	Survival probability (18)
1	6,695.38	0	0	6,336.14	359.24	1
2	7,560.11	7,485.50	523.98	14,547.34	1,022.25	0.846455
3	– 17,753.16	15,408.21	1,078.57	0	– 1,266.38	0.799163

[4½]

Year t	Profit signature (19)
1	359.24
2	865.29
3	– 1,012.04

[½]

Key to tables:

$(3) = \left[(1) - (2)\right] \times 0.07$

$(5)_t = 20{,}000\left(1.06\right)^{t-1} + 100$

$(6) = (4) \times (5)$

$(7) = \left[1 - (4)\right] \times \left[\text{surrender probability}\right]$

$(8)_1 = 8{,}000 \times 1.03 + 100$

$(8)_2 = 8{,}000\left(1.03^2 + 1.03\right) + 100$

$(9) = (7) \times (8)$

$(10)_3 = 20{,}000 \times 1.06^3 \times 1.1 + 100$

$(11) = 1 - (4) - (7)$

$(12) = (10) \times (11)$

$(13) = (1) - (2) + (3) - (6) - (9) - (12)$

$$(14)_t = 20{,}000 \times 1.06^{t-1} A^{4\%}_{57+t-1:\overline{3-(t-1)|}} - 20{,}000 \frac{A^{4\%}_{57:\overline{3|}}}{\ddot{a}^{4\%}_{57:\overline{3|}}} \ddot{a}^{4\%}_{57+t-1:\overline{3-(t-1)|}}$$

$(15) = (14) \times 0.07$

$(16)_t = (14)_{t+1} \times (11)_t$

$(17) = (13) + (14) + (15) - (16)$

$(18)_1 = 1$

$(18)_t = (18)_{t-1} \times (11)_{t-1}, \ t > 1$

$(19) = (17) \times (18)$ [Total 14]

(ii) ***Affordability of future bonuses***

Assuming 7% *pa* investment return, the first two years' cash flows accumulate to the following value at the end of year 3:

$$359.24 \times 1.07^2 + 865.29 \times 1.07 = £1,337.15 \qquad [1]$$

The outgo in the third year is £1,012.04. So the company can afford all the assumed bonuses during the policy with an additional profit of £325.11 at the end of the term. So, given these assumptions, it would be quite likely for the company to pay higher bonuses than these (though it depends upon how much profit is required for any shareholders). [1]

[Total 2]

Solution 4.17

(i) ***Completed table***

The completed table should read as follows:

Year	Premiums	Expenses	Interest	Expected cost of claims	Expected cost of increasing reserves	Profit vector
1	*1,530*	*–50*	88.80 (1)	–50 (2)	–1,569.80 (3)	*–51.00*
2	*1,530*	–13.30 (4)	91.00 (6)	–50 (2)	–1,536.70 (4)	*21.00*
3	*1,530*	–20.57 (5)	90.57 (6)	–5,000(2)	3,445.00	*45.00*

(Pre-calculated figures are shown in italics.) The missing figures can be derived using the following steps (which are indicated in brackets beside the figures in the table).

(1) Interest in the first year is $(1,530 - 50) \times 0.06$. [½]

(2) The expected cost of claims in Years 1 and 2 is $5,000 \times 0.01$. The expected cost of claims in Year 3 is 5,000 since all policies in force at the start of Year 3 will receive a benefit of 5,000 at the end of Year 3. These figures are shown as negative entries in the table as they are outgo for the insurer. [1]

(3) Let e_t denote the amount of the expense paid at time t and PRO_t be the profit vector term at time t. If the policyholder is aged x at time 0, then the recursive formula tells us that:

$$\left({}_tV + P - e_t\right)(1+i) = {}_{t+1}V\,p_{x+t} + S\,q_{x+t} + PRO_{t+1}$$

for $t = 0, 1$. This can also be written as:

$$PRO_{t+1} = \left(P - e_t\right)(1+i) - \underbrace{\left({}_{t+1}V\,p_{x+t} - {}_tV(1+i)\right)}_{\substack{\text{expected cost of increasing}\\ \text{the reserve at time } t+1}} - S\,q_{x+t}$$

So for Year 1, we have:

$$-51 = (1{,}530 - 50) \times 1.06 - \underbrace{\left({}_1V \times 0.99 - 0\right)}_{\substack{\text{expected cost of}\\ \text{increasing the}\\ \text{reserve at time 1}}} - 5{,}000 \times 0.01$$

and hence the expected cost of increasing the reserve at the end of Year 1 is 1,569.80 (which is shown in the table as a negative since it represents a cost to the insurer) … [1]

… and:

$${}_1V = \frac{1{,}569.80}{0.99} = 1{,}585.66$$ [½]

(4) For Year 2, the recursive formula is:

$$21 = \left(1{,}530 - e_1\right) \times 1.06 - \underbrace{\left({}_2V \times 0.99 - 1{,}585.66 \times 1.06\right)}_{\substack{\text{expected cost of}\\ \text{increasing the}\\ \text{reserve at time 2}}} - 5{,}000 \times 0.01$$

But we know that the interest earned on the reserve in the third year is 195, so:

$${}_2V \times 0.06 = 195 \Rightarrow {}_2V = 3{,}250$$ [½]

Hence the expected cost of increasing the reserve at the end of the second year is:

$$3,250 \times 0.99 - 1,585.66 \times 1.06 = 1,536.70 \qquad [1]$$

and:

$$e_1 = 1,530 - \frac{21 + 50 + 1,536.70}{1.06} = 13.30 \qquad [1]$$

(5) For Year 3, the recursive formula is:

$$45 = (1,530 - e_2) \times 1.06 - \underbrace{\left({}_3V \times 0.99 - {}_2V \times 1.06\right)}_{\substack{\text{expected cost of} \\ \text{increasing the} \\ \text{reserve at time 3}}} - 5,000$$

But ${}_3V = 0$, so the expected cost of increasing the reserve at time 3 is

$$-{}_2V \times 1.06 = -3,250 \times 1.06 = -3,445$$

Note that this is negative because the reserves that have been built up over the term of the policy are released at time 3 to cover the benefits.

We can summarise the expected cost of increasing reserves calculations in a table:

Year	Reserve at start of year	Interest	Expected reserve at end of year	Expected cost of increasing reserves
1	0	0	1,569.80	–1,569.80
2	1,585.66	95.14	3,217.50	– 1,536.70
3	3,250.00	195.00	0	3,445.00

So we have:

$$e_2 = 1,530 - \frac{45 + 5,000 - 3,445}{1.06} = 20.57 \qquad [½]$$

(6) So, finally, the interest earned on the fund in Years 2 and 3 is:

$$(1{,}530-13.30)\times 0.06 = 91.00 \quad \text{for Year 2}$$ [½]

$$(1{,}530-20.57)\times 0.06 = 90.57 \quad \text{for Year 3}$$ [½]

[Total 7]

(ii) ***Internal rate of return***

The internal rate of return is the interest rate that satisfies the equation:

$$-51v + 21\times 0.99v^2 + 45\times 0.99^2 v^3 = 0$$ [1]

(where the 0.99 factors represent persistency).

Dividing through by v gives:

$$-51 + 20.79v + 44.10v^2 = 0$$

Now for the quadratic equation $ax^2 + bx + c = 0$ we know the that roots will be given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Applying this quadratic formula gives (ignoring the negative root):

$$v = 0.8652 \Rightarrow i = \frac{1}{0.8652} - 1 = 15.6\%$$ [1]

[Total 2]

Alternatively, you could solve the equation of value using trial and error and/or interpolation.

(iii) ***Effect on internal rate of return***

The internal rate of return will increase. [½]

This is because the profits in the later years will be released sooner, which will increase their value. [1½]

[Total 2]

(iv) ***Net present value***

The net present value calculated using a risk discount rate of 7% is:

$$NPV = -51v + 20.79v^2 + 44.10v^3 = -47.66 + 18.16 + 36.00 = £6.50 \qquad \text{[Total 2]}$$

(v) ***Effect on net present value***

The net present value will increase. [½]

This is because the profits in the later years will be released sooner, which will increase their present value. [1½]
[Total 2]

Solution 4.18

Time selection can distort the results of an investigation through mortality rates changing over time. This distorts the results with the relative weights of different years' experience being combined. The resultant table cannot therefore be used as a wholly reliable guide for future use. [1]

For practical purposes, the period of a mortality investigation should be kept quite short to limit the effects of time selection. As short a time period as possible should be chosen as is compatible with a reasonable amount of data. There is thus a trade off between time selection (as a result of too long a period of investigation) and unreliable data (as a result of too short a period of investigation) through too small a sample size.
[2]

Four years is the period that most standard tables are based on. The investigation period should ideally cover an "average" period and take in one severe, one mild and two average winters, say. [1]

Class selection arises when groups of lives who will experience different mortality levels (due to some different permanent characteristics) are combined in a single experience. The resulting mortality rates are a weighted average of the sub-groups and should not be applied to situations where there will be different proportions of the various sub-groups. In practice it is desirable to divide an experience into groups that are as homogeneous as possible. Typical subdivisions are age, sex, type of contract, duration in force, and smoker / non-smoker. [2]
[Total 6]

Solution 4.19

We first find the mortality rates for each age group in the country as a whole:

$$^{s}m_Y = \frac{3,402}{10,200,000} = 0.000334$$

$$^{s}m_M = \frac{18,602}{16,800,000} = 0.001107$$

$$^{s}m_O = \frac{98,440}{12,900,000} = 0.007631$$

where $Y = \text{young}$, $M =$ middle-aged and $O = \text{old}$.

We can apply these rates to Regions A and B in order to determine the expected number of deaths in each region, if regional mortality had been in line with that of the country as a whole. The expected number of deaths in Region A is:

$$600,000 \times 0.000334 + 1,284,000 \times 0.001107 + 920,000 \times 0.007631 = 8,642.31$$

[1]

Comparing this with the actual total number of deaths for Region A, we get:

$$SMR = \frac{282 + 3,129 + 9,617}{8,642.31} \times 100 = 150.7$$ [1]

So the standardised mortality ratio for Region A is 150.7.

Repeating the process for Region B, we find that:

$$SMR = \frac{1,106 + 2,214 + 8,473}{11,671.56} \times 100 = 101.04$$ [1]

So the standardised mortality ratio for Region B is 101.0.

Mortality in Region B appears to be very much in line with that of the country as a whole. However, mortality in Region A is substantially higher at all ages. This is confirmed by the individual age-related mortality rates:

	Region A	Region B
Young	0.00047	0.000335
Middle-aged	0.00244	0.001127
Old	0.01045	0.007703

The next stage in the investigation might be to consider why rates are so much higher in Region *A*. Are there any obvious causes for the much higher mortality here? [2]
[Total 5]

Solution 4.20

(i) ***Selection definitions***

(a) ***Time selection***

Time selection refers to the fact that mortality changes over time. Thus a mortality investigation that covers a long time period (*eg* 10 years) will experience time selection. The resultant mortality rate will be a weighted average of the mortality rates over the period of investigation. [1]

If the period of investigation is kept short (say no more than four years) then the rate will have changed little and the estimated value will approximate the rate in any of the years of the experience. [1]

The British Offices' experience 1863-93 almost certainly suffered from time selection.

(b) ***Spurious selection***

Sometimes mortality in two classes (for particular ages and sexes) differs. This difference may reflect true mortality differences or, in part or in whole, differences due to the composition of the two classes. If the classes are divided by this compositional variable, the mortality differences between the classes will change. [1]

For example, North versus South group life rates change when we control for occupation. (Part of the North-South difference is due to the different occupational composition of the two regions.) [1]

Hence, spurious selection refers to the situation where selection appears to exist when it actually does not, or when the effect of selection is exaggerated or diminished. [1]

[Total 5]

(ii)(a) ***Class selection and temporary initial selection***

It is necessary to consider the values of q_{50} for each section of the data.

Year	Curtate duration 0			Curtate duration 1^+		
	E_{50}	θ_{50}	q_{50}	E_{50}	θ_{50}	q_{50}
			Works			*Works*
1998	10,000	20	0.00200	20,000	50	0.00250
2003	8,421	16	0.00190	20,000	48	0.00240
2008	5,000	9	0.00180	16,959	39	0.00190
			Staff			*Staff*
1998	10,000	16	0.00160	5,000	10	0.00200
2003	7,333	11	0.00150	14,211	27	0.00190
2008	2,143	3	0.00140	33,889	61	0,00180

Class selection

Class selection is demonstrated by the difference between the mortality rates of the works and staff employees. The mortality of staff employees is lighter than works employees for every comparable calendar year and duration. Staff employees will generally be paid more, have greater intelligence and hence look after themselves better. [2]

Temporary Initial Selection

Temporary initial selection is demonstrated by the mortality rates at duration 0 being less than those at duration 1 for every comparable class of employee and year. This will occur due to active service mortality, *ie* you will not be offered a job if you are not healthy at the time. Some companies also require you to pass a medical before you join. [2]

[Total 4]

(b) ***Spurious selection***

To achieve spurious selection, we need to mix two groups of lives and show that their combined mortality experience is different to that in the separate groups. Trends observed in the disaggregated data will be obscured by the aggregation. [1]

The data can be aggregated by:

(1) Year & category (*ie* ignore the split by duration)

Year	E_{50}	θ_{50}	q_{50}
		Works	
1998	30,000	70	0.00233
2003	28,421	64	0.00225
2008	21,959	48	0.00219
		Staff	
1998	15,000	26	0.00173
2003	21,544	38	0.00176
2008	36,032	64	0.00178

The figures above suggest the mortality of works employees is improving at a lesser rate than is really true, *ie* spurious selection is present. The figures suggest the mortality of staff is slightly deteriorating which is also not true, hence spurious selection is again present. [2]

The distortions are caused (as always with spurious selection) by the relative numbers in each different category.

(2) Category & duration (*ie* aggregate years)

	Curtate Duration 0			Curtate Duration 1^+		
	E_{50}	θ_{50}	q_{50}	E_{50}	θ_{50}	q_{50}
Works	23,421	45	0.00192	56,959	137	0.00241
Staff	19,476	30	0.00154	53,100	98	0.00185

The above figures diminish the effect of duration on mortality for staff, *ie* spurious selection is present. There is no evidence that this has occurred with the figures for the works employees. [2]

(3) Duration & year (*ie* aggregate staff with works)

Year	Curtate Duration 0			Curtate Duration 1+		
	E_{50}	θ_{50}	q_{50}	E_{50}	θ_{50}	q_{50}
1998	20,000	36	0.00180	25,000	60	0.00240
2003	15,754	27	0.00171	34,211	75	0.00219
2008	7,143	12	0.00168	50,848	100	0.00197

Again spurious selection is present. At duration 0 the improvement in mortality is understated by the above figures. At duration 1 the improvement in mortality over time is exaggerated. [2]

Three alternative methods of aggregating the data are:

(4) Mixing data for all times and both classes to give mortality rates of:

"duration 0" = 0.00175 "duration 1" = 0.00214

The pattern is the same as for the disaggregated rates, hence there is no evidence of spurious selection. [2]

(5) Mixing data for all durations and both classes to give mortality rates of:

"1998" = 0.00213 "2003" = 0.00204 "2008" = 0.00193

The pattern is the same as for the disaggregated rates, hence there is no evidence of spurious selection. [2]

(6) Mixing data for all durations and times to give mortality rates of:

"works" = 0.00226 "staff" = 0.00176

The pattern is the same as for the disaggregated rates, hence there is no evidence of spurious selection. [2]

[Maximum 7]

(c) ***Comment***

In 1998 there were 96 deaths from 45,000 lives giving $q_{50} = 0.00213$

This can be compared with the individual rates, *ie*

1998, duration 0:	q_{50} (staff) = 0.00160	q_{50} (works) = 0.00200
1998, duration 1:	q_{50} (staff) = 0.00200	q_{50} (works) = 0.00250

Hence this would produce cheap insurance for works employees of duration 1^+ which would be subsidised by the other groups. [2]

In 2008 there were 112 deaths from 57,991 lives giving $q_{50} = 0.00193$

2008, duration 0:	$q_{50}(\text{staff}) = 0.00140$	$q_{50}(\text{works}) = 0.00180$
2008, duration 1:	$q_{50}(\text{staff}) = 0.00180$	$q_{50}(\text{works}) = 0.00230$

The same comment can be made as applied to the 1998 rates, except this time the works employees at duration 1 comprise 29% of the people, as opposed to 44% in 1998. This suggests an increased overall margin. [2]

[Total 4]

Solution 4.21

(i) ***Types of selection***

Temporary initial selection arises when mortality changes with the passage of time from some event, quite apart from the change associated with varying age. However, after a certain period of time no further such change is evinced (or any such change is of negligible size). [1]

An example would be life assurance policyholders who have passed the medical underwriting tests of a life company, *eg* medical questionnaire and possible examination, before being accepted for their policy. After passing the test they have much better mortality than those who passed a similar test 10 or 20 years previously, but this superiority quickly wanes. [1]

Time selection arises when the mortality of a homogeneous group of lives changes over time. [1]

For instance, the mortality of a life company's policyholders in 2005 would probably be different to that of equivalently-aged policyholders in 1985, due to general improvements in mortality between these dates. [1]

Spurious selection arises when:

- we compare two groups of lives and observe different mortality,
- we then ascribe the variation to some apparent difference between the groups,
- but the mortality difference has arisen (at least partly) because the groups are heterogeneous with respect to some subtle factor which influences mortality,
- and the two groups have different proportions of these heterogeneous sub-groups.

[2]

An example would be measuring mortality of different age groups and ascribing all difference to age (*ie* class selection), when some of the difference in mortality is due to the older age groups' containing a higher proportion of smokers than the younger age groups. [1]

[Total 7]

(ii) ***Select period of 15 years***

If the underlying data indicate that, for instance, q_x is significantly different to $q_{[x-14]+14}$ then the period would seem statistically justifiable. [1]

However there are a number of riders:

- the amount of data in each cell of such an investigation could be very low, and lead to spurious results;
- the select period is so long that time selection will probably interfere; and,
- there may be an effect arising from selective withdrawals over such a long select period.

[1]

Even if the results are "theoretically" justifiable bearing in mind the above comments, whether they are justifiable in practice will depend on whether they have a significant impact on life policy premium and reserve calculations; if they do not then there is little point in using such a long period. Note that a long select period is likely to have significant impact only on term assurance business. [1]

It also seems intuitively unlikely that someone who passed a simple medical (perhaps just successful completion of standard health-related questions) 14 years ago should evince different mortality to someone who passed the same check 15 years ago, which is the implication of a 15 year select period. [1]

[Total 4]

Solution 4.22

(i) ***Explanations of the SMRs***

Electrical Engineers (so described): SMR=317

The ratio looks abnormally high which is very probably due to the lack of correspondence between the numerator and the denominator. The status of many electricians may have been elevated by the widow or widower to that of electrical engineer. The result would be to have too many deaths marked against this occupation. The exposed to risk will have been more accurately estimated from the census data, when the individual concerned would have given the information. [2]

Electrical Engineers (trained to university standard): SMR=42

This ratio reflects the relative wealth, education and socio-economic group of "real" electrical engineers. The lifestyle of this group will result in lower than average mortality. Misrecording of information may have resulted in the exposed to risk being overstated; however, we would still expect the ratio to be significantly less than 100%.
[2]

Publicans, innkeepers: SMR=155

The relative mortality of those involved with the sale of alcohol is higher than that of the general population. Reasons include the fact that these people are more likely to consume an excessive amount of alcohol, which is detrimental to health. Publicans will also spend a lot of their time in an unhealthy smoky atmosphere. It is possible that a previous occupation of a publican could contribute to poor health, although it is equally possible that the previous occupation could have been far more healthy. [2]

Physiotherapists: SMR=55

Physiotherapists are very aware of the factors affecting health, and this, combined with their education level and good income, contributes to the relatively low mortality rate. Physiotherapists in poor health would not be a good advert for their services and thus they make efforts to protect and enhance their health. [2]
[Total 8]

(ii) ***Other influences on mortality differentials***

Other social and economic factors that influence mortality differentials include:

- wealth and income [½]
- employment situation [½]
- housing [½]
- political situation [½]
- access to medical facilities [½]
- mental attitude [½]
- education [½]
- exercise [½]
- weight [½]
- intelligence [½]
- nutrition [½]
- smoking [½]
- alcohol consumption [½]
- drugs [½]

[Maximum 6]

(There are many minor factors. The list given above contains the more important factors.)

Solution 4.23

Class selection can arise in the following ways: [½]

- The mortality of lives that retire early but in good health, or at normal pension age, is likely to be lower than that of those lives who retired on grounds of ill health. [½]
- Different sections of a large pension scheme may exhibit different levels of mortality, *eg* managers and shop-floor workers. [½]
- Males and females experience different mortality. [½]

Temporary initial selection can arise in the following ways: [½]

- The mortality of those lives who retired on grounds of ill heath is likely to depend on duration since retirement for the first few years following retirement. Subsequently, it is likely to depend only on attained age. [½]
- Underwriting at the date of joining a pension scheme tends to be very limited. So there is only very slight temporary initial selection at this stage due to underwriting. But there is the "healthy worker" effect, *ie* people joining the scheme have usually just started work for the company and are probably in good health because they have just passed the employer's minimum entry requirement for health. [½]

Time selection can arise in the following way: [½]

- Withdrawal from a pension scheme can be associated with redundancy. If redundancy is the cause, withdrawal rates may vary markedly over time as economic conditions vary. [½]
- Mortality may be varying over time. [½]

[Maximum 4]

Solution 4.24

(i) ***Pros and cons of single figure indices***

An advantage of using a single figure index is that a single figure is more easily assimilated than a set of figures. It enables the reader to assess at a glance the relative mortality of the group in question. [1]

A disadvantage of using a single figure index is that the crude death rate obtained is heavily dependent on the age and sex structure of the population. A single figure index will miss any quirks in the age-sex specific rates that may exist. [1]

Another drawback is that if there are no age-sex specific rates available, any possible errors or quirks will not be investigated further and will remain in the data, thus producing results that may not reflect the true situation. A single figure index may not be adequate for all purposes. [1]

The most desirable option would be to have both the age-sex specific rates and the single figure index available. This would overcome the problems outlined above. [1]

[Total 4]

(ii) ***Calculations and comment***

(a) ***Crude death rates***

The crude death rates (total deaths/total exposed to risk) are as follows:

Region X $\dfrac{68,196}{3,600,000} = 0.01894 \;\; ie \;\; 18.9 \; per \; 1,000$

Region Y $\dfrac{21,422}{1,734,000} = 0.01235 \;\; ie \;\; 12.4 \; per \; 1,000$

Country $\dfrac{709,750}{50,000,000} = 0.01420 \;\; ie \;\; 14.2 \; per \; 1,000$

[2]

(b) ***SM rates and SM ratios***

The following table shows m_x for the two regions and for the country as a whole:

Age Group	Region X	Region Y	Country
0-14	0.00023	0.00026	0.00025
15-39	0.00084	0.00086	0.00083
40-59	0.00542	0.00542	0.00550
60-79	0.04900	0.04608	0.04700
80+	0.17200	0.16881	0.17000

The following table shows the population structure for the two regions and for the country as a whole:

Age Group	Region X	Region Y	Country
0-14	0.16389	0.23529	0.20400
15-39	0.27222	0.29412	0.33600
40-59	0.29167	0.29988	0.25800
60-79	0.24167	0.14994	0.17800
80+	0.03056	0.02076	0.02400

The standardised mortality rate weights the age-specific regional mortality rates by the population structure for the whole country. The **standardised mortality rate** for each region is thus:

Region X

$$\begin{aligned}&0.204 \times .00023 + \\ &0.336 \times .00084 + \\ &0.258 \times .00542 + \\ &0.178 \times .04900 + \\ &0.024 \times .17200\end{aligned} = \underline{0.01458} = 14.6 \text{ per } 1{,}000 \qquad [2]$$

Region Y

$$\begin{aligned}&0.204 \times .00026 + \\ &0.336 \times .00086 + \\ &0.258 \times .00542 + \\ &0.178 \times .04608 + \\ &0.024 \times .16881\end{aligned} = \underline{0.01399} = 14.0 \text{ per } 1{,}000 \qquad [2]$$

The standardised mortality ratio weights the country-wide mortality rates by the population structure for the particular region. The crude death rate for the region is divided by this number. The **standardised mortality ratio** for each region is thus actual deaths divided by expected deaths:

Region X

Actual deaths = 68,196

$$\text{Expected deaths} = 1{,}000 \times (590 \times .00025 + 980 \times .00083 + 1{,}050 \times .00550 + 870 \times .04700 + 110 \times 0.17000) = 66{,}326$$

$\Rightarrow$ SMR = $\underline{1.028}$ [1½]

Region Y

Actual deaths = 21,422

$$\text{Expected deaths} = 1{,}000 \times (408 \times .00025 + 510 \times .00083 + 520 \times .00550 + 260 \times .04700 + 36 \times 0.17000) = 21{,}725$$

$\Rightarrow$ SMR = $\underline{0.986}$ [1½]

(c) ***Comment***

The various indices are summarised below.

Region	X	Y
Crude rate (per 1,000)	18.9	12.4
Standardised rate (per 1,000)	14.6	14.0
SMR	1.028	0.986

The age specific death rates (and hence the standardised mortality rates) are similar for the two regions. (Region *X* has a slightly higher standardised rate due to the heavier mortality of its inhabitants aged 60 and over.) [1]

The standardised mortality ratio indicates differences between regions and assesses mortality relative to standard levels. Although the SMR gives no absolute measure of mortality, it indicates mortality levels relative to that experienced in the standard population. [1]

The crude death rates are very different due to the population structures of the two regions. Region X has a crude death rate over 50% higher than that of Region *Y*, due to its relatively older population. [1]

The results highlight the points made in part (i), *ie* single figure indices can be useful, but when used in isolation they can be particularly misleading. [1]

[Total 13]

Solution 4.25

(i) ***Unit reserves***

The following table shows the figures required:

Year	Unit reserve at start of year	Face value of units at end of year	Mgmt charge
1	10,000.00	10,682.00	218.00
2	10,682.00	11,410.51	232.87
3	11,410.51	12,188.71	248.75
4	12,188.71	13,019.98	265.71
5	13,019.98	13,907.94	283.84

[Total 3]

(ii) ***Net present value***

The following table shows the calculation of the profit signature.

Year	Expenses	Interest	Mgmt charge	In force profit	Prob in force	Profit
1	–570.00	–51.30	218.00	–403.30	1.000	–403.30
2	–21.00	–1.89	232.87	209.98	0.9453	198.48
3	–22.05	–1.98	248.75	224.71	0.8935	200.78
4	–23.15	–2.08	265.71	240.48	0.8446	203.10
5	–24.31	–2.19	283.84	257.34	0.7983	205.44

[3]

where probability in force at start of year $t = (0.95\times 0.995)^{t-1}$

The net present value of the contract at a risk discount rate of 12% is:

$$-\frac{403.30}{1.12}+\frac{198.48}{1.12^2}+\frac{200.78}{1.12^3}+\frac{203.10}{1.12^4}+\frac{205.44}{1.12^5}=186.69 \qquad [1]$$

[Total 4]

Part 5 – Revision Questions

This part contains 100 marks of questions testing the material from the whole course. You may like to try these questions under exam conditions as a mock exam.

Question 5.1

A certain pension scheme requires its members to contribute at a rate of 6½% of salary each month for a maximum period of 40 years. Calculate the value of future contributions for a member now aged exactly 30 with 5 years' past service whose earnings for the past year were £22,000. You may assume that interest rates, salary scales and decrements are the same as those underlying the pension fund functions in the *Tables*. [2]

Question 5.2

The future lifetime of a new-born person is defined to be the random variable T, which is continuously distributed on the interval $[0,\omega]$, where $0<\omega<\infty$. Assuming ELT15 (Females) mortality, calculate the following probabilities.

(a) $P(T>40\,|\,T>25)$

(b) $P(40<T\leq 70\,|\,T>25)$ [2]

Question 5.3

A level assurance policy pays a lump sum of £50,000 immediately on the death of a life aged 50, provided that he survives to his 60th birthday, and that his death occurs before that of another life currently aged 55.

Derive an expression in terms of assurance functions for the expected present value of this benefit, assuming that both lives are subject to the same mortality table. [4]

Question 5.4

A life insurance company sells a last survivor annuity of £500 *pa* to a man aged 65 and a woman aged 70. The single premium for the policy is £8,500. The benefit is paid annually in advance.

(i) Assuming that expenses can be ignored, write down a random variable that represents the present value of the profit made on one of these policies, as at the issue date of the policy, immediately before the premium is paid. [2]

(ii) Find the expected value of the random variable you have written down in part (i). [3]
[Total 5]

Basis: Mortality: PA92C20
Interest 4% *pa*

Question 5.5

The unit price for an insurer's unitised with-profits (UWP) contracts on 1st January of a particular year is £4.25. There is a guaranteed bonus interest rate of 1% per annum, and for the coming year (which is not a leap year) the discretionary bonus interest is to be 3% per annum, so that the overall increase in value would be 1.01×1.04 for the whole year. The bonus interest payments are credited daily to the policy by increasing the unit price at the equivalent daily interest rates. A regular charge of £3 is deducted on the last day of each month throughout the contract, paid for by cancelling units.

All unit transactions (allocations or cancellations) that take place on any individual day are calculated on the same unit price.

A particular policyholder is paying a regular premium of £500 per month, paid on the first day of every month.

On 1st March of this same year this policyholder holds 3,200 units at the start of the day, immediately prior to the payment of the monthly premium.

(i) Calculate the value of this policyholder's fund on 12th April of the same year. [5]

(ii) Assuming that the policyholder's fund on 12th April does have the value you have calculated in part (i), give two possible reasons why the death benefit payable on this day might be different from this amount. [1]
[Total 6]

Question 5.6

A special temporary annuity pays £2,000 annually in advance to a man aged exactly 60. The annuity will be paid for two years or until the policyholder's earlier death. If the policyholder survives to his 62nd birthday, he will receive an additional lump sum payment of £5,000.

Calculate the expected present value and the standard deviation of the present value of the benefits payable to the policyholder under this contract.

Basis: AM92 Ultimate mortality and 4% *pa* interest. [6]

Question 5.7

A multiple decrement table that allows for age retirements and deaths between the ages of 61 and 63 is as follows:

Age	$(al)_x$	$(ad)_x^r$	$(ad)_x^d$
61	10,000	1,291	494
62	8,215	1,471	662
63	6,082		

$(ad)_x^r$ refers to the number of retirements, and $(ad)_x^d$ to the number of deaths between the ages of x and $x+1$. Retirements and deaths occur continuously over each year of age.

Following improvements in the mortality experience, it is decided to construct a new table with the independent forces of mortality reduced by 40%.

Construct the new multiple decrement table, assuming that the forces of decrement are constant over each year of age. [8]

Question 5.8

A special ten-year endowment assurance pays £50,000 at the end of the year of death during the term, or £10,000 on survival to the end of the term. Premiums are paid annually in advance whilst the policy remains in force.

Calculate the retrospective net premium reserve for such a contract, issued to a man aged 50 at entry, immediately before the fourth premium payment is due.

Basis: AM92 Ultimate mortality, and 4% *pa* interest. [7]

Question 5.9

A life insurance company issued 1,000 identical single premium deferred annuity contracts to men aged 40 exact on 1 January 2005. The contract provides an annual pension of £10,000 payable annually in advance from 1 January 2025 and for the whole of life thereafter, or a return of the single premium immediately on death before that date.

(i) Calculate the net single premium.

Basis: before retirement assume AM92 Ultimate mortality and 4% *pa* interest; after retirement assume interest and mortality such that $a_{60} = 17$. [4]

(ii) On 1 January 2010, there were 976 of the original policies still in force; one year later the number in force had reduced to 973. Death was the only cause of exit during the year.

Calculate the company's mortality profit from these policies during the year 2010, assuming that the company calculates its reserves on the same basis as the premium basis. [5]

[Total 9]

Question 5.10

(i) What are the advantages of using a single figure index to measure mortality? [2]

(ii) What are the disadvantages of using a single figure index to measure mortality? [4]

(iii) What is the directly standardised mortality rate? Explain why standardised mortality indices are used. [3]

(iv) Write down the formula for the standardised mortality ratio, defining all the symbols that you use. [3]
[Total 12]

Question 5.11

A life insurance company sells 25-year with-profit endowment assurances to lives aged 30 exact. The basic sum assured is £50,000, and compound bonuses of 1.923% are added to the sum assured at the end of each year. The death benefit is payable at the end of the year of death, after the bonus amount for the current year has been added. Level premiums are payable monthly. The basis is as follows:

Mortality: AM92 Select

Interest: 6% *pa*

Expenses: Initial, 30% of the first year's premiums, payable at the start of the contract

Renewal, 5% of all premiums, including the first year, payable at the start of each year.

(i) Show that the monthly premium is £130.91. [6]

(ii) Find the gross premium prospective reserve just before the start of the tenth year of the policy, assuming that bonuses have been declared according to the initial assumptions. [6]
[Total 12]

Question 5.12

A man aged exactly 40, is a member of a pension scheme that pays a retirement pension of 1/60th of final pensionable salary for each past year of service. Final pensionable salary is defined to be the average salary earned over the previous three years. Retirement can take place at any age between 60 and 65. The member has earned £20,000 over the past year, and currently has 5 years' past service.

Defining all your terms, and stating any assumptions made, derive expressions in terms of suitable commutation functions for valuing the past service benefit and future service benefit for this member. [13]

Question 5.13

A life office is planning to issue a new series of 3-year unit linked endowment policies. Two designs of policy are under consideration, both having level annual premiums of £1,000.

Type A: 85% of the first year's premium and 101% of each subsequent premium is invested in units. On surrender the bid value of the units allocated is paid.

Type B: 95% of each premium is invested in units. On surrender the bid value of the units allocated is paid less a penalty of 10% of the total premiums outstanding under the policy.

There is a bid/offer spread in unit values, the bid price being 95% of the offer price. A fund management charge of 1% of the value of the policyholder's fund is deducted at the end of each policy year.

The death benefit, which is payable at the end of the year of death, and the maturity value are equal to the bid value of the units allocated. Surrenders are assumed to take place at the end of the year.

The office's expenses in respect of the policy are £100 at the start of the first year and £30 at the start of the second and third years.

The office holds unit reserves equal to the bid value of the units and zero non-unit reserves.

The dependent probability of mortality at each age is assumed to be 1% and the dependent probability of surrender at each duration is 5%.

The non-unit fund is assumed to grow at the rate of 7½% *pa*.

(i) Calculate the unit fund values at the end of each year assuming that the growth in unit value is 7½% *pa* and hence calculate the estimated maturity proceeds for each policy type. [6]

(ii) Calculate the net present value of the profit that is expected to arise under each policy type, using a discount rate of 10% *pa*. [8]
[Total 14]

Part 5 – Revision Solutions

Solution 5.1

The member entered at age 25, so he will complete his 40 years' service at age 65, the retirement age underlying the *Tables*. So the present value of his future contributions will be:

$$0.065 \times 22{,}000 \times \frac{{}^{s}\bar{N}_{30}}{s_{29}D_{30}} = 0.065 \times 22{,}000 \times \frac{680{,}611}{4.991 \times 7{,}874} = £24{,}766 \qquad [2]$$

Solution 5.2

(a) $P(T > 40 \mid T > 25) = {}_{15}p_{25} = \dfrac{l_{40}}{l_{25}} = \dfrac{97{,}952}{98{,}797} = 0.99145$ [1]

(b) $P(40 < T \leq 70 \mid T > 25) = {}_{15|30}q_{25} = \dfrac{l_{40} - l_{70}}{l_{25}}$

$$= \frac{97{,}952 - 79{,}970}{98{,}797} = 0.18201$$

[1]

[Total 2]

Solution 5.3

We first write the benefit in terms of an integral. If $x = 50$ and $y = 55$, we have:

$$EPV = 50,000\int_{10}^{\infty} v^t \, {}_t p_x \, \mu_{x+t} \; {}_t p_y \; dt \qquad [1]$$

Substituting $u = t - 10$, we get:

$$EPV = 50,000\int_{0}^{\infty} v^{u+10} \, {}_{u+10} p_x \; \mu_{x+u+10} \; {}_{u+10} p_y \; du \qquad [1]$$

Splitting up the survival factors, we get:

$$EPV = 50,000 \; {}_{10} p_{50} \; {}_{10} p_{55} \; v^{10} \int_{0}^{\infty} v^u \, {}_u p_{60} \; \mu_{60+u} \; {}_u p_{65} \; du \qquad [1]$$

We can now write this in terms of joint life assurance functions:

$$EPV = 50,000 A_{50:55:\overset{\;\;1}{\overline{10|}}} \; \bar{A}^{1}_{60:65} \qquad [1]$$

[Total 4]

Solution 5.4

(i) ***Present value random variable***

The present value of the profit will be (making no allowance for expenses):

$$X = 8,500 - 500\,\ddot{a}_{\overline{K_{\overline{65:70}}+1|}} \qquad [1]$$

where $K_{\overline{x:y}}$ is the curtate future lifetime of the last survivor status, *ie* the duration until the beginning of the year in which the second death occurs. [1]

[Total 2]

(ii) ***Expected present value***

The expected present value will be:

$$E[X] = 8,500 - 500E[\ddot{a}_{\overline{K_{\overline{65:70}}+1|}}] = 8,500 - 500\,\ddot{a}_{\overline{65:70}} \qquad [1]$$

We can find the value of the last survivor annuity as follows:

$$\ddot{a}_{\overline{65:65}} = \ddot{a}_{65(m)} + \ddot{a}_{70(f)} - \ddot{a}_{65:70} = 13.666 + 12.934 - 10.94 = 15.66 \qquad [1]$$

So the expected present value is:

$$8,500 - 500 \times 15.66 = £670 \qquad [1]$$

[Total 3]

Solution 5.5

(i) ***Value of fund on 12th April***

The unit price at the start of 1st March is:

$$4.25 \times (1.01 \times 1.03)^{(31+28)/365} = 4.25 \times 1.00010825^{59} = 4.27723 \qquad [2]$$

The value of the policyholder's fund on 1st March is then:

$$3,200 \times 4.27723 = 13,687.13 \qquad [½]$$

The value of the fund on 31st March is:

$$(13,687.13 + 500) \times 1.00010825^{30} - 3 = 14,230.28 \qquad [1]$$

On 1st April (before the premium is paid) the fund is worth:

$$14,230.28 \times 1.00010825 = 14,231.82 \qquad [½]$$

and by 12th April its value is:

$$(14,231.82 + 500) \times 1.00010825^{11} = 14,749.37 \qquad [1]$$

[Total 5]

(ii) ***Why the death benefit might be different***

The death benefit might be higher than the fund value if

- the insurance company is currently paying a terminal bonus on death [½]
- the contract has a minimum death benefit, which on 12th April is greater than the sum of the fund value plus any terminal bonus payable. [½]

[Total 1]

Solution 5.6

The present value of the benefits (PV) is:

$$2,000 \times \ddot{a}_{\overline{\min[K_{60}+1,\ 2]}|} + 5,000 \times v^2 \times S$$

where:

K_{60} = curtate future lifetime of the policyholder

$$S = \begin{cases} 1 & \text{if } K_{60} \geq 2 \\ 0 & \text{if } K_{60} < 2 \end{cases}$$

So:

$$E\left[PV\right] = 2,000 \times \frac{d_{60}}{l_{60}} + 2,000 \times \ddot{a}_{\overline{2}|} \times \frac{d_{61}}{l_{60}} + \left(2,000 \times \ddot{a}_{\overline{2}|} + 5,000v^2\right) \times \frac{l_{62}}{l_{60}} \quad [1½]$$

$$= \frac{1}{9,287.2164}\{2,000 \times 74.5020 + 2,000 \times 1.96154 \times 82.9973$$

$$+\left[2,000 \times 1.96154 + 5,000 \times 1.04^{-2}\right] \times 9,129.7170\}$$

$$= 8,452.03 \quad [1½]$$

$$E\left[PV^2\right] = 2,000^2 \times \frac{d_{60}}{l_{60}} + \left(2,000\ddot{a}_{\overline{2}|}\right)^2 \times \frac{d_{61}}{l_{60}} + \left(2,000\ddot{a}_{\overline{2}|} + 5,000v^2\right)^2 \times \frac{l_{62}}{l_{60}} \quad [1]$$

$$= \frac{1}{9,287.2164}\left\{2,000^2 \times 74.5020 + \left[2,000 \times 1.96154\right]^2 \times 82.9973\right.$$

$$\left. +\left[2,000 \times 1.96154 + 5,000 \times 1.04^{-2}\right]^2 \times 9,129.7170\right\}$$

$$= 71,962,793 \quad [1]$$

Hence:

$$sd(PV) = \sqrt{71,962,793 - 8,452.03^2} = £725.19 \quad [1]$$

[Total 6]

Solution 5.7

First we need to find the underlying forces of decrement. We can use the formula below since we are told that forces of decrement are constant over each year of age:

$$\bar{\mu}_x^j = \frac{(aq)_x^j}{(aq)_x}\left[-\ln(ap)_x\right] = \frac{(ad)_x^j}{(ad)_x}\left[-\ln\left(\frac{(al)_{x+1}}{(al)_x}\right)\right] \qquad [1]$$

where $\bar{\mu}_x^j$ is the (constant) force of decrement by cause *j* over the year of age $[x, x+1]$.

We then find the adjusted force of mortality as $\bar{\mu}_x^{d\prime} = \bar{\mu}_x^d \times 0.6$. Using these formulae we obtain:

x	$\bar{\mu}_x^r$	$\bar{\mu}_x^d$	$\bar{\mu}_x^{d\prime}$
61	0.142208	0.054416	0.032649
62	0.207325	0.093303	0.055982

[3]

Using the revised forces of decrement, the new independent probabilities can be found using:

$$(aq)_x^{r\prime} = \frac{\bar{\mu}_x^r}{\left(\bar{\mu}_x^r + \bar{\mu}_x^{d\prime}\right)}\left[1 - e^{-\left(\bar{\mu}_x^r + \bar{\mu}_x^{d\prime}\right)}\right]$$

and

$$(aq)_x^{d\prime} = \frac{\bar{\mu}_x^{d\prime}}{\left(\bar{\mu}_x^r + \bar{\mu}_x^{d\prime}\right)}\left[1 - e^{-\left(\bar{\mu}_x^r + \bar{\mu}_x^{d\prime}\right)}\right] \qquad [1]$$

which gives:

x	$(aq)_x^{r\prime}$	$(aq)_x^{d\prime}$
61	0.130469	0.029954
62	0.182276	0.049218

[2]

Multiplying these dependent probabilities by the appropriate $(al)_x$ terms we can calculate the number of decrements at each age and reconstruct the multiple decrement table.

The new multiple decrement table is:

Age	$(al)_x$	$(ad)_x^r$	$(ad)_x^d$
61	10,000	1,305	300
62	8,395	1,530	413
63	6,452		

[1]
[Total 8]

Solution 5.8

To find the net premium P:

$$P \times \ddot{a}_{50:\overline{10|}} = 50{,}000 \times A_{50:\overline{10|}} - 40{,}000 \times \frac{D_{60}}{D_{50}} \qquad [1]$$

$$\therefore P = \frac{50{,}000 \times 0.68024 - 40{,}000 \times \dfrac{882.85}{1{,}366.61}}{8.314} = £982.85 \qquad [1½]$$

To find the retrospective net premium reserve:

$${}_3V = \left[982.85 \times \ddot{a}_{50:\overline{3|}} - 50{,}000 \times A^{1}_{50:\overline{3|}} \right] \times \frac{D_{50}}{D_{53}} \qquad [2]$$

$$= \left[982.85 \times \left(\ddot{a}_{50} - \frac{D_{53}}{D_{50}} \ddot{a}_{53} \right) - 50{,}000 \times \left(A_{50} - \frac{D_{53}}{D_{50}} A_{53} \right) \right] \frac{D_{50}}{D_{53}} \qquad [1½]$$

$$= 982.85 \times \left(17.444 \times \frac{1{,}366.61}{1{,}204.65} - 16.524 \right)$$

$$-50{,}000 \times \left(0.32907 \times \frac{1{,}366.61}{1{,}204.65} - 0.36448 \right) \qquad [½]$$

$$= 982.85 \times 3.2653 - 50{,}000 \times 0.0088320 = 2{,}768 \qquad [½]$$

[Total 7]

Solution 5.9

(i) ***Calculating the net single premium***

The premium is given by:

$$P = 10{,}000\frac{D_{60}}{D_{40}}\ddot{a}_{60} + P\bar{A}^{\,1}_{40:\overline{20}|} \qquad [1½]$$

$$\therefore P\left[1-1.04^{½}\left(A_{40}-\frac{D_{60}}{D_{40}}A_{60}\right)\right] = 10{,}000\frac{D_{60}}{D_{40}}\left(a_{60}+1\right) \qquad [1]$$

$$\therefore P\left[1-1.04^{½}\left(0.23056-\frac{882.85}{2{,}052.96}\times 0.45640\right)\right] = 10{,}000\times\frac{882.85}{2{,}052.96}\times 18 \qquad [1]$$

$$\therefore P = \frac{77{,}407}{(1-0.034970)} = £80{,}212 \qquad [½]$$

[Total 4]

(ii) ***Calculating the mortality profit***

We need the reserve at the end of the year (*ie* at 31.12.2010):

$${}_6V = 10{,}000\times 18\times\frac{D_{60}}{D_{46}} + 80{,}212\times\bar{A}^{\,1}_{46:\overline{14}|} \qquad [1]$$

$$= 10{,}000\times 18\times\frac{882.85}{1{,}611.07} + 80{,}212\times 1.04^{½}\left(A_{46}-\frac{D_{60}}{D_{46}}A_{60}\right) \qquad [1]$$

$$= 98{,}638 + 80{,}212\times 1.04^{½}\left(0.28605-\frac{882.85}{1{,}611.07}\times 0.45640\right) = 101{,}579 \qquad [½]$$

Now calculate the death strain at risk (noting that the death benefit is paid half way through the year):

$$DSAR = 1.04^{½}\times 80{,}212 - 101{,}579 = -19{,}778 \qquad [½]$$

The expected death strain is:

$$EDS = 976\times q_{45}\times DSAR = 976\times 0.001465\times(-19{,}778) = -28{,}280 \qquad [1]$$

The actual death strain is:

$$ADS = 3 \times (-19,778) = -59,334 \qquad [½]$$

Therefore the mortality profit is:

$$EDS - ADS = -28,280 - (-59,334) = £31,054 \qquad [½]$$

[Total 5]

These calculations are quite sensitive to rounding.

Solution 5.10

(i) ***Advantages of single figure indices***

The advantage of using a single figure index is that a single figure is more easily assimilated than a set of figures. Thus you can see at a glance a summary statistic to give you a feel for the overall picture. [1]

Some single figure indices, *eg* the crude death rate, are easy to calculate too. [1]
[Total 2]

(ii) ***Disadvantages of single figure indices***

The disadvantages of using single figure indices are that some of them (*eg* the crude death rate) reflect differences between the age specific mortality rates and also other *compositional differences* between the groups being compared – in particular, differences in age and sex compositions. [1]

Standardisation removes age differences, and sex differences can be removed by comparing sexes separately. Other differences (*eg* occupational) may remain and are confounded with true mortality differences. [1]

Many single figure indices are heavily biased towards the mortality levels at the older ages and take very little account of mortality at the younger ages where changes are usually the most dramatic. [1]

A single figure index will miss any abnormalities in the age-sex specific rates that may exist. Any errors are less likely to be detected. [1]
[Total 4]

(iii) ***Directly standardised mortality rate***

The directly standardised mortality rate modifies the crude death rate to take account of the age-sex structure of the study group in question. It is the crude death rate that would be exhibited by a standard population if that standard population exhibited the age specific death rates of the study group. [2]

Alternatively, you could define the DSMR by its formula:

$$\frac{\sum_x {}^sE^c_{x,t}\, m_{x,t}}{\sum_x {}^sE^c_{x,t}}$$

defining all the symbols that you use.

Here ${}^sE^c_{x,t}$ *is the central exposed to risk for the standard population between the ages of* x *and* $x+t$*, and* $m_{x,t}$ *is the observed central rate of mortality for the study group between the ages of* x *and* $x+t$*.*

Standardised mortality indices are used so that more meaningful comparisons of mortality can be made between areas having very different age and sex distributions. [1]
[Total 3]

(iv) ***Formula for the standardised mortality ratio***

The standardised mortality ratio (SMR) is defined as:

$$SMR = \frac{\Sigma E^c_{x,t}\ m_{x,t}}{\Sigma E^c_{x,t}\ {}^s m_{x,t}} \qquad [1]$$

where:

$E^c_{x,t}$ is the central exposed to risk between ages x and $x+t$ [1]

$m_{x,t}$ is the central rate of mortality between ages x and $x+t$ [½]

and the superscript "*s*" refers to the standard population. [½]
[Total 3]

Solution 5.11

(i) ***Monthly premium***

Because the compound bonuses vest before the payment of the death benefit, the expected present value of the benefit has the following form:

$$\text{EPV benefit} = 50,000\left[1.01923v\frac{d_{[30]}}{l_{[30]}} + 1.01923^2 v^2 \frac{d_{[30]+1}}{l_{[30]}} + \cdots\right]$$

If we incorporate the factors of 1.01923 into the v's, we can write the expected present value as:

$$\text{EPV benefit} = 50,000\left[V\frac{d_{[30]}}{l_{[30]}} + V^2 \frac{d_{[30]+1}}{l_{[30]}} + \cdots\right]$$

where $V = \dfrac{1.01923}{1.06} = \dfrac{1}{1.04}$. So we can value the benefit as an endowment assurance using an interest rate of 4%. [1]

Let the annual amount of premium be P. The expected present value of the premiums is:

$$\text{EPV premiums} = P\,\ddot{a}^{(12)}_{[30]:\overline{25|}}$$

The expected present value of the expenses is:

$$\text{EPV expenses} = 0.3P + 0.05P\,\ddot{a}_{[30]:\overline{25|}}$$ [½]

where the annuity here is not a monthly annuity.

Calculating the relevant actuarial functions:

$$A^{4\%}_{[30]:\overline{25}|} = A_{[30]} - \frac{D_{55}}{D_{[30]}} A_{55} + \frac{D_{55}}{D_{[30]}}$$

$$= 0.16011 - \frac{1{,}105.41}{3{,}059.68}\left(0.38950 - 1\right) = 0.38067$$ [½]

$$\ddot{a}^{6\%}_{[30]:\overline{25}|} = \ddot{a}_{[30]} - v^{25}\frac{l_{55}}{l_{[30]}}\ddot{a}_{55} = 16.374 - \frac{1}{1.06^{25}} \times \frac{9{,}557.8179}{9{,}923.7497} \times 13.057$$ [1]

$$= 13.4439$$

$$\ddot{a}^{(12)}_{[30]:\overline{25}|} = \ddot{a}_{[30]:\overline{25}|} - \frac{11}{24}\left(1 - v^{25}\frac{l_{55}}{l_{[30]}}\right) = 13.4439 - \frac{11}{24}\left(1 - \frac{1}{1.06^{25}} \times \frac{9{,}557.8179}{9{,}923.7497}\right)$$

$$= 13.088$$

[1]

So the equation of value is:

$$13.088P = 50{,}000 \times 0.38067 + 0.3P + 0.05P \times 13.4439$$

We solve this to obtain $P = 1{,}570.96$. Dividing this by 12, we find that the monthly premium is £130.91. [2]

[Total 6]

(ii) ***Gross premium reserve***

We want the reserve at the start of the tenth year, *ie* nine years after the start of the policy. Death in the tenth year will result in a payment of $50{,}000 \times 1.01923^{10}$. So the expected present value of the future benefit is:

$$\text{EPV future benefit} = 50{,}000 \times 1.01923^{10}\frac{d_{39}}{l_{39}}v + 50{,}000 \times 1.01923^{11}\frac{d_{40}}{l_{39}} + \cdots$$

[½]

Matching the powers of the 1.01923 with the powers of v as before, we can write this as:

$$\text{EPV future benefit} = 50{,}000 \times 1.01923^{9} \times A^{4\%}_{39:\overline{16}|}$$

So the expression for the gross premium prospective reserve is:

$$ {}_9V = 50{,}000 \times 1.01923^9 \, A^{4\%}_{39:\overline{16|}} + 0.05P\,\ddot{a}^{6\%}_{39:\overline{16|}} - P\,\ddot{a}^{(12)\ @\ 6\%}_{39:\overline{16|}} \qquad [2] $$

Evaluating the functions as before:

$$ A^{4\%}_{39:\overline{16|}} = 0.22234 - \frac{1{,}105.41}{2{,}136.93}(0.38950 - 1) = 0.53814 \qquad [½] $$

$$ \ddot{a}^{6\%}_{39:\overline{16|}} = \ddot{a}_{39} - v^{16}\frac{l_{55}}{l_{39}}\ddot{a}_{55} = 15.602 - \frac{1}{1.06^{16}} \times \frac{9{,}557.8179}{9{,}864.8688} \times 13.057 $$

$$ = 10.6221 \qquad [1] $$

Also:

$$ \ddot{a}^{(12)}_{39:\overline{16|}} = \ddot{a}_{39:\overline{16|}} - \frac{11}{24}\left(1 - v^{16}\frac{l_{55}}{l_{39}}\right) = 10.6221 - \frac{11}{24}\left(1 - \frac{1}{1.06^{16}} \times \frac{9{,}557.8179}{9{,}864.8688}\right) $$

$$ = 10.339 \qquad [1] $$

So the reserve is:

$$ {}_9V = 50{,}000 \times 1.01923^9 \times 0.53814 + 0.05 \times 1{,}570.96 \times 10.6221 $$

$$ -1{,}570.96 \times 10.339 $$

$$ = 16{,}530.73 \qquad [1] $$

[Total 6]

Solution 5.12

First we need to define suitable functions. Let:

l_x be the number of lives with exact age x in a suitable service table

r_x be the number of service table retirements in the year of age x to $x+1$ ($x = 60,...,64$), and let r_{65} be the number of retirements at exact age 65

s_{x+t} / s_x
be the ratio of salary earned between ages $x+t$ and $x+t+1$ to that earned between ages x and $x+1$ in a suitable promotional salary scale, including a suitable allowance for inflationary increases

z_x be equal to $\frac{1}{3}(s_{x-1} + s_{x-2} + s_{x-3})$

$\bar{a}_x$ be the expected present value of a pension of £1 per annum paid to a life retiring at age x in accordance with the rules of the scheme, assuming mortality as appropriate for normal-age retired pensioners

i be the appropriate valuation interest rate

v be equal to $1/(1+i)$. [3½]

We assume that:

Retirements occur on average half way through the year of age, except for retirement at 65 which takes place at age 65 exact. [1]

Past service benefit

The life has 5 years past service. If he retires in the year of age 60 to 61, the expected present value now of the pension secured by this past service will be:

$$\frac{5}{60} \times 20{,}000 \times \frac{z_{60½}}{s_{39}} \times \frac{r_{60}}{l_{40}} \times v^{20½} \times \bar{a}_{60½} \qquad [1]$$

If we write down similar expressions for the value of his past service pension if he retires in the other possible years of age, we get an expression for the total past service as:

$$\frac{5}{60}\times 20,000\left[\frac{z_{60½}}{s_{39}}\times\frac{r_{60}}{l_{40}}\times v^{20½}\times\bar{a}_{60½}+\frac{z_{61½}}{s_{39}}\times\frac{r_{61}}{l_{40}}\times v^{21½}\times\bar{a}_{61½}\right.$$

$$+\frac{z_{62½}}{s_{39}}\times\frac{r_{62}}{l_{40}}\times v^{22½}\times\bar{a}_{62½}+\frac{z_{63½}}{s_{39}}\times\frac{r_{63}}{l_{40}}\times v^{23½}\times\bar{a}_{63½}$$

$$\left.+\frac{z_{64½}}{s_{39}}\times\frac{r_{64}}{l_{40}}\times v^{24½}\times\bar{a}_{64½}+\frac{z_{65}}{s_{39}}\times\frac{r_{65}}{l_{40}}\times v^{25}\times\bar{a}_{65}\right] \qquad [1]$$

Note that the expression for retirement at age 65 has a different form from that of the other terms.

Define:

(i) ${}^{z}C_x^{ra} = z_{x+½}\ r_x\ v^{x+½}\ \ \bar{a}_{x+½}\quad (x = 60, 61, ..., 64)$

(ii) ${}^{z}C_{65}^{ra} = z_{65}\ r_{65}\ v^{65}\ \ \bar{a}_{65}$

(iii) ${}^{s}D_x = v^x\ l_x\ s_{x-1}$

The past service can now be written:

$$\frac{5}{60}\times 20,000\left[\frac{{}^{z}C_{60}^{ra}+{}^{z}C_{61}^{ra}+{}^{z}C_{62}^{ra}+{}^{z}C_{63}^{ra}+{}^{z}C_{64}^{ra}+{}^{z}C_{65}^{ra}}{{}^{s}D_{40}}\right] \qquad [1]$$

If we now define:

$${}^{z}M_{60}^{ra} = {}^{z}C_{60}^{ra}+{}^{z}C_{61}^{ra}+{}^{z}C_{62}^{ra}+{}^{z}C_{63}^{ra}+{}^{z}C_{64}^{ra}+{}^{z}C_{65}^{ra}$$

the past service benefit can be written as:

$$\frac{5}{60}\times 20,000\times\frac{{}^{z}M_{60}^{ra}}{{}^{s}D_{40}} \qquad [1]$$

Future service benefit

We must now consider the benefit accrued in future years of service.

The life is currently aged 40. Consider first the service accrued in the year of age from 40 to 41. The value now of this year of accrual is (bearing in mind there are still 6 occasions on which he could retire in the future):

$$\frac{1}{60}\times 20{,}000\left[\frac{z_{60\frac{1}{2}}}{s_{39}}\times\frac{r_{60}}{l_{40}}\times v^{20\frac{1}{2}}\times\bar{a}_{60\frac{1}{2}}+\frac{z_{61\frac{1}{2}}}{s_{39}}\times\frac{r_{61}}{l_{40}}\times v^{21\frac{1}{2}}\times\bar{a}_{61\frac{1}{2}}\right.$$
$$+\frac{z_{62\frac{1}{2}}}{s_{39}}\times\frac{r_{62}}{l_{40}}\times v^{22\frac{1}{2}}\times\bar{a}_{62\frac{1}{2}}+\frac{z_{63\frac{1}{2}}}{s_{39}}\times\frac{r_{63}}{l_{40}}\times v^{23\frac{1}{2}}\times\bar{a}_{63\frac{1}{2}}$$
$$\left.+\frac{z_{64\frac{1}{2}}}{s_{39}}\times\frac{r_{64}}{l_{40}}\times v^{24\frac{1}{2}}\times\bar{a}_{64\frac{1}{2}}+\frac{z_{65}}{s_{39}}\times\frac{r_{65}}{l_{40}}\times v^{25}\times\bar{a}_{65}\right] \quad [2]$$

This can be written as $\frac{1}{60}\times 20{,}000\times\frac{{}^{z}M_{60}^{ra}}{{}^{s}D_{40}}$, as before.

So each future year of accrual up to the year of age 59 to 60 will give us the same expression. So this gives a future service benefit value of

$$\frac{20}{60}\times 20{,}000\times\frac{{}^{z}M_{60}^{ra}}{{}^{s}D_{40}} \quad [1]$$

The years of accrual between age 60 and 65 are different. For example, for the year of accrual from age 60 to 61, if he retires half way through this year, then he does not accrue a full year of benefit. So the total value of the service accruing in that year (on average) will be:

$$\frac{1}{60}\times 20{,}000\times\left[\frac{\frac{1}{2}\,{}^{z}C_{60}^{ra}+{}^{z}C_{61}^{ra}+{}^{z}C_{62}^{ra}+{}^{z}C_{63}^{ra}+{}^{z}C_{64}^{ra}+{}^{z}C_{65}^{ra}}{{}^{s}D_{40}}\right]$$

If we define ${}^{z}\bar{M}_{x}^{ra}$ to be equal to ${}^{z}\bar{M}_{x}^{ra}=\frac{1}{2}\,{}^{z}C_{x}^{ra}+{}^{z}C_{x+1}^{ra}+\cdots+{}^{z}C_{65}^{ra}$, then the full accrual value for the last five years' service is now:

$$\frac{1}{60}\times 20{,}000\left[\frac{{}^{z}\bar{M}_{60}^{ra}+{}^{z}\bar{M}_{61}^{ra}+{}^{z}\bar{M}_{62}^{ra}+{}^{z}\bar{M}_{63}^{ra}+{}^{z}\bar{M}_{64}^{ra}}{{}^{s}D_{40}}\right] \quad [1]$$

We can now write ${}^{z}\bar{R}_x^{ra} = {}^{z}\bar{M}_x^{ra} + \cdots + {}^{z}\bar{M}_{64}^{ra}$.

So the total value of the future service liability is now:

$$\frac{20}{60}\times 20,000\times\frac{{}^{z}M_{60}^{ra}}{{}^{s}D_{40}}+\frac{1}{60}\times 20,000\times\frac{{}^{z}\bar{R}_{60}^{ra}}{{}^{s}D_{40}}$$ [½]

[Total 13]

Solution 5.13

(i) ***Unit fund values and maturity proceeds***

The calculations of the build up of each fund are set out in the tables below:

TYPE A							
Policy year	Prem	Prem all'd	Cost of all'n	Fund b/f	Fund before charge	Annual charge	Fund c/f
1	1,000.00	850.00	807.50	0.00	868.06	–8.68	859.38
2	1,000.00	1,010.00	959.50	859.38	1,955.30	–19.55	1,935.75
3	1,000.00	1,010.00	959.50	1,935.75	3,112.39	–31.12	3,081.27

So the estimated maturity value for policy Type A is £3,081.27. [3]

TYPE B							
Policy year	Prem	Prem all'd	Cost of all'n	Fund b/f	Fund before charge	Annual charge	Fund c/f
1	1,000.00	950.00	902.50	0.00	970.19	–9.70	960.49
2	1,000.00	950.00	902.50	960.49	2,002.71	–20.03	1,982.68
3	1,000.00	950.00	902.50	1,982.68	3,101.57	–31.02	3,070.55

So the estimated maturity value for policy Type B is £3,070.55. [3]

[Total 6]

(ii) ***Net present value***

The calculations of the net present values of the profits are set out in the tables below:

TYPE A					
Policy year	Profit on allocation	Expenses	Non-unit interest	Annual charge	Profit in year
1	192.50	100.00	6.94	8.68	108.12
2	40.50	30.00	0.79	19.55	30.84
3	40.50	30.00	0.79	31.12	42.41

[2]

Since the dependent probabilities of mortality and withdrawal are 1% and 5%, out of every 100 policyholders at the start of each year, on average 1 will die and 5 will surrender, leaving 94 at the end of the year. So the probability of remaining in force until the end of the year is 0.94 (exactly). [1]

So the net present value for policy Type A is:

$$NPV_A = 108.12v + 30.84v^2 \times 0.94 + 42.41v^3 \times 0.94^2 = £150.40 \quad [1]$$

TYPE B						
Policy year	Profit on allocation	Expenses	Non-unit interest	Surrender Profit	Annual charge	Profit in year
1	97.50	100.00	–0.19	10.00	9.70	17.01
2	97.50	30.00	5.06	5.00	20.03	97.59
3	97.50	30.00	5.06	0.00	31.02	103.58

[3]

So the net present value for policy Type B is:

$$NPV_B = 17.01v + 97.59v^2 \times 0.94 + 103.58v^3 \times 0.94^2 = £160.03 \quad [1]$$

[Total 8]

Subject CT5: Assignment X1

2016 Examinations

Time allowed: 2½ hours

Instructions to the candidate

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Subject CT5: Assignment X1

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Score and grade for this assignment (to be completed by marker):

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Q12	Q13	Q14	Total	
$\frac{\_}{2}$	$\frac{\_}{5}$	$\frac{\_}{3}$	$\frac{\_}{6}$	$\frac{\_}{5}$	$\frac{\_}{5}$	$\frac{\_}{6}$	$\frac{\_}{3}$	$\frac{\_}{7}$	$\frac{\_}{4}$	$\frac{\_}{11}$	$\frac{\_}{5}$	$\frac{\_}{5}$	$\frac{\_}{13}$	$\frac{\_}{80}$	=_____%

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Notes on marker's section

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Question X1.1

If mortality follows the AM92 tables, calculate $\int_0^2 \mu_{[60]+t}\,dt$. [2]

Question X1.2

A man aged exactly 42 purchases a whole life annuity with a benefit of £5,000 *pa* payable continuously.

(i) Write down an expression using actuarial notation for the random variable representing the present value of this benefit. [1]

(ii) Show that the variance of this random variable is $\left(\dfrac{5,000}{\delta}\right)^2 \left({}^2\bar{A}_{42} - (\bar{A}_{42})^2\right)$. [2]

(iii) Calculate the variance of the annuity using AM92 ultimate mortality and 4% *pa* interest. [2]

[Total 5]

Question X1.3

A 20-year temporary annuity-due of 1 *pa* is issued to a life aged 50 exact.

(a) Express the expected present value of the annuity in terms of an assurance function.

(b) Hence calculate the value using the mortality table AM92 Ultimate with 4% interest. [3]

Question X1.4

A two-year term assurance policy is issued to a life aged x. The benefit amount is 100 if the life dies in the first year, and 200 if the life dies in the second year. Benefits are payable at the end of the year of death.

(i) Write down an expression for the present value random variable for this benefit. [2]

(ii) Calculate the standard deviation of the present value random variable assuming that $q_x = 0.025$, $q_{x+1} = 0.030$ and $i = 0.06$. [4]

[Total 6]

Question X1.5

The mortality of a group of independent lives follows the ELT15 (Females) table.

(i) Calculate the probability that 5 lives aged exactly 50 will all be alive in 20 years' time. [2]

(ii) If k is the largest integer such that $P\left(K_{50} \geq k\right) \geq 0.9$, calculate the value of k.

[3]

[Total 5]

Question X1.6

If $A_x = 0.20$, $A_{x+10} = 0.25$ and $A_{x:\overline{10}|} = 0.75$, calculate the values of:

(i) $A_{x:\overline{10}|}^{\ \ \ \ 1}$ [3]

(ii) $A_{x:\overline{10}|}^{1}$ [1]

(iii) $_{10|}A_x$ [1]

[Total 5]

Question X1.7

Estimate ${}_2p_{63.25}$ assuming ELT15 (Males) mortality at integer ages and:

(a) a uniform distribution of deaths between integer ages

(b) a constant force of mortality between integer ages. [6]

Question X1.8

Explain what is meant by the following expression, and calculate its value using AM92 mortality:

$${}_{3|}q_{[55]+1}$$ [3]

Question X1.9

In 20 years' time a sum of £20,000 is to be divided equally amongst the survivors of two lives now aged 30 and 40 and a charitable trust. Find the expected value and the variance of the net present value of the amount due to the charitable trust.

Basis: AM92 Ultimate, 4% *pa* interest. [7]

Question X1.10

A life insurance company issues an annuity to a life aged 60 exact. The purchase price is £200,000. The annuity is payable monthly in advance and is guaranteed to be paid for a period of 10 years and for the whole of life thereafter.

Calculate the annual annuity payment.

Basis:

Mortality: AM92 Ultimate

Interest: 6% *pa* [4]

Question X1.11

Let K denote the curtate future lifetime random variable of a life aged exactly x.

(i) Describe the benefit whose present value random variable is:

$$W = \begin{cases} 10{,}000\,\ddot{a}_{\overline{K+1}|} & \text{if } K < 10 \\ 10{,}000\,\ddot{a}_{\overline{10}|} & \text{if } K \geq 10 \end{cases}$$ [1]

(ii) Prove the premium conversion formula:

$$A_{x:\overline{n}|} = 1 - d\,\ddot{a}_{x:\overline{n}|}$$ [2]

(iii) Calculate the expected present value and the standard deviation of the present value of the benefit in (i), assuming:

- a force of interest of 0.04 *pa*
- the life is subject to a constant force of mortality of 0.02 *pa*. [8]

[Total 11]

Question X1.12

Calculate the probability of survival to age 60 exact using ELT15 (Males) for a life aged 45½ exact using two approximate methods. State any assumptions you make. [5]

Question X1.13

A population is subject to the force of mortality $\mu_x = e^{0.0002x} - 1$.

Calculate the probability that a life now aged 20 exact:

(i) survives to age 70 exact [2]

(ii) dies between ages 60 exact and 70 exact. [3]

[Total 5]

Question X1.14

A life insurance company issues identical deferred annuities to each of 100 women aged 63 exact. The benefit is £5,000 per annum payable continuously from a woman's 65th birthday, if still alive at that time, and for life thereafter.

(i) Write down an expression for the random variable for the present value of future benefits for one policy at outset. [3]

(ii) Calculate the total expected present value at outset of these annuities.

Basis: Mortality: PFA92C20
Interest: 4% per annum [2]

(iii) Calculate the total variance of the present value at outset of these annuities, using the same basis as in part (ii). [8]
[Total 13]

End of paper

Subject CT5: Assignment X2

2016 Examinations

Time allowed: 2½ hours

Instructions to the candidate

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Subject CT5: Assignment X2

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Score and grade for this assignment (to be completed by marker):

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	**Total**	
_ / 5	_ / 6	_ / 6	_ / 6	_ / 6	_ / 5	_ / 7	_ / 9	_ / 11	_ / 19	_ / 80	= _____%

Grade: A B C D E **Marker's initials:** ________

Please grade your Assignment X1 marker by ticking the appropriate box.

[ ] **Excellent** – the marker's comments were thorough and very helpful
[ ] **Good** – the marker's comments were generally helpful
[ ] **Acceptable** – please explain below how the marker could have been more helpful
[ ] **Poor** – the marker's comments were generally unhelpful; please give details below

Please give any additional comments here (especially if you rate the marker less than good):

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Feedback from marker

Notes on marker's section

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Question X2.1

A life insurance company issues a deferred annuity contract to a life aged exactly 40. Premiums are payable annually in advance for 25 years or until earlier death. On survival to age 65, an annuity of £15,000 *pa* is payable continuously for the whole of life. In the event of death during the deferment period, a lump sum is payable at the end of the year of death equal to the total amount of premiums paid to date, without interest. Calculate the annual premium using the following basis:

Mortality: AM92 Select before age 65
PMA92C20 after age 65

Interest: 4% *pa*

Expenses: none [5]

Question X2.2

(i) Describe the four different methods of allocating bonuses to conventional with-profits contracts in the UK. [3]

(ii) A conventional with-profits endowment assurance policy, with a twenty-year term, was issued four years ago to a man aged 40 exact at entry. The basic sum assured was £20,000, and level annual premiums are payable. Death benefits are paid at the end of the year of death.

The insurance company has declared compound reversionary bonuses of 4% per annum (vesting at the start of each year) over the last four years, and expects to carry on doing so into the foreseeable future.

Calculate the net premium reserve for this policy as at the end of the fourth policy year, using the following basis:

Interest: 4% *pa*
Mortality: AM92 Select [3]
[Total 6]

Question X2.3

(i) Outline the main features of a (non-unitised) accumulating with-profits contract. [4]

(ii) You are given the following details about a unitised with-profits contract:

- fund value on 11th March: £65,292
- monthly premium (payable on first day of month): £600
- annual bonus interest rate for the calendar year: 4.25%
- monthly policy fee (payable on fifteenth day of month): £3

The bonus interest is credited to the policy on a daily basis, by increasing the unit price at the appropriate daily effective interest rate.

Assuming there are no other charges on the contract, calculate the fund value for this policy as at 6th April of the same year. You should assume that there are 365 days in this calendar year. [2]
[Total 6]

Question X2.4

Calculate the level premium payable annually for 10 years by a life aged 50 in respect of a 15-year term assurance where the sum assured is £10,000 for the first 5 years and £15,000 thereafter. The sum assured is payable at the end of year of death.

Basis: AM92 Select mortality, 4% *pa* interest. Allow for expenses of 25% of the first premium and 5% of subsequent premiums. [6]

Question X2.5

An n-year term assurance with a sum assured of 1 payable at the end of the year of death is issued to a life aged x. Level premiums are payable annually in advance throughout the term of the policy or until the policyholder's earlier death. The premium includes an initial expense loading of I, and a renewal expense loading of e at the start of each policy year, including the first.

(i) Give expressions, in terms of standard actuarial functions, for:

(a) the gross premium

(b) the prospective gross premium reserve at (integer) time $t < n$

(c) the retrospective gross premium reserve, at (integer) time $t < n$.

[3]

(ii) Hence show that, if all three of the expressions in (i) are calculated on the same basis, the prospective and retrospective gross premium reserves are equal. [3]

[Total 6]

Question X2.6

A life insurance company issues annual premium whole life assurance policies with a sum assured of £100,000 payable at the end of the year of death to lives aged exactly 35.

Calculate the minimum premium the office could charge in order that the probability of making a loss on any one policy would be 1% or less. [5]

Basis: AM92 Select mortality, 6% *pa* interest, expenses of 5% of all premiums.

Question X2.7

A man aged 65 exact buys a whole life annuity that provides payments at the end of each policy year. The first payment is £10,000, and subsequent payments increase by 3% *pa* compound.

Let X denote the present value random variable for this annuity.

Interest is assumed to be 3% *pa* and mortality is assumed to follow the AM92 Ultimate table.

(i) Derive an expression for X and simplify this as much as possible. [2]

(ii) Calculate $E(X)$. [1]

(iii) Calculate the prospective reserve for this policy at time 5, assuming it is still in force. [2]

(iv) Calculate the probability that the present value of the benefit received by the policyholder is greater than £250,000. [2]

[Total 7]

Question X2.8

On 1 January 2008 an insurer issued a block of 25-year annual premium endowment policies that pay £120,000 at maturity, or £60,000 at the end of the year of earlier death to lives aged exactly 65. The premium basis assumed 4% interest, AM92 Select mortality and allowed for an initial expense of £200 and renewal expenses of 1% of each subsequent premium. Reserves are calculated on the same basis as the premiums.

The annual premium, calculated on the premium basis, was £3,071.40.

(i) Calculate the reserve required per policy at 31 December 2012. [3]

(ii) There were 197 policies in force on 1 January 2012. During 2012 there were 9 deaths, interest was earned at twice the rate expected and expenses were incurred at twice the rate expected. By considering the total reserve required at the start and end of the year, and all the cashflows during the year, calculate the profit or loss made by the insurer from all sources (not just from mortality) in respect of these policies for the 2012 calendar year. [6]

[Total 9]

Question X2.9

A life insurance company issued a with profits whole life policy to a life aged 20 exact, on 1 July 2002. Under the policy, the basic sum assured of £100,000 and attaching bonuses are payable immediately on death. The company declares simple reversionary bonuses at the start of each year. Level premiums are payable annually in advance under the policy.

(i) Give an expression for the gross future loss random variable under the policy at the outset. Define symbols where necessary. [3]

(ii) Calculate the annual premium, using the equivalence principle.

Basis:

Mortality		AM92 Select
Interest		6% per annum
Bonus loading		3% simple per annum
Expenses	Initial	£200
	Renewal	5% of each premium payable in the second and subsequent years

Assume bonus entitlement is earned immediately on payment of premium. [4]

(iii) On 30 June 2005 the policy is still in force. A total of £10,000 has been declared as a simple bonus to date on the policy.

The company calculates reserves for the policy using a gross premium prospective basis, with the following assumptions:

Mortality	AM92 Ultimate
Interest	4%
Bonus loading	4% per annum simple
Renewal expenses	5% of each premium

Calculate the reserve for the policy as at 30 June 2005. [4]

[Total 11]

Question X2.10

(i) Write down in the form of symbols, and also explain in words, the expressions "death strain at risk", "expected death strain" and "actual death strain". [6]

(ii) A life insurance company issues the following policies:

- 15-year term assurances with a sum assured of £150,000 where the death benefit is payable at the end of the year of death
- 15-year pure endowment assurances with a sum assured of £75,000
- 5-year single premium temporary immediate annuities with an annual benefit payable in arrear of £25,000

On 1 January 2010, the company sold 5,000 term assurance policies and 2,000 pure endowment policies to male lives aged 45 exact and 1,000 temporary immediate annuity policies to male lives aged 55 exact. For the term assurance and pure endowment policies, premiums are payable annually in advance. During the first two years, there were fifteen actual deaths from the term assurance policies written and five actual deaths from each of the other two types of policy written.

(a) Calculate the death strain at risk for each type of policy during 2012.

(b) During 2012, there were eight actual deaths from the term assurance policies written and one actual death from each of the other two types of policy written. Calculate the total mortality profit or loss to the office in the year 2012.

Basis:

Interest: 4% *pa*

Mortality: AM92 Ultimate for term assurances and pure endowments
PMA92C20 for annuities

[13]
[Total 19]

End of paper

Subject CT5: Assignment X3

2016 Examinations

Time allowed: 3 hours

Instructions to the candidate

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   - ***begin your answer to each question on a new page***
   - ***leave at least 2cm margin on all borders***
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2. *Please* ***do not****:*
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Subject CT5: Assignment X3

2016 Examinations

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Score and grade for this assignment (to be completed by marker):

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	**Total**	
_/5	_/5	_/4	_/5	_/11	_/18	_/7	_/10	_/8	_/9	_/18	_/100	=____%

Grade: A B C D E **Marker's initials:** ________

Please grade your Assignment X2 marker by ticking the appropriate box.

[ ] **Excellent** – the marker's comments were thorough and very helpful

[ ] **Good** – the marker's comments were generally helpful

[ ] **Acceptable** – please explain below how the marker could have been more helpful

[ ] **Poor** – the marker's comments were generally unhelpful; please give details below

Please give any additional comments here (especially if you rate the marker less than good):

Note: Giving feedback on your marker helps us to improve the quality of marking.

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Feedback from marker

Notes on marker's section

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A = Clear Pass   B = Probable Pass   C = Borderline   D = Probable Fail   E = Clear Fail

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Question X3.1

Define $\ddot{a}^{(12)}_{60:50:\overline{20|}}$ fully in words and calculate its value using PMA92C20 and PFA92C20 tables for the two lives respectively at 4% interest. [5]

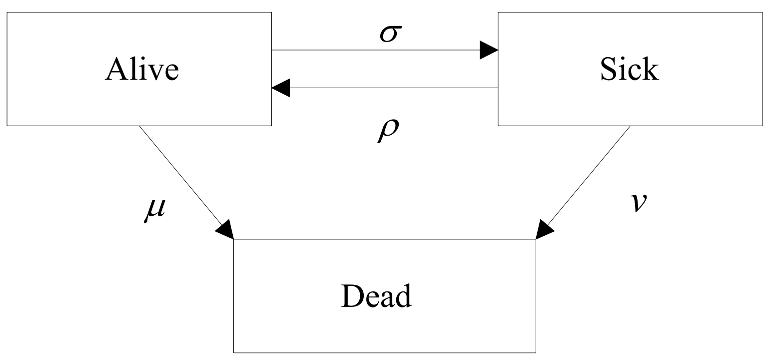
Question X3.2

A pension scheme provides a pension on ill-health retirement of 1/80th of Final Pensionable Salary for each year of pensionable service subject to a minimum pension of 20/80ths of Final Pensionable Salary. Final Pensionable Salary is defined as the average salary earned in the three years before retirement. Normal retirement age is 65 exact.

Derive a formula for the expected present value of the ill-health retirement benefit for a member currently aged 35 exact with exactly 10 years past service, and whose salary for the year before the calculation date is £20,000. [5]

Question X3.3

A three-state transition model is shown in the following diagram:



Assume that the transition probabilities are constant at all ages with $\mu = 2\%$, $\nu = 4\%$, $\rho = 1\%$ and $\sigma = 5\%$.

Calculate the expected present value of a sickness benefit of £2,000 *pa* paid continuously to a life now aged 40 exact and sick, for this period of sickness only, discounted at 4% *pa* and payable to a maximum age of 60 exact. [4]

Question X3.4

A joint life annuity of 1 *pa* is payable continuously to lives currently aged x and y while both lives are alive. The present value of the annuity payments is expressed as a random variable, in terms of the joint future lifetime of x and y.

Derive and simplify as far as possible expressions for the expected present value and the variance of the present value of the annuity. [5]

Question X3.5

A special 3-year term assurance issued to a man aged exactly 62 pays 50,000 immediately on death within the policy term. On survival to the end of the term, or immediately on earlier surrender, half of the total premiums paid to date (without interest) will be returned to the policyholder. A level annual premium is paid at the start of each year.

The insurance company uses the following basis to calculate its premiums:

Mortality:	independent probabilities as defined by the AM92 Select table
Surrender:	forces of surrender of 5%, 2.5% and 1% in policy years 1, 2, and 3 respectively
Interest:	3% *pa*
Expenses:	none

(i) Assuming that forces of decrement are constant over each year of age, construct a multiple decrement table that would be suitable for valuing the cashflows for this policy, using a radix of $(al)_{62} = 100,000$. [5]

(ii) Using the entries in the multiple decrement table, or otherwise, calculate the annual premium for this policy. [6]
[Total 11]

Question X3.6

A special term assurance policy is such that a sum of £20,000 is payable if a life (x) dies within a 20-year period. The sum assured will be payable immediately on (x)'s death if another life (y) dies before (x). However, if (y) is alive at the time of (x)'s death, payment of the sum assured will be deferred until the end of the 20-year period. A continuous level premium will be payable.

(i) State, with reasons, the appropriate annuity factor which should be used to calculate the expected present value of the premiums, as at policy outset. [3]

(ii) Calculate the annual rate of premium payable, assuming a constant annual force of interest of 0.05 throughout, and that both lives are subject to the same constant annual force of mortality of 0.005 at all ages. Ignore expenses. [9]

(iii) Assuming the same interest and mortality basis as in (ii), and that no benefit has yet been paid out under the policy, calculate the prospective reserve on this policy after exactly 4 years under each of the following scenarios:

(a) only life (x) is alive at that time

(b) only life (y) is alive at that time

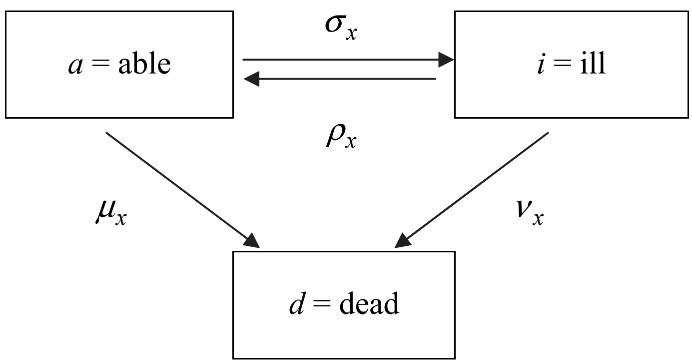
(c) both lives are dead by that time. [5]

(iv) Comment briefly on the differences between the answers you have obtained in (iii). [1]

[Total 18]

Question X3.7

Consider the following three-state illness-death model:



Let ${}_tp_x^{jk}$ denote the probability that a life in state j at age x will be in state k at age $x+t$, and let ${}_tp_x^{\overline{jj}}$ denote the probability that a life in state j at age x will remain in state j for at least t years. Given a constant force of interest of δ *pa*, write down integral expressions for the expected present values of each of the following benefits:

(i) A benefit of £50,000 payable immediately on the death of a life aged 50, provided that death occurs within the next 10 years. [2]

(ii) A benefit of £50,000 payable immediately on the death of a life aged 50, provided that death occurs within the next 10 years and the life has been sick for at least a year at the time of death. [3]

(iii) A sickness benefit of £5,000 *pa* payable continuously to a life aged 50 throughout any period of sickness. Benefits cease at age 60. [2]

[Total 7]

Question X3.8

A life insurance company issues an annuity contract to a man aged 65 exact and his wife aged 62 exact. Under the contract, an annuity of £20,000 *pa* is guaranteed payable for a period of 5 years and thereafter during the lifetime of the man. On the man's death, an annuity of £10,000 *pa* is payable to his wife, if she is then alive. This annuity commences on the monthly payment date next following, or coincident with, the date of his death or from the 5th policy anniversary, if later and is payable for the lifetime of his wife. Annuities are payable monthly in advance.

Calculate the single premium required for the contract.

Basis:

Mortality:	PMA92C20 for the male and PFA92C20 for the female
Interest:	4% *pa*
Expenses:	none

[10]

Question X3.9

A pension scheme provides a pension of 1/60th of final pensionable salary for each year of service (with part years counting proportionately) payable on normal retirement at age 65 or on voluntary early retirement before age 65. Final pensionable salary is defined to be the average pensionable salary during the 36 months immediately prior to retirement. Pensionable salary is defined to be the annual rate of salary less a fixed deduction of £2,000.

No benefit is payable on withdrawal or ill-health retirement.

Members' contributions of 5% of pensionable salary are deducted from members' monthly pay. Pay levels are reviewed on 1 January each year.

A group of 5 men all aged 35 nearest birthday joined the scheme (as part of a transfer from another company) on 1 May 2012. The total salary these members would have received during the year ending on 30 April 2012, if they had been working for the company during that period, would have been £75,000.

The trustees have asked the employer to meet the cost of future benefits for these members by making monthly contributions proportional to the amounts contributed by the members throughout the remainder of their service with the company.

Calculate the required employer's contribution rate for these members, assuming that the service table, salary scale, interest rate and other actuarial assumptions used in the *Tables* are appropriate in this case. [8]

Question X3.10

A life insurance company issues an annuity policy to two lives each aged 60 exact in return for a single premium. Under the policy, an annuity of £10,000 *pa* is payable annually in advance while at least one of the lives is alive.

(i) Write down an expression for the net future loss random variable at the outset for this policy. [2]

(ii) Calculate the single premium, using the equivalence principle.

Basis:

Mortality: PMA92C20 for the first life, PFA92C20 for the second life
Interest: 4% *pa*
Expenses: ignored [3]

(iii) Calculate the standard deviation of the net future loss random variable at the outset for this policy, using the basis in part (ii).

You are given that $\ddot{a}_{\overline{60:60}} = 11.957$ at a rate of interest 8.16% *pa*. [4]

[Total 9]

Question X3.11

Under the rules of a pension scheme, a member may retire due to age at any age from exact age 60 to exact age 65.

On age retirement, the scheme provides a pension of 1/60th of Final Pensionable Salary for each year of scheme service, subject to a maximum of 40/60ths of Final Pensionable Salary. Only complete years of service are taken into account.

Final Pensionable Salary is defined as the average salary over the three-year period before the date of retirement.

The pension scheme also provides a lump sum benefit of four times Pensionable Salary on death before retirement. The benefit is payable immediately on death and Pensionable Salary is defined as the annual rate of salary at the date of death.

You are given the following data in respect of a member:

Date of birth	1 January 1985
Date of joining the scheme	1 January 2006
Annual rate of salary at 1 January 2011	£50,000
Date of last salary increase	1 April 2010

(i) Derive commutation functions to value the past service and future service pension liability on age retirement for this member as at 1 January 2011. State any assumptions that you make and define all the symbols that you use. [12]

(ii) Derive commutation functions to value the liability in respect of the lump sum payable on death before retirement for this member as at 1 January 2011. State any assumptions that you make and define all the symbols that you use. [6]

[Total 18]

End of paper

Subject CT5: Assignment X4

2016 Examinations

Time allowed: 3 hours

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Subject CT5: Assignment X4

2016 Examinations

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Number of following pages: _______

Please put a tick in this box if you have solutions and a cross if you do not: ☐

Please tick here if you are allowed extra time or other special conditions in the profession's exams: ☐

Time to do assignment (see Note below): _____ hrs _____ mins

Under exam conditions (delete as applicable): yes / nearly / no

Note: If you take more than 3 hours, you should indicate how much you completed within this time so that the marker can provide useful feedback on your chances of success in the exam.

Score and grade for this assignment (to be completed by marker):

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Q12	**Total**	
_ / 6	_ / 5	_ / 4	_ / 3	_ / 9	_ / 7	_ / 10	_ / 2	_ / 11	_ / 13	_ / 18	_ / 12	_ / 100	=_____%

Grade: A B C D E **Marker's initials:** ________

Please grade your Assignment X3 marker by ticking the appropriate box.

[ ] **Excellent** – the marker's comments were thorough and very helpful
[ ] **Good** – the marker's comments were generally helpful
[ ] **Acceptable** – please explain below how the marker could have been more helpful
[ ] **Poor** – the marker's comments were generally unhelpful; please give details below

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[ ]	Numbered all pages of your script (excluding this cover sheet)?
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[ ]	Included your Marking Voucher or ordered Series X Marking?
[ ]	Rated your Assignment X3 marker?

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A = Clear Pass B = Probable Pass C = Borderline D = Probable Fail E = Clear Fail

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Question X4.1

You are given the following statistics in respect of the population of Urbania:

	Males		*Females*	
Age band	*Exposed to risk*	*Observed Mortality rate*	*Exposed to risk*	*Observed Mortality rate*
20–29	125,000	0.00356	100,000	0.00125
30–39	200,000	0.00689	250,000	0.00265
40–49	100,000	0.00989	200,000	0.00465
50–59	90,000	0.01233	150,000	0.00685

Calculate the directly and indirectly standardised mortality rates for the female lives, using the combined population as the standard population. [6]

Question X4.2

Explain how geographical location can affect mortality. [5]

Question X4.3

A 4-year conventional endowment assurance policy issued to lives aged exactly 61 has a sum assured of £10,000. The profit signature, calculated assuming AM92 Ultimate mortality and making no allowance for surrenders, is (–100, –20, 80, 140). Reserves have been calculated on a net premium basis using 6% *pa* interest.

The calculations are modified to allow for 10% of policies in force at the end of the first year to be surrendered with a surrender value of £1,500.

(i) Calculate the revised profit in the first year. [3]

(ii) Comment on the impact on the profit signature in years 2 to 4. [1]
[Total 4]

Question X4.4

A 10-year endowment assurance policy has a sum assured of £12,000 payable on survival or at the end of the year of earlier death. If the policy is surrendered, the policyholder will receive a return of premiums without interest. Surrenders can occur only at the end of a policy year.

A level premium of £1,100 *pa* is payable annually in advance.

For a policy in force at the start of the fifth year you are given the following details:

	(£)
Renewal expenses	40
Claim expenses on death or surrender	100
Reserve at the start of year	5,000
Reserve at end of year (per survivor)	6,500
Rate of interest	8% *pa*
Dependent probability of death	0.01
Dependent probability of surrender	0.07

Calculate the profit expected to emerge at the end of the fifth year, per policy in force at the start of that year. [3]

Question X4.5

Discuss the uses and shortcomings of the crude death rate, the standardised mortality rate and the standardised mortality ratio for comparing:

(a) the mortality, at different times, of the population of a given country, and

(b) the mortality, at a given time, of different occupational groups in the same population. [9]

Question X4.6

A unit-linked policy has the following profit vector:

Year	*In force profit*
1	–25
2	–12
3	–6
4	25
5	35

(i) Calculate the reserves required, in order to zeroise the losses occurring at the ends of years 2 and 3. Assume a rate of accumulation of 8% *pa*, and that $q_x = 0.01$ at each age. [2]

(ii) If the risk discount rate used is 10%, determine the net present value of the profits before and after zeroisation and state with reasons which of these figures you would expect to be greater. [5]
[Total 7]

Question X4.7

Define each of the following terms and give an example of each:

(a) class selection

(b) selective decrement

(c) spurious selection

(d) adverse selection

(e) temporary initial selection [10]

Question X4.8

Explain the difference between a profit vector and a profit signature. [2]

Question X4.9

(i) You are provided with the following results for two occupations A and B:

	All Occupations		Occupation A		Occupation B	
Age group	Population at risk (000s)	Deaths	Population at risk (000s)	Deaths	Population at risk (000s)	Deaths
16-34	360	360	21	21	12	36
35-44	390	780	42	84	44	88
45-54	430	2580	93	372	92	460
55-64	320	7680	78	2028	72	1512
Total	1500	11400	234	2505	220	2096

Calculate the crude death rates, standardised mortality rates and standardised mortality ratios for occupations A and B using the "All Occupations" experience as standard. [5]

(ii) It has been suggested that administration of the investigation would be simplified if, for the individual occupations, data on ages were supplied only for deaths. Within each age group, the actual deaths and the "All Occupations" mortality rate can be used to estimate the population. The ratio of total population estimated by this method to total actual population in each occupation provides an alternative index for the mortality experience.

Using the data in (i), calculate the value of this new index for each occupation, and comment on the results obtained in parts (i) and (ii). [6]

[Total 11]

Question X4.10

A 3-year unitised with-profits endowment is to be issued to a man aged exactly 55. The policy includes the following features:

- Allocation rate of 85% in year 1 and 100% thereafter.
- Premium of 5,000 paid at start of each year.
- Death benefit, paid at the end of the year of death, equal to the end year unit fund value plus terminal bonus, or 15,000, if higher.
- Maturity benefit equal to the end-year unit fund value plus terminal bonus.
- Surrender is permitted at the end of the first and second years, equal to the unit fund value plus terminal bonus less a surrender penalty of 80 per surrender.
- A policy fee is deducted at the start of each year except the first, equal to 1.5% of the unit fund value immediately **after** the premium for that year has been paid.

Calculate the net present value for this policy on the following assumptions:

- Initial expenses: 500
- Renewal expenses: 30
- Termination expenses: 50 per termination (death, surrender or maturity)
- Initial commission: 5% of the first year's premium
- Renewal commission: 1% of the second and third year's premiums
- Investment and actuarial management expenses:
  0.25% of the end-year unit fund value each year
- Mortality: 80% of AM92 Select
- Surrender probability: 10% of all policies in force at the end of each year
- Regular bonus interest: 4% per annum
- Terminal bonus rates: 1% of the unit fund value after 1 year
  3% of the unit fund value after 2 years
  6.5% of the unit fund value after 3 years
- Non-unit interest: 2% per annum
- Risk discount rate: 8% per annum
- All expected investment returns are assumed to be distributed to the policyholder through the regular and terminal bonuses.

[13]

Question X4.11

A life insurance company issues a 5-year with profits endowment assurance policy to a life aged 60 exact. The policy has a basic sum assured of £10,000. Simple reversionary bonuses are added at the start of each year, including the first. The sum assured (together with any bonuses attaching) is payable at maturity or at the end of year of death, if earlier. Level premiums are payable annually in advance throughout the term of the policy.

(i) Show that the annual premium is approximately £2,476.

Basis:	Mortality:	AM92 Select
	Interest:	6% per annum
	Initial expenses:	60% of the first premium
	Renewal expenses:	5% of the second and subsequent premiums
	Bonus Rates:	A simple reversionary bonus will be declared each year at a rate of 4% per annum

[5]

The office holds net premium reserves using a rate of interest of 4% per annum and AM92 Ultimate mortality.

In order to profit test this policy, the company assumes that it will earn interest at 7% per annum on its funds, mortality follows the AM92 Ultimate table and expenses and bonuses will follow the premium basis.

(ii) Calculate the expected profit margin on this policy using a risk discount rate of 9% per annum. [13]

[Total 18]

Question X4.12

A life insurance company issues a 3-year unit-linked endowment assurance contract to a female life aged 60 exact under which level premiums of £5,000 per annum are payable in advance. In the first year, 85% of the premium is allocated to units and 104% in the second and third years. The units are subject to a bid-offer spread of 5%, and an annual management charge of 0.75% of the bid value of the units is deducted at the end of each year.

If the policyholder dies during the term of the policy, the death benefit of £20,000 or the bid value of the units after the deduction of the management charge, whichever is higher, is payable at the end of the year of death. On survival to the end of the term, the bid value of the units is payable.

The company holds unit reserves equal to the full bid value of the units but does not set up non-unit reserves. It uses the following assumptions in carrying out profit tests of this contract:

Mortality:	AM92 Ultimate	
Surrenders:	None	
Expenses:	Initial:	600
	Renewal:	100 at the start of each of the second and third policy years
Unit fund growth rate:		6% per annum
Non-unit fund interest rate:		4% per annum
Risk discount rate:		10% per annum

(i) Calculate the expected net present value of the profit on this contract. [10]

(ii) State, with a reason, what the effect would be on the profit if the insurance company did hold non-unit reserves to zeroise negative cashflows, assuming it used a discount rate of 4% per annum for calculating those reserves. (You do not need to perform any further calculations.) [2]

[Total 12]

End of paper

For the session leading to the April 2016 exams – CT Subjects

Marking vouchers

Subjects	*Assignments*	*Mocks*
CT1, CT2, CT5, CT6	16 March 2016	22 March 2016
CT3, CT4, CT7, CT8	22 March 2016	30 March 2016

Series X Assignments

Subjects	*Assignment*	*Recommended submission date*	*Final deadline date*
CT1, CT2, CT5, CT6	**X1**	**18 November 2015**	13 January 2016
CT3, CT4, CT7, CT8		**25 November 2015**	20 January 2016
CT1, CT2, CT5, CT6	**X2**	**9 December 2015**	3 February 2016
CT3, CT4, CT7, CT8		**16 December 2015**	10 February 2016
CT1, CT2, CT5, CT6	**X3**	**20 January 2016**	24 February 2016
CT3, CT4, CT7, CT8		**27 January 2016**	2 March 2016
CT1, CT2, CT5, CT6	**X4**	**17 February 2016**	9 March 2016
CT3, CT4, CT7, CT8		**24 February 2016**	16 March 2016

Mock Exams

Subjects	*Recommended submission date*	*Final deadline date*
CT1, CT2, CT5, CT6	**9 March 2016**	22 March 2016
CT3, CT4, CT7, CT8	**16 March 2016**	30 March 2016

We encourage you to work to the recommended submission dates where possible.

We strongly recommend that you submit your mock exam electronically in order for us to return your marked script to you in plenty of time before your exam. If you submit your mock on the final deadline date you are likely to receive your script back less than a week before your exam.

For the session leading to the September/October 2016 exams – CT Subjects

Marking vouchers

Subjects	*Assignments*	*Mocks*
CT1, CT2, CT6, CT7	31 August 2016	7 September 2016
CT3, CT4, CT5, CT8	7 September 2016	14 September 2016

Series X Assignments

Subjects	*Assignment*	*Recommended submission date*	*Final deadline date*
CT1, CT2, CT6, CT7	**X1**	**8 June 2016**	6 July 2016
CT3, CT4, CT5, CT8		**15 June 2016**	13 July 2016
CT1, CT2, CT6, CT7	**X2**	**29 June 2016**	27 July 2016
CT3, CT4, CT5, CT8		**6 July 2016**	3 August 2016
CT1, CT2, CT6, CT7	**X3**	**20 July 2016**	10 August 2016
CT3, CT4, CT5, CT8		**27 July 2016**	17 August 2016
CT1, CT2, CT6, CT7	**X4**	**3 August 2016**	24 August 2016
CT3, CT4, CT5, CT8		**10 August 2016**	31 August 2016

Mock Exams

Subjects	*Recommended submission date*	*Final deadline date*
CT1, CT2, CT6, CT7	**24 August 2016**	7 September 2016
CT3, CT4, CT5, CT8	**31 August 2016**	14 September 2016

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