CT5 – P XS – 16

Series X Solutions

ActEd Study Materials: 2016 Examinations

Subject CT5

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Series X Solutions

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Assignment X1 – Solutions

Markers: This document does not necessarily give every possible approach to solving each of the questions. Please give credit for other valid approaches.

Solution X1.1

We can write:

$$\exp\left(-\int_0^2 \mu_{[60]+t}dt\right) = {}_2p_{[60]} = \frac{l_{62}}{l_{[60]}} = \frac{9129.717}{9263.1422} = 0.985596 \qquad [1]$$

So:

$$\int_0^2 \mu_{[60]+t}\,dt = -\log_e 0.985596 = 0.01451 \qquad [1]$$

[Total 2]

Solution X1.2

(i) ***Present value random variable***

The present value random variable is $5,000\bar{a}_{\overline{T_{42}|}}$. [1]

(ii) ***Variance formula***

For the variance we can write:

$$\text{var}\left(5,000\bar{a}_{\overline{T_{42}|}}\right) = 5,000^2\ \text{var}\left(\bar{a}_{\overline{T_{42}|}}\right)$$

Using the formula for a continuously-payable annuity gives:

$$5,000^2\ \text{var}\left(\bar{a}_{\overline{T_{42}|}}\right) = 5,000^2\ \text{var}\left(\frac{1-v^{T_{42}}}{\delta}\right) = \left(\frac{5,000}{\delta}\right)^2 \text{var}\left(v^{T_{42}}\right) \qquad [1]$$

Now $v^{T_{42}}$ represents the present value random variable of a whole of life assurance with a benefit of 1 payable immediately on death. Substituting in the formula for the variance of this benefit, which is on Page 36 of the Tables, we get:

$$\left(\frac{5,000}{\delta}\right)^2 \text{var}\left(v^{T_{42}}\right) = \left(\frac{5,000}{\delta}\right)^2 \left({}^2\bar{A}_{42} - (\bar{A}_{42})^2\right) \qquad [1]$$

[Total 2]

(iii) ***Variance calculation***

Evaluating this using an interest rate of 4%, we get:

$$\left(\frac{5,000}{\delta}\right)^2 \left((1+i_1)^{\frac{1}{2}} \left({}^2A_{42}\right) - \left((1.04)^{\frac{1}{2}} A_{42}\right)^2 \right) \qquad [1]$$

where $i_1 = 1.04^2 - 1$.

So the variance is:

$$\left(\frac{5,000}{\ln 1.04}\right)^2 \left(1.04 \times 0.07758 - \left((1.04)^{\frac{1}{2}} \times 0.24787\right)^2 \right) = 272,810,000$$

$$= (£16,517)^2 \qquad [1]$$

[Total 2]

Solution X1.3

This question is based on CT5 April 2005 Question 2.

(a) ***Expression for expected present value***

The expected present value of this temporary annuity-due is:

$$\ddot{a}_{50:\overline{20}|} = \frac{1 - A_{50:\overline{20}|}}{d} \qquad [1]$$

(b) ***Value of annuity***

From the *Tables*:

$$A_{50:\overline{20}|} = A^{1}_{50:\overline{20}|} + A_{50:\overline{20}|}^{\ \ \ 1} = A_{50} - \frac{D_{70}}{D_{50}} A_{70} + \frac{D_{70}}{D_{50}}$$

$$= 0.32907 - \frac{517.23}{1,366.61} \times 0.60097 + \frac{517.23}{1,366.61}$$

$$= 0.48009 \qquad [1]$$

So:

$$\ddot{a}_{50:\overline{20}|} = \frac{1 - 0.48009}{0.04/1.04} = 13.518 \qquad [1]$$

Alternatively:

$$\ddot{a}_{50:\overline{20}|} = \ddot{a}_{50} - \frac{D_{70}}{D_{50}} \ddot{a}_{70} = 17.444 - \frac{517.23}{1,366.61} \times 10.375 = 13.517$$

[Total 3]

Solution X1.4

(i) ***Present value random variable***

The present value random variable is:

$$PVRV = \begin{cases} 100v & \text{if } K_x = 0 \\ 200v^2 & \text{if } K_x = 1 \\ 0 & \text{if } K_x \geq 2 \end{cases}$$

[Total 2]

(ii) ***Standard deviation***

The EPV of the benefit is:

$$\begin{aligned} EPV &= 100v\, q_x + 200v^2\, p_x\, q_{x+1} \\ &= \frac{100}{1.06} \times 0.025 + \frac{200}{1.06^2} \times 0.975 \times 0.030 \\ &= 7.56497 \end{aligned}$$

[1]

and:

$$\begin{aligned} E\left(PV^2\right) &= (100v)^2\, q_x + \left(200v^2\right)^2\, p_x\, q_{x+1} \\ &= \left(\frac{100}{1.06}\right)^2 \times 0.025 + \left(\frac{200}{1.06^2}\right)^2 \times 0.975 \times 0.030 \\ &= 1{,}149.24870 \end{aligned}$$

[2]

So the variance of the present value random variable is:

$$\text{var}(PV) = E\left(PV^2\right) - (EPV)^2 = 1{,}092.01993$$

and the standard deviation is $\sqrt{1{,}092.01993} = 33.0457$ [1]

[Total 4]

Solution X1.5

(i) ***All alive in 20 years***

The probability that a life aged 50 is still alive in 20 years' time is:

$${}_{20}p_{50} = \frac{l_{70}}{l_{50}} = \frac{79,970}{96,247} = 0.830883 \qquad [1]$$

So the probability that 5 independent lives aged 50 are all alive in 20 years' time is:

$$0.830883^5 = 0.39600 \qquad [1]$$

[Total 2]

(ii) ***Value of k***

We want to find the largest integer k such that:

$$P(K_{50} \geq k) \geq 0.9$$

Now:

$$P(K_{50} \geq k) = {}_kp_{50} = \frac{l_{50+k}}{l_{50}}$$

Setting this equal to 0.9 gives:

$$l_{50+k} = 0.9\,l_{50} = 0.9 \times 96,247 = 86,622.3 \qquad [1]$$

From the *Tables*, we see that 86,622.3 lies between $l_{65}(=87,093)$ and $l_{66}(=85,875)$.

[1]

Since we require ${}_kp_{50} \geq 0.9$, we must have $k = 15$. [1]

[Total 3]

Solution X1.6

(i) We have:

$$\begin{aligned} A_{x:\overline{10|}}^{\;\;\,1} &= A_{x:\overline{10|}} - A^{1}_{x:\overline{10|}} \\ &= A_{x:\overline{10|}} - \left(A_x - v^{10}\ {}_{10}p_x\ A_{x+10}\right) \\ &= A_{x:\overline{10|}} - \left(A_x - A_{x:\overline{10|}}^{\;\;\,1}\ A_{x+10}\right) \\ &= 0.75 - 0.20 + 0.25 A_{x:\overline{10|}}^{\;\;\,1} \\ &= 0.55 + 0.25 A_{x:\overline{10|}}^{\;\;\,1} \end{aligned}$$ [2]

So:

$$0.75 A_{x:\overline{10|}}^{\;\;\,1} = 0.55 \Rightarrow A_{x:\overline{10|}}^{\;\;\,1} = \frac{55}{75} = 0.73333$$ [1]

[Total 3]

(ii) We now have:

$$A^{1}_{x:\overline{10|}} = A_{x:\overline{10|}} - A_{x:\overline{10|}}^{\;\;\,1} = 0.75 - 0.73333 = 0.01667$$ [Total 1]

(iii) Also:

$${}_{10|}A_x = A_{x:\overline{10|}}^{\;\;\,1} \times A_{x+10} = \frac{55}{75} \times 0.25 = 0.18333$$ [Total 1]

Alternatively, you could write:

$${}_{10|}A_x = A_x - A^{1}_{x:\overline{10|}} = 0.20 - 0.01667 = 0.18333$$

Solution X1.7

The survival probability can be written as:

$$_{2}p_{63.25} = {}_{0.75}p_{63.25} \times p_{64} \times {}_{0.25}p_{65} \qquad [1]$$

From the *Tables*:

$$p_{64} = 1 - q_{64} = 0.97801$$

You could also have calculated p_{64} *as* $\dfrac{l_{65}}{l_{64}}$.

(a) ***UDD***

Under the uniform distribution of deaths assumption:

$$_{0.25}p_{65} = 1 - 0.25q_{65} = 1 - 0.25 \times 0.02447 = 0.99388 \qquad [1]$$

and:

$$_{0.75}p_{63.25} = \frac{p_{63}}{_{0.25}p_{63}} = \frac{p_{63}}{1-0.25q_{63}} = \frac{0.98035}{1-0.25\times 0.01965} = 0.98519 \qquad [1]$$

So:

$$_{2}p_{63.25} = 0.98519 \times 0.97801 \times 0.99388 = 0.95763 \qquad [½]$$

Alternative solutions

You could also have used the UDD formula $_{t-s}q_{x+s} = \dfrac{(t-s)q_x}{1-s\,q_x}$ *to say that:*

$$_{0.75}q_{63.25} = \frac{0.75q_{63}}{1-0.25q_{63}} = \frac{0.75 \times 0.01965}{1-0.25\times 0.01965} = 0.01481$$

and hence:

$$_{0.75}p_{63.25} = 1 - 0.01481 = 0.98519$$

Or, another way to do this is to use:

$$_2p_{63.25} = \frac{l_{65.25}}{l_{63.25}}$$

and interpolate between the life table values from ELT15 (Males). We have:

$$\begin{aligned} l_{63.25} &= 0.25 \times l_{64} + 0.75 \times l_{63} \\ &= 0.25 \times 81{,}076 + 0.75 \times 82{,}701 \\ &= 82{,}294.75 \end{aligned}$$

and

$$\begin{aligned} l_{65.25} &= 0.25 \times l_{66} + 0.75 \times l_{65} \\ &= 0.25 \times 77{,}353 + 0.75 \times 79{,}293 \\ &= 78{,}808 \end{aligned}$$

So, $_2p_{63.25} = \dfrac{78{,}808}{82{,}294.75} = 0.95763$

Markers: please give credit for correct alternatives.

(b) ***CFM***

Under the constant force of mortality assumption:

$$_{0.25}p_{65} = (p_{65})^{0.25} = 0.97553^{0.25} = 0.99383$$ [1]

and:

$$_{0.75}p_{63.25} = (p_{63})^{0.75} = (0.98035)^{0.75} = 0.98523$$ [1]

So:

$$_2p_{63.25} = 0.98523 \times 0.97801 \times 0.99383 = 0.95761$$ [½]

[Total 6]

Solution X1.8

${}_{3|}q_{[55]+1}$ is the probability that a life aged exactly 56, who entered the select population 1 year ago, dies between the ages of 59 and 60. [1½]

Using AM92 mortality, we have:

$${}_{3|}q_{[55]+1} = \frac{d_{59}}{l_{[55]+1}} = \frac{66.7876}{9,513.9375} = 0.00702 \qquad [1½]$$

[Total 3]

Solution X1.9

There are three possible outcomes at the end of 20 years:

- both (30) and (40) alive
- just one of (30) and (40) alive
- neither of (30) and (40) alive

The probabilities of these events and the net present value of the amount due to the trust are:

Event	*Present Value*	*Probability*
(30), (40) alive	$\dfrac{20,000}{3}v_{4\%}^{20} = 3,042.58$	${}_{20}p_{30} \times {}_{20}p_{40} = 0.9785 \times 0.9423$ $= 0.9220$
only one of (30), (40) alive	$\dfrac{20,000}{2}v_{4\%}^{20} = 4,563.87$	${}_{20}p_{30}(1 - {}_{20}p_{40}) + {}_{20}p_{40}(1 - {}_{20}p_{30})$ $= 0.9785 \times 0.0577 + 0.9423 \times 0.0215$ $= 0.0767$
(30), (40) both dead	$20,000v_{4\%}^{20} = 9,127.74$	$(1 - {}_{20}p_{30})(1 - {}_{20}p_{40})$ $= 0.0577 \times 0.0215 = 0.0012$

[4]

So the required expected value is:

$$3,042.58 \times 0.9220 + 4,563.87 \times 0.0767 + 9,127.74 \times 0.0012 = £3,166 \qquad [1]$$

and the variance is:

$$3,042.58^2 \times 0.9220 + 4,563.87^2 \times 0.0767 + 9,127.74^2 \times 0.0012 - 3,166^2$$

$$= 207,578 \qquad [2]$$

[Total 7]

Solution X1.10

This question is CT5 April 2005 Question 4.

Let X denote the annual annuity payment. Then:

$$X\left(\ddot{a}_{\overline{10}|}^{(12)} + v^{10}\,{}_{10}p_{60}\,\ddot{a}_{70}^{(12)}\right) = 200{,}000 \qquad [1]$$

From Subject CT1:

$$\ddot{a}_{\overline{10}|}^{(12)} = \frac{i}{d^{(12)}}a_{\overline{10}|} = 1.032211 \times 7.3601 = 7.5972 \qquad [1]$$

The formula for a whole life annuity-due payable m thly in advance is given on Page 36 of the *Tables*. From this we have:

$$\ddot{a}_{70}^{(12)} = \ddot{a}_{70} - \frac{11}{24} = 9.140 - \frac{11}{24} = 8.682 \qquad [1]$$

So:

$$X\left(7.5972 + 1.06^{-10} \times \frac{8{,}054.0544}{9{,}287.2164} \times 8.682\right) = 200{,}000$$

$$\Rightarrow X = \frac{200{,}000}{11.8015} = £16{,}947 \qquad [1]$$

[Total 4]

Solution X1.11

(i) ***Description of benefit***

The benefit is an annuity of 10,000 *pa* payable annually in advance for a maximum of 10 years, but ceasing on earlier death. [1]

(ii) ***Proof***

We can write:

$$\ddot{a}_{x:\overline{n}|} = E\left(\ddot{a}_{\overline{\min\{K+1,n\}}|}\right) = E\left(\frac{1-v^{\min\{K+1,n\}}}{d}\right) = \frac{1-E\left(v^{\min\{K+1,n\}}\right)}{d} = \frac{1-A_{x:\overline{n}|}}{d}$$

[1½]

Rearranging this gives the required result:

$$A_{x:\overline{n}|} = 1 - d\,\ddot{a}_{x:\overline{n}|}$$ [½]

[Total 2]

(iii) ***Expected present value and standard deviation***

The expected present value is:

$$\begin{aligned}10{,}000\,\ddot{a}_{x:\overline{10}|} &= 10{,}000\sum_{k=0}^{9} v^k\ {}_kp_x \\ &= 10{,}000\sum_{k=0}^{9} e^{-0.04k}\,e^{-0.02k} \\ &= 10{,}000\left(\frac{1-e^{-0.06\times 10}}{1-e^{-0.06}}\right) \\ &= 77{,}476.56\end{aligned}$$ [2]

We have used the result that the sum to n terms of a geometric series $a + ar + ar^2 + \cdots + ar^{n-1}$ *is:*

$$s_n = \frac{a\left(1-r^n\right)}{1-r}$$

The variance of the present value random variable is:

$$\begin{aligned}\operatorname{var}(W) &= 10{,}000^2 \operatorname{var}\left[\ddot{a}_{\overline{\min\{K+1,10\}}|}\right] \\ &= 10{,}000^2 \operatorname{var}\left[\frac{1-v^{\min\{K+1,10\}}}{d}\right] \\ &= \frac{10{,}000^2}{d^2}\operatorname{var}\left[v^{\min\{K+1,10\}}\right]\end{aligned}$$ [1]

Now $v^{\min\{K+1,10\}}$ is the present value of a benefit of 1 paid at time 10 or at the end of the year of death of (x), whichever is sooner. So it is the present value of a 10-year endowment assurance on (x), and:

$$\operatorname{var}\left[v^{\min\{K+1,10\}}\right] = {}^2A_{x:\overline{10}|} - \left(A_{x:\overline{10}|}\right)^2$$ [1]

We can calculate the values of these assurances using premium conversion:

$$A_{x:\overline{10}|} = 1 - d\,\ddot{a}_{x:\overline{10}|} = 1-\left(1-e^{-0.04}\right)\times 7.747656 = 0.696210$$ [1]

Since ${}^2A_{x:\overline{10}|}$ is equal to $A_{x:\overline{10}|}$ evaluated using a force of interest of $2\delta = 0.08$, and:

$$\ddot{a}_{x:\overline{10}|} @ (\delta = 0.08) = \sum_{k=0}^{9} e^{-0.08k}\, e^{-0.02k} = \frac{1-e^{-0.1\times 10}}{1-e^{-0.1}} = 6.642533$$ [1]

it follows that:

$${}^2A_{x:\overline{10}|} = 1-\left(1-e^{-0.08}\right)\times 6.642533 = 0.489298$$ [1]

So:

$$\operatorname{var}\left[v^{\min\{K+1,10\}}\right] = 0.489298 - 0.696210^2 = 0.004589$$

The standard deviation of the present value random variable is therefore:

$$\frac{10,000}{d}\times\sqrt{0.004589}=\frac{10,000}{\left(1-e^{-0.04}\right)}\times\sqrt{0.004589}=17,277 \qquad [1]$$

[Total 8]

Solution X1.12

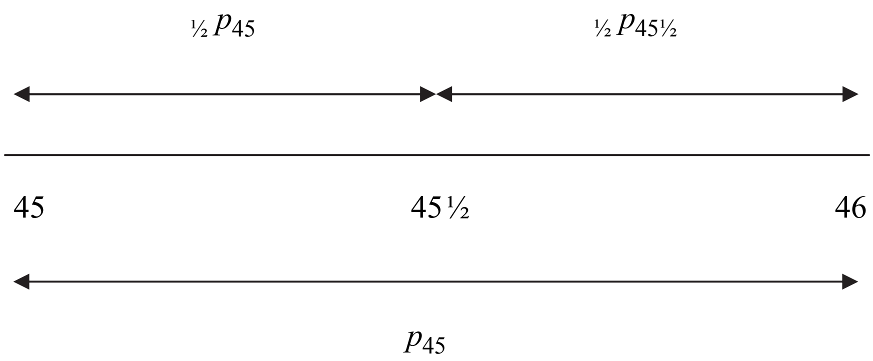
This question is CT5 April 2005 Question 6.

Uniform distribution of deaths

This assumption says that ${}_tq_x = t\,q_x$ for integer values of x and $0\le t\le 1$. Since we are starting at age $45\tfrac{1}{2}$, which is not an integer, we must first write ${}_{\frac12}p_{45\frac12}$ in terms of ${}_tp_{45}$. So we begin by writing:

$${}_{\frac12}p_{45\frac12}=\frac{p_{45}}{{}_{\frac12}p_{45}} \qquad [½]$$

This can be easily seen from the following diagram:



Then under the UDD assumption:

$${}_{\frac12}p_{45\frac12}=\frac{1-q_{45}}{1-{}_{\frac12}q_{45}}=\frac{1-q_{45}}{1-\frac12 q_{45}}=\frac{1-0.00266}{1-\frac12\times 0.00266}=0.99867 \qquad [1]$$

Alternative solutions

Alternatively, you could use the UDD formula ${}_{t-s}q_{x+s} = \dfrac{(t-s)\,q_x}{1-s\,q_x}$ *to say that:*

$${}_{½}q_{45½} = \frac{½\,q_{45}}{1-½\,q_{45}} = \frac{½\times 0.00266}{1-½\times 0.00266} = 0.00133$$

and hence ${}_{½}p_{45½} = 1-0.00133 = 0.99867$.

Also:

$${}_{14}p_{46} = \frac{l_{60}}{l_{46}} = \frac{86{,}714}{95{,}266} = 0.91023 \qquad [½]$$

So:

$${}_{14½}p_{45½} = {}_{½}p_{45½} \times {}_{14}p_{46} = 0.99867\times 0.91023 = 0.90902 \qquad [1]$$

Or, another way to do this is to use:

$${}_{14½}p_{45½} = \frac{l_{60}}{l_{45½}}$$

and interpolate between the life table values from ELT15 (Males). We have:

$$\begin{aligned} l_{45½} &= 0.5\times l_{45} + 0.5\times l_{46} \\ &= 0.5\times 95{,}521 + 0.5\times 95{,}266 \\ &= 95{,}393.5 \end{aligned}$$

So: $${}_{14½}p_{45½} = \frac{86{,}714}{95{,}393.5} = 0.90901$$

Markers: please give credit for correct alternatives.

Constant force of mortality

We now assume that μ is constant between integer ages.

In this case:

$$_{½}p_{45½} = \exp\left(-\int_0^{½} \mu_{45½+t}\, dt\right) = e^{-½\mu}$$

$$= \left(e^{-\mu}\right)^{½} = \left(p_{45}\right)^{½}$$

$$= \left(1-0.00266\right)^{½} = 0.99867 \qquad [1]$$

So:

$$_{14½}p_{45½} = {}_{½}p_{45½} \times {}_{14}p_{46} = 0.99867 \times 0.91023 = 0.90902 \qquad [1]$$

Note that the two assumptions give the same answer correct to 4dp.

Another possible assumption that you could have used here is the Balducci assumption, which you met in Subject CT4 or Subject 104. The Balducci formula is given on Page 33 of the Tables. It states that $_{1-t}q_{x+t} = \left(1-t\right)q_x$ for integer values of x and $0 \le t \le 1$. In this case:

$$_{½}p_{45½} = 1 - {}_{½}q_{45½} = 1 - ½\, q_{45} = 1 - ½ \times 0.00266 = 0.99867 \qquad \textit{[1]}$$

So:

$$_{14½}p_{45½} = {}_{½}p_{45½} \times {}_{14}p_{46} = 0.99867 \times 0.91023 = 0.90902 \qquad \textit{[1]}$$

as before.

[Maximum 5]

Markers: please award the full five marks for correct solutions using either method.

Solution X1.13

This question is CT5 April 2010 Question 5.

(i) ***Survival to age 70 exact***

We need:

$$ {}_{50}p_{20} = \exp\left[-\int_0^{50} \mu_{20+s}\, ds\right] = \exp\left[-\int_0^{50} e^{0.0002(20+s)} - 1\, ds\right] \qquad [½]$$

The integral can be evaluated as follows:

$$\int_0^{50} e^{0.0002(20+s)} - 1\, ds = \left[\frac{e^{0.004+0.0002s}}{0.0002} - s\right]_0^{50} = \left(\frac{e^{0.014}}{0.0002} - 50\right) - \left(\frac{e^{0.004}}{0.0002}\right)$$

$$= 5{,}000\left(e^{0.014} - e^{0.004}\right) - 50 = 0.45224 \qquad [1]$$

So we have:

$$ {}_{50}p_{20} = \exp[-0.45224] = 0.63620 \qquad [½]$$

[Total 2]

(ii) ***Death between age 60 and age 70***

By analogy with the first part of the question, we can see that:

$$\int_0^{40} e^{0.0002(20+s)} - 1\, ds = \left[\frac{e^{0.004+0.0002s}}{0.0002} - s\right]_0^{40} = \left(\frac{e^{0.012}}{0.0002} - 40\right) - \left(\frac{e^{0.004}}{0.0002}\right)$$

$$= 5{,}000\left(e^{0.012} - e^{0.004}\right) - 40 = 0.32139 \qquad [1]$$

So:

$$ {}_{40}p_{20} = \exp[-0.32139] = 0.72514 \qquad [½]$$

So we have that the probability that a life now aged 20 exact dies between 60 and 70 is:

$$ {}_{40}p_{20} - {}_{50}p_{20} = 0.72514 - 0.63620 = 0.08894 \qquad [1½]$$

[Total 3]

Solution X1.14

This question is CT5 September 2006, Question 11.

(i) ***Present value random variable***

The present value of the benefits takes the values:

$$\begin{cases} 0 & T_{63} < 2 \\ 5,000\left(\bar{a}_{\overline{T_{63}}|} - \bar{a}_{\overline{2}|}\right) & T_{63} \ge 2 \end{cases}$$ [3]

This can alternatively be expressed in any of the following ways: markers please give full credit for any correct expression.

- $5,000\left(\bar{a}_{\overline{T_{63}}|} - \bar{a}_{\overline{\min\{T_{63},2\}}|}\right)$
- $5,000\left(\bar{a}_{\overline{T_{63}}|} - \bar{a}_{\overline{\min\{T_{63},2\}}|}\right)$
- $\begin{cases} 0 & T_{63} < 2 \\ 5,000\bar{a}_{\overline{T_{63}-2}|}v^2 & T_{63} \ge 2 \end{cases}$
- $5,000\bar{a}_{\overline{\max\{T_{63}-2,0\}}|}v^2$

(ii) ***Expected present value***

The expected present value is:

$$\begin{aligned} EPV &= 100 \times 5,000\bar{a}_{65}\ {}_2p_{63}\, v^2 \\ &= 500,000\left(\ddot{a}_{65} - 0.5\right)\frac{l_{65}}{l_{63}}v^2 \\ &= 500,000 \times 14.371 \times \frac{9,703.708}{9,775.888} \times 1.04^{-2} \\ &= £6,594,347 \end{aligned}$$

[½ for each line, total 2]

(iii) ***Variance of the total present value***

The total variance is:

$$100\,\text{var}\left[PV\right] = 100\,\text{var}\left(5{,}000\bar{a}_{\overline{\max\{T_{63}-2,0\}}|}v^2\right)$$

$$= 100\times 5{,}000^2\, v^4\ \text{var}\left(\frac{1-v^{\max\{T_{63}-2,0\}}}{\delta}\right)$$

$$= \frac{100\times 5{,}000^2\, v^4}{\delta^2}\text{var}\left(v^{\max\{T_{63}-2,0\}}\right)$$

$$= \frac{100\times 5{,}000^2\, v^4}{\delta^2}\text{var}\left(\frac{v^{\max\{T_{63},2\}}}{v^2}\right)$$

$$= \frac{100\times 5{,}000^2}{\delta^2}\text{var}\left(v^{\max\{T_{63},2\}}\right)$$

[½ for each line, total 2½]

Now:

$$\text{var}\left(v^{\max\{T_{63},2\}}\right) = E\left[(v^2)^{\max\{T_{63},2\}}\right] - \left(E\left[v^{\max\{T_{63},2\}}\right]\right)^2$$ [1]

So:

$$\text{var}\left(v^{\max\{T_{63},2\}}\right) = \left(v^2\right)^2\, {}_2q_{63} + \left(v^2\right)^2\, {}_2p_{63}\, {}^2\bar{A}_{65} - \left(v^2\, {}_2q_{63} + v^2\, {}_2p_{63}\, \bar{A}_{65}\right)^2$$

$$= 1.04^{-4}\left(1-\frac{l_{65}}{l_{63}}\right) + 1.04^{-4}\frac{l_{65}}{l_{63}}\sqrt{1.04^2}\left({}^2A_{65}\right)$$

$$-\left(1.04^{-2}\left(1-\frac{l_{65}}{l_{63}}\right) + 1.04^{-2}\frac{l_{65}}{l_{63}}\left(1-\delta\bar{a}_{65}\right)\right)^2$$

[½ for each line, total 1½]

where ${}^2\bar{A}$ indicates a whole of life assurance calculated at an interest rate of $i = 8.16\%$.

[½]

Substituting in values from the *Tables* we get:

$$\text{var}\left(v^{\max\{T_{63},2\}}\right) = 1.04^{-4}\left(1 - \frac{9,703.708}{9,775.888}\right) + 1.04^{-3}\frac{9,703.708}{9,775.888}0.20847$$

$$-\left(1.04^{-2}\left(1 - \frac{9,703.708}{9,775.888}\right) + 1.04^{-2}\frac{l_{65}}{l_{63}}\left(1-(\ln 1.04)\times 14.371\right)\right)^2$$

$$= 0.0243901$$

[½ for each line, total 1½]

So, the total variance of the present value is:

$$\frac{100\times 5,000^2}{\delta^2}\text{var}\left(v^{\max\{T_{63},2\}}\right) = \frac{100\times 5,000^2}{(\ln 1.04)^2}\times 0.0243901$$

$$= 3.963894\times 10^{10}$$

$$= (£199,095)^2$$

[1]
[Total 8]

Assignment X2 – Solutions

Markers: This document does not necessarily give every possible approach to solving each of the questions. Please give credit for other valid approaches.

Solution X2.1

Let P denote the annual premium. Then:

$$\text{EPV premiums} = P\ddot{a}_{[40]:\overline{25|}} = 15.887P$$ [½]

There are two parts to the benefit. Let's consider the deferred annuity first.

$$\text{EPV annuity benefit} = 15{,}000\frac{D_{65}}{D_{[40]}}\bar{a}_{65}$$ [½]

The D/D factor is calculated using AM92 select mortality. So:

$$\frac{D_{65}}{D_{[40]}} = \frac{689.23}{2{,}052.54} = 0.33579$$ [½]

The annuity factor is calculated using PMA92C20 mortality. So:

$$\bar{a}_{65} \approx \ddot{a}_{65} - 0.5 = 13.666 - 0.5 = 13.166$$ [½]

Hence:

$$\text{EPV annuity benefit} = 15{,}000 \times 0.33579 \times 13.166 = 66{,}315.90$$ [½]

Let's now consider the death benefit.

$$\begin{aligned}\text{EPV death benefit} &= P(IA)^{1}_{[40]:\overline{25|}} \\ &= P\left\{(IA)_{[40]} - \frac{D_{65}}{D_{[40]}}\left[(IA)_{65} + 25A_{65}\right]\right\} \\ &= P\left[7.95835 - 0.33579(7.89442 + 25 \times 0.52786)\right] \\ &= 0.87615P\end{aligned}$$ [1½]

So the equation of value is:

$$15.887P = 66{,}315.90 + 0.87615P$$ [½]

So $P = £4{,}417.86$. [½]

[Total 5]

Solution X2.2

(i) ***Four bonus methods***

There are three methods of declaring a reversionary bonus, whereby the sum insured is increased and, once increased, cannot be decreased. [½]

- Simple reversionary bonus: the rate of bonus each year is a percentage of the initial sum insured. [½]
- Compound reversionary bonus: the rate of bonus each year is a percentage of the initial sum insured plus previously declared reversionary bonuses. [½]
- Super-compound reversionary bonus: there are two rates of bonus. One is applied to the basic sum insured, the other is applied to the previously declared bonuses. [½]

In addition there is the terminal bonus, whereby the sum insured is increased at maturity or on earlier claim. The terminal bonus rate is normally a percentage of final sum insured. [1]

[Total 3]

(ii) ***Net premium reserve***

The net premium reserve at time 4 is:

$${}_4V = 20{,}000\ {}_4V_{[40]:\overline{20}|} + B_4\ A_{44:\overline{16}|}$$ [1]

where:

$$B_4 = \text{bonuses declared up to time } 4 = 20{,}000\left[1.04^4 - 1\right] = 3{,}397.17$$ [½]

Note that net premium reserves for with-profits policies:

- *include the value of existing bonuses declared to date*
- *ignore all future bonuses*
- *assume a net premium that excludes **all** bonuses.*

Therefore:

$$ {}_4V = 20{,}000\left[1-\frac{\ddot{a}_{44:\overline{16}|}}{\ddot{a}_{[40]:\overline{20}|}}\right]+3{,}397.17\,A_{44:\overline{16}|} $$ [½]

$$ = 20{,}000\left[1-\frac{11.934}{13.930}\right]+3{,}397.17\times 0.54100 $$ [½]

$$ = £4{,}704 $$ [½]

[Total 3]

Alternatively, we could have calculated:

$$ {}_4V = 20{,}000(1.04)^4\,A_{44:\overline{16}|} - P\,\ddot{a}_{44:\overline{16}|} $$

where:

$$ P = 20{,}000\frac{A_{[40]:\overline{20}|}}{\ddot{a}_{[40]:\overline{20}|}} = 20{,}000\times\frac{0.46423}{13.930} = 666.52 $$

giving:

$$ {}_4V = 20{,}000(1.04)^4\times 0.54100 - 666.52\times 11.934 = £4{,}704 $$

Solution X2.3

(i) ***Features of accumulating with-profits***

The benefits take the form of an accumulating fund of premiums. [½]

The fund accumulates with interest, ... [½]

... where the interest rate may be partly guaranteed, ... [½]

... with the remainder being discretionary "bonus" interest, which can vary over time. [½]

Interest rates cannot be negative, ... [½]

... they will reflect the underlying profits made by the insurer (including investment profits), ... [½]

... but will be smoothed over time so as to produce a more stable progression compared to the underlying asset returns. [½]

On death or maturity, a terminal bonus can be paid out in addition to the accumulated fund value, ... [½]

... at the discretion of the insurance company. [½]

[Maximum 4]

(ii) ***Fund value on 6th April***

First we need the daily effective interest rate. This is:

$$i = 1.0425^{1/365} - 1 = 0.0114\%$$ [½]

The fund on 15th March (in 4 days time), after the deduction of the policy fee, will be:

$$65{,}292 \times (1+i)^4 - 3 = 65{,}318.79$$ [½]

The fund on 1st April (17 days later), after payment of the premium, will be:

$$65{,}318.79 \times (1+i)^{17} + 600 = 66{,}045.53$$ [½]

So by 6th April (5 days later) the fund value will be:

$$66{,}045.53 \times (1+i)^5 = 66{,}083.20$$ [½]

[Total 2]

Solution X2.4

If the annual premium is P, then:

$$EPV\ \ premiums = P\ddot{a}_{[50]:\overline{10}|} = 8.318P \qquad [1]$$

$$EPV\ \ benefits = 10,000A^{\ 1}_{[50]:\overline{5}|} + 15,000\,v^5\ {}_5p_{[50]}A^{\ 1}_{55:\overline{10}|}$$

$$= 10,000\left(A_{[50]} - \frac{D_{55}}{D_{[50]}}A_{55}\right) + 15,000\frac{D_{55}}{D_{[50]}}\left(A_{55} - \frac{D_{65}}{D_{55}}A_{65}\right)$$

$$= 10,000\left(0.32868 - \frac{1,105.41}{1,365.77}\times 0.38950\right)$$

$$+15,000\times\frac{1,105.41}{1,365.77}\left(0.38950 - \frac{689.23}{1,105.41}\times 0.52786\right)$$

$$= 867.31 \qquad [3]$$

$$\begin{aligned} EPV\ \ expenses &= 0.25P + 0.05P(\ddot{a}_{[50]:\overline{10}|} - 1) \\ &= (0.25 + 0.05\times 7.318)P \\ &= 0.6159P \qquad [1] \end{aligned}$$

So the premium equation is:

$$8.318P = 867.31 + 0.6159P$$

and:

$$P = 867.31/7.7021 = £112.61 \qquad [1]$$

[Total 6]

Note for markers: any candidate who uses ultimate instead of select mortality, but otherwise performs the calculations correctly, will get an answer of £113.36. Award 5 marks for this.

Solution X2.5

(i)(a) ***Gross premium***

Let P denote the gross annual premium. Then the equation of value is:

$$P\,\ddot{a}_{x:\overline{n}|} = A^{1}_{x:\overline{n}|} + I + e\,\ddot{a}_{x:\overline{n}|}$$

So:

$$P = \frac{A^{1}_{x:\overline{n}|} + I + e\,\ddot{a}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} \qquad [1]$$

(i)(b) ***Prospective gross premium reserve***

The prospective gross premium reserve at integer time t is:

$${}_tV^{pro} = A^{1}_{x+t:\overline{n-t}|} + e\,\ddot{a}_{x+t:\overline{n-t}|} - P\,\ddot{a}_{x+t:\overline{n-t}|} \qquad [1]$$

(i)(c) ***Retrospective gross premium reserve***

The retrospective gross premium reserve at integer time t is:

$${}_tV^{retro} = \left[P\,\ddot{a}_{x:\overline{t}|} - A^{1}_{x:\overline{t}|} - I - e\,\ddot{a}_{x:\overline{t}|}\right]\frac{(1+i)^t}{{}_tp_x} \qquad [1]$$

[Total 3]

(ii) ***Equality of reserves***

The premium equation is:

$$P\,\ddot{a}_{x:\overline{n}|} = A^{1}_{x:\overline{n}|} + I + e\,\ddot{a}_{x:\overline{n}|}$$

Splitting this up at time t, it is equivalent to:

$$P\left(\ddot{a}_{x:\overline{t}|} + v^t\;{}_tp_x\;\ddot{a}_{x+t:\overline{n-t}|}\right) = A^{1}_{x:\overline{t}|} + v^t\;{}_tp_x\;A^{1}_{x+t:\overline{n-t}|} + I + e\left(\ddot{a}_{x:\overline{t}|} + v^t\;{}_tp_x\;\ddot{a}_{x+t:\overline{n-t}|}\right)$$

[1]

Now, rearranging so that all the terms containing $v^t \; {}_t p_x$ are on the same side of the equation, we get:

$$P\ddot{a}_{x:\overline{t}|} - A^{1}_{x:\overline{t}|} - I - e\,\ddot{a}_{x:\overline{t}|} = v^t \; {}_t p_x \left(A^{\;1}_{x+t:\overline{n-t}|} + e\,\ddot{a}_{x+t:\overline{n-t}|} - P\ddot{a}_{x+t:\overline{n-t}|} \right) \qquad [1]$$

Dividing both sides through by $v^t \; {}_t p_x$ then gives:

$$\left(P\ddot{a}_{x:\overline{t}|} - A^{1}_{x:\overline{t}|} - I - e\,\ddot{a}_{x:\overline{t}|} \right) \frac{(1+i)^t}{{}_t p_x} = A^{\;1}_{x+t:\overline{n-t}|} + e\,\ddot{a}_{x+t:\overline{n-t}|} - P\ddot{a}_{x+t:\overline{n-t}|} \qquad [½]$$

The LHS of this equation is the retrospective reserve and the RHS is the prospective reserve. So the reserves are equal. [½]

[Total 3]

Solution X2.6

Let K denote the curtate future lifetime of a new policyholder. Then the insurer's loss random variable for the policy is:

$$L = 100,000v^{K+1} + 0.05P\ddot{a}_{\overline{K+1|}} - P\ddot{a}_{\overline{K+1|}}$$

$$= 100,000v^{K+1} - 0.95P\ddot{a}_{\overline{K+1|}} \qquad [1]$$

L will be positive if the policyholder dies "too soon". We want to find the value of t such that:

$$P(L > 0) = P(T < t) = 0.01$$

where T represents the policyholder's complete future lifetime.

In other words, we want to find t such that:

$$P(T \geq t) = {}_t p_{[35]} = 0.99 \qquad [1]$$

In terms of life table functions, we have:

$$\frac{l_{[35]+t}}{l_{[35]}} \geq 0.99 \quad \Rightarrow \quad l_{[35]+t} \geq 0.99\, l_{[35]} = 0.99 \times 9,892.9151 = 9,793.99 \qquad [½]$$

From the *Tables*: $l_{45} = 9,801.3123$ and $l_{46} = 9,786.9534$

So t lies somewhere between 10 and 11, and we set $K = 10$. [1]

So we need to find the "break even" premium P, assuming the benefit is paid at the end of year 11 and using 6% interest. This is given by the equation:

$$0.95P\ddot{a}_{\overline{11|}} = 100,000v^{11} \qquad [½]$$

Rearranging to find P:

$$P = \frac{100,000v^{11}}{0.95\ddot{a}_{\overline{11|}}} = \frac{100,000}{0.95\ddot{s}_{\overline{11|}}} = \frac{100,000}{0.95 \times 15.86994} = £6,632.86 \qquad [1]$$

[Total 5]

Solution X2.7

(i) ***Present value random variable***

If, for example, the man dies in the fourth policy year (*ie* if $K_{65} = 3$), the present value of his benefit is:

$$X = 10,000\left(v + 1.03v^2 + 1.03^2 v^3\right)$$
$$= 10,000\, v\left(1 + 1.03v + 1.03^2 v^2\right)$$

With $i = 0.03$, this simplifies to:

$$X = 10,000v \times 3$$

or, in general:

$$X = 10,000\, v\, K_{65} = \frac{10,000\, K_{65}}{1.03} \qquad [2]$$

(ii) ***Expected present value***

$$E\left(X\right) = \frac{10,000 \times E\left(K_{65}\right)}{1.03} = \frac{10,000\, e_{65}}{1.03} = \frac{10,000 \times 16.645}{1.03} = £161,602 \qquad [1]$$

(iii) ***Prospective reserve at time 5***

The future benefits are $10,000 \times 1.03^5$ at time 6, $10,000 \times 1.03^6$ at time 7, and so on. All of these payments are contingent on survival. The prospective reserve at time 5 is the EPV at time 5 of these future benefits.

$${}_5V^{pro} = 10,000 \times 1.03^5 \left(v\, p_{70} + 1.03\, v^2\, {}_2p_{70} + 1.03^2\, v^3\, {}_3p_{70} + \cdots\right) \qquad [1]$$

Assuming $i = 0.03$ and AM92 Ultimate mortality:

$$
\begin{aligned}
{}_5V^{pro} &= 10{,}000 \times 1.03^4 \left(p_{70} + {}_2p_{70} + {}_3p_{70} + \cdots \right) \\
&= 10{,}000 \times 1.03^4 \times e_{70} \\
&= 10{,}000 \times 1.03^4 \times 13.023 \\
&= £146{,}575
\end{aligned}
$$

[1]

[Total 2]

(iv) ***Probability***

From (i), the present value random variable is $X = \dfrac{10{,}000\,K_{65}}{1.03}$. So:

$$
\begin{aligned}
P\left(X > 250{,}000\right) &= P\left(K_{65} > \frac{250{,}000 \times 1.03}{10{,}000} \right) \\
&= P\left(K_{65} > 25.75\right) \\
&= P\left(K_{65} \geq 26\right)
\end{aligned}
$$

[1]

since K_{65} can only take non-negative integer values. Assuming AM92 Ultimate mortality:

$$
P\left(K_{65} \geq 26\right) = {}_{26}p_{65} = \frac{l_{91}}{l_{65}} = \frac{1{,}376.1906}{8{,}821.2612} = 0.15601
$$

[1]

[Total 2]

Solution X2.8

(i) ***Reserve***

The prospective reserve at the end of the 5th policy year is:

$$ {}_5V^{pro} = 60{,}000A^{1}_{70:\overline{20}|} + 120{,}000\frac{D_{90}}{D_{70}} + 0.01P\,\ddot{a}_{70:\overline{20}|} - P\,\ddot{a}_{70:\overline{20}|} \qquad [1] $$

We have:

$$ \frac{D_{90}}{D_{70}} = \frac{48.61}{517.23} = 0.09398 \qquad [½] $$

Also:

$$ A^{1}_{70:\overline{20}|} = A_{70} - \frac{D_{90}}{D_{70}}A_{90} = 0.60097 - 0.09398 \times 0.84196 = 0.52184 \qquad [½] $$

$$ \ddot{a}_{70:\overline{20}|} = \ddot{a}_{70} - \frac{D_{90}}{D_{70}}\ddot{a}_{90} = 10.375 - 0.09398 \times 4.109 = 9.9888 \qquad [½] $$

So:

$$ \begin{aligned} {}_5V^{pro} &= 60{,}000 \times 0.52184 + 120{,}000 \times 0.09398 - 0.99 \times 3{,}071.40 \times 9.9888 \\ &= £12{,}215.36 \end{aligned} \qquad [½] $$

[Total 3]

Alternatively, you could have calculated the reserve retrospectively using the formula:

$$ {}_5V^{retro} = \frac{D_{[65]}}{D_{70}}\left[P\ddot{a}_{[65]:\overline{5}|} - 60{,}000 \times A^{1}_{[65]:\overline{5}|} - 200 - 0.01P(\ddot{a}_{[65]:\overline{5}|} - 1)\right] \qquad [1] $$

We have:

$$ \frac{D_{70}}{D_{[65]}} = \frac{517.23}{685.44} = 0.75460 \qquad [½] $$

$$ \ddot{a}_{[65]:\overline{5}|} = \ddot{a}_{[65]} - \frac{D_{70}}{D_{[65]}}\ddot{a}_{70} = 12.337 - 0.75460 \times 10.375 = 4.5081 \qquad [½] $$

$$A^{1}_{[65]:\overline{5}|} = A_{[65]} - \frac{D_{70}}{D_{[65]}} A_{70} = 0.52550 - 0.75460 \times 0.60097 = 0.07201 \qquad [½]$$

and hence:

$${}_5V^{retro} = \frac{3,071.40 \times 4.5081 - 60,000 \times 0.07201 - 200 - 0.01 \times 3,071.40 \times 3.5081}{0.75460}$$

$$= £12,215.42 \qquad [½]$$

(Note that prospective and retrospective reserves are equal since they are both calculated on the same basis as the premium. The few pence difference is a result of using rounded values from the Tables.)

(ii) ***Insurer's profit***

The reserve per policy at the end of 2011 is:

$${}_4V^{pro} = 60,000 A^{1}_{69:\overline{21}|} + 120,000 \frac{D_{90}}{D_{69}} + 0.01 P \ddot{a}_{69:\overline{21}|} - P \ddot{a}_{69:\overline{21}|}$$

We have:

$$\frac{D_{90}}{D_{69}} = \frac{48.61}{550.14} = 0.08836$$

$$A^{1}_{69:\overline{21}|} = A_{69} - \frac{D_{90}}{D_{69}} A_{90} = 0.58638 - 0.08836 \times 0.84196 = 0.51198 \qquad [½]$$

$$\ddot{a}_{69:\overline{21}|} = \ddot{a}_{69} - \frac{D_{90}}{D_{69}} \ddot{a}_{90} = 10.754 - 0.08836 \times 4.109 = 10.3909 \qquad [½]$$

and hence:

$${}_4V^{pro} = 60,000 \times 0.51198 + 120,000 \times 0.08836 - 0.99 \times 3,071.40 \times 10.3909$$

$$= £9,726.66 \qquad [½]$$

The reserves required on 1 January 2012 total:

$$197 \times 9,726.66 = £1,916,152.02 \qquad [½]$$

The premiums received on 1 January 2012 total:

$$197 \times 3,071.40 = £605,065.80$$ [½]

Expenses incurred at the start of 2012 total:

$$2 \times 0.01 \times 197 \times 3,071.40 = £12,101.32$$ [1]

Interest earned during 2012 was:

$$2 \times 0.04 \times (1,916,152.02 + 605,065.80 - 12,101.32) = £200,729.32$$ [1]

There were 9 deaths during 2012. So the reserves required on 31 December 2012 (using the prospective reserve figure of £12,215.36) total:

$$(197 - 9) \times 12,215.36 = £2,296,487.68$$ [½]

So the profit earned in 2012 was:

$$\begin{aligned} Profit_{2012} &= 1,916,152.02 + 605,065.80 - 12,101.32 + 200,729.32 \\ &\qquad - 9 \times 60,000 - 2,296,487.68 \\ &= -£126,642 \end{aligned}$$

ie a loss of approximately £127,000. [1]

[Total 6]

Solution X2.9

This question is CT5 September 2005 Question 10.

(i) ***Gross future loss random variable***

Suppose that:

- b is the level of simple bonus, expressed as a percentage of the sum assured
- I is the initial expense
- e is the renewal expense, payable at the start of each year, including the first
- f is the termination expense, payable at the time a claim is made
- P is the annual premium
- K is the curtate future lifetime of the policyholder
- T is the complete future lifetime of the policyholder

[1 for all notation defined,
½ mark if up to three items missing]

Then the gross future loss random variable at the outset is:

$$L = 100,000\left[1 + b\left(K+1\right)\right]v^T + I + e\,\ddot{a}_{\overline{K+1|}} + f\,v^T - P\,\ddot{a}_{\overline{K+1|}} \qquad [2]$$

[Total 3]

(ii) ***Annual premium***

Note that the rate of interest is 6% pa in this part of the question.

The expected present value of the premiums is:

$$P\,\ddot{a}_{[20]} = 16.877P \qquad [½]$$

The expected present value of the expenses is:

$$200 + 0.05P\left(\ddot{a}_{[20]} - 1\right) = 200 + 0.79385P \qquad [½]$$

The expected present value of the benefits is:

$$100,000\bar{A}_{[20]} + 3,000(I\bar{A})_{[20]} \qquad [½]$$

Now:

$$\bar{A}_{[20]} \approx 1.06^{½} \times A_{[20]} = 1.06^{½} \times 0.04472 = 0.04604$$ [½]

and:

$$(I\bar{A})_{[20]} \approx 1.06^{½} \times (IA)_{[20]} = 1.06^{½} \times 2.00874 = 2.06812$$ [½]

So:

$$\text{EPV benefits} = 10,808.58$$ [½]

Using the principle of equivalence we have:

$$16.877P = 10,808.58 + 200 + 0.79385P$$

$$\Rightarrow P = £684.48$$ [1]

[Total 4]

Award full marks for the correct final premium here.

(iii) ***Reserve at time 3***

Note that the rate of interest is 4% pa in this part of the question.

The reserve at time 3 is:

$${}_3V = 110,000\bar{A}_{23} + 4,000(I\bar{A})_{23} - 0.95P\ddot{a}_{23}$$ [2]

$$= 110,000 \times 1.04^{½} \times 0.12469 + 4,000 \times 1.04^{½} \times 6.09644$$
$$-0.95 \times 684.48 \times 22.758$$

$$= £24,057.70$$

or £24,058 to the nearest £1. [2]

[Total 4]

Markers : award full marks for the correct final answer here. Award method marks appropriately.

Solution X2.10

This question is CT5 April 2005 Question 14 (with the dates changed).

(i) ***Definitions***

The death strain at risk for a policy issued t years ago when the policyholder was aged x, which provides a sum assured of S payable at the end of the year of death and provides no benefit on survival to time $t+1$ is given by:

$$DSAR = S - {}_{t+1}V \qquad [1]$$

It is the amount of money, over and above the reserve at time $t+1$, that has to be paid in respect of each death during the policy year $(t, t+1)$.

The expected death strain for such a policy is:

$$EDS = q_{x+t}\left(S - {}_{t+1}V\right) \qquad [1]$$

This is the amount that the life insurance company expects to pay extra to the year-end reserve for the policy. For a group of identical policies, the expected death strain is given by:

expected number of deaths $\times$ $DSAR$ [1]

The actual death strain is:

$$ADS = \begin{cases} 0 & \text{if the policyholder survives to time } t+1 \\ S - {}_{t+1}V & \text{if the policyholder dies in the year } (t, t+1) \end{cases} \qquad [1]$$

So it is the observed value of the indicator random variable:

$$D = \begin{cases} 0 & \text{if the policyholder survives to time } t+1 \\ 1 & \text{if the policyholder dies in the year } (t, t+1) \end{cases} \qquad [1]$$

multiplied by the death strain at risk. For a group of identical policies, the actual death strain is given by:

actual number of deaths $\times$ $DSAR$ [1]

[Total 6]

(ii)(a) ***Death strain at risk for each type of policy for calendar year 2004***

The end of calendar year 2012 is time 3, when time is measured in years from the start of the policies.

Term assurance

To calculate the reserve at time 3, we first need to calculate the annual premium for the policy. If we denote this by P, then:

$$P\ddot{a}_{45:\overline{15}|} = 150{,}000 A^{1}_{45:\overline{15}|} \qquad [½]$$

From the *Tables*:

$$\ddot{a}_{45:\overline{15}|} = 11.386 \qquad [½]$$

Also:

$$A^{1}_{45:\overline{15}|} = A_{45:\overline{15}|} - \frac{D_{60}}{D_{45}} = 0.56206 - \frac{882.85}{1{,}677.97} = 0.03592 \qquad [½]$$

Alternatively, you can calculate this as:

$$A^{1}_{45:\overline{15}|} = A_{45} - \frac{D_{60}}{D_{45}} A_{60} = 0.27605 - \frac{882.85}{1{,}677.97} \times 0.45640 = 0.03592$$

So:

$$P = \frac{150{,}000 \times 0.03592}{11.386} = £473.21 \qquad [½]$$

The reserve at time 3 is:

$$\begin{aligned} {}_3V &= 150{,}000 A^{1}_{48:\overline{12}|} - 473.21\,\ddot{a}_{48:\overline{12}|} \\ &= 150{,}000\left(0.63025 - \frac{882.85}{1{,}484.43}\right) - 473.21 \times 9.613 \\ &= £777.52 \end{aligned} \qquad [1]$$

Or, you could calculate the term assurance using:

$$A^{\ 1}_{48:\overline{12|}} = A_{48} - \frac{D_{60}}{D_{48}} \times A_{60}$$

$$= 0.30695 - \frac{882.85}{1,484.43} \times 0.45640$$

$$= 0.035511$$

which leads to the same answer.

The death strain at risk for each term assurance policy is then:

$$DSAR = S - {}_3V = 150,000 - 777.52 = £149,222 \qquad [½]$$

Pure endowment

Let the annual premium for the pure endowment be P'. Then:

$$P'\ddot{a}_{45:\overline{15|}} = 75,000\frac{D_{60}}{D_{45}} \Rightarrow 11.386P' = 75,000 \times \frac{882.85}{1,677.97} \Rightarrow P' = £3,465.71 \quad [1]$$

The reserve at time 3 is:

$$_3V = 75,000 A_{48:\overline{12|}}^{\ \ \ \ 1} - 3,465.71\ \ddot{a}_{48:\overline{12|}}$$

$$= 75,000 \times \frac{882.85}{1,484.43} - 3,465.71 \times 9.613$$

$$= £11,289.63 \qquad [1]$$

There is no death benefit if the policyholder dies during calendar year 2012, so the death strain at risk for each pure endowment policy is:

$$DSAR = 0 - {}_3V = -£11,290 \qquad [½]$$

Temporary annuity

Watch out here – these policyholders are aged 55 at entry and have PMA92C20 mortality.

The reserve at time 3 for the temporary annuity is:

$$
\begin{aligned}
{}_3V &= 25,000\, a_{58:\overline{2}|} \\
&= 25,000\left(v\, p_{58} + v^2\ {}_2p_{58}\right) \\
&= 25,000\left(\frac{1-0.001814}{1.04} + \frac{(1-0.001814)(1-0.002110)}{1.04^2}\right) \\
&= £47,018.15
\end{aligned}
$$
[1]

There is no death benefit for this policy. However, if the policyholder survives to time 3, there is a survival benefit of £25,000, which is not included in the reserve at time 3. [½]

So the death strain at risk for each temporary annuity is:

$$DSAR = 0 - \left({}_3V + 25,000\right) = -£72,018$$ [½]

This calculation is quite sensitive to rounding and to the method of calculation used. For example, if you had calculated the annuity as:

$$a_{58:\overline{2}|} = a_{58} - v^2\ {}_2p_{58}\, a_{60} = 15.356 - 1.04^{-2} \times \frac{9,826.131}{9,864.803} \times 14.632 = 1.881$$

then you would get:

$${}_3V = 47,023.16 \quad \textit{and} \quad DSAR = -72,023$$

(ii)(b) ***Total mortality profit or loss***

Term assurance policies

There are $5,000-15=4,985$ term assurance policies in force on 1 January 2012.

The expected death strain for this group of policies is:

$$EDS = 4,985\,q_{47} \times 149,222 = 4,985 \times 0.001802 \times 149,222 = £1,340,457$$ [½]

The actual death strain for this group of policies is:

$$ADS = 8 \times 149,222 = £1,193,776$$ [½]

So the mortality profit from this group of policies is:

$$MP = EDS - ADS = £146,681$$ [½]

Pure endowment policies

There are $2,000-5=1,995$ pure endowment policies in force on 1 January 2012.

The expected death strain for this group of policies is:

$$EDS = 1,995\,q_{47} \times (-11,290) = 1,995 \times 0.001802 \times (-11,290) = -£40,587$$ [½]

The actual death strain for this group of policies is:

$$ADS = 1 \times (-11,290) = -£11,290$$ [½]

So the mortality profit from this group of policies is:

$$MP = EDS - ADS = -£29,297$$ [½]

Temporary annuity policies

There are $1,000-5=995$ temporary annuity policies in force on 1 January 2012.

The expected death strain for this group of policies is:

$$EDS = 995\,q_{57} \times (-72,018) = 995 \times 0.001558 \times (-72,018) = -£111,643$$ [½]

The actual death strain for this group of policies is:

$$ADS = 1 \times (-72,018) = -£72,018$$ [½]

So the mortality profit from this group of policies is:

$$MP = EDS - ADS = -£39,625$$ [½]

Total mortality profit

The total mortality profit is then:

$$146,681 - 29,297 - 39,625 = £77,759$$ [½]

[Total 13]

Assignment X3 – Solutions

Markers: This document does not necessarily give every possible approach to solving each of the questions. Please give credit for other valid approaches.

Solution X3.1

This question is CT5 September 2005 Question 6.

The symbol $\ddot{a}^{(12)}_{60:50:\overline{20|}}$ represents the expected present value of an annuity of 1 *pa* payable monthly in advance while a life aged exactly 60 and a life aged exactly 50 are both alive, with payments being made for a maximum of 20 years from the outset. [2]

Using PA92C20 mortality and 4% *pa* interest:

$$\ddot{a}^{(12)}_{60:50:\overline{20|}} = \ddot{a}^{(12)}_{60:50} - v^{20}\ {}_{20}p^{(m)}_{60}\ {}_{20}p^{(f)}_{50}\ \ddot{a}^{(12)}_{80:70} \quad [1]$$

$$= \left(\ddot{a}_{60:50} - \frac{11}{24}\right) - v^{20}\ {}_{20}p^{(m)}_{60}\ {}_{20}p^{(f)}_{50}\left(\ddot{a}_{80:70} - \frac{11}{24}\right) \quad [1]$$

$$= \left(15.161 - \frac{11}{24}\right) - 1.04^{-20} \times \frac{6,953.536}{9,826.131} \times \frac{9,392.621}{9,952.697} \times \left(6.876 - \frac{11}{24}\right)$$

$$= 12.747 \quad [1]$$

[Total 5]

Solution X3.2

This question is Subject CT5, April 2005, Question 8.

Define:

$j =$ the valuation rate of interest [¼]

$v = \dfrac{1}{1+j}$ [¼]

$i_x =$ the number of ill-health retirements between x and $x+1$, $x \leq 64$ [¼]

$l_x =$ the number of active lives at age x exact [¼]

Both i_x and l_x must come from a suitable service table. [¼]

$a_x^i =$ the expected present value at age x of a pension of 1 *pa* payable on ill-health retirement at age x, and payable in accordance with the scheme rules [¼]

$\{s_x\}$ is a salary scale such that:

$$\frac{s_{x+t}}{s_x} = \frac{\text{expected salary earned in year of age } (x+t, x+t+1)}{\text{expected salary earned in year of age } (x, x+1)}$$ [¼]

$$z_x = \frac{s_{x-1} + s_{x-2} + s_{x-3}}{3}$$ [¼]

Assume that ill-health retirements occur uniformly over each year of age and part years of service count proportionately. [¼]

Past service benefit

The member has 10 years of past service, so is already entitled to 10/80ths of final pensionable salary when he retires. If he retires in the year of age $(y, y+1)$, we are assuming it occurs at age $y+½$, so his FPS will be:

$$20{,}000\frac{z_{y+½}}{s_{34}} \qquad [¼]$$

Note that we have s_{34} *in the denominator since he earned £20,000 between age 34 and 35.*

The expected present value of the past service benefit is:

$$\frac{10}{80}\times 20{,}000\left[\frac{i_{35}}{l_{35}}v^{½}\frac{z_{35½}}{s_{34}}a^i_{35½}+\frac{i_{36}}{l_{35}}v^{1½}\frac{z_{36½}}{s_{34}}a^i_{36½}+\cdots+\frac{i_{64}}{l_{35}}v^{29½}\frac{z_{64½}}{s_{34}}a^i_{64½}\right]$$

$$=\frac{10}{80}\times 20{,}000\left[\frac{i_{35}\,v^{35½}\,z_{35½}\,a^i_{35½}+\cdots+i_{64}\,v^{64½}\,z_{64½}\,a^i_{64½}}{s_{34}v^{35}l_{35}}\right] \qquad [¼]$$

Note that we will deal with the guarantee in the future service benefit.

Now define:

$$D_x = v^x\, l_x \qquad [¼]$$

$${}^zC^{ia}_x = i_x\, v^{x+½}\, z_{x+½}\, a^i_{x+½} \qquad [¼]$$

and: $${}^zM^{ia}_x = {}^zC^{ia}_x + {}^zC^{ia}_{x+1} + \cdots + {}^zC^{ia}_{64} \qquad [¼]$$

Then the expected present value of the past service benefit is:

$$\frac{10}{80}\times 20{,}000\left(\frac{{}^zC^{ia}_{35}+{}^zC^{ia}_{36}+\cdots+{}^zC^{ia}_{64}}{s_{34}\,D_{35}}\right)=\frac{10}{80}\times 20{,}000\times\frac{{}^zM^{ia}_{35}}{s_{34}\,D_{35}} \qquad [¼]$$

Future service benefit – years of service from age 35 to 45

On retirement in ill-health, whenever that occurs, the member will receive the minimum pension of 20/80ths of FPS. 10/80ths have already been accounted for in the past service benefit, so the 10 years of service from age 35 to 45 are covered by the remaining 10/80ths. [¼]

If we follow the same method as for the past service liability, we get:

$$\frac{10}{80}\times 20{,}000\left[\frac{i_{35}}{l_{35}}v^{½}\frac{z_{35½}}{s_{34}}a^{i}_{35½}+\frac{i_{36}}{l_{35}}v^{1½}\frac{z_{36½}}{s_{34}}a^{i}_{36½}+\cdots+\frac{i_{64}}{l_{35}}v^{29½}\frac{z_{64½}}{s_{34}}a^{i}_{64½}\right]$$

$$=\frac{10}{80}\times 20{,}000\left[\frac{i_{35}\,v^{35½}\,z_{35½}\,a^{i}_{35½}+\cdots+i_{64}\,v^{64½}\,z_{64½}\,a^{i}_{64½}}{s_{34}v^{35}l_{35}}\right]$$

$$=\frac{10}{80}\times 20{,}000\left(\frac{{}^{z}C^{ia}_{35}+{}^{z}C^{ia}_{36}+\cdots+{}^{z}C^{ia}_{64}}{s_{34}\,D_{35}}\right)=\frac{10}{80}\times 20{,}000\times\frac{{}^{z}M^{ia}_{35}}{s_{34}\,D_{35}} \qquad [¼]$$

If the guaranteed 20 years of service is valued altogether (rather than as past/future) then the expected present value of the guarantee is the sum of the past service expression and the future service expression from age 35 to 45:

$$\frac{20}{80}\times 20{,}000\left(\frac{{}^{z}C^{ia}_{35}+{}^{z}C^{ia}_{36}+\cdots+{}^{z}C^{ia}_{64}}{s_{34}\,D_{35}}\right)=\frac{20}{80}\times 20{,}000\times\frac{{}^{z}M^{ia}_{35}}{s_{34}\,D_{35}}$$

Future service benefit – years of service from age 45

Once age 45 is reached, the minimum service of 20 years is attained and any additional service beyond this age will give extra years of service for the member's ill health pension. [¼]

Consider the year of service from age 45 to age 46. If we follow the same method as for the past service liability, the expected present value of the benefit in respect of the year of future service from age 45 to age 46 is:

$$\frac{1}{80}\times 20{,}000\left[0.5\frac{i_{45}}{l_{35}}v^{10½}\frac{z_{45½}}{s_{34}}a^{i}_{45½}+\frac{i_{46}}{l_{35}}v^{11½}\frac{z_{46½}}{s_{34}}a^{i}_{46½}+\cdots+\frac{i_{64}}{l_{35}}v^{29½}\frac{z_{64½}}{s_{34}}a^{i}_{64½}\right]$$

$$=\frac{1}{80}\times 20{,}000\left[\frac{0.5i_{45}\,v^{45½}\,z_{45½}\,a^{i}_{45½}+\cdots+i_{64}\,v^{64½}\,z_{64½}\,a^{i}_{64½}}{s_{34}\,v^{35}\,l_{35}}\right] \qquad [¼]$$

Notice the main difference here; that the first term has 0.5 as the coefficient instead of 1 because, remember, if the member retires between 45 and 46, we assume it will occur halfway through the year.

Here, we define:

$${}^{z}\overline{M}_x^{ia} = {}^{z}M_x^{ia} - \tfrac{1}{2}\,{}^{z}C_x^{ia}$$ [¼]

So, the expected present value of the benefit in respect of the year of future service from age 45 to age 46 is:

$$\frac{1}{80} \times 20{,}000 \left[\frac{0.5\,{}^{z}C_{45}^{ia} + {}^{z}C_{46}^{ia} + \cdots + {}^{z}C_{64}^{ia}}{s_{34}\,D_{35}} \right]$$

$$= \frac{1}{80} \times 20{,}000 \frac{{}^{z}\overline{M}_{45}^{ia}}{s_{34}\,D_{35}}$$ [¼]

Similarly, for other ages x between 46 and 64, the expected present value of the benefit in respect of the year of future service from age x to age $x + 1$ is:

$$\frac{1}{80} \times 20{,}000 \frac{{}^{z}\overline{M}_{x}^{ia}}{s_{34}\,D_{35}}$$ [¼]

Here, we define:

$${}^{z}\overline{R}_x^{ia} = {}^{z}\overline{M}_x^{ia} + {}^{z}\overline{M}_{x+1}^{ia} + \cdots + {}^{z}\overline{M}_{64}^{ia}$$ [¼]

So, the expected present value of the benefit in respect of all the years of future service from age 45 is:

$$\frac{1}{80} \times 20{,}000 \left(\frac{{}^{z}\overline{M}_{45}^{ia} + {}^{z}\overline{M}_{46}^{ia} + \cdots + {}^{z}\overline{M}_{64}^{ia}}{s_{34}\,D_{35}} \right)$$

$$= \frac{1}{80} \times 20{,}000 \frac{{}^{z}\overline{R}_{45}^{ia}}{s_{34}\,D_{35}}$$ [¼]

Putting these two elements together, the expected present value of the future service benefit is:

$$\frac{1}{80}\times 20{,}000\left(\frac{10\,{}^{z}M_{35}^{ia}+{}^{z}\bar{R}_{45}^{ia}}{s_{34}\,D_{35}}\right) \qquad [\tfrac{1}{4}]$$

Combining this with the past service benefit gives a total expected present value of:

$$\frac{1}{80}\times 20{,}000\left(\frac{20\,{}^{z}M_{35}^{ia}+{}^{z}\bar{R}_{45}^{ia}}{s_{34}\,D_{35}}\right) \qquad [\tfrac{1}{4}]$$

[Maximum 5]

Alternative solution for the future service liability

Some people may prefer to view the future service liability as one component rather than the breakdown we have given above into that before age 45 and that after age 45.

If the member retires through ill health before age 45, he will receive the minimum pension of 20/80ths of FPS. 10/80ths have been accounted for in the past service benefit, so the remaining 10/80ths will form part of the future service benefit. *[¼]*

If the member retires between 45 and 46, we assume it will occur halfway through the year, so his future service pension would be 10½/80ths of FPS. Similarly, if he retires between 46 and 47, his future pension would be 11½/80ths of FPS, and so on. *[¼]*

So the future service benefit is:

$$\begin{aligned}\frac{1}{80}\times 20{,}000\Bigg[&10\frac{i_{35}}{l_{35}}v^{\frac{1}{2}}\frac{z_{35\frac{1}{2}}}{s_{34}}a^{i}_{35\frac{1}{2}}+10\frac{i_{36}}{l_{35}}v^{1\frac{1}{2}}\frac{z_{36\frac{1}{2}}}{s_{34}}a^{i}_{36\frac{1}{2}}+\cdots\\ &+10\frac{i_{44}}{l_{35}}v^{9\frac{1}{2}}\frac{z_{44\frac{1}{2}}}{s_{34}}a^{i}_{44\frac{1}{2}}+10\tfrac{1}{2}\frac{i_{45}}{l_{35}}v^{10\frac{1}{2}}\frac{z_{45\frac{1}{2}}}{s_{34}}a^{i}_{45\frac{1}{2}}\\ &+11\tfrac{1}{2}\frac{i_{46}}{l_{35}}v^{11\frac{1}{2}}\frac{z_{46\frac{1}{2}}}{s_{34}}a^{i}_{46\frac{1}{2}}+\cdots+29\tfrac{1}{2}\frac{i_{64}}{l_{35}}v^{29\frac{1}{2}}\frac{z_{64\frac{1}{2}}}{s_{34}}a^{i}_{64\frac{1}{2}}\Bigg]\end{aligned} \qquad [\tfrac{1}{2}]$$

This can be written as:

$$= \frac{1}{80} \times 20{,}000 \left[10 \left(\frac{i_{35}\, v^{35½}\, z_{35½}\, a^{i}_{35½} + \cdots + i_{44}\, v^{44½}\, z_{44½}\, a^{i}_{44½}}{s_{34}\, D_{35}} \right) \right.$$

$$+ \frac{10½\; i_{45}\, v^{45½}\, z_{45½}\, a^{i}_{45½}}{s_{34}\, D_{35}} + \frac{11½\; i_{46}\, v^{46½}\, z_{46½}\, a^{i}_{46½}}{s_{34}\, D_{35}} + \cdots$$

$$\left. + \frac{29½\; i_{64}\, v^{64½}\, z_{64½}\, a^{i}_{64½}}{s_{34}\, D_{35}} \right] \qquad [¼]$$

$$= \frac{1}{80} \times 20{,}000 \left[\frac{10\left({}^{z}C^{ia}_{35} + \cdots + {}^{z}C^{ia}_{44} \right) + 10½\; {}^{z}C^{ia}_{45} + 11½\; {}^{z}C^{ia}_{46} + \cdots + 29½\; {}^{z}C^{ia}_{64}}{s_{34}\, D_{35}} \right]$$

[¼]

The numerator in the bracketed term above is:

$$10\left({}^{z}C^{ia}_{35} + \cdots + {}^{z}C^{ia}_{44} \right) + 10½\; {}^{z}C^{ia}_{45} + 11½\; {}^{z}C^{ia}_{46} + \cdots + 29½\; {}^{z}C^{ia}_{64}$$

$$= 10\left({}^{z}C^{ia}_{35} + \cdots + {}^{z}C^{ia}_{64} \right) + ½\, {}^{z}C^{ia}_{45} + 1½\; {}^{z}C^{ia}_{46} + \cdots + 19½\; {}^{z}C^{ia}_{64}$$

$$= 10\, {}^{z}M^{ia}_{35} + \left(½\, {}^{z}C^{ia}_{45} + {}^{z}C^{ia}_{46} + \cdots + {}^{z}C^{ia}_{64} \right)$$

$$+ \left(½\, {}^{z}C^{ia}_{46} + {}^{z}C^{ia}_{47} + \cdots + {}^{z}C^{ia}_{64} \right)$$

$$+ \cdots$$

$$+ ½\, {}^{z}C^{ia}_{64}$$

$$= 10\, {}^{z}M^{ia}_{35} + \left({}^{z}M^{ia}_{45} - ½\, {}^{z}C^{ia}_{45} \right) + \left({}^{z}M^{ia}_{46} - ½\, {}^{z}C^{ia}_{46} \right) + \cdots + \left({}^{z}M^{ia}_{64} - ½\, {}^{z}C^{ia}_{64} \right) \qquad [½]$$

Defining ${}^{z}\overline{M}^{ia}_{x}$ *and* ${}^{z}\overline{R}^{ia}_{x}$ *in the same way as before ...* *[¼]*

... the above expression is:

$$10\, {}^{z}M^{ia}_{35} + {}^{z}\overline{M}^{ia}_{45} + {}^{z}\overline{M}^{ia}_{46} + \cdots + {}^{z}\overline{M}^{ia}_{64} = 10\, {}^{z}M^{ia}_{35} + {}^{z}\overline{R}^{ia}_{45} \qquad [¼]$$

This gives the same answer as before for the expected present value of the future service benefit, ie:

$$\frac{1}{80}\times 20{,}000\left(\frac{10\,{}^{z}M_{35}^{ia}+{}^{z}\overline{R}\,_{45}^{ia}}{s_{34}\,D_{35}}\right) \qquad [\tfrac{1}{4}]$$

Solution X3.3

This question is CT5 April 2005 Question 5, slightly adapted.

The probability of a life aged x, who is currently sick, staying in the sick state for at least t years is given by:

$${}_{t}p_{40}^{\overline{SS}}=\exp\left(-\int_0^t\left(\rho_{x+s}+\nu_{x+s}\right)ds\right).$$

Since the transition intensities are assumed to be constant, the expression simplifies to:

$${}_{t}p_{40}^{\overline{SS}}=e^{-t(\rho+\nu)}. \qquad [1]$$

The expected present value of the sickness benefit is then:

$$2{,}000\int_0^{20}e^{-\delta t}\ {}_{t}p_{40}^{\overline{SS}}\,dt=2{,}000\int_0^{20}e^{-(\delta+\rho+\nu)t}\ dt \qquad [1]$$

$$=\left[-\frac{2{,}000}{\delta+\rho+\nu}e^{-(\delta+\rho+\nu)t}\right]_0^{20} \qquad [1]$$

$$=\frac{2{,}000}{\ln 1.04+0.05}\left[1-e^{-20(\ln 1.04+0.05)}\right]$$

$$=£18{,}652.72 \qquad [1]$$

[Total 4]

Solution X3.4

This question is CT5 April 2005 Question 7.

The present value random variable for this annuity is:

$$\bar{a}_{\overline{T_{xy}|}} = \bar{a}_{\overline{\min\{T_x, T_y\}|}} \qquad [½]$$

The expected present value is:

$$E\left(\bar{a}_{\overline{T_{xy}|}}\right) = \bar{a}_{xy} \qquad [1]$$

Alternatively, we have:

$$\bar{a}_{\overline{T_{xy}|}} = \frac{1 - v^{T_{xy}}}{\delta} \Rightarrow E\left(\bar{a}_{\overline{T_{xy}|}}\right) = \frac{1 - \bar{A}_{xy}}{\delta}$$

The variance of the present value random variable is:

$$\text{var}\left(\bar{a}_{\overline{T_{xy}|}}\right) = \text{var}\left(\frac{1 - v^{T_{xy}}}{\delta}\right) = \frac{1}{\delta^2}\text{var}\left(v^{T_{xy}}\right) = \frac{1}{\delta^2}\left[E\left(v^{2T_{xy}}\right) - \left[E\left(v^{T_{xy}}\right)^2\right]\right] \qquad [1½]$$

Now:

$$E\left(v^{T_{xy}}\right) = \bar{A}_{xy} \qquad [½]$$

and:

$$E\left(v^{2T_{xy}}\right) = {}^2\bar{A}_{xy} \qquad [1]$$

where the superscript of 2 to the left of the assurance symbol indicates that the assurance is evaluated using twice the standard force of interest, which is equivalent to evaluating using the rate of interest $i' = (1+i)^2 - 1$. So the variance of the present value random variable is:

$$\text{var}\left(\bar{a}_{\overline{T_{xy}|}}\right) = \frac{1}{\delta^2}\left[{}^2\bar{A}_{xy} - \left(\bar{A}_{xy}\right)^2\right] \qquad [½]$$

[Total 5]

Solution X3.5

(i) ***Multiple decrement table***

First we need to calculate the dependent probabilities of decrement, using the implied underlying forces.

The (assumed constant) force of mortality for the year of age beginning at age $62+t$ is obtained from:

$$\bar{\mu}^d_{62+t} = -\ln\left(p^d_{[62]+t}\right) = -\ln\left(1-q^d_{[62]+t}\right)$$ [½]

for $t=0,1,2$, where $q^d_{[62]+t}$ is the mortality probability from the AM92 Select table.

This gives the following values:

Age x	Duration t	$q^d_{[62]+t}$	$\bar{\mu}^d_x$
62	0	0.007164	0.0071898
63	1	0.010815	0.0108739
64	2	0.012716	0.0127975

[1]

We now construct the dependent probabilities using:

$$(aq)^j_x \approx \frac{\bar{\mu}^j_x}{\bar{\mu}^d_x + \bar{\mu}^s_x}\left[1-e^{-(\bar{\mu}^d_x+\bar{\mu}^s_x)}\right]$$ [1]

for $j=d,s$ and $x=62, 63$ and 64, where *d* represents death and *s* represents surrender. This gives:

Age x	$(aq)^d_x$	$(aq)^s_x$
62	0.0069881	0.0485971
63	0.0106812	0.0245569
64	0.0126528	0.0098869

[1½]

The multiple decrement table is then constructed recursively using:

$$(ad)_x^j = (al)_x\,(aq)_x^j \qquad\qquad j = d, s$$

$$(al)_{x+1} = (al)_x - (ad)_x^d - (ad)_x^s$$

$$(al)_{62} = 100,000$$

The table is then:

x	$(al)_x$	$(ad)_x^d$	$(ad)_x^s$
62	100,000	698.81	4,859.71
63	94,441.48	1,008.75	2,319.19
64	91,113.55	1,152.84	900.83
65	89,059.88		

[1]
[Total 5]

(ii) ***Premium***

Assuming deaths occur on average half way through each year, the expected present value (EPV) of the death benefit is:

$$EPV[D] = \frac{50,000}{(al)_{62}}\left[v^{½}(ad)_{62}^d + v^{1½}(ad)_{63}^d + v^{2½}(ad)_{64}^d\right]$$

$$= \frac{50,000}{100,000}\left[\frac{698.81}{1.03^{½}} + \frac{1,008.75}{1.03^{1½}} + \frac{1,152.84}{1.03^{2½}}\right]$$

$$= 1,362.13$$ [2]

The EPV of the surrender and survival benefit is:

$$EPV[S] = \frac{0.5P}{(al)_{62}}\left[v^{½}(ad)_{62}^s + 2v^{1½}(ad)_{63}^s + 3v^{2½}(ad)_{64}^s + 3v^3(al)_{65}\right]$$

$$= \frac{0.5P}{100,000}\left[\frac{4,859.71}{1.03^{½}} + \frac{2\times 2,319.19}{1.03^{1½}} + \frac{3\times 900.83}{1.03^{2½}} + \frac{3\times 89,059.88}{1.03^3}\right]$$

$$= 1.28121P$$ [2]

The EPV of the premiums is:

$$EPV[P] = P\left[\frac{(al)_{62}}{(al)_{62}} + v\frac{(al)_{63}}{(al)_{62}} + v^2\frac{(al)_{64}}{(al)_{62}}\right]$$

$$= P\left[1 + \frac{0.9444148}{1.03} + \frac{0.9111355}{1.03^2}\right]$$

$$= 2.77574P \qquad [1]$$

The equation of value is therefore:

$$2.77574P = 1{,}362.13 + 1.28121P$$

$$\Rightarrow P = \frac{1{,}362.13}{2.77574 - 1.28121} = 911.41 \qquad [1]$$

[Total 6]

Alternatively, we could calculate the EPVs using the dependent probabilities directly, ie using:

$$EPV[D] = 50{,}000\left[v^{1/2}(aq)^d_{62} + v^{1\frac{1}{2}}(ap)_{62}(aq)^d_{63} + v^{2\frac{1}{2}}\,{}_2(ap)_{62}(aq)^d_{64}\right]$$

$$EPV[S] = 0.5P\left[v^{1/2}(aq)^s_{62} + 2v^{1\frac{1}{2}}(ap)_{62}(aq)^s_{63} + 3v^{2\frac{1}{2}}\,{}_2(ap)_{62}(aq)^s_{64} + 3v^3\,{}_3(ap)_{62}\right]$$

$$EPV[P] = P\left[1 + v\,{}_1(ap)_{62} + v^2\,{}_2(ap)_{62}\right]$$

Solution X3.6

(i) ***Premium annuity factor***

If (x) dies within the 20 years, and (y) is already dead by that point, then the benefit will be paid at the moment of (x)'s death and the contract will terminate, so premiums would cease at that point. [½]

If (x) dies within the 20 years, and (y) is still alive at that point, then the sum assured becomes guaranteed to be payable at time 20. This payment would be made whether or not any more premiums were paid, so it is appropriate to assume that premiums would cease at the moment of (x)'s death. [½]

If (x) is still alive at time 20, regardless of what has happened to (y), the contract ceases without any benefit being paid out, so premiums would also cease at this point. [½]

Taken together, these mean that the premiums would be paid for 20 years, or until the death of (x), if earlier, irrespective of what happens to (y). [½]

So, given that premiums are payable continuously, the appropriate annuity factor would be:

$$\bar{a}_{x:\overline{20}|} \qquad [1]$$

[Total 3]

(ii) ***Calculating the annual rate of premium***

The assurance is payable immediately on (x)'s death if (x) dies after (y) and within the 20 years. The expected present value of this is:

$$20{,}000\bar{A}^{\;2}_{x:y:\overline{20}|} = 20{,}000\bar{A}^{1}_{x:\overline{20}|} - 20{,}000\bar{A}^{1}_{x:y:\overline{20}|} \qquad [1]$$

Now:

$$\begin{aligned}
\bar{A}^{1}_{x:\overline{20}|} &= \int_0^{20} e^{-\delta t}\ {}_tp_x\ \mu_{x+t}\,dt \\
&= \int_0^{20} e^{-0.05t}\ e^{-0.005t}\ 0.005\,dt \\
&= \frac{0.005}{-0.055}\left[e^{-0.055\times 20} - e^{0}\right] \\
&= \frac{0.005}{0.055}\left[1-e^{-1.1}\right] = 0.060648 \qquad [1½]
\end{aligned}$$

$$\begin{aligned}
\bar{A}^{1}_{x:y:\overline{20}|} &= \int_0^{20} e^{-\delta t}\ {}_tp_{x:y}\ \mu_{x+t}\,dt \\
&= \int_0^{20} e^{-0.05t}\ e^{-0.01t}\ 0.005\,dt \\
&= \frac{0.005}{-0.06}\left[e^{-0.06\times 20} - e^{0}\right] \\
&= \frac{0.005}{0.06}\left[1-e^{-1.2}\right] = 0.058234 \qquad [1½]
\end{aligned}$$

The expected present value of this benefit is therefore:

$$20,000 \times (0.060648 - 0.058234) = £48.28$$ [½]

In addition, the assurance is payable if (x) dies while (y) is alive, in which case the payment is made at the end of the twenty year period. The value of this is:

$$20,000\, e^{-20\delta}\ {}_{20}q^{1}_{x:y}$$ [1]

The probability term is:

$$\begin{aligned} {}_{20}q^{1}_{x:y} &= \int_0^{20} {}_tp_{x:y}\, \mu_{x+t}\, dt \\ &= 0.005\int_0^{20} e^{-0.01t}\, dt \\ &= \frac{0.005}{0.01}\left(1-e^{-0.2}\right) \\ &= 0.090635 \end{aligned}$$

This gives a value of $20,000 \times 0.367879 \times 0.090635 = 666.86$. [1]

So the total expected present value of the benefit is £715.14. [½]

If P is the annual rate of premium, the equation of value is:

$$P\,\bar{a}_{x:\overline{20}|} = 715.14$$ [½]

where:

$$\begin{aligned} \bar{a}_{x:\overline{20}|} &= \int_0^{20} e^{-\delta t}\ {}_tp_x\, dt \\ &= \int_0^{20} e^{-0.05t}\, e^{-0.005t}\, dt \\ &= \frac{1}{-0.055}\left[e^{-0.055\times 20} - e^{0}\right] \\ &= \frac{1}{0.055}\left[1-e^{-1.1}\right] = 12.1296 \end{aligned}$$ [1]

So:

$$P = \frac{715.14}{12.1296} = 58.96$$ [½]

[Total 9]

(iii) ***Prospective reserve at time 4***

(a) *Only (x) alive*

The benefit will be paid immediately on the death of (x+4) if (x+4) dies within the next 16 years; premiums would cease in 16 years or on the earlier death of (x+4). So the prospective reserve would be:

$${}_4V^{(a)} = 20{,}000\,\bar{A}^{\;1}_{x+4:\overline{16}|} - 58.96\,\bar{a}_{x+4:\overline{16}|}$$ [1]

where:

$$\begin{aligned}\bar{A}^{\;1}_{x+4:\overline{16}|} &= \int_0^{16} e^{-\delta t}\ {}_tp_{x+4}\,\mu_{x+4+t}\,dt \\ &= \int_0^{16} e^{-0.05t}\,e^{-0.005t}\,0.005\,dt \\ &= \frac{0.005}{-0.055}\left[e^{-0.055\times 16} - e^{0}\right] \\ &= \frac{0.005}{0.055}\left[1 - e^{-0.88}\right] = 0.053202\end{aligned}$$ [1]

$$\begin{aligned}\bar{a}_{x+4:\overline{16}|} &= \int_0^{16} e^{-\delta t}\ {}_tp_{x+4}\,dt \\ &= \int_0^{16} e^{-0.05t}\,e^{-0.005t}\,dt \\ &= \frac{1}{-0.055}\left[e^{-0.055\times 16} - e^{0}\right] \\ &= \frac{1}{0.055}\left[1 - e^{-0.88}\right] = 10.640311\end{aligned}$$ [1]

Hence the reserve is:

$${}_4V^{(a)} = 20{,}000 \times 0.053202 - 58.96 \times 10.640311 = 436.70$$ [½]

(b) *Only (y) alive*

This means that (x) must have died some time in the past four years with (y) being alive at the time of (x)'s death. This means that the benefit will definitely be paid in 16 years time, without any further premiums being payable. The reserve at time 4 is therefore:

$$ {}_4V^{(b)} = 20,000\, e^{-16\delta} = 20,000\, e^{-16\times 0.05} = 8,986.58 \qquad [1]$$

(c) *Both dead*

As the benefit has not yet been paid out, then (x) must have died before (y) in the last four years. This is the same situation as (b), and so the reserve value is the same, *ie*:

$$ {}_4V^{(c)} = 8,986.58 \qquad [½]$$

[Total 5]

(iv) ***Comment***

The reserve is much larger for (b) and (c) than for (a) because in (b) and (c) the benefit is certain to be paid, whereas in (a) it is not certain to be paid. [1]

Solution X3.7

(i) $$50,000\int_0^{10} e^{-\delta t}\left({}_t p_{50}^{aa}\, \mu_{50+t} + {}_t p_{50}^{ai}\, \nu_{50+t}\right) dt \qquad [2]$$

This can be reasoned as follows. Suppose that the life dies at age $50+t$. This can be from either the able state or the ill state – the probability density functions (PDFs) of these are in the brackets. A benefit of £50,000 is then paid at time t, and this is discounted back to time 0. Integrating over all possible times t gives the required expression.

(ii) $$50,000\int_0^{9} e^{-\delta t}\, {}_t p_{50}^{aa}\, \sigma_{50+t} \left(\int_1^{10-t} e^{-\delta s}\, {}_s p_{50+t}^{\overline{ii}}\, \nu_{50+t+s}\, ds \right) dt \qquad [3]$$

This time, suppose that the life gets sick at time t. The PDF for this is ${}_t p_{50}^{aa}\, \sigma_{50+t}$. The life could get sick at any time, but if this happens after time 9, it will not lead to any benefit. So we integrate t between the limits of 0 and 9.

He has to stay sick for a year before any benefit is paid. If he remains sick for s (>1) years, and dies from the sick state at age $50+t+s$, then the benefit is paid at time $t+s$ and must be discounted back to time 0. The PDF of this happening is ${}_s p^{\overline{ii}}_{50+t}\, \upsilon_{50+t+s}$. Note that s must be at least 1 for any benefit to be paid, but the policy term is 10 years. However, given that the life falls sick at time t, the duration of sickness required for the payment of the benefit is between 1 and $10-t$. So we integrate s between these limits.

(iii) $5,000\int_0^{10} e^{-\delta t}\, {}_t p^{ai}_{50}\, dt$ [2]

If the life is sick at time t, which has probability ${}_t p^{ai}_{50}$, then he will receive benefit at the rate of £5,000 pa. This is multiplied by the discount factor $e^{-\delta t}$ to give the present value. Then integrate over all points in time t where a benefit could be paid.

[Total 7]

Solution X3.8

This question is CT5 September 2005 Question 8.

The annuity is payable monthly and is guaranteed for 5 years. It is then paid throughout the lifetime of the male and continues to be paid to the female, albeit at half the original annual amount, following the death of the male. However, the date of commencement of the payments to the female depends on when the male dies. If he dies before time 5, the payments to the female start at time 5, ie they just follow on from the guaranteed part. If he dies after time 5, the payments to the female start on the monthly payment date following his death.

EPV of the guaranteed annuity

The expected present value of the guaranteed annuity benefit is:

$$20,000\,\ddot{a}^{(12)}_{\overline{5}|} = 20,000 \times \frac{i}{d^{(12)}} \times a_{\overline{5}|} = 20,000 \times 1.021537 \times 4.4518$$

$$= 90,953.57 \qquad [1½]$$

EPV of the contingent benefit payable to the male (ie after the guarantee expires)

This is:

$$20{,}000v^5 \; {}_5p_{65}^{(m)} \; \ddot{a}_{70}^{(12)} = 20{,}000 \times 1.04^{-5} \times \frac{9{,}238.134}{9{,}647.797} \times \left(11.562 - \frac{11}{24}\right)$$

$$= 174{,}777.62 \qquad [1½]$$

EPV of the annuity payable to the female following death of the male, provided both are still alive at time 5

This is:

$$10{,}000v^5 \; {}_5p_{65}^{(m)} \; {}_5p_{62}^{(f)} \; \ddot{a}_{70(m)|67(f)}^{(12)} \qquad [1]$$

Now:

$$\ddot{a}_{70(m)|67(f)}^{(12)} = \ddot{a}_{67(f)}^{(12)} - \ddot{a}_{70(m):67(f)}^{(12)}$$

$$= \ddot{a}_{67(f)} - \frac{11}{24} - \left(\ddot{a}_{70(m):67(f)} - \frac{11}{24}\right)$$

$$= \ddot{a}_{67(f)} - \ddot{a}_{70(m):67(f)}$$

$$= 14.111 - 10.233$$

$$= 3.878 \qquad [1½]$$

So:

$$10{,}000v^5 \; {}_5p_{65}^{(m)} \; {}_5p_{62}^{(f)} \; \ddot{a}_{70(m)|67(f)}^{(12)}$$

$$= 10{,}000 \times 1.04^{-5} \times \frac{9{,}238.134}{9{,}647.797} \times \frac{9{,}605.483}{9{,}804.173} \times 3.878$$

$$= 29{,}902.36 \qquad [1½]$$

EPV of the annuity payable to female from time 5, provided she is alive and the male is dead

This is:

$$10{,}000v^5 \; {}_5q_{65}^{(m)} \; {}_5p_{62}^{(f)} \; \ddot{a}_{67}^{(12)}$$
$$=10{,}000\times 1.04^{-5}\times\left(1-\frac{9{,}238.134}{9{,}647.797}\right)\times\frac{9{,}605.483}{9{,}804.173}\times\left(14.111-\frac{11}{24}\right)$$
$$=4{,}668.29$$ [1½]

Total EPV

Summing all the parts above, we get the total expected present value to be:

$$90{,}953.57+174{,}777.62+29{,}902.36+4{,}668.29=£300{,}302$$ [1½]
[Total 10]

The final answer is quite sensitive to rounding.

Solution X3.9

The formula for the value of the future service benefits is:

$$\frac{75{,}000}{60}\frac{{}^z\bar{R}_{35}^{ra}}{s_{34}D_{35}}-\frac{5\times 2{,}000}{60}\frac{\bar{R}_{35}^{ra}}{D_{35}}$$ [2]

The benefit formula is based on the same definition of final average pay as is used in the *Tables*.

From the *Tables*:

$$s_{34}=6.389$$
$$D_{35}=4{,}781$$
$${}^z\bar{R}_{35}^{ra}=3{,}524{,}390$$
$$\bar{R}_{35}^{ra}=327{,}244$$

So the value of the future service benefits is:

$$\frac{75,000}{60}\times\frac{3,524,390}{6.389\times 4,781}-\frac{5\times 2,000}{60}\times\frac{327,244}{4,781}=132,818 \qquad [2]$$

The formula for the value of the future contributions is:

$$k\left(75,000\frac{{}^{s}\bar{N}_{35}}{s_{34}D_{35}}-5\times 2,000\frac{\bar{N}_{35}}{D_{35}}\right) \qquad [1]$$

From the *Tables*:

$${}^{s}\bar{N}_{35}=502,836 \qquad \bar{N}_{35}=59,914$$

So the value of future contributions of $100k$ % of pensionable pay is:

$$k\left(75,000\times\frac{502,836}{6.389\times 4,781}-5\times 2,000\times\frac{59,914}{4,781}\right)=1,109,311k \qquad [1]$$

So, in order to meet the cost of the benefits (£132,818), the total contribution rate (members and company combined) must be:

$$k=132,818/1,109,311=11.97\%$$

Since the members contribute 5%, the company must pay the remaining 6.97%. [2]

[Total 8]

Solution X3.10

This question is CT5 September 2005 Question 9.

(i) ***Net future loss random variable***

The net future loss random variable at the outset for this policy is:

$$L=10,000\,\ddot{a}_{\overline{K_{\overline{60:60}}+1}|}-P=10,000\ddot{a}_{\overline{\max\{K_{60},K_{60}\}+1}|}-P$$

where P is the single premium and K_{60} is the curtate future lifetime of a life aged 60.

[2]

(ii) ***Single premium***

The premium is:

$$P = 10,000\,\ddot{a}_{\overline{60:60}}$$
$$= 10,000\left(\ddot{a}_{60}^{(m)} + \ddot{a}_{60}^{(f)} - \ddot{a}_{60:60}\right) \qquad [1]$$
$$= 10,000(15.632 + 16.652 - 14.090) \qquad [1]$$
$$= 10,000 \times 18.194$$
$$= 181,940 \qquad [1]$$

[Total 3]

(iii) ***Standard deviation***

The variance of the net future loss random variable is:

$$\text{var}(L) = 10,000^2\left[\frac{{}^2A_{\overline{60:60}} - \left(A_{\overline{60:60}}\right)^2}{d^2}\right] \qquad [1]$$

This formula is derived as follows:

$$\text{var}(L) = \text{var}\left(10,000\,\ddot{a}_{\overline{K_{\overline{60:60}}+1}|} - P\right) = 10,000^2\,\text{var}\left(\frac{1 - v^{K_{\overline{60:60}}+1}}{d}\right)$$

$$= \frac{10,000^2}{d^2}\,\text{var}\left(v^{K_{\overline{60:60}}+1}\right) = \frac{10,000^2}{d^2}\left[{}^2A_{\overline{60:60}} - \left(A_{\overline{60:60}}\right)^2\right]$$

Using premium conversion and the result $\ddot{a}_{\overline{60:60}} = 18.194$ at 4% interest from part (ii), we have:

$$A_{\overline{60:60}} = 1 - d\,\ddot{a}_{\overline{60:60}} = 1 - \frac{0.04}{1.04} \times 18.194 = 0.30023 \qquad [1]$$

Also:

$${}^2A_{\overline{60:60}} = \left(1 - d\,\ddot{a}_{\overline{60:60}}\right)^{@8.16\%} = 1 - \frac{0.0816}{1.0816} \times 11.957 = 0.09792 \qquad [1]$$

So the standard deviation of L is:

$$\sqrt{10,000^2\left[\frac{{}^2A_{\overline{60:60}}-\left(A_{\overline{60:60}}\right)^2}{d^2}\right]}=£22,933 \qquad [1]$$

[Total 4]

The final answer is quite sensitive to rounding.

Solution X3.11

This question is CT5 September 2005 Question 13 with the years changed.

(i) ***Age retirement benefit***

Let:

i be the valuation rate of interest [¼]

$$v=\frac{1}{1+i} \qquad [¼]$$

r_x be the number of age retirements between x and $x+1$, $x \le 64$ [¼]

r_{65} be the number of age retirements at exact age 65 [¼]

l_x be the number of active lives at age x exact [¼]

All of the r_x and l_x values must come from a suitable service table. [¼]

a_x^r be the expected present value at age x of a pension of 1 *pa* payable on age retirement at age x, and payable in accordance with the scheme rules. [¼]

$\{s_x\}$ is a salary scale such that:

$$\frac{s_{x+t}}{s_x}=\frac{\text{expected salary earned in year of age }(x+t,x+t+1)}{\text{expected salary earned in year of age }(x,x+1)} \qquad [½]$$

$$z_x=\frac{s_{x-1}+s_{x-2}+s_{x-3}}{3} \qquad [¼]$$

Assume that age retirements before age 65 occur halfway between birthdays on average. [¼]

Note to markers: please give full credit for alternative notation provided it is clearly defined and used correctly.

Past service benefit

The member is aged exactly 26 on the valuation date and has 5 years of past service. So he is already entitled to 5/60ths of final pensionable salary when he retires. [½]

If he retires in the year of age $(y, y+1)$ for $y < 65$, we are assuming it occurs at age $y + 0.5$, so his FPS will be:

$$50{,}000\frac{z_{y+0.5}}{s_{25.25}} \qquad [1]$$

If he retires at exact age 65, his FPS will be $50{,}000\dfrac{z_{65}}{s_{25.25}}$. [½]

Note that we have $s_{25.25}$ in the denominator since he started to earn £50,000 on 1 April 2010, when he was aged exactly 25.25.

The expected present value of the past service benefit is:

$$\frac{5}{60}\times 50{,}000\left[\frac{r_{26}}{l_{26}}v^{0.5}\frac{z_{26.5}}{s_{25.25}}a^r_{26.5}+\frac{r_{27}}{l_{26}}v^{1.5}\frac{z_{27.5}}{s_{25.25}}a^r_{27.5}+\cdots\right.$$

$$\left.+\frac{r_{64}}{l_{26}}v^{38.5}\frac{z_{64.5}}{s_{25.25}}a^r_{64.5}+\frac{r_{65}}{l_{26}}v^{39}\frac{z_{65}}{s_{25.25}}a^r_{65}\right] \qquad [½]$$

$$=\frac{5}{60}\times 50{,}000\left[\frac{r_{26}\,v^{26.5}\,z_{26.5}\,a^r_{26.5}+\cdots+r_{64}\,v^{64.5}\,z_{64.5}\,a^r_{64.5}+r_{65}\,v^{65}\,z_{65}\,a^r_{65}}{s_{25.25}\,v^{26}l_{26}}\right] \qquad [½]$$

Note that $r_x = 0$ for $x < 60$ so it would also be correct to write the expected present value of the past service liability as:

$$\frac{5}{60}\times 50{,}000\left[\frac{r_{60}\,v^{60.5}\,z_{60.5}\,a^r_{60.5}+\cdots+r_{64}\,v^{64.5}\,z_{64.5}\,a^r_{64.5}+r_{65}\,v^{65}\,z_{65}\,a^r_{65}}{s_{25.25}\,v^{26}l_{26}}\right]$$

Now define:

$$D_x = v^x \, l_x \qquad [¼]$$

$$^zC_x^{ra} = r_x \, v^{x+0.5} \, z_{x+0.5} \, a_{x+0.5}^r \text{ for } x < 65 \qquad [¼]$$

$$^zC_{65}^{ra} = r_{65} \, v^{65} \, z_{65} \, a_{65}^r \qquad [¼]$$

and:

$$^zM_x^{ra} = {^zC_x^{ra}} + {^zC_{x+1}^{ra}} + \cdots + {^zC_{65}^{ra}} \qquad [¼]$$

Then the expected present value of the past service benefit is:

$$\frac{5}{60} \times 50{,}000 \left(\frac{^zC_{26}^{ra} + {^zC_{27}^{ra}} + \cdots + {^zC_{65}^{ra}}}{s_{25.25} \, D_{26}} \right) = \frac{5}{60} \times 50{,}000 \times \frac{^zM_{26}^{ra}}{s_{25.25} \, D_{26}} \qquad [1]$$

Note that:

$$\frac{5}{60} \times 50{,}000 \left(\frac{^zC_{60}^{ra} + {^zC_{61}^{ra}} + \cdots + {^zC_{65}^{ra}}}{s_{25.25} \, D_{26}} \right) = \frac{5}{60} \times 50{,}000 \times \frac{^zM_{60}^{ra}}{s_{25.25} \, D_{26}}$$

is also correct.

Future service benefit

Consider the year of service from age y to age $y+1$. If the member completes this year of service, he will accrue 1/60th of FPS towards his annual pension. [½]

If he does not complete the year, he will accrue nothing. [½]

So the expected present value of the benefit in respect of the year of future service from age y to age $y+1$ is:

$$\frac{1}{60} \times 50{,}000 \left[\frac{r_{y+1}}{l_{26}} v^{y+1.5-26} \frac{z_{y+1.5}}{s_{25.25}} a_{y+1.5}^r + \frac{r_{y+2}}{l_{26}} v^{y+2.5-26} \frac{z_{y+2.5}}{s_{25.25}} a_{y+2.5}^r + \cdots \right.$$
$$\left. + \frac{r_{64}}{l_{26}} v^{38.5} \frac{z_{64.5}}{s_{25.25}} a_{64.5}^r + \frac{r_{65}}{l_{26}} v^{39} \frac{z_{65}}{s_{25.25}} a_{65}^r \right] \qquad [½]$$

$$= \frac{1}{60} \times 50{,}000 \Big[r_{y+1}\, v^{y+1.5}\, z_{y+1.5}\, a^{r}_{y+1.5} + \cdots$$

$$+ r_{64}\, v^{64.5}\, z_{64.5}\, a^{r}_{64.5} + r_{65}\, v^{65}\, z_{65}\, a^{r}_{65} \Big] \Big/ s_{25.25}\, v^{26} l_{26} \qquad [½]$$

$$= \frac{1}{60} \times 50{,}000 \left[\frac{{}^{z}C^{ra}_{y+1} + \cdots + {}^{z}C^{ra}_{65}}{s_{25.25}\, D_{26}} \right] \qquad [¼]$$

$$= \frac{1}{60} \times 50{,}000 \times \frac{{}^{z}M^{ra}_{y+1}}{s_{25.25}\, D_{26}} \qquad [¼]$$

Note that ${}^{z}M^{ra}_{y+1} = {}^{z}M^{ra}_{60}$ *for* $y \le 59$.

Now we sum over all possible years of future service that would lead to accrual. Because the pension is subject to a maximum of 40 years of accrual and the member has already accrued 5 years, we sum over the years of service (26,27), (27,28), …, (60,61). So the expected present value of the future service benefit is:

$$\frac{1}{60} \times 50{,}000 \times \frac{{}^{z}M^{ra}_{27} + {}^{z}M^{ra}_{28} + \cdots + {}^{z}M^{ra}_{61}}{s_{25.25}\, D_{26}} \qquad [1]$$

Now defining:

$${}^{z}R^{ra}_{x} = {}^{z}M^{ra}_{x} + {}^{z}M^{ra}_{x+1} + \cdots + {}^{z}M^{ra}_{65} \qquad [¼]$$

the expected present value of the future service benefit is:

$$\frac{1}{60} \times 50{,}000 \times \frac{{}^{z}R^{ra}_{27} - {}^{z}R^{ra}_{62}}{s_{25.25}\, D_{26}} \qquad [½]$$

[Total 12]

Note that:

$$\frac{1}{60} \times 50{,}000 \times \frac{34\, {}^{z}M^{ra}_{60} + {}^{z}M^{ra}_{61}}{s_{25.25}\, D_{26}} = \frac{1}{60} \times 50{,}000 \times \frac{33\, {}^{z}M^{ra}_{60} + {}^{z}R^{ra}_{60} - {}^{z}R^{ra}_{62}}{s_{25.25}\, D_{26}}$$

is also correct.

(ii) ***Death in service benefit***

We now also define d_x to be the number of deaths between the ages of x and $x+1$, according to a suitable service table. [½]

We assume that deaths occur halfway between birthdays on average. [½]

So if the member dies in the year of age $(y, y+1)$, we assume that this happens at age $y+0.5$. [½]

We also assume that salary increases take place on 1 April each year, *ie* at ages 26.25, 27.25, 28.25, *etc*. [½]

The scheme provides a benefit of 4 times annual salary at the date of death on death before retirement. So if the member dies in the year of age $(y, y+1)$, then using our assumptions, the amount of benefit payable is $4 \times 50{,}000 \times \dfrac{s_{y+0.25}}{s_{25.25}}$. [1]

So the expected present value of the death benefit is:

$$4 \times 50{,}000 \left[\frac{d_{26}}{l_{26}} v^{0.5} \frac{s_{26.25}}{s_{25.25}} + \frac{d_{27}}{l_{26}} v^{1.5} \frac{s_{27.25}}{s_{25.25}} + \cdots + \frac{d_{64}}{l_{26}} v^{38.5} \frac{s_{64.25}}{s_{25.25}} \right] \quad [½]$$

$$= 4 \times 50{,}000 \times \frac{s_{26.25}\, d_{26}\, v^{26.5} + s_{27.25}\, d_{27}\, v^{27.5} + \cdots + s_{64.25}\, d_{64}\, v^{64.5}}{s_{25.25}\, l_{26}\, v^{26}} \quad [½]$$

If we now define:

$${}^{s}C_x^d = s_{x+0.25}\, d_x\, v^{x+0.5} \quad [½]$$

and:

$${}^{s}M_x^d = {}^{s}C_x^d + {}^{s}C_{x+1}^d + \cdots + {}^{s}C_{64}^d \quad [½]$$

then the expected present value of the death benefit is:

$$4 \times 50{,}000 \times \frac{{}^{s}C_{26}^d + {}^{s}C_{27}^d + \cdots + {}^{s}C_{64}^d}{s_{25.25}\, D_{26}} = 4 \times 50{,}000 \times \frac{{}^{s}M_{26}^d}{s_{25.25}\, D_{26}} \quad [1]$$

[Total 6]

Assignment X4 – Solutions

Markers: This document does not necessarily give every possible approach to solving each of the questions. Please give credit for other valid approaches.

Solution X4.1

This question is CT5 April 2005 Question 10.

Directly standardised mortality rate

The directly standardised mortality rate is defined as:

$$\frac{\sum_x {}^sE^c_{x,t}\, m_{x,t}}{\sum_x {}^sE^c_{x,t}}$$

where ${}^sE^c_{x,t}$ is the central exposed to risk between x and $x+t$ for the standard population, and $m_{x,t}$ is the central rate of mortality between the ages of x and $x+t$ for the study group.

The standard population is the combined population. So we have:

Age band	Exposed to risk for standard population
20-29	225,000
30-39	450,000
40-49	300,000
50-59	240,000
Total	1,215,000

[½]

The directly standardised mortality rate for the female lives is then:

$$\frac{225\times 0.00125+450\times 0.00265+300\times 0.00465+240\times 0.00685}{1,215}=0.00371 \quad [1]$$

Indirectly standardised mortality rate

This is given by:

$$F \times \text{crude death rate for the study group}$$

where F is the area comparability factor and is given by the formula:

$$F = \frac{\text{crude death rate for standard population}}{\text{crude death rate for study population assuming standard mortality}}$$

To calculate F, we need the age-specific mortality rates for the standard population.

From the given data, we can calculate the number of deaths in each age band and hence the mortality rates for the standard population:

Age band	**Male deaths**	**Female deaths**	**All deaths**	**Age-specific mortality rate for standard population**
20-29	445	125	570	0.00253
30-39	1,378	662.5	2,040.5	0.00453
40-49	989	930	1,919	0.00640
50-59	1,109.7	1,027.5	2,137.2	0.00891
Total	3,921.7	2,745	6,666.7	

The crude death rate for the standard population is then:

$$\frac{\text{total deaths in standard population}}{\text{total exposed to risk for standard population}} = \frac{6,666.7}{1,215,000} = 0.00549 \qquad [1]$$

The crude death rate for the female lives assuming standard mortality is:

$$\frac{\text{total female deaths assuming standard mortality}}{\text{total exposed to risk for females}}$$

$$= \frac{100 \times 0.00253 + 250 \times 0.00453 + 200 \times 0.00640 + 150 \times 0.00891}{100 + 250 + 200 + 150}$$

$$= 0.00572 \qquad [1]$$

So:

$$F = \frac{0.00549}{0.00572} = 0.95979 \qquad [1]$$

The crude death rate for the female lives (using female mortality) is:

$$\frac{\text{total female deaths}}{\text{total exposed to risk for females}} = \frac{2,745}{700,000} = 0.00392 \qquad [½]$$

The indirectly standardised mortality rate for the female lives is then:

$$0.95979 \times 0.00392 = 0.00376 \qquad [1]$$

[Total 6]

Solution X4.2

Geographical location can affect mortality in the following ways:

- The rainfall and temperature in a region may make the area prone to certain types of diseases, *eg* tropical diseases. [1]
- Natural disasters such as tidal waves, hurricanes, floods, drought and famines occur more frequently in certain countries than in others. [1]
- Availability of medical facilities, preventative screening and immunisation programmes will vary with geographical location. [1]
- Road accidents may be more common in cities. However, since the traffic speed is likely to be slower than elsewhere, road accidents are less likely to be fatal. [1]
- Countries at war or where there is a high level of violence and social unrest will have higher mortality rates. [1]

[Total 5]

Solution X4.3

(i) ***Revised profit in first year***

The net premium reserve per policy at the end of the first year can be calculated as:

$${}_1V^{net} = 10,000\left(1 - \frac{\ddot{a}_{62:\overline{3}|}}{\ddot{a}_{61:\overline{4}|}}\right) = 10,000\left(1 - \frac{2.805}{3.622}\right) = £2,256 \qquad [1½]$$

So the profit arising from surrenders in the first year is:

$$0.1 \times p_{61} \times (2,256 - 1,500) = £75 \qquad [1]$$

So the profit in the first year will be increased by £75, from –£100 to –£25. [½]
[Total 3]

(ii) ***Impact on profit signature***

In subsequent years, the numbers of policies remaining in force will be reduced by a factor of 0.9, leading to a corresponding reduction in the profits in the profit signature.
[1]

Solution X4.4

The expected profit for year 5 per policy in force at the beginning of the year is given by:

$$\left({}_4V + P - e\right)\left(1+i\right) - {}_5V\,(ap)_{x+4} - \left(S+f\right)(aq)^d_{x+4} - \left(5P+f\right)(aq)^s_{x+4} \qquad [1]$$

where e denotes the renewal expenses and f denotes the claims expenses.

Putting in the values gives an expected profit of:

$$\begin{aligned}&\left(5,000+1,100-40\right)\left(1.08\right)-6,500\left(1-0.01-0.07\right)\\&\qquad -\left(12,000+100\right)\times 0.01-\left(5\times 1,100+100\right)\times 0.07\\&=51.80\end{aligned} \qquad [2]$$

[Total 3]

Solution X4.5

This is a good test of how well you understand the single figure indices. It is not always enough to be able just to calculate them.

Crude Death Rate

- heavily influenced by mortality at older ages [½]
- beware epidemics distorting figures [½]

(a) OK if population structure reasonably stable [½]
hence beware mass immigration or emigration [¼]
easy to calculate [¼]

(b) don't do it! [½]
because of different age distributions [¼]
and different sex distributions in each occupational group [¼]

Standardised Mortality Rate

- influenced again by mortality at older ages [½]

(a) generally OK [½]
but practical constraints [½]
since need age/sex-specific mortality rates at each time point [¼]
no problems caused by changing population structure [¼]

(b) good [½]
copes well with age and sex variations provided age-specific rates are available for occupational groups [½]
but cannot allow for some occupations being selected against [½]
eg entering/leaving an occupation due to health reasons [½]

Standardised Mortality Ratio

- heavily influenced by relative mortality at older ages [1]

(a) generally OK [½]
the standard mortality rates must be the same figures each time otherwise you are not performing a valid comparison [½]

(b) fine [½]
except possible practical problems with the gathering of data [¼]
eg population age distributions in different occupational groups [¼]
[Maximum 9]

Solution X4.6

(i) ***Required reserves***

The reserves required at the end of Year 2 and Year 1 are:

$$ {}_2V = \frac{6}{1.08} = 5.56 \qquad [½]$$

and: $$ {}_1V = \frac{1}{1.08}\left(12 + 0.99 \times \frac{6}{1.08}\right) = 16.20 \qquad [1½]$$

[Total 2]

(ii) ***NPV of profits before and after zeroisation***

Before zeroisation, the net present value (based on a risk discount rate of 10%) is:

$$NPV = -\frac{25}{1.10} - 12 \times \frac{0.99}{1.10^2} - 6 \times \frac{0.99^2}{1.10^3} + 25 \times \frac{0.99^3}{1.10^4} + 35 \times \frac{0.99^4}{1.10^5} = 0.48 \qquad [1]$$

After zeroisation the profit in Year 1 will become:

$$\text{Profit in Year } 1 = -25 - 12 \times \frac{0.99}{1.08} - 6 \times \frac{0.99^2}{1.08^2} = -41.04 \qquad [1]$$

So the profit vector will become:

Year	*In force profit*
1	–41.04
2	0
3	0
4	25
5	35

[1]

So the NPV after zeroisation will be:

$$NPV = -\frac{41.04}{1.10} + 0 + 0 + 25 \times \frac{0.99^3}{1.10^4} + 35 \times \frac{0.99^4}{1.10^5} = 0.14 \qquad [1]$$

As expected, the NPV after zeroisation is smaller because the emergence of the profits has been deferred and the risk discount rate is greater than the accumulation rate. [1]

[Total 5]

Solution X4.7

(a) ***Class selection***

Class selection is where individuals are grouped according to characteristics that have permanent effects on mortality or morbidity. [1]

Examples include:

- sex
- smoker status
- policy type [1]

(b) ***Selective decrement***

This is where lives that exit from the population through a particular cause of decrement do not have the same mortality or morbidity characteristics as the population as a whole. The effect of the selective decrement is then to alter the average mortality or morbidity experience of those who remain in the population. [1]

For example, consider a population of life assurance policyholders. People who are in good health may withdraw their policies whilst those in poor health are much less likely to do so. As a result, the average mortality of those who withdraw tends to be lower than that of the policyholders as a whole, so that the average mortality of those that remain increases. Withdrawal is therefore acting as a selective decrement with respect to mortality. [1]

(c) ***Spurious selection***

This is where heterogeneity in a population leads to incorrect interpretations of mortality or morbidity differences being made. [1]

For example:

- A population of females may have a greater proportion of older lives, or of smokers, than a population of males. An analysis of the mortality of the two groups (*subdivided only by their sex*) would reveal that females have higher mortality than males. This is almost certainly an erroneous indication of the effect of sex on mortality.
- Increasing the strictness of underwriting for life insurance policies over time will lead to a lighter mortality experience. This may be falsely interpreted as an improvement in mortality over time.

[1]

(d) ***Adverse selection***

This is the name given to any type of selection that leads to an adverse effect on another party, *eg* a life insurance company. [1]

Examples include:

- Smokers looking for life insurance will favour companies that charge identical premiums to smokers and non-smokers, whereas non-smokers will favour companies that do differentiate as they will charge lower rates to non-smokers. As a result, companies of the first type will suffer from adverse selection, as they will end up with a higher proportion of bad risks.
- Selective withdrawal of healthy lives from life insurance policies will result in a higher average mortality experience for the remaining policyholders. [1]

(e) ***Temporary initial selection***

Temporary initial selection arises when mortality changes with the passage of time since some event, over and above the change associated with age. This type of selection is temporary because after a period of time, the length of time since the event took place becomes insignificant. [1]

An example of temporary initial selection is found amongst life assurance policyholders who passed some medical underwriting test before being accepted for insurance. Immediately after passing the test, these lives have much better mortality than lives of the same age who passed the test several years before. However, their mortality tends towards that of the overall group as time since the test increases. [1]

[Total 10]

Solution X4.8

This question is CT5 April 2005 Question 1.

A profit vector is a vector whose entries are the expected cashflows at the end of each policy year per policy in force at the beginning of the respective policy year. [1]

A profit signature is a vector whose entries are the expected cashflows at the end of each policy year, per policy in force at time 0. [½]

To calculate the profit signature from the profit vector, we multiply each entry in the profit vector by the probability that the policy is in force at the start of the corresponding policy year. The first entries in the profit vector and the profit signature are equal. [½]

[Total 2]

Solution X4.9

(i) ***Calculations***

Occupation A

Crude Death Rate

Deaths are 2,505 (given data)

divided by a population of 234,000 gives a crude death rate of 0.01071 [½]

Standardised Mortality Rate

We apply occupation-specific mortality rates to the "all occupations" population structure:

$$\frac{0.001\times 360+0.002\times 390+0.004\times 430+0.026\times 320}{1,500}=0.00745 \quad [1]$$

Standardised Mortality Ratio (SMR)

Expected deaths are calculated using the "all occupations mortality":

$$1,000\times(0.001\times 21+0.002\times 42+0.006\times 93+0.024\times 78)=2,535 \quad [½]$$

$$\frac{actual\ deaths}{expected\ deaths}=\frac{2,505}{2,535}=0.988 \quad [½]$$

Occupation B

Crude Death Rate

Deaths are given as 2,096

divided by a population of 220,000 gives a crude death rate of 0.00953 [½]

Standardised Mortality Rate

$$\frac{0.003\times 360+0.002\times 390+0.005\times 430+0.021\times 320}{1,500}=0.00715 \quad [1]$$

Standardised Mortality Ratio (SMR)

Expected deaths are calculated from:

$$1{,}000 \times (0.001 \times 12 + 0.002 \times 44 + 0.006 \times 92 + 0.024 \times 72) = 2{,}380$$ [½]

$$\frac{actual\ deaths}{expected\ deaths} = \frac{2{,}096}{2{,}380} = 0.881$$ [½]

[Total 5]

(ii) ***Calculations and comments***

Occupation A

$$\frac{21}{0.001} + \frac{84}{0.002} + \frac{372}{0.006} + \frac{2{,}028}{0.024} = 209{,}500$$

Hence the index gives $\dfrac{209{,}500}{234{,}000} = 0.895$ [1]

Occupation B

$$\frac{36}{0.001} + \frac{88}{0.002} + \frac{460}{0.006} + \frac{1{,}512}{0.024} = 219{,}667$$

Hence the index gives $\dfrac{219{,}667}{220{,}000} = 0.998$ [1]

Both the crude death rate and the standardised death rate of Occupation A exceed that of Occupation B. The difference in the standardised rates is smaller however, due to the slightly different population structures that serve to exaggerate the difference in the underlying mortality levels. [1]

The SMR for each occupation is lower than one, indicating that the occupations suffer lighter mortality than average. Again Occupation B has the lower figure of the two. [1]

The new index gives a slightly different picture suggesting that Occupation A suffers lighter mortality than Occupation B. (This index compares the age specific mortality rates and then weights them according to the size of the population in each age group. The weighting in both the SMR and the standardised rates are biased towards older ages.) [1]

These results show that while single figure indices provide useful summary information, they can be misleading when viewed in isolation as they only paint part of the picture.

[1]

[Total 6]

Solution X4.10

The unit fund values, plus the terminal bonus amounts, are shown in the following table:

Year	Allocated premium (1)	Total unit fund at start of year (2)	Policy fee (3)	Total unit fund at end of year (4)	Terminal bonus (5)	Unit fund plus TB at end of year (6)
1	4,250	4,250	0	4,420	44.20	4,464.20
2	5,000	9,420	141.30	9,649.85	289.50	9,939.34
3	5,000	14,649.85	219.75	15,007.30	975.48	15,982.78

[3]

Key to table:

$(3) = (2) \times 0.015$ in years 2 and 3

$(4) = [(2) - (3)] \times 1.04$

$(5) = (4) \times (\text{TB rate})$

The non-unit cashflows and net present value are calculated in the following table:

Year	Profit on allocation (7)	Policy fee (8)	Expenses and commission (9)	Interest (10)	AM92 select (11)	Mortality probability (12)
1	750	0	750	0	0.003358	0.002686
2	0	141.30	80	1.23	0.004903	0.003922
3	0	219.75	80	2.79	0.005650	0.004520

[3]

Year	Expected death cost (13)	Dependent surrender probability (14)	Expected surrender profit (15)	Investment and actuarial expenses (16)	Termination expenses (17)	Cashflow (18)
1	28.30	0.099731	7.98	11.05	5.12	– 36.50
2	19.85	0.099608	7.97	24.12	5.18	21.34
3	0	0	0	37.52	50	55.02

[4]

Year	Probability in force (19)	Profit signature (20)	Discount (21)	NPV (22)
1	1	– 36.50	0.925926	– 33.79
2	0.897582	19.16	0.857339	16.42
3	0.804655	44.28	0.793832	35.15
			Total	**17.78**

[3]

Key to table:

$(7) = 5,000 - (1)$

$(8) = (3)$

$(9)_1 = 500 + 0.05 \times 5,000; \qquad (9)_2 = (9)_3 = 30 + 0.01 \times 5,000$

$(10) = [(7) + (8) - (9)] \times 0.02$

$(12) = (11) \times 0.8$

$(13) = \max\left[15,000 - (6), 0\right] \times (12)$

$(14) = [1 - (12)] \times 0.1$

$(15) = 80 \times (14)$

$(16) = (4) \times 0.0025$

$(17)_1 = (17)_2 = 50 \times [(12) + (14)]$

$(18) = (7) + (8) - (9) + (10) - (13) + (15) - (16) - (17)$

$(19)_t = (19)_{t-1} \times \left[1 - (12)_{t-1} - (14)_{t-1}\right], \ t = 2, 3$

$(20) = (18) \times (19)$

$(21)_t = 1.08^{-t}$

$(22) = (20) \times (21)$

Solution X4.11

This question is CT5 April 2009 Question 14.

(i) ***Annual premium***

Let the annual premium be P. The expected present value of the premiums is:

$$EPV(\text{premiums}) = P\ddot{a}_{[60]:\overline{5}|} = 4.398P$$ [½]

The expected present value of the expenses is:

$$\begin{aligned} EPV(\text{expenses}) &= 0.6P + 0.05P\left(\ddot{a}_{[60]:\overline{5}|} - 1\right) \\ &= 0.6P + 0.05P \times 3.398 \\ &= 0.7699P \end{aligned}$$ [½]

Turning our attention to the benefits, simple bonuses of 4% *pa* are declared at the start of each year. This means that the sum assured increases by $£10,000 \times 0.04 = £400$ each time a bonus is awarded. [½]

Since bonuses are added at the start of the year, we must allow for a sum assured of £10,400 on death during the first year, £10,800 on death during the second year, and so on. The payment at the end of the fifth year will be £12,000, irrespective of whether the policyholder dies during the final year or survives to the end of the contract.

An expression for the expected present value of the benefits is:

$$EPV(\text{benefits}) = 10,000A_{[60]:\overline{5}|} + 400(IA)_{[60]:\overline{5}|}$$ [½]

Alternatively, if you split off the maturity benefit and treat the contract as a term assurance plus a pure endowment, you can write:

$$\begin{aligned} EPV(\textit{benefits}) &= 10,000A^{1}_{[60]:\overline{5}|} + 400(IA)^{1}_{[60]:\overline{5}|} + 12,000A_{[60]:\overline{5}|}^{\ \ \ \ \ \ 1} \\ &= 10,000 \times 0.039428 + 400 \times 0.126806 + 12,000 \times 0.711612 \end{aligned}$$

To evaluate the increasing assurance, we use the formula:

$$(IA)_{[60]:\overline{5}|} = (IA)_{[60]} - v^5\,{}_5p_{[60]}\left((IA)_{65} + 5A_{65}\right) + 5v^5\,{}_5p_{[60]}$$ [1]

Looking up values in the *Tables* at 6% interest:

$$(IA)_{[60]:\overline{5}|} = 5.4772 - (1.06)^{-5}\left(\frac{8,821.2612}{9,263.1422}\right)\left(5.50985 + 5\times 0.40177\right)$$
$$+ 5(1.06)^{-5}\left(\frac{8,821.2612}{9,263.1422}\right)$$

giving $(IA)_{[60]:\overline{5}|} = 3.68486$. [½]

Also, at 6% interest:

$$A_{[60]:\overline{5}|} = 0.75104$$ [½]

This gives:

$$EPV(\text{benefits}) = 10,000\times 0.75104 + 400\times 3.68486$$
$$= £8,984.35$$ [½]

The premium is found by solving:

$$4.398P = 8,984.35 + 0.7699P$$
$$\Rightarrow \quad P = £2,476.32$$

So, the premium is £2,476, as stated. [½]
[Total 5]

(ii) ***Profit margin***

To calculate a net premium reserve for a with-profits policy we need to use the following method:

1. *Calculate the net premium for the contract ignoring bonuses.*

2. *Calculate the net premium reserve. The expected present value of future benefits should include allowance for bonuses declared so far, but no allowance for future bonuses. The premium used should be the net premium calculated in 1.*

In order to carry out the profit test, we need to calculate the net premium reserve held at each duration.

The net premium, NP, for the contract, using 4% interest and AM92 Ultimate mortality is:

$$NP \times \ddot{a}_{60:\overline{5}|} = 10,000 A_{60:\overline{5}|} \Rightarrow NP = \frac{10,000 \times 0.82499}{4.550} = £1,813.16 \qquad [1]$$

By time 1, one bonus of £400 has been awarded, so the sum assured is £10,400. The net premium reserve at time 1 is therefore:

$$\begin{aligned} {}_1V &= 10,400 A_{61:\overline{4}|} - 1,813.16 \ddot{a}_{61:\overline{4}|} \\ &= 10,400 \times 0.85685 - 1,813.16 \times 3.722 \\ &= £2,162.66 \end{aligned} \qquad [½]$$

By time 2, two bonuses of £400 have been awarded, so the sum assured is £10,800. The net premium reserve at time 2 is therefore:

$$\begin{aligned} {}_2V &= 10,800 A_{62:\overline{3}|} - 1,813.16 \ddot{a}_{62:\overline{3}|} \\ &= 10,800 \times 0.89013 - 1,813.16 \times 2.857 \\ &= £4,433.21 \end{aligned} \qquad [½]$$

By time 3, three bonuses of £400 have been awarded, so the sum assured is £11,200. The net premium reserve at time 3 is therefore:

$$\begin{aligned} {}_3V &= 11,200 A_{63:\overline{2}|} - 1,813.16 \ddot{a}_{63:\overline{2}|} \\ &= 11,200 \times 0.92498 - 1,813.16 \times 1.951 \\ &= £6,822.30 \end{aligned} \qquad [½]$$

By time 4, four bonuses of £400 have been awarded, so the sum assured is £11,600. The net premium reserve at time 4 is therefore:

$$\begin{aligned} {}_4V &= 11,600 A_{64:\overline{1}|} - 1,813.16 \ddot{a}_{64:\overline{1}|} \\ &= 11,600 \times 0.96154 - 1,813.16 \times 1 \\ &= £9,340.70 \end{aligned} \qquad [½]$$

At time 5, the contract expires, so no reserves will be held at that point.

Alternatively, the net premium reserves may be calculated as follows:

$$
\begin{aligned}
{}_1V &= 10{,}400A_{61:\overline{4|}} - 1{,}813.16\ddot{a}_{61:\overline{4|}} \\
&= 10{,}000A_{61:\overline{4|}} - 1{,}813.16\ddot{a}_{61:\overline{4|}} + 400A_{61:\overline{4|}} \\
&= 10{,}000\,{}_1V_{60:\overline{5|}} + 400A_{61:\overline{4|}} \\
&= 10{,}000\left(1 - \frac{\ddot{a}_{61:\overline{4|}}}{\ddot{a}_{60:\overline{5|}}}\right) + 400A_{61:\overline{4|}}
\end{aligned}
$$

where the final equality uses the formula for the net premium reserve of an endowment assurance given on page 37 of the Tables. (You needn't show the first two lines of the above in the exam.)

The net premium reserve at time 2 is:

$$
{}_2V = 10{,}000\,{}_2V_{60:\overline{5|}} + 800A_{62:\overline{3|}} = 10{,}000\left(1 - \frac{\ddot{a}_{62:\overline{3|}}}{\ddot{a}_{60:\overline{5|}}}\right) + 800A_{62:\overline{3|}}
$$

with other reserves calculated similarly.

Traditionally, profit tests are set out with the years as row headings and the cashflows as column headings. It can be easier to set out these questions in an exam if you transpose the table, as shown below. Markers, please award full credit either way if the actual numbers are correct.

Carrying out the profit test:

Note	Year	1	2	3	4	5
	Premium (P)	2,476.32	2,476.32	2,476.32	2,476.32	2,476.32
(1)	Expenses	–1,485.79	–123.82	–123.82	–123.82	–123.82
(2)	Interest	69.34	164.68	164.68	164.68	164.68
(3)	Benefit	–83.43	–97.30	–113.25	–131.59	–12,000
(4)	Reserves at start of year	0	2,162.66	4,433.21	6,822.30	9,340.70
(5)	Interest on reserves	0	151.39	310.32	477.56	653.85
(6)	Expected reserves at end of year	–2,145.31	–4,393.27	–6,753.31	–9,234.74	0
(7)	Profit vector	–1,168.87	340.66	394.15	450.71	511.73
(8)	Probability of survival to start of year	1	0.991978	0.9830412	0.9731007	0.9620619
(9)	Profit signature	–1,168.87	337.93	387.47	438.59	492.32

[7, lose ½ for each incorrect value, subject to minimum of 0]

Markers, award full method marks here if the method has been followed through correctly, ie if one incorrect value early in the calculation has led to many incorrect values but the method is otherwise perfect, award 6.5 marks out of 7.

Notes:

(1) $-0.6P$ for year 1; $-0.05P$ for years 2 to 5

(2) $0.07(P+(1))$

(3) $-(10,000+400k)q_{60+k-1}$ for years $k=1,2,3,4$

$-12,000$ in year 5, as the benefit is paid irrespective of whether the policyholder dies in the final year or survives to the end of the term

(4) 0 for year 1; ${}_{k-1}V$ for years $k=2,3,4,5$

(5) $0.07\times(4)$

(6) $-{}_kV\times p_{60+k-1}$ for years $k=1,2,3,4$; 0 for year 5

(7) $P+(1)+(2)+(3)+(4)+(5)+(6)$

(8) ${}_{k-1}p_{60}$ for years $k=1,2,3,4,5$

(9) $(7)\times(8)$

The profit margin is defined as:

$$\frac{NPV}{EPV(premiums)}$$

where the calculations are performed at the risk discount rate.

Using the risk discount rate of 9% *pa,* the net present value of the profit signature is:

$$\begin{aligned} NPV &= -1{,}168.87v + 337.93v^2 + 387.47v^3 + 438.59v^4 + 492.32v^5 \\ &= £141.95 \end{aligned}$$ [1]

The expected present value of the premiums is:

$$\begin{aligned} EPV(premiums) &= 2{,}476.32\left(1 + vp_{60} + v^2\,{}_2p_{60} + v^3\,{}_3p_{60} + v^4\,{}_4p_{60}\right) \\ &= 2{,}476.32 \times 4.17044 \\ &= £10{,}327.34 \end{aligned}$$ [1]

The profit margin is:

$$\frac{141.95}{10{,}327.34} = 0.0137$$

ie 1.37%. [1]

[Total 13]

Solution X4.12

This question is CT5 September 2006 Question 9.

(i) ***Expected net present value of the profit***

Unit fund

The expected cashflows in the unit fund are given in the table below. Cashflows out of the fund are shown as negative entries.

Year	Premium	Cost of allocation	Fund at start of year	End fund before charge	Management charge	Fund at end of year
1	5,000	4,037.50	4,037.50	4,279.75	– 32.10	4,247.65
2	5,000	4,940	9,187.65	9,738.91	– 73.04	9,665.87
3	5,000	4,940	14,605.87	15,482.22	– 116.12	15,366.10

[4, lose ½ for each incorrect value, subject to minimum of 0]

Markers, award full method marks here if the method has been followed through correctly, ie if one incorrect value early in the calculation has led to many incorrect values but the method is otherwise perfect, award 3.5 marks out of 4.

Non-unit fund

The expected cashflows in the non-unit fund are:

Year	Premium less cost of allocation	Expenses	Interest	Expected benefit cost	Management charge	Profit vector
1	962.50	–600	14.50	–126.37	32.10	282.73
2	60	–100	–1.60	–93.10	73.04	–61.66
3	60	–100	–1.60	–46.86	116.12	27.66

[3, lose ½ for each incorrect value, subject to minimum of 0]

Markers: the marks for the expected benefit cost column are awarded separately below.

Markers, award full method marks here if the method has been followed through correctly, ie if one incorrect value early in the calculation has led to many incorrect values but the method is otherwise perfect, award 2.5 marks out of 3.

The expected benefit cost figures are calculated as follows.

If the policyholder dies in Year 1, a death benefit of £20,000 is payable at the end of the first year. The sum of £4,247.65 comes from the unit fund, and the remainder comes from the non-unit fund.

The expected death cost in Year 1 is:

$$(20,000-4,247.65)q_{60} = 15,752.35 \times 0.008022 = 126.37$$ [1]

The entries for Years 2 and 3 are calculated in a similar way:

$$(20,000-9,665.87)q_{61} = 10,334.13 \times 0.009009 = 93.10$$ [½]

and $$(20,000-15,366.10)q_{62} = 4,633.90 \times 0.010112 = 46.86$$ [½]

If the policyholder survives to the end of Year 3, she receives the full bid value of the units. This comes from the unit fund, so there is no cashflow from the non-unit fund.

The expected present value of the profit, discounted at the risk discount rate, is:

$$\begin{aligned}\text{EPV profit} &= \frac{282.73}{1.10} + p_{60}\frac{-61.66}{1.10^2} + {}_2p_{60}\frac{27.66}{1.10^3} \\ &= \frac{282.73}{1.10} + (1-0.008022)\frac{-61.66}{1.10^2} + \frac{9,129.717}{9,287.2164} \times \frac{27.66}{1.10^3} \\ &= 226.91\end{aligned}$$ [1]

[Total 10]

(ii) ***Effect of holding non-unit reserves to zeroise negative cashflows***

Holding non-unit reserves to offset the negative cashflow at time 2 defers the emergence of profit at time 1. Because we are discounting the profit flows at a higher rate (10%) than the interest earned on the reserves (4%), deferring the emergence of profit will reduce its expected present value. [2]