Institute and Faculty of Actuaries

Subject CT1
Financial Mathematics
Core Technical

Core Reading

for the 2016 exams

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SUBJECT CT1 CORE READING

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Accreditation

The Institute and Faculty of Actuaries would like to thank the numerous people who have helped in the development of this material and in the previous versions of Core Reading.

The following book has been used as the basis for several Units:

An introduction to the mathematics of finance. McCutcheon, J. J.; Scott, W. F. Heinemann, 1986. ISBN: 043491228X, by permission of the authors who are the holders of copyright of the book.

CORE READING

Introduction

The Core Reading manual has been produced by the Institute and Faculty of Actuaries. The purpose of the Core Reading is to assist in ensuring that tutors, students and examiners have clear shared appreciation of the requirements of the syllabus for the qualification examinations for Fellowship of the Institute and Faculty of Actuaries. The manual supports coverage of the syllabus in helping to ensure that both depth and breadth are re-enforced.

In examinations students will be expected to demonstrate their understanding of the concepts in Core Reading. Examiners will have this Core Reading manual when setting the papers. In preparing for examinations students are recommended to work through past examination questions and will find additional tuition helpful. The manual will be updated each year to reflect changes in the syllabus, to reflect current practice and in the interest of clarity.

UNIT 1 — GENERALISED CASHFLOW MODEL

Syllabus objective (i) Describe how to use a generalised cashflow model to describe financial transactions.

1. For a given cashflow process, state the inflows and outflows in each future time period and discuss whether the amount or the timing (or both) is fixed or uncertain.

2. Describe in the form of a cashflow model the operation of a zero coupon bond, a fixed interest security, an index-linked security, cash on deposit, an equity, an "interest only" loan, a repayment loan, and an annuity certain.

1 Cashflow process

The practical work of the actuary often involves the management of various *cashflows*. These are simply sums of money, which are paid or received at different times. The timing of the cashflows may be known or uncertain. The amount of the individual cashflows may also be known or unknown in advance. From a theoretical viewpoint one may also consider a **continuously** payable cashflow.

For example, a company operating a privately owned bridge, road or tunnel will receive toll payments. The company will pay out money for maintenance, debt repayment and for other management expenses. From the company's viewpoint the toll payments are positive cashflows (i.e. money received) while the maintenance, debt repayments and other expenses are negative cashflows (i.e. money paid out). Similar cashflows arise in all businesses. In some businesses, such as insurance companies, investment income will be received in relation to positive cashflows (premiums) received before the negative cashflows (claims and expenses).

Where there is uncertainty about the amount or timing of cashflows, an actuary can assign probabilities to both the amount and the existence of a cashflow. In this Subject we will assume that the existence of the future cashflows is certain.

2 Examples of cashflow scenarios

In this section some simple examples are given of practical situations which are readily described by cashflow models.

2.1 A zero-coupon bond

The term "zero-coupon bond" is used to describe a security that is simply a contract to provide a specified lump sum at some specified future date. For the investor there is a negative cashflow at the point of investment and a single known positive cashflow on the specified future date.

2.2 A fixed interest security

A body such as an industrial company, a local authority, or the government of a country may raise money by floating a loan on the stock exchange. In many instances such a loan takes the form of a fixed interest security, which is issued in bonds of a stated nominal amount. The characteristic feature of such a security in its simplest form is that the holder of a bond will receive a lump sum of specified amount at some specified future time together with a series of regular level interest payments until the repayment (or redemption) of the lump sum.

The investor has an initial negative cashflow, a single known positive cashflow on the specified future date, and a series of smaller known positive cashflows on a regular set of specified future dates.

2.3 An index-linked security

With a conventional fixed interest security the interest payments are all of the same amount. If inflationary pressures in the economy are not kept under control, the purchasing power of a given sum of money diminishes with the passage of time, significantly so when the rate of inflation is high. For this reason some investors are attracted by a security for which the actual cash amount of interest payments and of the final capital repayment are linked to an "index" which reflects the effects of inflation.

Here the initial negative cashflow is followed by a series of unknown positive cashflows and a single larger unknown positive cashflow, all on specified dates. However, it is known that the amounts of the future cashflows relate to the inflation index. Hence these cashflows are said to be known in "real" terms.

Note that in practice the operation of an index-linked security will be such that the cashflows do not relate to the inflation index at the time of payment, due to delays in calculating the index. It is also possible that the need of the borrower (or perhaps the investors) to know the amounts of the payments in advance may lead to the use of an index from an earlier period.

2.4 Cash on deposit

If cash is placed on deposit, the investor can choose when to disinvest and will receive interest additions during the period of investment. The interest additions will be subject to regular change as determined by the investment provider. These additions may only be known on a day-to-day basis. The amounts and timing of cashflows will therefore be unknown.

2.5 An equity

Equity shares (also known as shares or equities in the UK and as common stock in the USA) are securities that are held by the owners of an organisation. Equity shareholders own the company that issued the shares. For example if a company issues 4,000 shares and an investor buys 1,000, the investor owns 25 per cent of the company. In a small company all the equity shares may be held by a few individuals or institutions. In a large organisation there may be many thousands of shareholders.

Equity shares do not earn a fixed rate of interest as fixed interest securities do. Instead the shareholders are entitled to a share in the company's profits, in proportion to the number of shares owned.

The distribution of profits to shareholders takes the form of regular payments of *dividends*. Since they are related to the company profits that are not known in advance, dividend rates are variable. It is expected that company profits will increase over time. It is therefore expected also that dividends per share will increase — though there are likely to be fluctuations. This means that in order to construct a cashflow schedule for an equity it is necessary first to make an assumption about the growth of future dividends. It also means that the entries in the cashflow schedule are uncertain — they are estimates rather than known quantities.

In practice the relationship between dividends and profits is not a simple one. Companies will, from time to time, need to hold back some profits to provide funds for new projects or expansion. Companies may also hold back profits in good years to subsidise dividends in years with poorer profits. Additionally, companies may be able to distribute profits in a manner other than dividends, such as by buying back the shares issued to some investors.

Since equities do not have a fixed redemption date, but can be held in perpetuity, we may assume that dividends continue indefinitely (unless the investor sells the shares or the company buys them back), but it is important to bear in mind the risk that the company will fail, in which case the dividend income will cease and the shareholders would only be entitled to any assets which remain after creditors are paid. The future positive cashflows for the investor are therefore uncertain in amount and may even be lower, in total, than the initial negative cashflow.

2.6 An annuity certain

An annuity certain provides a series of regular payments in return for a single premium (i.e. a lump sum) paid at the outset. The precise conditions under which the annuity payments will be made will be clearly specified. In particular, the number of years for which the annuity is payable, and the frequency of payment, will be specified. Also, the payment amounts may be level or might be specified to vary — for example in line with an inflation index, or at a constant rate.

The cashflows for the investor will be an initial negative cashflow followed by a series of smaller regular positive cashflows throughout the specified term of payment. In the case of level annuity payments, the cashflows are similar to those for a fixed interest security. From the perspective of the annuity provider, there is an initial positive cashflow followed by a known number of regular negative cashflows.

In Subject CT5, Contingencies, the theory of this Subject will be extended to deal with annuities where the payment term is uncertain, that is, for which payments are made only so long as the annuity policyholder survives.

2.7 An "interest-only" loan

An "interest-only" loan is a loan that is repayable by a series of interest payments followed by a return of the initial loan amount.

In the simplest of cases, the cashflows are the reverse of those for a fixed interest security. The provider of the loan effectively buys a fixed interest security from the borrower.

In practice, however, the interest rate need not be fixed in advance. The regular cashflows may therefore be of unknown amounts.

It may also be possible for the loan to be repaid early. The number of cashflows and the timing of the final cashflows may therefore be uncertain.

2.8 A repayment loan (or mortgage)

A repayment loan is a loan that is repayable by a series of payments that include partial repayment of the loan capital in addition to the interest payments.

In its simplest form, the interest rate will be fixed and the payments will be of fixed equal amounts, paid at regular known times.

The cashflows are similar to those for an annuity certain.

As for the "interest-only" loan, complications may be added by allowing the interest rate to vary or the loan to be repaid early. Additionally, it is possible that the regular repayments could be specified to increase (or decrease) with time. Such changes could be smooth or discrete.

It is important to appreciate that with a repayment loan the breakdown of each payment into "interest" and "capital" changes significantly over the period of the loan. The first repayment will consist almost entirely of interest and will provide only a very small capital repayment. In contrast, the final repayment will consist almost entirely of capital and will have a small interest content.

E N D

UNIT 2 — THE TIME VALUE OF MONEY

Syllabus objectives

(ii) Describe how to take into account the time value of money using the concepts of compound interest and discounting.

1. Accumulate a single investment at a constant rate of interest under the operation of:

    - simple interest
    - compound interest

2. Define the present value of a future payment.

3. Discount a single investment under the operation of simple (commercial) discount at a constant rate of discount.

4. Describe how a compound interest model can be used to represent the effect of investing a sum of money over a period.

(iii) Show how interest rates or discount rates may be expressed in terms of different time periods.

1. Derive the relationship between the rates of interest and discount over one effective period arithmetically and by general reasoning.

1 The idea of interest

Interest may be regarded as a reward paid by one person or organisation (the **borrower**) for the use of an asset, referred to as **capital**, belonging to another person or organisation (the **lender**).

When the capital and interest are expressed in monetary terms, capital is also referred to as **principal**. The total received by the lender after a period of time is called the **accumulated value**. The difference between the principal and the accumulated value is called the **interest**. Note that we are assuming here that no other payments are made or incurred (e.g. charges, expenses).

If there is some risk of default (i.e. loss of capital or non-payment of interest) a lender would expect to be paid a higher rate of interest than would otherwise be the case. Another factor that may influence the rate of interest on any transaction is an allowance for the possible depreciation or appreciation in the value of the currency in which the transaction is carried out. This factor is very important in times of high inflation.

We will now consider two types of interest within the framework of a savings account.

1.1 Simple interest

The essential feature of **simple interest** is that interest, once credited to an account, does not itself earn further interest.

Suppose an amount C is deposited in an account that pays simple interest at the rate of $i \times 100\%$ per annum. Then after n years the deposit will have accumulated to:

$$C(1 + ni) \tag{1.1}$$

When n is not an integer, interest is paid on a pro-rata basis.

1.2 Compound (effective) interest

The essential feature of **compound interest** is that interest itself earns interest.

Suppose an amount C is deposited in an account that pays compound interest at the rate of $i \times 100\%$ per annum. Then after n years the deposit will have accumulated to:

$$C(1+i)^n \tag{1.2}$$

1.3 Accumulation factors

For $t_1 \le t_2$ we define $A(t_1, t_2)$ to be the accumulation at time t_2 of an investment of 1 at time t_1.

The number $A(t_1, t_2)$ is often called an accumulation factor, since the accumulation at time t_2 of an investment of C at time t_1 is, by proportion:

$$CA(t_1, t_2) \tag{1.3}$$

$A(n)$ is often used as an abbreviation for the accumulation factor $A(0,n)$.

1.4 The principle of consistency

Now let $t_0 \le t_1 \le t_2$ and consider an investment of 1 at time t_0. The proceeds at time t_2 will be $A(t_0, t_2)$ if one invests at time t_0 for term $t_2 - t_0$, or $A(t_0, t_1)\, A(t_1, t_2)$ if one invests at time t_0 for term $t_1 - t_0$ and then, at time t_1, reinvests the proceeds for term $t_2 - t_1$. In a consistent market these proceeds should not depend on the course of action taken by the investor. Accordingly, we say that under the **principle of consistency:**

$$A(t_0, t_n) = A(t_0, t_1)\, A(t_1, t_2) \ldots A(t_{n-1}, t_n) \tag{1.4}$$

2 Present values

It follows by formula 1.2 that an investment of

$$C/(1+i)^n \tag{2.1}$$

at time 0 (the present time) will give C at time $n \geq 0$.

This is called the **discounted present value** (or, more briefly, the **present value**) of C due at time $n \geq 0$.

We now define the function

$$v = \frac{1}{1+i} \tag{2.2}$$

It follows by formulae 2.1 and 2.2 that the discounted present value of C due at time $n \geq 0$ is:

$$Cv^n \tag{2.3}$$

3 Discount rates

An alternative way of obtaining the discounted value of a payment is to use discount rates.

3.1 Simple discount

As has been seen with simple interest, the interest earned is not itself subject to further interest. The same is true of simple discount, which is defined below.

Suppose an amount C is due after n years and a rate of **simple discount** of d per annum applies. Then the sum of money required to be invested now to amount to C after n years (i.e. the present value of C) is

$$C(1-nd) \tag{3.1}$$

In normal commercial practice, d is usually encountered only for periods of less than a year. If a lender bases his short-term transactions on a simple rate of discount d then, in return for a repayment of X after a period t ($t < 1$) he will lend $X(1 - td)$ at the start of the period. In this situation, d is also known as a rate of **commercial discount**.

3.2 Compound (effective) discount

As has been seen with compound interest, the interest earned is subject to further interest. The same is true of compound discount, which is defined below.

Suppose an amount C is due after n years and a rate of **compound** (or **effective**) **discount** of d per annum applies. Then the sum of money required to be invested now to accumulate to C after n years (i.e. the present value of C) is

$$C(1-d)^n \tag{3.2}$$

3.3 Discount factors

In the same way that the accumulation factor $A(n)$ gives the accumulation at time n of an investment of 1 at time 0, we define $v(n)$ to be the present value of a payment of 1 due at time n. Hence:

$$v(n)=\frac{1}{A(n)} \tag{3.3}$$

4 Effective rates of interest and discount

Effective rates are compound rates that have interest paid *once* per unit time either at the end of the period (effective interest) or at the beginning of the period (effective discount). This distinguishes them from nominal rates where interest is paid more frequently than once per unit time.

We can demonstrate the equivalence of compound and effective rates by an alternative way of considering effective rates.

4.1 Effective rate of interest

An investor will lend an amount 1 at time 0 in return for a repayment of $(1+i)$ at time 1. Hence we can consider i to be the interest paid at the *end* of the year. Accordingly i is called the **rate of interest** (or the **effective rate of interest**) per unit time.

So denoting the effective rate of interest during the nth period by i_n, we have

$$i_n=\frac{A(n)-A(n-1)}{A(n-1)}$$

If i is the compound rate of interest, we have:

$$i_n = \frac{(1+i)^n - (1+i)^{n-1}}{(1+i)^{n-1}} = (1+i) - 1 = i$$

Since this is independent of n, we see that the effective rate of interest is identical to the compound rate of interest we met earlier

4.2 Effective rate of discount

We can think of compound discount as an investor lending an amount $(1-d)$ at time 0 in return for a repayment of 1 at time 1. The sum of $(1-d)$ may be considered as a loan of 1 (to be repaid after 1 unit of time) on which interest of amount d is payable **in advance**. Accordingly d is called the **rate of discount** (or the **effective rate of discount**) per unit time.

We can also show that the effective rate of discount is identical to the compound rate of discount we met earlier.

5 Equivalent rates

Two rates of interest and/or discount are **equivalent** if a given amount of principal invested for the same length of time produces the same accumulated value under each of the rates.

Comparing formulae (2.3) and (3.2), we see that:

$$v = 1 - d \tag{5.1}$$

And from (2.2) and (5.1) we obtain the rearrangements:

$$d = iv \tag{5.2}$$

and:

$$d = \frac{i}{1+i} \tag{5.3}$$

Recall that d is the interest paid at time 0 on a loan of 1, whereas i is the interest paid at time 1 on the same loan. If the rates are equivalent then if we discount i from time 1 to time 0 we will obtain d. This is the interpretation of equations (5.2) and (5.3).

E N D

UNIT 3 — INTEREST RATES

Syllabus objective (iii) Show how interest rates or discount rates may be expressed in terms of different time periods.

2. Derive the relationships between the rate of interest payable once per effective period and the rate of interest payable p times per time period and the force of interest.

3. Explain the difference between nominal and effective rates of interest and derive effective rates from nominal rates.

4. Calculate the equivalent annual rate of interest implied by the accumulation of a sum of money over a specified period where the force of interest is a function of time.

1 Nominal rates of interest and discount

Recall from Unit 2 that "effective" rates of interest and discount have interest paid once per measurement period, either at the end of the period or at the beginning of the period.

"Nominal" is used where interest is paid more (or less) frequently than once per measurement period.

1.1 Nominal rates of interest

We denote the **nominal rate of interest *payable p* times per period** by $i^{(p)}$. This is also referred to as the rate of interest ***convertible p*thly** or ***compounded p*thly**.

A nominal rate of interest per period, payable *p*thly, $i^{(p)}$, is defined to be a rate of interest of $i^{(p)}/p$ applied for each *p*th of a period. For example, a nominal rate of interest of 6% p.a. convertible quarterly means an interest rate of $6/4 = 1.5\%$ per quarter.

Hence, by definition, $i^{(p)}$ is equivalent to a *p*thly *effective* rate of interest of $i^{(p)}/p$.

Therefore the effective interest rate i is obtained from:

$$1+i = \left(1+\frac{i^{(p)}}{p}\right)^p \tag{3.1}$$

Note that $i^{(1)} = i$.

The treatment of problems involving nominal rates of interest (or discount) is almost always considerably simplified by an appropriate choice of the time unit.

By choosing the basic time unit to be the period corresponding to the frequency with which the nominal rate of interest is convertible, we can use $i^{(p)}/p$ as the effective rate of interest per unit time. For example, if we have a nominal rate of interest of 18% per annum convertible monthly, we should take one month as the unit of time and 1½% as the rate of interest per unit time.

1.2 Nominal rates of discount

We denote the **nominal rate of discount *payable p* times per period** by $d^{(p)}$. This is also referred to as the rate of discount ***convertible p*thly** or ***compounded p*thly**.

A nominal rate of discount per period payable *p*thly, $d^{(p)}$, is defined as a rate of discount of $d^{(p)}/p$ applied for each *p*th of a period.

Hence, by definition, $d^{(p)}$ is equivalent to a *p*thly *effective* rate of discount of $d^{(p)}/p$.

Therefore the effective discount rate d is obtained from:

$$1 - d = \left(1 - \frac{d^{(p)}}{p}\right)^p \tag{3.2}$$

Note that $d^{(1)} = d$.

2 The force of interest

2.1 Derivation from nominal interest convertible *p*thly

We assume that for each value of i there is number, δ, such that:

$$\lim_{p \to \infty} i^{(p)} = \delta$$

δ is the nominal rate of interest per unit time convertible continuously (or momently). This is also referred to as the rate continuously compounded. We call it the **force of interest.**

Euler's rule states that:

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$$

Applying this to the right-hand-side of (3.1) gives:

$$\lim_{p\to\infty}\left(1+\frac{i^{(p)}}{p}\right)^p = e^{i^{(\infty)}}$$

Hence:

$$1+i = e^{\delta} \qquad (3.3)$$

Since $v=(1+i)^{-1}$, we have:

$$v = e^{-\delta} \qquad (3.4)$$

From equation (3.4) we have:

$$v^t = (e^{-\delta})^t = e^{-\delta t}$$

Hence, the discount factor for a force of interest δ is:

$$v(n) = e^{-\delta n}$$

2.1 Derivation from nominal discount convertible *p*thly

It can also be shown that:

$$\lim_{p\to\infty} d^{(p)} = \delta$$

However, $d^{(p)}$ tends to this limit from below whereas $i^{(p)}$ tends to this limit from above.

Hence, we have:

$$d < d^{(2)} < d^{(3)} < \cdots < \delta < \cdots < i^{(3)} < i^{(2)} < i$$

3 Relationships between effective, nominal and force of interest

3.1 An alternative way of considering nominal interest convertible *p*thly

Recall that effective interest i can be thought of as interest paid at the *end* of the period. Hence, an investor lending an amount 1 at time 0 receives a repayment of $(1+i)$ at time 1.

Similarly, nominal interest convertible *p*thly can be thought of as the total interest per unit of time paid on a loan of amount 1 at time 0, where interest is paid in p equal instalments at the *end* of each *p*th subinterval (i.e. at times $1/p, 2/p, 3/p, \ldots, 1$).

Since $i^{(p)}$ is the *total* interest paid and each interest payment is of amount $i^{(p)}/p$ then the accumulated value at time 1 of the interest payments is:

$$\frac{i^{(p)}}{p}(1+i)^{(p-1)/p}+\frac{i^{(p)}}{p}(1+i)^{(p-2)/p}+\cdots+\frac{i^{(p)}}{p}=i$$

Hence:

$$i^{(p)}=p\left[(1+i)^{1/p}-1\right]$$

3.2 An alternative way of considering nominal discount convertible *p*thly

Recall that effective discount d can be thought of as interest paid at the *start* of the period. Hence, an investor lending an amount 1 at time 0 receives a repayment of 1 at time 1, but d is paid at the start so a sum of $(1-d)$ is lent at time 0.

Similarly, $d^{(p)}$ is the total amount of interest per unit of time payable in equal instalments at the *start* of each *p*th subinterval (*ie* at times $0, 1/p, 2/p, \ldots, (p-1)/p$). As a consequence the present value at time 0 of the interest payments is:

$$\frac{d^{(p)}}{p}+\frac{d^{(p)}}{p}(1-d)^{1/p}+\cdots+\frac{d^{(p)}}{p}(1-d)^{(p-1)/p}=d$$

Hence:

$$d^{(p)}=p\left[1-(1-d)^{1/p}\right]$$

3.3 An alternative way of considering force of interest

Now δ is the total amount of interest payable as a continuous payment stream, *ie* an amount δdt is paid over an infinitesimally small period dt at time t.

As a consequence the accumulated value at time 1 of these interest payments is:

$$\int_0^1 \delta(1+i)^{1-t}\,dt$$

which, by symmetry, is equal to:

$$\int_0^1 \delta(1+i)^{t}\,dt = i$$

Hence:

$$\delta = \ln(1+i) \qquad \text{or} \qquad e^{\delta} = 1+i$$

It is essential to appreciate that, at force of interest δ per unit time, the five series of payments illustrated in Figure 3.1.1 below all have the same value.

	0	$\frac{1}{p}$	$\frac{2}{p}$	$\frac{3}{p}$	...	$\frac{p-1}{p}$	1	time
(1)	d							
(2)	$\frac{d^{(p)}}{p}$	$\frac{d^{(p)}}{p}$	$\frac{d^{(p)}}{p}$	$\frac{d^{(p)}}{p}$	...	$\frac{d^{(p)}}{p}$		
(3)		$\frac{i^{(p)}}{p}$	$\frac{i^{(p)}}{p}$	$\frac{i^{(p)}}{p}$	...	$\frac{i^{(p)}}{p}$	$\frac{i^{(p)}}{p}$	equivalent payments
(4)							i	
(5)	———	———	———	———	δ	———	———	

Figure 3.1.1 Equivalent payments

4 Force of interest as a function of time

4.1 Formal definition

The force of interest is the instantaneous change in the fund value, expressed as an annualized percentage of the current fund value.

So the **force of interest** at time t is defined to be:

$$\delta(t) = \frac{V_t'}{V_t}$$

where V_t is the value of the fund at time t and V_t' is the derivative of V_t with respect to t.

Hence:

$$\delta(t) = \frac{d}{dt} \ln V_t$$

Integrating this from t_1 to t_2 gives:

$$\int_{t_1}^{t_2} \delta(t) dt = \left[\ln V_t\right]_{t_1}^{t_2} = \ln V_{t_2} - \ln V_{t_1} = \ln\left(\frac{V_{t_2}}{V_{t_1}}\right)$$

$$\Rightarrow \quad \frac{V_{t_2}}{V_{t_1}} = e^{\int_{t_1}^{t_2} \delta(t)\, dt}$$

Hence:

$$A(t_1, t_2) = e^{\int_{t_1}^{t_2} \delta(t)\, dt}$$

4.2 Relationship to constant force of interest

For the case when the force of interest is constant, δ, between time 0 and time n, we have:

$$A(0,n) = e^{\int_0^n \delta\, dt} = e^{\delta n}$$

Hence:

$$(1+i)^n = e^{\delta n}$$

Therefore:

$$(1+i) = e^{\delta}$$

as before.

4.3 Applications of force of interest

Although the force of interest is a theoretical measure it is the most fundamental measure of interest (as all other interest rates can be derived from it). However, since the majority of transactions involve discrete processes we tend to use other interest rates in practice.

It still remains a useful conceptual and analytical tool and can be used as an approximation to interest paid very frequently, *e.g.* daily.

E N D

UNIT 4 — REAL AND MONEY INTEREST RATES

Syllabus objective (iv) Demonstrate a knowledge and understanding of real and money interest rates.

1 Definition of real and money interest rates

Accumulating an investment of 1 for a period of time t from time 0 produces a new total accumulated value $A(0, t)$, say.

Typically the investment of 1 will be a sum of money, say £1 or $1 or 1 Euro.

In this case, if we are given the information on the initial investment of 1 in the specified currency, the period of the investment, and the cash amount of money accumulated, then the underlying interest rate is termed a **money rate of interest.**

More generally, given any series of monetary payments accumulated over a period, a money rate of interest is that rate which will have been earned so as to produce the total amount of **cash** in hand at the end of the period of accumulation.

In practice, most such accumulations will take place in economies subject to inflation, where a given sum of money in the future will have less purchasing power than at the present day. It is often useful, therefore, to reconsider what the accumulated value is worth allowing for the eroding effects of inflation.

Returning to the initial example above, suppose the accumulation took place in an economy subject to inflation so that the cash $A(0, t)$ is effectively worth only $A^*(0, t)$ after allowing for inflation, where $A^*(0, t) < A(0, t)$. In this case, the rate of interest at which the original sum of 1 would have to be accumulated to produce the sum A^* is lower than the money rate of interest.

The sum $A^*(0, t)$ is referred to as the real amount accumulated, and the underlying interest rate, reduced for the effects of inflation, is termed a **real rate of interest.**

More generally, given any series of monetary payments accumulated over a period, a real rate of interest is that rate which will have been earned so as to produce the total amount of **cash** in hand at the end of the period of accumulation **reduced for the effects of inflation.**

Unit 11 of this Subject will describe ways of calculating real rates of interest given the money rates of interest (and vice versa).

2 Deflationary conditions

The above descriptions assume that the inflation rate is positive. Where the inflation rate is negative, termed "deflation" the above theory still applies and $A^*(0, t) > A(0, t)$, giving rise to the conclusion that the real rate of interest in such circumstances would be higher than the money rate of interest.

As might be expected, where there is no inflation $A^*(0, t) = A(0, t)$ and the real and money rates of interest are the same.

3 Usefulness of real and money interest rates

We assume here that we have a positive inflation rate.

Which of the two rates of interest, real or money, is the more useful will depend on two main factors:

- the purpose to which the rate will be put
- whether the underlying data has or has not already been adjusted for inflation.

The purpose to which the rate will be put

Generally, where the actuary is performing calculations to determine how much should be invested to provide for future outgo, the first step will be to determine whether the future outgo is real or monetary in nature. The type of interest rate to be assumed would then be, respectively, a real or a monetary rate.

For example, first suppose that an actuary was asked to calculate the sum to be invested by a person aged 40 to provide a round-the-world cruise, when the person reaches 60, and where the person says the cruise costs £25,000.

Unless the person has, for some reason, already made an allowance for inflation in suggesting a figure of £25,000 then that amount is probably today's cost of the cruise. In this case, the actuary would be wise to assume (checking his understanding with the person) an inflation rate and this could be achieved by assuming a real rate of interest.

As an alternative example, suppose that a person has a mortgage of £50,000 to be paid off in twenty years' time. Here, the party which granted the mortgage would contractually be entitled to only £50,000 in twenty years' time. Accordingly, in working out how much should be invested to repay the outgo in this case, a money rate of interest would be assumed.

Whether the underlying data has or has not already been adjusted for inflation

In the first example above, we see that the data may already have been adjusted for inflation and in that case it would not be appropriate to allow for inflation again. A money rate would then be assumed.

More generally in actuarial work, the nature of the data provided must be understood before choosing the type and amount of assumptions to be made.

E N D

UNIT 5 — DISCOUNTING AND ACCUMULATING

Syllabus objective (v) Calculate the present value and the accumulated value of a stream of equal or unequal payments using specified rates of interest and the net present value at a real rate of interest, assuming a constant rate of inflation.

1. Discount and accumulate a sum of money or a series (possibly infinite) of cashflows to any point in time where:

   - the rate of interest or discount is constant
   - the rate of interest or discount varies with time but is not a continuous function of time
   - either or both the rate of cashflow and the force of interest are continuous functions of time

2. Calculate the present value and accumulated value of a series of equal or unequal payments made at regular intervals under the operation of specified rates of interest where the first payment is:

   - deferred for a period of time
   - not deferred

Real rates of interest are dealt with in Unit 11.

1 Present values of cashflows

In many compound interest problems one must find the discounted present value of cashflows due in the future. It is important to distinguish between (a) **discrete** and (b) **continuous** payments.

1.1 Discrete cashflows

The present value of the sums $c_{t_1}, c_{t_2}, ..., c_{t_n}$ due at times t_1, t_2, ..., t_n (where $0 \le t_1 < t_2 < ... < t_n$) is,

$$c_{t_1} v(t_1) + c_{t_2} v(t_2) + ... + c_{t_n} v(t_n) = \sum_{j=1}^{n} c_{t_j} v(t_j) \qquad (1.1.1)$$

If the number of payments is infinite, the present value is defined to be

$$\sum_{j=1}^{\infty} c_{t_j}\, v(t_j) \tag{1.1.2}$$

provided that this series converges. It usually will in practical problems.

1.2 Continuously payable cashflows (payment streams)

Suppose that $T > 0$ and that between times 0 and T an investor will be paid money continuously, the rate of payment at time t being £$\rho(t)$ per unit time. What is the present value of this cashflow?

In order to answer this question it is essential to understand what is meant by the **rate of payment** of the cashflow at time t. If $M(t)$ denotes the **total** payment made between time 0 and time t, then **by definition**,

$$\rho(t) = M'(t) \qquad \text{for all } t \tag{1.2.1}$$

Then, if $0 \le \alpha < \beta \le T$, the total payment received between time α and time β is

$$\begin{aligned} M(\beta) - M(\alpha) &= \int_\alpha^\beta M'(t)dt \\ &= \int_\alpha^\beta \rho(t)dt \end{aligned} \tag{1.2.2}$$

Thus the rate of payment at any time is simply the derivative of the **total** amount paid up to that time, and the total amount paid between any two times is the integral of the rate of payments over the appropriate time interval.

Between times t and $t + dt$ the total payment received is $M(t + dt) - M(t)$. If dt is very small this is approximately $M'(t)dt$ or $\rho(t)dt$. Theoretically, therefore, we may consider the present value of the money received between times t and $t + dt$ as $v(t)\rho(t)dt$. The present value of the entire cashflow is obtained by integration as

$$\int_0^T v(t)\rho(t)dt \tag{1.2.3}$$

If T is infinite we obtain, by a similar argument, the present value

$$\int_0^\infty v(t)\rho(t)dt \tag{1.2.4}$$

By combining the results for discrete and continuous cashflows, we obtain the formula

$$\Sigma c_t\, v(t) + \int_0^\infty v(t)\rho(t)dt \tag{1.2.5}$$

for the present value of a general cashflow (the summation being over those values of t for which c_t, the discrete cashflow at time t, is non-zero).

So far we have assumed that all payments, whether discrete or continuous, are positive. If one has a series of income payments (which may be regarded as positive) and a series of outgoings (which may be regarded as negative) their **net present value** is defined as the difference between the value of the positive cashflow and the value of the negative cashflow.

2 Valuing cashflows

Consider times t_1 and t_2, where t_2 is not necessarily greater than t_1. The **value at time t_1 of the sum C due at time t_2** is defined as:

(a) If $t_1 \geq t_2$, the accumulation of C from time t_2 until time t_1; or

(b) If $t_1 < t_2$, the discounted value at time t_1 of C due at time t_2.

In both cases the value at time t_1 of C due at time t_2 is

$$C \exp\left[-\int_{t_1}^{t_2} \delta(t)dt\right] \tag{2.1.1}$$

(Note the convention that, if $t_1 > t_2$, $\int_{t_1}^{t_2} \delta(t)dt = -\int_{t_2}^{t_1} \delta(t)dt$.)

Since

$$\int_{t_1}^{t_2} \delta(t)dt = \int_0^{t_2} \delta(t)dt - \int_0^{t_1} \delta(t)dt$$

it follows immediately from equation 2.1.1 that the value at time t_1 of C due at time t_2 is

$$C \frac{v(t_2)}{v(t_1)} \tag{2.1.2}$$

The value at a general time t_1 of a discrete cashflow of c_t at time t (for various values of t) and a continuous payment stream at rate $\rho(t)$ per time unit may now be found, by the methods given in section 1, as

$$\Sigma c_t \frac{v(t)}{v(t_1)} + \int_{-\infty}^{\infty} \rho(t) \frac{v(t)}{v(t_1)}\, dt \tag{2.1.3}$$

where the summation is over those values of t for which $c_t \neq 0$. We note that in the special case when $t_1 = 0$ (the present time), the value of the cashflow is

$$\Sigma c_t \; v(t) + \int_{-\infty}^{\infty} \rho(t) v(t) dt \qquad (2.1.4)$$

where the summation is over those values of t for which $c_t \neq 0$. This is a generalisation of formula 1.2.5 to cover the past as well as present or future payments. If there are incoming and outgoing payments, the corresponding **net value** may be defined, as in section 1, as the difference between the value of the **positive** and the **negative** cashflows. If all the payments are due at or after time t_1, their value at time t_1 may also be called their **discounted value**, and if they are due at or before time t_1, their value may be referred to as their **accumulation**. It follows that any value may be expressed as the sum of a discounted value and an accumulation. This fact is helpful in certain problems. Also, if $t_1 = 0$ and all the payments are due at or after the present time, their value may also be described as their **(discounted) present value**, as defined by formula 1.2.5.

It follows from formula 2.1.3 that the value at any time t_1 of a cashflow may be obtained from its value at another time t_2 by applying the factor $v(t_2)/v(t_1)$, i.e.

$$\begin{bmatrix} \text{Value at time } t_1 \\ \text{of cashflow} \end{bmatrix} = \begin{bmatrix} \text{Value at time } t_2 \\ \text{of cashflow} \end{bmatrix} \left[\frac{v(t_2)}{v(t_1)} \right] \qquad (2.1.5)$$

or

$$\begin{bmatrix} \text{Value at time } t_1 \\ \text{of cashflow} \end{bmatrix} \left[v(t_1) \right] = \begin{bmatrix} \text{Value at time } t_2 \\ \text{of cashflow} \end{bmatrix} \left[v(t_2) \right] \qquad (2.1.6)$$

Each side of equation 2.1.6 is the value of the cashflow at the present time (time 0).

In particular, by choosing time t_2 as the present time and letting $t_1 = t$, we obtain the result:

$$\begin{bmatrix} \text{Value at time } t \\ \text{of cashflow} \end{bmatrix} = \begin{bmatrix} \text{Value at the present} \\ \text{time of cashflow} \end{bmatrix} \left[\frac{1}{v(t)} \right] \qquad (2.1.7)$$

These results are useful in many practical examples. The time 0 and the unit of time may be chosen so as to simplify the calculations.

3 Interest income

Consider now an investor who wishes not to accumulate money but to receive an income while keeping his capital fixed at C. If the rate of interest is fixed at i per time unit, and if the investor wishes to receive income at the end of each time unit, it is clear that the income will be iC per time unit, payable in arrear, until such time as the capital is withdrawn.

However, if interest is paid continuously with force of interest $\delta(t)$ at time t then the income received between times t and $t+dt$ will be $C\delta(t)dt$. So the total interest income from time 0 to time T will be:

$$I(T) = \int_0^T C\delta(t)\, dt \tag{3.1.1}$$

If the investor withdraws the capital at time T, the present values of the income and capital at time 0 are,

$$C \int_0^T \delta(t)v(t)dt \tag{3.1.2}$$

and

$$Cv(T) \tag{3.1.3}$$

respectively. Since

$$\begin{aligned}
\int_0^T \delta(t)v(t)dt &= \int_0^T \delta(t) \exp\left[-\int_0^t \delta(s)ds\right] dt \\
&= \left[-\exp\left(-\int_0^t \delta(s)ds\right)\right]_0^T \\
&= 1 - v(T)
\end{aligned}$$

we obtain

$$C = C \int_0^T \delta(t)v(t)dt + Cv(T) \tag{3.1.4}$$

as one would expect by general reasoning.

So far we have described the difference between money returned at the end of the term and the cash originally invested as "interest". In practice, however, this quantity may be divided into **interest income** and **capital gains**, the term **capital loss** being used for a negative capital gain.

E N D

UNIT 6 — COMPOUND INTEREST FUNCTIONS

Syllabus objective (vi) Define and use the more important compound interest functions including annuities certain.

1. Derive formulae in terms of i, v, n, d, δ, $i^{(p)}$ and $d^{(p)}$ for $a_{\overline{n}|}$, $s_{\overline{n}|}$, $a^{(p)}_{\overline{n}|}$, $s^{(p)}_{\overline{n}|}$, $\ddot{a}_{\overline{n}|}$, $\ddot{s}_{\overline{n}|}$, $\ddot{a}^{(p)}_{\overline{n}|}$, $\ddot{s}^{(p)}_{\overline{n}|}$, $\bar{a}_{\overline{n}|}$ and $\bar{s}_{\overline{n}|}$.

2. Derive formulae in terms of i, v n, d, δ, $i^{(p)}$ and $d^{(p)}$) for $_{m|}a_{\overline{n}|}$, $_{m|}a^{(p)}_{\overline{n}|}$, $_{m|}\ddot{a}_{\overline{n}|}$, $_{m|}\ddot{a}^{(p)}_{\overline{n}|}$ and $_{m|}\bar{a}_{\overline{n}|}$.

3. Derive formulae in terms of i, v, n, δ, $a_{\overline{n}|}$ and $\ddot{a}_{\overline{n}|}$ for $(Ia)_{\overline{n}|}$, $(I\ddot{a})_{\overline{n}|}$, $(I\bar{a})_{\overline{n}|}$, $(\bar{I}\bar{a})_{\overline{n}|}$ and the respective deferred annuities.

1 Annuities certain: present values and accumulations

1.1 Annual payments

Consider a series of n payments, each of amount 1, to be made at time intervals of one unit, the first payment being made at time $t + 1$.

	1	1	1	...	1	1	payment
t	$t + 1$	$t + 2$	$t + 3$	...	$t + n - 1$	$t + n$	time

Such a sequence of payments is illustrated in the diagram above, in which the rth payment is made at time $t + r$.

The value of this series of payments **one unit of time before the first payment is made** is denoted by $a_{\overline{n}|}$. Clearly, if $i = 0$, then $a_{\overline{n}|} = n$; otherwise

$$a_{\overline{n}|} = v + v^2 + v^3 + ... + v^n$$

$$= \frac{v(1-v^n)}{1-v}$$

$$= \frac{1-v^n}{v^{-1}-1}$$

$$= \frac{1-v^n}{i} \qquad (1.1.1)$$

If $n = 0$, $a_{\overline{n}|}$ is defined to be zero.

Thus $a_{\overline{n}|}$ is the value at the start of any period of length n of a series of n payments, each of amount 1, to be made **in arrear** at unit time intervals over the period. It is common to refer to such a series of payments, made in arrear, as an **immediate annuity certain** and to call $a_{\overline{n}|}$ the present value of the immediate annuity certain. When there is no possibility of confusion with a life annuity (i.e. a series of payments dependent on the survival of one or more human lives), the term **annuity** may be used as an alternative to annuity certain.

The value of this series of payments **at the time the first payment is made** is denoted by $\ddot{a}_{\overline{n}|}$. If $i = 0$, then $\ddot{a}_{\overline{n}|} = n$; otherwise

$$\ddot{a}_{\overline{n}|} = 1 + v + v^2 + ... + v^{n-1}$$

$$= \frac{1-v^n}{1-v}$$

$$= \frac{1-v^n}{d} \qquad (1.1.2)$$

Thus $\ddot{a}_{\overline{n}|}$ is the value at the start of any given period of length n of a series of n payments, each of amount 1, to be made **in advance** at unit time intervals over the period. It is common to refer to such a series of payments, made in advance, as an **annuity due** and to call $\ddot{a}_{\overline{n}|}$ the present value of the annuity due.

It follows directly from the above definitions that

$$\left.\begin{aligned} \ddot{a}_{\overline{n}|} &= (1+i)a_{\overline{n}|} \\ \text{and that, for } n \ge 2, \qquad \ddot{a}_{\overline{n}|} &= 1 + a_{\overline{n-1}|} \end{aligned}\right\} \quad (1.1.3)$$

The value of the series of payments **at the time the last payment is made** is denoted by $s_{\overline{n}|}$. The value **one unit of time after the last payment is made** is denoted by $\ddot{s}_{\overline{n}|}$. If $i = 0$ then $s_{\overline{n}|} = \ddot{s}_{\overline{n}|} = n$; otherwise

$$\begin{aligned} s_{\overline{n}|} &= (1+i)^{n-1} + (1+i)^{n-2} + (1+i)^{n-3} + \dots + 1 \\ &= (1+i)^n a_{\overline{n}|} \\ &= \frac{(1+i)^n - 1}{i} \end{aligned} \qquad (1.1.4)$$

and

$$\begin{aligned} \ddot{s}_{\overline{n}|} &= (1+i)^n + (1+i)^{n-1} + (1+i)^{n-2} + \dots + (1+i) \\ &= (1+i)^n \ddot{a}_{\overline{n}|} \\ &= \frac{(1+i)^n - 1}{d} \end{aligned} \qquad (1.1.5)$$

Thus $s_{\overline{n}|}$ and $\ddot{s}_{\overline{n}|}$ are the values at the end of any period of length n of a series of n payments, each of amount 1, made at unit time intervals over the period, where the payments are made in arrear and in advance respectively. Sometimes $s_{\overline{n}|}$ and $\ddot{s}_{\overline{n}|}$ are called the **accumulation** (or the **accumulated amount**) of an immediate annuity and an annuity due respectively. When $n = 0$, $s_{\overline{n}|}$ and $\ddot{s}_{\overline{n}|}$ are defined to be zero. It is an immediate consequence of the above definition that

$$\left.\begin{aligned} \ddot{s}_{\overline{n}|} &= (1+i)s_{\overline{n}|} \\ \text{and that} \qquad s_{\overline{n+1}|} &= 1 + \ddot{s}_{\overline{n}|} \\ \text{or} \qquad \ddot{s}_{\overline{n}|} &= s_{\overline{n+1}|} - 1 \end{aligned}\right\} \quad (1.1.6)$$

1.2 Continuously payable annuities

Let n be a non-negative number. The value at time 0 of an annuity payable continuously between time 0 and time n, where the rate of payment per unit time is constant and equal to 1, is denoted by $\bar{a}_{\overline{n}|}$. Clearly

$$\bar{a}_{\overline{n}|} = \int_0^n e^{-\delta t}\,dt$$

$$= \frac{1-e^{-\delta n}}{\delta}$$

$$= \frac{1-v^n}{\delta} \qquad \text{(if } \delta \neq 0\text{)} \qquad (1.2.1)$$

Note that $\bar{a}_{\overline{n}|}$ is defined even for non-integral values of n. If $\delta = 0$ (or, equivalently, $i = 0$), $\bar{a}_{\overline{n}|}$ is of course equal to n.

Since equation 1.2.1 may be written as

$$\bar{a}_{\overline{n}|} = \frac{i}{\delta}\left(\frac{1-v^n}{i}\right)$$

it follows immediately that, if n is an integer,

$$\bar{a}_{\overline{n}|} = \frac{i}{\delta}\,a_{\overline{n}|} \qquad \text{(if } \delta \neq 0\text{)} \qquad (1.2.2)$$

The **accumulated amount** of such an annuity at the time the payments cease is denoted by $\bar{s}_{\overline{n}|}$.

By definition, therefore,

$$\bar{s}_{\overline{n}|} = \int_0^n e^{\delta(n-t)}\,dt.$$

Hence

$$\bar{s}_{\overline{n}|} = (1+i)^n \,.\, \bar{a}_{\overline{n}|}$$

If the rate of interest is non-zero,

$$\bar{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{\delta}$$

$$= \frac{i}{\delta} \cdot s_{\overline{n}|}$$

1.3 Annuities payable *p*thly

If p and n are positive integers, the notation $a_{\overline{n}|}^{(p)}$ is used to denote the value at time 0 of a level annuity payable $p$thly in arrear at the rate of 1 per unit time over the time interval $[0, n]$. For this annuity the payments are made at times $1/p$, $2/p$, $3/p$, ..., n and the amount of each payment is $1/p$.

By definition, a series of p payments, each of amount $i^{(p)}/p$ in arrear at $p$thly subintervals over any unit time interval, has the same value as a single payment of amount i at the end of the interval. By proportion, p payments, each of amount $1/p$ in arrear at $p$thly subintervals over any unit time interval, have the same value as a single payment of amount $i/i^{(p)}$ at the end of the interval.

Consider now that annuity for which the present value is $a_{\overline{n}|}^{(p)}$. The remarks in the preceding paragraph show that the p payments after time $r - 1$ and not later than time r have the same value as a single payment of amount $i/i^{(p)}$ at time r. This is true for $r = 1, 2, ..., n$, so the annuity has the same value as a series of n payments, each of amount $i/i^{(p)}$, at times $1, 2, ..., n$. This means that

$$a_{\overline{n}|}^{(p)} = \frac{i}{i^{(p)}} a_{\overline{n}|} \tag{1.3.1}$$

An alternative approach, from first principles, is to write

$$a_{\overline{n}|}^{(p)} = \sum_{t=1}^{np} \frac{1}{p} v^{t/p}$$

$$= \frac{1}{p} \frac{v^{1/p}(1-v^n)}{1-v^{1/p}}$$

$$= \frac{1-v^n}{p[(1+i)^{1/p} - 1]}$$

$$= \frac{1 - v^n}{i^{(p)}} \tag{1.3.2}$$

which confirms equation 1.3.1.

Likewise we define $\ddot{a}^{(p)}_{\overline{n}|}$ to be the present value of a level annuity due payable *p*thly at the rate of 1 per unit time over the time interval [0, *n*]. (The annuity payments, each of amount $1/p$, are made at times 0, $1/p$, $2/p$, ..., $n - (1/p)$.)

By definition, a series of p payments, each of amount $d^{(p)}/p$, in advance at *p*thly subintervals over any unit time interval has the same value as a single payment of amount i at **end** of the interval. Hence, by proportion, p payments, each of amount $1/p$ in advance at *p*thly subintervals, have the same value as a single payment of amount $i/d^{(p)}$ at the **end** of the interval. This means (by an identical argument to that above) that

$$\ddot{a}^{(p)}_{\overline{n}|} = \frac{i}{d^{(p)}} a_{\overline{n}|} \tag{1.3.3}$$

Alternatively, from first principles, we may write

$$\ddot{a}^{(p)}_{\overline{n}|} = \sum_{t=1}^{np} \frac{1}{p} v^{(t-1)/p}$$

$$= \frac{1 - v^n}{d^{(p)}} \tag{1.3.4}$$

(on simplification), which confirms equation 1.3.3. Note that

$$a^{(p)}_{\overline{n}|} = v^{1/p} \ \ddot{a}^{(p)}_{\overline{n}|} \tag{1.3.5}$$

each expression being equal to $\dfrac{(1 - v^n)}{i^{(p)}}$.

Note that, since

$$\lim_{p \to \infty} i^{(p)} = \lim_{p \to \infty} d^{(p)} = \delta$$

it follows immediately from equation 1.3.2 and 1.3.4 that

$$\lim_{p \to \infty} a^{(p)}_{\overline{n}|} = \lim_{p \to \infty} \ddot{a}^{(p)}_{\overline{n}|} = \bar{a}_{\overline{n}|}$$

Similarly, we define $s_{\overline{n}|}^{(p)}$ and $\ddot{s}_{\overline{n}|}^{(p)}$ to be the accumulated amounts of the corresponding *p*thly immediate annuity and annuity due respectively. Thus

$$s_{\overline{n}|}^{(p)} = (1+i)^n \, a_{\overline{n}|}^{(p)}$$

$$= (1+i)^n \, \frac{i}{i^{(p)}} \, a_{\overline{n}|} \qquad \text{(by 1.3.1)}$$

$$= \frac{i}{i^{(p)}} \, s_{\overline{n}|} \qquad (1.3.6)$$

Also

$$\ddot{s}_{\overline{n}|}^{(p)} = (1+i)^n \, \ddot{a}_{\overline{n}|}^{(p)}$$

$$= (1+i)^n \, \frac{i}{d^{(p)}} \, a_{\overline{n}|} \qquad \text{(by 1.3.3)}$$

$$= \frac{i}{d^{(p)}} \, s_{\overline{n}|} \qquad (1.3.7)$$

The above proportional arguments may be applied to other varying series of payments. Consider, for example, an annuity payable annually in arrear for n years, the payment in the tth year being x_t. The present value of this annuity is obviously

$$a = \sum_{t=1}^{n} x_t \, v^t \qquad (1.3.8)$$

Consider now a second annuity, also payable for n years with the payment in the tth year, again of amount x_t, being made in p equal instalments in arrear over that year. If $a^{(p)}$ denotes the present value of this second annuity, by replacing the p payments for year t (each of amount x_t/p) by a single equivalent payment at the end of the year of amount $x_t[i/i^{(p)}]$, we immediately obtain

$$a^{(p)} = \frac{i}{i^{(p)}} \, a$$

where a is given by equation 1.3.8 above.

1.4 Annuities payable *p*thly where *p* < 1

In section 1.3 the symbol $a_{\overline{n}|}^{(p)}$ was introduced. Intuitively, with this notation one considers *p* to be an integer greater than 1 and assumes that the product *n.p* is also an integer. (This, of course, will be true when n itself is an integer, but one might for example, have $p = 4$ and $n = 5.75$ so that $n.p = 23$.) Then $a_{\overline{n}|}^{(p)}$ denotes the value at time 0 of *n.p* payments, each of amount $1/p$, at times $1/p, 2/p, ..., (np)/p$.

From a theoretical viewpoint it is perhaps worth noting that when *p* is the **reciprocal** of an integer and *n.p* is also an integer (e.g. when $p = 0.25$ and $n = 28$), $a_{\overline{n}|}^{(p)}$ still gives the value at time 0 of *n.p* payments, each of amount $1/p$, at times $1/p, 2/p, ..., (np)/p$.

For example, the value at time 0 of a series of seven payments, each of amount 4, at times 4, 8, 12 ,..., 28 may be denoted by $a_{\overline{28}|}^{(0.25)}$.

It follows that this value equals.

$$\frac{1-v^{28}}{(0.25)\,.[(1+i)^4-1]}$$

This last expression may be written in the form

$$\left[\frac{4}{\dfrac{(1+i)^4-1}{i}}\right].\frac{1-v^{28}}{i} = \frac{4}{s_{\overline{4}|}}.a_{\overline{28}|}$$

1.5 Non-integer values of *n*

Let *p* be a positive integer. Until now the symbol $a_{\overline{n}|}^{(p)}$ has been defined only when *n* is a positive integer. For certain non-integral values of *n* the symbol $a_{\overline{n}|}^{(p)}$ has an intuitively obvious interpretation. For example, it is not clear what meaning, if any, may be given to $a_{\overline{23.5}|}$, but the symbol $a_{\overline{23.5}|}^{(4)}$ ought to represent the present value of an immediate annuity of 1 per annum payable quarterly in arrear for 23.5 years (i.e. a total of 94 quarterly payments, each of amount 0.25). On the other hand, $a_{\overline{23.25}|}^{(2)}$ has no obvious meaning.

Suppose that n is an integer multiple of $1/p$, say $n = r/p$, where r is an integer. In this case we define $a_{\overline{n|}}^{(p)}$ to be the value at time 0 of a series of r payments, each of amount $1/p$, at times $1/p, 2/p, 3/p, \ldots, r/p = n$. If $i = 0$, then clearly $a_{\overline{n|}}^{(p)} = n$. If $i \neq 0$, then

$$a_{\overline{n|}}^{(p)} = \frac{1}{p}\left(v^{1/p} + v^{2/p} + v^{3/p} + \ldots + v^{r/p}\right) \quad (1.5.1)$$

$$= \frac{1}{p} v^{1/p} \left(\frac{1-v^{r/p}}{1-v^{1/p}}\right)$$

$$= \frac{1}{p}\left[\frac{1-v^{r/p}}{(1+i)^{1/p}-1}\right]$$

Thus

$$a_{\overline{n|}}^{(p)} = \begin{cases} \dfrac{1-v^n}{i^{(p)}} & \text{if } i \neq 0 \\ \\ n & \text{if } i = 0 \end{cases} \quad (1.5.2)$$

Note that, by working in terms of a new time unit equal to $1/p$ times the original time unit and with the equivalent effective interest rate of $i^{(p)}/p$ per new time unit, we see that

$$a_{\overline{n|}}^{(p)} \text{ at rate } i = \frac{1}{p} a_{\overline{np|}} \text{ at rate } i^{(p)}/p \quad (1.5.3)$$

This formula is useful when $i^{(p)}/p$ is a tabulated rate of interest. Note that the definition of $a_{\overline{n|}}^{(p)}$ given by equation 1.5.2 is mathematically meaningful for **all** non-negative values of n. For our present purpose, therefore, it is convenient to adopt equation 1.5.2 as a definition of $a_{\overline{n|}}^{(p)}$ for all n. If n is not an integer multiple of $\frac{1}{p}$, there is no universally recognised definition of $a_{\overline{n|}}^{(p)}$. For example, if $n = n_1 + f$, where n_1 is an integer multiple of $1/p$ and $0 < f < 1/p$, some writers define $a_{\overline{n|}}^{(p)}$ as

$$a_{\overline{n_1|}}^{(p)} + fv^n .$$

With this alternative definition

$$a^{(2)}_{\overline{23.75}|} = a^{(2)}_{\overline{23.5}|} + \tfrac{1}{4}v^{23.75}$$

which is the present value of an annuity of 1 per annum, payable half-yearly in arrear for 23.5 years, together with a final payment of 0.25 after 23.75 years. Note that this is **not** equal to the value obtained from definition 1.5.2.

If $i \neq 0$, we define for all non-negative n

$$\left.\begin{aligned}
\ddot{a}^{(p)}_{\overline{n}|} &= (1+i)^{1/p}\, a^{(p)}_{\overline{n}|} = \frac{1-v^n}{d^{(p)}} \\
s^{(p)}_{\overline{n}|} &= (1+i)^n\, a^{(p)}_{\overline{n}|} = \frac{(1+i)^n - 1}{i^{(p)}} \\
\ddot{s}^{(p)}_{\overline{n}|} &= (1+i)^n\, \ddot{a}^{(p)}_{\overline{n}|} = \frac{(1+i)^n - 1}{d^{(p)}}
\end{aligned}\right\} \quad (1.5.4)$$

If $i = 0$, each of these last three functions is defined to equal n.

Whenever n is an integer multiple of $1/p$, say $n = r/p$, then $\ddot{a}^{(p)}_{\overline{n}|}$, $s^{(p)}_{\overline{n}|}$, $\ddot{s}^{(p)}_{\overline{n}|}$ are values at different times of an annuity certain of r payments, each of amount $1/p$, at intervals of $1/p$ time unit.

As before, we use the simpler notations $a_{\overline{n}|}$, $\ddot{a}_{\overline{n}|}$, $s_{\overline{n}|}$ and $\ddot{s}_{\overline{n}|}$ to denote $a^{(1)}_{\overline{n}|}$, $\ddot{a}^{(1)}_{\overline{n}|}$, $s^{(1)}_{\overline{n}|}$, and $\ddot{s}^{(1)}_{\overline{n}|}$ respectively, thus extending the definition of $a_{\overline{n}|}$ etc. to all non-negative values of n. It is a trivial consequence of our definitions that the formulae

$$\left.\begin{aligned}
a^{(p)}_{\overline{n}|} &= \frac{i}{i^{(p)}}\, a_{\overline{n}|} \\
\ddot{a}^{(p)}_{\overline{n}|} &= \frac{i}{d^{(p)}}\, a_{\overline{n}|} \\
s^{(p)}_{\overline{n}|} &= \frac{i}{i^{(p)}}\, s_{\overline{n}|} \\
\ddot{s}^{(p)}_{\overline{n}|} &= \frac{i}{d^{(p)}}\, s_{\overline{n}|}
\end{aligned}\right\} \quad (1.5.5)$$

(valid when $i \neq 0$) now hold for **all** values of n.

1.6 Perpetuities

We can also consider an annuity that is payable forever. This is called a **perpetuity**. For example, consider an equity that pays a dividend of £10 at the end of each year. Equities are covered in more detail in Unit 10 Section 3. An investor who purchases the equity pays an amount equal to the present value of the dividends. The present value of the dividends is:

$$10v + 10v^2 + 10v^3 + \cdots$$

This can be summed using the formula for an infinite geometric progression:

$$10v + 10v^2 + 10v^3 + \cdots = \frac{10v}{1-v} = \frac{10}{i}$$

Recall the formula for the present value of an annuity of £10 p.a. that continues for n years:

$$10a_{\overline{n}|} = \frac{10(1-v^n)}{i}$$

We have let $n \to \infty$ in this expression in order to arrive at the formula $\frac{10}{i}$.

Note that this formula only holds when i is positive.

In general:

Perpetuity

The present value of payments of 1 p.a. payable at the end of each year forever is $\frac{1}{i}$. This present value is written as $a_{\overline{\infty}|}$, i.e. $a_{\overline{\infty}|} = \frac{1}{i}$.

The present value of payments of 1 p.a. payable at the start of each year forever is $\frac{1}{d}$. This present value is written as $\ddot{a}_{\overline{\infty}|}$, i.e. $\ddot{a}_{\overline{\infty}|} = \frac{1}{d}$.

Perpetuities payable pthly

The present value of payments of 1 p.a. payable in instalments of $\frac{1}{p}$ at the end of each $p$thly time period forever is:

$$a_{\overline{\infty}|}^{(p)} = \frac{1}{i^{(p)}}$$

The present value of payments of 1 p.a. payable in instalments of $\frac{1}{p}$ at the start of each $p$thly time period forever is:

$$\ddot{a}_{\overline{\infty}|}^{(p)} = \frac{1}{d^{(p)}}$$

2 Deferred annuities

2.1 Annual payments

Suppose that m and n are non-negative integers. The value at time 0 of a series of n payments, each of amount 1, due at times $(m + 1)$, $(m + 2)$, ..., $(m + n)$ is denoted by ${}_{m|}a_{\overline{n}|}$ (see the figure below).

			1	1	...	1	payment
0	1	... m	$m + 1$	$m + 2$	...	$m + n$	time

Such a series of payments may be considered as an immediate annuity, deferred for m time units. When $n > 0$,

$$\begin{aligned} {}_{m|}a_{\overline{n}|} &= v^{m+1} + v^{m+2} + v^{m+3} + \ldots + v^{m+n} \qquad (2.1.1) \\ &= (v + v^2 + v^3 + \ldots + v^{m+n}) - (v + v^2 + v^3 + \ldots + v^m) \\ &= v^m(v + v^2 + v^3 + \ldots + v^n) \end{aligned}$$

The last two equations show that

$$ {}_{m|}a_{\overline{n|}} = a_{\overline{m+n|}} - a_{\overline{m|}} \tag{2.1.2}$$

$$ = v^m \, a_{\overline{n|}} \tag{2.1.3}$$

Either of these two equations may be used to determine the value of a deferred immediate annuity. Together they imply that

$$ a_{\overline{m+n|}} = a_{\overline{m|}} + v^m \, a_{\overline{n|}} \tag{2.1.4}$$

We may define the corresponding deferred annuity due as

$$ {}_{m|}\ddot{a}_{\overline{n|}} = v^m \, \ddot{a}_{\overline{n|}} \tag{2.1.5}$$

2.2 Continuously payable annuities

If m is a non-negative number, we use the symbol ${}_{m|}\bar{a}_{\overline{n|}}$ to denote the present value of a continuously payable annuity of 1 per unit for n time units, deferred for m time units. Thus

$$ {}_{m|}\bar{a}_{\overline{n|}} = \int_m^{m+n} e^{-\delta t}\, dt$$

$$ = e^{-\delta m} \int_0^n e^{-\delta s}\, ds$$

$$ = \int_0^{m+n} e^{-\delta t}\, dt - \int_0^m e^{-\delta t}\, dt$$

Hence

$$ {}_{m|}\bar{a}_{\overline{n|}} = \bar{a}_{\overline{m+n|}} - \bar{a}_{\overline{m|}} \tag{2.2.1}$$

$$ = v^m \, \bar{a}_{\overline{n|}} \tag{2.2.2}$$

2.3 Annuities payable *p*thly

The present values of an immediate annuity and an annuity due, payable *p*thly at the rate of 1 per unit time for n time units and deferred for m time units, are denoted by

$$\left.\begin{aligned} {}_{m|}a_{\overline{n}|}^{(p)} &= v^m a_{\overline{n}|}^{(p)} \\ \text{and}\quad {}_{m|}\ddot{a}_{\overline{n}|}^{(p)} &= v^m \ddot{a}_{\overline{n}|}^{(p)} \end{aligned}\right\} \quad (2.3.1)$$

respectively.

2.4 Non-integer values of *n*

We may also extend the definitions of ${}_{m|}a_{\overline{n}|}^{(p)}$ and ${}_{m|}\ddot{a}_{\overline{n}|}^{(p)}$ to all values of n by the formulae

$$\left.\begin{aligned} {}_{m|}a_{\overline{n}|}^{(p)} &= v^m a_{\overline{n}|}^{(p)} \\ {}_{m|}\ddot{a}_{\overline{n}|}^{(p)} &= v^m \ddot{a}_{\overline{n}|}^{(p)} \end{aligned}\right\} \quad (2.4.1)$$

and so

$$\left.\begin{aligned} {}_{m|}a_{\overline{n}|}^{(p)} &= a_{\overline{n+m}|}^{(p)} - a_{\overline{m}|}^{(p)} \\ {}_{m|}\ddot{a}_{\overline{n}|}^{(p)} &= \ddot{a}_{\overline{n+m}|}^{(p)} - \ddot{a}_{\overline{m}|}^{(p)} \end{aligned}\right\} \quad (2.4.2)$$

3 Varying annuities

3.1 Annual payments

For an annuity in which the payments are not all of an equal amount it is a simple matter to find the present (or accumulated) value from first principles. Thus, for example, the present value of such an annuity may always be evaluated as

$$\sum_{i=1}^{n} X_i v^{t_i}$$

where the *i*th payment, of amount X_i, is made at time t_i.

In the particular case when $X_i = t_i = i$ the annuity is known as an **increasing annuity** and its present value is denoted by $(Ia)_{\overline{n}|}$. Thus

$$(Ia)_{\overline{n}|} = v + 2v^2 + 3v^3 + \ldots + nv^n \qquad (3.1.1)$$

Hence

$$(1+i)(Ia)_{\overline{n}|} = 1 + 2v + 3v^2 + \ldots + nv^{n-1}$$

By subtraction, we obtain

$$\begin{aligned} i(Ia)_{\overline{n}|} &= 1 + v + v^2 + \ldots + v^{n-1} - nv^n \\ &= \ddot{a}_{\overline{n}|} - nv^n \end{aligned}$$

so

$$(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i} \qquad (3.1.2)$$

The present value of any annuity payable in arrear for n time units for which the amounts of successive payments form an arithmetic progression can be expressed in terms of $a_{\overline{n}|}$ and $(Ia)_{\overline{n}|}$. If the first payment of such an annuity is P and the second payment is $(P + Q)$, the tth payment is $(P - Q) + Qt$, then the present value of the annuity is

$$(P - Q)\,a_{\overline{n}|} + Q(Ia)_{\overline{n}|} \qquad (3.1.3)$$

Alternatively, the present value of the annuity can be derived from first principles.

The notation $(I\ddot{a})_{\overline{n}|}$ is used to denote the present value of an increasing annuity due payable for n time units, the tth payment (of amount t) being made at time $t - 1$. Thus

$$\begin{aligned} (I\ddot{a})_{\overline{n}|} &= 1 + 2v + 3v^2 + \ldots + nv^{n-1} \\ &= (1+i)(Ia)_{\overline{n}|} \qquad (3.1.4) \\ &= 1 + a_{\overline{n-1}|} + (Ia)_{\overline{n-1}|} \qquad (3.1.5) \end{aligned}$$

3.2 Continuously payable annuities

For increasing annuities which are payable continuously it is important to distinguish between an annuity which has a constant rate of payment r (per unit time) throughout the rth period and an annuity which has a rate of payment t at time t. For the former the rate of payment is a step function taking the discrete values 1, 2, ... For the latter the rate of payment itself increases continuously. If the annuities are payable for n time units, their present values are denoted by $(I\bar{a})_{\overline{n}|}$ and $(\bar{I}\bar{a})_{\overline{n}|}$ respectively.

Clearly

$$(I\bar{a})_{\overline{n}|} = \sum_{r=1}^{n} \left(\int_{r-1}^{r} r\, v^t\, dt\right)$$

and

$$(\bar{I}\bar{a})_{\overline{n}|} = \int_0^n t\, v^t\, dt$$

and it can be shown that

$$(I\bar{a})_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{\delta} \qquad (3.2.1)$$

and

$$(\bar{I}\bar{a})_{\overline{n}|} = \frac{\bar{a}_{\overline{n}|} - nv^n}{\delta} \qquad (3.2.2)$$

The present values of deferred increasing annuities are defined in the obvious manner: for example,

$$_{m|}(Ia)_{\overline{n}|} = v^m (Ia)_{\overline{n}|}$$

E N D

UNIT 7 — EQUATIONS OF VALUE

Syllabus objective (vii) Define an equation of value.

1. Define an equation of value, where payment or receipt is certain.

2. Describe how an equation of value can be adjusted to allow for uncertain receipts or payments.

3. Understand the two conditions required for there to be an exact solution to an equation of value.

1 The equation of value and the yield on a transaction

Consider a transaction that provides that, in return for outlays of amount a_{t_1}, a_{t_2}, .. , a_{t_n} at time t_1, t_2, ... t_n, an investor will receive payments of b_{t_1}, b_{t_2}, ..., b_{t_n} at these times respectively. (In most situations only **one** of a_{t_r} and b_{t_r} will be non-zero.) At what force or rate of interest does the series of outlays have the same value as the series of receipts? At force of interest δ the two series are of equal value if and only if

$$\sum_{r=1}^{n} a_{t_r} e^{-\delta t_r} = \sum_{r=1}^{n} b_{t_r} e^{-\delta t_r} \qquad (1.1.1)$$

This equation may be written as

$$\sum_{r=1}^{n} c_{t_r} e^{-\delta t_r} = 0 \qquad (1.1.2)$$

where

$$c_{t_r} = b_{t_r} - a_{t_r}$$

is the amount of the **net cashflow** at time t_r. (We adopt the convention that a negative cashflow corresponds to a payment **by** the investor and a positive cashflow represents a payment **to** the investor.)

Equation 1.1.2, which expresses algebraically the condition that, at force of interest δ, the total value of the net cashflows is 0, is called the **equation of value** for the force of interest implied by the transaction. If we let $e^{\delta} = 1 + i$, the equation may be written as

$$\sum_{r=1}^{n} c_{t_r} (1+i)^{-t_r} = 0 \qquad (1.1.3)$$

The latter form is known as the equation of value for the rate of interest or the **yield equation**. Alternatively, the equation may be written as

$$\sum_{r=1}^{n} c_{t_r} v^{t_r} = 0$$

In relation to continuous payment streams, if we let $\rho_1(t)$ and $\rho_2(t)$ be the rates of paying and receiving money at time t respectively, we call $\rho(t) = \rho_2(t) - \rho_1(t)$ the **net rate of cashflow** at time t. The equation of value (corresponding to equation 1.1.2) for the force of interest is

$$\int_0^\infty \rho(t)\, e^{-\delta t}\, dt = 0 \tag{1.1.4}$$

When both discrete and continuous cashflows are present, the equation of value is

$$\sum_{r=1}^{n} c_{t_r} e^{-\delta t_r} + \int_0^\infty \rho(t)\, e^{-\delta t}\, dt = 0 \tag{1.1.5}$$

and the equivalent yield equation is

$$\sum_{r=1}^{n} c_{t_r} (1+i)^{-t_r} + \int_0^\infty \rho(t)\, (1+i)^{-t}\, dt = 0 \tag{1.1.6}$$

For any given transaction, equation 1.1.5 may have no roots, a unique root, or several roots. If there is a unique root, δ_0 say, it is known as the force of interest implied by the transaction, and the corresponding rate of interest $i_0 = e^{\delta_0} - 1$ is called the **yield** per unit time. (Alternative terms for the yield are the **internal rate of return** and the **money-weighted rate of return** for the transaction.)

Thus the yield is defined if and only if equation 1.1.6 has precisely one root greater than −1 and, when such a root exists, it is the yield.

The analysis of the equation of value for a given transaction may be somewhat complex depending on the shape of the function $f(i)$ denoting the left hand side of equation 1.1.6. However, when the equation $f(i) = 0$ is such that f is a monotonic function, its analysis is particularly simple.

The equation has a root if and only if we can find i_1 and i_2 with $f(i_1)$ and $f(i_2)$ of opposite sign.

In this case, the root is unique and lies between i_1 and i_2. By choosing i_1 and i_2 to be "tabulated" rates sufficiently close to each other, we may determine the yield to any desired degree of accuracy.

It should be noted that, after multiplication by $(1+i)^{t_0}$, equation 1.1.3 takes the equivalent form

$$\sum_{r=1}^{n} c_{t_r} (1+i)^{t_0 - t_r} = 0 \qquad (1.1.7)$$

This slightly more general form may be called **the equation of value at time t_0**. It is of course directly equivalent to the original equation (which is now seen to be the equation of value at time 0). In certain problems a particular choice of t_0 may simplify the solution.

2 Uncertain payment or receipt

If there is uncertainty about the payment or receipt of a cashflow at a particular time, allowance can be made in one of two ways:

- apply a probability of payment/receipt to the cashflow at each time
- use a higher rate of discount

2.1 Probability of cashflow

The probability of payment/receipt can be allowed for by adapting the earlier equations. For example, equation 1.1.6 can be revised to produce:

$$\sum_{r=1}^{n} p_{t_r}\, c_{t_r} (1+i)^{-t_r} + \int_0^\infty p(t)\, \rho(t)\, (1+i)^{-t}\, dt = 0 \qquad (2.1.1)$$

where p_{t_r} and $p(t_r)$ represent the probability of a cashflow at time t.

Where the force of interest is constant, and we can say that the probability is itself in the form of a discounting function, then equation 1.1.5 can be generalised as:

$$\sum_{r=1}^{n} c_{t_r} e^{-\delta t_r} e^{-\mu t_r} + \int_0^\infty \rho(t)\, e^{-\delta t}\, e^{-\mu t}\, dt = 0 \qquad (2.1.2)$$

where μ is a constant force, rather than rate, of the probability of a cashflow at time t.

These probabilities of cashflows may often be estimated by consideration of the past experience of similar cashflows. For example, this approach is used to assess the probabilities of cashflows that are dependent on the survival of a life — this is the theme of Subjects CT4, Models and CT5, Contingencies.

In other cases, there may be lack of data from which to determine an accurate probability for a cashflow. Instead a more approximate probability, or likelihood, may be determined after careful consideration of the risks.

In some cases, it may be spurious to attempt to determine the probability of each cashflow and so more approximate methods may be justified.

Wherever the uncertainty about the probability of the amount or timing of a cashflow could have significant financial effect, a sensitivity analysis may be performed. This involves calculations performed using different possible values for the likelihood and the amounts of the cashflows. Alternatively a stochastic approach could be used to indicate possible outcomes (see Unit 14 and Subject CT4, Models).

2.2 Higher discount rate

As the discounting functions and the probability functions in equations 2.1.1 and 2.1.2 are both dependent on time, they can be combined into a single time dependent function. In cases where there is insufficient information to objectively produce the probability functions, this combined function can be viewed as an adjusted discounting function that makes an implicit allowance for the probability of the cashflow.

Where the probability of the cashflow is a function that is of similar form to the discounting function, the combination can be treated as if a different discount rate were being used. For example, equation 2.1.2 becomes:

$$\sum_{r=1}^{n} c_{t_r} e^{-\delta' t_r} + \int_0^\infty \rho(t)\, e^{-\delta' t}\, dt = 0$$

where $\delta' = \delta + \mu$ The revised force of discount is therefore greater than the actual force of discount as μ must be positive in order to give a probability between 0 and 1. It can therefore be shown that the rate of discount that is effectively used is greater than the actual rate of discount before the implicit allowance for the probability of the cashflow.

E N D

UNIT 8 — LOAN SCHEDULES

Syllabus objective (viii) Describe how a loan may be repaid by regular instalments of interest and capital.

1. Describe flat rates and annual effective rates.

2. Calculate a schedule of repayments under a loan and identify the interest and capital components of annuity payments where the annuity is used to repay a loan for the case where annuity payments are made once per effective time period or p times per effective time period and identify the capital outstanding at any time.

1 Introduction

A very common transaction involving compound interest is a loan that is repaid by regular instalments, at a fixed rate of interest, for a predetermined term.

Consider a very simple example. Assume a bank lends an individual £1,000 for three years, in return for three payments of £X, say, one at the end of each year. The bank will charge an effective rate of interest of 7% per annum.

The equation of value for the transaction gives:

$$1000 = Xa_{\overline{3|}} \Rightarrow X = 381.05$$

So the borrower pays £381.05 at times t = 1, 2 and 3 in return for the loan of £1,000 at time 0. These three payments cover both the interest due and the £1,000 capital.

It is helpful to see how this works in detail:

At time 1 the interest due on the loan of £1000 is £70. The total payment made is £381.05. This leaves £311.05 that is available to repay some of the capital. The capital outstanding after this is then £(1000 – 311.05) = £688.95.

At time 2 the interest due is now only 7% of £688.95 = £48.22, as the borrower does not pay interest on the capital that is already repaid, only on the amount outstanding. This leaves £(381.05 – 48.22) = £332.83 available to repay capital. The capital outstanding after this is then £(688.95 – 332.83) = £356.12.

Finally, at time 3 the interest due is 7% of £356.12 = £24.93, leaving £381.05 – £24.93 = £356.12 available to pay the outstanding sum of £356.12, and the capital is precisely repaid.

One important point is that each repayment must pay first for interest due on the outstanding capital. The balance is then used to repay some of the capital outstanding. Each payment therefore comprises both interest and capital repayment. It may be necessary to identify the separate elements of the payments — for example if the tax treatment of interest and capital differs. Notice also that, where repayments are level, the interest component of the repayment instalments will decrease as capital is repaid, with the consequence that the capital payment will increase.

2 Calculating the capital outstanding

Let L_t be the amount of the loan outstanding at time $t = 0, 1, \ldots, n$, immediately after the repayment at t. The repayments are assumed to be in regular instalments, of amount X_t at time t, $t = 1, 2, 3, \ldots, n$. (Note that we are not assuming all instalments are the same amount.) Let i be the effective rate of interest, per time unit, charged on the loan. Let f_t be the capital repaid at t, and let b_t be the interest paid at t, so that $X_t = f_t + b_t$.

The equation of value for the loan at time 0 is:

$$L_0 = X_1 v + X_2 v^2 + \ldots + X_n v^n$$

We can find the loan outstanding at t *prospectively* or *retrospectively*.

2.1 Prospective loan calculation

Consider the loan transactions at time n, which is the end of the contract term. After the final instalment of capital and interest the loan is exactly repaid. So the final instalment, X_n must exactly cover the capital that remains outstanding after the instalment paid at $n - 1$, together with the interest due on that capital. That is:

$$b_n = iL_{n-1} \; ; \quad f_n = L_{n-1} \text{ so that}$$

$$X_n = iL_{n-1} + L_{n-1} = (1 + i) L_{n-1} \Rightarrow L_{n-1} = X_n v$$

Similarly, at any time $t + 1$, $t \le n - 2$ we know that the capital repaid is $L_t - L_{t+1}$, so that the instalment X_{t+1} is:

$$X_{t+1} = iL_t + (L_t - L_{t+1}) \Rightarrow L_t = (L_{t+1} + X_{t+1}) v$$

Similarly, $L_{t+1} = L_{t+2} + X_{t+2}\, v$, and working forward, successively substituting for L_{t+r} until we get to $L_n = 0$, we get:

$$\begin{aligned} L_t &= (L_{t+1} + X_{t+1})\, v \\ &= ((L_{t+2} + X_{t+2})\, v + X_{t+1})\, v = X_{t+1}\, v + X_{t+2}\, v^2 + L_{t+2}\, v^2 \\ &= X_{t+1}\, v + X_{t+2}\, v^2 + X_{t+3}\, v^3 + L_{t+3}\, v^3 \\ &= \vdots \\ &= X_{t+1}\, v + X_{t+2}\, v^2 + X_{t+3}\, v^3 + \ldots + X_n\, v^{n-t} \end{aligned}$$

This gives the **prospective method** for calculating the loan outstanding. What this equation tells us is that, for calculating the loan outstanding immediately after the repayment at t, say, we have:

Prospective Method: The loan outstanding at time *t* is the present (or discounted) value at time *t* of the future repayment instalments.

Note carefully the condition for this method — the present value must be calculated at a repayment date.

2.2 Retrospective loan calculation

At $t = 1$ the interest due is $b_1 = iL_0$, so the capital repaid is $f_1 = X_1 - iL_0$, leaving a loan outstanding of:

$$L_1 = L_0 - (X_1 - iL_0) = L_0(1 + i) - X_1$$

In general, at time $t \geq 1$ the interest due is $b_t = iL_{t-1}$, leaving capital repaid at t of $X_t - iL_{t-1}$, giving

$$L_t = L_{t-1}(1 + i) - X_t$$

Similarly, $L_{t-1} = L_{t-2}(1 + i) - X_{t-1}$ and, working back from t to 0 we have:

$$\begin{aligned} L_t &= L_{t-1}(1 + i) - X_t \\ &= (L_{t-2}(1 + i) - X_{t-1})\,(1 + i) - X_t = L_{t-2}(1 + i)^2 - X_{t-1}\,(1 + i) - X_t \\ &= L_0\,(1 + i)^t - (X_1(1 + i)^{t-1} + X_2(1 + i)^{t-2} + \dots + X_{t-1}(1 + i) + X_t) \end{aligned}$$

This gives the retrospective method of calculating the outstanding loan. This may be described in words as:

Retrospective Method: The loan outstanding at time *t* is the accumulated value at time *t* of the original loan less the accumulated value at time *t* of the repayments to date.

Both of these approaches are very useful in calculating the capital outstanding at any time. Neither result actually depends on the interest rate being constant. It may be useful to work through the equations assuming the interest charged on the loan in year $r - 1$ to r is i_r, say.

3 Calculating the interest and capital element of the repayments

Given the outstanding capital at any time we can calculate the interest and capital element of any instalment.

For example, consider the instalment X_t at time t. We can calculate the interest element contained in this payment by calculating the loan outstanding immediately after the previous instalment, at $t - 1$, L_{t-1}. The interest due on capital of L_{t-1} for one unit of time at effective rate i per time unit is iL_{t-1}, and this is the interest paid at t. The capital repaid may be found using $X_t - iL_{t-1}$, or by $L_{t-1} - L_t$.

Similarly, it is a simple matter to calculate the interest paid and capital repaid over several instalments. For example, consider the five instalments from $t + 1$ to $t + 5$, inclusive. The loan outstanding immediately before the first instalment is L_t. The loan outstanding after the fifth instalment is L_{t+5}. The total capital repaid is therefore $L_t - L_{t+5}$. The total capital and interest paid is $X_{t+1} + X_{t+2} + \dots + X_{t+5}$. Hence, the total interest paid is

$$\sum_{k=t+1}^{t+5} b_k = (X_{t+1} + X_{t+2} + \dots + X_{t+5}) - (L_t - L_{t+5}).$$

4 The loan schedule

The loan payments can be expressed in the form of a table, or "schedule", as follows.

Year $r \to r+1$	Loan outstanding at r	Instalment at $r+1$	Interest due at $r+1$	Capital repaid at $r+1$	Loan outstanding at $r+1$
$0 \to 1$	L_0	X_1	iL_0	$X_1 - iL_0$	L_1 $= L_0 - (X_1 - iL_0)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$t \to t+1$	L_t	X_{t+1}	iL_t	$X_{t+1} - iL_t$	L_{t+1} $= L_t - (X_{t+1} - iL_t)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1 \to n$	L_{n-1}	X_n	iL_{n-1}	$X_n - iL_{n-1}$	0

With spreadsheet software it is a simple matter to construct the entire schedule for any loan.

5 Repayment instalments payable more frequently than annually

Most loans will be repaid in quarterly, monthly or weekly instalments. No new principles are involved where payments are made more frequently than annually, but care needs to be taken in calculating the interest due at any instalment date.

If the rate of interest used is effective over the same time unit as the frequency of the repayment instalments, then the calculations proceed exactly as above, with the time unit redefined appropriately.

For the case where the interest is expressed as an effective annual rate, with repayment instalments payable $p$thly, we have the equation of value for the loan, given repayments of X_t at time $t = \frac{1}{p}, \frac{2}{p}, \frac{3}{p}, \ldots, n$,

$$L_0 = X_{1/p}\, v^{1/p} + X_{2/p}\, v^{2/p} + X_{3/p}\, v^{3/p} + \ldots + X_n\, v^n$$

It is easy to show that the two basic principles for calculating the loan outstanding hold when repayments are more frequent than annual. That is, the loan outstanding at any repayment date, immediately after an instalment has been paid, may still be calculated as the present value of the remaining repayment instalments, or as the accumulated value of the original loan less the repayments made to date.

Prospectively:

$$L_t = X_{t+1/p}\, v^{1/p} + X_{t+2/p}\, v^{2/p} + \dots + X_n\, v^{n-t}$$

Retrospectively:

$$L_t = L_0(1+i)^t -$$

$$\left(X_{\frac{1}{p}}(1+i)^{t-1/p} + X_{\frac{2}{p}}(1+i)^{t-2/p} + \dots + X_{t-1/p}(1+i)^{1/p} + X_t\right)$$

Given an annual effective rate of interest of i, the effective rate of interest over a period $\frac{1}{p}$ is $(1+i)^{1/p}-1$, which is equal to $i^{(p)}/p$. The interest due at $t+\frac{1}{p}$, given capital outstanding of L_t at some repayment date t, is therefore $b_{t+1/p} = ((1+i)^{1/p}-1)\, L_t$. The capital repaid at $t+\frac{1}{p}$ is then

$$f_{t+1/p} = X_{t+\frac{1}{p}} - ((1+i)^{1/p}-1)\, L_t = X_{t+\frac{1}{p}} - \frac{i^{(p)}}{p}\, L_t$$

6 Consumer credit: flat rates and APRs

Where the borrower is an individual, borrowing from an institution such as a bank, it is common to use the **flat rate of interest** as a measure of the interest charge. The flat rate of interest is defined as the total interest paid over the whole transaction, per unit of initial loan, per year of the loan. For example, if a loan of L_0 is repaid over two years by level monthly instalments of amount X, then the total capital and interest paid is $24X$. The total capital must be the amount of the original loan, so the total interest paid is $24X - L_0$. This gives the flat rate of interest per annum

$$F = \frac{24X - L_0}{2L_0}$$

The flat rate is a very simple calculation that ignores the details of the gradual repayment of capital over the term of a loan. Flat rates are only useful for comparing loans of equal term. Two loans of different terms calculated using the same effective rate of interest will have different flat rates. Since the flat rate ignores the repayment of capital over the term of the loan, it will be considerably lower than the true effective rate of interest charged on the loan.

To ensure that consumers can make informed judgements about the interest rates charged, lenders are required (in most circumstances) to give information about the effective rate of interest charged. In the UK this is in the form of the Annual Percentage Rate of charge, or APR, which is defined as the effective annual rate of interest, rounded to the nearer 1/10th of 1%.

E N D

UNIT 9 — PROJECT APPRAISAL

Syllabus objective (ix) Show how discounted cashflow techniques can be used in investment project appraisal.

1. Calculate the net present value and accumulated profit of the receipts and payments from an investment project at given rates of interest.

2. Calculate the internal rate of return implied by the receipts and payments from an investment project.

3. Describe payback period and discounted payback period and discuss their suitability for assessing the suitability of an investment project.

4. Determine the payback period and discounted payback period implied by the receipts and payments from an investment project.

5. Calculate the money-weighted rate of return, the time-weighted rate of return and the linked internal rate of return on an investment or a fund.

1 Introduction

Suppose an investor is considering the merits of an investment or business project. The investment or project will normally require an initial outlay and possibly other outlays in future, which will be followed by receipts, although in some cases the pattern of income and outgo is more complicated. The cashflows associated with the investment or business venture may be completely fixed (as in the case of a secure fixed-interest security maturing at a given date) or they may have to be estimated. The estimation of the cash inflows and outflows associated with a business project usually requires considerable experience and judgement. All the relevant factors (such as taxation and investment grants) and risks (such as construction delays) should be considered by the actuary, with assistance from experts in the relevant field (e.g. civil engineering for building projects). The identification and assessment of the risks may be done using the Risk Analysis and Management for Projects (RAMP) approach for risk analysis and management that has been developed by, and published on behalf of, the actuarial and civil engineering professions.

Considerable uncertainty will exist in the assessment of many of the risks, so it is prudent to perform calculations on more than one set of assumptions, e.g. on the basis of “optimistic”, “average”, and “pessimistic” forecasts respectively. More complicated techniques (using statistical theory) are available to deal with this kind of uncertainty. Precision is not attainable in the estimation of cashflows for many business projects and hence extreme accuracy is out of place in many calculations.

Net cashflow c_t at time t (measured in suitable time units) is

$$c_t = \text{cash inflow at time } t - \text{cash outflow at time } t \tag{1.1.1}$$

If any payments may be regarded as continuous then $\rho(t)$, the net rate of cashflow per unit time at time t, is defined as

$$\rho(t) = \rho_1(t) - \rho_2(t) \tag{1.1.2}$$

where $\rho_1(t)$ and $\rho_2(t)$ denote the rates of inflow and outflow at time t respectively.

2 Fixed interest rates

2.1 Net present values

Having ascertained or estimated the net cashflows of the investment or project under scrutiny, the investor will wish to measure its profitability in relation to other possible investments or projects. In particular, the investor may wish to determine whether or not it is prudent to borrow money to finance the venture.

Assume for the moment that the investor may borrow or lend money at a fixed rate of interest i per unit time. The investor could accumulate the net cashflows connected with the project in a separate account in which interest is payable or credited at this fixed rate. By the time the project ends (at time T, say), the balance in this account will be

$$\Sigma c_t\,(1+i)^{T-t} + \int_0^T \rho(t)\,(1+i)^{T-t}\,dt \tag{2.1.1}$$

where the summation extends over all t such that $c_t \neq 0$. The present value at rate of interest i of the net cashflows is called the **net present value** at rate of interest i of the investment or business project, and is usually denoted by $NPV(i)$. Hence

$$NPV(i) = \Sigma c_t\,(1+i)^{-t} + \int_0^T \rho(t)\,(1+i)^{-t}\,dt \tag{2.1.2}$$

(If the project continues indefinitely, the accumulation 2.1.1 is not defined, but the net present value may be defined by equation 2.1.2 with $T = \infty$.) If $\rho(t) = 0$, we obtain the simpler formula

$$NPV(i) = \Sigma c_t\, v^t \tag{2.1.3}$$

where $v = (1+i)^{-1}$.

Since the equation

$$NPV(i) = 0 \qquad (2.1.4)$$

is the equation of value for the project at the present time, the yield i_0 on the transaction is the solution of this equation, provided that a unique solution exists.

It may readily be shown that $NPV(i)$ is a smooth function of the rate of interest i and that $NPV(i) \to c_0$ as $i \to \infty$.

2.2 Internal rate of return

In economics and accountancy the yield per annum is often referred to as the **internal rate of return** (IRR) or the **yield to redemption**. The latter term is frequently used when dealing with fixed-interest securities, for which the "running" (or "flat") yield is also considered.

The practical interpretation of the net present value function $NPV(i)$ and the yield is as follows. Suppose that the investor may lend or borrow money at a fixed rate of interest i_1. Since, from equation 2.1.2, $NPV(i_1)$ is the present value at rate of interest i_1 of the net cashflows associated with the project, we conclude that the project will be profitable if and only if

$$NPV(i_1) > 0 \qquad (2.2.1)$$

Also, if the project ends at time T, then the profit (or, if negative, loss) at that time is (by expression 2.1.1)

$$NPV(i_1)(1 + i_1)^T \qquad (2.2.2)$$

Let us now assume that, as is usually the case in practice, the yield i_0 exists and $NPV(i)$ changes from positive to negative when $i = i_0$. Under these conditions it is clear that the project is profitable if and only if

$$i_1 < i_0 \qquad (2.2.3)$$

i.e. the yield exceeds that rate of interest at which the investor may lend or borrow money.

Many projects will need to provide a return to shareholders and so there will not be a specific fixed rate of interest that has to be exceeded. Instead a target, or hurdle, rate of return may be set for assessing whether a project is likely to be sufficiently profitable.

2.3 Accumulated value

The accumulated value, at time T, of a cashflow can be expressed as:

$$A(T) = \Sigma c_t(1 + i)^{T-t} + \int_0^T \rho(t)\,(1 + i)^{T-t}\,dt \qquad (2.3.1)$$

2.4 The comparison of two investment projects

Suppose now that an investor is comparing the merits of two possible investments or business ventures, which we call projects A and B respectively. We assume that the borrowing powers of the investor are not limited.

Let $NPV_A(i)$ and $NPV_B(i)$ denote the respective net present value functions and let i_A and i_B denote the yields (which we shall assume to exist). It might be thought that the investor should always select the project with the higher yield, but this is not invariably the best policy. A better criterion to use is **the profit at time** T (the date when the later of the two projects ends) or, equivalently, the **net present value**, calculated at the rate of interest i_1 at which the investor may lend or borrow money. This is because A is the more profitable venture if

$$NPV_A(i_1) > NPV_B(i_1) \qquad (2.4.1)$$

The fact that $i_A > i_B$ may not imply that $NPV_A(i_1) > NPV_B(i_1)$ is illustrated in Figure 2.4.1. Although i_A is larger than i_B, the $NPV(i)$ functions "cross over" at i'. It follows that $NPV_B(i_1) > NPV_A(i_1)$ for any $i_1 < i'$, where i' is the **cross-over rate**. There may even be more than one cross-over point, in which case the range of interest rates for which project *A* is more profitable than project *B* is more complicated.

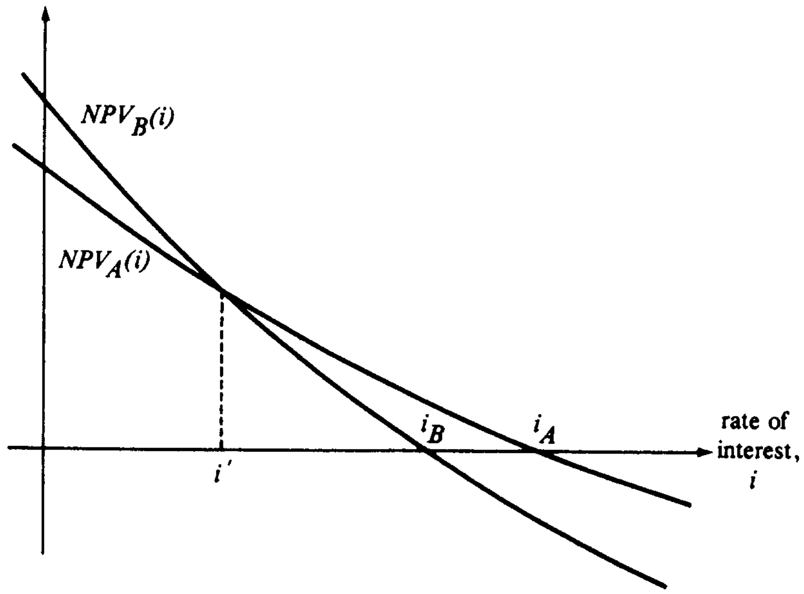

Figure 2.4.1 Investment comparison

Example

An investor is considering whether to invest in either or both of the following loans:

Loan A For a purchase price of £10,000, the investor will receive £1,000 per annum payable quarterly in arrear for 15 years.

Loan B For a purchase price of £11,000, the investor will receive an income of £605 per annum, payable annually in arrear for 18 years, and a return of his outlay at the end of this period.

The investor may lend or borrow money at 4% per annum. Would you advise the investor to invest in either loan, and, if so, which would be the more profitable?

Solution

We first consider loan A:

$$NPV_A(i) = -\,10{,}000 + 1{,}000 a_{\overline{15}|}^{(4)}$$

and the yield is found by solving the equation $NPV_A(i) = 0$, or $a_{\overline{15}|}^{(4)} = 10$, which gives $i_A \approx$ 5.88%.

For loan B we have

$$NPV_B(i) = -\,11{,}000 + 605\, a_{\overline{18}|} + 11{,}000 v^{18}$$

and the yield (i.e. the solution of $NPV_B(i) = 0$) is $i_B = 5.5\%$. The rate of interest at which the investor may lend or borrow money is 4% per annum, which is less than both i_A and i_B, so we compare $NPV_A(0.04)$ and $NPV_B(0.04)$.

Now $NPV_A(0.04) = £1{,}284$ and $NPV_B(0.04) = £2{,}089$, so it follows that, although the yield on loan B is less than on loan A, the investor will make a larger profit from loan B. We should therefore advise him that an investment in either loan would be profitable, but that, if only one of them is to be chosen, then loan B will give the higher profit.

The above example illustrates the fact that the choice of investment depends very much on the rate of interest i_1 at which the investor may lend or borrow money. If this rate of interest were 5¾%, say, then loan B would produce a loss to the investor, while loan A would give a profit.

3 Different interest rates for lending and borrowing

We have assumed so far that the investor may borrow or lend money at the same rate of interest i_1. In practice, however, the investor will probably have to pay a higher rate of interest (j_1, say) on borrowings than the rate (j_2, say) he receives on investments. The difference $j_1 - j_2$ between these rates of interest depends on various factors, including the credit-worthiness of the investor and the expense of raising a loan.

The concepts of net present value and yield are in general no longer meaningful in these circumstances. We must calculate the accumulation of net cashflows from first principles, the rate of interest depending on whether or not the investor's account is in credit. In many practical problems the balance in the investor's account (i.e. the accumulation of net cashflows) will be negative until a certain time t_1 and positive afterwards, except perhaps when the project ends.

In some cases the investor must finance his investment or business project by means of a fixed-term loan without an early repayment option. In these circumstances the investor cannot use a positive cashflow to repay the loan gradually, but must accumulate this money at the rate of interest applicable on lending, i.e. j_2.

3.1 Payback periods

In many practical problems the net cashflow changes sign only once, this change being from negative to positive. In these circumstances the balance in the investor's account will change from negative to positive at a unique time t_1, or it will always be negative, in which case the project is not viable. If this time t_1 exists, it is referred to as the **discounted payback period** (DPP). It is the smallest value of t such that $A(t) \geq 0$, where

$$A(t) = \sum_{s \leq t} c_s(1 + j_1)^{t-s} + \int_0^t \rho(s)\,(1 + j_1)^{t-s}\,ds \tag{3.1.1}$$

Note that t_1 does not depend on j_2 but only on j_1, the rate of interest applicable to the investor's borrowings. Suppose that the project ends at time T. If $A(T) < 0$ (or, equivalently, if $NPV(j_1) < 0$) the project has no discounted payback period and is not profitable. If the project is viable (i.e. there is a discounted payback period t_1) the **accumulated profit** when the project ends at time T is

$$P = A(t_1)\,(1 + j_2)^{T-t_1} + \sum_{t > t_1} c_t\,(1 + j_2)^{T-t}$$

$$+ \int_{t_1}^T \rho(t)\,(1 + j_2)^{T-t}\,dt \tag{3.1.2}$$

This follows since the net cashflow is accumulated at rate j_2 after the discounted payback period has elapsed.

If interest is ignored in formula 3.1.1 (i.e. if we put $j_1 = 0$), the resulting period is called the **payback period**. However, its use instead of the discounted payback period often leads to erroneous results and is therefore not to be recommended.

The discounted payback period is often employed when considering a single investment of C, say, in return for a series of payments each of R, say, payable annually in arrear for n years. The discounted payback period t_1 years is clearly the smallest integer t such that $A^*(t) \geq 0$, where

$$A^*(t) = -C(1 + j_1)^t + R s_{\overline{t|}} \qquad \text{at rate } j_1 \tag{3.1.3}$$

i.e. the smallest integer t such that

$$R a_{\overline{t|}} \geq C \qquad \text{at rate } j_1 \tag{3.1.4}$$

The project is therefore viable if $t_1 \le n$, in which case the accumulated profit after n years is clearly

$$P = A^*(t_1)\ (1+j_2)^{n-t_1} + Rs_{\overline{n-t_1}|} \qquad \text{at rate } j_2 \tag{3.1.5}$$

4 Measurement of investment performance

It is often necessary to be able to measure the investment performance of a fund (for example a pension fund, or the funds of an insurance company) over a period.

4.1 Money Weighted Rate of Return

One measure of the performance is the yield earned on the fund over the period. For example, consider a fund with value F_0 at time 0, with net cashflows C_{t_k} at times t_1, t_2, ..., t_n and fund value F_T at time $T \ge t_n$, then the equation of value, equating values at time T, is

$$F_0\,(1+i)^T + C_{t_1}(1+i)^{T-t_1} + C_{t_2}(1+i)^{T-t_2} + \ldots + C_{t_n}(1+i)^{T-t_n} = F_T$$

where i is the effective annual rate of interest earned by the fund in the interval $[0, T]$.

In this equation of value the left hand side is the value at time T of the fund at the start of the period plus or minus all the cashflows received or paid out in the interval.

The yield earned on the fund is also called the **money weighted rate of return** (MWRR).

As a measure of investment performance the money weighted rate of return is not entirely satisfactory, as it is sensitive to the amounts and timing of the net cashflows. If, say, we are assessing the skill of the fund manager, this is not ideal, as the fund manager does not control the timing or amount of the cashflows — he or she is merely responsible for investing the positive net cashflows and realising cash to meet the negative net cashflows.

4.2 Time Weighted Rate of Return

Define F_0, F_T, and C_{t_k} as above, and let C_0 be the cashflow (if any) at time $t = 0$. In addition let F_{t_k-} be the amount of the fund just before the cashflow due at time t_k, so that the amount of the fund just after the receipt of the net cashflow due at time t_k is $F_{t_k-} + C_{t_k}$. Then the **Time Weighted Rate of Return** (TWRR) is i per annum, where

$$(1+i)^T = \frac{F_{t_1-}}{F_0 + C_0}\,\frac{F_{t_2-}}{F_{t_1-} + C_{t_1}}\,\frac{F_{t_3-}}{F_{t_2-} + C_{t_2}} \cdots \frac{F_T}{F_{t_n-} + C_{t_n}}$$

Each factor on the right hand side gives the proportionate increase in the fund between cashflows. The product of these factors gives the notional accumulation factor for a single investment of 1 at time $t = 0$, invested until time T.

Using the TWRR eliminates the effect of the cashflow amounts and timing, and therefore gives a fairer basis for assessing the investment performance for the fund.

The disadvantages of both the time weighted and money weighted rates of return are that the calculation requires information about all the cashflows of the fund during the period of interest. In addition, the TWRR requires the fund values at all the cashflow dates. A disadvantage of the MWRR is that the equation may not have a unique solution — or indeed any solution. If the fund performance is reasonably stable in the period of assessment then the TWRR and the MWRR will give similar results.

4.3 Linked Internal Rate of Return

If the rate of return on a fund is measured over a series of intervals $(0, t_1)$, (t_1, t_2), (t_2, t_3), ..., (t_{n-1}, t_n), such that the annual effective rate of interest earned by a fund in the interval (t_{r-1}, t_r) is i_r (where i_1 is the annual rate earned in $(0, t_1)$) then the **Linked Internal Rate of Return** is i per annum, where

$$(1+i)^{t_n} = (1+i_1)^{t_1}\,(1+i_2)^{t_2-t_1}\,(1+i_3)^{t_3-t_2}\,...(1+i_n)^{t_n-t_{n-1}}$$

The linked internal rate of return will be equal to the TWRR if the sub intervals (t_{r-1}, t_r) are the same in each calculation. In practice, the yields i_r may be calculated by approximate methods, and then, if the sub intervals used are sufficiently short, the linked internal rate of return will be close to the TWRR.

E N D

UNIT 10 — INVESTMENTS

Syllabus objective (x) Describe the investment and risk characteristics of the following types of asset available for investment purposes:

- fixed interest government borrowings
- fixed interest borrowing by other bodies
- index-linked government borrowings
- shares and other equity-type finance
- derivatives

1 Fixed interest government borrowings

1.1 Fixed interest government bonds

A government or government body may raise money by floating a loan on a stock exchange. The terms of the issue are set out by the borrower and investors may be invited to subscribe to the loan at a given price (called the **issue price**), or the issue may be by tender, in which case investors are invited to nominate the price that they are prepared to pay and the loan is then issued to the highest bidders, subject to certain rules of allocation.

The annual interest payable to each holder, which is often but not invariably payable half-yearly, is found by multiplying the **nominal amount** of his holding N by the rate of interest per annum D, which is generally called the **coupon rate**.

The money payable at redemption is calculated by multiplying the nominal amount held N by the **redemption price** R per unit nominal (which is often quoted "per cent" in practice). If $R = 1$ the stock is said to be redeemable **at par**; if $R > 1$ the stock is said to be **redeemable above par** or at **a premium**; and if $R < 1$ the stock is said to be **redeemable below par** or at a **discount**. The **redemption date** is the set date on which the redemption money is due to be paid. Some bonds have variable redemption dates, in which case the redemption date may be chosen by the borrower (or perhaps the lender) as any interest date within a certain period, or any interest date on or after a given date. In the latter case the stock is said to have no final redemption date, or to be **undated**. Some banks allow the interest and redemption proceeds to be bought and sold separately, effectively creating bonds with no coupon and bonds redeemable at zero.

The coupon rate, redemption price and term to redemption of a fixed interest security serve to define the cash payments promised to a tax-free investor in return for his purchase price. If the investor is subject to taxation, appropriate deductions from the cashflow must be made. For example, if an investor is liable to income tax at rate t_1 on the interest payments, the annual income after tax will be $(1 - t_1)DN$.

In most developed economies, bonds issued by the government form the largest, most important and most liquid part of the bond market. Investors can therefore deal in large quantities with little (or no) impact on the price. Bonds issued by the governments of developed countries in their domestic currency are the most secure long-term investment available.

However, this security together with the low volatility of return relative to other long-term investments should lead to a low expected return, though this will be compensated for to an extent by very low dealing costs.

Relative to inflation, however, the income stream may be volatile. Some governments therefore issue bonds that provide interest and redemption payments that are linked to an inflation index.

However, indexation will need to be based on the movement of the inflation index with a time lag to allow for publication of the index figure and the need to calculate monetary amounts of coupons in advance. There is effectively no inflation protection during the lag period.

1.2 Government bills

Government bills are short-dated securities issued by governments to fund their short-term spending requirements. They are issued at a discount and redeemed at par with no coupon.

They are mostly denominated in the domestic currency, although issues can be made in other currencies.

The yield on government bills is typically quoted as a simple rate of discount for the term of the bill. For example, a 3-month bill may be quoted as being offered at a discount of 2%. This would mean that the initial investment required to buy the bill would be 2% less than the payment 3 months later.

Government bills are absolutely secure and often highly marketable, despite not being quoted. They are often used as a benchmark risk-free short-term investment.

2 Fixed interest borrowing by other bodies

2.1 Characteristics of corporate debt

Corporate bonds are, in many ways, similar to conventional government bonds in their characteristics. Here the debt is issued by a company rather than a government.

The major differences between corporate bonds and government bonds are:

- Corporate bonds are usually less secure than government bonds. The level of security depends on the type of bond, the company which has issued it, and the term.

- Corporate bonds are usually less marketable than government bonds, mainly because the sizes of issues are much smaller.

2.2 Debentures

Debentures are part of the loan capital of companies. The term loan capital usually refers to long-term borrowings rather than short-term. The issuing company provides some form of security to holders of the debenture.

Debenture stocks are considered more risky than government bonds and are usually less marketable. Accordingly the yield required by investors will be higher than for a comparable government bond.

2.3 Unsecured loan stocks

Unsecured loan stocks are issued by various companies. They are unsecured — holders rank alongside other unsecured creditors. Yields will be higher than on comparable debentures issued by the same company, to reflect the higher default risk.

2.4 Eurobonds

Eurobonds are a form of unsecured medium or long-term borrowing made by issuing bonds which pay regular interest payments and a final capital repayment at par. Eurobonds are issued and traded internationally and are often not denominated in a currency native to the country of the issuer.

Eurobonds are issued by large companies, governments and supra-national organisations. They are usually unsecured. Yields depend upon the issuer (and hence risk) and issue size (and hence marketability), but will typically be slightly lower than for the conventional unsecured loan stocks of the same issuer.

The features of Eurobonds vary a lot more than traditional bond issues. In the absence of any full-blown government control, issuers have been free to add novel features to their issues. They do this to make them appeal to different investors.

2.5 Certificates of deposit

A certificate of deposit is a certificate stating that some money has been deposited. They are issued by banks and building societies. Terms to maturity are usually in the range 28 days to 6 months. Interest is payable on maturity.

The degree of security and marketability will depend on the issuing bank. There is an active secondary market in certificates of deposit.

3 Shares and other equity-type borrowing

3.1 Ordinary shares

Ordinary shares — also called equities — are securities, issued by commercial undertakings and other bodies, which entitle their holders to receive all the net profits of the company after interest on loans and fixed interest stocks has been paid. The cash paid out each year is called the dividend, the remaining profits (if any) being retained as reserves or to finance the company's activities.

Ordinary shares are the principal way in which companies in many countries are financed. They offer investors high potential returns for high risk, particularly risk of capital losses.

Ordinary shares are the lowest ranking form of finance issued by companies. Dividends are not a legal obligation of the company but are paid at the discretion of the directors. The initial running yield on ordinary shares is low but dividends should increase with inflation and real growth in a company's earnings.

The expected overall future return on ordinary shares ought to be higher than for most other classes of security to compensate for the greater risk of default, and for the variability of returns.

The return on ordinary shares is made up of two components, the dividends received and any increase in the market price of the shares.

Marketability of ordinary shares varies according to the size of the company but will be better than for the loan capital of the same company if:

- the bulk of the company's capital is in the form of ordinary shares
- the loan capital is fragmented into several different issues
- investors buy and sell ordinary shares more frequently than they trade in loan capital, perhaps because the residual nature of ordinary shares makes them more sensitive to changes in investors' views about a company

Ordinary shareholders get voting rights in proportion to the number of shares held, so shareholders may have the ability to influence the decisions taken by the directors and managers of the company.

3.2 Preference shares

Preference shares are less common than ordinary shares. Assuming that the company makes sufficient profits, they offer a fixed stream of investment income. The investment characteristics are often more like those of unsecured loan stocks than ordinary shares.

The crucial difference between preference shares and ordinary shares is that preference share dividends are limited to a set amount which is almost always paid.

Preference shareholders rank above ordinary shareholders (both for dividends and, usually, on winding up), and only get voting rights if dividends are unpaid or if there is a matter which directly affects the rights of preference shareholders.

Preference dividends, like ordinary dividends, are only paid at the directors' discretion, but no ordinary dividend can be paid if there are any outstanding preference dividends. In most cases preference shares are cumulative, which means that unpaid dividends are carried forward. In a given company, the risk of preference shareholders not getting their dividends is greater than the risk of loan stockholders not being paid, but less than the risk of ordinary shareholders not being paid.

For all investors, the expected return on preference shares is likely to be lower than on ordinary shares because the risk of holding preference shares is lower. Preference shares rank higher on a winding-up, and the level of income payments is more certain.

Marketability of preference shares is likely to be similar to loan capital marketability.

3.3 Property

There are many different types of properties available for investment, for example: offices, shops and industrial properties (e.g. warehouses, factories).

The return from investing in property comes from rental income and from capital gains, which may be realised on sale. Property is a real investment and as such rents and capital values might be expected to increase broadly with inflation in the long term, which makes the returns from property similar in nature to those from ordinary shares. However, neither rental income nor capital values are guaranteed and there can be considerable fluctuations in capital values in particular, in real and nominal terms.

Rental terms are specified in lease agreements. Typically, it is agreed that rents are reviewed at specific intervals such as every three or five years. The rent is changed, at a review time, to be more or less equal to the market rent on similar properties at the time of the review. Some leases have clauses which specify upward-only adjustments of rents.

The following characteristics are particular to property investments:

(a) large unit sizes, leading to less flexibility than investment in shares
(b) each property is unique, so can be difficult to value. Valuation is expensive, because of the need to employ an experienced surveyor
(c) the actual value obtainable on sale is uncertain: values in property markets can fluctuate just as stock markets can
(d) buying and selling expenses are higher than for shares and bonds
(e) net rental income may be reduced by maintenance expenses
(f) there may be periods when the property is unoccupied, and no income is received

Marketability is poor because each property is unique and because buying and selling incur high costs.

The running yield from property investments will normally be higher than that for ordinary shares. The reasons for this are:

1. dividends usually increase annually, whereas rents are reviewed less often
2. property is much less marketable
3. expenses associated with property investment are much higher
4. large, indivisible units of property are much less flexible

5. on average, dividends will tend to increase more rapidly than rents, as dividends benefit from returns arising from the retention of profits and their reinvestment within the company.

4 Derivatives

A derivative is a financial instrument with a value dependent on the value of some other, underlying asset.

4.1 Futures

A futures contract is a standardised, exchange tradable contract between two parties to trade a specified asset on a set date in the future at a specified price.

Financial futures are based on an underlying financial instrument, rather than a physical commodity. They exist in four main categories:

- bond futures
- short interest rate futures
- stock index futures
- currency futures

Each party to a futures contract must deposit a sum of money known as *margin* with the clearing house. Margin payments act as a cushion against potential losses which the parties may suffer from future adverse price movements.

When the contract is first struck, *initial margin* is deposited with the clearing house. Additional payments of *variation margin* are made daily to ensure that the clearing house's exposure to credit risk is controlled. This exposure can increase after the contract is struck through subsequent adverse price movements.

4.1.1 Bond futures

For delivery, the contract requires physical delivery of a bond. If the contract were specified in terms of a particular bond then it would be possible simply to deliver the required amount of that stock. If the contract is specified in terms of a notional stock then there needs to be a linkage between it and the cash market. The bonds which are eligible for delivery are listed by the exchange. The party delivering the bond will choose the stock from the list which is *cheapest to deliver*. The price paid by the receiving party is adjusted to allow for the fact that the coupon may not be equal to that of the notional bond which underlies the contract settlement price.

4.1.2 Short interest rate futures

The way that the quotation is structured means that as interest rates fall the price rises, and vice versa. The price is stated as 100 minus the 3-month interest rate. For example, with an interest rate of 6.25% the future is priced as 93.75.

The contract is based on the interest paid on a notional deposit for a specified period from the expiry of the future. However no principal or interest changes hands. The contract is cash settled. On expiry the purchaser will have made a profit (or loss) related to the difference between the final settlement price and the original dealing price.

The party delivering the contract will have made a corresponding loss (or profit).

4.1.3 Stock index futures

The contract provides for a notional transfer of assets underlying a stock index at a specified price on a specified date.

4.1.4 Currency futures

The contract requires the delivery of a set amount of a given currency on the specified date.

4.2 Options

An option gives an investor the right, but not the obligation, to buy or sell a specified asset on a specified future date.

A call option gives the right, but not the obligation, to buy a specified asset on a set date in the future for a specified price. A put option gives the right, but not the obligation, to sell a specified asset on a set date in the future for a specified price.

An American style option is an option that can be exercised on any date before its expiry. A European style option is an option that can be exercised only at expiry.

4.2.1 Swaps

A swap is a contract between two parties under which they agree to exchange a series of payments according to a prearranged formula.

In the most common form of interest rate swap, one party agrees to pay to the other a regular series of fixed amounts for a certain term. In exchange, the second party agrees to pay a series of variable amounts based on the level of a short-term interest rate. Both sets of payments are in the same currency.

The fixed payments can be thought of as interest payments on a deposit at a fixed rate, while the variable payments are the interest on the same deposit at a floating rate. The deposit is purely a notional one and no exchange of principal takes place.

A currency swap is an agreement to exchange a fixed series of interest payments and a capital sum in one currency for a fixed series of interest payments and a capital sum in another.

The swap will be priced so that the present value of the cashflows is slightly negative for the investor and positive for the issuing organisation. The difference represents the price that the investor is prepared to pay for the advantages brought by the swap on the one hand, and the issuer's expected profit margin on the other.

Each counterparty to a swap faces two kinds of risk:

Market risk is the risk that market conditions will change so that the present value of the net outgo under the agreement increases. The market maker will often attempt to hedge market risk by entering into an offsetting agreement.

Credit risk is the risk that the other counterparty will default on its payments. This will only occur if the swap has a negative value to the defaulting party so the risk is not the same as the risk that the counterparty would default on a loan of comparable maturity.

4.3 Convertibles

Convertible forms of company securities are, almost invariably, unsecured loan stocks or preference shares that convert into ordinary shares of the issuing company.

The convertible will have a stated annual interest payment. The date of conversion might be a single date or, at the option of the holder, one of a series of specified dates.

The characteristics of a convertible security in the period prior to conversion are a cross between those of fixed interest stock and ordinary shares. As the likely date of conversion (or not) gets nearer, it becomes clearer whether the convertible will stay as loan stock or become ordinary shares. As this happens, its behaviour becomes closer to that of the security into which it converts.

Convertibles generally provide higher income than ordinary shares and lower income than conventional loan stock or preference shares.

There will generally be less volatility in the price of the convertible than in the share price of the underlying equity.

From the investor's point of view, convertible securities offer the opportunity to combine the lower risk of a debt security with the potential for large gains of an equity. The price paid for this is a lower running yield than on a normal loan stock or preference share. The option to convert will have *time value*, which will be reflected in the price of the stock.

$\boxed{\textbf{E N D}}$

UNIT 11 — ELEMENTARY COMPOUND INTEREST PROBLEMS

Syllabus objective (xi) Analyse elementary compound interest problems.

1. Calculate the present value of payments from a fixed interest security where the coupon rate is constant and the security is redeemed in one instalment.

2. Calculate upper and lower bounds for the present value of a fixed interest security that is redeemable on a single date within a given range at the option of the borrower.

3. Calculate the running yield and the redemption yield from a fixed interest security (as in 1.), given the price.

4. Calculate the present value or yield from an ordinary share and a property, given simple (but not necessarily constant) assumptions about the growth of dividends and rents.

5. Solve an equation of value for the real rate of interest implied by the equation in the presence of specified inflationary growth.

6. Calculate the present value or real yield from an index-linked bond, given assumptions about the rate of inflation.

7. Calculate the price of, or yield from, a fixed interest security where the investor is subject to deduction of income tax on coupon payments and redemption payments are subject to the deduction of capital gains tax.

8. Calculate the value of an investment where capital gains tax is payable, in simple situations, where the rate of tax is constant, indexation allowance is taken into account using specified index movements and allowance is made for the case where an investor can offset capital losses against capital gains.

This unit also deals with real rates of interest as required in syllabus objective (iv).

1 Fixed interest securities

As in other compound interest problems, one of two questions may be asked:

(1) What price *P* per unit nominal, should be paid by an investor to secure a net yield of *i* per annum?

(2) Given that the investor pays a price *P* per unit nominal, what net yield per annum will be obtained?

1.1 Price to be paid or yield to be obtained

The price, *P*, to be paid to achieve a yield of *i* per annum is equal to:

$$P = \begin{pmatrix} \text{Present value, at rate} \\ \text{of interest } i \text{ per annum,} \\ \text{of net interest payments} \end{pmatrix} + \begin{pmatrix} \text{Present value, at rate} \\ \text{of interest } i \text{ per annum,} \\ \text{of net capital payments} \end{pmatrix} \tag{1.1}$$

The yield available on a stock that can be bought at a given price, *P*, can be found by solving equation (1.1) for the net yield *i*.

If the investor is not subject to taxation the yield *i* is referred to as a **gross** yield. The yield on a security is sometimes referred to as the **yield to redemption** or the **redemption yield** to distinguish it from the **flat** (or **running**) yield, which is defined as *D*/*P*, the ratio of the coupon rate to the price per unit nominal of the stock.

1.2 No tax

Consider an *n* year fixed interest security which pays coupons of *D* per annum, payable *p*thly in arrear and has redemption amount *R*.

The price of this bond, at an effective rate of interest *i* per annum, with no allowance for tax (i.e. *i* represents the gross yield) is:

$$P = Da_{\overline{n|}}^{(p)} + Rv^n \quad \text{at rate } i \text{ per annum} \tag{1.2}$$

Note: One could also work with a period of half a year. The corresponding equation of value would then be

$$P = \frac{D}{2}a_{\overline{2n|}} + Rv^{2n} \quad \text{at rate } i' \text{ where } (1+i')^2 = 1+i$$

1.3 Income tax

Suppose an investor is liable to income tax at rate t_1 on the coupons, which is due at the time that the coupons are paid. The price, P', of this bond, at an effective rate of interest i per annum, where i now represents the net yield, is now:

$$P' = (1-t_1)Da_{\overline{n}|}^{(p)} + Rv^n \qquad \text{at rate } i \text{ per annum} \tag{1.3}$$

It is possible in some countries that the tax is paid at some later date, for example at the calendar year end.

This does not cause any particular problems as we follow the usual procedure — identify the cashflow amounts and dates and set out the equation of value.

For example, suppose that income tax on the bond is paid in a single instalment, due, say, k years after the second half-yearly coupon payment each year. Then the equation of value for a given net yield i and price (or value) P' is, immediately after a coupon payment,

$$P' = Da_{\overline{n}|}^{(p)} + Rv^n - t_1Dv^k a_{\overline{n}|}$$

Other arrangements may be dealt with similarly from first principles.

1.4 Capital gains tax

If the price paid for a bond is less than the redemption (or sale price if sold earlier) then the investor has made a capital gain.

Capital gains tax is a tax levied on the capital gain. In contrast to income tax, this tax is normally payable once only in respect of each disposal, at the date of sale or redemption.

1.4.1 Capital gains test

Consider an n year fixed interest security which pays coupons of D per annum, payable $p$thly in arrear and has redemption amount R. An investor, liable to income tax at rate t_1, purchases the bond at price P'. If $R > P'$ then there is a capital gain and from (1.3), we have:

$$R > (1-t_1)Da_{\overline{n}|}^{(p)} + Rv^n$$

$$R(1-v^n) > (1-t_1)D\frac{1-v^n}{i^{(p)}}$$

$$R > (1-t_1)\frac{D}{i^{(p)}}$$

$$i^{(p)} > (1-t_1)\frac{D}{R} \qquad (1.4)$$

If the investor is also subject to tax at rate t_2 $(0 < t_2 < 1)$ on the capital gains, then let the price payable, for a given net yield i, be P''.

If $i^{(p)} > (1-t_1)\dfrac{D}{R}$ then there is a capital gain. At the redemption date of the loan there is therefore an additional liability of $t_2(R-P'')$.

In this case:

$$P'' = (1-t_1)Da_{\overline{n}|}^{(p)} + Rv^n - t_2(R-P'')v^n \qquad \text{at rate } i \text{ per annum} \qquad (1.5)$$

Note that if a stock is sold before the final maturity date, the capital gains tax liability will in general be different, since it will be calculated with reference to the sale proceeds rather than the corresponding redemption amount.

If $i^{(p)} \le (1-t_1)\dfrac{D}{R}$ then there is no capital gain and no capital gains tax liability due at redemption. Hence $P'' = P'$ in (1.3). (We are assuming that it is not permissible to offset the capital loss against any other capital gain: see example 1.4.2).

1.4.2 Finding the yield when there is capital gains tax

An investor who is liable to capital gains tax may wish to determine the net yield on a particular transaction in which he has purchased a loan at a given price.

One possible approach is to determine the price on two different net yield bases and then estimate the actual yield by interpolation. This approach is not always the quickest method. Since the purchase price is known, so too is the amount of the capital gains tax, and the net receipts for the investment are thus known. In this situation one may more easily write down an equation of value which will provide a simpler basis for interpolation, as illustrated by the next example.

Example 1.4.1

A loan of £1,000 bears interest of 6% per annum payable yearly and will be redeemed at par after ten years. An investor, liable to income tax and capital gains tax at the rates of 40% and 30% respectively, buys the loan for £800. What is his net effective annual yield?

Solution

Note that the net income each year of £36 is 4.5% of the purchase price. Since there is a gain on redemption, the net yield is clearly greater than 4.5%.

The gain on redemption is £200, so that the capital gains tax payable will be £60 and the net redemption proceeds will be £940. The net effective yield p.a. is thus that value of i for which

$$800 = 36\, a_{\overline{10}|} + 940v^{10}$$

If the net gain on redemption (i.e. £140) were to be paid in equal instalments over the ten-year duration of the loan rather than as a lump sum, the net receipts each year would be £50 (i.e. £36 + £14). Since £50 is 6.25% of £800, the net yield actually achieved is less than 6.25%. When $i = 0.055$, the right-hand side of the above equation takes the value 821.66, and when $i = 0.06$ the value is 789.85. By interpolation, we estimate the net yield as

$$i = 0.055 + \frac{821.66 - 800}{821.66 - 789.85}\, 0.005 = 0.0584$$

The net yield is thus 5.84% per annum.

Alternatively, we may find the prices to give net yields of 5.5% and 6% per annum. These prices are £826.27 and £787.81, respectively. The yield may then be obtained by interpolation. However, this alternative approach is somewhat longer than the first method.

1.4.3 Offsetting capital losses against capital gains

Until now we have considered the effects of capital gains tax on the basis that it is **not** permitted to offset capital gains by capital losses. In some situations, however, it may be permitted to do so. This may mean that an investor, when calculating his liability for capital gains tax in any year, is allowed to deduct from his total capital gains for the year the total of his capital losses (if any). If the total capital losses exceed the total capital gains, no "credit" will generally be given for the overall net loss, but no capital gains tax will be payable.

Example 1.4.2

Suppose that in a particular tax year an investor sold the following two assets, both of which were purchased some years ago.

Asset A	Sold for £1,865 (Purchase price £1,300)
Asset B	Sold for £500 (Purchase price £900)

The sale of asset A produces a capital gain of £565 while for asset B there is a capital loss of £400.

In the absence of both the right to offset losses against gains and of indexation the sales of these two assets lead to an overall capital gain of £565. (The loss of £400 is simply regarded as a zero capital gain.)

If the offsetting of losses against gains is allowed, the sales of these two assets lead to an overall capital gain of £165 (i.e. £565 – £400).

1.4.4 The indexation of capital gains

When an asset is sold at a profit, it may be permitted to reduce the capital gain for taxation purposes by determining the amount of the capital gain not by reference to the actual purchase price but by reference to a (greater) “notional” purchase price. The notional purchase price is the actual purchase price increased in line with an approved index. This practice is generally referred to as the “indexation of gains”.

Example 1.4.3

In example 1.4.2, suppose that indexation is also allowed and that over the period during which the investor owned asset A the approved index increased by 18%. In this case the “notional” purchase price of asset A is £1,534 (i.e. £1,300 × 1.18) and the capital gain arising on the sale of this asset is reduced to £331 (i.e. £1,865 – £1,534). If no offsetting of losses is allowed, this last amount will be the total capital gain arising from the sale of the two assets. If, however, offsetting is permitted, there is zero overall capital gain — since the loss of £400 on the sale of asset B exceeds the (reduced) gain arising from the sale of asset A.

It is perhaps worth pointing out that indexation of losses is **not** usually permitted. Thus, for example, if over the period during which the investor owned asset B the approved index increased by 12%, the loss on the sale of this asset is still considered to be £400. It is **not** taken as £508 (i.e. £900 × 1.12 – £500).

The principles used here can be applied to other investments that are subject to such tax.

1.5 Optional redemption dates

Sometimes a security is issued without a fixed redemption date. In such cases the terms of issue may provide that the borrower can redeem the security **at the borrower's option** at any interest date on or after some specified date. Alternatively, the issue terms may allow the borrower to redeem the security at the borrower's option at any interest date on or between two specified dates (or possibly on any one of a series of dates between two specified dates).

The latest possible redemption date is called the **final redemption date** of the stock, and if there is no such date, then the stock is said to be **undated**. It is also possible for a loan to be redeemable between two specified interest dates, or on or after a specified interest date, at the option of the lender, but this arrangement is less common than when the borrower chooses the redemption date.

An investor who wishes to purchase a loan with redemption dates at the option of the borrower cannot, at the time of purchase, know how the market will move in the future and hence when the borrower will repay the loan. The investor thus cannot know the yield which will be obtained. However, by using (1.4) the investor can determine either:

(1) The maximum price to be paid, if the net yield is to be **at least** some specified value;

or

(2) The minimum net yield the investor will obtain, if the price is some specified value.

Consider a fixed interest security which pays coupons of D per annum, payable $p$thly in arrear and has redemption amount R. The security has an outstanding term of n years, which may be chosen by the borrower subject to the restriction that $n_1 \le n \le n_2$. (We assume that n_1 and n_2 are integer multiples of $1/p$.) Suppose that an investor, liable to income tax at rate t_1, wishes to achieve a net annual yield of *at least* i.

It follows from equations (1.3) and (1.4) that if $i^{(p)} > (1-t_1)\dfrac{D}{R}$ then the purchaser will receive a capital gain when the security is redeemed. From the investor's viewpoint, the sooner a capital gain is received the better. The investor will therefore obtain a greater yield on a security which is redeemed first. So to *ensure* the investor receives a net annual yield of *at least* i then they should assume the worst case result: that the redemption money is paid as late as possible, i.e. $n = n_2$.

Similarly if $i^{(p)} < (1-t_1)\dfrac{D}{R}$ then there will be a capital loss when the security is redeemed. The investor will wish to defer this loss as long as possible, and will therefore obtain a greater yield on a security which is redeemed later. So to *ensure* the investor receives a net annual yield of *at least* i then they should assume the worst case result: that the redemption money is paid as soon as possible, i.e. $n = n_1$.

Finally, if $i^{(p)} = (1-t_1)\dfrac{D}{R}$ then there is neither a capital gain nor a capital loss. So it will make no difference to the investor when the security is redeemed. The net annual yield will be i irrespective of the actual redemption date chosen.

Suppose, alternatively, that the price of the loan is given. The minimum net annual yield is obtained by again assuming the worst case result for the investor. So if:

(a) $P < R$, then the investor receives a capital gain when the security is redeemed. The worst case is that the redemption money is repaid at the **latest** possible date. If this does in fact occur, the net annual yield will be that calculated. If redemption takes place at an earlier date, the net annual yield will be greater than that calculated.

(b) $P > R$, then the investor receives a capital loss when the security is redeemed. The worst case is that the redemption money is repaid at the **earliest** possible date. The actual yield obtained will be at least the value calculated on this basis.

(c) $P = R$, then the investor receives neither a capital gain nor a capital loss. The net annual yield is i, where $i^{(p)} = (1-t_1)\dfrac{D}{R}$, irrespective of the actual redemption date chosen.

Note that a capital gains tax liability does not change any of this. For example, an investment which has a capital gain before allowing for capital gains tax must still have a net capital gain after allowing for the capital gains tax liability, so that the "worst case" for the investor is still the latest redemption. However, in some cases, for example if the redemption price varies, the simple strategy described above will not be adequate, and several values may need to be calculated to determine which is lowest.

2 Uncertain income securities

Securities with uncertain income include:

1. **Equities**, which have regular declarations of **dividends**. The dividends vary according to the performance of the company issuing the stocks and may be zero.

2. **Property** which carries regular payments of rent, which may be subject to regular review.

3. **Index-linked bonds** which carry regular coupon payments and a final redemption payment, all of which are increased in proportion to the increase in a relevant index of inflation.

For all of these investments investors may be interested in calculating the yield for a given price, or the price or value of the security for a given yield. In order to calculate the value or the yield it is necessary to make assumptions about the future income.

Given the uncertain nature of the future income, one method of modelling the cashflows is to assume statistical distributions for, say, the inflation or dividend growth rate. In this course however we will make simpler assumptions — for example that dividends increase at a constant rate. It is important to recognise that modelling random variables deterministically, ignoring the variability of the payments and the uncertainty about the expected growth rate, is not adequate for many purposes and stochastic methods will be required.

In all three cases, using this deterministic approach means that we estimate the future cashflows and then solve the equation of value using the estimated cashflows.

Index-linked bonds differ slightly from the other two in that the income is certain in *real* terms. These are therefore covered separately, in section 4.

2.1 Equities

Given deterministic assumptions about the growth of dividends, we can estimate the future dividends for any given equity, and then solve the equation of value using estimated cashflows for the yield or the price or value.

So, let the value of an equity just after a dividend payment be P, and let D be the amount of this dividend payment. Assume that dividends grow in such a way that the dividend due at time t is estimated to be D_t. We generally value the equity assuming dividends continue in perpetuity, and without explicit allowance for the possibility that the company will default and the dividend payments will cease. In this case, assuming annual dividends,

$$P = \sum_{t=1}^{\infty} D_t v_i^t$$

where i is the return on the share, given price P.

If we assume a constant dividend growth rate of g, say, then $D_t = D(1+g)^t$ and

$$P = Da_{\overline{\infty}|\,i'} \qquad \text{where } i' = \frac{1+i}{1+g} - 1$$

$$\Rightarrow \quad P = \frac{D(1+g)}{i-g}$$

At certain times close to the dividend payment date the equity may be offered for sale *excluding* the next dividend. This allows for the fact that there may not be time between the sale date and the dividend payment date for the company to adjust its records to ensure the buyer receives the dividend. An equity which is offered for sale without the next dividend is called **ex-dividend** or “xd”. The valuation of ex-dividend stocks requires no new principles.

2.2 Property

The valuation of property by discounting future income follows very similar principles to the valuation of equities. Both require some assumption about the increase in future income; both have income which is related to the rate of inflation (both property rents and company profits will be broadly linked to inflation, over the long term); in both cases we use a deterministic approach.

The major differences between the approach to the property equation of value, compared with the equity equation of value, are (1) property rents are generally fixed for a number of years at a time and (2) some property contracts may be fixed term, so that after a certain period the property income ceases and ownership passes back to the original owner (or another investor) with no further payments.

Let P be the price immediately after receipt of the periodic rental payment. Let m be the frequency of the rental payments each year. We estimate the future cashflows, such that D_t / m is the rental income at time t, $t = \frac{1}{m}, \frac{2}{m}, ...$ If the rents cease after some time n then clearly $D_t = 0$ for $t > n$.

Then the equation of value is:

$$P = \ P = \sum_{k=1}^{\infty} \frac{1}{m} D_{k/m} v^{\frac{k}{m}}$$

3 Real rates of interest

The idea of a real rate of interest, as distinct from a money rate of interest, was introduced in Unit 4. Ways of calculating real rates of interest will now be examined.

3.1 Inflation adjusted cashflows

The real rate of interest of a transaction is the rate of interest after allowing for the effect of inflation on a payment series.

The effect of inflation means that a unit of money at, say, time 0 has different purchasing power than a unit of money at any other time. We find the real rate of interest by first adjusting all payment amounts for inflation, so that they are all expressed in units of purchasing power at the same date.

As a simple example, consider a transaction represented by the following payment line:

Time:	0	1
Payment:	–100	120

That is, for an investment of 100 at time 0 an investor receives 120 at time 1.

The effective rate of interest on this transaction is clearly 20% per annum. The real rate of interest is found by first expressing both payments in units of the same purchasing power. Suppose that inflation over this one year period is 5% per annum. This means that 120 at time 1 has a value of 120/1.05 = 114.286 in terms of time 0 money units.

So, in "real" terms, that is, after adjusting for the rate of inflation, the transaction is represented as:

Time:	0	1
Payment:	–100	114.286

Hence, the real rate of interest is 14.286%.

3.2 Calculating real yields using an inflation index

Where the rates of inflation are known (that is, we are looking back in time at a transaction that is complete) we may adjust payments for the rate of inflation by reference to a relevant inflation index.

For example, assume we have an inflation index, $Q(t_k)$ at time t_k , and a payment series as follows:

Time, t:	0	1	2	3
Payment:	–100	8	8	108
$Q(t)$	150	156	166	175

Clearly the rate of interest on this transaction is 8%.

Now we can change all these amounts into time 0 money values by dividing the payment at time t by the proportional increase in the inflation index from 0 to t. For example the inflation-adjusted value of the payment of 8 at time 1 is $8 \div Q(1)/Q(0)$. The series of payments in time 0 money values is then as follows:

Time, t:	0	1	2	3
Payment:	–100	7.6923	7.2289	92.5714

This gives a yield equation for the real yield:

$$-100 + 7.6923\,v_{i'} + 7.2289\,v_{i'}^2 + 92.5714\,v_{i'}^3 = 0 \quad \text{where } i' \text{ is the real rate of interest}$$

which can be solved using numerical methods to give $i' = 2.63\%$.

In general, the real yield equation for a series of cashflows $\{C_{t_1}, C_{t_2}, ..., C_{t_n}\}$, given associated inflation index values $\{Q(0), Q(t_1), Q(t_2), ..., Q(t_n)\}$ is, using time 0 money units:

$$\sum_{k=1}^{n} C_{t_k} \frac{Q(0)}{Q(t_k)} v_{i'}^{t_k} = 0$$

$$\Rightarrow \sum_{k=1}^{n} \frac{C_{t_k}}{Q(t_k)} v_{i'}^{t_k} = 0$$

The second equation here, in which all terms are divided by $Q(0)$, demonstrates that the solution of the yield equation is independent of the date the payment units are adjusted to.

3.3 Calculating real yields given constant inflation assumptions

If we are considering future cashflows, the actual inflation experience will not be known, and some assumption about future inflation will be required. For example, if it is assumed that a constant rate of inflation of j per annum will be experienced, then a cashflow of, say, 100 due at t has value $100(1 + j)^{-t}$ in time 0 money values.
So, for a fixed net cashflow series $\{C_{t_k}\}$, $k = 0, 1, 2, ..., n$, assuming a rate of inflation of j per annum, the real, effective rate of interest, i', is the solution of the real yield equation:

$$\sum_{k=1}^{n} C_{t_k}\, v_j^{t_k}\, v_{i'}^{t_k} = 0$$

We also know that the effective rate of interest with no inflation adjustment which may be called the “money yield” to distinguish from the real yield, is i where

$$\sum_{k=1}^{n} C_{t_k} \, v_i^{t_k} = 0$$

So the relationship between the real yield i', the rate of inflation j and the money yield i is $v_i = v_j \, v_{i'}$

$$\Rightarrow i' = \frac{i-j}{1+j}$$

In some cases a combination of known inflation index values and an assumed future inflation rate may be used to find the real rate of interest.

Conversely, if we know the real yield i' which we have obtained from an equation of value using inflation-adjusted cashflows then we can calculate the money yield as follows:

$$i = i' + j(1 + i')$$

3.4 Payments related to the rate of inflation

Some contracts specify that the cashflows will be adjusted to allow for future inflation, usually in terms of a given inflation index. The index-linked government security is an example. The actual cashflows will be unknown until the inflation index at the relevant dates are known. The contract cashflows will be specified in terms of some nominal amount to be paid at time t, say c_t. If the inflation index at the base date is $Q(0)$ and the relevant value for the time t payment is $Q(t)$ then the actual cashflow is

$$C_t = c_t \, \frac{Q(t)}{Q(0)}$$

It is easy to show that if the real yield i' is calculated by reference to the same inflation index as is used to inflate the cashflows, then i' is the solution of the real yield equation:

$$\sum_{k=1}^{n} C_{t_k} \, \frac{Q(0)}{Q(t_k)} v_{i'}^{t_k} = 0$$

$$\Rightarrow \sum_{k=1}^{n} c_{t_k} \, v_{i'}^{t_k} = 0$$

In other words we can solve the yield equation using the nominal amounts.

However, it is *not* always the case that the index used to inflate the cashflows is the same as that used to calculate the real yield. For example the index-linked UK government security has coupons inflated by reference to the inflation index value 3 months before the payment is made. The real yield, however, is calculated using the inflation index at the actual payment dates.

3.5 The effects of inflation

Consider the simplest situation, in which an investor can lend and borrow money at the same rate of interest i_1. In certain economic conditions the investor may assume that some or all elements of the future cashflows should incorporate allowances for inflation (i.e. increases in prices and wages). The extent to which the various items in the cashflow are subject to inflation may differ. For example, wages may increase more rapidly than the prices of certain goods, or vice versa, and some items (such as the income from rent-controlled property) may not rise at all, even in highly inflationary conditions.

The case when **all** items of cashflow are subject to the same rate of escalation j per time unit is of special interest. In this case we find or estimate c_t^e and $\rho^e(t)$, the net cashflow and the net rate of cashflow allowing for escalation at rate j per unit time, by the formulae

$$c_t^e = (1+j)^t c_t \tag{3.1}$$

$$\rho^e(t) = (1+j)^t \rho(t) \tag{3.2}$$

where c_t and $\rho(t)$ are estimates of the net cashflow and the net rate of cashflow respectively at time t without any allowance for inflation. It follows that, with allowance for inflation at rate j per unit time, the net present value of an investment or business project at rate of interest i is

$$\begin{aligned} NPV_j(i) &= \sum c_t(1+j)^t(1+i)^{-t} + \int_0^\infty \rho(t)(1+j)^t(1+i)^{-t}\,dt \\ &= \sum c_t(1+i_0)^{-t} + \int_0^\infty \rho(t)(1+i_0)^{-t}\,dt \qquad (3.3) \end{aligned}$$

where:

$$1+i_0 = \frac{1+i}{1+j}$$

or:

$$i_0 = \frac{i-j}{1+j} \tag{3.4}$$

If j is not too large, one sometimes uses the approximation

$$i_0 \approx i - j \tag{3.5}$$

These results are of considerable practical importance, because projects which are apparently unprofitable when rates of interest are high may become highly profitable when even a modest allowance is made for inflation. It is, however, true that in many ventures the positive cashflow generated in the early years of the venture is insufficient to pay bank interest, so recourse must be had to further borrowing (unless the investor has adequate funds of their own). This in itself does not undermine the profitability of the project, but the investor would require the agreement of his lending institution before further loans could be obtained and this might cause difficulties in practice.

4 Index-linked bonds

Index-linked bond cashflows are described in Unit 10. The coupon and redemption payments are increased according to an index of inflation.

Given simple assumptions about the rate of future inflation it is possible to estimate the future payments. Given these assumptions we may calculate the price or yield by solving the equation of value using the estimated cashflows.

For example, let the nominal annual coupon rate for an n-year index-linked bond be D per £1 nominal face value with coupons payable half-yearly, and let the nominal redemption price be R per £1 nominal face value. We assume that payments are inflated by reference to an index with base value $Q(0)$, such that the coupon due at time t years is

$$\frac{D}{2}\frac{Q(t)}{Q(0)}$$

Then the equation of value, given an effective (money) yield of i per annum, and a present value or price P per £1 nominal at issue or immediately following a coupon payment, is

$$P = \sum_{k=1}^{2n} \frac{D}{2}\frac{Q(k/2)}{Q(0)} v_i^{\frac{k}{2}} + R\frac{Q(n)}{Q(0)} v_i^n$$

We estimate the unknown value of $Q(t)$ using some assumption about future inflation and using the latest known value — which may be $Q(0)$. For example, assume inflation increases at rate j_t per annum in the year $t-1$ to t, then we have

$$Q(½) = Q(0) \,.\, (1+j_1)^{½}$$

$$Q(1) = Q(0) \,.\, (1+j_1)$$

$$Q(1½) = Q(0) \,.\, (1+j_1)(1+j_2)^{½}$$

$$Q(2) = Q(0) \,.\, (1+j_1)(1+j_2)$$

etc.

It is important to bear in mind that the index used may not be the same as the actual inflation index value at time t that one would use, for example, to calculate the real (inflation-adjusted) yield. In the case of UK index-linked bonds, the payments are increased using the index values from 3 months before the payment date. Real yields would be calculated using the inflation index values *Pt* the payment date.

Like equities, index-linked bonds (and fixed interest bonds) may be offered for sale “ex-dividend”. No new principles are involved in the valuation of ex-dividend index-linked bonds.

E N D

UNIT 12 — THE “NO ARBITRAGE” ASSUMPTION AND FORWARD CONTRACTS

Syllabus objective (xii) Calculate the delivery price and the value of a forward contract using arbitrage free pricing methods.

1. Define “arbitrage” and explain why arbitrage may be considered impossible in many markets.

2. Calculate the price of a forward contract in the absence of arbitrage assuming:

   - no income or expenditure associated with the underlying asset during the term of the contract
   - a fixed income from the asset during the term
   - a fixed dividend yield from the asset during the term

3. Explain what is meant by “hedging” in the case of a forward contract.

4. Calculate the value of a forward contract at any time during the term of the contract in the absence of arbitrage, in the situations listed in 2 above.

1 The “No Arbitrage” assumption

1.1 Introduction

Arbitrage in financial mathematics is generally described as a risk-free trading profit. More accurately, an arbitrage opportunity exists if either

(a) an investor can make a deal that would give her or him an immediate profit, with no risk of future loss;

or

(b) an investor can make a deal that has zero initial cost, no risk of future loss, and a non-zero probability of a future profit.

The concept of arbitrage is very important because we generally assume that in modern developed financial markets arbitrage opportunities do not exist. This assumption is referred to as the “No Arbitrage” assumption, and is fundamental to modern financial mathematics.

If we assume that there are no arbitrage opportunities in a market, then it follows that any two securities or combinations of securities that give exactly the same payments must have the same price. This is sometimes called the “Law of One Price”. The ideas are demonstrated in the following example.

Example 1

Consider a very simple securities market, consisting of two securities, A and B. At time $t = 0$ the prices of the securities are P_0^A and P_0^B respectively. The term of both the securities is 1 year. At $t = 1$ there are two possible outcomes. Either the “market” goes up, in which case security A pays $P_1^A(u)$ and B pays $P_1^B(u)$, or it goes down, with payments $P_1^A(d)$ and $P_1^B(d)$ respectively.

Investors can buy securities, in which case they pay the time 0 price and receive the time 1 income, or they can sell securities, in which case they receive the time 0 price and must pay the time 1 outgo.

Now, assume first that we have the following payment table:

	Time 0 price	Market goes up	Market goes down
Security:	P_0	$P_1(u)$	$P_1(d)$
	£	£	£
A	6	7	5
B	11	14	10

There is an arbitrage opportunity here. An investor could buy one unit of security B and sell two units of security A. This would give income at time 0 of £12 from the sale of security A and an outgo of £11 from the purchase of security B — which gives a net income at time 0 of £1. At time 1 the outgo due on the portfolio of 2 units of security A exactly matches the income due from security B, whether the market moves up or down. Thus, the investor makes a profit at time 0, with no risk of a future loss.

It is clear that investment A is unattractive compared with investment B. This will cause pressure to reduce the price of A and to increase the price of B, as there will be no demand for A and an excessive demand for B. Ultimately we would achieve balance, when $P_0^A = P_0^B / 2$, when the arbitrage opportunity is eliminated, and the prices are consistent.

Another example is given in the following table:

Security:	Time 0 price P_0 £	Market goes up $P_1(u)$ £	Market goes down $P_1(d)$ £
A	6	7	5
B	6	7	4

An arbitrage opportunity exists, as an investor could buy one unit of A and sell one unit of B. The net income at time 0 is £0, as the income from the sale of B matches the outgo on the purchase of A. At time 1 the net income is £0 if the market goes up, and £1 if the market goes down. So, for a zero investment, the investor has a possibility of making a profit (assuming the probability that the market goes down is not zero) and no possibility of making a loss.

With these prices, investors will naturally choose to buy investment A and will want to sell investment B. This will put pressure on the price of A to increase, and on the price of B to decrease. The arbitrage opportunity is eliminated when $P_0^A > P_0^B$.

1.2 Why do we assume "No Arbitrage"?

The "No Arbitrage" assumption is very simple and very powerful. It enables us to find the price of complex instruments by "replicating" the payoffs. This means that if we can construct a portfolio of assets with exactly the same payments as the investment that we are interested in, then the price of the investment must be the same as the price of the "replicating portfolio".

In practice, in the major developed securities markets arbitrage opportunities, when they do arise are very quickly eliminated as investors spot them and trade on them. Such opportunities are so fleeting in nature, according to the empirical evidence, that it is sensible, realistic and prudent to assume that they do not exist. We also assume here that there are no transaction costs or taxes associated with buying, selling or holding assets. These are also idealised assumptions, but they enable us to develop a methodology that may be adapted to deal with these institutional features if necessary.

The "No Arbitrage" assumption will be used extensively in Subject CT8, Financial Economics. In this subject we introduce the ideas in the context of "Forward Contracts".

2 Forward contracts

2.1 Introduction

A **forward contract** is an agreement made at some time $t = 0$, say, between two parties under which one agrees to buy from the other a specified amount of an asset (denoted S) at a specified price on a specified future date. The investor agreeing to sell the asset is said to hold a *short forward position* in the asset, and the buyer is said to hold a *long forward position*.

Let S_r be the price of the underlying asset (for example, a unit of equity stock) at time r. The price will not generally be predictable — for example, we may contract to buy shares in a company in 6 months time. We know what the current price of the shares is, but the price will vary more or less continuously, so we do not know with certainty what the share price will be at any future date.

Let K be the price agreed at time $t = 0$ to be paid at time $t = T$, called the **forward price**. $t = 0$ is the time the forward contract is agreed, T is the time the contract matures (that is, when the sale actually happens). We also assume there is a known force of interest δ that is available on a risk-free investment over the term of the contract. This is known as the "risk-free" force of interest.

At time 0 when the agreement is made no money changes hands (except possibly a "good faith" deposit — we will ignore this here). The price K agreed at time $t = 0$ is determined such that the value of the forward contract at the time $t = 0$ is zero.

The forward contract **will** generally have non-zero value at time T; if $K > S_T$ then the seller receives K for an asset worth (at that time) S_T, and has made a profit at time T of $K - S_T$. Similarly, if $K < S_T$ then the buyer has paid K for an asset worth S_T, giving the buyer a profit at time T of $S_T - K$.

2.2 Calculating the forward price for a security with no income

One important question is how to determine the forward price, K. This is the price *agreed* at time $t = 0$ but not actually *paid* until the contract ends, at $t = T$.

The price S_r is uncertain for all $r > 0$. However, using the no arbitrage assumption we can find the forward price without having to make any assumption about the statistical properties of the process S_r. Instead, we can use a replication argument.

We assume at this stage that there are no payments or costs associated with holding the stock.

Consider the following two investment portfolios:

Portfolio A: Enter a forward contract to buy one unit of an asset S, with forward price K, maturing at time T; simultaneously invest an amount $Ke^{-\delta T}$ in the risk-free investment.

Portfolio B: Buy one unit of the asset, at the current price S_0 .

At time $t = 0$ the **price of Portfolio A** is $Ke^{-\delta T}$ for the risk-free investment; recall that the price of a forward contract is zero.

The **price of Portfolio B** is S_0.

At time $t = T$ the **cashflows for Portfolio A** are:

An amount K is received from the risk-free investment ($Ke^{-\delta T}$ invested at force of interest δ for T years gives an accumulated value of K). The same amount K is paid on the forward contract. Receive 1 unit of asset S.

The **payout from Portfolio B** is one unit of asset S.

Now, the future cashflows of portfolio A are identical to those of portfolio B — both give a net portfolio of one unit of the underlying asset S. The no arbitrage assumption states that when the future cashflows of two portfolios are identical, the price must also be the same — that is:

$$Ke^{-\delta T} = S_0 \Rightarrow K = S_0\, e^{\delta T}$$

The no arbitrage assumption gives the price for the forward contract with no need for any model of how the asset price S_t will actually move over the term of the contract.

2.3 Calculating the forward price for a security with fixed cash income

Assume now that at some time t_1, $0 \le t_1 < T$, the security underlying the forward contract provides a fixed amount c to the holder. For example, if the security is a government bond, there will be fixed coupon payments due every six months.

Now consider the following two portfolios:

Portfolio A: Enter a forward contract to buy one unit of an asset S, with forward price K, maturing at time T; simultaneously invest an amount $Ke^{-\delta T} + ce^{-\delta t_1}$ in the risk-free investment.

Portfolio B: Buy one unit of the asset, at the current price S_0. At time t_1 invest the income of c in the risk-free investment.

At time $t = 0$ the **price of Portfolio A** is $Ke^{-\delta T} + ce^{-\delta t_1}$ for the risk-free investment, and zero for the forward contract.

The **price of Portfolio B** is S_0.

At time $t = T$ the **payout from Portfolio A** is: Income of $K + ce^{\delta(T-t_1)}$ from the risk-free investment; Outgo of K on the forward contract. Receive 1 unit of asset, value S_T. The net portfolio at T is one unit of the asset S plus $ce^{\delta(T-t_1)}$ units of the risk-free security.

The **payout from Portfolio B** is one unit of asset, value S_T plus $ce^{\delta(T-t_1)}$ units of the risk-free security, from the invested coupon payment.

The net cashflows of portfolio A at time T are identical to those of portfolio B — both give a net portfolio of one unit of the underlying asset S plus $ce^{\delta(T-t_1)}$ units of the risk-free security. Using the no arbitrage assumption the prices must also be the same — that is:

$$Ke^{-\delta T} + ce^{-\delta t_1} = S_0 \Rightarrow K = S_0\, e^{\delta T} - ce^{\delta(T-t_1)}$$

For a long forward contract on a fixed interest security there may be more than one coupon payment. It is easy to adapt the above method to allow for this. If we let I denote the present value at time $t = 0$ of the fixed income payments due during the term of the forward contract, then the forward price at time $t = 0$ per unit of security S is

$$K = (S_0 - I)\, e^{\delta T}$$

2.4 Calculating the forward price for a security with known dividend yield

Let D be the known dividend yield per annum. We assume that dividends are received continuously, and are immediately reinvested in the security of S.

If we start with one unit of the security at time $t = 0$, the accumulated holding at time T would be e^{DT} units of the security. This is because the number of units owned is continuously compounding at rate D per annum for T years. If instead of 1 unit at time $t = 0$ we hold e^{-DT} units, reinvesting the dividend income, at time T we would hold $e^{DT}\, e^{-DT} = 1$ unit of the security.

Now consider the following two portfolios:

Portfolio A: Enter a forward contract to buy one unit of an asset S, with forward price K, maturing at time T; simultaneously invest an amount $Ke^{-\delta T}$ in the risk-free investment.

Portfolio B: Buy e^{-DT} units of the asset S, at the current price S_0. Reinvest dividend income in the security S immediately it is received.

At time $t = 0$ the **price of Portfolio A** is $Ke^{-\delta T}$ for the risk-free investment, and zero for the forward contract.

The **price of Portfolio B** is $e^{-DT} S_0$.

At time $t = T$ the **cashflows of Portfolio A** are: An amount K is received from the risk-free investment. Outgo K is paid on the forward contract. Receive 1 unit of asset S. The net portfolio at T is one unit of the asset S.

The **payout from Portfolio B** is one unit of the asset S.

The net cashflows of portfolio A at time T are identical to those of portfolio B — both give a net portfolio of one unit at the underlying asset S. Using the no arbitrage assumption the prices must also be the same — that is:

$$Ke^{-\delta T} = S_0\, e^{-DT} \Rightarrow K = S_0\, e^{(\delta - D)T}$$

It is simple to adjust the portfolios to get the forward price if the dividends are paid discretely.

The important principle for this case and the known income case is that, if the income is proportional to the underlying security, S, we assume the income is reinvested in the security. If the income is a fixed amount regardless of the price of the security at the payment date, then we assume it is invested in the risk-free security. This is because when the payment is proportional to the stock price (e.g. dividends) we know how many units of stock they will purchase, but we do not know how much cash is paid (as the stock price is unknown). So we can predict the amount of stock held at the end if we assume reinvestment in the stock. With a cash payment on the other hand, we would not know how much stock could be bought, but we do know how much the cash would accumulate to at the risk-free force of interest. Assuming dividends are reinvested in the security, but cash is invested at the risk-free (and known) force of interest enables us to predict the final portfolio without requiring any information about the price of the asset S during the course of the contract.

2.5 Hedging

Hedging is a general term which describes the use of financial instruments (including stocks, bonds, forward contracts and more complex financial contracts such as options) to reduce or eliminate a future risk of loss.

An investor who agrees to sell an asset at a given price in a forward contract need not hold the asset at the start of the contract. However, by the end of the contract he or she must own the asset ready to sell under the terms of the forward contract. If the investor waits until the end of the contract to buy the asset S the risk exists that the price will rise above the forward price, and they will have to pay more than the forward price K than they receive for the asset. On the other hand, if they buy the asset at the start of the forward contract, and hold it until the contract matures, there is a risk that the price will have fallen, and they have paid more than they needed to.

To hedge the risk the investor could borrow an amount $Ke^{-\delta T}$ at the risk-free force of interest, buy the asset S at the start of the contract, at the price S_0, and hold it until it is to be handed over at time T. The price of this "hedge portfolio" is $-Ke^{-\delta T} + S_0 = 0$. We are assuming here that there is no interest or dividend income associated with the asset S.

At time T the investor owes K that is exactly covered by the forward price received at T. He or she also owes one unit of asset S under the forward contract, which is also paid from the hedge portfolio.

This way, if the investor holds the hedge portfolio he or she is certain not to make a loss on the forward contract. There is also no chance of making a profit.

This is called a "static hedge" since the hedge portfolio, which consists of the asset to be sold plus the borrowed risk-free investment, does not change over the term of the contract. For more complex financial instruments, such as options, the hedge portfolio is more complex, and requires (in principle) continuous rebalancing to maintain. This is called a "dynamic hedge".

2.6 The value of a forward contract

2.6.1 With no interest or dividend income

Consider a forward contract agreed at time $t = 0$, with a forward price K_0, for one unit of security S. The maturity date of the contract is time T.

At the start of the contract the value, to buyer and seller of the asset S, is 0. At the maturity date the value of the contract to the seller of the asset is $K_0 - S_T$ and to the buyer is $-(K_0 - S_T)$.

It is of interest to find the value of the contract at intermediate times.

Suppose at time $r > 0$ an investor holds a long forward contract — that is, holds a contract agreeing to buy an asset S at $T > r$ at a price agreed at time $t = 0$ of K_0. The investor wants to know the value of this contract during the term at time r.

Consider the following two portfolios purchased at time r:

Portfolio A: Consists of the existing long forward contract (bought at time 0) with current value V_l. Invest $K_0\, e^{-\delta(T-r)}$ at time r in the risk-free investment for $T - r$ years.

Portfolio B: Buy a new long forward contract at time r, with maturity at T, forward price $K_r = S_r\, e^{\delta(T-r)}$. The price of a forward contract at issue is zero. Also, invest $K_r\, e^{-\delta(T-r)}$ in the risk-free investment for $T - r$ years.

The price of portfolio A at r is $V_l + K_0\, e^{-\delta(T-r)}$.

The price of portfolio B at r is $K_r\, e^{-\delta(T-r)}$.

The payout from portfolio A at T is one unit of the asset S; the investment of $K_0\, e^{-\delta(T-r)}$ accumulates to K_0, which matches the outgo on the forward contract.

The payout from portfolio B at T is also one unit of the asset S; again, the risk-free investment accumulates to K_r, which meets the forward price required.
By the no arbitrage assumption we have:

$$V_l + K_0\, e^{-\delta(T-r)} = K_r\, e^{-\delta(T-r)} \Rightarrow V_l = (K_r - K_0)\, e^{-\delta(T-r)}$$

We may substitute K_r and K_0 to get the value of the forward contract in terms of the asset price,

$$V_l = S_r - S_0\, e^{\delta r}$$

which gives the value of the long forward contract at time r. The value of a short forward contract may be determined by similar arguments, to be $V_s = S_0\, e^{\delta r} - S_r$, that is $V_s = -V_l$.

The value of forward contracts where there is some interest or dividend income associated with the underlying asset may be determined easily using similar arguments.

2.7 Note

We have simplified the calculations by assuming that the risk-free force of interest is independent of the time or duration of the investment. In fact, as shown in Unit 13 there is a *term structure* to interest rates — that is, the interest rate earned on an investment depends on both the time a sum is invested and on the length of time for which it is invested. The results above may be adjusted to allow for this, replacing δ with the appropriate *spot* or *forward* force of interest.

E N D

UNIT 13 — TERM STRUCTURE OF INTEREST RATES

Syllabus objective (xiii) Show an understanding of the term structure of interest rates.

1. Describe the main factors influencing the term structure of interest rates.

2. Explain what is meant by the par yield and yield to maturity.

3. Explain what is meant by, derive the relationships between and evaluate:

   - discrete spot rates and forward rates
   - continuous spot rates and forward rates

4. Define the duration and convexity of a cashflow sequence, and illustrate how these may be used to estimate the sensitivity of the value of the cashflow sequence to a shift in interest rates.

5. Evaluate the duration and convexity of a cashflow sequence.

6. Explain how duration and convexity are used in the (Redington) immunisation of a portfolio of liabilities.

1 Introduction

So far in this course it has generally been assumed that the interest rate i or force of interest δ earned on an investment are independent of the term of that investment. In practice the interest rate offered on investments does usually vary according to the term of the investment. It is often important to take this variation into consideration.

In investigating this variation we make use of **unit zero coupon bond** prices. A unit zero coupon bond of term n, say, is an agreement to pay £1 at the end of n years. No coupon payments are paid. It is also called a **pure discount bond**.

We denote the price at issue of a unit zero coupon bond maturing in n years by P_n.

2 Discrete time

2.1 Discrete time spot rates

The yield on a unit zero coupon bond with term n years, y_n , is called the ***n*-year spot rate of interest**. Using the equation of value for the zero coupon bond we find the yield on the bond y_n from

$$P_n = \frac{1}{(1+y_n)^n} \Rightarrow (1 + y_n) = P_n^{-\frac{1}{n}}$$

Since rates of interest differ according to the term of the investment, in general $y_s \neq y_t$ for $s \neq t$. Every fixed interest investment may be regarded as a combination of (perhaps notional) zero coupon bonds. For example, a bond paying coupons of D every year for n years, with a final redemption payment of R at time n may be regarded as a combined investment of n zero coupon bonds with maturity value D, with terms of 1 year, 2 years ..., n years, plus a zero coupon bond of nominal value R with term n years.

Defining $v_{y_t} = (1 + y_t)^{-1}$, the price of the bond is:

$$\begin{aligned} A &= D(P_1 + P_2 + ... + P_n) + RP_n \\ &= D(v_{y_1} + v_{y_2}^2 + ... + v_{y_n}^n) + Rv_{y_n}^n \end{aligned}$$

This is actually a consequence of "no arbitrage"; the portfolio of zero coupon bonds has the same payouts as the fixed interest bond, and the prices must therefore be the same.

The variation by term of interest rates is often referred to as the **term structure of interest rates**. The curve of spot rates $\{y_t\}$ is an example of a **yield curve**.

2.2 Discrete time forward rates

The discrete time forward rate, $f_{t,r}$, is the annual interest rate agreed at time 0 for an investment made at time $t > 0$ for a period of r years.

That is, if an investor agrees at time 0 to invest £100 at time t for r years, the accumulated investment at time $t + r$ is

$$100(1 + f_{t,r})^r$$

Forward rates, spot rates and zero-coupon bond prices are all connected. The accumulation at time t of an investment of 1 at time 0 is $(1 + y_t)^t$. If we agree at time 0 to invest the amount $(1 + y_t)^t$ at time t for r years, we will earn an annual rate of $f_{t,r}$. So we know that £1 invested for $t + r$ years will accumulate to $(1 + y_t)^t (1 + f_{t,r})^r$. But we also know from the $(t + r)$ spot rates that £1 invested for $t + r$ years accumulates to $(1 + y_{t+r})^{t+r}$, and we also know from the zero coupon bond prices that £1 invested for $t + r$ years accumulates to P_{t+r}^{-1}. Hence we know that

$$(1 + y_t)^t (1 + f_{t,r})^r = (1 + y_{t+r})^{t+r} = P_{t+r}^{-1}$$

from which we find that

$$(1 + f_{t,r})^r = \frac{(1 + y_{t+r})^{t+r}}{(1 + y_t)^t} = \frac{P_t}{P_{t+r}}$$

so that the full term structure may be determined given the spot rates, the forward rates or the zero coupon bond prices.

One-period forward rates are of particular interest. The one-period forward rate at time t (agreed at time 0) is denoted $f_t = f_{t,1}$. We define $f_0 = y_1$. Comparing an amount of £1 invested for t years at the spot rate y_t, and the same investment invested 1 year at a time with proceeds reinvested at the appropriate one-year forward rate, we have

$$(1 + y_t)^t = (1 + f_0)(1 + f_1)(1 + f_2) \ldots (1 + f_{t-1})$$

3 Continuous time rates

3.1 Continuous time spot rates

Let P_t be the price of a unit zero coupon bond of term t. Then the t-year spot *force of interest* is Y_t where

$$P_t = e^{-Y_t t} \Rightarrow Y_t = -\frac{1}{t} \log P_t$$

This is also called the continuously compounded spot rate of interest or the continuous-time spot rate. Y_t and its corresponding discrete annual rate y_t are connected in the same way as δ and i; an investment of £1 for t years at a discrete spot rate y_t accumulates to $(1 + y_t)^t$; at the continuous time rate it accumulates to $e^{Y_t t}$; these must be equal, so $y_t = e^{Y_t} - 1$.

3.2 Continuous time forward rates

The continuous time forward rate $F_{t,r}$ is the force of interest equivalent to the annual forward rate of interest $f_{t,r}$.

A £1 investment of duration r years, starting at time t, agreed at time $0 \le t$ accumulates using the annual forward rate of interest to $(1 + f_{t,r})^r$ at time $t + r$.

Using the equivalent forward force of interest the same investment accumulates to $e^{F_{t,r}r}$. Hence the annual rate and continuous-time rate are related as

$$f_{t,r} = e^{F_{t,r}} - 1$$

The relationship between the continuous time spot and forward rates may be derived by considering the accumulation of £1 at a continuous time spot rate of Y_t for t years, followed by the continuous time forward rate of $F_{t,r}$ for r years. Compare this with an investment of £1 at a continuous time spot rate of Y_{t+r} for $t + r$ years. The two investments are equivalent, so the accumulated values must be the same. Hence

$$e^{tY_t} \, e^{rF_{t,r}} = e^{(t+r)Y_{t+r}}$$

$$\Rightarrow \; tY_t + rF_{t,r} = (t + r)\, Y_{t+r}$$

$$\Rightarrow \; F_{t,r} = \frac{(t + r)\, Y_{t+r} - tY_t}{r}$$

Also, using $Y_t = -\frac{1}{t} \log P_t$, we have

$$F_{t,r} = \frac{1}{r} \log\left(\frac{P_t}{P_{t+r}}\right)$$

3.3 Instantaneous forward rates

The instantaneous forward rate F_t is defined as

$$F_t = \lim_{r \to 0} F_{t,r}$$

The instantaneous forward rate may broadly be thought of as the forward force of interest applying in the instant of time $t \to t + \Delta t$.

$$F_t = \lim_{r \to 0} \frac{1}{r} \log\left(\frac{P_t}{P_{t+r}}\right) \qquad (1)$$

$$= -\lim_{r \to 0} \frac{\log P_{t+r} - \log P_t}{r} \qquad (2)$$

$$= -\frac{d}{dt}\log P_t \tag{3}$$

$$= -\frac{1}{P_t}\frac{d}{dt}P_t \tag{4}$$

We also find, by integrating both sides of (3) and using the fact that $P_0 = 1$ (as the price of a unit zero coupon bond of term zero years must be 1), that

$$P_t = e^{-\int_0^t F_s ds}$$

<u>Note</u>

We have described in this unit the **initial term structure**, where everything is fixed at time 0. In practice the term structure varies rapidly over time, and the 5-year spot rate tomorrow may be quite different from the 5-year spot rate today. In more sophisticated treatments we model the change in term structure over time.

In this case all the variables we have used, i.e.

$$P_t \quad y_t \quad f_{t,r} \quad Y_t \quad F_{t,r}$$

need another argument, v, say, to give the "starting point". For example, $y_{v,t}$ would be the t-year discrete spot rate of interest applying **at time** v; $F_{v,t,r}$ would be the force of interest agreed at time v, applying to an amount invested at time $v + t$ for the r-year period to time $v + t + r$.

4 Theories of the term structure of interest rates

4.1 Introduction

Some examples of typical (spot rate) yield curves are given below.

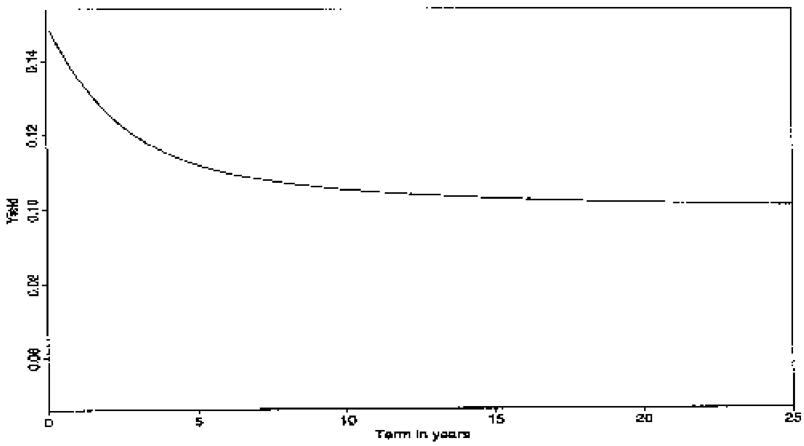

Figure 1: Decreasing yield curve

In Figure 1 the long-term bond yields are lower than the short-term bonds. Since price is a decreasing function of yield, an interpretation is that long-term bonds are more expensive than short-term bonds. There are several possible explanations — for example it is possible that investors believe that they will get a higher overall return from long-term bonds, despite the lower current yields, and the higher demand for long-term bonds has pushed up the price, which is equivalent to pushing down the yield, compared with short-term bonds. Other explanations for different yield curve shapes are given below.

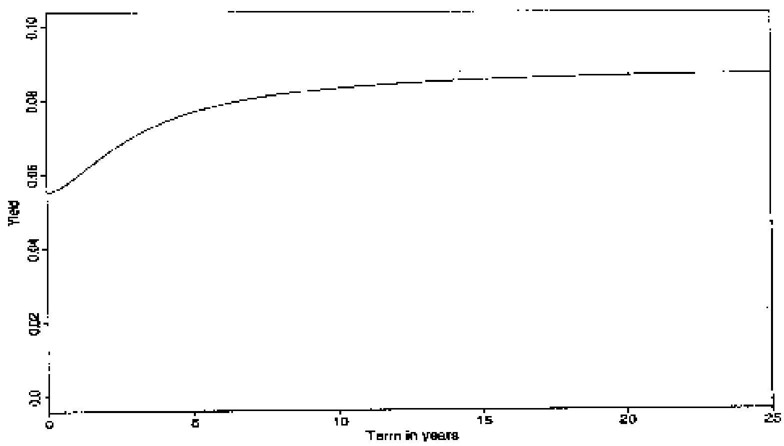

Figure 2: Increasing yield curve

In Figure 2 the long-term bonds are higher yielding (or cheaper) than the short-term bonds.

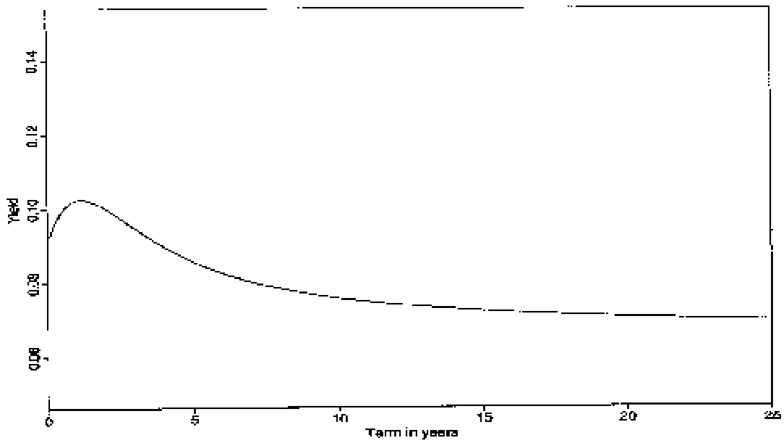

Figure 3: Humped yield curve

In Figure 3 the short-term bonds are generally cheaper than the long bonds, but the very short rates (with terms less than 1 year) are lower than the 1 year rates.

The three most popular explanations for the fact that interest rates vary according to the term of the investment are:

1. Expectations Theory

2. Liquidity Preference

3. Market Segmentation

Expectations Theory The relative attraction of short and longer term investments will vary according to expectations of future movements in interest rates. An expectation of a fall in interest rates will make short-term investments less attractive and longer term investments more attractive. In these circumstances yields on short-term investments will rise and yields on long-term investments will fall. An expectation of a rise in interest rates will have the converse effect.

In Figure 1 it appears that the demand for long-term bonds may be greater than for short, implying an expectation that interest rates will fall. By buying long-term bonds investors can continue getting higher rates after a future fall in interest rates, for the duration of the long bond.

In Figure 2 the demand is higher for short-term bonds — perhaps indicating an expectation of a rise in interest rates.

Liquidity Preference Longer dated bonds are more sensitive to interest rate movements than short dated bonds. It is assumed that risk averse investors will require compensation (in the form of higher yields) for the greater risk of loss on longer bonds. This might explain some of the excess return offered on long-term bonds over short-term bonds in Figure 2.

Market Segmentation Bonds of different terms are attractive to different investors, who will choose assets that are similar in term to their liabilities. The liabilities of banks, for example, are very short-term (investors may withdraw a large proportion of the funds at very short notice); hence banks invest in very short-term bonds. Many pension funds have liabilities that are very long-term, so pension funds are more interested in the longest dated bonds. The demand for bonds will therefore differ for different terms. The supply of bonds will also vary by term, as governments and companies' strategies may not correspond to the investors' requirements. The market segmentation hypothesis argues that the term structure emerges from these different forces of supply and demand.

These theories are covered in more detail in Subject CA1, Core Applications Concepts.

4.2 Yields to maturity

The yield to maturity for a coupon paying bond (also called the redemption yield) has been defined as the effective rate of interest at which the discounted value of the proceeds of a bond equal the price. It is widely used, but has the disadvantage that it depends on the coupon rate of the bond, and therefore does not give a simple model of the relationship between term and yield.

In the UK, yield curves plotting the average (smoothed) yield to maturity of coupon paying bonds are produced separately for "low coupon", "medium coupon" and "high coupon" bonds.

4.3 Par yields

The ***n*-year par yield** represents the coupon per £1 nominal that would be payable on a bond with term n years, which would give the bond a current price under the current term structure of £1 per £1 nominal, assuming the bond is redeemed at par.

That is, if yc_n is the n-year par yield,

$$1 = (yc_n)\ (v_{y_1} + v_{y_2}^2 + v_{y_3}^3 + \ldots + v_{y_n}^n) + 1v_{y_n}^n$$

The par yields give an alternative measure of the relationship between the yield and term of investments. The difference between the par yield rate and the spot rate is called the **coupon bias**.

5 Duration, convexity and immunisation

In this section we consider simple measures of vulnerability to interest rate movements. For simplicity we assume a flat yield curve, and that when interest rates change, all change by the same amount, so that the curve stays flat. A flat yield curve implies that $y_t = f_{t,r} = i$ for all t, r and $Y_t = F_{t,r} = F_t = \delta$ for all t, r.

5.1 Interest rate risk

Suppose an institution holds assets of value V_A, to meet liabilities of value V_L. Since both V_A and V_L represent the discounted value of future cashflows, both are sensitive to the rate of interest. We assume that the institution is healthy at time 0 so that currently $V_A \geq V_L$.

If rates of interest fall, both V_A and V_L will increase. If rates of interest rise then both will decrease. We are concerned with the risk that following a downward movement in interest rates the value of assets increases by less than the value of liabilities, or that, following an upward movement in interest rates the value of assets decreases by more than the value of the liabilities.

In order to examine the impact of interest rate movements on different cashflow sequences we will use changes in the yield to maturity to represent changes in the underlying term structure. This is approximately (but not exactly) the same as assuming a constant movement of similar magnitude in the one-period forward rates.

5.2 Effective duration

One measure of the sensitivity of a series of cashflows, to movements in the interest rates, is the **effective duration** (or volatility). Consider a series of cashflows $\{C_{t_k}\}$ for k = 1, 2, ..., n. Let A be the present value of the payments at rate (yield to maturity) i, so that

$$A = \sum_{k=1}^{n} C_{t_k} \, v_i^{t_k}$$

Then the effective duration is defined to be

$$v(i) = -\frac{1}{A}\frac{d}{di} A = -\frac{A'}{A} \tag{5.2.1}$$

$$= \left(\frac{1}{\sum_{k=1}^{n} C_{t_k} v_i^{t_k}} \right) \left(\sum_{k=1}^{n} C_{t_k} \; t_k \; v_i^{t_k+1} \right)$$

This is a measure of the rate of change of value of A with i, which is independent of the size of the present value. Equation 5.2.1 assumes that the cashflows do not depend on the rate of interest.

For a small movement ε in interest rates, from i to $i + \varepsilon$, the relative change in value of the present value is approximately $-\varepsilon v(i)$ so the new present value is approximately $A(1 - \varepsilon v(i))$.

5.3 Duration

Another measure of interest rate sensitivity is the **duration**, also called **Macauley Duration** or discounted mean term. This is the mean term of the cashflows $\{C_{t_k}\}$, weighted by present value. That is, at rate *i*, the duration of the cashflow sequence $\{C_{t_k}\}$ is

$$\tau = \frac{\sum_{k=1}^{n} t_k \; C_{t_k} \; v_i^{t_k}}{\sum_{k=1}^{n} C_{t_k} \; v_i^{t_k}}$$

Comparing this expression with the equation for the effective duration it is clear that

$$\tau = (1 + i)\, v(i)$$

Another way of deriving the Macauley duration is in terms of the force of interest, δ:

$$\tau = -\frac{1}{A}\frac{d}{d\delta} A = \frac{di}{d\delta}\; v(i)$$

$$i = e^{\delta} - 1 \Rightarrow \frac{di}{d\delta} = e^{\delta}$$

$$\Rightarrow \tau = e^{\delta}\, v(i) = (1 + i)\, v(i)$$

The equation for τ in terms of the cashflows C_{t_k} may be found by differentiating *A* with respect to δ, recalling that $v_i^{t_k} = e^{-\delta t_k}$.

The duration of an *n*-year coupon paying bond, with coupons of *D* payable annually, redeemed at *R*, is

$$\tau = \frac{D(Ia)_{\overline{n}|} + Rnv^n}{Da_{\overline{n}|} + Rv^n}$$

The duration of an *n*-year zero coupon bond of nominal amount 100, say, is

$$\tau = \frac{100nv^n}{100v^n} = n$$

Note that another definition of duration exists; the **modified duration**. This is covered in more detail in Subjects ST5 and ST6 (Finance and Investment Specialist Technical A and B) but can be expressed in terms of the Macauley Duration as

$$\frac{\tau}{1+\dfrac{i^{(p)}}{p}}$$

Where $i^{(p)}$ and p are as defined in Unit 3.

5.4 Convexity

The **convexity** of the cashflow series $\{C_{t_k}\}$ is defined as

$$c(i) = \frac{1}{A}\frac{d^2}{di^2}A = \frac{A''}{A}$$

$$= \left(\frac{1}{\sum\limits_{k=1}^{n} C_{t_k} v_i^{t_k}}\right)\left(\sum\limits_{t=1}^{n} C_{t_k}\, t_k\, (t_k+1)\, v_i^{t_k+2}\right)$$

Combining convexity and duration gives a more accurate approximation to the change in A following a small change in interest rates. For small ε

$$\frac{A(i+\varepsilon)-A(i)}{A} = \frac{\partial A}{\partial i}\times\frac{1}{A}\times\varepsilon$$

$$+ ½ \times \frac{\partial^2 A}{\partial i^2}\times\frac{1}{A}\times\varepsilon^2$$

$$+ \ldots$$

$$\approx \quad -\varepsilon\nu(i) + \varepsilon^2 \times ½ \times c(i)$$

Convexity gives a measure of the change in duration of a bond when the interest rate changes. Positive convexity implies that $\tau(i)$ is a decreasing function of i. This means, for example, that A increases more when there is a decrease in interest rates than it falls when there is an increase of the same magnitude in interest rates.

5.5 Immunisation

Consider a fund with asset cashflows $\{A_{t_k}\}$ and liability cashflows $\{L_{t_k}\}$. Let $V_A(i)$ be the present value of the assets at effective rate of interest i and let $V_L(i)$ be the present value of the liabilities at rate i; let $v_A(i)$ and $v_L(i)$ be the volatility of the asset and liability cashflows respectively, and let $c_A(i)$ and $c_L(i)$ be the convexity of the asset and liability cashflows respectively.

At rate of interest i_0 the fund is immunised against small movements in the rate of interest of ε if and only if $V_A(i_0) = V_L(i_0)$ and $V_A(i_0 + \varepsilon) \geq V_L(i_0 + \varepsilon)$.

Then consider the surplus $S(i) = V_A(i) - V_L(i)$. From Taylor's theorem:

$$S(i_0 + \varepsilon) = S(i_0) + \varepsilon S'(i_0) + \frac{\varepsilon^2}{2} S''(i_0) + \ldots$$

Consider the terms on the right hand side. We know that $S(i_0) = 0$.

The second term, $\varepsilon S'(i_0)$, will be equal to zero for any values of ε (positive or negative) if and only if $S'(i_0) = 0$, that is if $V_A'(i_0) = V_L'(i_0)$. This is equivalent to requiring that $v_A(i) = v_L(i)$ or (equivalently) that the durations of the two cashflow series are the same.

In the third term, $\frac{\varepsilon^2}{2}$ is always positive, regardless of the sign of ε. Thus, if we ensure that $S''(i_0) > 0$, then the third term will also always be positive.

This is equivalent to requiring that $V_A''(i_0) > V_L''(i_0)$, which is equivalent to requiring that $c_A(i) > c_L(i)$.

For small $|\varepsilon|$ the fourth and subsequent terms in the Taylor expansion will be very small. Hence, given the three conditions above, the fund is protected against small movements in interest rates. This result is known as **Redington's immunisation** after the British actuary who developed the theory.

The conditions for Redington's immunisation may be summarised as follows:

1. $V_A(i_0) = V_L(i_0)$ — that is, the value of the assets at the starting rate of interest is equal to the value of the liabilities.

2. The volatilities of the asset and liability cashflow series are equal, that is, $v_A(i_0) = v_L(i_0)$.

3. The convexity of the asset cashflow series is greater than the convexity of the liability cashflow series — that is, $c_A(i_0) > c_L(i_0)$.

In practice there are difficulties with implementing an immunisation strategy based on these principles. For example the method requires continuous rebalancing of portfolios to keep the asset and liability volatilities equal. There may be options or other uncertainties in the assets or in the liabilities, making the assessment of the cashflows approximate rather than known. Assets may not exist to provide the necessary overall asset volatility to match the liability volatility. Despite these problems, immunisation theory remains an important consideration in the selection of assets.

E N D

UNIT 14 — STOCHASTIC INTEREST RATE MODELS

Syllabus objective (xiv) Show an understanding of simple stochastic models for investment returns.

1. Describe the concept of a stochastic interest rate model and the fundamental distinction between this and a deterministic model.

2. Derive algebraically, for the model in which the annual rates of return are independently and identically distributed and for other simple models, expressions for the mean value and the variance of the accumulated amount of a single premium.

3. Derive algebraically, for the model in which the annual rates of return are independently and identically distributed, recursive relationships which permit the evaluation of the mean value and the variance of the accumulated amount of an annual premium.

4. Derive analytically, for the model in which each year the random variable $(1 + i)$ has an independent log-normal distribution, the distribution functions for the accumulated amount of a single premium and for the present value of a sum due at a given specified future time.

5. Apply the above results to the calculation of the probability that a simple sequence of payments will accumulate to a given amount at a specific future time.

1 An introduction to stochastic interest rate models

1.1 Preliminary remarks

Financial contracts are often of a long-term nature. Accordingly, at the outset of many contracts there may be considerable uncertainty about the economic and investment conditions which will prevail over the duration of the contract. Thus, for example, if it is desired to determine premium rates on the basis of one fixed rate of interest, it is nearly always necessary to adopt a conservative basis for the rate to be used in any calculations.

An alternative approach to recognising the uncertainty that in reality exists is provided by the use of **stochastic interest rate models**. In such models no single interest rate is used. Variations in the rate of interest are allowed for by the application of probability theory. Possibly one of the simplest models is that in which each year the rate of interest obtained is independent of the rates of interest in all previous years and takes one of a finite set of values, each value having a constant probability of being the actual rate for the year. Alternatively, the rate of interest may take any value within a specified range, the actual value for the year being determined by some given probability density function.

1.2 An introductory example

At this stage we consider briefly an elementary example, which — although necessarily artificial — provides a simple introduction to the probabilistic ideas implicit in the use of stochastic interest rate models.

Suppose that an investor wishes to invest a lump sum of P into a fund which grows under the action of compound interest at a constant rate for n years. This constant rate of interest is not known **now**, but will be determined immediately after the investment has been made.

The accumulated value of the sum will, of course, be dependent on the rate of interest. In assessing this value before the interest rate is known, it could be assumed that the mean interest rate will apply. However, the accumulated value using the mean rate of interest will not equal the mean accumulated value. In algebraic terms:

$$P\Big(1+\sum_{j=1}^{k}(i_j\, p_j)\Big)^n \neq P\sum_{j=1}^{k}\big(p_j\,(1+i_j)^n\big)$$

where i_j is the jth of k possible rates of interest
p_j is the probability of the rate of interest i_j .

1.3 Independent annual rates of return

In our previous example the effective annual rate of interest was fixed **throughout the duration** of the investment. A more flexible model is provided by assuming that over each **single year** the annual yield on invested funds will be one of a specified set of values or lie within some specified range of values, the yield in any particular year being independent of the yields in all previous years and being determined by a given probability distribution.

Measure time in years. Consider the time interval $[0, n]$ subdivided into successive periods $[0, 1], [1, 2], ..., [n-1, n]$. For $t = 1, 2, ..., n$ let i_t be the yield obtainable over the tth year, i.e. the period $[t-1, t]$. Assume that money is invested only at the beginning of each year. Let F_t denote the accumulated amount at time t of all money invested before time t and let P_t be the amount of money invested at time t. Then, for $t = 1, 2, 3, ...$,

$$F_t = (1+i_t)(F_{t-1}+P_{t-1}) \qquad (1.3.1)$$

It follows from this equation that a single investment of 1 at time 0 will accumulate at time n to

$$S_n = (1 + i_1)(1 + i_2) \dots (1 + i_n) \tag{1.3.2}$$

Similarly a series of annual investments, each of amount 1, at times 0, 1, 2, ..., $n - 1$ will accumulate at time n to

$$\begin{aligned} A_n = \; & (1 + i_1)(1 + i_2)(1 + i_3) \dots (1 + i_n) \\ & + (1 + i_2)(1 + i_3) \dots (1 + i_n) \\ & + \quad \vdots \\ & + (1 + i_{n-1})(1 + i_n) \\ & + (1 + i_n) \end{aligned} \tag{1.3.3}$$

Note that A_n and S_n are random variables, each with its own probability distribution function. For example, if the yield each year is 0.02, 0.04, or 0.06 and each value is equally likely, the value of S_n will be between 1.02^n and 1.06^n. Each of these extreme values will occur with probability $(1/3)^n$.

In general, a theoretical analysis of the distribution functions for A_n and S_n is somewhat difficult. It is often more useful to use simulation techniques in the study of practical problems. However, it is perhaps worth noting that the moments of the random variables A_n and S_n can be found relatively simply in terms of the moments of the distribution for the yield each year. This may be seen as follows.

Moments of S_n

From equation 1.3.2 we obtain

$$(S_n)^k = \prod_{t=1}^{n} (1 + i_t)^k$$

and hence

$$E[S_n^k] = E\left[\prod_{t=1}^{n} (1 + i_t)^k\right]$$

$$= \prod_{t=1}^{n} E[(1 + i_t)^k] \tag{1.3.4}$$

since (by hypothesis) $i_1, i_2, ..., i_n$ are independent. Using this last expression and given the moments of the annual yield distribution, we may easily find the moments of S_n.

For example, suppose that the yield each year has mean j and variance s^2. Then, letting $k = 1$ in equation 1.3.4, we have

$$\begin{aligned} E[S_n] &= \prod_{t=1}^{n} E[1 + i_t] \\ &= \prod_{t=1}^{n} (1 + E[i_t]) \\ &= (1 + j)^n \qquad (1.3.5) \end{aligned}$$

since, for each value of t, $E[i_t] = j$.

With $k = 2$ in equation 1.3.4 we obtain

$$\begin{aligned} E[S_n^2] &= \prod_{t=1}^{n} E[1 + 2i_t + i_t^2] \\ &= \prod_{t=1}^{n} (1 + 2E[i_t] + E[i_t^2]) \\ &= (1 + 2j + j^2 + s^2)^n \qquad (1.3.6) \end{aligned}$$

since, for each value of t,

$$E[i_t^2] = (E[i_t])^2 + \text{var}[i_t] = j^2 + s^2$$

The variance of S_n is

$$\begin{aligned} \text{var}[S_n] &= E[S_n^2] - (E[S_n])^2 \\ &= (1 + 2j + j^2 + s^2)^n - (1 + j)^{2n} \qquad (1.3.7) \end{aligned}$$

from equations 1.3.5 and 1.3.6.

These arguments are readily extended to the derivation of the higher moments of S_n in terms of the higher moments of the distribution of the annual rate of interest.

Moments of A_n

It follows from equation 1.3.3 (or from equation 1.3.1) that, for $n \geq 2$,

$$A_n = (1 + i_n)(1 + A_{n-1}) \tag{1.3.8}$$

The usefulness of this equation lies in the fact that, since A_{n-1} depends only on the values $i_1, i_2, ..., i_{n-1}$, the random variables i_n and A_{n-1} are independent. (By assumption the yields each year are independent of one another.) Accordingly, equation 1.3.8 permits the development of a recurrence relation from which may be found the moments of A_n. We illustrate this approach by obtaining the mean and variance of A_n.

Let

$$\mu_n = E[A_n]$$

and let

$$m_n = E[A_n^2]$$

Since

$$A_1 = 1 + i_1$$

it follows that

$$\mu_1 = 1 + j \tag{1.3.9}$$

and

$$m_1 = 1 + 2j + j^2 + s^2 \tag{1.3.10}$$

where, as before, j and s^2 are the mean and variance of the yield each year.

Taking expectations of equation 1.3.8, we obtain (since i_n and A_{n-1} are independent)

$$\mu_n = (1 + j)(1 + \mu_{n-1}) \qquad n \geq 2 \tag{1.3.11}$$

This equation, combined with initial value μ_1, implies that, for all values of n,

$$\mu_n = \ddot{s}_{\overline{n|}} \qquad \text{at rate } j \tag{1.3.12}$$

Thus the expected value of A_n is simply $\ddot{s}_{\overline{n|}}$, calculated at the mean rate of interest.

Since

$$A_n^2 = (1 + 2i_n + i_n^2)(1 + 2A_{n-1} + A_{n-1}^2)$$

by taking expectations we obtain, for $n \geq 2$,

$$m_n = (1 + 2j + j^2 + s^2)(1 + 2\mu_{n-1} + m_{n-1}) \qquad (1.3.13)$$

As the value of μ_{n-1} is known (by equation 1.3.12), equation 1.3.13 provides a recurrence relation for the calculation successively of m_2, m_3, m_4, ... The variance of A_n may be obtained as

$$\begin{aligned} \text{var}[A_n] &= E[A_n^2] - (E[A_n])^2 \\ &= m_n - \mu_n^2 \end{aligned} \qquad (1.3.14)$$

In principle the above arguments are fairly readily extended to provide recurrence relations for the higher moments of A_n.

Example 1.3.1

A company considers that on average it will earn interest on its funds at the rate of 4% p.a. However, the investment policy is such that in any one year the yield on the company's funds is equally likely to take any value between 2% and 6%.

For both single and annual premium accumulations with terms of 5, 10, 15, 20, and 25 years and single (or annual) investment of £1, find the mean accumulation and the standard deviation of the accumulation at the maturity date. (Ignore expenses.)

Solution

The annual rate of interest is uniformly distributed on the interval [0.02, 0.06]. The corresponding probability density function is constant and equal to 25 (i.e. 1/(0.06 – 0.02)). The mean annual rate of interest is clearly

$$j = 0.04$$

and the variance of the annual rate of interest is

$$s^2 = \frac{1}{12}(.06 - .02)^2 = \frac{4}{3} \times 10^{-4}$$

We are required to find $E[A_n]$, $(\text{var}[A_n])^{1/2}$, $E[S_n]$, and $(\text{var}[S_n])^{1/2}$ for n = 5, 10, 15, 20, and 25.

Substituting the above values of j and s^2 in equations 1.3.5 and 1.3.7, we immediately obtain the results for the single premiums. For the annual premiums we must use the recurrence relation 1.3.13 (with $\mu_{n-1} = \ddot{s}_{\overline{n-1}|}$ at 4%) together with equation 1.3.14.

The results are summarised in table 1.3.1. It should be noted that, for both annual and single premiums, the standard deviation of the accumulation increases rapidly with the term.

Table 1.3.1 Accumulations for example 1.3.1

Term (years)	**Single investment £1**		**Annual investment £1**	
	Mean accumulation (£)	Standard deviation (£)	Mean accumulation (£)	Standard deviation (£)
5	1.21665	0.03021	5.63298	0.09443
10	1.48024	0.05198	12.48635	0.28353
15	1.80094	0.07748	20.82453	0.57899
20	2.19112	0.10886	30.96920	1.00476
25	2.66584	0.14810	43.31174	1.59392

The log-normal distribution

In general a theoretical analysis of the distribution functions for A_n and S_n is somewhat difficult, even in the relatively simple situation when the yields each year are independent and identically distributed. There is, however, one special case for which an exact analysis of the distribution function for S_n is particularly simple.

Suppose that the random variable $\log(1 + i_t)$ is normally distributed with mean μ and variance σ^2. In this case, the variable $(1 + i_t)$ is said to have a **log-normal** distribution with parameters μ and σ^2.

Equation 1.3.2 is equivalent to

$$\log S_n = \sum_{t=1}^{n} \log(1 + i_t)$$

The sum of a set of independent normal random variables is itself a normal random variable. Hence, when the random variables $(1 + i_t)$ $(t \geq 1)$ are independent and each has a log-normal distribution with parameters μ and σ^2, the random variable S_n has a log-normal distribution with parameters $n\mu$ and $n\sigma^2$.

Since the distribution function of a log-normal variable is readily written down in terms of its two parameters, in the particular case when the distribution function for the yield each year is log-normal we have a simple expression for the distribution function of S_n .

Similarly for the present value of a sum of 1 due at the end of n years
$= V_n = (1 + i_1)^{-1} \dots (1 + i_n)^{-1}$:

$$\log V_n = -\log(1 + i_1) - \dots - \log(1 + i_n)$$

Since, for each value of t, $\log(1 + i_t)$ is normally distributed with mean μ and variance σ^2, each term on the right hand side of the above equation is normally distributed with mean $-\mu$ and variance σ^2. Also the terms are independently distributed. So, $\log V_n$ is normally distributed with mean $-n\mu$ and variance $n\sigma^2$. That is, V_n has log-normal distribution with parameters $-n\mu$ and $n\sigma^2$.

By statistically modelling V_n it is possible to answer questions such as:

- to a given point in time, for a specified confidence interval, what is the range of values for an accumulated investment

- what is the maximum loss which will be incurred with a given level of probability

Although outside of the scope of this Subject, it is interesting to note that such techniques may be extended readily to model and predict the behaviour of portfolios of investments. These techniques are referred to as “Value at Risk” or “VaR” methods and are covered in more detail in subjects CT8, Financial Economics and ST6 Finance & Investment Specialist Technical B.

One possible definition of Value at Risk is a portfolio’s maximum loss from an adverse market movement, within a specified confidence interval and over a defined period of time.

As with all statistical modelling techniques, the results of VaR can only be as good as the statistical model of the performance of the underlying investments. In all investment markets, even seemingly efficient ones, it continues to prove very difficult to choose a reliable statistical model which is robust over even short periods of time.

E N D

Institute and Faculty of Actuaries

Subject CT1
Financial Mathematics
Core Technical

Syllabus with Cross referencing to the Core Reading

for the 2016 exams

1 June 2015

Aim

The aim of the Financial Mathematics subject is to provide a grounding in financial mathematics and its simple applications.

Links to other subjects

Subject CT2 – Finance and Financial Reporting: develops the use of the asset types introduced in this subject.

Subject CT4 – Models: develops the idea of stochastic interest rates.

Subject CT5 – Contingencies: develops some of the techniques introduced in this subject in situations where cashflows are dependent on survival.

Subject CT7 – Business Economics: develops the behaviour of interest rates.

Subject CT8 – Financial Economics: develops the principles further.

Subjects CA1 – Actuarial Risk Management, CA2 – Model Documentation, Analysis and Reporting and the Specialist Technical and Specialist Applications subjects: use the principles introduced in this subject.

Objectives

On completion of the subject the trainee actuary will be able to:

(i) Describe how to use a generalised cashflow model to describe financial transactions. (Unit 1)

1. For a given cashflow process, state the inflows and outflows in each future time period and discuss whether the amount or the timing (or both) is fixed or uncertain.

2. Describe in the form of a cashflow model the operation of a zero coupon bond, a fixed interest security, an index-linked security, cash on deposit, an equity, an “interest only” loan, a repayment loan, and an annuity certain.

(ii) Describe how to take into account the time value of money using the concepts of compound interest and discounting. (Unit 2)

1. Accumulate a single investment at a constant rate of interest under the operation of:

   - simple interest
   - compound interest

2. Define the present value of a future payment.

3. Discount a single investment under the operation of simple (commercial) discount at a constant rate of discount.

4. Describe how a compound interest model can be used to represent the effect of investing a sum of money over a period.

(iii) Show how interest rates or discount rates may be expressed in terms of different time periods. (Units 2 and 3)

1. Derive the relationship between the rates of interest and discount over one effective period arithmetically and by general reasoning.

2. Derive the relationships between the rate of interest payable once per effective period and the rate of interest payable *p* times per time period and the force of interest.

3. Explain the difference between nominal and effective rates of interest and derive effective rates from nominal rates.

4. Calculate the equivalent annual rate of interest implied by the accumulation of a sum of money over a specified period where the force of interest is a function of time.

(iv) Demonstrate a knowledge and understanding of real and money interest rates. (Unit 4)

(v) Calculate the present value and the accumulated value of a stream of equal or unequal payments using specified rates of interest and the net present value at a real rate of interest, assuming a constant rate of inflation. (Unit 5)

1. Discount and accumulate a sum of money or a series (possibly infinite) of cashflows to any point in time where:

   - the rate of interest or discount is constant

   - the rate of interest or discount varies with time but is not a continuous function of time

   - either or both the rate of cashflow and the force of interest are continuous functions of time

2. Calculate the present value and accumulated value of a series of equal or unequal payments made at regular intervals under the operation of specified rates of interest where the first payment is:

   - deferred for a period of time
   - not deferred

(vi) Define and use the more important compound interest functions including annuities certain. (Unit 6)

1. Derive formulae in terms of i, v, n, d, δ, $i^{(p)}$ and $d^{(p)}$ for $a_{\overline{n|}}$, $s_{\overline{n|}}$, $a_{\overline{n|}}^{(p)}$, $s_{\overline{n|}}^{(p)}$, $\ddot{a}_{\overline{n|}}$, $\ddot{s}_{\overline{n|}}$, $\ddot{a}_{\overline{n|}}^{(p)}$, $\ddot{s}_{\overline{n|}}^{(p)}$, $\bar{a}_{\overline{n|}}$ and $\bar{s}_{\overline{n|}}$.

2. Derive formulae in terms of i, v, n, d, δ, $i^{(p)}$ and $d^{(p)}$ for ${}_{m|}a_{\overline{n|}}$, ${}_{m|}a_{\overline{n|}}^{(p)}$, ${}_{m|}\ddot{a}_{\overline{n|}}$, ${}_{m|}\ddot{a}_{\overline{n|}}^{(p)}$ and ${}_{m|}\bar{a}_{\overline{n|}}$.

3. Derive formulae in terms of i, v, n, δ, $a_{\overline{n|}}$ and $\ddot{a}_{\overline{n|}}$ for $(Ia)_{\overline{n|}}$, $(I\ddot{a})_{\overline{n|}}$, $(I\bar{a})_{\overline{n|}}$, $(\bar{I}\bar{a})_{\overline{n|}}$ and the respective deferred annuities.

(vii) Define an equation of value. (Unit 7)

1. Define an equation of value, where payment or receipt is certain.

2. Describe how an equation of value can be adjusted to allow for uncertain receipts or payments.

3. Understand the two conditions required for there to be an exact solution to an equation of value.

(viii) Describe how a loan may be repaid by regular instalments of interest and capital. (Unit 8)

1. Describe flat rates and annual effective rates.

2. Calculate a schedule of repayments under a loan and identify the interest and capital components of annuity payments where the annuity is used to repay a loan for the case where annuity payments are made once per effective time period or *p* times per effective time period and identify the capital outstanding at any time.

(ix) Show how discounted cashflow techniques can be used in investment project appraisal. (Unit 9)

1. Calculate the net present value and accumulated profit of the receipts and payments from an investment project at given rates of interest.

2. Calculate the internal rate of return implied by the receipts and payments from an investment project.

3. Describe payback period and discounted payback period and discuss their suitability for assessing the suitability of an investment project.

4. Determine the payback period and discounted payback period implied by the receipts and payments from an investment project.

5. Calculate the money-weighted rate of return, the time-weighted rate of return and the linked internal rate of return on an investment or a fund.

(x) Describe the investment and risk characteristics of the following types of asset available for investment purposes: (Unit 10)

- fixed interest government borrowings
- fixed interest borrowing by other bodies
- index-linked government borrowings
- shares and other equity-type finance
- derivatives

(xi) Analyse elementary compound interest problems. (Unit 11)

1. Calculate the present value of payments from a fixed interest security where the coupon rate is constant and the security is redeemed in one instalment.

2. Calculate upper and lower bounds for the present value of a fixed interest security that is redeemable on a single date within a given range at the option of the borrower.

3. Calculate the running yield and the redemption yield from a fixed interest security (as in 1.), given the price.

4. Calculate the present value or yield from an ordinary share and a property, given simple (but not necessarily constant) assumptions about the growth of dividends and rents.

5. Solve an equation of value for the real rate of interest implied by the equation in the presence of specified inflationary growth.

6. Calculate the present value or real yield from an index-linked bond, given assumptions about the rate of inflation.

7. Calculate the price of, or yield from, a fixed interest security where the investor is subject to deduction of income tax on coupon payments and redemption payments are subject to the deduction of capital gains tax.

8. Calculate the value of an investment where capital gains tax is payable, in simple situations, where the rate of tax is constant, indexation allowance is taken into account using specified index movements and allowance is made for the case where an investor can offset capital losses against capital gains.

(xii) Calculate the delivery price and the value of a forward contract using arbitrage free pricing methods. (Unit 12)

1. Define “arbitrage” and explain why arbitrage may be considered impossible in many markets.

2. Calculate the price of a forward contract in the absence of arbitrage assuming:

   - no income or expenditure associated with the underlying asset during the term of the contract

   - a fixed income from the asset during the term

   - a fixed dividend yield from the asset during the term.

3. Explain what is meant by “hedging” in the case of a forward contract.

4. Calculate the value of a forward contract at any time during the term of the contract in the absence of arbitrage, in the situations listed in 2 above.

(xiii) Show an understanding of the term structure of interest rates. (Unit 13)

1. Describe the main factors influencing the term structure of interest rates.

2. Explain what is meant by the par yield and yield to maturity.

3. Explain what is meant by, derive the relationships between and evaluate:

   - discrete spot rates and forward rates
   - continuous spot rates and forward rates

4. Define the duration and convexity of a cashflow sequence, and illustrate how these may be used to estimate the sensitivity of the value of the cashflow sequence to a shift in interest rates.

5. Evaluate the duration and convexity of a cashflow sequence.

6. Explain how duration and convexity are used in the (Redington) immunisation of a portfolio of liabilities.

(xiv) Show an understanding of simple stochastic models for investment returns. (Unit 14)

1. Describe the concept of a stochastic interest rate model and the fundamental distinction between this and a deterministic model.

2. Derive algebraically, for the model in which the annual rates of return are independently and identically distributed and for other simple models, expressions for the mean value and the variance of the accumulated amount of a single premium.

3. Derive algebraically, for the model in which the annual rates of return are independently and identically distributed, recursive relationships which permit the evaluation of the mean value and the variance of the accumulated amount of an annual premium.

4. Derive analytically, for the model in which each year the random variable $(1 + i)$ has an independent log-normal distribution, the distribution functions for the accumulated amount of a single premium and for the present value of a sum due at a given specified future time.

5. Apply the above results to the calculation of the probability that a simple sequence of payments will accumulate to a given amount at a specific future time.

END OF SYLLABUS