Series

A **finite series** is the sum of the terms of a finite sequence. Thus, if

$$a_1,\ a_2,\ \ldots,\ a_n$$

is a sequence of n terms, then the corresponding series is

$$a_1 + a_2 + \cdots + a_n.$$

The number a_k is referred to as the kth term of the series.

We often use the **sigma notation** for series. For example, if we have the series

$$2 + 4 + 6 + \cdots + 100$$

in which the kth term is given by $2k$, then we can write this series as

$$\sum_{k=1}^{50} 2k.$$

Note that the variable k here is a **dummy variable**. This means that we could also write the series as

$$\sum_{i=1}^{50} 2i \qquad \text{or} \qquad \sum_{j=1}^{50} 2j.$$

Exercise 6

By writing out the terms, find the sum

$$\sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right).$$

An **infinite series** is the ‘formal sum’ of the terms of an infinite sequence:

$$a_1 + a_2 + a_3 + a_4 + \cdots.$$

For example, the sequence of odd numbers gives the infinite series $1+3+5+7+\cdots$.

We can sum an infinite series to a finite number of terms. The sum of the first n terms of an infinite series is often written as

$$S_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k.$$

This is sometimes called the **nth partial sum** of the infinite series.

Given a formula for the sum of the first n terms of a series, we can recover a formula for the nth term by a simple subtraction, as follows. Starting from

$$S_n = a_1 + a_2 + \cdots + a_{n-1} + a_n$$

$$S_{n-1} = a_1 + a_2 + \cdots + a_{n-1},$$

by subtracting we obtain

$$S_n - S_{n-1} = a_n.$$

For example, if the sum of the first n terms of a series is given by $S_n = n^2$, then the nth term is

$$a_n = S_n - S_{n-1} = n^2 - (n-1)^2 = 2n - 1.$$

So the terms form the sequence of odd numbers. Hence, we have found a formula for the sum of the first n odd numbers:

$$1 + 3 + 5 + \cdots + (2n-1) = n^2.$$

In general, it can be difficult to find a simple formula for the sum of a series to n terms. For the rest of this section, we restrict our attention to arithmetic and geometric series.

Arithmetic series

An **arithmetic series** is a series in which the terms form an arithmetic sequence. That is, each term is obtained from the preceding one by adding a constant.

The series

$$1 + 2 + 3 + \cdots + n$$

is an arithmetic series with common difference 1. There is an easy way to find the sum of this series. We write the series forwards and then backwards:

$$\begin{array}{cccccccccccccc} S_n = & 1 & + & 2 & + & 3 & + & \cdots & + & (n-1) & + & n \\ S_n = & n & + & (n-1) & + & (n-2) & + & \cdots & + & 2 & + & 1. \end{array}$$

Adding downwards in pairs, we obtain

$$2S_n = (1+n) + (2+n-1) + (3+n-2) + \cdots + (n-1+2) + (n+1).$$

Each of the n terms on the right-hand side simplifies to $n+1$. Thus $2S_n = n(n+1)$, and so we have shown that

$$1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1).$$

For example,

$$1 + 2 + 3 + \cdots + 100 = \frac{1}{2} \times 100 \times 101 = 5050.$$

Legend has it that the famous mathematician Gauss discovered this at the age of nine!

This 'trick' works for any arithmetic series, and gives a formula for the sum S_n of the first n terms of an arithmetic series with first term $a_1 = a$ and last term $a_n = \ell$. The formula is

$$S_n = \frac{n}{2}(a + \ell).$$

Exercise 7

Use the method of writing the arithmetic series

$$a + (a+d) + (a+2d) + \cdots + (\ell - d) + \ell$$

forwards and backwards to derive the formula $S_n = \frac{n}{2}(a+\ell)$ given above.

Since the last term ℓ can be written as $a_n = a + (n-1)d$, where d is the common difference, we also have

$$\begin{aligned} S_n &= \frac{n}{2}(a + \ell) \\ &= \frac{n}{2}\big(a + a + (n-1)d\big) \\ &= \frac{n}{2}\big(2a + (n-1)d\big). \end{aligned}$$

Example

Find the formula for the sum of the first n terms of the arithmetic sequence

1. $2, 5, 8, \dots$
2. $107, 98, 89, \dots.$

Solution

1. Here $a = 2$ and $d = 3$, so

$$S_n = \frac{n}{2}\big(4 + (n-1) \times 3\big) = \frac{n}{2}\big(3n+1\big).$$

Alternatively, we can find the nth term of the sequence, which is $a_n = 3n - 1$, and use the formula

$$S_n = \frac{n}{2}\big(a + \ell\big) = \frac{n}{2}\big(2 + (3n-1)\big) = \frac{n}{2}\big(3n+1\big).$$

2. Here $a = 107$ and $d = -9$, so

$$S_n = \frac{n}{2}\big(2 \times 107 + (n-1) \times -9\big) = \frac{n}{2}\big(223 - 9n\big).$$

For both parts of the previous example, we can substitute $n = 1$ and check this gives the first term of the series. Note that, since the formula for the sum is a quadratic, checking the three cases $n = 1$, $n = 2$, $n = 3$ is sufficient to prove that the answer is correct.

Exercise 8

Sum the arithmetic series

$$\log_2 3 + \log_2 9 + \log_2 27 + \cdots$$

to n terms.

Geometric series

A **geometric series** is a series in which the terms form a geometric sequence. That is, each term is obtained from the preceding one by multiplying by a constant.

For example,

$$2 + 8 + 32 + 128 + \cdots$$

is a geometric series with first term 2 and common ratio 4. The nth term is $a_n = 2 \times 4^{n-1}$.

We can find a formula for the sum of the first n terms of this series, again using a little trick. We multiply the series by the common ratio 4 and subtract the original, as follows. Starting from

$$\begin{aligned} S_n &= 2 + 8 + 32 + 128 + \cdots + 2 \times 4^{n-1} \\ 4S_n &= 8 + 32 + 128 + \cdots + 2 \times 4^{n-1} + 2 \times 4^n, \end{aligned}$$

we subtract to obtain

$$4S_n - S_n = 2 \times 4^n - 2,$$

and so

$$S_n = \frac{1}{3}\big(2 \times 4^n - 2\big) = \frac{2(4^n - 1)}{3}.$$

This 'trick' works for any geometric series, and gives a formula for the sum S_n of the first n terms of a geometric series with first term a and common ratio r. The formula is

$$S_n = \frac{a(r^n - 1)}{r - 1}, \quad \text{for } r \neq 1.$$

Note that this can also be written as

$$S_n = \frac{a(1 - r^n)}{1 - r}, \quad \text{for } r \neq 1.$$

The second formula is often more convenient to use when r lies between -1 and 1.

In the case when $r = 1$, the sum of the series is clearly na, since all the terms are identical.

Exercise 9

Use the method of multiplying the geometric series

$$a + ar + ar^2 + \cdots + ar^{n-1}$$

by r and subtracting to derive the formula for S_n given above.

Example

Find the formula for the sum of the first n terms of the geometric sequence

1. $2, 6, 18, \ldots$
2. $486, 162, 54, \ldots.$

Solution

1 Here $a = 2$ and $r = 3$, so

$$S_n = \frac{2(3^n - 1)}{3 - 1} = 3^n - 1.$$

2 Here $a = 486$ and $r = \frac{1}{3}$, so

$$S_n = \frac{486\big(1 - (\frac{1}{3})^n\big)}{1 - \frac{1}{3}} = 729\big(1 - (\tfrac{1}{3})^n\big).$$

For both parts of the previous example, we can put $n = 1$ and check that we obtain the first term of the sequence.

Exercise 10

Find the sum to n terms of the geometric series

$$\sqrt{3} + 6 + 12\sqrt{3} + \cdots.$$

Summary

Arithmetic sequence	$a,\ a+d,\ a+2d,\ a+3d,\ \ldots$ The nth term is $a_n = a + (n-1)d$, where a is the first term and d is the common difference.
Arithmetic series	$a + (a+d) + (a+2d) + (a+3d) + \cdots$ The sum of the first n terms is $$S_n = \frac{n}{2}\big(2a + (n-1)d\big),$$ where a is the first term and d is the common difference. This can also be written $S_n = \frac{n}{2}(a + \ell)$, where ℓ is the nth term a_n.
Geometric sequence	$a,\ ar,\ ar^2,\ ar^3,\ \ldots$ The nth term is $a_n = ar^{n-1}$, where a is the first term and r is the common ratio.
Geometric series	$a + ar + ar^2 + ar^3 + \cdots$ The sum of the first n terms is $$S_n = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}, \quad \text{for } r \neq 1,$$ where a is the first term and r is the common ratio.

Telescoping series

Most series are neither arithmetic nor geometric. Some of these series can be summed by expressing the summand as a difference.

Example

1 Find the sum

$$\sum_{k=2}^{n} \frac{2}{k^2-1}.$$

2 Does the infinite series

$$\sum_{k=2}^{\infty} \frac{2}{k^2-1}$$

have a limiting sum? If so, what is its value?

Solution

1 We can factor k^2-1 and split the summand into

$$\frac{2}{k^2-1} = \frac{1}{k-1} - \frac{1}{k+1}.$$

Thus,

$$\begin{aligned}\sum_{k=2}^{n} \frac{2}{k^2-1} &= \sum_{k=2}^{n} \Big(\frac{1}{k-1} - \frac{1}{k+1}\Big) \\ &= \sum_{k=2}^{n} \frac{1}{k-1} - \sum_{k=2}^{n} \frac{1}{k+1}.\end{aligned}$$

If we write out the terms of these two sums, we have

$$\Big(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n-2}+\frac{1}{n-1}\Big)-\Big(\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n-2}+\frac{1}{n-1}+\frac{1}{n}+\frac{1}{n+1}\Big).$$

Most of the terms cancel out (telescope), giving

$$\begin{aligned}\sum_{k=2}^{n} \frac{2}{k^2-1} &= 1+\frac{1}{2}-\frac{1}{n}-\frac{1}{n+1} \\ &= \frac{3}{2}-\frac{1}{n}-\frac{1}{n+1}.\end{aligned}$$

2 Since the terms $\frac{1}{n}$ and $\frac{1}{n+1}$ go to zero as n goes to infinity, the series has a limiting sum of $\frac{3}{2}$.

The harmonic series

The **harmonic series** is

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots.$$

There is no simple expression for the sum of the first n terms of this series. Does the series have a limiting sum? The following argument shows that the answer is no.

We can group the terms of the series as follows:

$$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{10}+\cdots+\frac{1}{16}\right)+\cdots.$$

Each term is greater than or equal to the last term in its bracket, and so we can write

$$\begin{aligned}
&1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots\\
&\qquad\geq 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\left(\frac{1}{16}+\frac{1}{16}+\cdots+\frac{1}{16}\right)+\cdots\\
&\qquad= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots,
\end{aligned}$$

which grows without bound. So the harmonic series does not have a limiting sum.

On the other hand, if we square each term and look at the series

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots,$$

then it can be shown (although it is not all that easy) that this series has a limiting sum of $\frac{\pi^2}{6}$. This result was proven by Euler in the 18th century.