Sequences

Motivation

We encounter sequences at the very beginning of our mathematical experience. The list of even numbers

2, 4, 6, 8, 10, …

and the list of odd numbers

1, 3, 5, 7, 9, …

are examples. We can 'predict' what the 20th term of each sequence will be just by using common sense.

Another sequence of great historical interest is the **Fibonacci sequence**

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, …

in which each term is the sum of the two preceding terms; for example, $55 = 21 + 34$. In this case it is somewhat more difficult to predict the 20th term, without listing all the previous ones.

Sequences arise in many areas of mathematics, including finance. For example, we can invest $1000 at an interest rate of 5% per annum, compounded annually, and list the sequence consisting of the value of the investment each year:

$1000, $1050, $1102.50, $1157.63, $1215.51, …

(rounded to the nearest cent).

Sequences can be either **finite** or **infinite**. For example,

$$2,\ 4,\ 6,\ 8,\ 10$$

is a finite sequence with five terms, whereas

$$2,\ 4,\ 6,\ 8,\ 10,\ \dots$$

continues without bound and is an infinite sequence. We usually use … to denote that the sequence continues without bound.

For a given infinite sequence, we can ask the questions:

- Can we find a formula for the general term of the sequence?
- Does the sequence have a limit, that is, do the numbers in the sequence get as close as we like to some number?

For example, we can see intuitively that the terms in the infinite sequence

$$1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \dots$$

whose general term is $\frac{1}{n}$, are approaching 0 as n becomes very large.

A **finite series** arises when we add the terms of a finite sequence. For example,

$$2 + 4 + 6 + 8 + \dots + 20$$

is the series formed from the sequence $2, 4, 6, 8, \dots, 20$.

An **infinite series** is the 'formal sum' of the terms of an infinite sequence. For example,

$$1 + 3 + 5 + 7 + 9 + \dots$$

is the series formed from the sequence of odd numbers. We can spot an interesting pattern in this series. The sum of the first two terms is 4, the sum of the first three terms is 9, and the sum of the first four terms is 16. So we guess that, in general, the sum of the first n terms is n^2.

For a given infinite series, we can ask the questions:

- Can we find a formula for the sum of the first n terms of the series?
- Does the series have a limit, that is, if we add the first n terms of the series, does this sum get as close as we like to some number as n becomes larger?

If it exists, this limit is often referred to as the **limiting sum** of the infinite series. In this module, we examine limiting sums for one special but commonly occurring type of series, known as a geometric series.

Sequences and series are very important in mathematics and also have many useful applications, in areas such as finance, physics and statistics.

Content

Sequences

The list of positive odd numbers

1, 3, 5, 7, 9, …

is an example of a typical **infinite sequence**. The dots indicate that the sequence continues forever, with no last term. We will use the symbol a_n to denote the *n***th term** of a given sequence. Thus, in this example, $a_1 = 1$, $a_2 = 3$, $a_3 = 5$ and so on; the first term is $a_1 = 1$, but there is no last term.

The list of positive odd numbers less than 100

1, 3, 5, 7, …, 99

is an example of a typical **finite sequence**. The first term of this sequence is 1 and the last term is 99. This sequence contains 50 terms.

There are several ways to display a sequence:

- write out the first few terms
- give a formula for the general term
- give a recurrence relation.

Writing out the first few terms is not a good method, since you have to 'believe' there is some clearly defined pattern, and there may be many such patterns present. For example, if we simply write

1, 2, 4, …

then the next term might be 8 (powers of two), or possibly 7 (*Lazy Caterer's sequence*), or perhaps even 23 if there is some more complicated pattern going on. Hence, if the first few terms only are given, some rule should also be given as to how to uniquely determine the next term in the sequence.

A much better way to describe a sequence is to give a formula for the *n*th term a_n. This is also called a formula for the **general term**. For example,

$$a_n = 2n - 1$$

is a formula for the general term in the sequence of odd numbers $1, 3, 5, \ldots$. From the formula, we can, for example, write down the 10th term, since $a_{10} = 2 \times 10 - 1 = 19$.

In some cases it is not easy, or even possible, to give an explicit formula for a_n. In such cases, it may be possible to determine a particular term in the sequence in terms of some of the preceding terms. This relationship is often referred to as a **recurrence**. For example, the sequence of positive odd numbers may be defined by

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = a_n + 2, \text{ for } n \geq 1.$$

The initial term is $a_1 = 1$, and the recurrence tells us that we need to add two to each term to obtain the next term.

The Fibonacci sequence comprises the numbers

$$1, \ 1, \ 2, \ 3, \ 5, \ 8, \ 13, \ 21, \ 34, \ \ldots$$

where each term is the sum of the two preceding terms. This can be described by setting $a_1 = a_2 = 1$ and $a_{n+2} = a_{n+1} + a_n$, for $n \geq 1$.

Exercise 1

Consider the recurrence

$$a_1 = 2 \quad \text{and} \quad a_n = (a_{n-1})^2 + 1, \text{ for } n \geq 2.$$

Write down the first five terms of this sequence.

The general term of a sequence can sometimes be found by 'pattern matching'.

Exercise 2

Give a formula for the general term of

a the sequence $2, 4, 6, 8, \ldots$ of even numbers

b the sequence $1, 4, 9, 16, \ldots$ of squares.

In general, however, finding a formula for the general term of a sequence can be difficult. Consider, for example, the Fibonacci sequence:

$$1, \ 1, \ 2, \ 3, \ 5, \ 8, \ 13, \ 21, \ 34, \ \ldots .$$

We will discuss in the *History and applications* section how to show that the nth term of the Fibonacci sequence is given by

$$a_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right).$$

This is a very surprising result! (It is not even obvious that this formula will give an integer result for each n.) You might like to check that this formula works for $n = 1, 2, 3$.

Sequences can also be used to approximate real numbers. Thus, for example, the terms in the sequence

$$1, \ 1.4, \ 1.41, \ 1.414, \ 1.4142, \ \dots$$

give approximations to the real number $\sqrt{2}$.

Arithmetic sequences

We will limit our attention for the moment to one particular type of sequence, known as an **arithmetic sequence** (or **arithmetic progression**). This is a sequence of the form

$$a, \ a+d, \ a+2d, \ a+3d, \ \dots$$

where each term is obtained from the preceding one by adding a constant, called the **common difference** and often represented by the symbol d. Note that d can be positive, negative or zero.

Thus, the sequence of even numbers

$$2, \ 4, \ 6, \ 8, \ 10, \ \dots$$

is an arithmetic sequence in which the common difference is $d = 2$.

It is easy to see that the formula for the nth term of an arithmetic sequence is

$$a_n = a + (n-1)d.$$

Example

Find the formula for the nth term of the arithmetic sequence

1 2, 5, 8, …

2 107, 98, 89, ….

Solution

1 Here $a = 2$ and $d = 3$, so

$$a_n = 2 + (n-1) \times 3 = 3n - 1.$$

2 Here $a = 107$ and $d = -9$, so

$$a_n = 107 + (n-1) \times -9 = 116 - 9n.$$

Exercise 3

Find the nth term of the arithmetic sequence

$$\log_5 2, \ \log_5 4, \ \log_5 8, \ \dots$$

We can also check whether a given number belongs to a given arithmetic sequence.

Example

Does the number 203 belong to the arithmetic sequence $3, 7, 11, \dots$?

Solution

Here $a = 3$ and $d = 4$, so $a_n = 3 + (n-1) \times 4 = 4n - 1$. We set $4n - 1 = 203$ and find that $n = 51$. Hence, 203 is the 51st term of the sequence.

Exercise 4

Show that 12 is not a term of the arithmetic sequence $210, 197, 184, \dots$.

Geometric sequences

A **geometric sequence** has the form

$$a, \ ar, \ ar^2, \ ar^3, \ \dots$$

in which each term is obtained from the preceding one by multiplying by a constant, called the **common ratio** and often represented by the symbol r. Note that r can be positive, negative or zero. The terms in a geometric sequence with negative r will oscillate between positive and negative.

The doubling sequence

$$1, \ 2, \ 4, \ 8, \ 16, \ 32, \ 64, \ \dots$$

is an example of a geometric sequence with first term 1 and common ratio $r = 2$, while

$$3, \ -6, \ 12, \ -24, \ 48, \ -96, \ \dots$$

is an example of a geometric sequence with first term 3 and common ratio $r = -2$.

It is easy to see that the formula for the nth term of a geometric sequence is

$$a_n = ar^{n-1}.$$

Example

Find the formula for the nth term of the geometric sequence

1   2, 6, 18, …

2   486, 162, 54, ….

Solution

1   Here $a = 2$ and $r = 3$, so $a_n = 2 \times 3^{n-1}$.

2   Here $a = 486$ and $r = \frac{1}{3}$, so $a_n = 486 \times \left(\frac{1}{3}\right)^{n-1}$.

Exercise 5

Find the nth term of the geometric sequence

$$\sqrt{6}, \ 2\sqrt{3}, \ 2\sqrt{6}, \ \dots .$$

We can also check whether a given number belongs to a given geometric sequence.

Example

Does the number 48 belong to the geometric sequence

3072, 1536, 768, …?

Solution

Here $a = 3072$ and $r = \frac{1}{2}$, so $a_n = 3072 \times \left(\frac{1}{2}\right)^{n-1}$.

We set $3072 \times \left(\frac{1}{2}\right)^{n-1} = 48$. This gives $\left(\frac{1}{2}\right)^{n-1} = \frac{1}{64}$, that is, $2^{n-1} = 64 = 2^6$, and so $n = 7$.

Hence, 48 is the 7th term of the sequence.

Example

Does the number 6072 belong to the geometric sequence

3, −6, 12, −24, 48, …?

> ### Solution
>
> Here $a = 3$ and $r = -2$, so $a_n = 3 \times (-2)^{n-1}$.
>
> We set $3 \times (-2)^{n-1} = 6072$. This gives $(-2)^{n-1} = 2024$.
>
> But 2024 is not a power of 2, and so 6072 does not belong to the sequence.