SOME RESULTS ON FREE RESOLUTIONS

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Abstract. In this article we prove some results about free resolutions of modules over commutative rings with unity. In Section 3 we discuss some notions which are additive in the sense of short exact sequences, and prove some results about finiteness of free resolutions in some special cases. We then discuss resolutions of local rings with nonzero annihilators. Section 4 contains some general results on free resolutions. In Section 5, we construct free resolutions of monomial ideals in a bivariate polynomial ring over a field.

One of the goals of this article is to document the process of discovery through which we arrived at the various ideas and results starting from examples. The appendix describes this process.

1. Background

This article is the documentation of the work done by the six authors on the first exploratory project on Free Resolutions in MA839 Advanced Commutative Algebra course which was taught by Prof. H. Ananthnarayan in Spring 2021 at IIT Bombay. At the beginning of the project work all six authors knew the contents of what is usually covered in a first course in commutative algebra. In particular, the authors were familiar with the notions of local rings, Noetherian and Artinian modules, structure theorem for modules over a PID, localisation, tensor products, exact sequences and flatness. The authors were introduced to the construction of a free resolution at the beginning of the project.

2. Definitions and Notation

By a *ring*, we shall always refer to a commutative ring with unity. We shall also implicitly assume that the ring is nonzero.
For a positive integer n, we denote by $R^{\oplus n}$ the direct sum of n copies of R, with the convention that $R^{\oplus 0} = \{0\}$.

Definition 2.1. A *multiplicative subset* $A \subset R$ is a subset such that $0 \notin A$, $1 \in A$, and if $a, b \in A$, then $ab \in A$. We shall denote the localization of R with respect to A by $A^{-1}R$. Similarly, if M is an R-module, we shall denote the localization of M with respect to A by $A^{-1}M$.

Definition 2.2. A *free R-module (of finite rank)* is an R-module F such that $F \cong R^{\oplus n}$ for some $n \geq 0$.

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Note: Throughout this document, we simply use "free R-module" to refer to a free R-module of finite rank.

Definition 2.3. Given an R-module M, a *free resolution of M over R* (or an *R-free resolution of M*) is an exact sequence of the form

$$\cdots \xrightarrow{d_{n+2}} F_{n+1} \xrightarrow{d_{n+1}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0,$$

where each F_i is a free R-module.

The resolution is said to have *finite length* (or said to be *finite*) if there exists N such that $F_n = 0$ for all $n > N$. Otherwise, it is said to have *infinite length* (or be *infinite*).

Note that as a consequence of exactness at M, it follows that M is finitely generated since F_0 has finite rank (by our convention).

Example 2.4. Consider the ring $R = \mathbb{Z}/6\mathbb{Z}$ and $M = \mathbb{Z}/2\mathbb{Z}$.
Since M is generated by the single element $\bar{1}$, we define $F_0 = R^{\oplus 1}$ and d_0 to be the map $\bar{1} \mapsto \bar{1}$.
Then, $\ker(d_0)$ is the submodule of F_0 generated by $\bar{2}$. Thus, we define $F_1 = R^{\oplus 1}$ and d_1 to be the map $\bar{1} \mapsto \bar{2}$.
Proceeding similarly, we see that each stage, the kernel is generated by one element and thus, we can keep mapping $R^{\oplus 1}$ onto the kernel to get the following resolution.

$$\cdots \to R^{\oplus 1} \xrightarrow{\cdot \bar{3}} R^{\oplus 1} \xrightarrow{\cdot \bar{2}} R^{\oplus 1} \xrightarrow{\cdot \bar{3}} R^{\oplus 1} \xrightarrow{\cdot \bar{2}} R^{\oplus 1} \xrightarrow{\bar{1} \mapsto \bar{1}} M \to 0.$$

This is an example of an infinite free resolution of M over R. Note that at each step, we did everything "minimally" and thus, one would expect that there is no finite free resolution.

3. Existence of Finite Free Resolutions

3.1. **Additive functions over short exact sequences.**

Question 3.1. *We had seen a free resolution of $\mathbb{Z}/2\mathbb{Z}$ over $\mathbb{Z}/6\mathbb{Z}$ which was infinite. Does there exist a finite free resolution?*

Question 3.2. *Does there exist a finite free resolution of k as a $\mathsf{k}[x]/\langle x^2 \rangle$ module?*

Instead of answering the particular questions above, we answer it in a more general setting.

Proposition 3.3. *Let R be a finite ring and M a finitely generated R-module. If $|M|$ is not a power of $|R|$, then M has no finite free resolution over R.*

Note that $|M|$ is finite by our assumption of R being finite.

Proof. Note that at an arbitrary stage of the free resolution, we have a diagram as follows.

$$\begin{tikzcd}
& 0 \arrow[dr] & & & & & \\
& & K_n \arrow[dr] & & & & \\
\cdots \arrow[r] & F_{n+1} \arrow[ur] \arrow[rr, "d_{n+1}"'] & & F_n \arrow[rr, "d_n"] \arrow[dr] & & F_{n-1} \arrow[r] & \cdots, \\
& & & & K_{n-1} \arrow[ur] \arrow[dr] & & \\
& & & & & 0 &
\end{tikzcd}$$

where $K_n := \ker(d_n)$.

Note that the diagonal is exact and hence, we have

$$|F_n| = |K_n||K_{n-1}|.$$

(By our assumption, all the modules appearing above do have finite cardinality.)

Now, note that the left hand side is a power of $|R|$. Thus, we see that $|K_n|$ is a power of $|R|$ *if and only if* $|K_{n-1}|$ is.

Note that to have a *finite* free resolution, we must have that some K_m is free. In particular, $|K_m|$ must be a power of $|R|$. By induction, this is possible only if $|K_0|$ was a power of R. However, we note that we have

$$0 \longrightarrow K_0 \longrightarrow F_0 \longrightarrow M \longrightarrow 0.$$

By the same logic as earlier, $|K_0|$ is *not* a power of $|R|$ since $|M|$ is not. This finishes the proof. ☐

Corollary 3.4. *There is no finite free resolution of $\mathbb{Z}/2\mathbb{Z}$ over $\mathbb{Z}/6\mathbb{Z}$.*
In general, we may consider $\mathbb{Z}/d\mathbb{Z}$ as a module over $\mathbb{Z}/n\mathbb{Z}$, where $0 < d < n$ and $d \mid n$. We have shown that this module does not have a finite free resolution.

This also answers Question 3.2 when k is finite.

Corollary 3.5. *Let k be a finite field. Then, k has no finite free resolution over $\mathsf{k}[x]/\langle x^n \rangle$ for $n > 1$.*

Proof. Let $q := |\mathsf{k}|$. Then, the ring has cardinality q^n whereas the module has q. ☐

Generalization 3.6. *Suppose that R is Artinian (and hence, Noetherian and thus, of finite length). If $\lambda_R(R) \nmid \lambda_R(M)$, then M has no finite free resolution.*

Note that if M is a finitely generated R-module, then $\lambda_R(M)$ is indeed finite.

Proof. The idea is the same as earlier. Given a short exact sequence

$$0 \to N \to F \to L \to 0$$

of R-modules, we have $\lambda_R(F) = \lambda_R(N) + \lambda_R(L)$. If F is free, then additivity of lengths tells us that $\lambda_R(F) = \operatorname{rank}(F)\lambda_R(R)$. In particular, $\lambda_R(R) \mid \lambda_R(F)$. Thus, $\lambda_R(N)$ is a multiple of $\lambda_R(R)$ if and only if $\lambda_R(L)$ is.

The argument of earlier now goes through. □

With the above, we can now answer Question 3.2 even when k is infinite.

Corollary 3.7. *Let k be any field. Then, k has no finite free resolution over $\mathsf{k}[x]/\langle x^n\rangle$ for $n > 1$.*

Proof. Letting $R := \mathsf{k}[x]/\langle x^n\rangle$. Then, $\lambda_R(R) > 1$ since $\langle x\rangle/\langle x^n\rangle$ is a R-submodule strictly between 0 and R.
On the other hand, any R-submodule of k must necessarily be a k-vector space and thus, $\lambda_R(\mathsf{k}) = 1$. □

Generalization 3.8. *Instead of length, we could work with any number $\nu_R(M)$ associated to a module M which is "additive" in the sense of short exact sequences. In fact, the original proposition was working with $\nu_R(M) = \log_{|R|}|M|$.*

Note that the above is a bit imprecise. We don't necessarily define $\nu_R(M)$ for all R and all M. For example, $\log_{|R|}|M|$ only makes sense for finitely generated modules over finite rings. On the other hand $\lambda_R(M)$ only made sense for finitely generated modules over Artinian rings.
Also, note that $\log_{|R|}|M|$ is not integer valued. By "$a \mid b$", we still mean that b is an integer multiple of a.

Generalization 3.9. *Let R be Artinian and M a finitely generated R module. If there exists a multiplicative subset $A \subset R$ such that $\lambda_{A^{-1}R}(A^{-1}R) \nmid \lambda_{A^{-1}R}(A^{-1}M)$, then there is no finite free resolution of M over R.*

Note that $A^{-1}R$ is again Artinian and so the above makes sense.

Proof. Suppose that there were a finite free resolution of M over R. Since localization is an exact functor, localizing the resolution with respect to A gives a finite free resolution of $A^{-1}M$ over $A^{-1}R$. However, this is a contradiction, by the previous generalization. □

Generalization 3.10. *Let S be an R-algebra which is an Artinian ring and* flat *as an R-module. Let M be a finitely generated R-module. If $\lambda_S(S) \nmid \lambda_S(S \otimes_R M)$, then M has no finite R-free resolution.*

Proof. The same proof as earlier goes through by the assumption of flatness. □

Remark 3.11. We have seen that length is one example of an additive function on modules. We note that for some specific rings the rank of a module turns out to be additive function on modules.

Let R be an integral domain. Let $S = R\setminus\{0\}$ be a multiplicatively closed set. Let $k = S^{-1}R$ be the field of fractions of R.

Definition 3.12 (**Rank of a module**). Let M be an R-module where R is an integral domain. The rank of M is defined as the cardinality of the largest R-linearly independent subset of M.

Remark 3.13. Let R be an integral domain and M be an R-module such that $\{x_1, x_2, \cdots, x_n\}$ is an R-linearly independent subset. Then, $\{x_1/1, x_2/1, \cdots, x_n/1\}$ is a k-linearly independent subset of $S^{-1}M$ as a k-vector space. Similarly, if $\{x_1/s_1, x_2/s_2, \cdots, x_n/s_n\}$ is a k-linearly independent set of $S^{-1}M$ then $\{x_1, x_2, \cdots, x_n\}$ is an R-linearly independent subset of M. Since a maximal linearly independent set of a vector space is a basis thus for modules of finite rank over an integral domain $\text{rank}_R\ M = \dim_k\ S^{-1}M$.

Theorem 3.14 (**Additivity of rank**). *Consider the following exact sequences of finitely generated R-modules where R is an integral domain.*

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$$

Then, we have that $Rank(M_1) - Rank(M) + Rank(M_2) = 0$.

Proof. We localize the given exact sequence with respect to $S = R \setminus \{0\}$ to get an exact sequence of vector spaces.

$$0 \longrightarrow S^{-1}M_1 \longrightarrow S^{-1}M_2 \longrightarrow S^{-1}M_3 \longrightarrow 0$$

Using the above remark that $\text{rank}_R\ M = \dim_k\ S^{-1}M$ and the additivity of dimension of vector spaces we get the result. □

Remark 3.15. The above result is analogous to the rank nullity theorem for vector spaces.

3.2. **Local rings with nonzero annihilators.** Propositions 3.3 and 3.6 fail to give an answer when $|M|$ is actually a power of $|R|$. For example, the following question cannot be answered by these propositions.

Question 3.16. *Does there exist a finite free resolution of $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ as a $\mathbb{Z}/4\mathbb{Z}$ module?*

Trying to compute the resolution optimally at each step suggests that the answer is "no." Indeed, that is the case. As before, we give a more general result.

Proposition 3.17. *Let $(R, \mathfrak{m})$ be a local ring. Further assume that $\mathfrak{m} = \langle x \rangle$ for some $x \in \mathcal{N}(R) \setminus \{0\}$. Let M be a finitely generated R-module. Then, M has a finite free resolution over R if and only if M is free.*

We can generalize further.

Proposition 3.18. *Let $(R, \mathfrak{m})$ be a local ring such that the annihilator of the maximal ideal is non trivial i.e.* $\text{ann}_R(\mathfrak{m}) \neq 0$. *Let M be a finitely generated R-module. Then, M has a finite resolution if and only if M is free.*

Proof. ($\Rightarrow$) If M is not free, consider a finite free resolution of M, minimum with respect to length

$$0 \longrightarrow F_n \xrightarrow{\phi_n} F_{n-1} \xrightarrow{\phi_{n-1}} \cdots \longrightarrow F_0 \xrightarrow{\phi_0} M \to 0.$$

Since M is not free, we have $n \geq 1$.

Let $\{e_1, \ldots, e_d\}$ and $\{f_1, \ldots, f_k\}$ be bases of F_n and F_{n-1} respectively. Let A be the $k \times d$ matrix of ϕ_n with respect to these bases.

Claim: *Each column of A contains at least one unit.*

Proof of claim. Choose $y \in \operatorname{ann}_R(\mathfrak{m}) \setminus \{0\}$. For the sake of contradiction, suppose that the j^{th} column of A contains no unit, that is, assume that the j^{th} column is $\{a_1, \ldots, a_k\} \subset \mathfrak{m}$. Then

$$\phi_n(ye_j) = y\phi_n(e_j) = \sum_{i=1}^{k} y a_i f_i = 0,$$

which contradicts the injectivity of ϕ_n. This proves the claim. □

Let the first column of A be $\{a_1, \ldots, a_k\}$, and suppose a_l is a unit. Therefore, using a scaling operation $(e_1 \to (a_l)^{-1} e_1)$, a row exchange $(f_l \longleftrightarrow f_1)$, and row and column transformations $(e_i \to e_i - a e_1,\ f_j \to f_j - a f_1)$, we can convert A to a matrix of the form

$$\begin{bmatrix} 1 & 0 \\ 0 & B \end{bmatrix}$$

Note that applying the above claim, each column of B has a unit too. Repeating the process, we can transform the bases such that the corresponding matrix is of the form

$$I \text{ or } \begin{bmatrix} I \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} I & 0 \end{bmatrix}.$$

The third case is ruled out by injectivity of ϕ_n (or the above claim). Call the transformed bases $\{e_1', \ldots, e_d'\}$ and $\{f_1', \ldots, f_k'\}$. Then $\phi_n(e_i') = f_i'\ \forall i \in \{1, \ldots, d\}$, and $f_j' \notin \operatorname{im}(\phi_n) = \ker(\phi_{n-1})\ \forall j > d$.

We now have another free resolution of M:

$$0 \longrightarrow F_{n-1}' \xrightarrow{\phi_{n-1}'} F_{n-2} \xrightarrow{\phi_{n-2}} \cdots \to F_0 \xrightarrow{\phi_0} M \to 0,$$

where F_{n-1}' is the free submodule of F_{n-1} generated by $\{f_{d+1}', \ldots, f_k'\}$, and ϕ_{n-1}' is the restriction of ϕ_{n-1} to F_{n-1}'. This contradicts the minimality of our original free resolution.

$(\Leftarrow)$ If M is free, it has a free resolution of length 0. □

The above proof technique[1] also yields the following result about rings.

Proposition 3.19. *Let S be a nonzero ring, and $\phi : S^{\oplus m} \to S^{\oplus n}$ be injective. Then $m \leq n$.*

Proof. We will prove this by induction on n. The base case $n = 0$ is trivially true. Suppose $m > n$, then we have an inclusion from $\psi : S^{\oplus n} \to S^{\oplus m}$.

We first note that every ring contains a subring generated by 1 and it is Noetherian. Let R be the subring of A generated by 1 and the elements of the coefficients of ϕ. Then, by Hilbert basis theorem, R is Noetherian.

[1]We had seen this technique in `https://math.stackexchange.com/a/1174915/` and later found out that it is a standard technique.

By restricting ϕ to $R^{\oplus m}$ we have map $\phi : R^{\oplus m} \to R^{\oplus n}$ which is R-linear. Every ring has a minimal prime ideal (Zorn's lemma), and localizing at a minimal prime ideal of R gives us a Noetherian local ring with unique prime ideal. So, we may assume that R is Noetherian and local with a unique prime ideal, which is the nilradical of R. Consider the maps $\phi : R^{\oplus m} \to R^{\oplus n}$ and $\psi : R^{\oplus n} \to R^{\oplus m}$.

Let $f = \psi \circ \phi$. Consider the coefficients of the matrix of f. If all the entries belong to the unique prime ideal then the matrix f is nilpotent. But f is composition of two injective maps and hence it cannot be nilpotent.

Hence, the matrix of f must contain at least one coefficient which does not belong to the unique prime ideal. Thus, the matrix of ϕ must contain at least one entry which is a unit. Then, by a sequence of row and column operations we can reduce the matrix of ϕ to the following form $\begin{bmatrix} 1 & 0 \\ 0 & A' \end{bmatrix}$.

Thus, the map ϕ can be viewed as a map from $R \oplus R^{\oplus(m-1)} \to R \oplus R^{\oplus(n-1)}$ and by induction it follows that $m - 1 \leq n - 1$ contradicting $m > n$. $\square$

Remark 3.20. As a consequence, we get the result that if $R^{\oplus n} \cong R^{\oplus m}$, then $n = m$. Thus, we have shown that every nonzero ring has *invariant basis number.*

4. Some General Results

Question 4.1. *If M is finitely generated and $M \cong F_1/K_1 \cong F_2/K_2$ for $F_1 = R^{\oplus n_1}$ and $F_2 = R^{\oplus n_2}$, then*

(1) is $n_1 = n_2$?

(2) is $K_1 \cong K_2$?

Answer. The answer to both is "no," in a trivial manner. Indeed, given $M \cong F_1/K_1$, we can define $F_2 := F_1 \oplus R$.

Then, we have the natural maps

$$F_1 \oplus R \xrightarrow{\pi} F_1 \xrightarrow{\varphi} M.$$

Then, putting $K_2 := \ker(\varphi \circ \pi)$ gives us that $M \cong F_2/K_2$. But, $F_2 \ncong F_1$ and $K_2 \cong K_1 \oplus R$. One can easily find examples of when $K_1 \oplus R \ncong K_1$. For example, if K_1 is finitely generated.

To be very explicit, we may take $M = R = F_1$ and $F_2 = R \oplus R$. Then, with the natural maps, $K_1 = 0 \ncong R \cong K_2$. $\square$

Question 4.2. *Suppose that $n_1 = n_2$ above. Is it then necessary that $K_1 \cong K_2$?*

Equivalently, if $F = R^{\oplus n}$ and $F/K_1 \cong F/K_2$, is it necessary that $K_1 \cong K_2$?

Before we give the general answer, we first see two cases where the question has a positive answer, namely when $n = 1$ and when R is a PID.

Proposition 4.3. *Let I_1 and I_2 be submodules of R such that $R/I_1 \cong R/I_2$ as R-modules. Then, $I_1 = I_2$.*

Proof. Note that if M and N are isomorphic R-modules, then $\operatorname{ann}_R M = \operatorname{ann}_R N$. Moreover, a submodule $I \subset R$ is simply an ideal and $\operatorname{ann}_R(R/I) = I$. Thus, we note

$$R/I_1 \cong R/I_2 \implies \operatorname{ann}_R(R/I_1) = \operatorname{ann}_R(R/I_2) \implies I_1 = I_2. \qquad \square$$

Proposition 4.4. *Let R be a PID and $n \geq 1$. Let K_1 and K_2 be submodules of $R^{\oplus n}$ such that $R^{\oplus n}/K_1 \cong R^{\oplus n}/K_2$. Then, $K_1 \cong K_2$.*

Proof. Consider a short exact sequence of the form

$$0 \to K \to R^{\oplus n} \to M \to 0.$$

Since R is a PID, we know that K is again a free module of rank $m \leq n$. Consider the multiplicative subset $A = R \setminus \{0\}$.

Now, localizing the exact sequence

$$0 \to R^{\oplus m} \to R^{\oplus n} \to M \to 0$$

with respect to A gives us

$$0 \to A^{-1}\left(R^{\oplus m}\right) \to A^{-1}\left(R^{\oplus n}\right) \to A^{-1}M \to 0. \tag{1}$$

Note that the above is an exact sequence of $\mathsf{k} := A^{-1}R$ modules. (Note that k is the field of fractions of R.) The exactness is a consequence of localization being an exact functor.

Also, recall that

$$\operatorname{rank}_R F = \dim_{\mathsf{k}}(A^{-1}F)$$

for any free R-module F. Thus, we see that the left two vector k-vector spaces in (1) have dimensions m and n, respectively. As a result, we get that $A^{-1}M$ is finite dimensional as a k-vector space and moreover,

$$m = n - \dim_{\mathsf{k}}(A^{-1}M).$$

Now, if we have another exact sequence of R-modules of the form

$$0 \to R^{\oplus m'} \to R^{\oplus n} \to M' \to 0$$

with $M \cong M'$, we see that

$$\begin{aligned} m' &= n - \dim_{\mathsf{k}}(A^{-1}M') \\ &= n - \dim_{\mathsf{k}}(A^{-1}M) \\ &= m. \end{aligned}$$

And thus, the kernels $R^{\oplus m}$ and $R^{\oplus m'}$ are isomorphic. $\square$

Generalization 4.5. *Instead of assuming R to be a PID, we may assume that R is an integral domain and that K_1 and K_2 are free.*

For $n > 1$, the answer to Question 4.2 is "no" in general. An example showing this can be given with the help of stably free modules. An R-module P is called *stably free* if $P \oplus R^{\oplus n} \cong R^{\oplus m}$ for some natural numbers m, n. Not all stably free modules are free[2]. Thus, for a stably free module P which is not free with $P \oplus R^{\oplus n} \cong R^{\oplus m}$, both the quotients $R^{\oplus m}/R^{\oplus(m-n)}$ and $R^{\oplus m}/P$ are isomorphic to $R^{\oplus n}$ but $P \ncong R^{\oplus(m-n)}$.

[2]One such example can be found at `https://kconrad.math.uconn.edu/blurbs/linmultialg/stablyfree.pdf`

Note that if we drop the "finite rank" condition, then the question is not so difficult as the next example shows.

Example 4.6. Consider the free module

$$F = \bigoplus_{i=1}^{\infty} R$$

and the R-linear map $\varphi : F \to F$ given by

$$\varphi(r_1, r_2, r_3, \ldots) = (r_2, r_3, \ldots).$$

Clearly, φ is surjective and $\ker(\varphi) \cong R$; this gives us that $F/\ker(\varphi) \cong F \cong F/0$ but $\ker(\varphi) \ncong 0$.

Note that if we drop the "free"-ness, it is easy to come up with examples of isomorphic quotients without isomorphic kernels as the next example shows.

Example 4.7. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Let $N_1 = \mathbb{Z}/4\mathbb{Z} \oplus 0$ and $N_2 = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Note that N_1 and N_2 are $\mathbb{Z}$-submodules of M with $N_1 \ncong N_2$. However, we have $M/N_1 \cong M/N_2$.

5. Resolutions of Monomial Ideals in Two Variables

We now compute free resolutions of $\mathsf{k}[x, y]/I$ over $\mathsf{k}[x, y]$, where I is a monomial ideal in $\mathsf{k}[x, y]$. First, we mention two examples which indicated how to proceed in the general case.

Example 5.1. Let $I_1 = \langle x, y \rangle$.
Note that if $f(x, y)x + g(x, y)y = 0$ for some $f(x, y), g(x, y) \in \mathsf{k}[x, y]$, then there exists $h(x, y) \in \mathsf{k}[x, y]$ such that

$$f(x, y) = yh(x, y) \quad \text{and} \quad g(x, y) = -xh(x, y).$$

Therefore, the following is a free resolution of $\mathsf{k}[x, y]/I_1$ over $\mathsf{k}[x, y]$.

$$0 \to \mathsf{k}[x, y] \xrightarrow{\left[\begin{smallmatrix} y \\ -x \end{smallmatrix}\right]} (\mathsf{k}[x, y])^{\oplus 2} \xrightarrow{[x\ y]} \mathsf{k}[x, y] \to \mathsf{k}[x, y]/I_1 \to 0.$$

Example 5.2. Let $I_2 = \langle x^2, xy, y^2 \rangle$.
Note that if

$$a_1 x^2 + a_2 xy + a_3 y^2 = 0$$

for some $a_1, a_2, a_3 \in \mathsf{k}[x, y]$, then we have $a_1 = b_1 y$ and $a_3 = b_3 x$ for some $b_1, b_3 \in \mathsf{k}[x, y]$. Therefore, $b_1 x + a_2 + b_3 y = 0$ which gives $a_2 = -b_1 x - b_3 y$ and hence,

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = b_1 \begin{bmatrix} y \\ -x \\ 0 \end{bmatrix} - b_3 \begin{bmatrix} 0 \\ y \\ -x \end{bmatrix}.$$

Also, the elements $\begin{bmatrix} y \\ -x \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ y \\ -x \end{bmatrix}$ of $(\mathsf{k}[x, y])^{\oplus 3}$ are linearly independent.

Thus, following is a free resolution of $\mathsf{k}[x, y]/I_2$ over $\mathsf{k}[x, y]$.

$$0 \to (\mathsf{k}[x, y])^{\oplus 2} \xrightarrow{\left[\begin{smallmatrix} y & 0 \\ -x & y \\ 0 & -x \end{smallmatrix}\right]} (\mathsf{k}[x, y])^{\oplus 3} \xrightarrow{[x^2\ xy\ y^2]} \mathsf{k}[x, y] \to \mathsf{k}[x, y]/I_2 \to 0.$$

In both the examples above we see that for the monomial ideal I_n of $\mathsf{k}[x, y]$ generated by all $n + 1$ monomials of the degree n, there is a resolution of R/I_n of the form

$$0 \to R^{\oplus n} \to R^{\oplus(n+1)} \to R \to R/I_n \to 0.$$

Indeed, this is always the case, and we have the following proposition.

Proposition 5.3. *Let $n \in \mathbb{N}$ and I_n be the monomial ideal of $R = \mathsf{k}[x, y]$ generated by all $n + 1$ monomials of degree n. That is,*

$$I_n = \langle x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n \rangle.$$

Then there exists an R-free resolution of R/I_n of the form

$$0 \to R^{\oplus n} \to R^{\oplus(n+1)} \to R \to R/I \to 0.$$

We separate out the main part of the proof in the following lemma.

Lemma 5.4. *There exists a R-free resolution of I_n of the form*

$$0 \to R^{\oplus n} \to R^{\oplus(n+1)} \to I_n \to 0.$$

Proof. Let $\{e_1, \ldots, e_{n+1}\}$ denote the standard basis of $R^{\oplus n+1}$, and $\{f_1, \ldots, f_n\}$ denote the standard basis of $R^{\oplus n}$. Define $v_1, \ldots, v_n \in R^{\oplus n+1}$ by $v_i = ye_i - xe_{i+1}$. In matrix form, note that we have

$$v_1 = \begin{bmatrix} y \\ -x \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ y \\ -x \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, v_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ y \\ -x \end{bmatrix}.$$

(Note that the length of each column is $n + 1$.)

Claim. $\{v_1, \ldots, v_n\}$ is linearly independent.
Suppose $a_1v_1 + \cdots + a_nv_n = 0$ for some $a_i \in R$. In matrix form, we have

$$a_1 \begin{bmatrix} y \\ -x \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ y \\ -x \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + a_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ y \\ -x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Comparing the entries from top to bottom, we inductively see that $a_1 = \cdots = a_n = 0$.

Our aim now is to show that the v_i essentially capture all the relations between the generators of I_n.

Consider the R-linear maps ϕ and ψ as in

$$0 \to R^{\oplus n} \xrightarrow{\phi} R^{\oplus n+1} \xrightarrow{\psi} I \to 0$$

defined by

$$\phi(f_i) = v_i \quad \text{and} \quad \psi(e_j) = x^{n+1-j}y^{j-1}.$$

ϕ is injective since $\{v_i\}_i$ is linearly independent and ψ is surjective since $I = \langle x^{n+1-j}y^{j-1} \mid 1 \leq j \leq n+1\rangle$. We now show exactness at the middle.

Note that

$$\psi(\phi(f_i)) = \psi(v_i) = \psi(ye_i - xe_{i+1}) = (yx^{n+1-i}y^i) - (x)(x^{n+1-i-1}y^i) = 0$$

and hence, $\mathrm{im}(\phi) \subset \ker(\psi)$. By construction, this was expected since $(y, -x)$ was capturing the obvious relation between $x^{i+1}y^j$ and x^iy^{j+1}. We now show the reverse containment.

Let $w = a_1e_1 + \cdots + a_ne_n \in \ker(\psi)$. We shall inductively choose coefficients for v_i's to show that w can be written as a R-linear combination of v_i's, which would imply that $w \in \mathrm{im}(\phi)$. For $1 \leq i \leq n+1$, let $\pi_i : R^{\oplus n+1} \to R^{\oplus n+1}$ be the unique R-linear map such that

$$\pi_i(e_j) = \begin{cases} e_j & \text{if } 1 \leq j \leq i, \\ 0 & \text{if } i < j. \end{cases}$$

(Thinking about it in terms of column vectors, we are simply restricting the column of length $n+1$ to its first i entries.)

As $\psi(a_1e_1 + \cdots + a_ne_n) = 0$ we have

$$a_1x^n + a_2x^{n-1}y^1 + \cdots + a_{n+1}y^n = 0.$$

This means that $y \mid a_1$ or $a_1 = yb_1$ for some $b_1 \in \mathsf{k}[x, y] = R$. Thus, we get

$$\pi_1(w - b_1v_1) = \pi_1(w) - \pi_1(b_1v_1) = a_1 - b_1y = 0.$$

Inductively assume that for some $j \leq n-1$ there exist $b_1, \ldots, b_j \in R$ such that

$$\pi_j(w - (b_1v_1 + \cdots + b_jv_j)) = 0.$$

Then $w - (b_1v_1 + \cdots + b_jv_j)$ is of the form $w_1 = c_{j+1}e_{j+1} + \cdots + c_{n+1}e_{n+1}$, for some $c_k \in R$. We apply ψ to $w - (b_1v_1 + \cdots + b_jv_j)$ to conclude that

$$c_{j+1}x^{n-j}y^j + \cdots + c_{n+1}y^n = 0$$

(as v_i's and w are in the kernel and ψ is R-linear). Now, dividing the above equation by y^j, we conclude that $y \mid c_{j+1}$.
Therefore, $c_{j+1} = yb_{j+1}$ for some $b_{j+1} \in R$. Now, observe that

$$\begin{aligned} w - (b_1v_1 + \cdots + b_jv_j + b_{j+1}v_{j+1}) &= w_1 - b_{j+1}v_{j+1} \\ &= c_{j+1}e_{j+1} + \cdots + c_{n+1}e_{n+1} - yb_{j+1}e_{j+1} + xb_{j+1}e_{j+2} \\ &= c_{j+2}e_{j+2} + \cdots + c_{n+1}e_{n+2} + xb_{j+1}e_{j+2}. \end{aligned}$$

Thus, we see that

$$w - (b_1v_1 + \cdots + b_jv_j + b_{j+1}v_{j+1}) \in \langle e_{j+2}, \ldots, e_{n+1}\rangle \subset \ker(\pi_{j+1}).$$

Therefore, by induction we conclude that there exist $b_1, \ldots, b_n \in R$ such that $w - (b_1v_1 + \cdots + b_nv_n)$ is of the form $b_{n+1}e_{n+1}$. By applying ψ we get $b_{n+1}y^n = 0$, i.e., $b_{n+1} = 0$. Therefore $w = b_1v_1 + \cdots + b_nv_n \in \mathrm{im}(\phi)$, and hence $\ker(\psi) \subset \mathrm{im}(\phi)$. Therefore the sequence

$$0 \to R^{\oplus n} \xrightarrow{\phi} R^{\oplus(n+1)} \xrightarrow{\psi} I \to 0.$$

is exact. This completes the proof. $\square$

Using the lemma we now complete the proof of the proposition.

Proof. By the lemma proved above, we have a free resolution of I_n of the form

$$0 \to R^{\oplus n} \xrightarrow{\phi_2} R^{\oplus(n+1)} \xrightarrow{\phi_1} I_n \to 0.$$

Therefore, R/I has a free resolution

$$0 \to R^{\oplus n} \xrightarrow{\phi_2} R^{\oplus(n+1)} \xrightarrow{\phi_1} R \xrightarrow{\phi_0} R/I_n \to 0,$$

where $\phi_0(1) = \bar{1}$. $\square$

Generalization 5.5. *The above generalizes to monomial ideals generated by monomials of the same degree, even if we don't take all possible monomials. That is, let $k > 1$ and consider the ideal*

$$I = \langle x^{r_1}y^{n-r_1}, \ldots, x^{r_k}y^{n-r_k} \rangle$$

for some integers $n \geq r_1 > \cdots > r_k \geq 0$. Then, we have the following R-free resolution:

$$0 \to R^{\oplus(k-1)} \to R^{\oplus k} \to I \to 0.$$

Sketch of proof. The proof is almost identical to the earlier case. Instead of $v_i = ye_i - xe_{i+1}$, we now define

$$v_i := y^{r_i - r_{i+1}}e_i - x^{r_i - r_{i+1}}e_{i+1}$$

for $1 \leq i \leq k-1$.

Again, looking at the matrix form of v_i's shows that they are linearly independent. As before, each v_i is capturing an obvious relation between two generators and hence, is in the kernel. The work is now to show that these are all the relations.

The base case of the induction again follows by taking y^{n-r_1} common from

$$a_1x^{r_1}y^{n-r_1} + \cdots + a_kx^{r_k}y^{n-r_k} = 0$$

and noting that $y^{r_1-r_2}$ divides all but the first term and hence, $y^{r_1-r_2} \mid a_1$.

The inductive step is carried out similarly. $\square$

Generalization 5.6. *The above generalizes even further to finitely generated monomial ideals, with monomial generators not necessarily of same degree. That is, let $k > 1$ and consider the ideal*

$$I = \langle x^{r_1}y^{s_1}, \ldots, x^{r_k}y^{s_k} \rangle \tag{2}$$

such that

$$r_1 > \cdots > r_k \geq 0,$$
$$0 \leq s_1 < \cdots < s_k.$$

Then, we have the following R-free resolution:

$$0 \to R^{\oplus(k-1)} \to R^{\oplus k} \to I \to 0.$$

Sketch of proof. Things work as earlier by defining

$$v_i := y^{s_{i+1}-s_i}e_i - x^{r_i-r_{i+1}}e_{i+1}$$

for $1 \leq i \leq k-1$. □

Remark 5.7. Note that given any ideal which is generated by finitely many monomials of the form $x^{a_i}y^{b_i}$, we can always put it in the form (2), by removing redundancies.

Indeed, if $a_i = a_j$, then one of $x^{a_i}y^{b_i}$ or $x^{a_j}y^{b_j}$ divides the other. Thus, we may assume $a_i > a_j$. In this case, we must have $b_i < b_j$ or else $x^{a_j}y^{b_j}$ would divide $x^{a_i}y^{b_i}$.

Remark 5.8. Since $\mathsf{k}[x,y]$ is Noetherian, we see that any monomial ideal is actually generated by finitely many monomials. Thus, by the previous remark, we see that the "finitely generated" hypothesis in Generalization 5.6 can be dropped and we actually have the result for all monomial ideals.

In view of the earlier remarks, we have actually proven the following.

Theorem 5.9. *Let* k *be a field and* $I \subset \mathsf{k}[x,y]$ *be a nonzero monomial ideal. Then,* R/I *has a free resolution of the form*

$$0 \to R^{\oplus n} \to R^{\oplus(n+1)} \to R \to R/I \to 0.$$

Appendix A. The Process - by H. Ananthnarayan

The mathematical content of this article is the outcome of an exploratory project carried out by the first six authors, as a part of an advanced course in commutative algebra taught by me. The aim of this project was to get the students comfortable with the notion of a free resolution by working with various examples, to make them realize the kind of questions that can come up, and to figure out some techniques to answer these questions. This article is a combined report of two groups, who had come up with similar (and some overlapping) results.

At the beginning of the project, the students were introduced to construction of free resolutions as a consequence of the fact that every module can be written as the quotient of a free module. After discussing examples of free resolutions over fields and PIDs, they were given specific examples, so that they could compare and contrast, come up with questions or conjectures, and try to answer as many of those as they could. Some open-ended questions were also asked to give them some direction. Some of the examples and questions are listed below:

Notation: k is a field, and R is a commutative ring with unity.

(1) Find a free resolution of k as a module over:
(i) k (ii) $\mathsf{k}[X]$ (iii) $\mathsf{k}[[X]]$ (iv) $\mathsf{k}[X]/\langle X^2\rangle$
(v) $\mathsf{k}[X,Y]$ (vi) $\mathsf{k}[X,Y]/\langle X^2, XY\rangle$ (vii) $\mathsf{k}[X,Y,Z]$

(2) Find a free resolution of $\mathbb{Z}/2\mathbb{Z}$ as a module over (i) $\mathbb{Z}$ (ii) $\mathbb{Z}/2\mathbb{Z}$ (iii) $\mathbb{Z}/4\mathbb{Z}$ (iv) $\mathbb{Z}/6\mathbb{Z}$.

(3) Find a free resolution of $\mathsf{k}[X,Y]/I$ over $\mathsf{k}[X,Y]$, where $I =$
(i) $\langle X,Y\rangle$ (ii) $\langle X^2,Y^2\rangle$ (iii) $\langle X^2,XY\rangle$ (iv) $\langle X^2,XY,Y^2\rangle$ (v) $\langle X\rangle$
Do the same over $\mathsf{k}[X,Y,Z]$ for $I =$ (i) $\langle X,Y\rangle$ (ii) $\langle X^2,XY,Y^2\rangle$ and (iii) $\langle X,Y,Z\rangle$.

(4) If $a \in R$ is not a unit, is $0 \to R \xrightarrow{\cdot a} R \to R/\langle a \rangle \to 0$ an R-free resolution of $R/\langle a \rangle$? Why or why not?

(5) Based on Q(1) - Q(4), make conjectures about R-free resolutions of special classes of R-modules, e.g.,
(i) R/I where I is principal, or generated by 2 elements. (ii) A quotient of $\mathsf{k}[X, Y]$.
(iii) Any other class.
Do your conjectures have a positive, or a negative answer?

(6) Let M be a finitely generated R-module, $F_1 \simeq R^{\oplus n_1}$, $F_2 \simeq R^{\oplus n_2}$, and $K_j \subset F_j$ be such that $M \simeq F_j/K_j$, for $j = 1, 2$.

  (a) Find an example to show that n_1 and n_2 need not be equal.

  (b) If $n_1 = n_2$, is it necessary that $K_1 \simeq K_2$?

(7) Given free resolutions for two of the three modules in a short exact sequence of R-modules, can we construct a free resolution for the third?

After a week of group discussions, new groups, called jigsaw groups, were formed for the purpose of sharing information. Each jigsaw group consisted of one member from each of the original groups, where they shared their questions and results with the other groups. This interaction was useful in validating some of their questions and results, and also gave them new ideas and generalizations to pursue. After this exercise, they went back to the original groups, and worked for another week before they wrote a preliminary report. The jigsaw groups led to two groups being influenced by each other, which led to this joint report.

Throughout the process, the students also kept updating me about their results, and I would ask more questions based on what they had. For example, when one group came up with Proposition 3.3, with suitable hints, they realized that cardinality could be replaced by length (Generalization 3.6), and further led to the contents of Subsection 3.1. Similarly, with a few leading questions, Proposition 3.17 was generalized to Proposition 3.18, which is the base case of the celebrated *Auslander-Buchsbaum formula*.

The contents of Section 5, in which resolutions of monomial ideals in a bivariate polynomial ring $\mathsf{k}[X, Y]$ are computed, are the consequence of looking at examples. The idea for constructing the resolution for the ideal $\langle X, Y \rangle^n$ for any $n \in \mathbb{N}$ came from looking at the ideals $\langle X, Y \rangle$, and $\langle X^2, XY, Y^2 \rangle$. The students further realized that this can be used to give resolutions for any monomial ideal in $\mathsf{k}[X, Y]$. Given time, and some direction, the students could have used these ideas to come up the *Taylor resolution*, a resolution for a monomial ideal in a polynomial ring in any finite number of variables over a field.

The upshot is that:
(i) the initial set of examples gave the students a feel for constructing resolutions, and to come up with questions, and test their ideas.
(ii) the few open-ended questions gave some initial direction to those ones who did not know where to begin.
(iii) the jigsaw groups gave them an opportunity to validate their results and see other related questions.
(iv) discussions with me helped them explore the limits of how much they could push their results.

I was pleasantly surprised by the results that the students were able to reach, given where they started in terms of background. This topic had the right mix of the known and the unknown, gave them scope for constructing examples, for making and testing conjectures, while also giving a feel for collaborative research work. Based on the feedback, these points made it a productive, and fun, venture for the students, and could possibly be adapted to projects in other courses as well.

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