Master's Thesis Stage 1

Homological Algebra

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Contents

Chapter 1

Graded Rings, Modules and Resolutions

1.1 Graded Rings and Modules

Definition 1.1.1. *Let H be a cancellative monoid under addition. A ring R is said to be **H-graded** if $R = \bigoplus_{i \in H} R_i$, where, each R_i is an abelian group and $R_i R_j \subseteq R_{i+j}$, for all $i, j \in H$. For each i, R_i is called the **homogeneous component of degree** i of R and the nonzero elements of R_i are called **homogeneous elements of degree i**.*

Remark 1.1.2.
(i) For a cancellative monoid H, we denote its associated group by G.
(ii) By an ordered monoid we mean a cancellative monoid H with an order $<$ satisfying: whenever $a < b$ in H, we have $a + c < b + c$ for all $c \in H$.
(iii) If H is an ordered monoid, then we say that it is well ordered if every nonempty subset S of H which is bounded below has the least element in S.

Definition 1.1.3. *A module M is called as a **graded module** over a graded ring R if $M = \bigoplus_{i \in G} M_i$, as a direct sum of subgroups of M and for all $i \in H, j \in G$, $R_i M_j \subseteq M_{i+j}$.*

Definition 1.1.4. *An ideal J of a graded ring R is said to be **graded** if it satifies any of the following equivalent conditions:*
(i) If $f \in J$, then every homogeneous component of f is in J.
(ii) $J = \oplus_{i \in N} J_i$, where $J_i = R_i \bigcap J$.
(iii) If J' is the ideal generated by all homogeneous elements in J, then $J = J'$.
(iv) J has a system of homogeneous generators.

Proposition 1.1.5. *Given a graded ideal I in a graded ring R, every associated prime of I is also graded.*

Proof. Suppose $J = (I : x)$ is a prime ideal for some x in R. Let $x = x_l + x_{l+1} + \cdots + x_k$ where $x_i \in R_i$, $l < k$ and x_l, x_k are non-zero.

Let $y = y_t + y_{t+1} + \cdots + y_s \in J$, where $y_i \in R_i$, $t < s$ and x_t, x_s are non-zero. If we show that $y_t \in J$, we are done by induction on $s - t$.
To see this, observe that we have $xy \in I$ and since I is graded, the lowest graded component of xy, which is $x_l y_t$, belongs to I. Similarly, $x_{l+1}y_t + x_l y_{t+1} \in I$, and on multiplying by y_t, we get that $x_{l+1}y_t^2 \in I$. Continuing in this manner, we get that $x_{l+i}y_t^{i+1} \in I$ for all $i = 0, 1, \ldots, k-l$, which implies that $y_t^{k-l+1}x \in I$ and hence, $y_t^{k-l+1} \in J$. Since J is prime, $y_t \in J$. Hence, J is a graded ideal. □

Definition 1.1.6. *Let R be a H-graded ring and $M = \bigoplus_{i \in G} M_i$ be a finitely generated R-module. Then we define an R-module $M(d)$ by $M(d) = \bigoplus_{i \in G} M_{i+d}$. $M(d)$ is called a **shifted** R**-module**.*

Definition 1.1.7. *Let $M = \bigoplus_{i \in G} M_i$, $M' = \bigoplus_{i \in G} M'_n$ be graded modules over R. An R-linear map $f : M \to M'$ is said to be a **graded map of degree** d if $f(M_i) \subseteq M'_{i+d}$ for all $i \in G$. If f has degree zero, we simply say that f is a graded R-module homomorphism.*

Proposition 1.1.8. *Let R be nonnegatively graded, M, N be graded R-modules and $\phi : M \to N$ be a graded homomorphism of degree d. Then*
(i) $\ker(\phi)$ *is a graded submodule of M.*
(ii) $\mathrm{Im}(\phi)$ *is a graded submodule of N.*

Proof. (i) It is clear that $\ker(\phi)$ is a submodule of M considered without grading. To show that $\ker(\phi)$ is graded, it suffices to show that if $x = x_r + \cdots + x_s$, is in $\ker(\phi)$, then each x_i is in $\ker(\phi)$. We show that $x_r \in \ker(\phi)$ and by induction we will get that $x_i \in \ker(\phi)$ for all i. Note that $\phi(x_i) \in N_{i+d}$. Therefore $\phi(x_r) \in N_{r+d} \cap (N_{(r+1)+d} \oplus \cdots \oplus N_{s+d}) = 0$. This shows that $\phi(x_r) = 0$ as desired.
(ii) It is clear that $\mathrm{Im}(\phi)$ is a submodule of N considered without grading. To show that $\mathrm{Im}(\phi)$ is graded, it suffices to show that if $y = y_r + \cdots + y_s$, is in $\mathrm{Im}(\phi)$, then each y_i is in $\mathrm{Im}(\phi)$. Since $\phi(M_i) \subseteq N_{i+d}$ and $y \in \mathrm{Im}(\phi)$, there exists $x = x_{r-d} + \cdots + x_{s-d} \in M$ such that $\phi(x) = y$ and $\phi(x_{i-d}) = y_i$. This shows that $y_i \in \mathrm{Im}(\phi)$. This completes the proof. □

Remark 1.1.9.
(i) If I is a graded ideal of R, then we have $R_i I_j \subseteq I_{i+j}$.
(ii) If I is a graded ideal of R, then the quotient ring R/I inherits the grading from R by $(R/I)_i = R_i/I_i$.
(iii) If N is a graded submodule of a graded module M, then M/N is graded with the grading given by $(M/N)_i = M_i/N_i$.

Proposition 1.1.10. *Tensor products of graded R-modules is graded, i.e., if M and N are graded R-modules, then $M \otimes N$ is graded R-module.*

Proof. We know that $M \otimes N$ is an R-module. We give grading to $M \otimes N$ as follows:
Define $(M \otimes N)_i$ to be generated (as a $\mathbb{Z}$-module) by all the elements in $M \otimes N$ of the form $m \otimes n$,

where $\deg(m)+\deg(n)=i$. Then we have $M\otimes N=\bigoplus_{i\in G}(M\otimes N)_i$. Moreover, for any $r_i\in R_i$ and $m\oplus n\in(M\otimes N)_j$, we have $r(m\otimes n)=(rm)\otimes n$. Therefore

$$\deg(r(m\otimes n))=(i+\deg(m))+\deg(n)=i+j.$$

This shows that $R_i(M\otimes N)_j\subseteq(M\otimes N)_{i+j}$. Hence $M\otimes N$ is graded.

□

Let $\mathrm{Hom}_i(M,N)=\{\phi:M\to N\mid deg(\phi)=i\}$. Then we define ${}^*\mathrm{Hom}(M,N)=\bigoplus_{i\in G}\mathrm{Hom}_i(M,N)$.

Remark 1.1.11. In general, ${}^*\mathrm{Hom}(M,N)\neq\mathrm{Hom}(M,N)$. However, we have the equality in a special case which we will prove shortly.

Lemma 1.1.12. *Let $M=\bigoplus_{i=1}^{m}R(n_i)$ and N be graded R-modules. Then ${}^*\mathrm{Hom}(M,N)\cong\mathrm{Hom}(M,N)$ with grading forgotten.*

Proof. It is clear that every $\phi=\phi_r+\cdots+\phi_s\in{}^*\mathrm{Hom}(M,N)$ is in $\mathrm{Hom}(M,N)$, and hence ${}^*\mathrm{Hom}(M,N)\subseteq\mathrm{Hom}(M,N)$. To show the other inclusion assume that $\phi\in\mathrm{Hom}(M,N)$. Let $e_j=(0,\ldots,0,1,0,\ldots,0)$ where 1 occurs at jth place. Then M is a free R-module with basis $\{e_1,\ldots,e_m\}$. If $\phi(e_j)=y_{j1}+\cdots+y_{jr_j}\in N$, then we have

$$\phi=\phi_{11}+\cdots\phi_{1r_1}+\cdots+\phi_{m1}+\cdots+\phi_{mr_m}$$

where $\phi_{js}:M\to N$ is given by $\phi_{js}(e_j)=y_{js}$ and $\phi_{js}(e_i)=0$ for all $i\neq j$. Note that each ϕ_{js} is well defined since $\{e_1,\ldots,e_m\}$ is a basis for M. Moreover ϕ_{js} is a graded homomorphism of degree $\deg(y_{js})+n_j$. Therefore $\phi\in{}^*\mathrm{Hom}(M,N)$. This completes the proof. □

Proposition 1.1.13. *Let R be a graded Noetherian ring, M be a finitely generated graded R-module and N be any graded R-module. Then ${}^*\mathrm{Hom}(M,N)=\mathrm{Hom}(M,N)$ with grading forgotten.*

Proof. It is clear that every $\phi=\phi_r+\cdots+\phi_s\in{}^*\mathrm{Hom}(M,N)$ is in $\mathrm{Hom}(M,N)$, and hence we have an inclusion ${}^*\mathrm{Hom}(M,N)\xrightarrow{i}\mathrm{Hom}(M,N)$.
Since M is finitely generated and R is Noetherian, we get an exact sequence of graded modules $G\to F\to M\to 0$ for some $F=\bigoplus_{j=1}^{n}R(n_j)$ and $G=\bigoplus_{j=1}^{m}R(m_j)$. By the previous lemma we have ${}^*\mathrm{Hom}(F,N)=\mathrm{Hom}(F,N)$, ${}^*\mathrm{Hom}(G,N)=\mathrm{Hom}(G,N)$. Thus we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & {}^*\mathrm{Hom}(M,N) \arrow[r] \arrow[d, "i"] & {}^*\mathrm{Hom}(F,N) \arrow[r] \arrow[d, equal] & {}^*\mathrm{Hom}(G,N) \arrow[d, equal] \\
0 \arrow[r] & \mathrm{Hom}(M,N) \arrow[r] & \mathrm{Hom}(F,N) \arrow[r] & \mathrm{Hom}(G,N)
\end{tikzcd}$$

Thus by five lemma, we get that the inclusion i is an isomorphism. □

Lemma 1.1.14 (Graded Nakayama Lemma)**.** *Let H be an ordered monoid such that $i > 0$ for all $i \in H \setminus \{0\}$ and $R = \bigoplus_{i \in H} R_i$ be a graded ring. Let $M = \bigoplus_{i \in G} M_i$ be an R-module such that there exists $n \in G$ with $M_i = 0$ for all $i < n$. Further assume that G is well ordered. If $R_+ = \bigoplus_{i \in H \setminus \{0\}} R_i$ and $R_+ M = M$ then $M = 0$.*

Proof. Let, if possible, $M \neq 0$. Let m be the smallest element of G such that for all $i < m$, we have $M_i = 0$ and $M_m \neq 0$. But then, $M = R_+ M \subseteq \bigoplus_{i\,m} M_i$, which has m^{th} component equal to 0. This contradiction shows that $M = 0$. $\square$

Corollary 1.1.15. *Let R be a non negatively graded ring and M be a finitely generated $\mathbb{Z}$-graded R-module. If $R_+ M = M$ then $M = 0$.*

Proof. Let $\{m_1, \ldots, m_r\}$ be a generating set for M and d=min$\{\deg(m_i) \mid 1 \leq i \leq r\}$. Since R is graded by $\mathbb{N} \cup \{0\}$, we get that $M_n = 0$, for every $n < d$. Thus, applying graded Nakayama lemma proved above, we get $M = 0$. $\square$

1.2 Graded Resolutions

From now on we assume that R is a graded ring with $R_0 = \mathsf{k}$, a field. We will mostly consider $R = \mathsf{k}[x_1, \ldots, x_r]$.

Definition 1.2.1. *Let M be a graded R-module and*

$$F_\bullet : \cdots \to F_n \xrightarrow{\phi_n} F_{n-1} \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \to 0$$

*be a free resolution of M. If all F_i's are graded R-modules and all ϕ_i's are graded maps of degree zero, then we say that $F_\bullet$ is a **graded free resolution** of M.*

Definition 1.2.2. Let $R = \mathsf{k}[x_1, \ldots, x_n]$ and M be a graded R-module. A graded free resolution

$$F_\bullet : \quad \cdots \to F_n \xrightarrow{\phi_n} \cdots \to F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \to 0$$

is said to be *minimal* if $\phi_i(F_i) \subseteq \langle x_1, \ldots, x_r \rangle F_{i-1}$ for all $i \geq 1$.

Example 1.2.3. Let $I = \langle x^2, y^2 \rangle$ and $R = \mathsf{k}[x, y]$. Then

$$F_\bullet : 0 \leftarrow R/I \xleftarrow{\phi_0} R \xleftarrow{\phi_1} R(-2) \oplus R(-2) \xleftarrow{\phi_2} R(-4) \leftarrow 0,$$

where $\phi_0(1) = 1$, $\phi_1(1, 0) = x^2$, $\phi_1(0, 1) = y^2$, $\phi_2(1) = (-y^2, x^2)$ is a minimal graded free resolution of R/I over R.

Example 1.2.4. Let $I = \langle x^3, y^2 \rangle$ and $R = \mathsf{k}[x, y]$. Then

$$F_\bullet : 0 \leftarrow R/I \xleftarrow{\phi_0} R \xleftarrow{\phi_1} R(-3) \oplus R(-2) \xleftarrow{\phi_2} R(-5) \leftarrow 0,$$

where $\phi_0(1) = 1$, $\phi_1(1, 0) = x^3$, $\phi_1(0, 1) = y^2$, $\phi_2(1) = (-y^2, x^3)$ is a minimal graded free resolution of R/I over R.

Definition 1.2.5. *Let $R = \mathsf{k}[x_1, \ldots, x_n]$ and M be a graded R-module.*

$$F_\bullet : \cdots \to F_n \xrightarrow{\phi_n} F_{n-1} \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \to 0$$

*be a minimal graded free resolution of M, where $F_i = \bigoplus\limits_j R(-j)^{\beta_{i,j}(M)}$. Then the numbers $\beta_{i,j}(M)$ are called **graded Betti numbers** of M. $\beta_i(M) = \sum\limits_j \beta_{i,j}(M)$ is called the total ith Betti number of M.*

Definition 1.2.6. *Let $\beta_{i,j}$ be graded Betti numbers of M. Then **Betti table** of M is written as*

$j \backslash i$	*0*	*1*	$\cdots$	p	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
0	$\beta_{0,0}$	$\beta_{1,1}$	$\cdots$	$\beta_{p,p}$	$\vdots$
1	$\beta_{0,1}$	$\beta_{1,2}$	$\cdots$	$\beta_{p,p+1}$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$

Definition 1.2.7. *Let k be a field and $M = \bigoplus\limits_{n \in \mathbb{Z}} M_n$ be a finitely generated graded module over the polynomial ring $\mathsf{k}[x_1, \ldots, x_r]$. Then the function $H_M : \mathbb{Z} \to \mathbb{Z}$, given by $H_M(j) = \dim_k(M_j)$ is called as **Hilbert function of M**.*

Let k be a field and $R = \mathsf{k}[x_1, \ldots, x_n]$. Let $a \in R \setminus \{0\}$ be such that $\deg(a) = d$. Since a is a nonzerodivisor on R, we get an exact sequence of R-modules

$$0 \to R(-d) \xrightarrow{\cdot a} R \to R/\langle a \rangle \to 0.$$

Since R is graded, for each i, we have an exact sequence of k-vector spaces

$$0 \to R(-d)_i \xrightarrow{\cdot a} R_i \to [R/\langle a \rangle]_i \to 0.$$

Now, using rank-nullity theorem for vector spaces, we get

$$\dim_k(R_i) = \dim_k((R(-d))_i) + \dim_k((R/\langle a \rangle)_i),$$

i.e.,

$$H_R(i) = H_{R(-d)}(i) + H_{R/\langle a \rangle}(i).$$

Therefore, $H_R(i) = H_R(i - d) + H_{R/\langle a \rangle}(i)$ or $H_{R/\langle a \rangle}(i) = H_R(i) - H_R(i - d)$.

Definition 1.2.8. *Given* k, M *as above, define the* ***Hilbert series*** *of* M *as* $H_M(t) = \sum_{j\geq 0} H_M(j)t^j$.

The next corollary follows from the above definition.

Corollary 1.2.9. $H_{R/\langle a\rangle}(t) = H_R(t)/(1-t)^d$.

Example 1.2.10. Let $R = \mathsf{k}[x, y]$ and $a = x^2$. In this case, for all $i \geq 0$, we have $H_R(i) = i + 1$. This is because the ith graded component of R, as a k-vector space is has a basis $\{x^r y^{i-r} \mid 0 \leq r \leq i\}$. For the element x^2, we have $\deg(x^2) = 2$. Hence, by the formula above, we must have $H_{R/\langle x^2\rangle}(i) = (i + 1) - (i - 1) = 2$; which is true as $\{\overline{xy^{i-1}}, \overline{y^i}\}$ form a k-vector space basis of $(R/\langle x^2\rangle)_i$.

Proposition 1.2.11. *Let* M, N *be graded* R*-modules. Then* $\mathrm{Tor}_i^R(M, N)$ *is graded for all* i.

Proof. Consider a graded free resolution of M as follows:

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0.$$

Tensoring with N gives a complex of graded modules

$$\cdots \to F_2 \otimes N \to F_1 \otimes N \to F_0 \otimes N \to M \otimes N \to 0.$$

Since $\mathrm{Tor}_i^R(M, N)$ is quotient of a graded submodule of a graded module by a graded submodule, we conclude that $\mathrm{Tor}_i^R(M, N)$ is graded for all i. □

Remark 1.2.12. If $F_\bullet$ is a graded free resolution of M then we define ${}^*\mathrm{Ext}_R^i(M, N) \cong H^i({}^*\mathrm{Hom}_R(F_\bullet, N))$. Then, by Proposition 1.1.13, if R is Noetherian local ring and M is finitely generated R-module, then ${}^*\mathrm{Ext}_R^i(M, N) \cong \mathrm{Ext}_R^i(M, N)$.

Chapter 2

Gröbner Bases and Schreyer's Algorithm

Let k be a field and $S = \mathsf{k}[x_1, \ldots, x_r]$.
If $a = (a_1, \ldots, a_r)$, x^a will denote the monomial $x_1^{a_1} \ldots x_r^{a_r}$. As is convention, an ideal of S generated by monomials will be referred to as a monomial ideal.

Definition 2.0.1. *Let F be a finitely generated free module over S with basis $\{e_1, \ldots, e_n\}$.*
*A **monomial** in F is an element of the form $m = x^a e_i$ for some i. We say that such an m involves the basis element e_i.*
*A **monomial submodule** of F is a submodule generated by elements of this form. Any monomial submodule M of F may be written as*

$$M = \oplus I_j e_j \subseteq \oplus S e_j = F,$$

with I_j the monomial ideal generated by those monomials m such that $me_j \in M$.
*A **term** in F is a monomial multiplied by a scalar.*

Definition 2.0.2. *Let F be a finitely generated free module over S with basis $\{e_1, \ldots, e_n\}$.*
*If m, n are monomials of S, $u, v \in k$, and $v \neq 0$, then we say that the term ume_i is divisible by the term vne_j if $i = j$ and m is **divisible** by n in S; the quotient is then $um/vn \in S$.*

Definition 2.0.3. *The set of monomials in M that are minimal elements in the partial order by divisibility on the monomials of F are referred as **minimal generators of** M.*

2.1 Hilbert Function of Monomial Submodules

Let F be a free S-module with basis $\{e_i : i = 1, \ldots, n\}$, and let $M \subseteq F$ be a monomial submodule. Since, as seen before, $M = \oplus I_j e_j$, we have $F/M = \oplus S/I_j$ and, since the Hilbert function is additive, it suffices to handle the case $F = S$ and $M = I$, where I is a monomial ideal.
Choosing one of the monomial generators f of I, and letting I' be the monomial ideal generated by the remaining generators, we have the following graded exact sequence:

$$0 \to S/(I' : f)(-d) \xrightarrow{\cdot f} S/I' \to S/I \to 0,$$

where d is the degree of f. If $I' = (f_1, f_2 \ldots, f_t)$, then

$$(I' : f) = (f_1/GCD(f_1, f), f_2/GCD(f_2, f), \ldots, f_n/GCD(f_t, f)).$$

For every integer n,

$$H_{S/I}(n) = H_{S/I'}(n) - H_{S/(I':f)}(n).$$

Note that both I' and $(I' : f)$ have fewer minimal generators than I, and hence, using induction, we can compute an expression for the Hilbert function or polynomial of I.

By choosing f sensibly, we can make the process much faster: If f contains the largest power of some variable x_1 of any of the minimal generators of I, then the minimal generators of the resulting ideal $(I' : f)$ will not involve x_1 at all. They will thus involve strictly fewer of the variables than the number involved in the minimal generators of I.

2.2 Syzygies of Monomial Submodules

Let F be a free module and let M be a submodule of F generated by monomials $m_1, \ldots, m_t$. Define

$$\phi : \oplus_{j=1}^{t} S\epsilon_j \to F; \phi(\epsilon_j) = m_j.$$

For each pair of indices i, j such that m_i and m_j involve the same basis element of F, we define

$$m_{ij} = m_i/\mathrm{GCD}(m_i, m_j),$$

and we define σ_{ij} to be the element of $\ker(\phi)$ given by

$$\sigma_{ij} = m_{ji}\epsilon_i - m_{ij}\epsilon_j.$$

Lemma 2.2.1. *With notation as above,* $\ker(\phi)$ *is generated by the set of all* σ_{ij}*, wherever defined.*

Proof. As a vector space over k, $\ker(\phi) = \oplus_f \ker(\phi)_f$, where

$$\ker(\phi)_f = \left\{ \sum_i a_i f_i \epsilon_i \in \ker(\phi) : m_i \textit{ divides } f,\ f_i = f/m_i,\ a_i \in \mathsf{k} \right\}.$$

Indeed, let

$$\sigma = \sum_i p_i \epsilon_i \in \ker(\phi).$$

For any monomial f that occurs in one of the $p_j m_j$, and for each i, let $p_{i,f}$ be the term of p_i such that $p_{i,f} m_i$ is a scalar times f. Then,

$$\sum_i p_i m_i = 0 \implies \sum_i \sum_f p_{i,f} m_i = 0 \implies \sum_f \sum_i p_{i,f} m_i = 0 \implies \forall f, \sum_i p_{i,f} m_i = 0.$$

Therefore, for all monomials f, $\sum_i p_{i,f}\epsilon_i \in \ker(\phi)$.
We may now assume $\sigma = \sum_i a_i f_i \epsilon_i$ for some monomial f of F. If $\sigma = 0$, σ lies in the module generated by σ_{ij}. If $\sigma \neq 0$, at least two of the $a_i f_i$ must be non-zero, since $\sum_i a_i f_i m_i = 0$. This implies that for some i, j, both m_i and m_j must divide f and in fact, $m_i f_i = m_j f_j = f$, which implies that $m_{ji} = m_j/\mathrm{GCD}(m_i, m_j)$ divides f_i. Let $k = f_i/m_{ji}$, then $k\sigma_{ij} \in ker(\phi)_f$, and $\sigma - a_i k \sigma_{ij}$ has fewer non-zero terms than σ. Hence, the proof is complete by induction on number of non-zero terms of σ. □

Example 2.2.2. *Let $S = \mathsf{k}[x, y]$, $F = S^2$, $M = \langle (x^2, 0), (0, xy), (0, y^3) \rangle$. Then we have*

$$\phi : \oplus_{j=1}^{3} S\epsilon_j \to F; \phi(\epsilon_1) = (x^2, 0), \phi(\epsilon_2) = (0, xy), \phi(\epsilon_3) = (0, y^3).$$

Suppose for some $a_1, a_2, a_3 \in S$, $\phi(a_1\epsilon_1 + a_2\epsilon_2 + a_3\epsilon_3) = 0$, then we have $(a_1x^2, a_2xy + a_3y^3) = 0$, and hence, $a_1 = 0$, $a_2 = by^2$, $a_3 = -bx$. Thus, $a_1\epsilon_1 + a_2\epsilon_2 + a_3\epsilon_3 = b(0, y^2 - x) = b\sigma_{23}$.

2.3 Monomial Orders

Let I be an ideal of S, J be a monomial ideal of S and B be the set of all monomials not in J. Then, the elements of B are k-linearly independent modulo I if and only if J contains at least one monomial from every polynomial in I.
Indeed, suppose J contains no monomial of $f \in I$, $f \neq 0$. Then $f \in \mathrm{Span}(B) \cap I$, which implies that the elements of B are linearly dependent modulo I. Conversely, suppose there exist $a_1, \ldots, a_n \in \mathsf{k}$ and $m_1, \ldots, m_n \in B$ such that $\sum_{i=1}^n a_i m_i \in I$, then $\sum_{i=1}^n a_i m_i$ is a polynomial in I for which no monomials belong to J.
Moreover, if B is a basis of S/I, J must be a minimal monomial ideal containing at least one monomial from every polynomial in I. Indeed, suppose J contains at least one monomial from each polynomial in I, but is not a minimal ideal satisfying this condition. Let $J_1 \subsetneq J$ satisfying the condition, and let $f \in J \setminus J_1$, where f is a monomial. Suppose $f \in \mathrm{Span}(B)$, that is, there exist $a_1, \ldots, a_n \in k$ and $m_1, \ldots, m_n \in B$ such that $f - \sum_{i=1}^n a_i m_i \in I$. Since J_1 contains at least one monomial of every polynomial in I, we have a contradiction. Hence, B cannot span S/I if J is not the minimal monomial ideal containing one monomial from each polynomial in I.

Definition 2.3.1. *Let F be a free S-module. A* ***monomial order*** *on F is a total order τ on the monomials of F such that the following two conditions are satisfied:*
(i) if m_1 is a monomial of F and $f \neq 1$ is a monomial of S, then $fm_1 >_\tau m_1$.
(ii) if m_1, m_2 are monomials of F and $f \neq 1$ is a monomial of S, then $m_1 >_\tau m_2$ implies $fm_1 >_\tau fm_2$.

Lemma 2.3.2 (Well-Ordering Property)**.** *Let F be a free S-module. The set of monomials in F is well-ordered with respect to any monomial order, that is, every non-empty subset of monomials in F has a least element.*

Proof. Let $X \subseteq F$ be a set of monomials. Since S is Noetherian, the submodule of F generated by X must be generated by a finite subset of X, say, Y. Since Y is a finite set of monomials, it must have a least element with respect to a monomial order. The least element of Y must be the least element of X because every element of X is an element in Y multiplied by a monomial in S. □

We will extend this notation to terms: If um_1 and vm_2 are terms with $0 \neq u, v \in \mathsf{k}$, and m_1, m_2 are monomials with $m_1 >_\tau m_2$ then we say $um_1 >_\tau vm_2$.

Definition 2.3.3. *Let F be a free S-module and τ be a monomial order on F. For any $f \in F$, we define the **initial term** of f, denoted by $\mathrm{in}_\tau(f)$ to be the greatest term of f with respect to the order τ. Given a submodule M of F, define the **initial submodule** of M, denoted by $\mathrm{in}_\tau(M)$, to be the monomial submodule generated by $\mathrm{in}_\tau(f)$ for all $f \in M$.*

Theorem 2.3.4 (Macaulay)**.** *Let F be a free S-module and M be a submodule of F. For any monomial order τ on F, the set B of all monomials not in $\mathrm{in}_\tau(M)$ forms a k-basis for F/M.*

Proof. Suppose the set B is not linearly independent. Then there exist distinct $m_1, \ldots, m_t \in B$ and $(a_1, \ldots, a_t) \in \mathsf{k}^\mathsf{t} \backslash \{\mathsf{0}\}$ such that $f := a_1 m_1 + \cdots + a_t m_t \in M$. Since $\mathrm{in}(f) \in \mathrm{in}(M)$, there must exist $i \in \{1, \ldots, t\}$ such that $m_i \in \mathrm{in}(M)$, which is a contradiction.
Suppose B does not span F/M. Let $f \in F \backslash (M + \mathrm{Span}(B))$ such that f has minimal initial term among all elements of $F \backslash (M + \mathrm{Span}(B))$. We can choose such an f by the well-ordering property. If $\mathrm{in}(f) \in \mathrm{Span}(B)$, $f - \mathrm{in}(f) \in F \backslash (M + \mathrm{Span}(B))$ has smaller initial term than f. Hence, $\mathrm{in}(f) \in \mathrm{in}(M)$. However, this implies that there exists $g \in M$ such that $\mathrm{in}(f) = \mathrm{in}(g)$, and $f - g \in F \backslash (M + \mathrm{Span}(B))$ has smaller initial term than f, leading to a contradiction. □

Corollary 2.3.5. *Given F, M, τ as above,* $\dim_\mathsf{k}(F/M) = \dim_\mathsf{k}(F/in_\tau(M))$.

Corollary 2.3.6. *Given monomial orders τ, γ on S and an ideal $I \in S$ such that $\mathrm{in}_\tau(I) \subset \mathrm{in}_\gamma(I)$, we have $\mathrm{in}_\tau(I) = \mathrm{in}_\gamma(I)$.*

Proof. If $\mathrm{in}_\tau(I) \subsetneq \mathrm{in}_\gamma(I)$, the set of monomials in $S \backslash \mathrm{in}_\gamma(I)$ is a proper subset of the set of monomials in $S \backslash \mathrm{in}_\tau(I)$. However, both these sets of monomials form a K-basis of S/I, which is a contradiction. □

Here are some important examples of monomial orders when $F = S$. Let $a = (a_1, \ldots, a_r), b = (b_1, \ldots, b_r)$ and $m = x^a, m' = x^b$

Lexicographic order: $m >_{lex} m'$ if and only if $a_i > b_i$ for the smallest i such that $a_i \neq b_i$.
Graded lexicographic order: $m >_{grlex} m'$ if and only if $\deg(m) > \deg(n)$ or $\deg(m) = \deg(n)$ and $a_i > b_i$ for the smallest i such that $a_i \neq b_i$.
Reverse graded lexicographic order: $m >_{grevlex} m'$ if and only if $\deg(m) > \deg(n)$ or $\deg(m) = \deg(n)$ and $a_i < b_i$ for the largest i such that $a_i \neq b_i$.

Remark 2.3.7. A "reverse lexicographic order" is not a monomial order, because 1 is not the least monomial. In fact, 1 is the largest monomial.

Definition 2.3.8. *A* ***Gröbner basis*** *with respect to an order τ on a free module F is a set of elements $g_1, \dots, g_t \in F$ such that if M is the submodule of F generated by $g_1, ..., g_t$, then $\mathrm{in}_\tau(g_1), ..., \mathrm{in}_\tau(g_t)$ generate $\mathrm{in}_\tau(M)$. We then say that $g_1, ..., g_t$ is a* ***Gröbner basis of*** *M.*

There is a Gröbner basis of any submodule M of F, with respect to any monomial order: if $g_1, \dots, g_t$ is a set of generators of M which is not a Gröbner basis, we can adjoin $g_{t+1}, \dots, g_{t'}$ until $\mathrm{in}(g_1), \dots, \mathrm{in}(g_{t'})$ generate $\mathrm{in}(M)$ (note that the Hilbert basis theorem implies that this can be done).

Lemma 2.3.9. *Let $N \subset M \subset F$ be submodules such that $\mathrm{in}(N) = \mathrm{in}(M)$ with respect to a given monomial order. Then, $N = M$.*

Proof. Suppose $N \neq M$, then, by the well-ordering property, there exists $f \in M \backslash N$ such that f has the least initial term among all the elements of M not in N. Since $f \in M$, we have $\mathrm{in}(f) \in \mathrm{in}(M) = \mathrm{in}(N)$, which implies the existence of $g \in N$ such that $\mathrm{in}(f) = \mathrm{in}(g)$. Note that $f - g \in M \backslash N$, but has smaller initial term than f, which is a contradiction to the choice of f. $\square$

The above lemma tells us that if $\langle \mathrm{in}(g_1), \dots, \mathrm{in}(g_t) \rangle = \mathrm{in}(M)$ for $g_1, \dots, g_t \in M$, then $\langle g_1, \dots, g_t \rangle = M$. This follows since $\langle \mathrm{in}_(g_1), \dots, \mathrm{in}_(g_t) \rangle \subset \mathrm{in}(\langle g_1, \dots, g_t \rangle) \subset \mathrm{in}(M)$.

2.4 Computing Syzygies

Proposition 2.4.1 (Division Algorithm)**.** *Let F be a free S-module with monomial order τ. If $f, g_1, ..., g_t \in F$, then there is an expression*

$$f = \sum_{i=1}^{t} f_i g_i + f' \textit{ with } f' \in F, f_i \in S,$$

where none of the monomials of f' is in $\langle \mathrm{in}(g_1), \dots, \mathrm{in}(g_t) \rangle$ and $\mathrm{in}(f) \geq_\tau \mathrm{in}(f_i g_i)$ for every i.

Definition 2.4.2. *With notation as above, any such f' is called a* ***remainder*** *of f with respect to $g_1, ..., g_t$, and an expression $f = \sum f_i g_i + f'$ satisfying the condition of the proposition is called a* ***standard expression*** *for f in terms of the g_i.*

The proof outlines an algorithm to attain a standard expression for any $f \in F$.

Proof. If $f, g_1, \dots, g_t \in F$, then we may produce a standard expression

$$f = \sum m_u g_{s_u} + f'$$

for f with respect to $g_1, \ldots, g_t$ by defining the indices s_u and the terms m_u inductively. Having chosen $s_1, \ldots, s_p$ and $m_1, \ldots, m_p$, if

$$f'_p := f - \sum_{u=1}^{p} m_u g_{s_u} \neq 0$$

and m is the maximal term of f'_p; that is divisible by $\mathrm{in}(g_i)$ for some i, then choose $s_{p+1} = i, m_{p+1} = m/\mathrm{in}(g_i)$. This process terminates when either $f'_p = 0$ or no $\mathrm{in}(g_i)$ divides a monomial of f; the remainder f' is then the last f'_p produced.
Note that the well-ordering property guarantees that this process must terminate, because the maximal term of f'_p divisible by some g_i decreases at each step. □

Fix the following notation:
Let F be a free module over S with monomial order τ. Let $g_1, \ldots, g_t$ be non-zero elements of F, and let $\oplus S\epsilon_i$ be a free module with basis $\{\epsilon_1, \ldots, \epsilon_t\}$.
For two terms $m_1, m_2 \in F$, $m_1 < m_2$ denotes that the monomial corresponding to m_1 is less than the monomial corresponding to m_2 with respect to the order τ.
For each pair of indices i, j such that $\mathrm{in}(g_i)$ and $\mathrm{in}(g_j)$ involve the same basis element of F, we define

$$m_{ij} = \mathrm{in}(g_i)/\mathrm{GCD}(\mathrm{in}(g_i), \mathrm{in}(g_j)) \in S,$$

and we set $\sigma_{ij} = m_{ji}\epsilon_i - m_{ij}\epsilon_j$ for $i < j$.
For each such pair i, j, choose a standard expression

$$m_{ji}g_i - m_{ij}g_j = \sum_{u=1}^{t} f_u^{(ij)} g_u + h_{ij}$$

for $m_{ji}g_i - m_{ij}g_j$ with respect to $g_1, \ldots, g_t$. Note that $\mathrm{in}(f_u^{(ij)} g_u) < \mathrm{in}(m_{ji}g_i)$.
Set $h_{ij} = 0$ if $\mathrm{in}(g_i)$ and $\mathrm{in}(g_j)$ involve different basis elements of F.
Define $\phi : \oplus S\epsilon_i \to F$, $\phi(\epsilon_i) = g_i$. Then, the set of σ_{ij} generate the syzygies on the module generated by the elements $\mathrm{in}(g_i)$ (by Lemma 2.2.1). Note that $\phi(\sigma_{ij}) = m_{ji}g_i - m_{ij}g_j$.

Theorem 2.4.3 (Buchberger's Criterion)**.** *The elements $g_1, \ldots, g_t$ form a Gröbner basis if and only if $h_{ij} = 0$ for all i and j.*

Proof. Let $M = \langle g_1, \ldots, g_t \rangle \subset F$. The expression for h_{ij} implies that $h_{ij} \in M$, and hence $\mathrm{in}(h_{ij}) \in \mathrm{in}(M)$. However, if $g_1, \ldots, g_t$ is a Gröbner basis, the definition of a standard expression forces $h_{ij} = 0$ for all i, j.
Conversely, suppose that $h_{ij} = 0$ for all i, j. Let $f = \sum_{i=1}^{t} h_i g_i \in M$, where, among all possible $h_1, \ldots, h_t$ such that $f = \sum_{i=1}^{t} h_i g_i$, $h_1, \ldots, h_t$ are chosen such that $\max\{\mathrm{in}(h_i g_i) : 1 \leq i \leq t\}$ is minimal. We prove that $\mathrm{in}(f) \in \langle \mathrm{in}(g_1), \ldots, \mathrm{in}(g_t) \rangle$.
If $\mathrm{in}(f) = \mathrm{in}(h_i g_i)$ for some i, $\mathrm{in}(g_i) | \mathrm{in}(f) \Rightarrow \mathrm{in}(f) \in \langle \mathrm{in}(g_1), \ldots, \mathrm{in}(g_t) \rangle$.

Hence, let $\text{in}(f) < \max\{\text{in}(h_i g_i) : 1 \le i \le t\} = m$. Define an equivalence relation $\equiv$ on terms as follows: $m_1 \equiv m_2$ if there exists $\lambda \in \mathsf{k}\backslash\{0\}$ such that $m_1 = \lambda m_2$. Without loss of generality, suppose $\text{in}(h_i g_i) \equiv m$ for $i = 1, \dots, t_1$ and $\text{in}(h_i g_i) < m$ for $i = t_1 + 1, \dots, t$

$$\begin{aligned} f = \sum_{i=1}^{t} h_i g_i &= \sum_{i=1}^{t_1} h_i g_i + \sum_{i=t_1}^{t} h_i g_i \\ &= \sum_{i=1}^{t_1} \text{in}(h_i) g_i + \sum_{i=1}^{t_1} (h_i - \text{in}(h_i)) g_i + \sum_{i=t_1+1}^{t} h_i g_i. \end{aligned}$$

Note that $\sum_{i=1}^{t_1} \text{in}(h_i)\text{in}(g_i) = 0$.
Define $\phi_1 : \oplus S\epsilon_j \to M, \phi_1(\epsilon_j) = \text{in}(g_j)$ and $\phi_2 : \oplus S\epsilon_j \to M, \phi_2(\epsilon_j) = g_j$. Note that $\sum_{i=1}^{t_1} \text{in}(h_i)\epsilon_i \in \ker(\phi_1)$. Therefore, by Lemma 2.2.1,

$$\sum_{i=1}^{t_1} \text{in}(h_i)\epsilon_i = \sum_{i<j} k_{ij}\sigma_{ij}, \text{[1]}$$

where $k_{ij} = a_{ij} m / \text{LCM}(\text{in}(g_i), \text{in}(g_j))$ for some $a_{ij} \in \mathsf{k}$.
Note that $\phi_2(\sum_{i=1}^{t_1} \text{in}(h_i)\epsilon_i) = \sum_{i=1}^{t_1} \text{in}(h_i) g_i$.
Hence,

$$\sum_{i=1}^{t_1} \text{in}(h_i) g_i = \sum_{i<j} k_{ij}(m_{ji} g_i - m_{ij} g_j) = \sum_{i<j} k_{ij} \sum_{u=1}^{t} f_u^{(ij)} g_u,$$

since $h_{ij} = 0$ for all i, j. Note that since $\text{in}(f_u^{(ij)} g_u) < \text{in}(m_{ji} g_i)$, we have $\text{in}(k_{ij} f_u^{(ij)}) < m$.
Hence, we have an expression for $f = \sum_i h'_i g_i$, where $\max\{\text{in}(h'_i g_i) : 1 \le i \le t\} < m$, which is a contradiction. $\square$

This result gives us an effective method for computing Gröbner bases.

Buchberger's Algorithm: In the situation of Theorem 2.4.3, suppose that M is a submodule of F, and let $g_1, \dots, g_t \in M$ be a set of generators of M. Compute the remainders h_{ij}. If all the $h_{ij} = 0$, then $\{g_1, \dots, g_t\}$ forms a Gröbner basis of M. If some $h_{ij} \ne 0$, then replace $g_1, \dots, g_t$ with $g_1, \dots, g_t, h_{ij}$, and repeat the process. As the submodule generated by the initial forms of $g_1, \dots, g_t, h_{ij}$ is strictly larger than that generated by the initial forms of $g_1, \dots, g_t$, this process must terminate after finitely many steps.

The next theorem shows that if $\{g_1, \dots, g_t\}$ is a Gröbner basis of M, the equations $h_{ij} = 0$ generate the first syzygy of M.

[1] let $k_{ij} = 0$ and $\sigma_{ij} = 0$ for i, j where σ_{ij} is not originally defined

For $i < j$ such that $\text{in}(g_i)$ and $\text{in}(g_j)$ involve the same basis element of F, we set

$$w_{ij} = m_{ji}\epsilon_i - m_{ij}\epsilon_j - \sum_{u=1}^{t} f_u^{(ij)}\epsilon_u.$$

Let W be the set of all such w_{ij}.

Theorem 2.4.4 (Schreyer)**.** *With notation as above, suppose that $\{g_1, \ldots, g_t\}$ is a Gröbner basis of M. Let γ be the monomial order on $\oplus_{j=1}^{t} S\epsilon_j$ defined by taking $m\epsilon_u > n\epsilon_v$ if and only if*

$$\text{in}(mg_u) >_\tau \text{in}(ng_v) \textit{ with respect to the given order } \tau \textit{ on } F$$

or

$$\text{in}(mg_u) \equiv \text{in}(ng_v), \textit{ but } u < v.$$

W generates the first syzygy of M. Moreover, W forms a Gröbner basis of the syzygies with respect to the order γ, and $\text{in}_\gamma(w_{ij}) = m_{ji}\epsilon_i$.

Proof. We first prove that $\text{in}_\gamma(w_{ij}) = m_{ji}\epsilon_i$. Since

$$\text{in}(m_{ji}g_i) = \text{in}(m_{ij}g_j),$$

and these terms are by hypothesis greater than any that appear in the $\sum_{u=1}^{t} f_u^{(ij)} g_u$, $\text{in}(w_{ij})$ must be either $m_{ji}\epsilon_i$ or $-m_{ij}\epsilon_j$. Since $i < j$, $\text{in}_\gamma(w_{ij}) = m_{ji}\epsilon_i$.
To show that W forms a Gröbner basis, let $w = \sum_{i=1}^{t} f_i\epsilon_i$. Let $\text{in}(f_i) = h_i$ for all i. The theorem is proved once we show that $\text{in}_\gamma(w) \in \langle \text{in}_\gamma(v) : v \in W\rangle$. Note that $\text{in}_\gamma(w) = \text{in}_\gamma(f_j\epsilon_j) = h_j\epsilon_j$ for some j. Let

$$\sigma = \sum_{i : h_i\text{in}(g_i) \equiv h_j \text{in}(g_j)} f_i\epsilon_i.$$

σ is a syzygy on $\{\text{in}(g_i) : i \geq j\}$, because if $h_i\text{in}(g_i) \equiv_\gamma h_j\text{in}(g_j)$, we must have $i \geq j$. Hence, by Lemma 2.2.1, σ is generated by σ_{uv} for $u, v \geq j$, and ϵ_j only appears in σ_{jv} for $j < v$. This implies that h_j is a k-linear combination of $\{m_{vj} : j < v\}$ and thus, $\text{in}_\gamma(w)$ is a k-linear combination of $\{m_{vj}\epsilon_j : j < v\}$, which proves the theorem. $\square$

Corollary 2.4.5. *With notation as in Theorem 2.4.4, suppose that the g_i are arranged such that whenever $\text{in}(g_i)$ and $\text{in}(g_j)$ involve the same basis vector e of F, say $\text{in}(g_i) = m_ie, \text{in}(g_j) = m_je$ with $m_i, m_j \in S$,*

$$i < j \implies m_i > m_j \textit{ in lexicographic order.}$$

If the variables $x_1, \ldots, x_s$ are missing from $\text{in}(g_i)$ for all i, then the variables $x_1, \ldots, x_{s+1}$ are missing from $\text{in}_\gamma(w_{ij})$ for all $i < j$ for which w_{ij} is defined. Further, $F/\langle g_1, \ldots, g_t\rangle$ has a free resolution of length $\leq r - s$.

Proof. If the variables $x_1, \dots, x_s$ are missing from $\mathrm{in}(g_i)$ for all i, then, due to the stipulated arrangement of $\{g_1, \dots, g_t\}$, for $i < j$ such that $\mathrm{in}(g_i)$ and $\mathrm{in}(g_j)$ involve the same basis element, the variable x_{s+1} must appear in g_i with at least as high a power as in g_j. As a result, the variable x_{s+1} does not appear in m_{ji}, and hence, does not appear in $\mathrm{in}_\gamma(w_{ij}) = m_{ji}\epsilon_i$.
We now show that $F/(g_1, \dots, g_t)$ has a free resolution of length $\leq r - s$ by induction on $r - s$. Suppose first that $r - s = 0$, so that none of the variables $x_1, \dots, x_r$ appears in the terms $\mathrm{in}(g_i)$. Since none of the variables appear in $\mathrm{in}(g_i)$ for all i, $\mathrm{in}(g_i)$ must be a scalar times a basis element of F. Let F' be the free submodule spanned by all the e_j which do not appear in $\mathrm{in}(g_i)$ for any i. By Theorem 2.3.4, F' is isomorphic to $F/(g_1, \dots, g_t)$.
Suppose $r - s > 0$. By the first statement of the theorem, the variables $x_1, \dots, x_{s+1}$ are missing from $\mathrm{in}_\gamma(w_{ij})$ for all i, j. Order the w_{ij} to satisfy the same hypothesis as on the g_i. Then, by the induction hypothesis, $F/\langle W \rangle$ has a free resolution of length $\leq r - s - 1$. Combining this with the natural map $\phi : \oplus S\epsilon_i \to F$, we get a free resolution of $F/\langle g_1, \dots, g_t \rangle$ of length $\leq r - s$. □

Example 2.4.6. Let $F = S$ and $I = \langle x^3 - yz, y^2 - xz, x^2y - z^2 \rangle$. Let $g_1 = x^3 - yz, g_2 = y^2 - xz, g_3 = x^2y - z^2$. In this example, we consider the lexicographic order on S. Thus, we have

$$\mathrm{in}(g_1) = x^3, \mathrm{in}(g_2) = -xz, \mathrm{in}(g_3) = x^2y.$$

Let $S_{ij} = m_{ji}g_i = m_{ij}g_j$. Then,

$$\begin{aligned} S_{12} &= \frac{-xz}{x}(x^3 - yz) - \frac{x^3}{x}(y^2 - xz) \\ &= yz^2 - x^2y^2 \\ &= -yg_3, \end{aligned}$$

and hence, $h_{12} = 0$. Similarly, $S_{23} = xy^3 - z^3 = h_{23}$. Thus, we add $g_4 = h_{23}$ to the original basis $\{g_1, g_2, g_3\}$. For the basis $\{g_1, g_2, g_3, g_4\}$, we immediately have $h_{12} = h_{23} = 0$. Calculation also reveals that $S_{13} = -zg_2$ and $S_{14} = -z(y^2 + xz)g_2$, which implies that $h_{13} = h_{14} = 0$. However, $S_{24} = y^5 - z^4 = h_{24}$. For the new basis $\{g_1, g_2, g_3, g_4, g_5\}$, where $g_5 = y^5 - z^4$, we instantly have $h_{12} = h_{23} = h_{13} = h_{14} = h_{24} = 0$. Further,

$$S_{34} = -z^2g_2, S_{15} = -z(y^4 + xy^2z + x^2z^2)g_2, S_{25} = z^4g_2 + y^2g_5, S_{35} = -z^2(y^2 + xz)g_2, S_{45} = -z^3g_2.$$

This shows that $\{g_1, g_2, g_3, g_4, g_5\}$ is a Gröbner basis of I.
Rearranging the basis to satisfy the hypothesis of the corollary, we have $I = \langle x^3 - yz, x^2y - z^2, xy^3 - z^3, xz - y^2, y^5 - z^4 \rangle$. Hence,

$$
\begin{aligned}
w_{12} &= y\epsilon_1 - x\epsilon_2 - z\epsilon_4 \\
w_{13} &= y^3\epsilon_1 - x^2\epsilon_2 - z\epsilon_4 \\
w_{14} &= z\epsilon_1 - x^2\epsilon_4 - z(y^2 + xz)\epsilon_4 \\
w_{15} &= y^5\epsilon_1 - x^3\epsilon_5 - z(y^4 + xy^2z + x^2z^2)\epsilon_4 \\
w_{23} &= y^2\epsilon_2 - x\epsilon_3 - z^2\epsilon_4 \\
w_{24} &= z\epsilon_2 - xy\epsilon_4 - \epsilon_3 \\
w_{25} &= y^4\epsilon_2 - x^2\epsilon_5 - z^2(y^2 + xz)\epsilon_4 \\
w_{34} &= z\epsilon_3 - y^3\epsilon_4 + \epsilon_5 \\
w_{35} &= y^2\epsilon_3 - x\epsilon_5 - z^3\epsilon_4 \\
w_{45} &= (y^5 - z^4)\epsilon_2 + (y^2 - xz)\epsilon_5
\end{aligned}
$$

Note that x is missing from the initial terms of all the w_{ij}, as it should be, according to the previous corollary.

Chapter 3

Comparison between an ideal and its initial ideal

3.1 Gradings defined by weights

Definition 3.1.1. *Let* $\mathbf{w} = (w_1, \ldots, w_r) \in \mathbb{N}^r$*. We call this vector a* ***weight*** *and set* $\deg_{\mathbf{w}} x_i = w_i$ *for* $i = 1, \ldots, n$*. Then, for* $(a_1, \ldots, a_r) \in \mathbb{N}^r$*,*

$$\deg_{\mathbf{w}} x_1^{a_1} \ldots x_r^{a_r} = \sum_{i=1}^{r} a_i w_i.$$

A polynomial $f \in S$ *is called* ***homogeneous of degree*** j *with respect to the weight* $\mathbf{w}$ *if the degree of all homogeneous components of* f *is* j*.*

Fix a weight $\mathbf{w}$ and let S_j be the k-vector space spanned by all homogeneous polynomials of degree j. Then, S_j is finite dimensional and the monomials u with $\deg_{\mathbf{w}} u = j$ form a k-basis. It follows that

$$S = \oplus_j S_j.$$

Thus, note that we have defined a new grading on S.

Definition 3.1.2. *Each polynomial* $f \in S$ *can be uniquely written as* $f = \sum_j f_j$ *with* $f_j \in S_j$*. The summands* f_j *are called the* ***homogeneous components*** *of* f *with respect to* $\mathbf{w}$*.*
The ***degree*** *of* f *with respect to* $\mathbf{w}$ *is defined to be* $\deg_{\mathbf{w}} f = \max\{j : f_j \neq 0\}$*, and if* $i = \deg_{\mathbf{w}} f$*, then* f_i *is called the* ***initial term*** *of* f *with respect to* $\mathbf{w}$ *and is denoted by* $\mathrm{in}_{\mathbf{w}}(f)$*.*

Note that $\mathrm{in}_{\mathbf{w}}(f)$ need not be a monomial.

Definition 3.1.3. *Let* $I \subset S$ *be an ideal. We define the* ***initial ideal*** *of* I *with respect to* $\mathbf{w}$ *as*

$$\mathrm{in}_{\mathbf{w}}(I) = \langle \mathrm{in}_{\mathbf{w}}(f) : f \in I \rangle.$$

A set of polynomials $f_1, \ldots, f_n \in I$ *such that* $\mathrm{in}_{\mathbf{w}}(I) = \langle \mathrm{in}_{\mathbf{w}}(f_1), \ldots, \mathrm{in}_{\mathbf{w}}(f_n) \rangle$ *is called a standard basis of* I *with respect to* $\mathbf{w}$*.*

The following lemma shows that a standard basis of I with respect to a weight generates I.

Lemma 3.1.4. *Let $J \subset I$ be ideals in S. If $\mathrm{in}_{\mathbf{w}}(J) = \mathrm{in}_{\mathbf{w}}(I)$, then $I = J$.*

Proof. Suppose $I \neq J$. Let $f \in I \backslash J$ such that $\deg_{\mathbf{w}} f$ is minimum among all elements in $I \backslash J$. Since $\mathrm{in}_{\mathbf{w}}(f) \in \mathrm{in}_{\mathbf{w}}(I) = \mathrm{in}_{\mathbf{w}}(J)$ and $\mathrm{in}_{\mathbf{w}}(J)$ is a homogeneous ideal with respect to the grading given by $\mathbf{w}$, there must exist $g \in J$ such that $\mathrm{in}_{\mathbf{w}}(f) = \mathrm{in}_{\mathbf{w}}(g)$. Note that $f - g \in I \backslash J$, and $\deg_{\mathbf{w}}(f - g) < \deg_{\mathbf{w}}(f)$, which is a contradiction. □

The following lemma is proved in [2].

Lemma 3.1.5. *Given a monomial order τ and pairs of monomials $(g_1, h_1), \ldots, (g_m, h_m)$ such that $g_i >_\tau h_i$ for all i, there exists a weight $\mathbf{w}$ such that $\deg_{\mathbf{w}} g_i > \deg_{\mathbf{w}} h_i$ for all i.*

Theorem 3.1.6. *Given an ideal I and a monomial order τ, there exists a weight $\mathbf{w}$ such that $\mathrm{in}_\tau(I) = \mathrm{in}_{\mathbf{w}}(I)$.*

Proof. Let $\{g_1, \ldots, g_n\}$ be a Gröbner basis of I with respect to the monomial order τ. For all i, define K_i to be the set of all monomials appearing in g_i, and denote the monomial corresponding to $\mathrm{in}_\tau(g_i)$ as m_i. Define $K = \sqcup_i (g_i, K_i \backslash \{m_i\}) \in S^2$. By the previous lemma, there exists a weight $\mathbf{w}$ such that $g > h$ for all $(g, h) \in K$. Observe that $\mathrm{in}_{\mathbf{w}}(g_i) = \mathrm{in}_\tau(g_i)$ for all I. Hence,

$$\mathrm{in}_\tau(I) = \langle \mathrm{in}_\tau(g_1), \ldots, \mathrm{in}_\tau(g_n) \rangle \subset \mathrm{in}_{\mathbf{w}}(I).$$

Define a monomial order $\tau_{\mathbf{w}}$ as $m_1 <_{\tau_{\mathbf{w}}} m_2$ if (i) $\deg_{\mathbf{w}}(m_1) < \deg_{\mathbf{w}}(m_2)$ or (ii) $\deg_{\mathbf{w}}(m_1) = \deg_{\mathbf{w}}(m_2)$ and $m_1 <_\tau m_2$. Thus, we have

$$\mathrm{in}_\tau(I) = \mathrm{in}_\tau(\mathrm{in}_\tau(I)) \subset \mathrm{in}_\tau(\mathrm{in}_{\mathbf{w}}(I)) = \mathrm{in}_{\tau_{\mathbf{w}}}(I).$$

Corollary 2.3.6 implies that $\mathrm{in}_\tau(I) = \mathrm{in}_{\tau_{\mathbf{w}}}(I)$. We show that $\mathrm{in}_{\tau_{\mathbf{w}}}(I) \supset \mathrm{in}_{\mathbf{w}}(I)$ to complete the proof.
Observe that $\mathrm{in}_{\tau_{\mathbf{w}}}(g_i) = \mathrm{in}_\tau(g_i) = \mathrm{in}_{\mathbf{w}}(g_i)$ for all i and hence, $\{g_1, \ldots, g_n\}$ is a Gröbner basis of I with respect to $\tau_{\mathbf{w}}$ as well.
Let $f \in I$ and $f = f_1 g_1 + \cdots + f_n g_n$ be a standard expression for f in terms of $\{g_1, \ldots, g_n\}$. Since $\mathrm{in}_{\tau_{\mathbf{w}}}(f) \geq_{\tau_{\mathbf{w}}} \mathrm{in}_{\tau_{\mathbf{w}}}(f_i g_i)$ for all i, we have $\deg_{\mathbf{w}} f \geq \deg_{\mathbf{w}}(f_i g_i)$. Let $L = \{i \in \{1, \ldots, n\} : \deg_{\mathbf{w}} f = \deg_{\mathbf{w}}(f_i g_i)\}$. Then,

$$\mathrm{in}_{\mathbf{w}}(f) = \sum_{i \in L} \mathrm{in}_{\mathbf{w}}(f_i g_i) = \sum_{i \in L} \mathrm{in}_{\mathbf{w}}(f_i) \mathrm{in}_{\mathbf{w}}(g_i) = \sum_{i \in L} \mathrm{in}_{\mathbf{w}}(f_i) \mathrm{in}_{\tau_{\mathbf{w}}}(g_i) \in \mathrm{in}_{\tau_{\mathbf{w}}}(I).$$

□

3.2 Homogenization

Definition 3.2.1. *Fix a weight* $\mathbf{w}$*. Let* f *be a non-zero polynomial in* S *with homogeneous components* f_j *(with respect to the weight* $\mathbf{w}$*). We introduce a new variable* t *and define the* ***homogenization*** *of* f *with respect to* $\mathbf{w}$ *as the polynomial*

$$f^h = \sum_j f_j t^{\deg_{\mathbf{w}} f - j} \in S[t].$$

Note that f^h is homogeneous in $S[t]$ with respect to the extended weight $(w_1, \ldots, w_r, 1) \in \mathbb{N}^{r+1}$.

Definition 3.2.2. *Let* $I \subset S$ *be an ideal. The* ***homogenization*** *of* I *is defined to be the ideal*

$$I^h = \langle f^h : f \in I \rangle \subset S[t].$$

For any homogeneous polynomial $g \in S[t]$, let $\overline{g}$ denote the polynomial in S obtained by substituting $t = 1$.

Lemma 3.2.3. *Let* $f \in S[t]$ *be homogeneous with respect to the weight* $(w_1, \ldots, w_r, 1)$*. Then* $f \in I^h$ *iff* $f = t^n g^h$ *for some* $g \in I$ *and some* $n \in \mathbb{Z}_{\geq 0}$*. Further, in this case,* $g = \overline{f}^h$*.*

Proof. It is clear that $f \in I^h$ if $f = t^n g^h$ for some $g \in I$ and some $n \in \mathbb{Z}_+$.
Suppose $f \in I^h$ is homogeneous. Then, there exist $f_1, \ldots, f_s \in I$ and $g_1, \ldots, g_s \in S[t]$ such that $f = \sum_{i=1}^s g_i f_i^h$.
We have

$$\overline{f} = \sum_{i=1}^s \overline{g_i} \overline{f_i^h} = \sum_{i=1}^s \overline{g_i} f_i \in I.$$

We claim that $f = t^n \overline{f}^h$ for some non-negative integer n. To observe this, let $f = g_l(x_1, \ldots, x_r)t^l + \cdots + g_k(x_1, \ldots, x_r)t^k$ such that $l \leq k$ and $g_l, g_k \neq 0$. Then, $\overline{f} = g_l(x_1, \ldots, x_r) + \cdots + g_k(x_1, \ldots, x_r)$ and

$$\overline{f}^h = g_l(x_1, \ldots, x_r) + g_{l+1}(x_1, \ldots, x_r)t + \cdots + g_k(x_1, \ldots, x_r)t^{k-l},$$

which implies that $f = t^l \overline{f}^h$ and completes the proof. □

Remark 3.2.4. *Observe that in the above proof, we have also shown that if* f *is homogeneous in* I^h*, then* $\overline{f} \in I$*.*

Definition 3.2.5. *A monomial order* τ *on* S *is said to respect* $\mathbf{w}$ *if for all* $m_1, m_2 \in S$ *such that* $\deg_{\mathbf{w}} m_1 < \deg_{\mathbf{w}} m_2$*, we have* $m_1 <_\tau m_2$*.*

Example 3.2.6. *The graded lexicographic order and reverse graded lexicographic order respect the standard grading on* S*. More generally, the order* $<_{\mathbf{w}}$ *respects* $\mathbf{w}$*.*

For a monomial order τ which respects $\mathbf{w}$, define a natural extension τ' to $S[t]$ as follows: $x^a t^c <_{\tau'} x^b t^d$ iff (i) $x^a <_\tau x^b$ or (ii) $x^a = x^b$ and $c < d$, where, as usual, x^a denotes $x_1^{a_1} \dots x_r^{a_r}$.
This monomial order has the property that $\mathrm{in}_\tau(g) = \mathrm{in}_{\tau'}(g^h)$ for all non-zero $g \in S$.

Proposition 3.2.7. *Let $I \subset S$ be an ideal, and let $\{g_1, \dots, g_n\}$ be a Gröbner basis of I with respect to a monomial order τ which respects $\mathbf{w}$. Then, $\{g_1^h, \dots, g_n^h\}$ is a Gröbner basis of I^h with respect to τ'.*

Proof. Note that since I^h is a homogeneous ideal with respect to the extended weight $(w_1, \dots, w_r, 1)$, it is sufficient to prove that if $f \in I^h$ is homogeneous with respect to $(w_1, \dots, w_r, 1)$, then $\mathrm{in}_{\tau'}(f) \in \langle \mathrm{in}_{\tau'}(g_1^h), \dots, \mathrm{in}_{\tau'}(g_n^h) \rangle$.
Let $f \in I^h$, be homogeneous. Then, by the previous lemma, there exist $g \in I$ and $m \in \mathbb{Z}_+$ such that $f = t^m g^h$. Hence,

$$\mathrm{in}_{\tau'}(f) = t^m \mathrm{in}_{\tau'}(g^h) = t^m \mathrm{in}_\tau(g).$$

There exist $u \in S$ and $i \in \{1, \dots, n\}$ such that $\mathrm{in}_\tau(g) = u\mathrm{in}_\tau(g_i) = u\mathrm{in}_{\tau'}(g_i^h)$. Thus, $\mathrm{in}_{\tau'} f = ut^m \mathrm{in}_{\tau'}(g_i^h)$. □

Proposition 3.2.8. *Given an ideal $I \subset S$, $S[t]/I^h$ is a free $\mathsf{k}[t]$-module.*

Proof. Let $\{g_1, \dots, g_n\}$ be a Gröbner basis of I with respect to a monomial order τ graded with respect to $\mathbf{w}$. Then, $\{g_1^h, \dots, g_n^h\}$ is a Gröbner basis of I^h with respect to τ'. It follows from Theorem 2.3.4 that the set of all monomials in $S[t]$ not in $\langle \mathrm{in}_{\tau'}(g_1^h), \dots, \mathrm{in}_{\tau'}(g_n^h) \rangle$ forms a k-basis of $S[t]/I^h$. Since $\mathrm{in}_{\tau'}(g_i^h) = \mathrm{in}_\tau(g_i)$, we have $\langle \mathrm{in}_{\tau'}(g_1^h), \dots, \mathrm{in}_{\tau'}(g_n^h) \rangle = \langle \mathrm{in}_\tau(g_1), \dots, \mathrm{in}_\tau(g_n) \rangle S[t]$ and hence, the set of all monomials in S not in $\langle \mathrm{in}_\tau(g_1), \dots, \mathrm{in}_\tau(g_n) \rangle$ forms a $\mathsf{k}[t]$-basis of $S[t]/I^h$. □

Lemma 3.2.9. *Let R be a ring and consider $\phi : R[t] \to R$, a ring homomorphism with $\phi|_R = Id$, or equivalently, an R-linear ring homomorphism. Given an ideal $I \in R[t]$, ϕ naturally induces an R-linear ring homomorphism $\overline{\phi} : R[t]/I \to R/\phi(I)$ given by $\overline{\phi}(\overline{f}) = \overline{\phi(f)}$, and $\ker(\overline{\phi}) = (t - \phi(t))R[t]/I$.*

Proof. Clearly, $\overline{\phi}(f)$ is well-defined and $(t - \phi(t))R[t]/I \subset \ker(\overline{\phi})$.
Let $f \in R[t]$ such that $\overline{f} \in \ker(\overline{\phi})$. There exist $a \in R$ and $g \in R[t]$ such that $f = a + (t - \phi(t))g$, which implies that $\overline{\phi}(\overline{f}) = \overline{a}$. Thus, we have $a \in \phi(I)$. Let $h \in I$ such that $\phi(h) = a$, that is, $h = a + (t - \phi(t))h'$. Then,

$$f - h \in (t - \phi(t))R[t] \implies f \in I + (t - \phi(t))R[t],$$

which completes the proof. □

Proposition 3.2.10. *Given an ideal $I \subset S$ and a weight $\mathbf{w}$ on S, we have the following S-linear ring isomorphisms:*

$$\frac{S[t]/I^h}{tS[t]/I^h} \cong S/\mathrm{in}_{\mathbf{w}}(I) \ \text{and} \ \frac{S[t]/I^h}{(t-a)S[t]/I^h} \cong S/I \ \forall a \in S \setminus \{0\}.$$

Proof. For all $a \in \mathsf{k}$, define an S-linear map $\phi_a : S[t] \to S$ as $\phi_a(1) = 1$ and $\phi_a(t) = a$. We claim that $\phi_0(I^h) = \text{in}_{\mathbf{w}}(I)$.
Given $f \in I$, $\phi_0(f^h) = \text{in}_{\mathbf{w}}(f)$. Since $I^h = \langle f^h : f \in I \rangle$, it follows that $\phi_0(I^h) = \text{in}_{\mathbf{w}}(I)$. From the previous lemma, we have $\frac{S[t]/I^h}{tS[t]/I^h} \cong S\text{in}_{\mathbf{w}}(I)$.
For $a \neq 0$, define a ring homomorphism $\psi_a : S \to S$ as $\psi_a(x_i) = a^{w_i}x_i$ for all i and $\psi_a|_{\mathsf{k}} = Id$. We claim that $\psi_a\phi_a(I^h) = I$. Then, according to the previous lemma, $\frac{S[t]/I^h}{(t-a)S[t]/I^h} \cong S/\phi_a(I^h)$ as S-modules and since $a \neq 0$, ψ_a is a ring isomorphism and $S/\phi_a(I^h) \cong S/I$ as rings.
By Proposition 3.2.7, there exists a Gröbner basis $\{g_1, \ldots, g_n\}$ of I such that $\{g_1^h, \ldots, g_n^h\}$ is a Gröbner basis of I^h. Let $g_i = \sum_j g_{ij}$ where g_{ij} denotes the homogeneous component of g_i of degree j (with respect to $\mathbf{w}$). Then,

$$\phi_a(g_i^h) = \sum_j a^{\deg_{\mathbf{w}} g_i - j} g_{ij},$$

and

$$\psi_a(\phi_a(g_i^h)) = a^{\deg_{\mathbf{w}} g_i} g_i.$$

Since $a \neq 0$, we are done. $\square$

We now compare the Betti numbers of an ideal with those of its initial ideal.
Let $I \subset S$ be a graded ideal with respect to the standard grading on S, and fix a weight $\mathbf{w}$ on S. Let $\{g_1, \ldots, g_n\}$ be a Gröbner basis of I with respect to a monomial order which respects $\mathbf{w}$, and further, such that g_i is homogeneous with respect to the standard grading for all i. Then, $\{g_1^h, \ldots, g_n^h\}$ is a system of generators (in fact, a Gröbner basis) of I^h.
If we assign to each x_i the bidegree $(w_i, 1)$ and to t the bidegree $(1, 0)$, then all the generators g_i^h are bihomogeneous, and hence I^h is a bigraded ideal. Therefore $S[t]/I^h$ has a bigraded minimal free $S[t]$-resolution,

$$F_\bullet : 0 \to F_p \to F_{p-1} \to \cdots \to F_0 \to S[t]/I^h \to 0,$$

where $F_i = \oplus_{j,k}(S[t](-k, -j))^{\beta_{ijk}}$. Note that the minimality of the resolution is equivalent to the condition that all entries in the matrices describing the maps must belong to $\langle x_1, \ldots, x_r, t \rangle$.
Note that as $S[t]/I^h$ is a free $\mathsf{k}[t]$-module, $t - a$ is a non-zero divisor on $S[t]/I^h$ for all $a \in \mathsf{k}$. Since t is a non-zero divisor on $S[t]/I^h$ and on $S[t]$, and $t \in \langle x_1, \ldots, x_r, t \rangle$, $F_\bullet/tF_\bullet$ is a bigraded minimal free S-resolution of $\frac{S[t]/I^h}{tS[t]/I^h} \cong S/\text{in}_{\mathbf{w}}(I)$. Observe that the bigraded shifts of $F_\bullet/tF_\bullet$ are the same as those in $F_\bullet$ and in particular, the second component of the shifts in the resolution are the ordinary shifts of the standard graded ideal $\text{in}_{\mathbf{w}}(I)$. Thus, we have

$$\beta_{ij}(S/\text{in}_{\mathbf{w}}(I)) = \sum_k \beta_{ijk} \textit{ for all } i, j.$$

On the other hand, since $t - 1$ is also a non-zero divisor on $S[t]/I^h$ and on $S[t]$, $F_\bullet/(t-1)F_\bullet$ is a free S-resolution of $\frac{S[t]/I^h}{(t-1)S[t]/I^h} \cong S/I$. Note that $t - 1$ is homogeneous with respect to the second component of the bidegree and hence the second components of the shifts in the resolution $F_\bullet$ are preserved. However, $t-1$ does not belong to $\langle x_1,\ldots,x_r,t\rangle$ and hence $F_\bullet/(t-1)F_\bullet$ need not be a minimal resolution. Therefore, we have

$$\beta_{ij}(S/I)\le\sum_k\beta_{ijk}\ for\ all\ i,j.$$

We have thus proved the following theorem.

Theorem 3.2.11. *Let $I\subset S$ be a graded ideal and $\mathbf{w}$ be a weight. Then*

$$\beta_{ij}(I)\le\beta_{ij}(\mathrm{in}_{\mathbf{w}}(I))\ for\ all\ i,j.$$

Theorem 3.2.11 and Thereom 3.1.6 yield the following corollary.

Corollary 3.2.12. *Let $I\subset S$ be a graded ideal and τ be a monomial order on S. Then*

$$\beta_{ij}(I)\le\beta_{ij}(\mathrm{in}_{\tau}(I))\ for\ all\ i,j.$$

Corollary 3.2.13. *Given I,τ as above,*
(i) $\mathrm{pdim}(S/I)\le\mathrm{pdim}(S/\mathrm{in}_\tau(I))$*;*
(ii) $\mathrm{depth}(S/I)\ge\mathrm{depth}(S/\mathrm{in}_\tau(I))$.

Proof. Corollary 3.2.12 directly implies (a). (b) follows from (a) and the Auslander-Buchsbaum formula. □

Proposition 3.2.14. *Let $I\subset S$ be a graded ideal. Then,*
(i) If $\mathrm{in}_{\mathbf{w}}(I)$ is a prime ideal, so is I.
(ii) If $\mathrm{in}_{\mathbf{w}}(I)$ is a radical ideal, so is I.

Proof. Let $I^h\in S[t]$ be the homogenization of I with respect to the weight $\mathbf{w}$. We claim that I is prime (resp. radical) if I^h is prime (resp. radical).
$\phi(f^h)=t^{\deg_{\mathbf{w}} f}f$.
Suppose I^h is prime. Consider $f,g\in S\backslash\{0\}$ such that $fg\in I$. Then, $(fg)^h=f^hg^h\in I^h$, which implies that $f^h\in I^h$ or $g^h\in I^h$. Without loss of generality, let $f^h\in I^h$. Then, by Remark 3.2.4, note that $f=\overline{(f^h)}\in I$.
Similarly, suppose I^h is radical. Consider $f\in S\backslash\{0\}$ such that $f^n\in I$ for $n\in\mathbb{N}$. Then, $(f^n)^h=(f^h)^n\in I^h$ and hence, $f^h\in I^h$. Proceeding as above, we have $f\in I$.
The following lemma along with Proposition 3.2.10 proves that if $\mathrm{in}_{\mathbf{w}}(I)$ is prime (resp. radical), so is I^h. □

Lemma 3.2.15. *Let R be a finitely generated positively graded* k*-algebra and let $s\in R$ be a homogeneous non-zero divisor of R such that R/sR is a domain (resp. a reduced ring) and* $\deg(s)>0$*. Then R is also a domain (resp. a reduced ring).*

Proof. Suppose R/sR is a domain and there exist $a,b\in R\backslash\{0\}$ such that $ab=0$. By the Krull Intersection Theorem, $\cap_{k\ge0}\langle s\rangle^k=0$ and hence, there exist $n_a,n_b\in\mathbb{Z}_{\ge0}$ such that $a\in\langle s\rangle^{n_a},b\in\langle s\rangle^{n_b}$ and $a\notin\langle s\rangle^{n_a+1},b\notin\langle s\rangle^{n_b+1}$. Let $a=a's^{n_a},b=b's^{n_b}$ where $a',b'\notin\langle s\rangle$. Then, $a'b'=0$ and hence $\overline{a'b'}=0$, which implies that $a'\in\langle s\rangle$ or $b'\in\langle s\rangle$, a contradiction.
Similarly, suppose R/sR is a reduced ring and there exists $a\in R\backslash\{0\}$ such that $a^n=0$. Let n_a and a' be as above. Then, $a'^n=0$ and hence $\overline{a'}^n=0$, which implies that $a'\in\langle s\rangle$, a contradiction. □

Chapter 4

Polarization

As usual, let $S = \mathsf{k}[x_1, \ldots, x_r]$.

Lemma 4.0.1. *Let $I \subset S$ be a monomial ideal with minimal generating set of monomials $\{m_1, \ldots, m_n\}$, where $m_i = \prod_{j=1}^n x_j^{a_{ij}}$ for $i = 1, \ldots, n$. Fix an integer $j \in [n]$ and suppose that $a_{ij} > 1$ for at least one $i \in [r]$. Let $T = S[y]$ and let $J \subset T$ be the monomial ideal with minimal generating set of monomials $\{m'_1, \ldots, m'_n\}$, where*

$$m'_i = \begin{cases} m_i & a_{ij} = 0 \\ (m_i/x_j)y & a_{ij} \geq 1. \end{cases}$$

Then $y - x_j$ is a non-zero divisor in T/J and

$$\frac{T/J}{(y - x_j)T/J} \cong S/I$$

as S-modules.

Proof. Suppose $y - x_j$ is a zero divisor in T/J. Then $y - x_j \in P$ for some $P \in \mathrm{Ass}(J)$. By applying Proposition 1.1.5 on the $\mathbb{N}^r$-grading, P is a monomial ideal, and hence $y, x_j \in P$. Thus, there exists a monomial $f \in T \backslash J$ such that $yf, x_j f \in J$. Then there exist m'_k, m'_l and monomials $f_1, f_2 \in T$ such that $yf = m'_k f_1$ and $x_j f = m'_l f_2$.
Since $f \notin J$, x_j divides m'_l and hence, by the construction of J, y divides m'_l. This implies that y divides f. Note that y does not divide f_1 because $f \notin J$. This forces y^2 to divide m'_k, which is a contradiction to the construction of J.
Define a ring homomorphism $\phi : T \to S$ such that $\phi|_S = Id$ and $\phi(y) = x_j$. Then, $\phi(J) = I$ and by Lemma 3.2.9, we have the required isomorphism. $\square$

Motivated by Lemma 4.0.1, we define the polarization of a monomial ideal I.
Let $I \subset S$ be a monomial ideal with minimal generating set of monomials $\{m_1, \ldots, m_n\}$, where $m_i = \prod_{j=1}^n x_j^{a_{ij}}$ for $i = 1, \ldots, n$. For all $j = 1, \ldots, r$, define $a_j = \max\{a_{ij} : i = 1, \ldots, n\}$.

Let $T = \mathsf{k}[x_{11}, x_{12} \ldots, x_{1a_1}, x_{21}, \ldots, x_{2a_2}, \ldots, x_{n1}, \ldots, x_{na_n}]$. Define J to be a monomial ideal in T with generating set $\{m_1', \ldots, m_n'\}$ where

$$m_i' = \prod_{j=1}^{n} \prod_{k=1}^{a_{ij}} x_{jk}$$

for all $i \in [n]$.

Definition 4.0.2. *The monomial ideal J is called the **polarization** of I.*

Example 4.0.3. *Consider the ideal $\langle x_1 x_2^2, x_2^4 \rangle \subset \mathsf{k}[x_1, x_2]$. The polarisation of I is*

$$J = \langle x_{11}, x_{21}x_{22}, x_{21}x_{22}x_{23}x_{24} \rangle \subset \mathsf{k}[x_{11}, x_{21}, x_{22}, x_{23}, x_{24}].$$

Proposition 4.0.4. *Let $I \subset S$ be a monomial ideal and $J \subset T$ be its polarization. Then the sequence $\mathbf{z}$ given by*

$$x_{n1} - x_{na_n}, \ldots, x_{n1} - x_{n2}, \ldots, x_{21} - x_{2a_2}, \ldots, x_{21} - x_{22}, \ldots, x_{11} - x_{1a_1}, \ldots, x_{11} - x_{12}$$

is a regular sequence on T/J and

$$\frac{T/J}{(z)T/J} \cong S/I$$

as graded k-algebras.

Proof. Firstly, replace x_i in S by x_{i1} for all $i \in [r]$. Let the minimal generating set of monomials of I be $\{m_1^{(11)}, \ldots, m_n^{(11)}\}$ Now, let $T_{12} = S[x_{12}]$ and define $m_i^{(12)} = m_i^{(11)}$ if x_{11} does not appear in $m_i^{(11)}$ and $m_i^{(12)} = (m_i^{(11)}/x_{11})x_{12}$ otherwise. Let $J_{12} = \langle m_1^{(12)}, \ldots, m_n^{(12)} \rangle$. By Lemma 4.0.1, $x_{11} - x_{12}$ is a non-zero divisor on T_{12}/J_{12} and

$$\frac{T_{12}/J_{12}}{(x_{11} - x_{12})T_{12}/J_{12}} \cong S/I.$$

Similarly, let $T_{13} = T_{12}[x_{13}]$ and define $J_{13} = \langle m_1^{(13)}, \ldots, m_n^{(13)} \rangle$ $m_i^{(13)} = m_i^{(11)}$ if x_{11} does not appear in $m_i^{(11)}$ and $m_i^{(13)} = (m_i^{(11)}/x_{11})x_{13}$ otherwise. Note that

$$\frac{T_{13}/J_{13}}{(x_{11} - x_{13}, x_{11} - x_{12})T_{13}/J_{13}} \cong \frac{T_{12}/J_{12}}{(x_{11} - x_{12})T_{12}/J_{12}} \cong S/I.$$

Continue the process until T_{1a_1}. Then, let $T_{22} = T_{1a_1}[x_{22}]$. We eventually get $T_{na_n} = T$. Repeated application of Lemma 4.0.1 completes the proof. □

Corollary 4.0.5. *Let $I \subset S$ be a monomial ideal and $J \subset T$ be its polarization. Then*
(i) $\beta_{ij}(I) = \beta_{ij}(J)$ for all i, j;
(ii) $H_{S/I}(t) = (1-t)^{\delta} H_{T/J}(t)$ where $\delta = \dim T - \dim S$;
(iii) $\operatorname{pdim}(S/I) = \operatorname{pdim}(S/J)$ and $\operatorname{reg}(S/I) = \operatorname{reg}(T/J)$.

Proof. (i) Follows from the fact that $\mathbf{z}$ is a regular sequence on T/J.
(ii) Follows from Corollary 1.2.9.
(iii) Follows from (i). □

Chapter 5

The lexsegment ideal

Given a graded ideal $I \subset S$, our aim is to show the existence of a special ideal, the lexsegment ideal of I, denoted by I^{lex}, such that S/I and S/I^{lex} have the same Hilbert function.
By Corollary 2.3.5, S/I and $S/\text{in}_\tau(I)$ have the same Hilbert function for any monomial order τ on S. Thus, we can assume that I is a monomial ideal. By Theorem 2.3.4, the monomials in S not belonging to I form a k-basis of I and since this k-basis determines the Hilbert functions of S/I, the Hilbert function of S/I does not depend on the base field k. We can therefore assume that $\text{char}(\mathsf{k}) = 0$.
We denote by $M_d(S)$ the set of all monomials of S of degree d.

Definition 5.0.1. *A set $\mathcal{L} \subset M_d(S)$ is called a* ***lexsegment*** *if for all $m \in \mathcal{L}$, we have that $m' \in \mathcal{L}$ for all $m' \in M_d(S)$ such that $m' \geq_{\text{lex}} m$.*

Definition 5.0.2. *A set $\mathcal{L} \subset M_d(S)$ is called* ***strongly stable*** *if $x_i(m/x_j) \in \mathcal{L}$ for all $m \in \mathcal{L}$ and all pairs (i, j) such that $i < j$ and x_j divides m.*

For a monomial $m \in S$, we set $\gamma(m) = \max\{i : x_i \, \textit{divides} \, m\}$.

Definition 5.0.3. *A set $\mathcal{L} \subset M_d(S)$ is called* ***stable*** *if $x_i(m/x_j) \in \mathcal{L}$ for all $m \in \mathcal{L}$ and all $i < \gamma(m)$.*

Definition 5.0.4. *A monomial ideal I is said to be a lexsegment ideal or a (strongly) stable monomial ideal, if for each d the monomials of degree d in I form a lexsegment, or a (strongly) stable set of monomials respectively.*

Remark 5.0.5. Note that every lexsegment set is strongly stable, and every strongly stable set is stable.

Example 5.0.6. *Let $S = \mathsf{k}[x, y, z, w]$.*
Suppose I_1 is the smallest lexsegment ideal containing xyz. Then $I_1 = \langle xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$.
Suppose I_2 is the smallest strongly stable ideal containing xyz. Then $I_2 = \langle xyz, xy^2, x^2z, x^2y, x^3\rangle$.
Suppose I_3 is the smallest stable ideal containing xyz. Then $I_3 = \langle xyz, xy^2, x^2y, x^3\rangle$.

Now we have that S/I and $S/\text{gin}_\tau(I)$ have the same Hilbert function, and that $gin_\tau(I)$ is a strongly stable ideal [2]. Hence, we can assume that I is a strongly stable ideal.

Theorem 5.0.7. *Let $I \subset S$ be a graded ideal. There exists a unique lexsegment ideal, denoted I^{lex}, such that S/I and S/I^{lex} have the same Hilbert function.*

Given a graded ideal I, with j^{th} graded component I_j, denote by I_j^{lex} the k-vector space spanned by the unique lexsegment $\mathcal{L}_j$ with $|\mathcal{L}_j| = \dim_{\mathsf{k}} I_j$. Define $I^{\text{lex}} = \oplus_j I_j^{\text{lex}}$.
Note that if I^{lex} as defined above is an ideal, it is the only possible lexsegment ideal such that S/I and S/I^{lex} have the same Hilbert function. Therefore, we only need to show that I^{lex} is an ideal to prove Theorem 5.0.7. It is sufficient to show that $\{x_1, \dots, x_r\}\mathcal{L}_j \subset \mathcal{L}_{j+1}$.

Definition 5.0.8. *Let $\mathcal{N}$ be a set of monomials in S. Then the shadow of $\mathcal{N}$ is said to be the set*

$$\text{Shad}(\mathcal{N}) = \{x_1, \dots, x_r\}\mathcal{N} = \{x_i u : u \in N, i = 1, \dots, n\}.$$

Lemma 5.0.9. *If $\mathcal{N} \subset M_d(S)$ is stable, strongly stable or lexsegment, then so is* $\text{Shad}(\mathcal{N})$*.*

Given $\mathcal{N} \subset M_d(S)$, we denote by $\gamma_i(N)$ the number of elements $\gamma(m) = i$ and set $\gamma_{\le i}(\mathcal{N}) = \sum_{j=1}^{i} \gamma_i(\mathcal{N})$.

Lemma 5.0.10. *Let $\mathcal{N} \subset M_d(S)$ be a stable set of monomials. Then* $\text{Shad}(N)$ *is a stable set and*
(i) $\gamma_i(\text{Shad}(N)) = \gamma_{\le i}(\mathcal{N})$*;*
(ii) $|\text{Shad}(N)| = \sum_{i=1}^{r} \gamma_{\le i}(\mathcal{N})$.

Proof. (ii) follows directly from (i). To prove (i), define the map

$$\phi : \{m \in \mathcal{N} : \gamma(m) \le i\} \to \{m \in \text{Shad}(\mathcal{N}) : \gamma(m) = i\},\ m \to m x_i.$$

ϕ is clearly injective. Let $m' \in \text{Shad}(\mathcal{N})$ such that $\gamma(m') = i$. There exists $j \in [r]$ and $m \in \mathcal{N}$ such that $m' = x_j m$. We must have $\gamma(m) \le i$. If $j = i$, then we are done. If $j < i$, then $\gamma(m) = i$ and since $\mathcal{N}$ is stable, $m_1 = x_j(m/x_i) \in \mathcal{N}$. Hence, we have $m' = x_i m_1$ for $m_1 \in \mathcal{N}$. This proves that ϕ is a bijection, which implies (i). □

Theorem 5.0.11 (Bayer)**.** *Let $\mathcal{L} \subset M_d(S)$ be a lexsegment and $\mathcal{N} \subset M_d(S)$ be a strongly stable set of monomials with $|\mathcal{L}| \le |\mathcal{N}|$. Then $\gamma_{\le i}(\mathcal{L}) \le \gamma_{\le i}(\mathcal{N})$ for $i = 1, \dots, r$.*

Proof. Observe that we can write $\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_1 x_r \cup \cdots \cup \mathcal{N}_d x_r^d$ where each $\mathcal{N}_j$ is a strongly stable set of monomials of degree $d - j$ in the variables $x_1, \dots, x_{r-1}$. The lexsegment $\mathcal{L}$ has a similar decomposition $\mathcal{L}_0 \cup \cdots \cup \mathcal{L}_r x_r$, where each $\mathcal{L}_j$ is a lexsegment.
We prove the theorem by induction on the number of variables. If $r = 1$, we have that $\gamma_{\le 1}(\mathcal{L}) = |\mathcal{L}| \le |\mathcal{N}| = \gamma_{\le 1}(\mathcal{N})$.
Let $r > 1$. We have that $\gamma_{\le r}(\mathcal{L}) = |\mathcal{L}|$ and $\gamma_{\le r}(\mathcal{N}) = |\mathcal{N}|$ and hence, $\gamma_{\le r}(\mathcal{L}) \le \gamma_{\le r}(\mathcal{N})$. Note that for $i < r$, $\gamma_{\le i}(\mathcal{L}) = \gamma_{\le i}(\mathcal{L}_0)$ and $\gamma_{\le i}(\mathcal{N}) = \gamma_{\le i}(\mathcal{N}_0)$. Hence, if we show that $|\mathcal{L}_0| \le |\mathcal{N}_0|$, the proof is done by induction.

For each j, let $\mathcal{N}_j^*$ be the lexsegment in $M_{d-j}(\mathsf{k}[x_1,\ldots,x_{r-1}])$ with $|\mathcal{N}_j^*| = |\mathcal{N}_j|$ and let $\mathcal{N}^* = \mathcal{N}_0^* \cup \mathcal{N}_1^* x_r \cup \cdots \cup \mathcal{N}_d^* x_r^d$. We claim that $\mathcal{N}^*$ is a strongly stable set of monomials.
Observe that it suffices to show that $\{x_1,\ldots,x_{r-1}\}\mathcal{N}_j^* \subset \mathcal{N}_{j-1}^*$. By using that $\mathcal{N}$ is a strongly stable set, we have that $\{x_1,\ldots,x_r\}\mathcal{N}_j \subset \mathcal{N}_{j-1}$. Then, by Lemma 5.0.10 and the induction hypothesis, we have that

$$|\{x_1,\ldots,x_{r-1}\}\mathcal{N}_j^*| = \sum_{i=1}^{r-1} \gamma_{\leq i}(\mathcal{N}_j^*) \leq \sum_{i=1}^{r-1} \gamma_{\leq i}(\mathcal{N}_j) = |\{x_1,\ldots,x_{r-1}\}\mathcal{N}_j| \leq |\mathcal{N}_{j-1}| = |\mathcal{N}_{j-1}^*|.$$

The fact that $|\{x_1,\ldots,x_{r-1}\}\mathcal{N}_j^*|$ and $|\mathcal{N}_{j-1}^*|$ are both lexsegments forces $|\{x_1,\ldots,x_{r-1}\}\mathcal{N}_j^*| \subset |\mathcal{N}_{j-1}^*|$, which implies that $\mathcal{N}^*$ is a strongly stable set of monomials.
Now, given a monomial $m = \prod_{i=1}^r x_i^{a_i}$, we set $\overline{m} = (x_{n-1}/x_n)^{a_n} m$. Observe that if $m_1 \leq m_2$ in the lexicographic order, then $\overline{m_1} \leq \overline{m_2}$.
Let $m_1 = \min \mathcal{L}$ and $m_2 = \min \mathcal{N}^*$. Since $\mathcal{N}^*$ is strongly stable, $\overline{m_2} \in \mathcal{N}_0^*$ and $\overline{m_2} \geq \min(\mathcal{N}_0^*)$. Further, $\min(\mathcal{N}_0^*) \geq m_2$, which implies that $\overline{\min(\mathcal{N}_0^*)} = \min(\mathcal{N}_0^*) \geq \overline{m_2}$. Hence, $\min(\mathcal{N}_0^*) = \overline{m_2}$ and similarly, $\min(\mathcal{L}_0^*) = \overline{m_1}$.
Since $|\mathcal{L}| \leq |\mathcal{N}| = |\mathcal{N}^*|$, we have that $m_1 \geq m_2$ and hence, $\overline{m_1} \geq \overline{m_2}$. As $\mathcal{L}_0$ and $\mathcal{N}_0^*$ are lexsegments, we get that $|\mathcal{L}_0| \leq |\mathcal{N}_0^*| = |\mathcal{N}_0|$, which completes the proof. $\square$

We now complete the proof of Theorem 5.0.7.
Recall that we may assume that I is strongly stable. Let $\mathcal{N}_j$ be the strongly stable set of monomials which spans the k-vector space I_j. Since $|\mathcal{L}_j| = |\mathcal{N}_j|$, Bayer's theorem together with Lemma 5.0.10 implies that

$$|\mathrm{Shad}(\mathcal{L}_j)| = \sum_{i=1}^{r} \gamma_{\leq i}(\mathcal{L}_j) \leq \sum_{i=1}^{r} \gamma_{\leq i}(\mathcal{N}_j) = |\mathrm{Shad}(\mathcal{N}_j)|.$$

Since I is an ideal, we have that $\mathrm{Shad}(\mathcal{N}_j) \subset \mathcal{N}_{j+1}$. Hence,

$$|\mathrm{Shad}(\mathcal{L}_j)| \leq |\mathrm{Shad}(\mathcal{N}_j)| \leq |\mathcal{N}_{j+1}| = |\mathcal{L}_{j+1}|.$$

Since $\mathrm{Shad}(\mathcal{L}_j)$ and $\mathcal{L}_{j+1}$ are both lexsegments, $|\mathrm{Shad}(\mathcal{L}_j)| \leq |\mathcal{L}_{j+1}|$ implies that $\mathrm{Shad}(\mathcal{L}_j) \subset \mathcal{L}_{j+1}$, as desired.

Chapter 6

The Auslander-Buchsbaum-Serre Theorem

Theorem 6.0.1. *Let $(R, \mathfrak{m}, \mathsf{k})$ be a Noetherian local ring and $\mu(\mathfrak{m}) = n$. Then* $\operatorname{pdim}_R(\mathsf{k}) \geq n$ *and* $\beta_i^R(\mathsf{k}) \geq \binom{n}{i}$ *for all $i \in \{0, \ldots, n\}$. In particular, if R is a regular local ring, then* $\operatorname{depth}(R) \geq \mu(\mathfrak{m})$.

Proof. Let $\{x_1, \ldots, x_n\}$ be a minimal generating set of $\mathfrak{m}$. Therefore, $x_i \in \mathfrak{m} \setminus \mathfrak{m}^2$ for all i. We construct a minimal free resolution of k step by step.
Let $F_1 := \wedge^1 R^n$ denote $\oplus_{i=1}^n Re_i$. Define $\phi_1 : \wedge^1 R^n \to R$ as $\phi_1(e_i) = x_i$. Let

$$v_{ij} := x_i e_j - x_j e_i \in F_1, \text{ for all } 1 \leq i < j \leq n.$$

Note that each $v_{ij} \in \ker(\phi_1)$. We claim that the set $\{v_{ij} | 1 \leq i < j \leq n\}$ can be extended to a minimal generating set of $K_1 := \ker(\phi_1)$. Indeed, suppose there exists $\{a_{ij} \in R \mid 1 \leq i < j \leq n\}$ such that $\sum\limits_{1 \leq i < j \leq n} a_{ij} v_{ij} \in \mathfrak{m} K_1$. Since F_1 maps minimally onto $\mathfrak{m}$, $K_1 \subseteq \mathfrak{m} F_1$, and hence $\mathfrak{m} K_1 \subseteq \mathfrak{m}^2 F_1$. Suppose, for some $1 \leq i' < j' \leq n$ we have $a_{i'j'} \notin \mathfrak{m}$. Observe that the coefficient of $e_{j'}$ in $\sum\limits_{1 \leq i < j \leq n} a_{ij} v_{ij}$ is $\sum\limits_{i=1}^{j'-1} a_{ij'} x_i - \sum\limits_{i=j'+1}^{n} a_{j'i} x_i$, and hence $\sum\limits_{i=1}^{j'-1} a_{ij'} x_i - \sum\limits_{i=j'+1}^{n} a_{j'i} x_i \in \mathfrak{m}^2$. Since $a_{i'j'} \notin \mathfrak{m}$, it is a unit. Hence,

$$\{x_1, \ldots, x_{i'-1}, \sum_{i=1}^{j'-1} a_{ij'} x_i - \sum_{i=j'+1}^{n} a_{j'i} x_i, x_{i'+1}, \ldots, x_n\}$$

is also a minimal generating set of $\mathfrak{m}$ with one of the elements in $\mathfrak{m}^2$, which is a contradiction. Therefore, $a_{ij} \in \mathfrak{m}$ for all $1 \leq i < j \leq n$, which proves the claim.
Let $F_2 = R^{\beta_2^R(\mathsf{k})}$ be a free module mapping minimally onto $\ker(\phi_1)$. From what we have seen above, $\operatorname{rank}(F_2) = \beta_2^R(\mathsf{k}) \geq \binom{n}{2}$. Thus, we write $F_2 = \wedge^2 R^n \oplus G_2$.
Inductively assume that $\{v_{i_1 \ldots i_r} \mid 1 \leq i < j \leq n\}$ form a part of minimal generating set of $\ker(\phi_{r-1})$, where

$$v_{i_1 \ldots i_r} = \sum_{k=1}^{r} (-1)^{k-1} x_{i_k} e_{i_1} \wedge \cdots \wedge e_{i_{k-1}} \wedge e_{i_{k+1}} \wedge \cdots \wedge e_r \in F_{r-1} = \wedge^{r-1} R^n \oplus G_{r-1}.$$

Let F_r denote a free module mapping minimally onto $\ker(\phi_{r-1})$. Then we have $\operatorname{rank}(F_r) = \beta_r^R(\mathsf{k}) \geq \binom{n}{r}$. We can decompose F_r as $\wedge^r R^n \bigoplus G_r$, where G_r is an R-free module of rank $\beta_r^R(\mathsf{k}) - \binom{n}{r}$. Let $\phi_r : F_r \to F_{r-1}$ be such that

$$\phi_r(e_{i_1} \wedge \cdots \wedge e_{i_r}) = v_{i_1 \dots i_r}.$$

For all $1 \leq i_1 < \cdots < i_{r+1} \leq n$, let

$$v_{i_1 \dots i_{r+1}} = \sum_{k=1}^{r+1} (-1)^{k-1} x_{i_k} e_{i_1} \wedge \cdots \wedge e_{i_{k-1}} \wedge e_{i_{k+1}} \wedge \cdots \wedge e_{i_{r+1}}.$$

Note that for all $1 \leq i_1 < \cdots < i_{r+1} \leq n$, $v_{i_1 \dots i_{r+1}} \in \ker(\phi_r)$. Let K_r denote $\ker(\phi_r)$. We claim that

$$\{v_{i_1 \dots i_{r+1}} \mid 1 \leq i_1 < \cdots < i_{r+1} \leq n\}$$

forms a part of minimal generating set of K_r. To prove the claim, suppose that

$$\{a_{i_1 \dots i_{r+1}} \in R | 1 \leq i_1 < \cdots < i_{r+1} \leq n\}$$

be such that $\sum a_{i_1 \dots i_{r+1}} v_{i_1 \dots i_{r+1}} \in \mathfrak{m} K_r$. Since F_r maps minimally onto $\ker(\phi_{r-1})$, we must have $K_r \subseteq \mathfrak{m} F_r$ and hence, $\mathfrak{m} K_r \subseteq \mathfrak{m}^2 F_r$. Suppose, for some $1 \leq i'_1 < \cdots < i'_{r+1} \leq n$, $a_{i'_1 \dots i'_{r+1}} \notin \mathfrak{m}$. Observe that the coefficient of $e_{i'_2} \wedge \dots \wedge e_{i'_{r+1}}$ in $\sum a_{i_1 \dots i_{r+1}} v_{i_1 \dots i_{r+1}}$ is

$$\sum_{i_1=1}^{i'_2-1} a_{i_1 i'_2 i'_3 \dots i'_{r+1}} x_{i_1} - \sum_{i_1=i'_2+1}^{i'_3-1} a_{i'_2 i_1 i'_3 \dots i'_{r+1}} x_{i_1} + \cdots + (-1)^r \sum_{i_1=i'_{r+1}+1}^{n} a_{i'_2 i'_3 \dots i'_{r+1} i_1} x_{i_1},$$

which must belong to $\mathfrak{m}^2$. Since $a_{i'_1 i'_2 \dots i'_{r+1}} \notin \mathfrak{m}$, this contradicts the assumption that $\{x_1, \dots, x_n\}$ is a minimal generating set of $\mathfrak{m}$. Therefore, the set $\{v_{i_1 \dots i_{r+1}} \mid 1 \leq i_1 < \cdots < i_{r+1} \leq n\}$ can be extended to a minimal generating set of K_r. Hence, $\operatorname{rank}(F_{r+1}) \geq \binom{n}{r+1}$.
If R is regular local, by the Auslander-Buchsbaum formula, $\operatorname{depth}(R) = \operatorname{pdim}_R(\mathsf{k})$ and hence $\operatorname{depth}(R) \geq \mu(\mathfrak{m})$. □

Recall the following result.

Lemma 6.0.2. *Let $(R, \mathfrak{m}, \mathsf{k})$ be a Noetherian local ring. Let $\{b_1, \dots, b_n\}$ be a minimal generating set of $\mathfrak{m}$, and $K_\bullet(b_1, \dots, b_n)$ be the corresponding Koszul complex. Then* $\operatorname{depth}(R) = \min\{j \mid H_{n-j}(K_\bullet(b_1, \dots, b_n)) \neq 0\}$.

Corollary 6.0.3. *Let $(R, \mathfrak{m}, \mathsf{k})$ be a regular local ring. Then* $\operatorname{depth}(R) = \mu(\mathfrak{m})$.

Proof. Note that the above lemma implies that $\operatorname{depth}(R) \leq \mu(\mathfrak{m})$. Hence, we have $\operatorname{depth}(R) = \mu(\mathfrak{m})$. □

Corollary 6.0.4. *Let $(R, \mathfrak{m}, \mathsf{k})$ be a regular local ring. Then every minimal generating set of $\mathfrak{m}$ is a regular sequence.*

Proof. From the above lemma, since $\operatorname{depth}(R) = \mu(\mathfrak{m})$, we have that the Koszul complex corresponding to a minimal generating set of $\mathfrak{m}$ must be exact. $\square$

Theorem 6.0.5 (Auslander-Buchsbaum-Serre)**.** *Let $(R, \mathfrak{m}, \mathsf{k})$ be a Noetherian local ring with $\operatorname{depth}(R) = d$. The following statements are equivalent:*
(i) $\operatorname{pdim}_R(\mathsf{k}) < \infty$.
(ii) $\operatorname{pdim}_R(M) < \infty$ *for all finitely generated R-modules M.*
(iii) $\mathfrak{m}$ *is generated by a regular sequence.*
(iv) $\mathfrak{m}$ *is generated by d elements.*
(v) $\operatorname{pdim}_R(\mathsf{k}) = d$.

Proof. (i) $\Longrightarrow$ (ii) Let $\operatorname{pdim}_R(\mathsf{k}) = n < \infty$. Then we have $\operatorname{Tor}_i^R(M, \mathsf{k}) = 0$ for all $i > n$ and for all finitely generated R-modules M. Therefore, $\operatorname{pdim}_R(M) < \infty$ for all finitely generated R-modules M.
(ii) $\Longrightarrow$ (i) is obvious since k is a finitely generated R-module.
(i) $\Longrightarrow$ (iii) follows from Corollary 6.0.4.
(iii) $\Longrightarrow$ (i) If $\mathfrak{m}$ is generated by a regular sequence $x_1, \ldots, x_n$, then the Koszul complex on $x_1, \ldots, x_n$ gives a free resolution of k of finite length, which proves $\operatorname{pdim}_R(\mathsf{k}) = n < \infty$.
(i) $\Longrightarrow$ (iv) From Theorem 6.0.1 we know that $\operatorname{depth}(R) \geq \mu(\mathfrak{m})$. Hence, $d \geq \mu(\mathfrak{m})$. Thus $\mathfrak{m}$ is generated by d elements.
(i) $\Longrightarrow$ (v) If $\operatorname{pdim}_R(\mathsf{k}) < \infty$, then by Auslander-Buchsbaum formula we get $\operatorname{pdim}_R(\mathsf{k}) = \operatorname{depth}(R) = d$.
(v) $\Longrightarrow$ (i) is obvious.
(iv) $\Longrightarrow$ (i) Let $\mathfrak{m}$ be generated by d elements, that is, let $\mu(\mathfrak{m}) \leq \operatorname{depth}(R)$. From Lemma 6.0.2 we have $\operatorname{depth}(R) \leq \mu(\mathfrak{m})$. Therefore $\operatorname{depth}(R) = \mu(\mathfrak{m}) = d$. If $x_1, \ldots, x_d$ is a minimal generating set of $\mathfrak{m}$, then by Lemma 6.0.2 we have that the Koszul complex on $x_1, \ldots, x_d$ is exact, and hence is a free resolution of k of length d. This shows that $\operatorname{pdim}_R(\mathsf{k}) = n < \infty$. $\square$

Proposition 6.0.6. *Let $(R, \mathfrak{m}, \mathsf{k})$ be a regular local ring such that* $\operatorname{depth}(R) = 2$*. Then $\mu(\mathfrak{m}) = 2$, and $\mathfrak{m}$ is generated by a regular sequence.*

Proof. By Auslander-Buchsbaum formula we have $\operatorname{depth}(R) = \operatorname{pdim}_R(\mathsf{k}) = 2$. We know that $\mu(\mathfrak{m}) \neq 1$, otherwise we would have $\operatorname{depth}(R) = 1$. Hence, $\mu(\mathfrak{m}) \geq 2$, and by Theorem 6.0.1 we have $\mu(\mathfrak{m}) = 2$. Suppose that $\mathfrak{m} = \langle x_1, x_2 \rangle$. We show that x_1, x_2 form a regular sequence by showing that the Koszul complex $K(x_1, x_2)$ is exact. Consider the Koszul complex on x_1, x_2 as follows:

$$0 \to R \xrightarrow{\phi_2} R^2 \xrightarrow{\phi_1} R \to R/\mathfrak{m} \to 0.$$

Since $\operatorname{pdim}_R(\mathsf{k}) = 2$ and since R^2 maps minimally onto $\mathfrak{m}$, we see that $\ker(\phi_1)$ is free. By the Hilbert-Burch theorem, $\operatorname{rank}(\ker(\phi_1)) = 1$. Let $\ker(\phi_1) = \langle (a_1, a_2) \rangle$, where $a_1, a_2 \in \mathfrak{m}$. Since $(-x_2, x_1) \in \ker(\phi_1)$, there exists $c \in R$ such that $(-x_2, x_1) = c(a_1, a_2)$. Since $x_1, x_2 \in \mathfrak{m} \setminus \mathfrak{m}^2$, we must have $c \notin \mathfrak{m}$, and hence $\ker(\phi_1) = \langle (-x_2, x_1) \rangle$. Therefore, the Koszul complex $K(x_1, x_2)$ is exact, and hence x_1, x_2 is regular. $\square$

Proposition 6.0.7. *Let R be a UFD, and let I be an ideal of R such that $\mu(R) = 2$. Then,* $\beta_2^R(R/I) = 1$.

Proof. Let $I = \{x_1, x_2\}$, and let $a_1, a_2 \in R\backslash\{0\}$ such that $a_1x_1 + a_2x_2 = 0$. Let $a = gcd(a_1, a_2)$, and $b_1 = a_1/a, b_2 = a_2/a$. Note that $b_1x_1 + b_2x_2 = 0$, and b_1, b_2 are coprime.
Suppose there exist $k_1, k_2 \in R$ such that $k_1x_1 + k_2x_2 = 0$.

$$k_1x_1 + k_2x_2 = 0 \implies b_1k_1x_1 + b_1k_2x_2 = 0 \implies -b_2k_1x_2 + b_1k_2x_2 = 0 \implies b_2k_1 = b_1k_2.$$

Since b_1 and b_2 are co-prime, $b_1|k_1$. Let $k_1 = kb_1$, and thus $k_2 = kb_2$. Hence, the kernel of the map from R_2 to R, which maps the basis elements of R_2 to a minimal generating set of I must be a cyclic R-module. This implies that $\beta_2^R = 1$. □

Chapter 7

Existence of bounds on projective dimension and regularity

7.1 Burch's construction

To start with, I recall some results which will be instrumental in the following proof.

Lemma 7.1.1. *Let $(R, \mathfrak{m}, \mathsf{k})$ be a Noetherian local ring. Let I be an ideal whose associated primes are minimal over I. If $\mathfrak{P} \in \mathrm{Ass}(I)$, $I_P \cap R$ is the P-primary component in the minimal irredundant primary decomposition of I.*

Theorem 7.1.2 (Burch)**.** *Let $(R, \mathfrak{m}, \mathsf{k})$ be a Noetherian local ring and let s be an integer such that $1 \leq s \leq \mathrm{depth}(R)$. Then there exists an ideal I_s of R, generated by three or fewer elements of R, such that $\mathrm{pdim}(R/I_s) = s$.*

Proof. If $\mathrm{depth}(R) \leq 3$, let I_s be the ideal generated by a regular sequence of length s.
Suppose $\mathrm{depth}(R) = d > 3$. We inductively construct a sequence $\{g_1, \ldots, g_{2(d-2)}\}$ of elements of R.
Let g_1, g_2, g_3, g_4 be a regular sequence in R. Consider the short exact sequence

$$0 \to (g_1, g_2) \cap (g_3, g_4) \to (g_1, g_2) \oplus (g_3, g_4) \to (g_1, g_2, g_3, g_4) \to 0.$$

Since $\mathrm{pdim}(R/(g_1, g_2)) = \mathrm{pdim}(R/(g_3, g_4)) = 2$ and $\mathrm{pdim}(R/(g_1, g_2, g_3, g_4)) = 4$, we have that $\mathrm{pdim}(R/(g_1, g_2) \cap (g_3, g_4)) = 3$ (follows directly by observing the induced long exact sequence of Tor modules).
Suppose we have chosen $g_1, g_2, \ldots, g_{2k}$ $(k < d - 2)$ satisfying the conditions that
(i) $g_{2i-1}, g_{2i}, g_{2j-1}, g_{2j}$ is a regular sequence for all $i \leq j \leq k$.
(ii) $\mathrm{pdim}(R/(g_1, g_2) \cap \cdots \cap (g_{2k-1}, g_{2k})) = k + 1$.
By the Auslander-Buchsbaum formula, $\mathrm{depth}(R/(g_1, g_2) \cap \cdots \cap (g_{2k-1}, g_{2k})) = d - k - 1 \geq 2$. Let g_{2k+1} be a nonzerodivisor and non-unit in $R/(g_1, g_2) \cap \cdots \cap (g_{2k-1}, g_{2k})$. Then,

$$\mathrm{depth} \frac{R}{(g_1, g_2) \cap \cdots \cap (g_{2k-1}, g_{2k}) + (g_{2k+1})} \geq 1.$$

Hence, the ideal $(g_1, g_2) \cap \cdots \cap (g_{2k-1}, g_{2k}) + (g_{2k+1})$ has no $\mathfrak{m}$-primary component. Also, since $\operatorname{depth}(R) > 3$, for any $1 \leq i \leq k$, $\operatorname{depth}(R/(g_{2i-1}, g_{2i}, g_{2k+1}) > 0$ and $(g_{2i-1}, g_{2i}, g_{2k+1})$ has no $\mathfrak{m}$-primary component. Since R is Noetherian, all ideals of R have finitely many associated primes, and we can pick g_{2k+2} to be a non-unit in no associated prime of $(g_1, g_2) \cap \cdots \cap (g_{2k-1}, g_{2k}) + (g_{2k+1})$ and in no associated prime of $(g_{2i-1}, g_{2i}, g_{2k+1})$ for any $i \leq k$. Then,

$$\operatorname{pdim} \frac{R}{(g_1, g_2) \cap \cdots \cap (g_{2k-1}, g_{2k}) + (g_{2k+1}, g_{2k+2})} = k + 3,$$

and hence, by observing the induced long exact sequence of Tor modules, we have that

$$\operatorname{pdim} \frac{R}{(g_1, g_2) \cap \cdots \cap (g_{2k-1}, g_{2k}) \cap (g_{2k+1}, g_{2k+2})} = k + 2.$$

Thus, we can construct a sequence $\{g_1, \ldots, g_{2d-4}\}$ such that $\{g_1, \ldots, g_{2k}\}$ satisfies the above conditions for all $k \leq d - 2$.

Observe that $(g_{2i-1}, g_{2i}) : (g_j) = (g_{2i-1}, g_{2i})$ for all $i \leq 2d - 4$ and $j \neq 2i, 2i - 1$. Hence, for each associated prime $\mathfrak{P}$ of (g_{2i-1}, g_{2i}),

$$(g_1 g_3 \ldots g_{2d-5}, g_2 g_4 \ldots g_{2d-4})_{\mathfrak{P}} = (g_{2i-1}, g_{2i})_{\mathfrak{P}}.$$

It follows that $\mathfrak{P} R_{\mathfrak{P}}$ is an associated prime of $(g_1 g_3 \ldots g_{2d-5}, g_2 g_4 \ldots g_{2d-4})_{\mathfrak{P}}$ in $R_{\mathfrak{P}}$ and thus, $\mathfrak{P}$ is an associated prime of $(g_1 g_3 \ldots g_{2d-5}, g_2 g_4 \ldots g_{2d-4})$ in R. However, $\prod_{j \leq 2s-4, j \notin \{2i-1, 2i\}} g_j$ is a nonzerodivisor in $R/(g_{2i-1}, g_{2i})$ and a zerodivisor in $R/(g_1 g_3 \ldots g_{2d-5}, g_2 g_4 \ldots g_{2d-4})$. This implies that $(g_1 g_3 \ldots g_{2d-5}, g_2 g_4 \ldots g_{2d-4})$ has an associated prime which is not an associated prime of (g_{2i-1}, g_{2i}).

Note that $g_1 g_3 \ldots g_{2d-5}, g_2 g_4 \ldots g_{2d-4}$ is a regular sequence. Indeed, if $g_1 g_3 \ldots g_{2d-5} h_1 = g_2 g_4 \ldots g_{2d-4} h_2$ for $h_1, h_2 \in R$, then $g_1 g_3 \ldots g_{2d-7} h_1 = g_{2d-4} h_3$, as $\{g_{2d-5}, g_{2d-4}\}$ is a regular sequence. Since $\{g_{2d-7}, g_{2d-4}\}$ is also a regular sequence, $g_1 g_3 \ldots g_{2d-9} h_1 = g_{2d-4} h_4$. Proceeding in this manner, we get $h_1 = g_{2d-4} f_1$ for some $f_1 \in R$ and similarly, $h_2 = g_{2d-5} f_2$ for some $f_2 \in R$. Hence, $g_1 g_3 \ldots g_{2d-7} f_1 = g_2 g_4 \ldots g_{2d-6} f_2$ and we are done by induction.

Hence, all associated primes of $(g_1 g_3 \ldots g_{2d-5}, g_2 g_4 \ldots g_{2d-4})$ are minimal over $(g_1 g_3 \ldots g_{2d-5}, g_2 g_4 \ldots g_{2d-4})$ and have height two. Also note that g_j is not contained in any associated prime of (g_{2i-1}, g_{2i}) for any i, which forces that no associated prime of (g_{2i-1}, g_{2i}) is an associated prime of (g_{2j-1}, g_{2j}). Hence,

$$(g_1 g_3 \ldots g_{2d-5}, g_2 g_4 \ldots g_{2d-4}) = (g_1, g_2) \cap \cdots \cap (g_{2d-5}, g_{2d-4}) \cap J$$

where $J = \cap_{\mathfrak{P} \in \Lambda}((g_1 g_3 \ldots g_{2d-5}, g_2 g_4 \ldots g_{2d-4})_{\mathfrak{P}} \cap R)$, where $\Lambda = \operatorname{Ass}((g_1 g_3 \ldots g_{2d-5}, g_2 g_4 \ldots g_{2d-4})) \setminus \cup_{1 \leq i \leq d-2} \operatorname{Ass}(g_{2i-1}, g_{2i})$.

Fix $i \leq d - 2$. Observe that since every associated prime of J has height 2 and is not an associated prime of (g_{2i-1}, g_{2i}), it must contain a nonzerodivisor in $R/(g_{2i-1}, g_{2i})$. Since every primary component of J contains a power of an associated prime of J, it also contains a nonzerodivisor in $R/(g_{2i-1}, g_{2i})$. The product of the nonzerodivisors corresponding to every primary

component of J produces an element in J which is a nonzerodivisor in $R/(g_{2i-1}, g_{2i})$. Hence, $(g_{2i-1}, g_{2i}) : J = (g_{2i-1}, g_{2i})$ for $1 \leq i \leq d-2$.
By prime avoidance, there exists $x_d \in J$ such that x_d is a nonzerodivisor in $R/(g_{2i-1}, g_{2i})$ for $1 \leq i \leq d-2$. Then,

$$(g_1 g_3 \dots g_{2d-5}, g_2 g_4 \dots g_{2d-4}) : x_d = (g_1, g_2) \cap \cdots \cap (g_{2d-5}, g_{2d-4}).$$

Since the projective dimension of $(g_1 g_3 \dots g_{2d-5}, g_2 g_4 \dots g_{2d-4})$ is two, the short exact sequence

$$0 \to \frac{R}{I : x_d} \xrightarrow{x_d} \frac{R}{I} \to \frac{R}{(I, x_d)} \to 0$$

(where $I = (g_1 g_3 \dots g_{2d-5}, g_2 g_4 \dots g_{2d-4})$) gives us that

$$\operatorname{pdim} \frac{R}{(g_1 g_3 \dots g_{2d-5}, g_2 g_4 \dots g_{2d-4}, x_d}) = s.$$

□

The above theorem tells us that we cannot hope to bound projective dimension as a function of the number of generators. This raises the question of whether we can achieve bounds as functions of the degrees of the generators.

7.2 Stillman's question and existence of bounds on regularity

Stillman's question: Let k be a field. Does there exist a bound, independent of n, on the projective dimension of an ideal in $S = \mathsf{k}[X_1, \dots, X_n]$ which is generated by N forms of degrees $d_1, \dots, d_N$?
Stillman's question was answered in the affirmative by Ananyan and Hochster [8]. However, the bounds they produce are far from optimal. Optimal bounds have been given in the some cases, such as the following:

1. When I is minimally generated by N quadrics and $\operatorname{ht}(I) = 2$, $\operatorname{pdim}(S/I) \leq 2N - 2$ [9].
2. When I is minimally generated by four quadrics, $\operatorname{pdim}(S/I) \leq 6$ [10].
3. When I is minimally generated by three cubics, $\operatorname{pdim}(S/I) \leq 5$. [11].

Definition 7.2.1. *Let R be a polynomial ring over a field and M be a finitely generated graded R-module. The* ***Castelnuovo-Mumford regularity*** *of M is defined as* $\operatorname{reg}_R(M) = \max\{j - i : \beta_{ij}(M) \neq 0\}$.

Note that $\text{reg}(R/I) = \text{reg}(I) - 1$ for an ideal $I \subset R$.
A question similar to Stillman's question can be asked on bounds on regularity.

Question 1: Let k be a field. Does there exist a bound, independent of n, on the regularity of an ideal in $S = \mathsf{k}[X_1, \ldots, X_n]$ which is generated by N forms of degrees $d_1, \ldots, d_N$?
In fact, as outlined below, question 1 is equivalent to Stillman's question, if k is infinite. In fact, even if k is finite, we can consider the algebraic closure of k and arrive at the same conclusion.

Suppose Stillman's question has an affirmative answer, that is, there is a bound $B = B(N, d_1, \ldots, d_N)$ such that $\text{pdim}(R/I) \leq B$ for any ideal $I \subset S = \mathsf{k}[x_1, \ldots, x_n]$ which is minimally generated by N forms of degree $d_1, \ldots, d_N$. By the Auslander-Buchsbaum formula, $\text{depth}(S/I) \geq n - B$. Let $\bar{f} = f_1, \ldots, f_{n-B}$ be a sequence of linear forms in S which is regular in S/I. Such a sequence can be chosen because k is infinite. Since S is a domain and $f_1, \ldots, f_{n-B}$ are linear forms, $f_1, \ldots, f_{n-B}$ is a regular sequence in R as well. Hence, $\text{reg}_S(S/I) = \text{reg}_{S/(\bar{f})}(S/(I + (\bar{f})))$.
Now, $S/(\bar{f})$ is a polynomial ring in B variables. There exists a bound on the regularity of $S/(I + (\bar{f}))$ in terms of $d(J) = \max\{d_1, \ldots d_N\}$ and the number of variables B ([12], Theorem 3.8).

Conversely, assume that question 1 can be answered in the positive, that is, there exists a bound $B = B(N, d_1, \ldots, d_N)$ such that $\text{reg}(I) \leq B$ for any ideal $I \subset S$ which is minimally generated by N forms of degree $d_1, \ldots, d_N$. Consider $\text{gin}_{\text{grevlex}}(I)$, the generic initial ideal of I with respect to the graded reverse lexicographic order. By a theorem of Bayer and Stillman ([1], Corollaries 19.11 and 20.21),

$$\text{pdim}(S/I) = \text{pdim}(S/\text{gin}_{\text{grevlex}}(I)), \text{reg}(S/I) = \text{reg}(S/\text{gin}_{\text{grevlex}}(I)).$$

Moreover, the projective dimension of $S/\text{gin}_{\text{grevlex}}(I)$ is the number of distinct variables appearing in all the monomials minimally generating $\text{gin}_{\text{grevlex}}(I)$. Observe that for any ideal J of S, we have that $d(J) \leq J$, where $d(J)$ denotes the maximal degree of a minimal generator of M. Hence,

$$\begin{aligned}
\text{pdim}(S/I) &= \text{pdim}(R/\text{gin}_{\text{grevlex}}(I)) \\
&= \textit{number of distinct variables appearing in generators of } \text{gin}_{\text{grevlex}}(I) \\
&\leq \textit{sum of degrees of generators of } \text{gin}_{\text{grevlex}}(I) \\
&\leq (\textit{number of generators of } \text{gin}_{\text{grevlex}}(I))d(\text{gin}_{\text{grevlex}}(I)) \\
&\leq (\textit{number of generators of } \text{gin}_{\text{grevlex}}(I))\text{reg}(\text{gin}_{\text{grevlex}}(I)) \\
&= (\textit{number of generators of } \text{gin}_{\text{grevlex}}(I))\textit{reg}(I) \\
&\leq (\textit{number of generators of } \text{gin}_{\text{grevlex}}(I))B(N, d_1, \ldots, d_N).
\end{aligned} \tag{7.1}$$

The ideal $\text{gin}_{\text{grevlex}}(I)$ is generated by the initial terms of the elements of a Gröbner basis of I, after a generic change of co-ordinates. Note that a change of co-ordinates on I does not change the number of generators of I or the degrees of those generators. Hence, without loss of generality, we can assume that I is in generic co-ordinates. To complete the proof, we need to bound the cardinality of a Gröbner basis of I in terms of $N, d_1, \ldots, d_N$.

In a process similar to Buchberger's algorithm, we can attain a Gröbner basis by adjoining S-pairs of the form

$$S(f,g) = \frac{\mathrm{LCM}(\mathrm{in}(f),\mathrm{in}(g))}{\mathrm{in}(f)}f - \frac{\mathrm{LCM}(\mathrm{in}(f),\mathrm{in}(g))}{\mathrm{in}(g)}g.$$

Starting with N generators, the maximum number of elements adjoined to the generating set on each iteration is a polynomial function in N. Further, $\deg(S(f,g)) \geq \max\{\deg(f),\deg(g)\}$ and this inequality is strict unless $\mathrm{in}(f)$ divides $\mathrm{in}(g)$ or vice-versa. On the other hand,

$$\begin{aligned} \deg(S(f,g)) &= \deg(\mathrm{in}(S(f,g))) \\ &= d(\mathrm{gin}_{\mathrm{grevlex}}(I)) \\ &\leq \mathrm{reg}(\mathrm{gin}_{\mathrm{grevlex}}(I)) \\ &= \mathrm{reg}(I) \\ &\leq B. \end{aligned} \tag{7.2}$$

This limits the possible iterations in terms of $N, d_1, \ldots, d_N$. The proof is thus complete.

7.3 Regularity of modules over a Koszul algebra

Lemma 7.3.1. *Let R and M be as in Definition 7.2.1. Then,* $\mathrm{reg}(M(-d)) = \mathrm{reg}(M) + d$.

The proof follows immediately from the projective resolution of $M(-d)$.
Observe that one can think of regularity of a module as the height of its Betti table. The fact that the Betti table of $M(-d)$ is d rows of zeroes above the Betti table of M gives us another method of verifying the above lemma.
Note that $\mathrm{reg}(M) = \max\{r : \exists i \textit{ such that } \mathrm{Tor}_i^R(M,\mathsf{k})_{i+r} \neq 0\}$, where $\mathsf{k} = R/R_+$.

Lemma 7.3.2. *Let R be a non-negatively graded ring and $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules. Then*
(i) $\mathrm{reg}(B) \leq \max\{\mathrm{reg}(A), \mathrm{reg}(C)\}$.
(ii) $\mathrm{reg}(C) \leq \max\{\mathrm{reg}(B), \mathrm{reg}(A) - 1\}$.
(iii) $\mathrm{reg}(A) \leq \max\{\mathrm{reg}(B), \mathrm{reg}(C) + 1\}$.

Proof. (i) Set $t = \mathrm{reg}(B)$. Then, there exists i such that $\mathrm{Tor}_i^R(B,\mathsf{k})_{i+t} \neq 0$. The induced long exact sequence on Tor modules gives us the following exact sequence

$$\mathrm{Tor}_{i+1}^R(C,\mathsf{k})_{i+t} \to \mathrm{Tor}_i^R(A,\mathsf{k})_{i+t} \to \mathrm{Tor}_i^R(B,\mathsf{k})_{i+t} \to \mathrm{Tor}_i^R(C,\mathsf{k})_{i+t} \to \mathrm{Tor}_{i-1}^R(A,\mathsf{k})_{i+t}.$$

Since $\mathrm{Tor}_i^R(B,\mathsf{k})_{i+t} \neq 0$, $\mathrm{Tor}_i^R(A,\mathsf{k})_{i+t} \neq 0$ or $\mathrm{Tor}_i^R(C,\mathsf{k})_{i+t} \neq 0$. Hence, $\mathrm{reg}(A) \geq t$ or $\mathrm{reg}(C) \geq t$.
(ii) Set $t = \mathrm{reg}(C)$. There exists i such that $\mathrm{Tor}_i^R(C,\mathsf{k})_{i+t} \neq 0$. From the above exact sequence, $\mathrm{Tor}_{i-1}^R(A,k)_{i+t} \neq 0$ or $\mathrm{Tor}_i^R(B,\mathsf{k})_{i+t} \neq 0$. Hence, $t + 1 \leq \mathrm{reg}(A)$ or $t \leq \mathrm{reg}(B)$.
(iii) Set $t = \mathrm{reg}(A)$. There exists i such that $\mathrm{Tor}_i^R(A,\mathsf{k})_{i+t} \neq 0$. From the above exact sequence, $\mathrm{Tor}_{i+1}^R(C,\mathsf{k})_{i+t} \neq 0$ or $\mathrm{Tor}_i^R(B,\mathsf{k})_{i+t} \neq 0$. Hence, $t - 1 \leq \mathrm{reg}(C)$ or $t \leq \mathrm{reg}(B)$. ∎

Lemma 7.3.3. *Let R be a non-negatively graded ring and M be a graded R-module. Suppose $x \in R_1$ is a nonzerodivisor on M. Then,* $\operatorname{reg}(M) = \operatorname{reg}(M/xM)$.

Proof. Consider the short exact sequence

$$0 \to M(-1) \xrightarrow{x} M \to M/xM \to 0.$$

Let $t = \operatorname{reg}(M)$ and $s = \operatorname{reg}(M/xM)$. Then, by Lemma 7.3.1, $\operatorname{reg}(M(-1)) = t + 1$. By (ii) of Lemma 7.3.2, $s \leq t$. Similarly, by (iii) of Lemma 7.3.2, $t + 1 \leq s + 1$. Hence, $s = t$. $\square$

If M is a graded $\mathsf{k}[x_1, \ldots, x_n]$-module of finite length, let $\max(M) = \max\{r : M_r \neq 0\}$.

Proposition 7.3.4. *Let $S = \mathsf{k}[x_1, \ldots, x_n]$, and let M be a graded S-module of finite length. Then,*

$$\operatorname{reg}_S(M) = \max\{r : M_r \neq 0\}.$$

Moreover, if $s = \operatorname{reg}_S(M)$,

$$\operatorname{Tor}_n^S(M, \mathsf{k})_{n+s} \neq 0.$$

Proof. Consider the Koszul complex as a resolution of k,

$$0 \to S(-n)^{b_n} \to S(-n+1)^{b_{n-1}} \to \cdots \to S(-1)^{b_1} \to S \to \mathsf{k} \to 0,$$

where $b_i = \binom{n}{i}$.
Let $s = \max(M)$. We have $\operatorname{Tor}_i^S(M, \mathsf{k}) \subset M(-i)^{b_i}$. Hence, $\max(\operatorname{Tor}_i^S(M, \mathsf{k})) \leq \max(M(-i)) = s + i$. Thus, $\operatorname{reg}(M) \leq s$. Further, note that

$$\operatorname{Tor}_n^S(M, \mathsf{k}) = \ker\left(M(-n) \xrightarrow{\begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix}} M(-n+1)^n\right).$$

Observe that $S_1 M_s \subset M_{s+1} = 0$. Hence, $0 \neq M(-n)_{s+n} \subset \operatorname{Tor}_n^S(M, \mathsf{k})$, which implies that $\operatorname{reg}_S(M) = s$ and $\operatorname{Tor}_n^S(M, \mathsf{k})_{n+s} \neq 0$. $\square$

Definition 7.3.5. *A **Koszul algebra** R is a graded k-algebra over which the residue field k has a linear resolution, that is,* $\operatorname{reg}_R(\mathsf{k}) = 0$.

Theorem 7.3.6. *Let R be a Koszul algebra, and let $S = \mathsf{k}[R_1] = \mathsf{k}[x_1, \ldots, x_n]$. The regularity of any module M over R is finite. In fact,*

$$\operatorname{reg}_R(M) \leq \operatorname{reg}_Q(M).$$

Proof. We first prove the theorem in the case that M has finite length. We proceed by induction on length of M.
In this case, $R/R_+ \cong \mathsf{k}$ injects into M via multiplication by an element of M, say, x of degree d. Let N be the cokernel of this map. We have the short exact sequence

$$0 \to k(-d) \xrightarrow{x} M \to N \to 0.$$

By Lemma 7.3.2, $\operatorname{reg}_R(M) \leq \operatorname{reg}_R(N)$ or $\operatorname{reg}_R(M) \leq \operatorname{reg}_R(k(-d)) = d$ (by Lemma 7.3.1). If $\operatorname{reg}_R(M) \leq \operatorname{reg}_R(N)$, we can apply the induction hypothesis to conclude that

$$\operatorname{reg}_R(M) \leq \operatorname{reg}_R(N) \leq \operatorname{reg}_Q(N) = \max(N) \leq \max(M) = \operatorname{reg}_Q(M).$$

On the other hand, if $\operatorname{reg}_R(M) \leq d$, then $\operatorname{reg}_R(M) \leq \max(M) = \operatorname{reg}_Q(M)$.

In the general case, we use Noetherian induction on the poset of submodules of M ordered by reverse inclusion. Hence, to prove that $\operatorname{reg}_R(M) \leq \operatorname{reg}_Q(M)$, it is sufficient to prove the following statement: Given a submodule $N \subset M$, if $\operatorname{reg}_R(M/N_1) \leq \operatorname{reg}_Q(M/N_1)$ for all $N_1 \supset N$, then $\operatorname{reg}_R(M/N) \leq \operatorname{reg}_Q(M/N)$. Without loss of generality, let $N = 0$.

If R_+ is not associated to M, then supposing as we may that k is infinite, there exists an element $x \in R_1$ such that x is a nonzerodivisor on M. The result now follows from the induction hypothesis and 7.3.3.

If M is not of finite length, but R_+ is associated to M, let M' be a maximal submodule of finite length contained in M and let $M'' = M/M'$ (note that $M' \neq 0$ because k injects into M). Then, R_+ is not associated to M''. Indeed, if R_+ was associated to M'', k would inject into M'' and hence M'' would contain a simple module, contradicting the maximality of M'. As in the proof of Proposition 7.3.4, $\operatorname{Tor}_n^S(M'', \mathsf{k}) = \operatorname{ann}''_M(S_+)(-n) = 0$. This implies that $\operatorname{Tor}_n^S(M, \mathsf{k}) = \operatorname{Tor}_n^S(M', \mathsf{k})$ and since $\operatorname{Tor}_n^S(M', \mathsf{k})_{n+\operatorname{reg}_S(M')} = 0$, we have $\operatorname{reg}_Q(M') \leq \operatorname{reg}_Q(M)$.

If $\operatorname{reg}_R(M'') \leq \operatorname{reg}_Q(M')$, by Lemma 7.3.2 and the finite length case treated above, we have that

$$\begin{aligned}\operatorname{reg}_R(M) &\leq \max\{\operatorname{reg}_R(M'), \operatorname{reg}_R(M'')\} \\ &\leq \operatorname{reg}_Q(M') \\ &\leq \operatorname{reg}_Q(M).\end{aligned} \tag{7.3}$$

If $\operatorname{reg}_R(M'') > \operatorname{reg}_Q(M')$, then by the induction hypothesis and the finite length case above,

$$\operatorname{reg}_R(M') \leq \operatorname{reg}_Q(M') < \operatorname{reg}_R(M'') \leq \operatorname{reg}_Q(M'').$$

Since $\operatorname{reg}_R(M') \leq \operatorname{reg}_R(M'')$, $\operatorname{reg}_R(M) \leq \operatorname{reg}_R(M'')$ by Lemma 7.3.2(i) and $\operatorname{reg}_R(M'') \leq \operatorname{reg}_R(M)$ by Lemma 7.3.2(ii). Hence, $\operatorname{reg}_R(M) = \operatorname{reg}_R(M'')$ and similarly, $\operatorname{reg}_Q(M) = \operatorname{reg}_Q(M'')$. By the induction hypothesis, we are done. $\square$

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