

Master's Thesis Stage 1

Homological Algebra

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Chapter 1

Graded Rings, Modules and Resolutions

1.1 Graded Rings and Modules

Definition 1.1.1. Let H be a cancellative monoid under addition. A ring R is said to be **H -graded** if $R = \bigoplus_{i \in H} R_i$, where, each R_i is an abelian group and $R_i R_j \subseteq R_{i+j}$, for all $i, j \in H$.

For each i , R_i is called the **homogeneous component of degree i** of R and the nonzero elements of R_i are called **homogeneous elements of degree i** .

Remark 1.1.2.

- (i) For a cancellative monoid H , we denote its associated group by G .
- (ii) By an ordered monoid we mean a cancellative monoid H with an order $<$ satisfying: whenever $a < b$ in H , we have $a + c < b + c$ for all $c \in H$.
- (iii) If H is an ordered monoid, then we say that it is well ordered if every nonempty subset S of H which is bounded below has the least element in S .

Definition 1.1.3. A module M is called as a **graded module** over a graded ring R if $M = \bigoplus_{i \in G} M_i$, as a direct sum of subgroups of M and for all $i \in H, j \in G$, $R_i M_j \subseteq M_{i+j}$.

Definition 1.1.4. An ideal J of a graded ring R is said to be **graded** if it satisfies any of the following equivalent conditions:

- (i) If $f \in J$, then every homogeneous component of f is in J .
- (ii) $J = \bigoplus_{i \in N} J_i$, where $J_i = R_i \cap J$.
- (iii) If J' is the ideal generated by all homogeneous elements in J , then $J = J'$.
- (iv) J has a system of homogeneous generators.

Proposition 1.1.5. Given a graded ideal I in a graded ring R , every associated prime of I is also graded.

Proof. Suppose $J = (I : x)$ is a prime ideal for some x in R . Let $x = x_l + x_{l+1} + \cdots + x_k$ where $x_i \in R_i$, $l < k$ and x_l, x_k are non-zero.

Let $y = y_t + y_{t+1} + \cdots + y_s \in J$, where $y_i \in R_i$, $t < s$ and x_t, x_s are non-zero. If we show that $y_t \in J$, we are done by induction on $s - t$.

To see this, observe that we have $xy \in I$ and since I is graded, the lowest graded component of xy , which is $x_t y_t$, belongs to I . Similarly, $x_{l+1} y_t + x_l y_{t+1} \in I$, and on multiplying by y_t , we get that $x_{l+1} y_t^2 \in I$. Continuing in this manner, we get that $x_{l+i} y_t^{i+1} \in I$ for all $i = 0, 1, \dots, k - l$, which implies that $y_t^{k-l+1} x \in I$ and hence, $y_t^{k-l+1} \in J$. Since J is prime, $y_t \in J$. Hence, J is a graded ideal. \square

Definition 1.1.6. Let R be a H -graded ring and $M = \bigoplus_{i \in G} M_i$ be a finitely generated R -module. Then we define an R -module $M(d)$ by $M(d) = \bigoplus_{i \in G} M_{i+d}$. $M(d)$ is called a **shifted R -module**.

Definition 1.1.7. Let $M = \bigoplus_{i \in G} M_i$, $M' = \bigoplus_{i \in G} M'_i$ be graded modules over R . An R -linear map $f : M \rightarrow M'$ is said to be a **graded map of degree d** if $f(M_i) \subseteq M'_{i+d}$ for all $i \in G$. If f has degree zero, we simply say that f is a **graded R -module homomorphism**.

Proposition 1.1.8. Let R be nonnegatively graded, M, N be graded R -modules and $\phi : M \rightarrow N$ be a graded homomorphism of degree d . Then

- (i) $\ker(\phi)$ is a graded submodule of M .
- (ii) $\text{Im}(\phi)$ is a graded submodule of N .

Proof. (i) It is clear that $\ker(\phi)$ is a submodule of M considered without grading. To show that $\ker(\phi)$ is graded, it suffices to show that if $x = x_r + \cdots + x_s$, is in $\ker(\phi)$, then each x_i is in $\ker(\phi)$. We show that $x_r \in \ker(\phi)$ and by induction we will get that $x_i \in \ker(\phi)$ for all i . Note that $\phi(x_i) \in N_{i+d}$. Therefore $\phi(x_r) \in N_{r+d} \cap (N_{(r+1)+d} \oplus \cdots \oplus N_{s+d}) = 0$. This shows that $\phi(x_r) = 0$ as desired.

(ii) It is clear that $\text{Im}(\phi)$ is a submodule of N considered without grading. To show that $\text{Im}(\phi)$ is graded, it suffices to show that if $y = y_r + \cdots + y_s$, is in $\text{Im}(\phi)$, then each y_i is in $\text{Im}(\phi)$. Since $\phi(M_i) \subseteq N_{i+d}$ and $y \in \text{Im}(\phi)$, there exists $x = x_{r-d} + \cdots + x_{s-d} \in M$ such that $\phi(x) = y$ and $\phi(x_{i-d}) = y_i$. This shows that $y_i \in \text{Im}(\phi)$. This completes the proof. \square

Remark 1.1.9.

- (i) If I is a graded ideal of R , then we have $R_i I_j \subseteq I_{i+j}$.
- (ii) If I is a graded ideal of R , then the quotient ring R/I inherits the grading from R by $(R/I)_i = R_i/I_i$.
- (iii) If N is a graded submodule of a graded module M , then M/N is graded with the grading given by $(M/N)_i = M_i/N_i$.

Proposition 1.1.10. Tensor products of graded R -modules is graded, i.e., if M and N are graded R -modules, then $M \otimes N$ is graded R -module.

Proof. We know that $M \otimes N$ is an R -module. We give grading to $M \otimes N$ as follows: Define $(M \otimes N)_i$ to be generated (as a \mathbb{Z} -module) by all the elements in $M \otimes N$ of the form $m \otimes n$,

where $\deg(m) + \deg(n) = i$. Then we have $M \otimes N = \bigoplus_{i \in G} (M \otimes N)_i$. Moreover, for any $r_i \in R_i$ and $m \oplus n \in (M \otimes N)_j$, we have $r(m \otimes n) = (rm) \otimes n$. Therefore

$$\deg(r(m \otimes n)) = (i + \deg(m)) + \deg(n) = i + j.$$

This shows that $R_i(M \otimes N)_j \subseteq (M \otimes N)_{i+j}$. Hence $M \otimes N$ is graded. \square

Let $\text{Hom}_i(M, N) = \{\phi : M \rightarrow N \mid \deg(\phi) = i\}$. Then we define ${}^*\text{Hom}(M, N) = \bigoplus_{i \in G} \text{Hom}_i(M, N)$.

Remark 1.1.11. In general, ${}^*\text{Hom}(M, N) \neq \text{Hom}(M, N)$. However, we have the equality in a special case which we will prove shortly.

Lemma 1.1.12. Let $M = \bigoplus_{i=1}^m R(n_i)$ and N be graded R -modules. Then ${}^*\text{Hom}(M, N) \cong \text{Hom}(M, N)$ with grading forgotten.

Proof. It is clear that every $\phi = \phi_r + \cdots + \phi_s \in {}^*\text{Hom}(M, N)$ is in $\text{Hom}(M, N)$, and hence ${}^*\text{Hom}(M, N) \subseteq \text{Hom}(M, N)$. To show the other inclusion assume that $\phi \in \text{Hom}(M, N)$. Let $e_j = (0, \dots, 0, 1, 0, \dots, 0)$ where 1 occurs at j th place. Then M is a free R -module with basis $\{e_1, \dots, e_m\}$. If $\phi(e_j) = y_{j1} + \cdots + y_{jr_j} \in N$, then we have

$$\phi = \phi_{11} + \cdots \phi_{1r_1} + \cdots + \phi_{m1} + \cdots + \phi_{mr_m}$$

where $\phi_{js} : M \rightarrow N$ is given by $\phi_{js}(e_j) = y_{js}$ and $\phi_{js}(e_i) = 0$ for all $i \neq j$. Note that each ϕ_{js} is well defined since $\{e_1, \dots, e_m\}$ is a basis for M . Moreover ϕ_{js} is a graded homomorphism of degree $\deg(y_{js}) + n_j$. Therefore $\phi \in {}^*\text{Hom}(M, N)$. This completes the proof. \square

Proposition 1.1.13. Let R be a graded Noetherian ring, M be a finitely generated graded R -module and N be any graded R -module. Then ${}^*\text{Hom}(M, N) = \text{Hom}(M, N)$ with grading forgotten.

Proof. It is clear that every $\phi = \phi_r + \cdots + \phi_s \in {}^*\text{Hom}(M, N)$ is in $\text{Hom}(M, N)$, and hence we have an inclusion ${}^*\text{Hom}(M, N) \xrightarrow{i} \text{Hom}(M, N)$.

Since M is finitely generated and R is Noetherian, we get an exact sequence of graded modules $G \rightarrow F \rightarrow M \rightarrow 0$ for some $F = \bigoplus_{j=1}^n R(n_j)$ and $G = \bigoplus_{j=1}^m R(m_j)$. By the previous lemma we have ${}^*\text{Hom}(F, N) = \text{Hom}(F, N)$, ${}^*\text{Hom}(G, N) = \text{Hom}(G, N)$. Thus we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & {}^*\text{Hom}(M, N) & \longrightarrow & {}^*\text{Hom}(F, N) & \longrightarrow & {}^*\text{Hom}(G, N) \\ & & \downarrow i & & \parallel & & \parallel \\ 0 & \longrightarrow & \text{Hom}(M, N) & \longrightarrow & \text{Hom}(F, N) & \longrightarrow & \text{Hom}(G, N) \end{array}$$

Thus by five lemma, we get that the inclusion i is an isomorphism. \square

Lemma 1.1.14 (Graded Nakayama Lemma). *Let H be an ordered monoid such that $i > 0$ for all $i \in H \setminus \{0\}$ and $R = \bigoplus_{i \in H} R_i$ be a graded ring. Let $M = \bigoplus_{i \in G} M_i$ be an R -module such that there exists $n \in G$ with $M_i = 0$ for all $i < n$. Further assume that G is well ordered. If $R_+ = \bigoplus_{i \in H \setminus \{0\}} R_i$ and $R_+M = M$ then $M = 0$.*

Proof. Let, if possible, $M \neq 0$. Let m be the smallest element of G such that for all $i < m$, we have $M_i = 0$ and $M_m \neq 0$. But then, $M = R_+M \subseteq \bigoplus_{i < m} M_i$, which has m^{th} component equal to 0. This contradiction shows that $M = 0$. \square

Corollary 1.1.15. *Let R be a non negatively graded ring and M be a finitely generated \mathbb{Z} -graded R -module. If $R_+M = M$ then $M = 0$.*

Proof. Let $\{m_1, \dots, m_r\}$ be a generating set for M and $d = \min\{\deg(m_i) \mid 1 \leq i \leq r\}$. Since R is graded by $\mathbb{N} \cup \{0\}$, we get that $M_n = 0$, for every $n < d$. Thus, applying graded Nakayama lemma proved above, we get $M = 0$. \square

1.2 Graded Resolutions

From now on we assume that R is a graded ring with $R_0 = \mathbf{k}$, a field. We will mostly consider $R = \mathbf{k}[x_1, \dots, x_r]$.

Definition 1.2.1. *Let M be a graded R -module and*

$$F_\bullet : \dots \rightarrow F_n \xrightarrow{\phi_n} F_{n-1} \xrightarrow{\phi_{n-1}} \dots \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \rightarrow 0$$

*be a free resolution of M . If all F_i 's are graded R -modules and all ϕ_i 's are graded maps of degree zero, then we say that F_\bullet is a **graded free resolution** of M .*

Definition 1.2.2. Let $R = \mathbf{k}[x_1, \dots, x_n]$ and M be a graded R -module. A graded free resolution

$$F_\bullet : \dots \rightarrow F_n \xrightarrow{\phi_n} \dots \rightarrow F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \rightarrow 0$$

is said to be *minimal* if $\phi_i(F_i) \subseteq \langle x_1, \dots, x_r \rangle F_{i-1}$ for all $i \geq 1$.

Example 1.2.3. Let $I = \langle x^2, y^2 \rangle$ and $R = \mathbf{k}[x, y]$. Then

$$F_\bullet : 0 \leftarrow R/I \xleftarrow{\phi_0} R \xleftarrow{\phi_1} R(-2) \oplus R(-2) \xleftarrow{\phi_2} R(-4) \leftarrow 0,$$

where $\phi_0(1) = 1$, $\phi_1(1, 0) = x^2$, $\phi_1(0, 1) = y^2$, $\phi_2(1) = (-y^2, x^2)$ is a minimal graded free resolution of R/I over R .

Example 1.2.4. Let $I = \langle x^3, y^2 \rangle$ and $R = \mathbb{k}[x, y]$. Then

$$F_{\bullet} : 0 \leftarrow R/I \xleftarrow{\phi_0} R \xleftarrow{\phi_1} R(-3) \oplus R(-2) \xleftarrow{\phi_2} R(-5) \leftarrow 0,$$

where $\phi_0(1) = 1$, $\phi_1(1, 0) = x^3$, $\phi_1(0, 1) = y^2$, $\phi_2(1) = (-y^2, x^3)$ is a minimal graded free resolution of R/I over R .

Definition 1.2.5. Let $R = \mathbb{k}[x_1, \dots, x_n]$ and M be a graded R -module.

$$F_{\bullet} : \dots \rightarrow F_n \xrightarrow{\phi_n} F_{n-1} \xrightarrow{\phi_{n-1}} \dots \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \rightarrow 0$$

be a minimal graded free resolution of M , where $F_i = \bigoplus_j R(-j)^{\beta_{i,j}(M)}$. Then the numbers $\beta_{i,j}(M)$ are called **graded Betti numbers** of M . $\beta_i(M) = \sum_j \beta_{i,j}(M)$ is called the total i th Betti number of M .

Definition 1.2.6. Let $\beta_{i,j}$ be graded Betti numbers of M . Then **Betti table** of M is written as

$j \backslash i$	0	1	\dots	p	\dots
\vdots	\vdots	\vdots	\dots	\vdots	\vdots
0	$\beta_{0,0}$	$\beta_{1,1}$	\dots	$\beta_{p,p}$	\vdots
1	$\beta_{0,1}$	$\beta_{1,2}$	\dots	$\beta_{p,p+1}$	\vdots
\vdots	\vdots	\vdots	\dots	\vdots	\vdots

Definition 1.2.7. Let \mathbb{k} be a field and $M = \bigoplus_{n \in \mathbb{Z}} M_n$ be a finitely generated graded module over the polynomial ring $\mathbb{k}[x_1, \dots, x_r]$. Then the function $H_M : \mathbb{Z} \rightarrow \mathbb{Z}$, given by $H_M(j) = \dim_{\mathbb{k}}(M_j)$ is called as **Hilbert function of M** .

Let \mathbb{k} be a field and $R = \mathbb{k}[x_1, \dots, x_n]$. Let $a \in R \setminus \{0\}$ be such that $\deg(a) = d$. Since a is a nonzerodivisor on R , we get an exact sequence of R -modules

$$0 \rightarrow R(-d) \xrightarrow{a} R \rightarrow R/\langle a \rangle \rightarrow 0.$$

Since R is graded, for each i , we have an exact sequence of \mathbb{k} -vector spaces

$$0 \rightarrow R(-d)_i \xrightarrow{a} R_i \rightarrow [R/\langle a \rangle]_i \rightarrow 0.$$

Now, using rank-nullity theorem for vector spaces, we get

$$\dim_{\mathbb{k}}(R_i) = \dim_{\mathbb{k}}((R(-d))_i) + \dim_{\mathbb{k}}((R/\langle a \rangle)_i),$$

i.e.,

$$H_R(i) = H_{R(-d)}(i) + H_{R/\langle a \rangle}(i).$$

Therefore, $H_R(i) = H_R(i-d) + H_{R/\langle a \rangle}(i)$ or $H_{R/\langle a \rangle}(i) = H_R(i) - H_R(i-d)$.

Definition 1.2.8. Given k, M as above, define the **Hilbert series** of M as $H_M(t) = \sum_{j \geq 0} H_M(j)t^j$.

The next corollary follows from the above definition.

Corollary 1.2.9. $H_{R/\langle a \rangle}(t) = H_R(t)/(1-t)^d$.

Example 1.2.10. Let $R = k[x, y]$ and $a = x^2$. In this case, for all $i \geq 0$, we have $H_R(i) = i + 1$. This is because the i th graded component of R , as a k -vector space is has a basis $\{x^r y^{i-r} \mid 0 \leq r \leq i\}$. For the element x^2 , we have $\deg(x^2) = 2$. Hence, by the formula above, we must have $H_{R/\langle x^2 \rangle}(i) = (i + 1) - (i - 1) = 2$; which is true as $\{\overline{xy^{i-1}}, \overline{y^i}\}$ form a k -vector space basis of $(R/\langle x^2 \rangle)_i$.

Proposition 1.2.11. Let M, N be graded R -modules. Then $\text{Tor}_i^R(M, N)$ is graded for all i .

Proof. Consider a graded free resolution of M as follows:

$$\cdots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0.$$

Tensoring with N gives a complex of graded modules

$$\cdots \rightarrow F_2 \otimes N \rightarrow F_1 \otimes N \rightarrow F_0 \otimes N \rightarrow M \otimes N \rightarrow 0.$$

Since $\text{Tor}_i^R(M, N)$ is quotient of a graded submodule of a graded module by a graded submodule, we conclude that $\text{Tor}_i^R(M, N)$ is graded for all i . \square

Remark 1.2.12. If F_\bullet is a graded free resolution of M then we define ${}^*\text{Ext}_R^i(M, N) \cong H^i({}^*\text{Hom}_R(F_\bullet, N))$. Then, by Proposition 1.1.13, if R is Noetherian local ring and M is finitely generated R -module, then ${}^*\text{Ext}_R^i(M, N) \cong \text{Ext}_R^i(M, N)$.

Chapter 2

Gröbner Bases

Let k be a field and $S = k[x_1, \dots, x_r]$.

If $a = (a_1, \dots, a_r)$, x^a will denote the monomial $x_1^{a_1} \dots x_r^{a_r}$. As is convention, an ideal of S generated by monomials will be referred to as a monomial ideal.

Definition 2.0.1. Let F be a finitely generated free module over S with basis $\{e_1, \dots, e_n\}$.

A **monomial** in F is an element of the form $m = x^a e_i$ for some i . We say that such an m involves the basis element e_i .

A **monomial submodule** of F is a submodule generated by elements of this form. Any monomial submodule M of F may be written as

$$M = \oplus I_j e_j \subseteq \oplus S e_j = F,$$

with I_j the monomial ideal generated by those monomials m such that $m e_j \in M$.

A **term** in F is a monomial multiplied by a scalar.

Definition 2.0.2. Let F be a finitely generated free module over S with basis $\{e_1, \dots, e_n\}$.

If m, n are monomials of S , $u, v \in k$, and $v \neq 0$, then we say that the term $u m e_i$ is divisible by the term $v n e_j$ if $i = j$ and m is **divisible** by n in S ; the quotient is then $u m / v n \in S$.

Definition 2.0.3. The set of monomials in M that are minimal elements in the partial order by divisibility on the monomials of F . We will refer to the monomials in this set as **minimal generators** of M .

2.1 Hilbert Function of Monomial Submodules

Let F be a free S -module with basis $\{e_i : i = 1, \dots, n\}$, and let $M \subseteq F$ be a monomial submodule. Since, as seen before, $M = \oplus I_j e_j$, we have $F/M = \oplus S/I_j$ and, since the Hilbert function is additive, it suffices to handle the case $F = S$ and $M = I$, where I is a monomial ideal.

Choosing one of the monomial generators f of I , and letting I' be the monomial ideal generated by the remaining generators, we have the following graded exact sequence:

$$0 \rightarrow S/(I' : f)(-d) \xrightarrow{f} S/I' \rightarrow S/I \rightarrow 0,$$

where d is the degree of f . If $I' = (f_1, f_2, \dots, f_t)$, then

$$(I' : f) = (f_1/\text{GCD}(f_1, f), f_2/\text{GCD}(f_2, f), \dots, f_t/\text{GCD}(f_t, f)).$$

For every integer n ,

$$H_{S/I}(n) = H_{S/I'}(n) - H_{S/(I':f)}(n).$$

Note that both I' and $(I' : f)$ have fewer minimal generators than I , and hence, using induction, we can compute an expression for the Hilbert function or polynomial of I .

By choosing f sensibly, we can make the process much faster: If f contains the largest power of some variable x_1 of any of the minimal generators of I , then the minimal generators of the resulting ideal $(I' : f)$ will not involve x_1 at all. They will thus involve strictly fewer of the variables than the number involved in the minimal generators of I .

2.2 Syzygies of Monomial Submodules

Let F be a free module and let M be a submodule of F generated by monomials m_1, \dots, m_t . Define

$$\phi : \bigoplus_{j=1}^t S\epsilon_j \rightarrow F; \phi(\epsilon_j) = m_j.$$

For each pair of indices i, j such that m_i and m_j involve the same basis element of F , we define

$$m_{ij} = m_i/\text{GCD}(m_i, m_j),$$

and we define σ_{ij} to be the element of $\ker(\phi)$ given by

$$\sigma_{ij} = m_{ji}\epsilon_i - m_{ij}\epsilon_j.$$

Lemma 2.2.1. *With notation as above, $\ker(\phi)$ is generated by the set of all σ_{ij} , wherever defined.*

Proof. As a vector space over k , $\ker(\phi) = \bigoplus_f \ker(\phi)_f$, where

$$\ker(\phi)_f = \left\{ \sum_i a_i f_i \epsilon_i \in \ker(\phi) : m_i \text{ divides } f, f_i = f/m_i, a_i \in k \right\}.$$

Indeed, let

$$\sigma = \sum_i p_i \epsilon_i \in \ker(\phi).$$

For any monomial f that occurs in one of the $p_j m_j$, and for each i , let $p_{i,f}$ be the term of p_i such that $p_{i,f} m_i$ is a scalar times f . Then,

$$\sum_i p_i m_i = 0 \implies \sum_i \sum_f p_{i,f} m_i = 0 \implies \sum_f \sum_i p_{i,f} m_i = 0 \implies \forall f, \sum_i p_{i,f} m_i = 0.$$

Therefore, for all monomials f , $\sum_i p_{i,f} \epsilon_i \in \ker(\phi)$.

We may now assume $\sigma = \sum_i a_i f_i \epsilon_i$ for some monomial f of F . If $\sigma = 0$, σ lies in the module generated by σ_{ij} . If $\sigma \neq 0$, at least two of the $a_i f_i$ must be non-zero, since $\sum_i a_i f_i m_i = 0$. This implies that for some i, j , both m_i and m_j must divide f and in fact, $m_i f_i = m_j f_j = f$, which implies that $m_{ji} = m_j / \text{GCD}(m_i, m_j)$ divides f_i . Let $k = f_i / m_{ji}$, then $k \sigma_{ij} \in \ker(\phi)_f$, and $\sigma - a_i k \sigma_{ij}$ has fewer non-zero terms than σ . Hence, the proof is complete by induction on number of non-zero terms of σ . \square

Example 2.2.2. Let $S = \mathbf{k}[x, y]$, $F = S^2$, $M = \langle (x^2, 0), (0, xy), (0, y^3) \rangle$. Then we have

$$\phi : \oplus_{j=1}^3 S \epsilon_j \rightarrow F; \phi(\epsilon_1) = (x^2, 0), \phi(\epsilon_2) = (0, xy), \phi(\epsilon_3) = (0, y^3).$$

Suppose for some $a_1, a_2, a_3 \in S$, $\phi(a_1 \epsilon_1 + a_2 \epsilon_2 + a_3 \epsilon_3) = 0$, then we have $(a_1 x^2, a_2 xy + a_3 y^3) = 0$, and hence, $a_1 = 0$, $a_2 = by^2$, $a_3 = -bx$. Thus, $a_1 \epsilon_1 + a_2 \epsilon_2 + a_3 \epsilon_3 = b(0, y^2 - x) = b \sigma_{23}$.

2.3 Monomial Orders

Let I be an ideal of S , J be a monomial ideal of S and B be the set of all monomials not in J . Then, the elements of B are \mathbf{k} -linearly independent modulo I if and only if J contains at least one monomial from every polynomial in I .

Indeed, suppose J contains no monomial of $f \in I$, $f \neq 0$. Then $f \in \text{Span}(B) \cap I$, which implies that the elements of B are linearly dependent modulo I . Conversely, suppose there exist $a_1, \dots, a_n \in \mathbf{k}$ and $m_1, \dots, m_n \in B$ such that $\sum_{i=1}^n a_i m_i \in I$, then $\sum_{i=1}^n a_i m_i$ is a polynomial in I for which no monomials belong to J .

Moreover, if B is a basis of S/I , J must be a minimal monomial ideal containing at least one monomial from every polynomial in I . Indeed, suppose J contains at least one monomial from each polynomial in I , but is not a minimal ideal satisfying this condition. Let $J_1 \subsetneq J$ satisfying the condition, and let $f \in J \setminus J_1$, where f is a monomial. Suppose $f \in \text{Span}(B)$, that is, there exist $a_1, \dots, a_n \in \mathbf{k}$ and $m_1, \dots, m_n \in B$ such that $f - \sum_{i=1}^n a_i m_i \in I$. Since J_1 contains at least one monomial of every polynomial in I , we have a contradiction. Hence, B cannot span S/I if J is not the minimal monomial ideal containing one monomial from each polynomial in I .

Definition 2.3.1. Let F be a free S -module. A **monomial order** on F is a total order τ on the monomials of F such that the following two conditions are satisfied:

- (i) if m_1 is a monomial of F and $f \neq 1$ is a monomial of S , then $f m_1 >_\tau m_1$.
- (ii) if m_1, m_2 are monomials of F and $f \neq 1$ is a monomial of S , then $m_1 >_\tau m_2$ implies $f m_1 >_\tau f m_2$.

Lemma 2.3.2 (Well-Ordering Property). *Let F be a free S -module. The set of monomials in F is well-ordered with respect to any monomial order (every subset has a least element).*

Proof. Let $X \subseteq F$ be a set of monomials. Since S is Noetherian, the submodule of F generated by X must be generated by a finite subset of X , say, Y . Since Y is a finite set of monomials, it must have a least element with respect to a monomial order. The least element of Y must be the least element of X because every element of X is an element in Y multiplied by a monomial in S . \square

We will extend this notation to terms: If um_1 and vm_2 are terms with $0 \neq u, v \in \mathbf{k}$, and m_1, m_2 are monomials with $m_1 >_\tau m_2$ then we say $um_1 >_\tau vm_2$.

Definition 2.3.3. *Let F be a free S -module and τ be a monomial order on F . For any $f \in F$, we define the **initial term** of f , denoted by $\text{in}_\tau(f)$ to be the greatest term of f with respect to the order τ . Given a submodule M of F , define the **initial submodule** of M , denoted by $\text{in}_\tau(M)$, to be the monomial submodule generated by $\text{in}_\tau(f)$ for all $f \in M$.*

Theorem 2.3.4 (Macaulay). *Let F be a free S -module and M be a submodule of F . For any monomial order τ on F , the set B of all monomials not in $\text{in}_\tau(M)$ forms a \mathbf{k} -basis for F/M .*

Proof. Suppose the set B is not linearly independent. Then there exist distinct $m_1, \dots, m_t \in B$ and $(a_1, \dots, a_t) \in \mathbf{k}^t \setminus \{0\}$ such that $f := a_1m_1 + \dots + a_tm_t \in M$. Since $\text{in}(f) \in \text{in}(M)$, there must exist $i \in \{1, \dots, t\}$ such that $m_i \in \text{in}(M)$, which is a contradiction.

Suppose B does not span F/M . Let $f \in F \setminus (M + \text{Span}(B))$ such that f has minimal initial term among all elements of $F \setminus (M + \text{Span}(B))$. We can choose such an f by the well-ordering property. If $\text{in}(f) \in \text{Span}(B)$, $f - \text{in}(f) \in F \setminus (M + \text{Span}(B))$ has smaller initial term than f . Hence, $\text{in}(f) \in \text{in}(M)$. However, this implies that there exists $g \in M$ such that $\text{in}(f) = \text{in}(g)$, and $f - g \in F \setminus (M + \text{Span}(B))$ has smaller initial term than f , leading to a contradiction. \square

Corollary 2.3.5. *Given F, M, τ as above, $\dim_{\mathbf{k}}(F/M) = \dim_{\mathbf{k}}(F/\text{in}_\tau(M))$.*

Corollary 2.3.6. *Given monomial orders τ, γ on S and an ideal $I \in S$ such that $\text{in}_\tau(I) \subset \text{in}_\gamma(I)$, we have $\text{in}_\tau(I) = \text{in}_\gamma(I)$.*

Proof. If $\text{in}_\tau(I) \subsetneq \text{in}_\gamma(I)$, the set of monomials in $S \setminus \text{in}_\gamma(I)$ is a proper subset of the set of monomials in $S \setminus \text{in}_\tau(I)$. However, both these sets of monomials form a K -basis of S/I , which is a contradiction. \square

Here are some important examples of monomial orders when $F = S$. Let $a = (a_1, \dots, a_r), b = (b_1, \dots, b_r)$ and $m = x^a, m' = x^b$

Lexicographic order: $m >_{\text{lex}} m'$ if and only if $a_i > b_i$ for the smallest i such that $a_i \neq b_i$.

Graded lexicographic order: $m >_{\text{grlex}} m'$ if and only if $\deg(m) > \deg(n)$ or $\deg(m) = \deg(n)$ and $a_i > b_i$ for the smallest i such that $a_i \neq b_i$.

Reverse graded lexicographic order: $m >_{\text{grevlex}} m'$ if and only if $\deg(m) > \deg(n)$ or $\deg(m) = \deg(n)$ and $a_i < b_i$ for the largest i such that $a_i \neq b_i$.

Remark 2.3.7. A “reverse lexicographic order” is not a monomial order, because 1 is not the least monomial. In fact, 1 is the largest monomial.

Definition 2.3.8. A **Gröbner basis** with respect to an order τ on a free module F is a set of elements $g_1, \dots, g_t \in F$ such that if M is the submodule of F generated by g_1, \dots, g_t , then $\text{in}_\tau(g_1), \dots, \text{in}_\tau(g_t)$ generate $\text{in}_\tau(M)$. We then say that g_1, \dots, g_t is a **Gröbner basis of M** .

There is a Gröbner basis of any submodule M of F , with respect to any monomial order: if g_1, \dots, g_t is a set of generators of M which is not a Gröbner basis, we can adjoin $g_{t+1}, \dots, g_{t'}$ until $\text{in}(g_1), \dots, \text{in}(g_{t'})$ generate $\text{in}(M)$ (note that the Hilbert basis theorem implies that this can be done).

Lemma 2.3.9. Let $N \subset M \subset F$ be submodules such that $\text{in}(N) = \text{in}(M)$ with respect to a given monomial order. Then, $N = M$.

Proof. Suppose $N \neq M$, then, by the well-ordering property, there exists $f \in M \setminus N$ such that f has the least initial term among all the elements of M not in N . Since $f \in M$, we have $\text{in}(f) \in \text{in}(M) = \text{in}(N)$, which implies the existence of $g \in N$ such that $\text{in}(f) = \text{in}(g)$. Note that $f - g \in M \setminus N$, but has smaller initial term than f , which is a contradiction to the choice of f . \square

The above lemma tells us that if $\langle \text{in}(g_1), \dots, \text{in}(g_t) \rangle = \text{in}(M)$, then $\langle g_1, \dots, g_t \rangle = M$.

2.4 Computing Syzygies

Proposition 2.4.1 (Division Algorithm). Let F be a free S -module with monomial order τ . If $f, g_1, \dots, g_t \in F$, then there is an expression

$$f = \sum_{i=1}^t f_i g_i + f' \text{ with } f' \in F, f_i \in S,$$

where none of the monomials of f' is in $\langle \text{in}(g_1), \dots, \text{in}(g_t) \rangle$ and $\text{in}(f) \geq_\tau \text{in}(f_i g_i)$ for every i .

Definition 2.4.2. With notation as above, any such f' is called a **remainder** of f with respect to g_1, \dots, g_t , and an expression $f = \sum f_i g_i + f'$ satisfying the condition of the proposition is called a **standard expression** for f in terms of the g_i .

The proof outlines an algorithm to attain a standard expression for any $f \in F$.

Proof. If $f, g_1, \dots, g_t \in F$, then we may produce a standard expression

$$f = \sum m_u g_{s_u} + f'$$

for f with respect to g_1, \dots, g_t by defining the indices s_u and the terms m_u inductively. Having chosen s_1, \dots, s_p and m_1, \dots, m_p , if

$$f'_p := f - \sum_{u=1}^p m_u g_{s_u} \neq 0$$

and m is the maximal term of f'_p ; that is divisible by $\text{in}(g_i)$ for some i , then choose $s_{p+1} = i$, $m_{p+1} = m/\text{in}(g_i)$. This process terminates when either $f'_p = 0$ or no $\text{in}(g_i)$ divides a monomial of f ; the remainder f' is then the last f'_p produced.

Note that the well-ordering property guarantees that this process must terminate, because the maximal term of f'_p divisible by some g_i decreases at each step. \square

Fix the following notation:

Let F be a free module over S with monomial order τ . Let g_1, \dots, g_t be non-zero elements of F , and let $\oplus S\epsilon_i$ be a free module with basis $\{\epsilon_1, \dots, \epsilon_t\}$.

For two terms $m_1, m_2 \in F$, $m_1 < m_2$ denotes that the monomial corresponding to m_1 is less than the monomial corresponding to m_2 with respect to the order τ .

For each pair of indices i, j such that $\text{in}(g_i)$ and $\text{in}(g_j)$ involve the same basis element of F , we define

$$m_{ij} = \text{in}(g_i) / \text{GCD}(\text{in}(g_i), \text{in}(g_j)) \in S,$$

and we set $\sigma_{ij} = m_{ji}\epsilon_i - m_{ij}\epsilon_j$ for $i < j$.

For each such pair i, j , choose a standard expression

$$m_{ji}g_i - m_{ij}g_j = \sum_{u=1}^t f_u^{(ij)} g_u + h_{ij}$$

for $m_{ji}g_i - m_{ij}g_j$ with respect to g_1, \dots, g_t . Note that $\text{in}(f_u^{(ij)} g_u) < \text{in}(m_{ji}g_i)$.

Set $h_{ij} = 0$ if $\text{in}(g_i)$ and $\text{in}(g_j)$ involve different basis elements of F .

Define $\phi : \oplus S\epsilon_i \rightarrow F$, $\phi(\epsilon_i) = g_i$. Then, the set of σ_{ij} generate the syzygies on the module generated by the elements $\text{in}(g_i)$ (by Lemma 2.2.1). Note that $\phi(\sigma_{ij}) = m_{ji}g_i - m_{ij}g_j$.

Theorem 2.4.3 (Buchberger's Criterion). *The elements g_1, \dots, g_t form a Gröbner basis if and only if $h_{ij} = 0$ for all i and j .*

Proof. Let $M = \langle g_1, \dots, g_t \rangle \subset F$. The expression for h_{ij} implies that $h_{ij} \in M$, and hence $\text{in}(h_{ij}) \in \text{in}(M)$. However, if g_1, \dots, g_t is a Gröbner basis, the definition of a standard expression forces $h_{ij} = 0$ for all i, j .

Conversely, suppose that $h_{ij} = 0$ for all i, j . Let $f = \sum_{i=1}^t h_i g_i \in M$, where, among all possible h_1, \dots, h_t such that $f = \sum_{i=1}^t h_i g_i$, h_1, \dots, h_t are chosen such that $\max\{\text{in}(h_i g_i) : 1 \leq i \leq t\}$ is minimal. We prove that $\text{in}(f) \in \langle \text{in}(g_1), \dots, \text{in}(g_t) \rangle$.

If $\text{in}(f) = \text{in}(h_i g_i)$ for some i , $\text{in}(g_i) | \text{in}(f) \Rightarrow \text{in}(f) \in \langle \text{in}(g_1), \dots, \text{in}(g_t) \rangle$.

Hence, let $\text{in}(f) < \max\{\text{in}(h_i g_i) : 1 \leq i \leq t\} = m$. Define an equivalence relation \equiv on terms as follows: $m_1 \equiv m_2$ if there exists $\lambda \in \mathbf{k} \setminus \{0\}$ such that $m_1 = \lambda m_2$. Without loss of generality, suppose $\text{in}(h_i g_i) \equiv m$ for $i = 1, \dots, t_1$ and $\text{in}(h_i g_i) < m$ for $i = t_1 + 1, \dots, t$

$$\begin{aligned} f &= \sum_{i=1}^t h_i g_i = \sum_{i=1}^{t_1} h_i g_i + \sum_{i=t_1+1}^t h_i g_i \\ &= \sum_{i=1}^{t_1} \text{in}(h_i) g_i + \sum_{i=1}^{t_1} (h_i - \text{in}(h_i)) g_i + \sum_{i=t_1+1}^t h_i g_i. \end{aligned}$$

Note that $\sum_{i=1}^{t_1} \text{in}(h_i) \text{in}(g_i) = 0$.

Define $\phi_1 : \oplus S \epsilon_j \rightarrow M$, $\phi_1(\epsilon_j) = \text{in}(g_j)$ and $\phi_2 : \oplus S \epsilon_j \rightarrow M$, $\phi_2(\epsilon_j) = g_j$. Note that $\sum_{i=1}^{t_1} \text{in}(h_i) \epsilon_i \in \ker(\phi_1)$. Therefore, by Lemma 2.2.1,

$$\sum_{i=1}^{t_1} \text{in}(h_i) \epsilon_i = \sum_{i < j} k_{ij} \sigma_{ij},^1$$

where $k_{ij} = a_{ij} m / \text{LCM}(\text{in}(g_i), \text{in}(g_j))$ for some $a_{ij} \in \mathbf{k}$.

Note that $\phi_2(\sum_{i=1}^{t_1} \text{in}(h_i) \epsilon_i) = \sum_{i=1}^{t_1} \text{in}(h_i) g_i$.

Hence,

$$\sum_{i=1}^{t_1} \text{in}(h_i) g_i = \sum_{i < j} k_{ij} (m_{ji} g_i - m_{ij} g_j) = \sum_{i < j} k_{ij} \sum_{u=1}^t f_u^{(ij)} g_u,$$

since $h_{ij} = 0$ for all i, j . Note that since $\text{in}(f_u^{(ij)} g_u) < \text{in}(m_{ji} g_i)$, we have $\text{in}(k_{ij} f_u^{(ij)}) < m$.

Hence, we have an expression for $f = \sum_i h'_i g_i$, where $\max\{\text{in}(h'_i g_i) : 1 \leq i \leq t\} < m$, which is a contradiction. \square

This result gives us an effective method for computing Gröbner bases.

Buchberger's Algorithm: In the situation of Theorem 2.4.3, suppose that M is a submodule of F , and let $g_1, \dots, g_t \in M$ be a set of generators of M . Compute the remainders h_{ij} . If all the $h_{ij} = 0$, then $\{g_1, \dots, g_t\}$ forms a Gröbner basis of M . If some $h_{ij} \neq 0$, then replace g_1, \dots, g_t with g_1, \dots, g_t, h_{ij} , and repeat the process. As the submodule generated by the initial forms of g_1, \dots, g_t, h_{ij} is strictly larger than that generated by the initial forms of g_1, \dots, g_t , this process must terminate after finitely many steps.

The next theorem shows that if $\{g_1, \dots, g_t\}$ is a Gröbner basis of M , the equations $h_{ij} = 0$ generate the first syzygy of M .

¹let $k_{ij} = 0$ and $\sigma_{ij} = 0$ for i, j where σ_{ij} is not originally defined

For $i < j$ such that $\text{in}(g_i)$ and $\text{in}(g_j)$ involve the same basis element of F , we set

$$w_{ij} = m_{ji}\epsilon_i - m_{ij}\epsilon_j - \sum_{u=1}^t f_u^{(ij)}\epsilon_u.$$

Let W be the set of all such w_{ij} .

Theorem 2.4.4 (Schreyer). *With notation as above, suppose that $\{g_1, \dots, g_t\}$ is a Gröbner basis of M . Let γ be the monomial order on $\oplus_{j=1}^t S\epsilon_j$ defined by taking $m\epsilon_u > n\epsilon_v$ if and only if*

$$\text{in}(mg_u) >_{\tau} \text{in}(ng_v) \text{ with respect to the given order } \tau \text{ on } F$$

or

$$\text{in}(mg_u) \equiv \text{in}(ng_v), \text{ but } u < v.$$

W generates the first syzygy of M . Moreover, W forms a Gröbner basis of the syzygies with respect to the order γ , and $\text{in}_{\gamma}(w_{ij}) = m_{ji}\epsilon_i$.

Proof. We first prove that $\text{in}_{\gamma}(w_{ij}) = m_{ji}\epsilon_i$. Since

$$\text{in}(m_{ji}g_i) = \text{in}(m_{ij}g_j),$$

and these terms are by hypothesis greater than any that appear in the $\sum_{u=1}^t f_u^{(ij)}g_u$, $\text{in}(w_{ij})$ must be either $m_{ji}\epsilon_i$ or $-m_{ij}\epsilon_j$. Since $i < j$, $\text{in}_{\gamma}(w_{ij}) = m_{ji}\epsilon_i$.

To show that W forms a Gröbner basis, let $w = \sum_{i=1}^t f_i\epsilon_i$. Let $\text{in}(f_i) = h_i$ for all i . The theorem is proved once we show that $\text{in}_{\gamma}(w) \in \langle \text{in}_{\gamma}(v) : v \in W \rangle$. Note that $\text{in}_{\gamma}(w) = \text{in}_{\gamma}(f_j\epsilon_j) = h_j\epsilon_j$ for some j . Let

$$\sigma = \sum_{i: h_i \text{in}(g_i) \equiv h_j \text{in}(g_j)} f_i\epsilon_i.$$

σ is a syzygy on $\{\text{in}(g_i) : i \geq j\}$, because if $h_i \text{in}(g_i) \equiv_{\gamma} h_j \text{in}(g_j)$, we must have $i \geq j$. Hence, by Lemma 2.2.1, σ is generated by σ_{uv} for $u, v \geq j$, and ϵ_j only appears in σ_{jv} for $j < v$. This implies that h_j is a \mathbf{k} -linear combination of $\{m_{vj} : j < v\}$ and thus, $\text{in}_{\gamma}(w)$ is a \mathbf{k} -linear combination of $\{m_{vj}\epsilon_j : j < v\}$, which proves the theorem. \square

Corollary 2.4.5. *With notation as in Theorem 2.4.4, suppose that the g_i are arranged such that whenever $\text{in}(g_i)$ and $\text{in}(g_j)$ involve the same basis vector e of F , say $\text{in}(g_i) = m_i e, \text{in}(g_j) = m_j e$ with $m_i, m_j \in S$,*

$$i < j \implies m_i > m_j \text{ in lexicographic order.}$$

If the variables x_1, \dots, x_s are missing from $\text{in}(g_i)$ for all i , then the variables x_1, \dots, x_{s+1} are missing from $\text{in}_{\gamma}(w_{ij})$ for all $i < j$ for which w_{ij} is defined. Further, $F/\langle g_1, \dots, g_t \rangle$ has a free resolution of length $\leq r - s$.

Proof. If the variables x_1, \dots, x_s are missing from $\text{in}(g_i)$ for all i , then, due to the stipulated arrangement of $\{g_1, \dots, g_t\}$, for $i < j$ such that $\text{in}(g_i)$ and $\text{in}(g_j)$ involve the same basis element, the variable x_{s+1} must appear in g_i with at least as high a power as in g_j . As a result, the variable x_{s+1} does not appear in m_{ji} , and hence, does not appear in $\text{in}_\gamma(w_{ij}) = m_{ji}\epsilon_i$.

We now show that $F/(g_1, \dots, g_t)$ has a free resolution of length $\leq r - s$ by induction on $r - s$. Suppose first that $r - s = 0$, so that none of the variables x_1, \dots, x_r appears in the terms $\text{in}(g_i)$. Since none of the variables appear in $\text{in}(g_i)$ for all i , $\text{in}(g_i)$ must be a scalar times a basis element of F . Let F' be the free submodule spanned by all the e_j which do not appear in $\text{in}(g_i)$ for any i . By Theorem 2.3.4, F' is isomorphic to $F/(g_1, \dots, g_t)$.

Suppose $r - s > 0$. By the first statement of the theorem, the variables x_1, \dots, x_{s+1} are missing from $\text{in}_\gamma(w_{ij})$ for all i, j . Order the w_{ij} to satisfy the same hypothesis as on the g_i . Then, by the induction hypothesis, $F/\langle W \rangle$ has a free resolution of length $\leq r - s - 1$. Combining this with the natural map $\phi : \oplus S\epsilon_i \rightarrow F$, we get a free resolution of $F/\langle g_1, \dots, g_t \rangle$ of length $\leq r - s$. \square

Example 2.4.6. Let $F = S$ and $I = \langle x^3 - yz, y^2 - xz, x^2y - z^2 \rangle$. Let $g_1 = x^3 - yz, g_2 = y^2 - xz, g_3 = x^2y - z^2$. In this example, we consider the lexicographic order on S . Thus, we have

$$\text{in}(g_1) = x^3, \text{in}(g_2) = -xz, \text{in}(g_3) = x^2y.$$

Let $S_{ij} = m_{ji}g_i = m_{ij}g_j$. Then,

$$\begin{aligned} S_{12} &= \frac{-xz}{x}(x^3 - yz) - \frac{x^3}{x}(y^2 - xz) \\ &= yz^2 - x^2y^2 \\ &= -yg_3, \end{aligned}$$

and hence, $h_{12} = 0$. Similarly, $S_{23} = xy^3 - z^3 = h_{23}$. Thus, we add $g_4 = h_{23}$ to the original basis $\{g_1, g_2, g_3\}$. For the basis $\{g_1, g_2, g_3, g_4\}$, we immediately have $h_{12} = h_{23} = 0$. Calculation also reveals that $S_{13} = -zg_2$ and $S_{14} = -z(y^2 + xz)g_2$, which implies that $h_{13} = h_{14} = 0$. However, $S_{24} = y^5 - z^4 = h_{24}$. For the new basis $\{g_1, g_2, g_3, g_4, g_5\}$, where $g_5 = y^5 - z^4$, we instantly have $h_{12} = h_{23} = h_{13} = h_{14} = h_{24} = 0$. Further,

$$S_{34} = -z^2g_2, S_{15} = -z(y^4 + xy^2z + x^2z^2)g_2, S_{25} = z^4g_2 + y^2g_5, S_{35} = -z^2(y^2 + xz)g_2, S_{45} = -z^3g_2.$$

This shows that $\{g_1, g_2, g_3, g_4, g_5\}$ is a Gröbner basis of I .

Rearranging the basis to satisfy the hypothesis of the corollary, we have $I = \langle x^3 - yz, x^2y - z^2, xy^3 - z^3, xz - y^2, y^5 - z^4 \rangle$. Hence,

$$\begin{aligned}
w_{12} &= y\epsilon_1 - x\epsilon_2 - z\epsilon_4 \\
w_{13} &= y^3\epsilon_1 - x^2\epsilon_2 - z\epsilon_4 \\
w_{14} &= z\epsilon_1 - x^2\epsilon_4 - z(y^2 + xz)\epsilon_4 \\
w_{15} &= y^5\epsilon_1 - x^3\epsilon_5 - z(y^4 + xy^2z + x^2z^2)\epsilon_4 \\
w_{23} &= y^2\epsilon_2 - x\epsilon_3 - z^2\epsilon_4 \\
w_{24} &= z\epsilon_2 - xy\epsilon_4 - \epsilon_3 \\
w_{25} &= y^4\epsilon_2 - x^2\epsilon_5 - z^2(y^2 + xz)\epsilon_4 \\
w_{34} &= z\epsilon_3 - y^3\epsilon_4 + \epsilon_5 \\
w_{35} &= y^2\epsilon_3 - x\epsilon_5 - z^3\epsilon_4 \\
w_{45} &= (y^5 - z^4)\epsilon_2 + (y^2 - xz)\epsilon_5
\end{aligned}$$

Note that x is missing from the initial terms of all the w_{ij} , as it should be, according to the previous corollary.

Chapter 3

Comparison between an ideal and its initial ideal

3.1 Gradings defined by weights

Definition 3.1.1. Let $\mathbf{w} = (w_1, \dots, w_r) \in \mathbb{N}^r$. We call this vector a **weight** and set $\deg_{\mathbf{w}} x_i = w_i$ for $i = 1, \dots, n$. Then, for $(a_1, \dots, a_r) \in \mathbb{N}^r$,

$$\deg_{\mathbf{w}} x_1^{a_1} \dots x_r^{a_r} = \sum_{i=1}^r a_i w_i.$$

A polynomial $f \in S$ is called **homogeneous of degree j** with respect to the weight \mathbf{w} if the degree of all homogeneous components of f is j .

Fix a weight \mathbf{w} and let S_j be the \mathbf{k} -vector space spanned by all homogeneous polynomials of degree j . Then, S_j is finite dimensional and the monomials u with $\deg_{\mathbf{w}} u = j$ form a \mathbf{k} -basis. It follows that

$$S = \bigoplus_j S_j.$$

Thus, note that we have defined a new grading on S .

Definition 3.1.2. Each polynomial $f \in S$ can be uniquely written as $f = \sum_j f_j$ with $f_j \in S_j$. The summands f_j are called the **homogeneous components** of f with respect to \mathbf{w} . The **degree** of f with respect to \mathbf{w} is defined to be $\deg_{\mathbf{w}} f = \max\{j : f_j \neq 0\}$, and if $i = \deg_{\mathbf{w}} f$, then f_i is called the **initial term** of f with respect to \mathbf{w} and is denoted by $\text{in}_{\mathbf{w}}(f)$.

Note that $\text{in}_{\mathbf{w}}(f)$ need not be a monomial.

Definition 3.1.3. Let $I \subset S$ be an ideal. We define the **initial ideal** of I with respect to \mathbf{w} as

$$\text{in}_{\mathbf{w}}(I) = \langle \text{in}_{\mathbf{w}}(f) : f \in I \rangle.$$

A set of polynomials $f_1, \dots, f_n \in I$ such that $\text{in}_{\mathbf{w}}(I) = \langle \text{in}_{\mathbf{w}}(f_1), \dots, \text{in}_{\mathbf{w}}(f_n) \rangle$ is called a **standard basis** of I with respect to \mathbf{w} .

The following lemma shows that a standard basis of I with respect to a weight generates I .

Lemma 3.1.4. *Let $J \subset I$ be ideals in S . If $\text{in}_{\mathbf{w}}(J) = \text{in}_{\mathbf{w}}(I)$, then $I = J$.*

Proof. Suppose $I \neq J$. Let $f \in I \setminus J$ such that $\deg_{\mathbf{w}} f$ is minimum among all elements in $I \setminus J$. Since $\text{in}_{\mathbf{w}}(f) \in \text{in}_{\mathbf{w}}(I) = \text{in}_{\mathbf{w}}(J)$ and $\text{in}_{\mathbf{w}}(J)$ is a homogeneous ideal with respect to the grading given by \mathbf{w} , there must exist $g \in J$ such that $\text{in}_{\mathbf{w}}(f) = \text{in}_{\mathbf{w}}(g)$. Note that $f - g \in I \setminus J$, and $\deg_{\mathbf{w}}(f - g) < \deg_{\mathbf{w}}(f)$, which is a contradiction. \square

The following lemma is proved in [2].

Lemma 3.1.5. *Given a monomial order τ and pairs of monomials $(g_1, h_1), \dots, (g_m, h_m)$ such that $g_i >_{\tau} h_i$ for all i , there exists a weight \mathbf{w} such that $\deg_{\mathbf{w}} g_i > \deg_{\mathbf{w}} h_i$ for all i .*

Theorem 3.1.6. *Given an ideal I and a monomial order τ , there exists a weight \mathbf{w} such that $\text{in}_{\tau}(I) = \text{in}_{\mathbf{w}}(I)$.*

Proof. Let $\{g_1, \dots, g_n\}$ be a Gröbner basis of I with respect to the monomial order τ . For all i , define K_i to be the set of all monomials appearing in g_i , and denote the monomial corresponding to $\text{in}_{\tau}(g_i)$ as m_i . Define $K = \sqcup_i (g_i, K_i \setminus \{m_i\}) \in S^2$. By the previous lemma, there exists a weight \mathbf{w} such that $g > h$ for all $(g, h) \in K$. Observe that $\text{in}_{\mathbf{w}}(g_i) = \text{in}_{\tau}(g_i)$ for all i . Hence,

$$\text{in}_{\tau}(I) = \langle \text{in}_{\tau}(g_1), \dots, \text{in}_{\tau}(g_n) \rangle \subset \text{in}_{\mathbf{w}}(I).$$

Define a monomial order $\tau_{\mathbf{w}}$ as $m_1 <_{\tau_{\mathbf{w}}} m_2$ if (i) $\deg_{\mathbf{w}}(m_1) < \deg_{\mathbf{w}}(m_2)$ or (ii) $\deg_{\mathbf{w}}(m_1) = \deg_{\mathbf{w}}(m_2)$ and $m_1 <_{\tau} m_2$. Thus, we have

$$\text{in}_{\tau}(I) = \text{in}_{\tau}(\text{in}_{\tau}(I)) \subset \text{in}_{\tau}(\text{in}_{\mathbf{w}}(I)) = \text{in}_{\tau_{\mathbf{w}}}(I).$$

Corollary 2.3.6 implies that $\text{in}_{\tau}(I) = \text{in}_{\tau_{\mathbf{w}}}(I)$. We show that $\text{in}_{\tau_{\mathbf{w}}}(I) \supset \text{in}_{\mathbf{w}}(I)$ to complete the proof.

Observe that $\text{in}_{\tau_{\mathbf{w}}}(g_i) = \text{in}_{\tau}(g_i) = \text{in}_{\mathbf{w}}(g_i)$ for all i and hence, $\{g_1, \dots, g_n\}$ is a Gröbner basis of I with respect to $\tau_{\mathbf{w}}$ as well.

Let $f \in I$ and $f = f_1 g_1 + \dots + f_n g_n$ be a standard expression for f in terms of $\{g_1, \dots, g_n\}$. Since $\text{in}_{\tau_{\mathbf{w}}}(f) \geq_{\tau_{\mathbf{w}}} \text{in}_{\tau_{\mathbf{w}}}(f_i g_i)$ for all i , we have $\deg_{\mathbf{w}} f \geq \deg_{\mathbf{w}}(f_i g_i)$. Let $L = \{i \in \{1, \dots, n\} : \deg_{\mathbf{w}} f = \deg_{\mathbf{w}}(f_i g_i)\}$. Then,

$$\text{in}_{\mathbf{w}}(f) = \sum_{i \in L} \text{in}_{\mathbf{w}}(f_i g_i) = \sum_{i \in L} \text{in}_{\mathbf{w}}(f_i) \text{in}_{\mathbf{w}}(g_i) = \sum_{i \in L} \text{in}_{\mathbf{w}}(f_i) \text{in}_{\tau_{\mathbf{w}}}(g_i) \in \text{in}_{\tau_{\mathbf{w}}}(I).$$

\square

3.2 Homogenization

Definition 3.2.1. Fix a weight \mathbf{w} . Let f be a non-zero polynomial in S with homogeneous components f_j (with respect to the weight \mathbf{w}). We introduce a new variable t and define the **homogenization** of f with respect to \mathbf{w} as the polynomial

$$f^h = \sum_j f_j t^{\deg_{\mathbf{w}} f - j} \in S[t].$$

Note that f^h is homogeneous in $S[t]$ with respect to the extended weight $(w_1, \dots, w_r, 1) \in \mathbb{N}^{r+1}$.

Definition 3.2.2. Let $I \subset S$ be an ideal. The **homogenization** of I is defined to be the ideal

$$I^h = \langle f^h : f \in I \rangle \subset S[t].$$

For any homogeneous polynomial $g \in S[t]$, let \bar{g} denote the polynomial in S obtained by substituting $t = 1$.

Lemma 3.2.3. Let $f \in S[t]$ be homogeneous with respect to the weight $(w_1, \dots, w_r, 1)$. Then $f \in I^h$ iff $f = t^n g^h$ for some $g \in I$ and some $n \in \mathbb{Z}_{\geq 0}$. Further, in this case, $g = \bar{f}^h$.

Proof. It is clear that $f \in I^h$ if $f = t^n g^h$ for some $g \in I$ and some $n \in \mathbb{Z}_+$.

Suppose $f \in I^h$ is homogeneous. Then, there exist $f_1, \dots, f_s \in I$ and $g_1, \dots, g_s \in S[t]$ such that $f = \sum_{i=1}^s g_i f_i^h$.

We have

$$\bar{f} = \sum_{i=1}^s \bar{g}_i \bar{f}_i^h = \sum_{i=1}^s \bar{g}_i f_i \in I.$$

We claim that $f = t^n \bar{f}^h$ for some non-negative integer n . To observe this, let $f = g_l(x_1, \dots, x_r)t^l + \dots + g_k(x_1, \dots, x_r)t^k$ such that $l \leq k$ and $g_l, g_k \neq 0$. Then, $\bar{f} = g_l(x_1, \dots, x_r) + \dots + g_k(x_1, \dots, x_r)$ and

$$\bar{f}^h = g_l(x_1, \dots, x_r) + g_{l+1}(x_1, \dots, x_r)t + \dots + g_k(x_1, \dots, x_r)t^{k-l},$$

which implies that $f = t^l \bar{f}^h$ and completes the proof. \square

Remark 3.2.4. Observe that in the above proof, we have also shown that if f is homogeneous in I^h , then $\bar{f} \in I$.

Definition 3.2.5. A monomial order τ on S is said to respect \mathbf{w} if for all $m_1, m_2 \in S$ such that $\deg_{\mathbf{w}} m_1 < \deg_{\mathbf{w}} m_2$, we have $m_1 <_{\tau} m_2$.

Example 3.2.6. The graded lexicographic order and reverse graded lexicographic order respect the standard grading on S . More generally, the order $<_{\mathbf{w}}$ respects \mathbf{w} .

For a monomial order τ which respects \mathbf{w} , define a natural extension τ' to $S[t]$ as follows: $x^a t^c <_{\tau'} x^b t^d$ iff (i) $x^a <_{\tau} x^b$ or (ii) $x^a = x^b$ and $c < d$, where, as usual, x^a denotes $x_1^{a_1} \dots x_r^{a_r}$. This monomial order has the property that $\text{in}_{\tau}(g) = \text{in}_{\tau'}(g^h)$ for all non-zero $g \in S$.

Proposition 3.2.7. *Let $I \subset S$ be an ideal, and let $\{g_1, \dots, g_n\}$ be a Gröbner basis of I with respect to a monomial order τ which respects \mathbf{w} . Then, $\{g_1^h, \dots, g_n^h\}$ is a Gröbner basis of I^h with respect to τ' .*

Proof. Note that since I^h is a homogeneous ideal with respect to the extended weight $(w_1, \dots, w_r, 1)$, it is sufficient to prove that if $f \in I^h$ is homogeneous with respect to $(w_1, \dots, w_r, 1)$, then $\text{in}_{\tau'}(f) \in \langle \text{in}_{\tau'}(g_1^h), \dots, \text{in}_{\tau'}(g_n^h) \rangle$.

Let $f \in I^h$, be homogeneous. Then, by the previous lemma, there exist $g \in I$ and $m \in \mathbb{Z}_+$ such that $f = t^m g^h$. Hence,

$$\text{in}_{\tau'}(f) = t^m \text{in}_{\tau'}(g^h) = t^m \text{in}_{\tau}(g).$$

There exist $u \in S$ and $i \in \{1, \dots, n\}$ such that $\text{in}_{\tau}(g) = u \text{in}_{\tau}(g_i) = u \text{in}_{\tau'}(g_i^h)$. Thus, $\text{in}_{\tau'} f = u t^m \text{in}_{\tau'}(g_i^h)$. \square

Proposition 3.2.8. *Given an ideal $I \subset S$, $S[t]/I^h$ is a free $\mathbf{k}[t]$ -module.*

Proof. Let $\{g_1, \dots, g_n\}$ be a Gröbner basis of I with respect to a monomial order τ graded with respect to \mathbf{w} . Then, $\{g_1^h, \dots, g_n^h\}$ is a Gröbner basis of I^h with respect to τ' . It follows from Theorem 2.3.4 that the set of all monomials in $S[t]$ not in $\langle \text{in}_{\tau'}(g_1^h), \dots, \text{in}_{\tau'}(g_n^h) \rangle$ forms a \mathbf{k} -basis of $S[t]/I^h$. Since $\text{in}_{\tau'}(g_i^h) = \text{in}_{\tau}(g_i)$, we have $\langle \text{in}_{\tau'}(g_1^h), \dots, \text{in}_{\tau'}(g_n^h) \rangle = \langle \text{in}_{\tau}(g_1), \dots, \text{in}_{\tau}(g_n) \rangle S[t]$ and hence, the set of all monomials in S not in $\langle \text{in}_{\tau}(g_1), \dots, \text{in}_{\tau}(g_n) \rangle$ forms a $\mathbf{k}[t]$ -basis of $S[t]/I^h$. \square

Lemma 3.2.9. *Let R be a ring and consider $\phi : R[t] \rightarrow R$, a ring homomorphism with $\phi|_R = \text{Id}$, or equivalently, an R -linear ring homomorphism. Given an ideal $I \in R[t]$, ϕ naturally induces an R -linear ring homomorphism $\bar{\phi} : R[t]/I \rightarrow R/\phi(I)$ given by $\bar{\phi}(\bar{f}) = \overline{\phi(f)}$, and $\ker(\bar{\phi}) = (t - \phi(t))R[t]/I$.*

Proof. Clearly, $\bar{\phi}(f)$ is well-defined and $(t - \phi(t))R[t]/I \subset \ker(\bar{\phi})$.

Let $f \in R[t]$ such that $\bar{f} \in \ker(\bar{\phi})$. There exist $a \in R$ and $g \in R[t]$ such that $f = a + (t - \phi(t))g$, which implies that $\bar{\phi}(\bar{f}) = \bar{a}$. Thus, we have $a \in \phi(I)$. Let $h \in I$ such that $\phi(h) = a$, that is, $h = a + (t - \phi(t))h'$. Then,

$$f - h \in (t - \phi(t))R[t] \implies f \in I + (t - \phi(t))R[t],$$

which completes the proof. \square

Proposition 3.2.10. *Given an ideal $I \subset S$ and a weight \mathbf{w} on S , we have the following S -linear ring isomorphisms:*

$$\frac{S[t]/I^h}{tS[t]/I^h} \cong S/\text{in}_{\mathbf{w}}(I) \text{ and } \frac{S[t]/I^h}{(t-a)S[t]/I^h} \cong S/I \quad \forall a \in S \setminus \{0\}.$$

Proof. For all $a \in \mathbf{k}$, define an S -linear map $\phi_a : S[t] \rightarrow S$ as $\phi_a(1) = 1$ and $\phi_a(t) = a$. We claim that $\phi_0(I^h) = \text{in}_{\mathbf{w}}(I)$.

Given $f \in I$, $\phi_0(f^h) = \text{in}_{\mathbf{w}}(f)$. Since $I^h = \langle f^h : f \in I \rangle$, it follows that $\phi_0(I^h) = \text{in}_{\mathbf{w}}(I)$. From the previous lemma, we have $\frac{S[t]/I^h}{tS[t]/I^h} \cong S/\text{in}_{\mathbf{w}}(I)$.

For $a \neq 0$, define a ring homomorphism $\psi_a : S \rightarrow S$ as $\psi_a(x_i) = a^{w_i}x_i$ for all i and $\psi_a|_{\mathbf{k}} = \text{Id}$. We claim that $\psi_a\phi_a(I^h) = I$. Then, according to the previous lemma, $\frac{S[t]/I^h}{(t-a)S[t]/I^h} \cong S/\phi_a(I^h)$ as S -modules and since $a \neq 0$, ψ_a is a ring isomorphism and $S/\phi_a(I^h) \cong S/I$ as rings.

By Proposition 3.2.7, there exists a Gröbner basis $\{g_1, \dots, g_n\}$ of I such that $\{g_1^h, \dots, g_n^h\}$ is a Gröbner basis of I^h . Let $g_i = \sum_j g_{ij}$ where g_{ij} denotes the homogeneous component of g_i of degree j (with respect to \mathbf{w}). Then,

$$\phi_a(g_i^h) = \sum_j a^{\deg_{\mathbf{w}} g_i - j} g_{ij},$$

and

$$\psi_a(\phi_a(g_i^h)) = a^{\deg_{\mathbf{w}} g_i} g_i.$$

Since $a \neq 0$, we are done. \square

We now compare the Betti numbers of an ideal with those of its initial ideal.

Let $I \subset S$ be a graded ideal with respect to the standard grading on S , and fix a weight \mathbf{w} on S . Let $\{g_1, \dots, g_n\}$ be a Gröbner basis of I with respect to a monomial order which respects \mathbf{w} , and further, such that g_i is homogeneous with respect to the standard grading for all i . Then, $\{g_1^h, \dots, g_n^h\}$ is a system of generators (in fact, a Gröbner basis) of I^h .

If we assign to each x_i the bidegree $(w_i, 1)$ and to t the bidegree $(1, 0)$, then all the generators g_i^h are bihomogeneous, and hence I^h is a bigraded ideal. Therefore $S[t]/I^h$ has a bigraded minimal free $S[t]$ -resolution,

$$F_{\bullet} : 0 \rightarrow F_p \rightarrow F_{p-1} \rightarrow \dots \rightarrow F_0 \rightarrow S[t]/I^h \rightarrow 0,$$

where $F_i = \oplus_{j,k} (S[t](-k, -j))^{\beta_{ijk}}$. Note that the minimality of the resolution is equivalent to the condition that all entries in the matrices describing the maps must belong to $\langle x_1, \dots, x_r, t \rangle$.

Note that as $S[t]/I^h$ is a free $\mathbf{k}[t]$ -module, $t - a$ is a non-zero divisor on $S[t]/I^h$ for all $a \in \mathbf{k}$. Since t is a non-zero divisor on $S[t]/I^h$ and on $S[t]$, and $t \in \langle x_1, \dots, x_r, t \rangle$, F_{\bullet}/tF_{\bullet} is a bigraded minimal free S -resolution of $\frac{S[t]/I^h}{tS[t]/I^h} \cong S/\text{in}_{\mathbf{w}}(I)$. Observe that the bigraded shifts of F_{\bullet}/tF_{\bullet} are the same as those in F_{\bullet} and in particular, the second component of the shifts in the resolution are the ordinary shifts of the standard graded ideal $\text{in}_{\mathbf{w}}(I)$. Thus, we have

$$\beta_{ij}(S/\text{in}_{\mathbf{w}}(I)) = \sum_k \beta_{ijk} \text{ for all } i, j.$$

On the other hand, since $t - 1$ is also a non-zero divisor on $S[t]/I^h$ and on $S[t]$, $F_{\bullet}/(t - 1)F_{\bullet}$ is a free S -resolution of $\frac{S[t]/I^h}{(t-1)S[t]/I^h} \cong S/I$. Note that $t - 1$ is homogeneous with respect to the second component of the bidegree and hence the second components of the shifts in the resolution F_{\bullet} are

preserved. However, $t - 1$ does not belong to $\langle x_1, \dots, x_r, t \rangle$ and hence $F_\bullet / (t - 1)F_\bullet$ need not be a minimal resolution. Therefore, we have

$$\beta_{ij}(S/I) \leq \sum_k \beta_{ijk} \text{ for all } i, j.$$

We have thus proved the following theorem.

Theorem 3.2.11. *Let $I \subset S$ be a graded ideal and \mathbf{w} be a weight. Then*

$$\beta_{ij}(I) \leq \beta_{ij}(\text{in}_{\mathbf{w}}(I)) \text{ for all } i, j.$$

Theorem 3.2.11 and Theorem 3.1.6 yield the following corollary.

Corollary 3.2.12. *Let $I \subset S$ be a graded ideal and τ be a monomial order on S . Then*

$$\beta_{ij}(I) \leq \beta_{ij}(\text{in}_\tau(I)) \text{ for all } i, j.$$

Corollary 3.2.13. *Given I, τ as above,*

- (i) $\text{pdim}(S/I) \leq \text{pdim}(S/\text{in}_\tau(I))$;
- (ii) $\text{depth}(S/I) \geq \text{depth}(S/\text{in}_\tau(I))$.

Proof. Corollary 3.2.12 directly implies (a). (b) follows from (a) and the Auslander-Buchsbaum formula. \square

Proposition 3.2.14. *Let $I \subset S$ be a graded ideal. Then,*

- (i) *If $\text{in}_{\mathbf{w}}(I)$ is a prime ideal, so is I .*
- (ii) *If $\text{in}_{\mathbf{w}}(I)$ is a radical ideal, so is I .*

Proof. Let $I^h \in S[t]$ be the homogenization of I with respect to the weight \mathbf{w} . We claim that I is prime (resp. radical) if I^h is prime (resp. radical).

$$\phi(f^h) = t^{\deg_{\mathbf{w}} f} f.$$

Suppose I^h is prime. Consider $f, g \in S \setminus \{0\}$ such that $fg \in I$. Then, $(fg)^h = f^h g^h \in I^h$, which implies that $f^h \in I^h$ or $g^h \in I^h$. Without loss of generality, let $f^h \in I^h$. Then, by Remark 3.2.4, note that $f = \overline{(f^h)} \in I$.

Similarly, suppose I^h is radical. Consider $f \in S \setminus \{0\}$ such that $f^n \in I$ for $n \in \mathbb{N}$. Then, $(f^n)^h = (f^h)^n \in I^h$ and hence, $f^h \in I^h$. Proceeding as above, we have $f \in I$.

The following lemma along with Proposition 3.2.10 proves that if $\text{in}_{\mathbf{w}}(I)$ is prime (resp. radical), so is I^h . \square

Lemma 3.2.15. *Let R be a finitely generated positively graded \mathbf{k} -algebra and let $s \in R$ be a homogeneous non-zero divisor of R such that R/sR is a domain (resp. a reduced ring) and $\deg(s) > 0$. Then R is also a domain (resp. a reduced ring).*

Proof. Suppose R/sR is a domain and there exist $a, b \in R \setminus \{0\}$ such that $ab = 0$. By the Krull Intersection Theorem, $\cap_{k \geq 0} \langle s \rangle^k = 0$ and hence, there exist $n_a, n_b \in \mathbb{Z}_{\geq 0}$ such that $a \in \langle s \rangle^{n_a}, b \in \langle s \rangle^{n_b}$ and $a \notin \langle s \rangle^{n_a+1}, b \notin \langle s \rangle^{n_b+1}$. Let $a = a' s^{n_a}, b = b' s^{n_b}$ where $a', b' \notin \langle s \rangle$. Then, $a' b' = 0$ and hence $\overline{a' b'} = 0$, which implies that $a' \in \langle s \rangle$ or $b' \in \langle s \rangle$, a contradiction.

Similarly, suppose R/sR is a reduced ring and there exists $a \in R \setminus \{0\}$ such that $a^n = 0$. Let n_a and a' be as above. Then, $a'^{n_a} = 0$ and hence $\overline{a'^{n_a}} = 0$, which implies that $a' \in \langle s \rangle$, a contradiction. \square

Chapter 4

Polarization

As usual, let $S = \mathbf{k}[x_1, \dots, x_r]$.

Lemma 4.0.1. *Let $I \subset S$ be a monomial ideal with minimal generating set of monomials $\{m_1, \dots, m_n\}$, where $m_i = \prod_{j=1}^n x_j^{a_{ij}}$ for $i = 1, \dots, n$. Fix an integer $j \in [n]$ and suppose that $a_{ij} > 1$ for at least one $i \in [r]$. Let $T = S[y]$ and let $J \subset T$ be the monomial ideal with minimal generating set of monomials $\{m'_1, \dots, m'_n\}$, where*

$$m'_i = \begin{cases} m_i & a_{ij} = 0 \\ (m_i/x_j)y & a_{ij} \geq 1. \end{cases}$$

Then $y - x_j$ is a non-zero divisor in T/J and

$$\frac{T/J}{(y - x_j)T/J} \cong S/I$$

as S -modules.

Proof. Suppose $y - x_j$ is a zero divisor in T/J . Then $y - x_j \in P$ for some $P \in \text{Ass}(J)$. By applying Proposition 1.1.5 on the \mathbb{N}^r -grading, P is a monomial ideal, and hence $y, x_j \in P$. Thus, there exists a monomial $f \in T \setminus J$ such that $yf, x_j f \in J$. Then there exist m'_k, m'_l and monomials $f_1, f_2 \in T$ such that $yf = m'_k f_1$ and $x_j f = m'_l f_2$.

Since $f \notin J$, x_j divides m'_l and hence, by the construction of J , y divides m'_l . This implies that y divides f . Note that y does not divide f_1 because $f \notin J$. This forces y^2 to divide m'_k , which is a contradiction to the construction of J .

Define a ring homomorphism $\phi : T \rightarrow S$ such that $\phi|_S = \text{Id}$ and $\phi(y) = x_j$. Then, $\phi(J) = I$ and by Lemma 3.2.9, we have the required isomorphism. \square

Motivated by Lemma 4.0.1, we define the polarization of a monomial ideal I .

Let $I \subset S$ be a monomial ideal with minimal generating set of monomials $\{m_1, \dots, m_n\}$, where $m_i = \prod_{j=1}^n x_j^{a_{ij}}$ for $i = 1, \dots, n$. For all $j = 1, \dots, r$, define $a_j = \max\{a_{ij} : i = 1, \dots, n\}$.

Let $T = \mathbf{k}[x_{11}, x_{12}, \dots, x_{1a_1}, x_{21}, \dots, x_{2a_2}, \dots, x_{n1}, \dots, x_{na_n}]$. Define J to be a monomial ideal in T with generating set $\{m'_1, \dots, m'_n\}$ where

$$m'_i = \prod_{j=1}^n \prod_{k=1}^{a_{ij}} x_{jk}$$

for all $i \in [n]$.

Definition 4.0.2. The monomial ideal J is called the **polarization** of I .

Example 4.0.3. Consider the ideal $\langle x_1 x_2^2, x_2^4 \rangle \subset \mathbf{k}[x_1, x_2]$. The polarisation of I is

$$J = \langle x_{11}, x_{21} x_{22}, x_{21} x_{22} x_{23} x_{24} \rangle \subset \mathbf{k}[x_{11}, x_{21}, x_{22}, x_{23}, x_{24}].$$

Proposition 4.0.4. Let $I \subset S$ be a monomial ideal and $J \subset T$ be its polarization. Then the sequence \mathbf{z} given by

$$x_{n1} - x_{na_n}, \dots, x_{n1} - x_{n2}, \dots, x_{21} - x_{2a_2}, \dots, x_{21} - x_{22}, \dots, x_{11} - x_{1a_1}, \dots, x_{11} - x_{12}$$

is a regular sequence on T/J and

$$\frac{T/J}{(z)T/J} \cong S/I$$

as graded \mathbf{k} -algebras.

Proof. Firstly, replace x_i in S by x_{i1} for all $i \in [r]$. Let the minimal generating set of monomials of I be $\{m_1^{(11)}, \dots, m_n^{(11)}\}$. Now, let $T_{12} = S[x_{12}]$ and define $m_i^{(12)} = m_i^{(11)}$ if x_{11} does not appear in $m_i^{(11)}$ and $m_i^{(12)} = (m_i^{(11)}/x_{11})x_{12}$ otherwise. Let $J_{12} = \langle m_1^{(12)}, \dots, m_n^{(12)} \rangle$. By Lemma 4.0.1, $x_{11} - x_{12}$ is a non-zero divisor on T_{12}/J_{12} and

$$\frac{T_{12}/J_{12}}{(x_{11} - x_{12})T_{12}/J_{12}} \cong S/I.$$

Similarly, let $T_{13} = T_{12}[x_{13}]$ and define $J_{13} = \langle m_1^{(13)}, \dots, m_n^{(13)} \rangle$ $m_i^{(13)} = m_i^{(11)}$ if x_{11} does not appear in $m_i^{(11)}$ and $m_i^{(13)} = (m_i^{(11)}/x_{11})x_{13}$ otherwise. Note that

$$\frac{T_{13}/J_{13}}{(x_{11} - x_{13}, x_{11} - x_{12})T_{13}/J_{13}} \cong \frac{T_{12}/J_{12}}{(x_{11} - x_{12})T_{12}/J_{12}} \cong S/I.$$

Continue the process until T_{1a_1} . Then, let $T_{22} = T_{1a_1}[x_{22}]$. We eventually get $T_{na_n} = T$. Repeated application of Lemma 4.0.1 completes the proof. \square

Corollary 4.0.5. Let $I \subset S$ be a monomial ideal and $J \subset T$ be its polarization. Then

- (i) $\beta_{ij}(I) = \beta_{ij}(J)$ for all i, j ;
- (ii) $H_{S/I}(t) = (1-t)^\delta H_{T/J}(t)$ where $\delta = \dim T - \dim S$;
- (iii) $\text{pdim}(S/I) = \text{pdim}(S/J)$ and $\text{reg}(S/I) = \text{reg}(T/J)$.

Proof. (i) Follows from the fact that \mathbf{z} is a regular sequence on T/J .

(ii) Follows from Corollary 1.2.9.

(iii) Follows from (i). \square

Chapter 5

The lexsegment ideal

Given a graded ideal $I \subset S$, our aim is to show the existence of a special ideal, the lexsegment ideal of I , denoted by I^{lex} , such that S/I and S/I^{lex} have the same Hilbert function.

By Corollary 2.3.5, S/I and $S/\text{in}_\tau(I)$ have the same Hilbert function for any monomial order τ on S . Thus, we can assume that I is a monomial ideal. By Theorem 2.3.4, the monomials in S not belonging to I form a \mathbf{k} -basis of S/I and since this \mathbf{k} -basis determines the Hilbert functions of S/I , the Hilbert function of S/I does not depend on the base field \mathbf{k} . We can therefore assume that $\text{char}(\mathbf{k}) = 0$.

We denote by $M_d(S)$ the set of all monomials of S of degree d .

Definition 5.0.1. A set $\mathcal{L} \subset M_d(S)$ is called a **lexsegment** if for all $m \in \mathcal{L}$, we have that $m' \in \mathcal{L}$ for all $m' \in M_d(S)$ such that $m' \geq_{\text{lex}} m$.

Definition 5.0.2. A set $\mathcal{L} \subset M_d(S)$ is called **strongly stable** if $x_i(m/x_j) \in \mathcal{L}$ for all $m \in \mathcal{L}$ and all pairs (i, j) such that $i < j$ and x_j divides m .

For a monomial $m \in S$, we set $\gamma(m) = \max\{i : x_i \text{ divides } m\}$.

Definition 5.0.3. A set $\mathcal{L} \subset M_d(S)$ is called **stable** if $x_i(m/x_j) \in \mathcal{L}$ for all $m \in \mathcal{L}$ and all $i < \gamma(m)$.

Definition 5.0.4. A monomial ideal I is said to be a **lexsegment ideal** or a **(strongly) stable monomial ideal**, if for each d the monomials of degree d in I form a lexsegment, or a (strongly) stable set of monomials respectively.

Remark 5.0.5. Note that every lexsegment set is strongly stable, and every strongly stable set is stable.

Example 5.0.6. Let $S = \mathbf{k}[x, y, z, w]$.

Suppose I_1 is the smallest lexsegment ideal containing xyz . Then $I_1 = \langle xyz, xy^2, x^2w, x^2z, x^2y, x^3 \rangle$.

Suppose I_2 is the smallest strongly stable ideal containing xyz . Then $I_2 = \langle xyz, xy^2, x^2z, x^2y, x^3 \rangle$.

Suppose I_3 is the smallest stable ideal containing xyz . Then $I_3 = \langle xyz, xy^2, x^2y, x^3 \rangle$.

Now we have that S/I and $S/\text{gin}_\tau(I)$ have the same Hilbert function, and that $\text{gin}_\tau(I)$ is a strongly stable ideal [2]. Hence, we can assume that I is a strongly stable ideal.

Theorem 5.0.7. *Let $I \subset S$ be a graded ideal. There exists a unique lexsegment ideal, denoted I^{lex} , such that S/I and S/I^{lex} have the same Hilbert function.*

Given a graded ideal I , with j^{th} graded component I_j , denote by I_j^{lex} the \mathbf{k} -vector space spanned by the unique lexsegment \mathcal{L}_j with $|\mathcal{L}_j| = \dim_{\mathbf{k}} I_j$. Define $I^{\text{lex}} = \bigoplus_j I_j^{\text{lex}}$.

Note that if I^{lex} as defined above is an ideal, it is the only possible lexsegment ideal such that S/I and S/I^{lex} have the same Hilbert function. Therefore, we only need to show that I^{lex} is an ideal to prove Theorem 5.0.7. It is sufficient to show that $\{x_1, \dots, x_r\}\mathcal{L}_j \subset \mathcal{L}_{j+1}$.

Definition 5.0.8. *Let \mathcal{N} be a set of monomials in S . Then the shadow of \mathcal{N} is said to be the set*

$$\text{Shad}(\mathcal{N}) = \{x_1, \dots, x_r\}\mathcal{N} = \{x_i u : u \in \mathcal{N}, i = 1, \dots, n\}.$$

Lemma 5.0.9. *If $\mathcal{N} \subset M_d(S)$ is stable, strongly stable or lexsegment, then so is $\text{Shad}(\mathcal{N})$.*

Given $\mathcal{N} \subset M_d(S)$, we denote by $\gamma_i(\mathcal{N})$ the number of elements $\gamma(m) = i$ and set $\gamma_{\leq i}(\mathcal{N}) = \sum_{j=1}^i \gamma_j(\mathcal{N})$.

Lemma 5.0.10. *Let $\mathcal{N} \subset M_d(S)$ be a stable set of monomials. Then $\text{Shad}(\mathcal{N})$ is a stable set and*

- (i) $\gamma_i(\text{Shad}(\mathcal{N})) = \gamma_{\leq i}(\mathcal{N})$;
- (ii) $|\text{Shad}(\mathcal{N})| = \sum_{i=1}^r \gamma_{\leq i}(\mathcal{N})$.

Proof. (ii) follows directly from (i). To prove (i), define the map

$$\phi : \{m \in \mathcal{N} : \gamma(m) \leq i\} \rightarrow \{m \in \text{Shad}(\mathcal{N}) : \gamma(m) = i\}, m \rightarrow mx_i.$$

ϕ is clearly injective. Let $m' \in \text{Shad}(\mathcal{N})$ such that $\gamma(m') = i$. There exists $j \in [r]$ and $m \in \mathcal{N}$ such that $m' = x_j m$. We must have $\gamma(m) \leq i$. If $j = i$, then we are done. If $j < i$, then $\gamma(m) = i$ and since \mathcal{N} is stable, $m_1 = x_j(m/x_i) \in \mathcal{N}$. Hence, we have $m' = x_i m_1$ for $m_1 \in \mathcal{N}$. This proves that ϕ is a bijection, which implies (i). \square

Theorem 5.0.11 (Bayer). *Let $\mathcal{L} \subset M_d(S)$ be a lexsegment and $\mathcal{N} \subset M_d(S)$ be a strongly stable set of monomials with $|\mathcal{L}| \leq |\mathcal{N}|$. Then $\gamma_{\leq i}(\mathcal{L}) \leq \gamma_{\leq i}(\mathcal{N})$ for $i = 1, \dots, r$.*

Proof. Observe that we can write $\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_1 x_r \cup \dots \cup \mathcal{N}_d x_r^d$ where each \mathcal{N}_j is a strongly stable set of monomials of degree $d - j$ in the variables x_1, \dots, x_{r-1} . The lexsegment \mathcal{L} has a similar decomposition $\mathcal{L}_0 \cup \dots \cup \mathcal{L}_r x_r$, where each \mathcal{L}_j is a lexsegment.

We prove the theorem by induction on the number of variables. If $r = 1$, we have that $\gamma_{\leq 1}(\mathcal{L}) = |\mathcal{L}| \leq |\mathcal{N}| = \gamma_{\leq 1}(\mathcal{N})$.

Let $r > 1$. We have that $\gamma_{\leq r}(\mathcal{L}) = |\mathcal{L}|$ and $\gamma_{\leq r}(\mathcal{N}) = |\mathcal{N}|$ and hence, $\gamma_{\leq r}(\mathcal{L}) \leq \gamma_{\leq r}(\mathcal{N})$. Note that for $i < r$, $\gamma_{\leq i}(\mathcal{L}) = \gamma_{\leq i}(\mathcal{L}_0)$ and $\gamma_{\leq i}(\mathcal{N}) = \gamma_{\leq i}(\mathcal{N}_0)$. Hence, if we show that $|\mathcal{L}_0| \leq |\mathcal{N}_0|$, the proof is done by induction.

For each j , let \mathcal{N}_j^* be the lexsegment in $M_{d-j}(\mathbf{k}[x_1, \dots, x_{r-1}])$ with $|\mathcal{N}_j^*| = |\mathcal{N}_j|$ and let $\mathcal{N}^* = \mathcal{N}_0^* \cup \mathcal{N}_1^* x_r \cup \dots \cup \mathcal{N}_d^* x_r^d$. We claim that \mathcal{N}^* is a strongly stable set of monomials.

Observe that it suffices to show that $\{x_1, \dots, x_{r-1}\}\mathcal{N}_j^* \subset \mathcal{N}_{j-1}^*$. By using that \mathcal{N} is a strongly stable set, we have that $\{x_1, \dots, x_r\}\mathcal{N}_j \subset \mathcal{N}_{j-1}$. Then, by Lemma 5.0.10 and the induction hypothesis, we have that

$$|\{x_1, \dots, x_{r-1}\}\mathcal{N}_j^*| = \sum_{i=1}^{r-1} \gamma_{\leq i}(\mathcal{N}_j^*) \leq \sum_{i=1}^{r-1} \gamma_{\leq i}(\mathcal{N}_j) = |\{x_1, \dots, x_{r-1}\}\mathcal{N}_j| \leq |\mathcal{N}_{j-1}| = |\mathcal{N}_{j-1}^*|.$$

The fact that $|\{x_1, \dots, x_{r-1}\}\mathcal{N}_j^*|$ and $|\mathcal{N}_{j-1}^*|$ are both lexsegments forces $|\{x_1, \dots, x_{r-1}\}\mathcal{N}_j^*| \subset |\mathcal{N}_{j-1}^*|$, which implies that \mathcal{N}^* is a strongly stable set of monomials.

Now, given a monomial $m = \prod_{i=1}^r x_i^{a_i}$, we set $\overline{m} = (x_{n-1}/x_n)^{a_n} m$. Observe that if $m_1 \leq m_2$ in the lexicographic order, then $\overline{m}_1 \leq \overline{m}_2$.

Let $m_1 = \min \mathcal{L}$ and $m_2 = \min \mathcal{N}^*$. Since \mathcal{N}^* is strongly stable, $\overline{m}_2 \in \mathcal{N}_0^*$ and $\overline{m}_2 \geq \min(\mathcal{N}_0^*)$. Further, $\min(\mathcal{N}_0^*) \geq m_2$, which implies that $\min(\mathcal{N}_0^*) = \min(\mathcal{N}_0^*) \geq \overline{m}_2$. Hence, $\min(\mathcal{N}_0^*) = \overline{m}_2$ and similarly, $\min(\mathcal{L}_0^*) = \overline{m}_1$.

Since $|\mathcal{L}| \leq |\mathcal{N}| = |\mathcal{N}^*|$, we have that $m_1 \geq m_2$ and hence, $\overline{m}_1 \geq \overline{m}_2$. As \mathcal{L}_0 and \mathcal{N}_0^* are lexsegments, we get that $|\mathcal{L}_0| \leq |\mathcal{N}_0^*| = |\mathcal{N}_0|$, which completes the proof. \square

We now complete the proof of Theorem 5.0.7.

Recall that we may assume that I is strongly stable. Let \mathcal{N}_j be the strongly stable set of monomials which spans the \mathbf{k} -vector space I_j . Since $|\mathcal{L}_j| = |\mathcal{N}_j|$, Bayer's theorem together with Lemma 5.0.10 implies that

$$|\text{Shad}(\mathcal{L}_j)| = \sum_{i=1}^r \gamma_{\leq i}(\mathcal{L}_j) \leq \sum_{i=1}^r \gamma_{\leq i}(\mathcal{N}_j) = |\text{Shad}(\mathcal{N}_j)|.$$

Since I is an ideal, we have that $\text{Shad}(\mathcal{N}_j) \subset \mathcal{N}_{j+1}$. Hence,

$$|\text{Shad}(\mathcal{L}_j)| \leq |\text{Shad}(\mathcal{N}_j)| \leq |\mathcal{N}_{j+1}| = |\mathcal{L}_{j+1}|.$$

Since $\text{Shad}(\mathcal{L}_j)$ and \mathcal{L}_{j+1} are both lexsegments, $|\text{Shad}(\mathcal{L}_j)| \leq |\mathcal{L}_{j+1}|$ implies that $\text{Shad}(\mathcal{L}_j) \subset \mathcal{L}_{j+1}$, as desired.

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