Master's Thesis Stage 1

Homological Algebra

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Contents

Chapter 1

Graded Rings, Modules and Resolutions

1.1 Graded Rings and Modules

Definition 1.1.1. *Let H be a cancellative monoid under addition. A ring R is said to be* ***H-graded*** *if $R = \bigoplus\limits_{i \in H} R_i$, where, each R_i is an abelian group and $R_i R_j \subseteq R_{i+j}$, for all $i, j \in H$. For each i, R_i is called the* ***homogeneous component of degree*** *i of R and the nonzero elements of R_i are called* ***homogeneous elements of degree i****.*

Remark 1.1.2.
(i) For a cancellative monoid H, we denote its associated group by G.
(ii) By an ordered monoid we mean a cancellative monoid H with an order $<$ satisfying: whenever $a < b$ in H, we have $a + c < b + c$ for all $c \in H$.
(iii) If H is an ordered monoid, then we say that it is well ordered if every nonempty subset S of H which is bounded below has the least element in S.

Definition 1.1.3. *A module M is called as a* ***graded module*** *over a graded ring R if $M = \bigoplus\limits_{i \in G} M_i$, as a direct sum of subgroups of M and for all $i \in H, j \in G$, $R_i M_j \subseteq M_{i+j}$.*

Definition 1.1.4. *An ideal J of a graded ring R is said to be* ***graded*** *if it satifies any of the following equivalent conditions:*
(i) If $f \in J$, then every homogeneous component of f is in J.
(ii) $J = \oplus_{i \in N} J_i$, where $J_i = R_i \bigcap J$.
(iii) If J' is the ideal generated by all homogeneous elements in J, then $J = J'$.
(iv) J has a system of homogeneous generators.

Proposition 1.1.5. *Given a graded ideal I in a graded ring R, every associated prime of I is also graded.*

Proof. Suppose $J = (I : x)$ is a prime ideal for some x in R. Let $x = x_l + x_{l+1} + \cdots + x_k$ where $x_i \in R_i$, $l < k$ and x_l, x_k are non-zero.

Let $y = y_t + y_{t+1} + \cdots + y_s \in J$, where $y_i \in R_i$, $t < s$ and x_t, x_s are non-zero. If we show that $y_t \in J$, we are done by induction on $s - t$.
To see this, observe that we have $xy \in I$ and since I is graded, the lowest graded component of xy, which is $x_l y_t$, belongs to I. Similarly, $x_{l+1}y_t + x_l y_{t+1} \in I$, and on multiplying by y_t, we get that $x_{l+1}y_t^2 \in I$. Continuing in this manner, we get that $x_{l+i}y_t^{i+1} \in I$ for all $i = 0, 1, \ldots, k-l$, which implies that $y_t^{k-l+1}x \in I$ and hence, $y_t^{k-l+1} \in J$. Since J is prime, $y_t \in J$. Hence, J is a graded ideal. □

Definition 1.1.6. *Let R be a H-graded ring and $M = \bigoplus_{i \in G} M_i$ be a finitely generated R-module. Then we define an R-module $M(d)$ by $M(d) = \bigoplus_{i \in G} M_{i+d}$. $M(d)$ is called a **shifted** R**-module**.*

Definition 1.1.7. *Let $M = \bigoplus_{i \in G} M_i$, $M' = \bigoplus_{i \in G} M'_n$ be graded modules over R. An R-linear map $f : M \to M'$ is said to be a **graded map of degree** d if $f(M_i) \subseteq M'_{i+d}$ for all $i \in G$. If f has degree zero, we simply say that f is a graded R-module homomorphism.*

Proposition 1.1.8. *Let R be nonnegatively graded, M, N be graded R-modules and $\phi : M \to N$ be a graded homomorphism of degree d. Then*
(i) $\ker(\phi)$ *is a graded submodule of M.*
(ii) $\mathrm{Im}(\phi)$ *is a graded submodule of N.*

Proof. (i) It is clear that $\ker(\phi)$ is a submodule of M considered without grading. To show that $\ker(\phi)$ is graded, it suffices to show that if $x = x_r + \cdots + x_s$, is in $\ker(\phi)$, then each x_i is in $\ker(\phi)$. We show that $x_r \in \ker(\phi)$ and by induction we will get that $x_i \in \ker(\phi)$ for all i. Note that $\phi(x_i) \in N_{i+d}$. Therefore $\phi(x_r) \in N_{r+d} \cap (N_{(r+1)+d} \oplus \cdots \oplus N_{s+d}) = 0$. This shows that $\phi(x_r) = 0$ as desired.
(ii) It is clear that $\mathrm{Im}(\phi)$ is a submodule of N considered without grading. To show that $\mathrm{Im}(\phi)$ is graded, it suffices to show that if $y = y_r + \cdots + y_s$, is in $\mathrm{Im}(\phi)$, then each y_i is in $\mathrm{Im}(\phi)$. Since $\phi(M_i) \subseteq N_{i+d}$ and $y \in \mathrm{Im}(\phi)$, there exists $x = x_{r-d} + \cdots + x_{s-d} \in M$ such that $\phi(x) = y$ and $\phi(x_{i-d}) = y_i$. This shows that $y_i \in \mathrm{Im}(\phi)$. This completes the proof. □

Remark 1.1.9.
(i) If I is a graded ideal of R, then we have $R_i I_j \subseteq I_{i+j}$.
(ii) If I is a graded ideal of R, then the quotient ring R/I inherits the grading from R by $(R/I)_i = R_i/I_i$.
(iii) If N is a graded submodule of a graded module M, then M/N is graded with the grading given by $(M/N)_i = M_i/N_i$.

Proposition 1.1.10. *Tensor products of graded R-modules is graded, i.e., if M and N are graded R-modules, then $M \otimes N$ is graded R-module.*

Proof. We know that $M \otimes N$ is an R-module. We give grading to $M \otimes N$ as follows:
Define $(M \otimes N)_i$ to be generated (as a $\mathbb{Z}$-module) by all the elements in $M \otimes N$ of the form $m \otimes n$,

where $\deg(m)+\deg(n)=i$. Then we have $M\otimes N=\bigoplus_{i\in G}(M\otimes N)_i$. Moreover, for any $r_i\in R_i$ and $m\oplus n\in(M\otimes N)_j$, we have $r(m\otimes n)=(rm)\otimes n$. Therefore

$$\deg(r(m\otimes n))=(i+\deg(m))+\deg(n)=i+j.$$

This shows that $R_i(M\otimes N)_j\subseteq(M\otimes N)_{i+j}$. Hence $M\otimes N$ is graded.

$\square$

Let $\mathrm{Hom}_i(M,N)=\{\phi:M\to N\mid deg(\phi)=i\}$. Then we define ${}^*\mathrm{Hom}(M,N)=\bigoplus_{i\in G}\mathrm{Hom}_i(M,N)$.

Remark 1.1.11. In general, ${}^*\mathrm{Hom}(M,N)\neq\mathrm{Hom}(M,N)$. However, we have the equality in a special case which we will prove shortly.

Lemma 1.1.12. *Let $M=\bigoplus_{i=1}^{m}R(n_i)$ and N be graded R-modules. Then ${}^*\mathrm{Hom}(M,N)\cong\mathrm{Hom}(M,N)$ with grading forgotten.*

Proof. It is clear that every $\phi=\phi_r+\cdots+\phi_s\in{}^*\mathrm{Hom}(M,N)$ is in $\mathrm{Hom}(M,N)$, and hence ${}^*\mathrm{Hom}(M,N)\subseteq\mathrm{Hom}(M,N)$. To show the other inclusion assume that $\phi\in\mathrm{Hom}(M,N)$. Let $e_j=(0,\dots,0,1,0,\dots,0)$ where 1 occurs at jth place. Then M is a free R-module with basis $\{e_1,\dots,e_m\}$. If $\phi(e_j)=y_{j1}+\cdots+y_{jr_j}\in N$, then we have

$$\phi=\phi_{11}+\cdots\phi_{1r_1}+\cdots+\phi_{m1}+\cdots+\phi_{mr_m}$$

where $\phi_{js}:M\to N$ is given by $\phi_{js}(e_j)=y_{js}$ and $\phi_{js}(e_i)=0$ for all $i\neq j$. Note that each ϕ_{js} is well defined since $\{e_1,\dots,e_m\}$ is a basis for M. Moreover ϕ_{js} is a graded homomorphism of degree $\deg(y_{js})+n_j$. Therefore $\phi\in{}^*\mathrm{Hom}(M,N)$. This completes the proof. $\square$

Proposition 1.1.13. *Let R be a graded Noetherian ring, M be a finitely generated graded R-module and N be any graded R-module. Then ${}^*\mathrm{Hom}(M,N)=\mathrm{Hom}(M,N)$ with grading forgotten.*

Proof. It is clear that every $\phi=\phi_r+\cdots+\phi_s\in{}^*\mathrm{Hom}(M,N)$ is in $\mathrm{Hom}(M,N)$, and hence we have an inclusion ${}^*\mathrm{Hom}(M,N)\xrightarrow{i}\mathrm{Hom}(M,N)$.
Since M is finitely generated and R is Noetherian, we get an exact sequence of graded modules $G\to F\to M\to 0$ for some $F=\bigoplus_{j=1}^{n}R(n_j)$ and $G=\bigoplus_{j=1}^{m}R(m_j)$. By the previous lemma we have ${}^*\mathrm{Hom}(F,N)=\mathrm{Hom}(F,N)$, ${}^*\mathrm{Hom}(G,N)=\mathrm{Hom}(G,N)$. Thus we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & {}^*\mathrm{Hom}(M,N) \arrow[r] \arrow[d, "i"] & {}^*\mathrm{Hom}(F,N) \arrow[r] \arrow[d, equal] & {}^*\mathrm{Hom}(G,N) \arrow[d, equal] \\
0 \arrow[r] & \mathrm{Hom}(M,N) \arrow[r] & \mathrm{Hom}(F,N) \arrow[r] & \mathrm{Hom}(G,N)
\end{tikzcd}$$

Thus by five lemma, we get that the inclusion i is an isomorphism. $\square$

Lemma 1.1.14 (Graded Nakayama Lemma)**.** *Let H be an ordered monoid such that $i > 0$ for all $i \in H \setminus \{0\}$ and $R = \bigoplus_{i \in H} R_i$ be a graded ring. Let $M = \bigoplus_{i \in G} M_i$ be an R-module such that there exists $n \in G$ with $M_i = 0$ for all $i < n$. Further assume that G is well ordered. If $R_+ = \bigoplus_{i \in H \setminus \{0\}} R_i$ and $R_+ M = M$ then $M = 0$.*

Proof. Let, if possible, $M \neq 0$. Let m be the smallest element of G such that for all $i < m$, we have $M_i = 0$ and $M_m \neq 0$. But then, $M = R_+ M \subseteq \bigoplus_{i\ m} M_i$, which has m^{th} component equal to 0. This contradiction shows that $M = 0$. □

Corollary 1.1.15. *Let R be a non negatively graded ring and M be a finitely generated $\mathbb{Z}$-graded R-module. If $R_+ M = M$ then $M = 0$.*

Proof. Let $\{m_1, \ldots, m_r\}$ be a generating set for M and d=min$\{\deg(m_i) \mid 1 \leq i \leq r\}$. Since R is graded by $\mathbb{N} \cup \{0\}$, we get that $M_n = 0$, for every $n < d$. Thus, applying graded Nakayama lemma proved above, we get $M = 0$. □

1.2 Graded Resolutions

From now on we assume that R is a graded ring with $R_0 = \mathsf{k}$, a field. We will mostly consider $R = \mathsf{k}[x_1, \ldots, x_r]$.

Definition 1.2.1. *Let M be a graded R-module and*

$$F_\bullet : \cdots \to F_n \xrightarrow{\phi_n} F_{n-1} \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \to 0$$

be a free resolution of M. If all F_i's are graded R-modules and all ϕ_i's are graded maps of degree zero, then we say that $F_\bullet$ is a ***graded free resolution*** *of M.*

Definition 1.2.2. Let $R = \mathsf{k}[x_1, \ldots, x_n]$ and M be a graded R-module. A graded free resolution

$$F_\bullet : \quad \cdots \to F_n \xrightarrow{\phi_n} \cdots \to F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \to 0$$

is said to be *minimal* if $\phi_i(F_i) \subseteq \langle x_1, \ldots, x_r \rangle F_{i-1}$ for all $i \geq 1$.

Example 1.2.3. Let $I = \langle x^2, y^2 \rangle$ and $R = \mathsf{k}[x, y]$. Then

$$F_\bullet : 0 \leftarrow R/I \xleftarrow{\phi_0} R \xleftarrow{\phi_1} R(-2) \oplus R(-2) \xleftarrow{\phi_2} R(-4) \leftarrow 0,$$

where $\phi_0(1) = 1$, $\phi_1(1,0) = x^2$, $\phi_1(0,1) = y^2$, $\phi_2(1) = (-y^2, x^2)$ is a minimal graded free resolution of R/I over R.

Example 1.2.4. Let $I = \langle x^3, y^2 \rangle$ and $R = \mathsf{k}[x, y]$. Then

$$F_\bullet : 0 \leftarrow R/I \xleftarrow{\phi_0} R \xleftarrow{\phi_1} R(-3) \oplus R(-2) \xleftarrow{\phi_2} R(-5) \leftarrow 0,$$

where $\phi_0(1) = 1$, $\phi_1(1,0) = x^3$, $\phi_1(0,1) = y^2$, $\phi_2(1) = (-y^2, x^3)$ is a minimal graded free resolution of R/I over R.

Definition 1.2.5. *Let $R = \mathsf{k}[x_1, \ldots, x_n]$ and M be a graded R-module.*

$$F_\bullet : \cdots \to F_n \xrightarrow{\phi_n} F_{n-1} \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \to 0$$

*be a minimal graded free resolution of M, where $F_i = \bigoplus_j R(-j)^{\beta_{i,j}(M)}$. Then the numbers $\beta_{i,j}(M)$ are called **graded Betti numbers** of M. $\beta_i(M) = \sum_j \beta_{i,j}(M)$ is called the total ith Betti number of M.*

Definition 1.2.6. *Let $\beta_{i,j}$ be graded Betti numbers of M. Then **Betti table** of M is written as*

$j \backslash i$	0	1	$\cdots$	p	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
0	$\beta_{0,0}$	$\beta_{1,1}$	$\cdots$	$\beta_{p,p}$	$\vdots$
1	$\beta_{0,1}$	$\beta_{1,2}$	$\cdots$	$\beta_{p,p+1}$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$

Definition 1.2.7. *Let k be a field and $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a finitely generated graded module over the polynomial ring $\mathsf{k}[x_1, \ldots, x_r]$. Then the function $H_M : \mathbb{Z} \to \mathbb{Z}$, given by $H_M(j) = \dim_k(M_j)$ is called as **Hilbert function of M**.*

Let k be a field and $R = \mathsf{k}[x_1, \ldots, x_n]$. Let $a \in R \setminus \{0\}$ be such that $\deg(a) = d$. Since a is a nonzerodivisor on R, we get an exact sequence of R-modules

$$0 \to R(-d) \xrightarrow{\cdot a} R \to R/\langle a \rangle \to 0.$$

Since R is graded, for each i, we have an exact sequence of k-vector spaces

$$0 \to R(-d)_i \xrightarrow{\cdot a} R_i \to [R/\langle a \rangle]_i \to 0.$$

Now, using rank-nullity theorem for vector spaces, we get

$$\dim_k(R_i) = \dim_k((R(-d))_i) + \dim_k((R/\langle a \rangle)_i),$$

i.e.,

$$H_R(i) = H_{R(-d)}(i) + H_{R/\langle a \rangle}(i).$$

Therefore, $H_R(i) = H_R(i-d) + H_{R/\langle a \rangle}(i)$ or $H_{R/\langle a \rangle}(i) = H_R(i) - H_R(i-d)$.

Definition 1.2.8. *Given* k, M *as above, define the* ***Hilbert series*** *of* M *as* $H_M(t) = \sum_{j \geq 0} H_M(j)t^j$.

The next corollary follows from the above definition.

Corollary 1.2.9. $H_{R/\langle a \rangle}(t) = H_R(t)/(1-t)^d$.

Example 1.2.10. Let $R = \mathsf{k}[x, y]$ and $a = x^2$. In this case, for all $i \geq 0$, we have $H_R(i) = i + 1$. This is because the ith graded component of R, as a k-vector space is has a basis $\{x^r y^{i-r} \mid 0 \leq r \leq i\}$. For the element x^2, we have $\deg(x^2) = 2$. Hence, by the formula above, we must have $H_{R/\langle x^2 \rangle}(i) = (i+1) - (i-1) = 2$; which is true as $\{\overline{xy^{i-1}}, \overline{y^i}\}$ form a k-vector space basis of $(R/\langle x^2 \rangle)_i$.

Proposition 1.2.11. *Let* M, N *be graded* R*-modules. Then* $\operatorname{Tor}_i^R(M, N)$ *is graded for all* i.

Proof. Consider a graded free resolution of M as follows:

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0.$$

Tensoring with N gives a complex of graded modules

$$\cdots \to F_2 \otimes N \to F_1 \otimes N \to F_0 \otimes N \to M \otimes N \to 0.$$

Since $\operatorname{Tor}_i^R(M, N)$ is quotient of a graded submodule of a graded module by a graded submodule, we conclude that $\operatorname{Tor}_i^R(M, N)$ is graded for all i. $\square$

Remark 1.2.12. If $F_\bullet$ is a graded free resolution of M then we define ${}^*\operatorname{Ext}_R^i(M, N) \cong H^i({}^*\operatorname{Hom}_R(F_\bullet, N))$. Then, by Proposition 1.1.13, if R is Noetherian local ring and M is finitely generated R-module, then ${}^*\operatorname{Ext}_R^i(M, N) \cong \operatorname{Ext}_R^i(M, N)$.

Chapter 2

Gröbner Bases and Schreyer's Algorithm

Let k be a field and $S = \mathsf{k}[x_1, \ldots, x_r]$.
If $a = (a_1, \ldots, a_r)$, x^a will denote the monomial $x_1^{a_1} \ldots x_r^{a_r}$. As is convention, an ideal of S generated by monomials will be referred to as a monomial ideal.

Definition 2.0.1. *Let F be a finitely generated free module over S with basis $\{e_1, \ldots, e_n\}$.*
*A **monomial** in F is an element of the form $m = x^a e_i$ for some i. We say that such an m involves the basis element e_i.*
*A **monomial submodule** of F is a submodule generated by elements of this form. Any monomial submodule M of F may be written as*

$$M = \oplus I_j e_j \subseteq \oplus S e_j = F,$$

with I_j the monomial ideal generated by those monomials m such that $me_j \in M$.
*A **term** in F is a monomial multiplied by a scalar.*

Definition 2.0.2. *Let F be a finitely generated free module over S with basis $\{e_1, \ldots, e_n\}$.*
*If m, n are monomials of S, $u, v \in k$, and $v \neq 0$, then we say that the term ume_i is divisible by the term vne_j if $i = j$ and m is **divisible** by n in S; the quotient is then $um/vn \in S$.*

Definition 2.0.3. *The set of monomials in M that are minimal elements in the partial order by divisibility on the monomials of F. We will refer to the monomials in this set as **minimal generators of** M.*

2.1 Hilbert Function of Monomial Submodules

Let F be a free S-module with basis $\{e_i : i = 1, \ldots, n\}$, and let $M \subseteq F$ be a monomial submodule. Since, as seen before, $M = \oplus I_j e_j$, we have $F/M = \oplus S/I_j$ and, since the Hilbert function is additive, it suffices to handle the case $F = S$ and $M = I$, where I is a monomial ideal.

Choosing one of the monomial generators f of I, and letting I' be the monomial ideal generated by the remaining generators, we have the following graded exact sequence:

$$0 \to S/(I' : f)(-d) \xrightarrow{\cdot f} S/I' \to S/I \to 0,$$

where d is the degree of f. If $I' = (f_1, f_2 \ldots, f_t)$, then

$$(I' : f) = (f_1/GCD(f_1, f), f_2/GCD(f_2, f), \ldots, f_n/GCD(f_t, f)).$$

For every integer n,

$$H_{S/I}(n) = H_{S/I'}(n) - H_{S/(I':f)}(n).$$

Note that both I' and $(I' : f)$ have fewer minimal generators than I, and hence, using induction, we can compute an expression for the Hilbert function or polynomial of I.
By choosing f sensibly, we can make the process much faster: If f contains the largest power of some variable x_1 of any of the minimal generators of I, then the minimal generators of the resulting ideal $(I' : f)$ will not involve x_1 at all. They will thus involve strictly fewer of the variables than the number involved in the minimal generators of I.

2.2 Syzygies of Monomial Submodules

Let F be a free module and let M be a submodule of F generated by monomials $m_1, \ldots, m_t$. Define

$$\phi : \oplus_{j=1}^{t} S\epsilon_j \to F; \phi(\epsilon_j) = m_j.$$

For each pair of indices i, j such that m_i and m_j involve the same basis element of F, we define

$$m_{ij} = m_i/\mathrm{GCD}(m_i, m_j),$$

and we define σ_{ij} to be the element of $\ker(\phi)$ given by

$$\sigma_{ij} = m_{ji}\epsilon_i - m_{ij}\epsilon_j.$$

Lemma 2.2.1. *With notation as above,* $\ker(\phi)$ *is generated by the set of all* σ_{ij}*, wherever defined.*

Proof. As a vector space over k, $\ker(\phi) = \oplus_f \ker(\phi)_f$, where

$$\ker(\phi)_f = \left\{ \sum_i a_i f_i \epsilon_i \in \ker(\phi) : m_i \, divides \, f, \; f_i = f/m_i, \; a_i \in \mathsf{k} \right\}.$$

Indeed, let

$$\sigma = \sum_i p_i \epsilon_i \in \ker(\phi).$$

For any monomial f that occurs in one of the $p_j m_j$, and for each i, let $p_{i,f}$ be the term of p_i such that $p_{i,f} m_i$ is a scalar times f. Then,

$$\sum_i p_i m_i = 0 \implies \sum_i \sum_f p_{i,f} m_i = 0 \implies \sum_f \sum_i p_{i,f} m_i = 0 \implies \forall f, \sum_i p_{i,f} m_i = 0.$$

Therefore, for all monomials f, $\sum_i p_{i,f}\epsilon_i \in \ker(\phi)$.
We may now assume $\sigma = \sum_i a_i f_i \epsilon_i$ for some monomial f of F. If $\sigma = 0$, σ lies in the module generated by σ_{ij}. If $\sigma \neq 0$, at least two of the $a_i f_i$ must be non-zero, since $\sum_i a_i f_i m_i = 0$. This implies that for some i, j, both m_i and m_j must divide f and in fact, $m_i f_i = m_j f_j = f$, which implies that $m_{ji} = m_j / \mathrm{GCD}(m_i, m_j)$ divides f_i. Let $k = f_i / m_{ji}$, then $k\sigma_{ij} \in ker(\phi)_f$, and $\sigma - a_i k \sigma_{ij}$ has fewer non-zero terms than σ. Hence, the proof is complete by induction on number of non-zero terms of σ. □

Example 2.2.2. *Let $S = \mathsf{k}[x, y]$, $F = S^2$, $M = \langle (x^2, 0), (0, xy), (0, y^3)\rangle$. Then we have*

$$\phi : \oplus_{j=1}^3 S\epsilon_j \to F; \phi(\epsilon_1) = (x^2, 0), \phi(\epsilon_2) = (0, xy), \phi(\epsilon_3) = (0, y^3).$$

Suppose for some $a_1, a_2, a_3 \in S$, $\phi(a_1\epsilon_1 + a_2\epsilon_2 + a_3\epsilon_3) = 0$, then we have $(a_1x^2, a_2xy + a_3y^3) = 0$, and hence, $a_1 = 0$, $a_2 = by^2$, $a_3 = -bx$. Thus, $a_1\epsilon_1 + a_2\epsilon_2 + a_3\epsilon_3 = b(0, y^2 - x) = b\sigma_{23}$.

2.3 Monomial Orders

Let I be an ideal of S, J be a monomial ideal of S and B be the set of all monomials not in J. Then, the elements of B are k-linearly independent modulo I if and only if J contains at least one monomial from every polynomial in I.
Indeed, suppose J contains no monomial of $f \in I$, $f \neq 0$. Then $f \in \mathrm{Span}(B) \cap I$, which implies that the elements of B are linearly dependent modulo I. Conversely, suppose there exist $a_1, \dots, a_n \in \mathsf{k}$ and $m_1, \dots, m_n \in B$ such that $\sum_{i=1}^n a_i m_i \in I$, then $\sum_{i=1}^n a_i m_i$ is a polynomial in I for which no monomials belong to J.
Moreover, if B is a basis of S/I, J must be a minimal monomial ideal containing at least one monomial from every polynomial in I. Indeed, suppose J contains at least one monomial from each polynomial in I, but is not a minimal ideal satisfying this condition. Let $J_1 \subsetneq J$ satisfying the condition, and let $f \in J \setminus J_1$, where f is a monomial. Suppose $f \in \mathrm{Span}(B)$, that is, there exist $a_1, \dots, a_n \in k$ and $m_1, \dots, m_n \in B$ such that $f - \sum_{i=1}^n a_i m_i \in I$. Since J_1 contains at least one monomial of every polynomial in I, we have a contradiction. Hence, B cannot span S/I if J is not the minimal monomial ideal containing one monomial from each polynomial in I.

Definition 2.3.1. *Let F be a free S-module. A **monomial order** on F is a total order τ on the monomials of F such that the following two conditions are satisfied:*
(i) if m_1 is a monomial of F and $f \neq 1$ is a monomial of S, then $fm_1 >_\tau m_1$.
(ii) if m_1, m_2 are monomials of F and $f \neq 1$ is a monomial of S, then $m_1 >_\tau m_2$ implies $fm_1 >_\tau fm_2$.

Lemma 2.3.2 (Well-Ordering Property)**.** *Let F be a free S-module. The set of monomials in F is well-ordered with respect to any monomial order (every subset has a least element).*

Proof. Let $X \subseteq F$ be a set of monomials. Since S is Noetherian, the submodule of F generated by X must be generated by a finite subset of X, say, Y. Since Y is a finite set of monomials, it must have a least element with respect to a monomial order. The least element of Y must be the least element of X because every element of X is an element in Y multiplied by a monomial in S. □

We will extend this notation to terms: If um_1 and vm_2 are terms with $0 \neq u, v \in \mathsf{k}$, and m_1, m_2 are monomials with $m_1 >_\tau m_2$ then we say $um_1 >_\tau vm_2$.

Definition 2.3.3. *Let F be a free S-module and τ be a monomial order on F. For any $f \in F$, we define the* ***initial term*** *of f, denoted by $\mathrm{in}_\tau(f)$ to be the greatest term of f with respect to the order τ. Given a submodule M of F, define the* ***initial submodule*** *of M, denoted by $\mathrm{in}_\tau(M)$, to be the monomial submodule generated by $\mathrm{in}_\tau(f)$ for all $f \in M$.*

Theorem 2.3.4 (Macaulay)**.** *Let F be a free S-module and M be a submodule of F. For any monomial order τ on F, the set B of all monomials not in $\mathrm{in}_\tau(M)$ forms a k-basis for F/M.*

Proof. Suppose the set B is not linearly independent. Then there exist distinct $m_1, \ldots, m_t \in B$ and $(a_1, \ldots, a_t) \in \mathsf{k}^\mathsf{t} \backslash \{\mathsf{0}\}$ such that $f := a_1 m_1 + \cdots + a_t m_t \in M$. Since $\mathrm{in}(f) \in \mathrm{in}(M)$, there must exist $i \in \{1, \ldots, t\}$ such that $m_i \in \mathrm{in}(M)$, which is a contradiction.
Suppose B does not span F/M. Let $f \in F \backslash (M + \mathrm{Span}(B))$ such that f has minimal initial term among all elements of $F \backslash (M + \mathrm{Span}(B))$. We can choose such an f by the well-ordering property. If $\mathrm{in}(f) \in \mathrm{Span}(B)$, $f - \mathrm{in}(f) \in F \backslash (M + \mathrm{Span}(B))$ has smaller initial term than f. Hence, $\mathrm{in}(f) \in \mathrm{in}(M)$. However, this implies that there exists $g \in M$ such that $\mathrm{in}(f) = \mathrm{in}(g)$, and $f - g \in F \backslash (M + \mathrm{Span}(B))$ has smaller initial term than f, leading to a contradiction. □

Corollary 2.3.5. *Given F, M, τ as above,* $\dim_\mathsf{k}(F/M) = \dim_\mathsf{k}(F/in_\tau(M))$.

Corollary 2.3.6. *Given monomial orders τ, γ on S and an ideal $I \in S$ such that $\mathrm{in}_\tau(I) \subset \mathrm{in}_\gamma(I)$, we have $\mathrm{in}_\tau(I) = \mathrm{in}_\gamma(I)$.*

Proof. If $\mathrm{in}_\tau(I) \subsetneq \mathrm{in}_\gamma(I)$, the set of monomials in $S \backslash \mathrm{in}_\gamma(I)$ is a proper subset of the set of monomials in $S \backslash \mathrm{in}_\tau(I)$. However, both these sets of monomials form a K-basis of S/I, which is a contradiction. □

Here are some important examples of monomial orders when $F = S$. Let $a = (a_1, \ldots, a_r), b = (b_1, \ldots, b_r)$ and $m = x^a, m' = x^b$

Lexicographic order: $m >_{lex} m'$ if and only if $a_i > b_i$ for the smallest i such that $a_i \neq b_i$.
Graded lexicographic order: $m >_{grlex} m'$ if and only if $\deg(m) > \deg(n)$ or $\deg(m) = \deg(n)$ and $a_i > b_i$ for the smallest i such that $a_i \neq b_i$.
Reverse graded lexicographic order: $m >_{grevlex} m'$ if and only if $\deg(m) > \deg(n)$ or $\deg(m) = \deg(n)$ and $a_i < b_i$ for the largest i such that $a_i \neq b_i$.

Remark 2.3.7. A "reverse lexicographic order" is not a monomial order, because 1 is not the least monomial. In fact, 1 is the largest monomial.

Definition 2.3.8. *A* ***Gröbner basis*** *with respect to an order τ on a free module F is a set of elements $g_1, \dots, g_t \in F$ such that if M is the submodule of F generated by $g_1, ..., g_t$, then $\mathrm{in}_\tau(g_1), ..., \mathrm{in}_\tau(g_t)$ generate $\mathrm{in}_\tau(M)$. We then say that $g_1, ..., g_t$ is a* ***Gröbner basis of*** *M.*

There is a Gröbner basis of any submodule M of F, with respect to any monomial order: if $g_1, \dots, g_t$ is a set of generators of M which is not a Gröbner basis, we can adjoin $g_{t+1}, \dots, g_{t'}$ until $\mathrm{in}(g_1), \dots, \mathrm{in}(g_{t'})$ generate $\mathrm{in}(M)$ (note that the Hilbert basis theorem implies that this can be done).

Lemma 2.3.9. *Let $N \subset M \subset F$ be submodules such that $\mathrm{in}(N) = \mathrm{in}(M)$ with respect to a given monomial order. Then, $N = M$.*

Proof. Suppose $N \neq M$, then, by the well-ordering property, there exists $f \in M \backslash N$ such that f has the least initial term among all the elements of M not in N. Since $f \in M$, we have $\mathrm{in}(f) \in \mathrm{in}(M) = \mathrm{in}(N)$, which implies the existence of $g \in N$ such that $\mathrm{in}(f) = \mathrm{in}(g)$. Note that $f - g \in M \backslash N$, but has smaller initial term than f, which is a contradiction to the choice of f. □

The above lemma tells us that if $\langle \mathrm{in}(g_1), \dots, \mathrm{in}(g_t) \rangle = \mathrm{in}(M)$, then $\langle g_1, \dots, g_t \rangle = M$.

2.4 Computing Syzygies

Proposition 2.4.1 (Division Algorithm)**.** *Let F be a free S-module with monomial order τ. If $f, g_1, ..., g_t \in F$, then there is an expression*

$$f = \sum_{i=1}^{t} f_i g_i + f' \text{ with } f' \in F, f_i \in S,$$

where none of the monomials of f' is in $\langle \mathrm{in}(g_1), \dots, \mathrm{in}(g_t) \rangle$ and $\mathrm{in}(f) \geq_\tau \mathrm{in}(f_i g_i)$ for every i.

Definition 2.4.2. *With notation as above, any such f' is called a* ***remainder*** *of f with respect to $g_1, ..., g_t$, and an expression $f = \sum f_i g_i + f'$ satisfying the condition of the proposition is called a* ***standard expression*** *for f in terms of the g_i.*

The proof outlines an algorithm to attain a standard expression for any $f \in F$.

Proof. If $f, g_1, \dots, g_t \in F$, then we may produce a standard expression

$$f = \sum m_u g_{s_u} + f'$$

for f with respect to $g_1, \ldots, g_t$ by defining the indices s_u and the terms m_u inductively. Having chosen $s_1, \ldots, s_p$ and $m_1, \ldots, m_p$, if

$$f'_p := f - \sum_{u=1}^{p} m_u g_{s_u} \neq 0$$

and m is the maximal term of f'_p; that is divisible by $\text{in}(g_i)$ for some i, then choose $s_{p+1} = i, m_{p+1} = m/\text{in}(g_i)$. This process terminates when either $f'_p = 0$ or no $\text{in}(g_i)$ divides a monomial of f; the remainder f' is then the last f'_p produced.
Note that the well-ordering property guarantees that this process must terminate, because the maximal term of f'_p divisible by some g_i decreases at each step. □

Fix the following notation:
Let F be a free module over S with monomial order τ. Let $g_1, \ldots, g_t$ be non-zero elements of F, and let $\oplus S\epsilon_i$ be a free module with basis $\{\epsilon_1, \ldots, \epsilon_t\}$.
For two terms $m_1, m_2 \in F$, $m_1 < m_2$ denotes that the monomial corresponding to m_1 is less than the monomial corresponding to m_2 with respect to the order τ.
For each pair of indices i, j such that $\text{in}(g_i)$ and $\text{in}(g_j)$ involve the same basis element of F, we define

$$m_{ij} = \text{in}(g_i)/\text{GCD}(\text{in}(g_i), \text{in}(g_j)) \in S,$$

and we set $\sigma_{ij} = m_{ji}\epsilon_i - m_{ij}\epsilon_j$ for $i < j$.
For each such pair i, j, choose a standard expression

$$m_{ji}g_i - m_{ij}g_j = \sum_{u=1}^{t} f_u^{(ij)} g_u + h_{ij}$$

for $m_{ji}g_i - m_{ij}g_j$ with respect to $g_1, \ldots, g_t$. Note that $\text{in}(f_u^{(ij)} g_u) < \text{in}(m_{ji}g_i)$.
Set $h_{ij} = 0$ if $\text{in}(g_i)$ and $\text{in}(g_j)$ involve different basis elements of F.
Define $\phi : \oplus S\epsilon_i \to F$, $\phi(\epsilon_i) = g_i$. Then, the set of σ_{ij} generate the syzygies on the module generated by the elements $\text{in}(g_i)$ (by Lemma 2.2.1). Note that $\phi(\sigma_{ij}) = m_{ji}g_i - m_{ij}g_j$.

Theorem 2.4.3 (Buchberger's Criterion)**.** *The elements $g_1, \ldots, g_t$ form a Gröbner basis if and only if $h_{ij} = 0$ for all i and j.*

Proof. Let $M = \langle g_1, \ldots, g_t \rangle \subset F$. The expression for h_{ij} implies that $h_{ij} \in M$, and hence $\text{in}(h_{ij}) \in \text{in}(M)$. However, if $g_1, \ldots, g_t$ is a Gröbner basis, the definition of a standard expression forces $h_{ij} = 0$ for all i, j.
Conversely, suppose that $h_{ij} = 0$ for all i, j. Let $f = \sum_{i=1}^{t} h_i g_i \in M$, where, among all possible $h_1, \ldots, h_t$ such that $f = \sum_{i=1}^{t} h_i g_i$, $h_1, \ldots, h_t$ are chosen such that $\max\{\text{in}(h_i g_i) : 1 \leq i \leq t\}$ is minimal. We prove that $\text{in}(f) \in \langle \text{in}(g_1), \ldots, \text{in}(g_t) \rangle$.
If $\text{in}(f) = \text{in}(h_i g_i)$ for some i, $\text{in}(g_i) | \text{in}(f) \Rightarrow \text{in}(f) \in \langle \text{in}(g_1), \ldots, \text{in}(g_t) \rangle$.

Hence, let $\text{in}(f) < \max\{\text{in}(h_i g_i) : 1 \le i \le t\} = m$. Define an equivalence relation $\equiv$ on terms as follows: $m_1 \equiv m_2$ if there exists $\lambda \in \mathsf{k}\backslash\{0\}$ such that $m_1 = \lambda m_2$. Without loss of generality, suppose $\text{in}(h_i g_i) \equiv m$ for $i = 1, \ldots, t_1$ and $\text{in}(h_i g_i) < m$ for $i = t_1 + 1, \ldots, t$

$$\begin{aligned} f = \sum_{i=1}^{t} h_i g_i &= \sum_{i=1}^{t_1} h_i g_i + \sum_{i=t_1}^{t} h_i g_i \\ &= \sum_{i=1}^{t_1} \text{in}(h_i) g_i + \sum_{i=1}^{t_1} (h_i - \text{in}(h_i)) g_i + \sum_{i=t_1+1}^{t} h_i g_i. \end{aligned}$$

Note that $\sum_{i=1}^{t_1} \text{in}(h_i)\text{in}(g_i) = 0$.
Define $\phi_1 : \oplus S\epsilon_j \to M, \phi_1(\epsilon_j) = \text{in}(g_j)$ and $\phi_2 : \oplus S\epsilon_j \to M, \phi_2(\epsilon_j) = g_j$. Note that $\sum_{i=1}^{t_1} \text{in}(h_i)\epsilon_i \in \ker(\phi_1)$. Therefore, by Lemma 2.2.1,

$$\sum_{i=1}^{t_1} \text{in}(h_i)\epsilon_i = \sum_{i<j} k_{ij}\sigma_{ij},$$ [1]

where $k_{ij} = a_{ij} m / \text{LCM}(\text{in}(g_i), \text{in}(g_j))$ for some $a_{ij} \in \mathsf{k}$.
Note that $\phi_2(\sum_{i=1}^{t_1} \text{in}(h_i)\epsilon_i) = \sum_{i=1}^{t_1} \text{in}(h_i) g_i$.
Hence,

$$\sum_{i=1}^{t_1} \text{in}(h_i) g_i = \sum_{i<j} k_{ij}(m_{ji} g_i - m_{ij} g_j) = \sum_{i<j} k_{ij} \sum_{u=1}^{t} f_u^{(ij)} g_u,$$

since $h_{ij} = 0$ for all i, j. Note that since $\text{in}(f_u^{(ij)} g_u) < \text{in}(m_{ji} g_i)$, we have $\text{in}(k_{ij} f_u^{(ij)}) < m$. Hence, we have an expression for $f = \sum_i h'_i g_i$, where $\max\{\text{in}(h'_i g_i) : 1 \le i \le t\} < m$, which is a contradiction. □

This result gives us an effective method for computing Gröbner bases.

Buchberger's Algorithm: In the situation of Theorem 2.4.3, suppose that M is a submodule of F, and let $g_1, \ldots, g_t \in M$ be a set of generators of M. Compute the remainders h_{ij} . If all the $h_{ij} = 0$, then $\{g_1, \ldots, g_t\}$ forms a Gröbner basis of M. If some $h_{ij} \neq 0$, then replace $g_1, \ldots, g_t$ with $g_1, \ldots, g_t, h_{ij}$, and repeat the process. As the submodule generated by the initial forms of $g_1, \ldots, g_t, h_{ij}$ is strictly larger than that generated by the initial forms of $g_1, \ldots, g_t$, this process must terminate after finitely many steps.

The next theorem shows that if $\{g_1, \ldots, g_t\}$ is a Gröbner basis of M, the equations $h_{ij} = 0$ generate the first syzygy of M.

[1] let $k_{ij} = 0$ and $\sigma_{ij} = 0$ for i, j where σ_{ij} is not originally defined

For $i < j$ such that $\mathrm{in}(g_i)$ and $\mathrm{in}(g_j)$ involve the same basis element of F, we set

$$w_{ij} = m_{ji}\epsilon_i - m_{ij}\epsilon_j - \sum_{u=1}^{t} f_u^{(ij)}\epsilon_u.$$

Let W be the set of all such w_{ij}.

Theorem 2.4.4 (Schreyer)**.** *With notation as above, suppose that $\{g_1, \ldots, g_t\}$ is a Gröbner basis of M. Let γ be the monomial order on $\oplus_{j=1}^{t} S\epsilon_j$ defined by taking $m\epsilon_u > n\epsilon_v$ if and only if*

$$\mathrm{in}(mg_u) >_\tau \mathrm{in}(ng_v) \textit{ with respect to the given order } \tau \textit{ on } F$$

or

$$\mathrm{in}(mg_u) \equiv \mathrm{in}(ng_v), \textit{ but } u < v.$$

W generates the first syzygy of M. Moreover, W forms a Gröbner basis of the syzygies with respect to the order γ, and $\mathrm{in}_\gamma(w_{ij}) = m_{ji}\epsilon_i$.

Proof. We first prove that $\mathrm{in}_\gamma(w_{ij}) = m_{ji}\epsilon_i$. Since

$$\mathrm{in}(m_{ji}g_i) = \mathrm{in}(m_{ij}g_j),$$

and these terms are by hypothesis greater than any that appear in the $\sum_{u=1}^{t} f_u^{(ij)} g_u$, $\mathrm{in}(w_{ij})$ must be either $m_{ji}\epsilon_i$ or $-m_{ij}\epsilon_j$. Since $i < j$, $\mathrm{in}_\gamma(w_{ij}) = m_{ji}\epsilon_i$.
To show that W forms a Gröbner basis, let $w = \sum_{i=1}^{t} f_i\epsilon_i$. Let $\mathrm{in}(f_i) = h_i$ for all i. The theorem is proved once we show that $\mathrm{in}_\gamma(w) \in \langle \mathrm{in}_\gamma(v) : v \in W\rangle$. Note that $\mathrm{in}_\gamma(w) = \mathrm{in}_\gamma(f_j\epsilon_j) = h_j\epsilon_j$ for some j. Let

$$\sigma = \sum_{i:h_i\mathrm{in}(g_i) \equiv h_j\mathrm{in}(g_j)} f_i\epsilon_i.$$

σ is a syzygy on $\{\mathrm{in}(g_i) : i \geq j\}$, because if $h_i\mathrm{in}(g_i) \equiv_\gamma h_j\mathrm{in}(g_j)$, we must have $i \geq j$. Hence, by Lemma 2.2.1, σ is generated by σ_{uv} for $u, v \geq j$, and ϵ_j only appears in σ_{jv} for $j < v$. This implies that h_j is a k-linear combination of $\{m_{vj} : j < v\}$ and thus, $\mathrm{in}_\gamma(w)$ is a k-linear combination of $\{m_{vj}\epsilon_j : j < v\}$, which proves the theorem. $\square$

Corollary 2.4.5. *With notation as in Theorem 2.4.4, suppose that the g_i are arranged such that whenever $\mathrm{in}(g_i)$ and $\mathrm{in}(g_j)$ involve the same basis vector e of F, say $\mathrm{in}(g_i) = m_ie, \mathrm{in}(g_j) = m_je$ with $m_i, m_j \in S$,*

$$i < j \implies m_i > m_j \textit{ in lexicographic order.}$$

If the variables $x_1, \ldots, x_s$ are missing from $\mathrm{in}(g_i)$ for all i, then the variables $x_1, \ldots, x_{s+1}$ are missing from $\mathrm{in}_\gamma(w_{ij})$ for all $i < j$ for which w_{ij} is defined. Further, $F/\langle g_1, \ldots, g_t\rangle$ has a free resolution of length $\leq r - s$.

Proof. If the variables $x_1, \ldots, x_s$ are missing from $\text{in}(g_i)$ for all i, then, due to the stipulated arrangement of $\{g_1, \ldots, g_t\}$, for $i < j$ such that $\text{in}(g_i)$ and $\text{in}(g_j)$ involve the same basis element, the variable x_{s+1} must appear in g_i with at least as high a power as in g_j. As a result, the variable x_{s+1} does not appear in m_{ji}, and hence, does not appear in $\text{in}_\gamma(w_{ij}) = m_{ji}\epsilon_i$.
We now show that $F/(g_1, \ldots, g_t)$ has a free resolution of length $\leq r - s$ by induction on $r - s$. Suppose first that $r - s = 0$, so that none of the variables $x_1, \ldots, x_r$ appears in the terms $\text{in}(g_i)$. Since none of the variables appear in $\text{in}(g_i)$ for all i, $\text{in}(g_i)$ must be a scalar times a basis element of F. Let F' be the free submodule spanned by all the e_j which do not appear in $\text{in}(g_i)$ for any i. By Theorem 2.3.4, F' is isomorphic to $F/(g_1, \ldots, g_t)$.
Suppose $r - s > 0$. By the first statement of the theorem, the variables $x_1, \ldots, x_{s+1}$ are missing from $\text{in}_\gamma(w_{ij})$ for all i, j. Order the w_{ij} to satisfy the same hypothesis as on the g_i. Then, by the induction hypothesis, $F/\langle W\rangle$ has a free resolution of length $\leq r - s - 1$. Combining this with the natural map $\phi : \oplus S\epsilon_i \to F$, we get a free resolution of $F/\langle g_1, \ldots, g_t\rangle$ of length $\leq r - s$. □

Example 2.4.6. Let $F = S$ and $I = \langle x^3 - yz, y^2 - xz, x^2y - z^2\rangle$. Let $g_1 = x^3 - yz, g_2 = y^2 - xz, g_3 = x^2y - z^2$. In this example, we consider the lexicographic order on S. Thus, we have

$$\text{in}(g_1) = x^3, \text{in}(g_2) = -xz, \text{in}(g_3) = x^2y.$$

Let $S_{ij} = m_{ji}g_i = m_{ij}g_j$. Then,

$$\begin{aligned} S_{12} &= \frac{-xz}{x}(x^3 - yz) - \frac{x^3}{x}(y^2 - xz) \\ &= yz^2 - x^2y^2 \\ &= -yg_3, \end{aligned}$$

and hence, $h_{12} = 0$. Similarly, $S_{23} = xy^3 - z^3 = h_{23}$. Thus, we add $g_4 = h_{23}$ to the original basis $\{g_1, g_2, g_3\}$. For the basis $\{g_1, g_2, g_3, g_4\}$, we immediately have $h_{12} = h_{23} = 0$. Calculation also reveals that $S_{13} = -zg_2$ and $S_{14} = -z(y^2 + xz)g_2$, which implies that $h_{13} = h_{14} = 0$. However, $S_{24} = y^5 - z^4 = h_{24}$. For the new basis $\{g_1, g_2, g_3, g_4, g_5\}$, where $g_5 = y^5 - z^4$, we instantly have $h_{12} = h_{23} = h_{13} = h_{14} = h_{24} = 0$. Further,

$$S_{34} = -z^2g_2, S_{15} = -z(y^4 + xy^2z + x^2z^2)g_2, S_{25} = z^4g_2 + y^2g_5, S_{35} = -z^2(y^2 + xz)g_2, S_{45} = -z^3g_2.$$

This shows that $\{g_1, g_2, g_3, g_4, g_5\}$ is a Gröbner basis of I.
Rearranging the basis to satisfy the hypothesis of the corollary, we have $I = \langle x^3 - yz, x^2y - z^2, xy^3 - z^3, xz - y^2, y^5 - z^4\rangle$. Hence,

$$
\begin{aligned}
w_{12} &= y\epsilon_1 - x\epsilon_2 - z\epsilon_4 \\
w_{13} &= y^3\epsilon_1 - x^2\epsilon_2 - z\epsilon_4 \\
w_{14} &= z\epsilon_1 - x^2\epsilon_4 - z(y^2 + xz)\epsilon_4 \\
w_{15} &= y^5\epsilon_1 - x^3\epsilon_5 - z(y^4 + xy^2z + x^2z^2)\epsilon_4 \\
w_{23} &= y^2\epsilon_2 - x\epsilon_3 - z^2\epsilon_4 \\
w_{24} &= z\epsilon_2 - xy\epsilon_4 - \epsilon_3 \\
w_{25} &= y^4\epsilon_2 - x^2\epsilon_5 - z^2(y^2 + xz)\epsilon_4 \\
w_{34} &= z\epsilon_3 - y^3\epsilon_4 + \epsilon_5 \\
w_{35} &= y^2\epsilon_3 - x\epsilon_5 - z^3\epsilon_4 \\
w_{45} &= (y^5 - z^4)\epsilon_2 + (y^2 - xz)\epsilon_5
\end{aligned}
$$

Note that x is missing from the initial terms of all the w_{ij}, as it should be, according to the previous corollary.

Chapter 3

Comparison between an ideal and its initial ideal

3.1 Gradings defined by weights

Definition 3.1.1. *Let* $\mathbf{w} = (w_1, \ldots, w_r) \in \mathbb{N}^r$*. We call this vector a* ***weight*** *and set* $\deg_{\mathbf{w}} x_i = w_i$ *for* $i = 1, \ldots, n$*. Then, for* $(a_1, \ldots, a_r) \in \mathbb{N}^r$*,*

$$\deg_{\mathbf{w}} x_1^{a_1} \ldots x_r^{a_r} = \sum_{i=1}^{r} a_i w_i.$$

A polynomial $f \in S$ *is called* ***homogeneous of degree*** j *with respect to the weight* $\boldsymbol{w}$ *if the degree of all homogeneous components of* f *is* j*.*

Fix a weight $\mathbf{w}$ and let S_j be the k-vector space spanned by all homogeneous polynomials of degree j. Then, S_j is finite dimensional and the monomials u with $\deg_{\mathbf{w}} u = j$ form a k-basis. It follows that

$$S = \oplus_j S_j.$$

Thus, note that we have defined a new grading on S.

Definition 3.1.2. *Each polynomial* $f \in S$ *can be uniquely written as* $f = \sum_j f_j$ *with* $f_j \in S_j$*. The summands* f_j *are called the* ***homogeneous components*** *of* f *with respect to* $\mathbf{w}$*.*
The ***degree*** *of* f *with respect to* $\mathbf{w}$ *is defined to be* $\deg_{\mathbf{w}} f = \max\{j : f_j \neq 0\}$*, and if* $i = \deg_{\mathbf{w}} f$*, then* f_i *is called the* ***initial term*** *of* f *with respect to* $\mathbf{w}$ *and is denoted by* $\mathrm{in}_{\mathbf{w}}(f)$*.*

Note that $\mathrm{in}_{\mathbf{w}}(f)$ need not be a monomial.

Definition 3.1.3. *Let* $I \subset S$ *be an ideal. We define the* ***initial ideal*** *of* I *with respect to* $\mathbf{w}$ *as*

$$\mathrm{in}_{\mathbf{w}}(I) = \langle \mathrm{in}_{\mathbf{w}}(f) : f \in I \rangle.$$

A set of polynomials $f_1, \ldots, f_n \in I$ *such that* $\mathrm{in}_{\mathbf{w}}(I) = \langle \mathrm{in}_{\mathbf{w}}(f_1), \ldots, \mathrm{in}_{\mathbf{w}}(f_n) \rangle$ *is called a standard basis of* I *with respect to* $\mathbf{w}$*.*

The following lemma shows that a standard basis of I with respect to a weight generates I.

Lemma 3.1.4. *Let $J \subset I$ be ideals in S. If $\mathrm{in}_{\mathbf{w}}(J) = \mathrm{in}_{\mathbf{w}}(I)$, then $I = J$.*

Proof. Suppose $I \neq J$. Let $f \in I \backslash J$ such that $\deg_{\mathbf{w}} f$ is minimum among all elements in $I \backslash J$. Since $\mathrm{in}_{\mathbf{w}}(f) \in \mathrm{in}_{\mathbf{w}}(I) = \mathrm{in}_{\mathbf{w}}(J)$ and $\mathrm{in}_{\mathbf{w}}(J)$ is a homogeneous ideal with respect to the grading given by $\mathbf{w}$, there must exist $g \in J$ such that $\mathrm{in}_{\mathbf{w}}(f) = \mathrm{in}_{\mathbf{w}}(g)$. Note that $f - g \in I \backslash J$, and $\deg_{\mathbf{w}}(f-g) < \deg_{\mathbf{w}}(f)$, which is a contradiction. □

The following lemma is proved in [2].

Lemma 3.1.5. *Given a monomial order τ and pairs of monomials $(g_1, h_1), \ldots, (g_m, h_m)$ such that $g_i >_\tau h_i$ for all i, there exists a weight $\mathbf{w}$ such that $\deg_{\mathbf{w}} g_i > \deg_{\mathbf{w}} h_i$ for all i.*

Theorem 3.1.6. *Given an ideal I and a monomial order τ, there exists a weight $\mathbf{w}$ such that $\mathrm{in}_\tau(I) = \mathrm{in}_{\mathbf{w}}(I)$.*

Proof. Let $\{g_1, \ldots, g_n\}$ be a Gröbner basis of I with respect to the monomial order τ. For all i, define K_i to be the set of all monomials appearing in g_i, and denote the monomial corresponding to $\mathrm{in}_\tau(g_i)$ as m_i. Define $K = \sqcup_i (g_i, K_i \backslash \{m_i\}) \in S^2$. By the previous lemma, there exists a weight $\mathbf{w}$ such that $g > h$ for all $(g, h) \in K$. Observe that $\mathrm{in}_{\mathbf{w}}(g_i) = \mathrm{in}_\tau(g_i)$ for all I. Hence,

$$\mathrm{in}_\tau(I) = \langle \mathrm{in}_\tau(g_1), \ldots, \mathrm{in}_\tau(g_n) \rangle \subset \mathrm{in}_{\mathbf{w}}(I).$$

Define a monomial order $\tau_{\mathbf{w}}$ as $m_1 <_{\tau_{\mathbf{w}}} m_2$ if (i) $\deg_{\mathbf{w}}(m_1) < \deg_{\mathbf{w}}(m_2)$ or (ii) $\deg_{\mathbf{w}}(m_1) = \deg_{\mathbf{w}}(m_2)$ and $m_1 <_\tau m_2$. Thus, we have

$$\mathrm{in}_\tau(I) = \mathrm{in}_\tau(\mathrm{in}_\tau(I)) \subset \mathrm{in}_\tau(\mathrm{in}_{\mathbf{w}}(I)) = \mathrm{in}_{\tau_{\mathbf{w}}}(I).$$

Corollary 2.3.6 implies that $\mathrm{in}_\tau(I) = \mathrm{in}_{\tau_{\mathbf{w}}}(I)$. We show that $\mathrm{in}_{\tau_{\mathbf{w}}}(I) \supset \mathrm{in}_{\mathbf{w}}(I)$ to complete the proof.
Observe that $\mathrm{in}_{\tau_{\mathbf{w}}}(g_i) = \mathrm{in}_\tau(g_i) = \mathrm{in}_{\mathbf{w}}(g_i)$ for all i and hence, $\{g_1, \ldots, g_n\}$ is a Gröbner basis of I with respect to $\tau_{\mathbf{w}}$ as well.
Let $f \in I$ and $f = f_1 g_1 + \cdots + f_n g_n$ be a standard expression for f in terms of $\{g_1, \ldots, g_n\}$. Since $\mathrm{in}_{\tau_{\mathbf{w}}}(f) \geq_{\tau_{\mathbf{w}}} \mathrm{in}_{\tau_{\mathbf{w}}}(f_i g_i)$ for all i, we have $\deg_{\mathbf{w}} f \geq \deg_{\mathbf{w}}(f_i g_i)$. Let $L = \{i \in \{1, \ldots, n\} : \deg_{\mathbf{w}} f = \deg_{\mathbf{w}}(f_i g_i)\}$. Then,

$$\mathrm{in}_{\mathbf{w}}(f) = \sum_{i \in L} \mathrm{in}_{\mathbf{w}}(f_i g_i) = \sum_{i \in L} \mathrm{in}_{\mathbf{w}}(f_i) \mathrm{in}_{\mathbf{w}}(g_i) = \sum_{i \in L} \mathrm{in}_{\mathbf{w}}(f_i) \mathrm{in}_{\tau_{\mathbf{w}}}(g_i) \in \mathrm{in}_{\tau_{\mathbf{w}}}(I).$$

□

3.2 Homogenization

Definition 3.2.1. *Fix a weight* $\mathbf{w}$*. Let* f *be a non-zero polynomial in* S *with homogeneous components* f_j *(with respect to the weight* $\mathbf{w}$*). We introduce a new variable* t *and define the* ***homogenization*** *of* f *with respect to* $\mathbf{w}$ *as the polynomial*

$$f^h = \sum_j f_j t^{\deg_{\mathbf{w}} f - j} \in S[t].$$

Note that f^h is homogeneous in $S[t]$ with respect to the extended weight $(w_1, \ldots, w_r, 1) \in \mathbb{N}^{r+1}$.

Definition 3.2.2. *Let* $I \subset S$ *be an ideal. The* ***homogenization*** *of* I *is defined to be the ideal*

$$I^h = \langle f^h : f \in I \rangle \subset S[t].$$

For any homogeneous polynomial $g \in S[t]$, let $\overline{g}$ denote the polynomial in S obtained by substituting $t = 1$.

Lemma 3.2.3. *Let* $f \in S[t]$ *be homogeneous with respect to the weight* $(w_1, \ldots, w_r, 1)$*. Then* $f \in I^h$ *iff* $f = t^n g^h$ *for some* $g \in I$ *and some* $n \in \mathbb{Z}_{\geq 0}$*. Further, in this case,* $g = \overline{f}^h$*.*

Proof. It is clear that $f \in I^h$ if $f = t^n g^h$ for some $g \in I$ and some $n \in \mathbb{Z}_+$.
Suppose $f \in I^h$ is homogeneous. Then, there exist $f_1, \ldots, f_s \in I$ and $g_1, \ldots, g_s \in S[t]$ such that $f = \sum_{i=1}^{s} g_i f_i^h$.
We have

$$\overline{f} = \sum_{i=1}^{s} \overline{g_i}\,\overline{f_i^h} = \sum_{i=1}^{s} \overline{g_i} f_i \in I.$$

We claim that $f = t^n \overline{f}^h$ for some non-negative integer n. To observe this, let $f = g_l(x_1, \ldots, x_r)t^l + \cdots + g_k(x_1, \ldots, x_r)t^k$ such that $l \leq k$ and $g_l, g_k \neq 0$. Then, $\overline{f} = g_l(x_1, \ldots, x_r) + \cdots + g_k(x_1, \ldots, x_r)$ and

$$\overline{f}^h = g_l(x_1, \ldots, x_r) + g_{l+1}(x_1, \ldots, x_r)t + \cdots + g_k(x_1, \ldots, x_r)t^{k-l},$$

which implies that $f = t^l \overline{f}^h$ and completes the proof. $\square$

Remark 3.2.4. *Observe that in the above proof, we have also shown that if* f *is homogeneous in* I^h*, then* $\overline{f} \in I$*.*

Definition 3.2.5. *A monomial order* τ *on* S *is said to respect* $\mathbf{w}$ *if for all* $m_1, m_2 \in S$ *such that* $\deg_{\mathbf{w}} m_1 < \deg_{\mathbf{w}} m_2$*, we have* $m_1 <_\tau m_2$*.*

Example 3.2.6. *The graded lexicographic order and reverse graded lexicographic order respect the standard grading on* S*. More generally, the order* $<_{\mathbf{w}}$ *respects* $\mathbf{w}$*.*

For a monomial order τ which respects $\mathbf{w}$, define a natural extension τ' to $S[t]$ as follows: $x^a t^c <_{\tau'} x^b t^d$ iff (i) $x^a <_\tau x^b$ or (ii) $x^a = x^b$ and $c < d$, where, as usual, x^a denotes $x_1^{a_1} \dots x_r^{a_r}$.
This monomial order has the property that $\mathrm{in}_\tau(g) = \mathrm{in}_{\tau'}(g^h)$ for all non-zero $g \in S$.

Proposition 3.2.7. *Let $I \subset S$ be an ideal, and let $\{g_1, \dots, g_n\}$ be a Gröbner basis of I with respect to a monomial order τ which respects $\mathbf{w}$. Then, $\{g_1^h, \dots, g_n^h\}$ is a Gröbner basis of I^h with respect to τ'.*

Proof. Note that since I^h is a homogeneous ideal with respect to the extended weight $(w_1, \dots, w_r, 1)$, it is sufficient to prove that if $f \in I^h$ is homogeneous with respect to $(w_1, \dots, w_r, 1)$, then $\mathrm{in}_{\tau'}(f) \in \langle \mathrm{in}_{\tau'}(g_1^h), \dots, \mathrm{in}_{\tau'}(g_n^h) \rangle$.
Let $f \in I^h$, be homogeneous. Then, by the previous lemma, there exist $g \in I$ and $m \in \mathbb{Z}_+$ such that $f = t^m g^h$. Hence,

$$\mathrm{in}_{\tau'}(f) = t^m \mathrm{in}_{\tau'}(g^h) = t^m \mathrm{in}_\tau(g).$$

There exist $u \in S$ and $i \in \{1, \dots, n\}$ such that $\mathrm{in}_\tau(g) = u\mathrm{in}_\tau(g_i) = u\mathrm{in}_{\tau'}(g_i^h)$. Thus, $\mathrm{in}_{\tau'} f = ut^m \mathrm{in}_{\tau'}(g_i^h)$. □

Proposition 3.2.8. *Given an ideal $I \subset S$, $S[t]/I^h$ is a free $\mathsf{k}[t]$-module.*

Proof. Let $\{g_1, \dots, g_n\}$ be a Gröbner basis of I with respect to a monomial order τ graded with respect to $\mathbf{w}$. Then, $\{g_1^h, \dots, g_n^h\}$ is a Gröbner basis of I^h with respect to τ'. It follows from Theorem 2.3.4 that the set of all monomials in $S[t]$ not in $\langle \mathrm{in}_{\tau'}(g_1^h), \dots, \mathrm{in}_{\tau'}(g_n^h) \rangle$ forms a k-basis of $S[t]/I^h$. Since $\mathrm{in}_{\tau'}(g_i^h) = \mathrm{in}_\tau(g_i)$, we have $\langle \mathrm{in}_{\tau'}(g_1^h), \dots, \mathrm{in}_{\tau'}(g_n^h) \rangle = \langle \mathrm{in}_\tau(g_1), \dots, \mathrm{in}_\tau(g_n) \rangle S[t]$ and hence, the set of all monomials in S not in $\langle \mathrm{in}_\tau(g_1), \dots, \mathrm{in}_\tau(g_n) \rangle$ forms a $\mathsf{k}[t]$-basis of $S[t]/I^h$. □

Lemma 3.2.9. *Let R be a ring and consider $\phi : R[t] \to R$, a ring homomorphism with $\phi|_R = Id$, or equivalently, an R-linear ring homomorphism. Given an ideal $I \in R[t]$, ϕ naturally induces an R-linear ring homomorphism $\overline{\phi} : R[t]/I \to R/\phi(I)$ given by $\overline{\phi}(\overline{f}) = \overline{\phi(f)}$, and $\ker(\overline{\phi}) = (t - \phi(t))R[t]/I$.*

Proof. Clearly, $\overline{\phi}(f)$ is well-defined and $(t - \phi(t))R[t]/I \subset \ker(\overline{\phi})$.
Let $f \in R[t]$ such that $\overline{f} \in \ker(\overline{\phi})$. There exist $a \in R$ and $g \in R[t]$ such that $f = a + (t - \phi(t))g$, which implies that $\overline{\phi}(\overline{f}) = \overline{a}$. Thus, we have $a \in \phi(I)$. Let $h \in I$ such that $\phi(h) = a$, that is, $h = a + (t - \phi(t))h'$. Then,

$$f - h \in (t - \phi(t))R[t] \implies f \in I + (t - \phi(t))R[t],$$

which completes the proof. □

Proposition 3.2.10. *Given an ideal $I \subset S$ and a weight $\mathbf{w}$ on S, we have the following S-linear ring isomorphisms:*

$$\frac{S[t]/I^h}{tS[t]/I^h} \cong S/\mathrm{in}_{\mathbf{w}}(I) \text{ and } \frac{S[t]/I^h}{(t-a)S[t]/I^h} \cong S/I \;\; \forall a \in S \setminus \{0\}.$$

Proof. For all $a \in \mathsf{k}$, define an S-linear map $\phi_a : S[t] \to S$ as $\phi_a(1) = 1$ and $\phi_a(t) = a$. We claim that $\phi_0(I^h) = \mathrm{in}_{\mathbf{w}}(I)$.
Given $f \in I$, $\phi_0(f^h) = \mathrm{in}_{\mathbf{w}}(f)$. Since $I^h = \langle f^h : f \in I\rangle$, it follows that $\phi_0(I^h) = \mathrm{in}_{\mathbf{w}}(I)$. From the previous lemma, we have $\frac{S[t]/I^h}{tS[t]/I^h} \cong S\mathrm{in}_{\mathbf{w}}(I)$.
For $a \neq 0$, define a ring homomorphism $\psi_a : S \to S$ as $\psi_a(x_i) = a^{w_i}x_i$ for all i and $\psi_a|_{\mathsf{k}} = Id$. We claim that $\psi_a\phi_a(I^h) = I$. Then, according to the previous lemma, $\frac{S[t]/I^h}{(t-a)S[t]/I^h} \cong S/\phi_a(I^h)$ as S-modules and since $a \neq 0$, ψ_a is a ring isomorphism and $S/\phi_a(I^h) \cong S/I$ as rings.
By Proposition 3.2.7, there exists a Gröbner basis $\{g_1, \dots, g_n\}$ of I such that $\{g_1^h, \dots, g_n^h\}$ is a Gröbner basis of I^h. Let $g_i = \sum_j g_{ij}$ where g_{ij} denotes the homogeneous component of g_i of degree j (with respect to $\mathbf{w}$). Then,

$$\phi_a(g_i^h) = \sum_j a^{\deg_{\mathbf{w}} g_i - j} g_{ij},$$

and

$$\psi_a(\phi_a(g_i^h)) = a^{\deg_{\mathbf{w}} g_i} g_i.$$

Since $a \neq 0$, we are done. □

We now compare the Betti numbers of an ideal with those of its initial ideal.
Let $I \subset S$ be a graded ideal with respect to the standard grading on S, and fix a weight $\mathbf{w}$ on S. Let $\{g_1, \dots, g_n\}$ be a Gröbner basis of I with respect to a monomial order which respects $\mathbf{w}$, and further, such that g_i is homogeneous with respect to the standard grading for all i. Then, $\{g_1^h, \dots, g_n^h\}$ is a system of generators (in fact, a Gröbner basis) of I^h.
If we assign to each x_i the bidegree $(w_i, 1)$ and to t the bidegree $(1, 0)$, then all the generators g_i^h are bihomogeneous, and hence I^h is a bigraded ideal. Therefore $S[t]/I^h$ has a bigraded minimal free $S[t]$-resolution,

$$F_\bullet : 0 \to F_p \to F_{p-1} \to \cdots \to F_0 \to S[t]/I^h \to 0,$$

where $F_i = \oplus_{j,k}(S[t](-k, -j))^{\beta_{ijk}}$. Note that the minimality of the resolution is equivalent to the condition that all entries in the matrices describing the maps must belong to $\langle x_1, \dots, x_r, t\rangle$.
Note that as $S[t]/I^h$ is a free $\mathsf{k}[t]$-module, $t - a$ is a non-zero divisor on $S[t]/I^h$ for all $a \in \mathsf{k}$. Since t is a non-zero divisor on $S[t]/I^h$ and on $S[t]$, and $t \in \langle x_1, \dots, x_r, t\rangle$, $F_\bullet/tF_\bullet$ is a bigraded minimal free S-resolution of $\frac{S[t]/I^h}{tS[t]/I^h} \cong S/\mathrm{in}_{\mathbf{w}}(I)$. Observe that the bigraded shifts of $F_\bullet/tF_\bullet$ are the same as those in $F_\bullet$ and in particular, the second component of the shifts in the resolution are the ordinary shifts of the standard graded ideal $\mathrm{in}_{\mathbf{w}}(I)$. Thus, we have

$$\beta_{ij}(S/\mathrm{in}_{\mathbf{w}}(I)) = \sum_k \beta_{ijk} \, for \; all \; i, j.$$

On the other hand, since $t - 1$ is also a non-zero divisor on $S[t]/I^h$ and on $S[t]$, $F_\bullet/(t-1)F_\bullet$ is a free S-resolution of $\frac{S[t]/I^h}{(t-1)S[t]/I^h} \cong S/I$. Note that $t - 1$ is homogeneous with respect to the second component of the bidegree and hence the second components of the shifts in the resolution $F_\bullet$ are preserved. However, $t-1$ does not belong to $\langle x_1, \ldots, x_r, t\rangle$ and hence $F_\bullet/(t-1)F_\bullet$ need not be a minimal resolution. Therefore, we have

$$\beta_{ij}(S/I) \leq \sum_k \beta_{ijk} \; for \; all \; i, j.$$

We have thus proved the following theorem.

Theorem 3.2.11. *Let $I \subset S$ be a graded ideal and $\mathbf{w}$ be a weight. Then*

$$\beta_{ij}(I) \leq \beta_{ij}(\mathrm{in}_{\mathbf{w}}(I)) \; for \; all \; i, j.$$

Theorem 3.2.11 and Thereom 3.1.6 yield the following corollary.

Corollary 3.2.12. *Let $I \subset S$ be a graded ideal and τ be a monomial order on S. Then*

$$\beta_{ij}(I) \leq \beta_{ij}(\mathrm{in}_\tau(I)) \; for \; all \; i, j.$$

Corollary 3.2.13. *Given I, τ as above,*
(i) $\mathrm{pdim}(S/I) \leq \mathrm{pdim}(S/\mathrm{in}_\tau(I))$*;*
(ii) $\mathrm{depth}(S/I) \geq \mathrm{depth}(S/\mathrm{in}_\tau(I))$.

Proof. Corollary 3.2.12 directly implies (a). (b) follows from (a) and the Auslander-Buchsbaum formula. □

Proposition 3.2.14. *Let $I \subset S$ be a graded ideal. Then,*
(i) If $\mathrm{in}_{\mathbf{w}}(I)$ is a prime ideal, so is I.
(ii) If $\mathrm{in}_{\mathbf{w}}(I)$ is a radical ideal, so is I.

Proof. Let $I^h \in S[t]$ be the homogenization of I with respect to the weight $\mathbf{w}$. We claim that I is prime (resp. radical) if I^h is prime (resp. radical).
$\phi(f^h) = t^{\deg_{\mathbf{w}} f} f$.
Suppose I^h is prime. Consider $f, g \in S\backslash\{0\}$ such that $fg \in I$. Then, $(fg)^h = f^h g^h \in I^h$, which implies that $f^h \in I^h$ or $g^h \in I^h$. Without loss of generality, let $f^h \in I^h$. Then, by Remark 3.2.4, note that $f = \overline{(f^h)} \in I$.
Similarly, suppose I^h is radical. Consider $f \in S\backslash\{0\}$ such that $f^n \in I$ for $n \in \mathbb{N}$. Then, $(f^n)^h = (f^h)^n \in I^h$ and hence, $f^h \in I^h$. Proceeding as above, we have $f \in I$.
The following lemma along with Proposition 3.2.10 proves that if $\mathrm{in}_{\mathbf{w}}(I)$ is prime (resp. radical), so is I^h. □

Lemma 3.2.15. *Let R be a finitely generated positively graded k-algebra and let $s \in R$ be a homogeneous non-zero divisor of R such that R/sR is a domain (resp. a reduced ring) and $\deg(s) > 0$. Then R is also a domain (resp. a reduced ring).*

Proof. Suppose R/sR is a domain and there exist $a, b \in R\backslash\{0\}$ such that $ab = 0$. By the Krull Intersection Theorem, $\cap_{k\geq 0}\langle s\rangle^k = 0$ and hence, there exist $n_a, n_b \in \mathbb{Z}_{\geq 0}$ such that $a \in \langle s\rangle^{n_a}, b \in \langle s\rangle^{n_b}$ and $a \notin \langle s\rangle^{n_a+1}, b \notin \langle s\rangle^{n_b+1}$. Let $a = a's^{n_a}, b = b's^{n_b}$ where $a', b' \notin \langle s\rangle$. Then, $a'b' = 0$ and hence $\overline{a'b'} = 0$, which implies that $a' \in \langle s\rangle$ or $b' \in \langle s\rangle$, a contradiction.
Similarly, suppose R/sR is a reduced ring and there exists $a \in R\backslash\{0\}$ such that $a^n = 0$. Let n_a and a' be as above. Then, $a'^n = 0$ and hence $\overline{a'}^n = 0$, which implies that $a' \in \langle s\rangle$, a contradiction. □

Chapter 4

Polarization

As usual, let $S = \mathsf{k}[x_1, \dots, x_r]$.

Lemma 4.0.1. *Let $I \subset S$ be a monomial ideal with minimal generating set of monomials $\{m_1, \dots, m_n\}$, where $m_i = \prod_{j=1}^{n} x_j^{a_{ij}}$ for $i = 1, \dots, n$. Fix an integer $j \in [n]$ and suppose that $a_{ij} > 1$ for at least one $i \in [r]$. Let $T = S[y]$ and let $J \subset T$ be the monomial ideal with minimal generating set of monomials $\{m'_1, \dots, m'_n\}$, where*

$$m'_i = \begin{cases} m_i & a_{ij} = 0 \\ (m_i/x_j)y & a_{ij} \geq 1. \end{cases}$$

Then $y - x_j$ is a non-zero divisor in T/J and

$$\frac{T/J}{(y - x_j)T/J} \cong S/I$$

as S-modules.

Proof. Suppose $y - x_j$ is a zero divisor in T/J. Then $y - x_j \in P$ for some $P \in \mathrm{Ass}(J)$. By applying Proposition 1.1.5 on the $\mathbb{N}^r$-grading, P is a monomial ideal, and hence $y, x_j \in P$. Thus, there exists a monomial $f \in T \backslash J$ such that $yf, x_j f \in J$. Then there exist m'_k, m'_l and monomials $f_1, f_2 \in T$ such that $yf = m'_k f_1$ and $x_j f = m'_l f_2$.
Since $f \notin J$, x_j divides m'_l and hence, by the construction of J, y divides m'_l. This implies that y divides f. Note that y does not divide f_1 because $f \notin J$. This forces y^2 to divide m'_k, which is a contradiction to the construction of J.
Define a ring homomorphism $\phi : T \to S$ such that $\phi|_S = Id$ and $\phi(y) = x_j$. Then, $\phi(J) = I$ and by Lemma 3.2.9, we have the required isomorphism. $\square$

Motivated by Lemma 4.0.1, we define the polarization of a monomial ideal I.
Let $I \subset S$ be a monomial ideal with minimal generating set of monomials $\{m_1, \dots, m_n\}$, where $m_i = \prod_{j=1}^{n} x_j^{a_{ij}}$ for $i = 1, \dots, n$. For all $j = 1, \dots, r$, define $a_j = \max\{a_{ij} : i = 1, \dots, n\}$.

Let $T = \mathsf{k}[x_{11}, x_{12} \dots, x_{1a_1}, x_{21}, \dots, x_{2a_2}, \dots, x_{n1}, \dots, x_{na_n}]$. Define J to be a monomial ideal in T with generating set $\{m'_1, \dots, m'_n\}$ where

$$m'_i = \prod_{j=1}^{n} \prod_{k=1}^{a_{ij}} x_{jk}$$

for all $i \in [n]$.

Definition 4.0.2. *The monomial ideal J is called the **polarization** of I.*

Example 4.0.3. *Consider the ideal $\langle x_1x_2^2, x_2^4\rangle \subset \mathsf{k}[x_1, x_2]$. The polarisation of I is*

$$J = \langle x_{11}, x_{21}x_{22}, x_{21}x_{22}x_{23}x_{24}\rangle \subset \mathsf{k}[x_{11}, x_{21}, x_{22}, x_{23}, x_{24}].$$

Proposition 4.0.4. *Let $I \subset S$ be a monomial ideal and $J \subset T$ be its polarization. Then the sequence $\mathbf{z}$ given by*

$$x_{n1} - x_{na_n}, \dots, x_{n1} - x_{n2}, \dots, x_{21} - x_{2a_2}, \dots, x_{21} - x_{22}, \dots, x_{11} - x_{1a_1}, \dots, x_{11} - x_{12}$$

is a regular sequence on T/J and

$$\frac{T/J}{(z)T/J} \cong S/I$$

as graded k-algebras.

Proof. Firstly, replace x_i in S by x_{i1} for all $i \in [r]$. Let the minimal generating set of monomials of I be $\{m_1^{(11)}, \dots, m_n^{(11)}\}$ Now, let $T_{12} = S[x_{12}]$ and define $m_i^{(12)} = m_i^{(11)}$ if x_{11} does not appear in $m_i^{(11)}$ and $m_i^{(12)} = (m_i^{(11)}/x_{11})x_{12}$ otherwise. Let $J_{12} = \langle m_1^{(12)}, \dots, m_n^{(12)}\rangle$. By Lemma 4.0.1, $x_{11} - x_{12}$ is a non-zero divisor on T_{12}/J_{12} and

$$\frac{T_{12}/J_{12}}{(x_{11} - x_{12})T_{12}/J_{12}} \cong S/I.$$

Similarly, let $T_{13} = T_{12}[x_{13}]$ and define $J_{13} = \langle m_1^{(13)}, \dots, m_n^{(13)}\rangle$ $m_i^{(13)} = m_i^{(11)}$ if x_{11} does not appear in $m_i^{(11)}$ and $m_i^{(13)} = (m_i^{(11)}/x_{11})x_{13}$ otherwise. Note that

$$\frac{T_{13}/J_{13}}{(x_{11} - x_{13}, x_{11} - x_{12})T_{13}/J_{13}} \cong \frac{T_{12}/J_{12}}{(x_{11} - x_{12})T_{12}/J_{12}} \cong S/I.$$

Continue the process until T_{1a_1}. Then, let $T_{22} = T_{1a_1}[x_{22}]$. We eventually get $T_{na_n} = T$. Repeated application of Lemma 4.0.1 completes the proof. □

Corollary 4.0.5. *Let $I \subset S$ be a monomial ideal and $J \subset T$ be its polarization. Then*
(i) $\beta_{ij}(I) = \beta_{ij}(J)$ for all i, j;
(ii) $H_{S/I}(t) = (1-t)^{\delta} H_{T/J}(t)$ where $\delta = \dim T - \dim S$;
(iii) $\mathrm{pdim}(S/I) = \mathrm{pdim}(S/J)$ and $\mathrm{reg}(S/I) = \mathrm{reg}(T/J)$.

Proof. (i) Follows from the fact that $\mathbf{z}$ is a regular sequence on T/J.
(ii) Follows from Corollary 1.2.9.
(iii) Follows from (i). □

Chapter 5

The lexsegment ideal

Given a graded ideal $I \subset S$, our aim is to show the existence of a special ideal, the lexsegment ideal of I, denoted by I^{lex}, such that S/I and S/I^{lex} have the same Hilbert function.
By Corollary 2.3.5, S/I and $S/\text{in}_\tau(I)$ have the same Hilbert function for any monomial order τ on S. Thus, we can assume that I is a monomial ideal. By Theorem 2.3.4, the monomials in S not belonging to I form a k-basis of I and since this k-basis determines the Hilbert functions of S/I, the Hilbert function of S/I does not depend on the base field k. We can therefore assume that $\text{char}(\mathsf{k}) = 0$.
We denote by $M_d(S)$ the set of all monomials of S of degree d.

Definition 5.0.1. *A set $\mathcal{L} \subset M_d(S)$ is called a **lexsegment** if for all $m \in \mathcal{L}$, we have that $m' \in \mathcal{L}$ for all $m' \in M_d(S)$ such that $m' \geq_{\text{lex}} m$.*

Definition 5.0.2. *A set $\mathcal{L} \subset M_d(S)$ is called **strongly stable** if $x_i(m/x_j) \in \mathcal{L}$ for all $m \in \mathcal{L}$ and all pairs (i, j) such that $i < j$ and x_j divides m.*

For a monomial $m \in S$, we set $\gamma(m) = \max\{i : x_i \, divides \, m\}$.

Definition 5.0.3. *A set $\mathcal{L} \subset M_d(S)$ is called **stable** if $x_i(m/x_j) \in \mathcal{L}$ for all $m \in \mathcal{L}$ and all $i < \gamma(m)$.*

Definition 5.0.4. *A monomial ideal I is said to be a lexsegment ideal or a (strongly) stable monomial ideal, if for each d the monomials of degree d in I form a lexsegment, or a (strongly) stable set of monomials respectively.*

Remark 5.0.5. Note that every lexsegment set is strongly stable, and every strongly stable set is stable.

Example 5.0.6. *Let $S = \mathsf{k}[x, y, z, w]$.*
Suppose I_1 is the smallest lexsegment ideal containing xyz. Then $I_1 = \langle xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$.
Suppose I_2 is the smallest strongly stable ideal containing xyz. Then $I_2 = \langle xyz, xy^2, x^2z, x^2y, x^3\rangle$.
Suppose I_3 is the smallest stable ideal containing xyz. Then $I_3 = \langle xyz, xy^2, x^2y, x^3\rangle$.

Now we have that S/I and $S/gin_\tau(I)$ have the same Hilbert function, and that $gin_\tau(I)$ is a strongly stable ideal [2]. Hence, we can assume that I is a strongly stable ideal.

Theorem 5.0.7. *Let $I \subset S$ be a graded ideal. There exists a unique lexsegment ideal, denoted I^{lex}, such that S/I and S/I^{lex} have the same Hilbert function.*

Given a graded ideal I, with j^{th} graded component I_j, denote by I_j^{lex} the k-vector space spanned by the unique lexsegment $\mathcal{L}_j$ with $|\mathcal{L}_j| = \dim_{\mathsf{k}} I_j$. Define $I^{\text{lex}} = \oplus_j I_j^{\text{lex}}$.
Note that if I^{lex} as defined above is an ideal, it is the only possible lexsegment ideal such that S/I and S/I^{lex} have the same Hilbert function. Therefore, we only need to show that I^{lex} is an ideal to prove Theorem 5.0.7. It is sufficient to show that $\{x_1, \dots, x_r\}\mathcal{L}_j \subset \mathcal{L}_{j+1}$.

Definition 5.0.8. *Let $\mathcal{N}$ be a set of monomials in S. Then the shadow of $\mathcal{N}$ is said to be the set*

$$\text{Shad}(\mathcal{N}) = \{x_1, \dots, x_r\}\mathcal{N} = \{x_i u : u \in N, i = 1, \dots, n\}.$$

Lemma 5.0.9. *If $\mathcal{N} \subset M_d(S)$ is stable, strongly stable or lexsegment, then so is* $\text{Shad}(\mathcal{N})$*.*

Given $\mathcal{N} \subset M_d(S)$, we denote by $\gamma_i(N)$ the number of elements $\gamma(m) = i$ and set $\gamma_{\leq i}(\mathcal{N}) = \sum_{j=1}^{i} \gamma_i(\mathcal{N})$.

Lemma 5.0.10. *Let $\mathcal{N} \subset M_d(S)$ be a stable set of monomials. Then* $\text{Shad}(N)$ *is a stable set and*
(i) $\gamma_i(\text{Shad}(N)) = \gamma_{\leq i}(\mathcal{N})$;
(ii) $|\text{Shad}(N)| = \sum_{i=1}^{r} \gamma_{\leq i}(\mathcal{N})$.

Proof. (ii) follows directly from (i). To prove (i), define the map

$$\phi : \{m \in \mathcal{N} : \gamma(m) \leq i\} \to \{m \in \text{Shad}(\mathcal{N}) : \gamma(m) = i\},\ m \to mx_i.$$

ϕ is clearly injective. Let $m' \in \text{Shad}(\mathcal{N})$ such that $\gamma(m') = i$. There exists $j \in [r]$ and $m \in \mathcal{N}$ such that $m' = x_j m$. We must have $\gamma(m) \leq i$. If $j = i$, then we are done. If $j < i$, then $\gamma(m) = i$ and since $\mathcal{N}$ is stable, $m_1 = x_j(m/x_i) \in \mathcal{N}$. Hence, we have $m' = x_i m_1$ for $m_1 \in \mathcal{N}$. This proves that ϕ is a bijection, which implies (i). □

Theorem 5.0.11 (Bayer)**.** *Let $\mathcal{L} \subset M_d(S)$ be a lexsegment and $\mathcal{N} \subset M_d(S)$ be a strongly stable set of monomials with $|\mathcal{L}| \leq |\mathcal{N}|$. Then $\gamma_{\leq i}(\mathcal{L}) \leq \gamma_{\leq i}(\mathcal{N})$ for $i = 1, \dots, r$.*

Proof. Observe that we can write $\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_1 x_r \cup \dots \cup \mathcal{N}_d x_r^d$ where each $\mathcal{N}_j$ is a strongly stable set of monomials of degree $d - j$ in the variables $x_1, \dots, x_{r-1}$. The lexsegment $\mathcal{L}$ has a similar decomposition $\mathcal{L}_0 \cup \dots \cup \mathcal{L}_r x_r$, where each $\mathcal{L}_j$ is a lexsegment.
We prove the theorem by induction on the number of variables. If $r = 1$, we have that $\gamma_{\leq 1}(\mathcal{L}) = |\mathcal{L}| \leq |\mathcal{N}| = \gamma_{\leq 1}(\mathcal{N})$.
Let $r > 1$. We have that $\gamma_{\leq r}(\mathcal{L}) = |\mathcal{L}|$ and $\gamma_{\leq r}(\mathcal{N}) = |\mathcal{N}|$ and hence, $\gamma_{\leq r}(\mathcal{L}) \leq \gamma_{\leq r}(\mathcal{N})$. Note that for $i < r$, $\gamma_{\leq i}(\mathcal{L}) = \gamma_{\leq i}(\mathcal{L}_0)$ and $\gamma_{\leq i}(\mathcal{N}) = \gamma_{\leq i}(\mathcal{N}_0)$. Hence, if we show that $|\mathcal{L}_0| \leq |\mathcal{N}_0|$, the proof is done by induction.

For each j, let $\mathcal{N}_j^*$ be the lexsegment in $M_{d-j}(\mathsf{k}[x_1,\ldots,x_{r-1}])$ with $|\mathcal{N}_j^*| = |\mathcal{N}_j|$ and let $\mathcal{N}^* = \mathcal{N}_0^* \cup \mathcal{N}_1^* x_r \cup \cdots \cup \mathcal{N}_d^* x_r^d$. We claim that $\mathcal{N}^*$ is a strongly stable set of monomials.
Observe that it suffices to show that $\{x_1,\ldots,x_{r-1}\}\mathcal{N}_j^* \subset \mathcal{N}_{j-1}^*$. By using that $\mathcal{N}$ is a strongly stable set, we have that $\{x_1,\ldots,x_r\}\mathcal{N}_j \subset \mathcal{N}_{j-1}$. Then, by Lemma 5.0.10 and the induction hypothesis, we have that

$$|\{x_1,\ldots,x_{r-1}\}\mathcal{N}_j^*| = \sum_{i=1}^{r-1} \gamma_{\leq i}(\mathcal{N}_j^*) \leq \sum_{i=1}^{r-1} \gamma_{\leq i}(\mathcal{N}_j) = |\{x_1,\ldots,x_{r-1}\}\mathcal{N}_j| \leq |\mathcal{N}_{j-1}| = |\mathcal{N}_{j-1}^*|.$$

The fact that $|\{x_1,\ldots,x_{r-1}\}\mathcal{N}_j^*|$ and $|\mathcal{N}_{j-1}^*|$ are both lexsegments forces $|\{x_1,\ldots,x_{r-1}\}\mathcal{N}_j^*| \subset |\mathcal{N}_{j-1}^*|$, which implies that $\mathcal{N}^*$ is a strongly stable set of monomials.
Now, given a monomial $m = \prod_{i=1}^r x_i^{a_i}$, we set $\overline{m} = (x_{n-1}/x_n)^{a_n} m$. Observe that if $m_1 \leq m_2$ in the lexicographic order, then $\overline{m_1} \leq \overline{m_2}$.
Let $m_1 = \min \mathcal{L}$ and $m_2 = \min \mathcal{N}^*$. Since $\mathcal{N}^*$ is strongly stable, $\overline{m_2} \in \mathcal{N}_0^*$ and $\overline{m_2} \geq \min(\mathcal{N}_0^*)$. Further, $\min(\mathcal{N}_0^*) \geq m_2$, which implies that $\overline{\min(\mathcal{N}_0^*)} = \min(\mathcal{N}_0^*) \geq \overline{m_2}$. Hence, $\min(\mathcal{N}_0^*) = \overline{m_2}$ and similarly, $\min(\mathcal{L}_0^*) = \overline{m_1}$.
Since $|\mathcal{L}| \leq |\mathcal{N}| = |\mathcal{N}^*|$, we have that $m_1 \geq m_2$ and hence, $\overline{m_1} \geq \overline{m_2}$. As $\mathcal{L}_0$ and $\mathcal{N}_0^*$ are lexsegments, we get that $|\mathcal{L}_0| \leq |\mathcal{N}_0^*| = |\mathcal{N}_0|$, which completes the proof. $\square$

We now complete the proof of Theorem 5.0.7.
Recall that we may assume that I is strongly stable. Let $\mathcal{N}_j$ be the strongly stable set of monomials which spans the k-vector space I_j. Since $|\mathcal{L}_j| = |\mathcal{N}_j|$, Bayer's theorem together with Lemma 5.0.10 implies that

$$|\mathrm{Shad}(\mathcal{L}_j)| = \sum_{i=1}^{r} \gamma_{\leq i}(\mathcal{L}_j) \leq \sum_{i=1}^{r} \gamma_{\leq i}(\mathcal{N}_j) = |\mathrm{Shad}(\mathcal{N}_j)|.$$

Since I is an ideal, we have that $\mathrm{Shad}(\mathcal{N}_j) \subset \mathcal{N}_{j+1}$. Hence,

$$|\mathrm{Shad}(\mathcal{L}_j)| \leq |\mathrm{Shad}(\mathcal{N}_j)| \leq |\mathcal{N}_{j+1}| = |\mathcal{L}_{j+1}|.$$

Since $\mathrm{Shad}(\mathcal{L}_j)$ and $\mathcal{L}_{j+1}$ are both lexsegments, $|\mathrm{Shad}(\mathcal{L}_j)| \leq |\mathcal{L}_{j+1}|$ implies that $\mathrm{Shad}(\mathcal{L}_j) \subset \mathcal{L}_{j+1}$, as desired.

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