

NP-Hardness of the 3-SAT Problem

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Problem Statement

Is the 3-SAT (3-CNF-SAT) problem NP-Hard? Explain and justify your answer.

Background on SAT and 3-SAT

The Boolean Satisfiability Problem (SAT) asks whether there exists an assignment of truth values to variables that makes a Boolean formula true.

A Boolean formula is said to be in:

- **Conjunctive Normal Form (CNF)** if it is a conjunction of clauses
- Each clause is a disjunction of literals

The **3-SAT problem** is a restricted version of SAT where:

- Each clause contains exactly three literals

Despite this restriction, 3-SAT remains computationally difficult.

Why 3-SAT Belongs to NP

To show that a problem is in NP, we must show that a proposed solution can be verified in polynomial time.

Given:

- A 3-SAT formula
- A truth assignment for all variables

We can verify the solution by:

- Evaluating each clause in constant time
- Checking all clauses in time linear in the number of clauses

Thus, verification takes polynomial time, and:

$$3\text{-SAT} \in \text{NP}$$

Understanding NP-Hardness

A problem is said to be **NP-Hard** if every problem in NP can be reduced to it in polynomial time.

In practice, it is sufficient to show that one known NP-Complete problem can be reduced to the problem under consideration.

Reduction from SAT to 3-SAT

The general SAT problem is known to be NP-Complete.

Any SAT formula with clauses of arbitrary length can be transformed into an equivalent 3-SAT formula using a systematic method:

- Clauses with fewer than 3 literals are padded using repeated variables
- Clauses with more than 3 literals are split into multiple 3-literal clauses using new auxiliary variables

This transformation has the following properties:

- It preserves satisfiability (the new formula is satisfiable if and only if the original one is)
- The size of the formula grows only linearly
- The transformation runs in polynomial time

Hence:

$$\text{SAT} \leq_p \text{3-SAT}$$

Why This Implies NP-Hardness

Since:

- SAT is NP-Complete
- SAT can be reduced to 3-SAT in polynomial time

it follows that 3-SAT is at least as hard as SAT.

Therefore:

$$\text{3-SAT is NP-Hard}$$

Final Conclusion

Yes, the 3-SAT (3-CNF-SAT) problem is NP-Hard.

Furthermore, because 3-SAT is both NP-Hard and a member of NP, we conclude that:

3-SAT is NP-Complete

Significance

The NP-Hardness of 3-SAT makes it a cornerstone problem in computational complexity theory. Many NP-Hard problems in scheduling, graph theory, and optimization are proven difficult via polynomial-time reductions from 3-SAT.