Proof That Leaf Nodes in a Binary Heap Occupy the Last Indices

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Problem Statement

In an array of size n representing a binary heap, prove that all **leaf nodes** are located at indices:

$$\left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ to } n$$

Background

A binary heap is a complete binary tree stored in an array. For a node stored at index i:

- Left child index $= 2i$
- Right child index $= 2i + 1$

Array indexing is assumed to start from 1.

Method 1: Index-Based Mathematical Proof

A node is a **leaf node** if it has no children.

For a node at index i to have at least one child, its left child must exist:

$$2i \leq n$$

If:

$$2i > n$$

then the node has no children and is therefore a leaf node.

Dividing both sides by 2:

$$i > \frac{n}{2}$$

Since indices are integers:

$$i \geq \left\lfloor \frac{n}{2} \right\rfloor + 1$$

The maximum index in the array is n.

Conclusion (Method 1)

All leaf nodes lie in the index range:

$$\left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n$$

Method 2: Proof Using Properties of Complete Binary Trees

A binary heap is a **complete binary tree**, which means:

- All levels are fully filled except possibly the last level.
- The last level is filled from left to right.

Observation

- Internal nodes must appear before leaf nodes in the array.
- Every internal node has at least one child.

Counting Internal Nodes

A node can be an internal node only if it has at least one child:

$$2i \leq n$$

This implies:

$$i \leq \left\lfloor \frac{n}{2} \right\rfloor$$

Thus, the number of internal nodes is at most:

$$\left\lfloor \frac{n}{2} \right\rfloor$$

Identifying Leaf Nodes

All remaining nodes must be leaf nodes.

Hence, leaf nodes occupy the indices:

$$\left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ to } n$$

Conclusion (Method 2)

All leaf nodes in a binary heap are located from $\left\lfloor \frac{n}{2} \right\rfloor + 1$ to n

Final Conclusion

Both methods independently prove that in an array representation of a binary heap of size n, all leaf nodes are located at indices:

$$\boxed{\left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ to } n}$$