LU Decomposition Using Gaussian Elimination

Manav Mantry

Introduction

LU decomposition is a matrix factorization technique in which a given square matrix A is expressed as the product of two matrices:

$$A = LU$$

where:

- L is a **lower triangular matrix** with unit diagonal elements.
- U is an **upper triangular matrix**.

LU decomposition is commonly obtained using the steps of **Gaussian Elimination** and is widely used for solving systems of linear equations efficiently.

Gaussian Elimination Overview

Gaussian Elimination transforms a matrix into an upper triangular form by applying elementary row operations to eliminate variables below the pivot elements.

These same elimination steps can be systematically recorded to construct the L and U matrices.

Steps of LU Decomposition Using Gaussian Elimination

Consider a square matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Step 1: Initialize U

Begin Gaussian elimination on matrix A. The matrix obtained after all elimination steps will become the upper triangular matrix U.

Step 2: Eliminate Entries Below the First Pivot

Use the pivot element a_{11} to eliminate entries below it.

The multipliers used are:

$$l_{21} = \frac{a_{21}}{a_{11}}, \quad l_{31} = \frac{a_{31}}{a_{11}}$$

These multipliers are stored in the corresponding positions of matrix L.

Step 3: Eliminate Entries Below the Second Pivot

After the first elimination, use the second pivot element u_{22} to eliminate the entry below it.

The multiplier used is:

$$l_{32} = \frac{u_{32}}{u_{22}}$$

This value is also stored in matrix L.

Step 4: Form the L Matrix

The lower triangular matrix L is formed by placing:

- The computed multipliers below the diagonal
- Ones on the diagonal
- Zeros above the diagonal

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

Step 5: Form the U Matrix

The upper triangular matrix U consists of the matrix obtained after completing Gaussian elimination:

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Final Result

After completing all steps:

$$A = LU$$

where:

- L contains the elimination multipliers
- U is the row-echelon (upper triangular) form of A

Advantages of LU Decomposition

- Efficient solution of multiple systems $Ax = b$ with different b
- Reduces computational cost compared to repeated Gaussian elimination
- Useful in matrix inversion and determinant computation

Conclusion

LU decomposition provides a systematic way to factor a matrix using Gaussian Elimination. By separating the elimination process into lower and upper triangular matrices, it simplifies and accelerates the solution of linear systems.

$$\boxed{A = LU}$$