Structure of the Symmetric Difference of Two Matchings

Manav Mantry

Problem Statement

Prove that every connected component of the symmetric difference of two matchings in a graph G is either a path or an even-length cycle.

Definitions

Let $G = (V, E)$ be an undirected graph.

- A **matching** is a set of edges such that no two edges share a common vertex.
- Let M_1 and M_2 be two matchings in G.
- The **symmetric difference** of M_1 and M_2 is defined as:

$$M_1 \oplus M_2 = (M_1 \setminus M_2) \cup (M_2 \setminus M_1)$$

Key Observations

Consider the subgraph $H = (V, M_1 \oplus M_2)$.

Observation 1: Degree Constraint

Since both M_1 and M_2 are matchings:

- Each vertex is incident to at most one edge from M_1.
- Each vertex is incident to at most one edge from M_2.

Therefore, in the graph H, every vertex has degree at most:

$$\deg_H(v) \leq 2$$

Observation 2: Alternating Edges

At any vertex with degree 2 in H:

- One incident edge belongs to M_1
- The other incident edge belongs to M_2

Thus, edges in H alternate between M_1 and M_2.

Structure of Connected Components

Because every vertex in H has degree at most 2, each connected component of H must be one of the following:

- A path
- A cycle

We analyze both cases.

Case 1: Path Components

If a connected component contains a vertex of degree 1, then it must be a path. Such a path alternates between edges of M_1 and M_2.

The path may be of either even or odd length.

Case 2: Cycle Components

If every vertex in a connected component has degree exactly 2, then the component is a cycle.

Because edges alternate between M_1 and M_2, returning to the starting vertex requires an even number of edges.

Hence, every cycle in $M_1 \oplus M_2$ must be of even length.

Conclusion

Every connected component of the symmetric difference $M_1 \oplus M_2$ is either:

- A path, or
- An even-length cycle

$$\boxed{\text{Every connected component of } M_1 \oplus M_2 \text{ is either a path or an even-length cycle.}}$$