LU Decomposition of Positive-Definite Matrices Without Pivoting

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Problem Statement

Prove that positive-definite matrices are suitable for LU decomposition and do not require pivoting to avoid division by zero in the recursive Gaussian elimination strategy.

Definition of Positive-Definite Matrix

A real symmetric matrix $A \in \mathbb{R}^{n \times n}$ is said to be **positive-definite** if:

$$x^T A x > 0 \quad \text{for all non-zero vectors } x \in \mathbb{R}^n$$

Important properties of positive-definite matrices include:

- All eigenvalues of A are strictly positive.
- All leading principal minors of A are positive.
- A is non-singular.

LU Decomposition and Pivoting

LU decomposition factorizes a matrix as:

$$A = LU$$

where:

- L is a lower triangular matrix with unit diagonal
- U is an upper triangular matrix

During Gaussian elimination, pivoting is usually required to avoid division by zero or numerical instability when a pivot element becomes zero.

We show that this situation never arises for positive-definite matrices.

Role of Leading Principal Minors

A necessary and sufficient condition for the existence of LU decomposition *without pivoting* is that all leading principal minors of A are non-zero.

For a positive-definite matrix:

$$\det(A_k) > 0 \quad \text{for all } k = 1, 2, \ldots, n$$

where A_k is the leading $k \times k$ principal submatrix of A.

Hence, all leading principal minors are strictly positive and therefore non-zero.

Implication for Gaussian Elimination

In the recursive Gaussian elimination process:

- At step k, the pivot element is the diagonal entry u_{kk}.
- This pivot is equal to the ratio of successive leading principal minors.

Since all leading principal minors are positive:

$$u_{kk} \neq 0 \quad \text{for all } k$$

Thus:

- Division by zero never occurs.
- No row exchanges (pivoting) are required.

Recursive Schur Complement Argument

At each step of Gaussian elimination, the remaining submatrix is a Schur complement of the previous pivot block.

A fundamental property of positive-definite matrices is that their Schur complements are also positive-definite.

Hence, at every recursive step:

- The reduced matrix remains positive-definite.
- The next pivot element remains strictly positive.

This guarantees the success of the recursive elimination strategy.

Connection to Cholesky Decomposition

In fact, for positive-definite matrices, an even stronger factorization exists:

$$A = LL^T$$

known as the **Cholesky decomposition**, which also requires no pivoting.

LU decomposition can be viewed as a generalization of Cholesky decomposition for non-symmetric matrices.

Conclusion

Since positive-definite matrices have:

- Positive leading principal minors
- Non-zero pivots at every elimination step
- Positive-definite Schur complements at each recursion

they always admit LU decomposition without pivoting and do not suffer from division by zero in the recursive Gaussian elimination strategy.

$$\boxed{\text{Positive-definite matrices are always suitable for LU decomposition without pivoting.}}$$