Non-Singularity of the Schur Complement

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Problem Statement

Prove that if a matrix A is non-singular, then its Schur complement is also non-singular.

Preliminaries and Definitions

A square matrix is said to be **non-singular** if it is invertible, or equivalently, if its determinant is non-zero.

Let the matrix A be partitioned into block form as:

$$A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$$

where:

- B is a square matrix of order k
- E is a square matrix of order $n - k$
- B is assumed to be invertible

Definition of Schur Complement

The **Schur complement** of block B in matrix A is defined as:

$$S = E - DB^{-1}C$$

Our goal is to prove that S is non-singular provided A is non-singular.

Key Idea of the Proof

The core idea is that Gaussian elimination on a block matrix can be used to eliminate the off-diagonal block D without changing the singularity of the matrix. This elimination naturally produces the Schur complement.

Block Gaussian Elimination

Consider multiplying A on the left by the block lower triangular matrix:

$$L = \begin{bmatrix} I & 0 \\ -DB^{-1} & I \end{bmatrix}$$

This matrix L is invertible since its diagonal blocks are identity matrices.
Now compute:

$$LA = \begin{bmatrix} I & 0 \\ -DB^{-1} & I \end{bmatrix} \begin{bmatrix} B & C \\ D & E \end{bmatrix} = \begin{bmatrix} B & C \\ 0 & E - DB^{-1}C \end{bmatrix}$$

Thus, we obtain:

$$LA = \begin{bmatrix} B & C \\ 0 & S \end{bmatrix}$$

Effect on Singularity

Since:

- L is invertible
- Multiplication by an invertible matrix does not change singularity

it follows that:

$$A \text{ is non-singular} \iff LA \text{ is non-singular}$$

Determinant of the Transformed Matrix

The matrix LA is block upper triangular. The determinant of a block triangular matrix is the product of the determinants of its diagonal blocks:

$$\det(LA) = \det(B)\det(S)$$

Since:

- A is non-singular $\Rightarrow \det(A) \neq 0$
- L is invertible $\Rightarrow \det(L) \neq 0$

we have:

$$\det(LA) = \det(L)\det(A) \neq 0$$

Hence:

$$\det(B)\det(S) \neq 0$$

Because B is invertible, $\det(B) \neq 0$, which implies:

$$\det(S) \neq 0$$

Final Conclusion

Since the determinant of the Schur complement S is non-zero, it is invertible and therefore non-singular.

$$\boxed{\text{If } A \text{ is non-singular and } B \text{ is invertible, then the Schur complement } S = E - DB^{-1}C \text{ is also non-singula}}$$

Intuitive Explanation (Optional)

Non-singularity of A means the linear system $Ax = b$ has a unique solution. Eliminating variables associated with block B using Gaussian elimination leaves a reduced system governed by the Schur complement. If this reduced system were singular, the original system would not have a unique solution, contradicting the non-singularity of A.