Assignment No. 2

Manav_17

Q.4. Find all possible Laurent's series expansion and specify the domain of convergence.

$\frac{1}{z^2+4}$ about $z=-i$

→ we can express z^2+4 as $(z+2i)\cdot(z-2i)$

$\therefore \frac{1}{z^2+4} = \frac{1}{(z+2i)\cdot(z-2i)}$

we have to find series about $z=-i$. Let $u=z+i$

$\therefore f(z) = \frac{1}{(z+2i)(z-2i)}$

By partial fraction

$\frac{1}{z^2+4} = \frac{A}{z+2i} + \frac{B}{z-2i}$

By solving the eqns

$A = -\frac{1}{4i}$ and $B = \frac{1}{4i}$

$\therefore f(z) = \frac{-1}{4i[(z+i)+i]} + \frac{1}{4i[(z+i)-3i]}$

$u=z+i$

$= \frac{-1}{4i(u+i)} + \frac{1}{4i(u-3i)}$

$\therefore f(z) = \frac{1}{4i}\left[\frac{1}{u-3i} - \frac{1}{u+i}\right]$

a. Case 1: $|u-3i|<3$

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Su.
→

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$$\therefore f(z) = \frac{1}{4i}\left[\frac{1}{3i\left(\frac{u}{3i}-1\right)} - \frac{1}{u\left(1+\frac{i}{u}\right)}\right]$$

$$= \frac{1}{4i}\left[\frac{1}{3i}\left(\frac{u}{3i}-\right)^{-1} - \frac{1}{u}\left(1+\frac{i}{u}\right)^{-1}\right]$$

$$\therefore f(z) = \frac{1}{4i}\left[\frac{-1}{3i}\left(1+\frac{u}{3i}+\left(\frac{u}{3i}\right)^2+\cdots\right) - \frac{1}{u}\left(1-\frac{i}{u}+\left(\frac{i}{u}\right)^2-\cdots\right)\right]$$

b. case 2 $|u+i|<1$

$$f(z) = \frac{1}{4i}\left[\frac{1}{u}\left(1+\frac{3i}{u}+\left(\frac{3i}{u}\right)^2+\cdots\right) - \frac{1}{i}\left(1-\frac{u}{i}+\left(\frac{u}{i}\right)^2-\cdots\right)\right]$$

c. case 3 $|u-3i|>3$ and $|u+i|>1$

$$\therefore f(z) = \frac{1}{4i}\left[\frac{1}{u(1-3i/u)} - \frac{1}{u(1+i/u)}\right]$$

$$\therefore f(z) = \frac{1}{4i}\left[\frac{1}{u}\left(1+\frac{3i}{u}+\left(\frac{3i}{u}\right)^2+\cdots\right) - \frac{1}{u}\left(1-\frac{i}{u}+\left(\frac{i}{u}\right)^2+\cdots\right)\right]$$

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Q.6 Find the laurent's series expansion of $\frac{z+5}{(z+1)^2(z-2)}$ convergent in region

i. $0<|z+1|<3$ 2. $|z+1|>3$ 3. $1<|z|<2$

u. $|z-2|>3$ 5. $1<|z-1|<2$

→ Solution

$f(z) = \frac{z+5}{(z+1)^2(z-2)}$ is not analytic at $z=-1,2$.

$f(z) = \frac{z+5}{(z+1)^2(z-2)} = \frac{A}{(z+1)} + \frac{B}{(z+1)^2} + \frac{C}{(z-2)}$ ...(1)

$z+5 = A(z+1)(z-2) + B(z-2) + C(z+1)^2$

for $z=2$, $7 = C\times 3^2$

$\therefore C = \frac{7}{9}$

For $z=-1$, $4 = B\times(-3)$ $\therefore B = \frac{-4}{3}$

coefficient of z^2 $0 = A+C = A+\frac{7}{9}$ $\therefore A = \frac{-7}{9}$

Now, eqn (1), becomes

$f(z) = -\frac{7}{9}\times\frac{1}{(z+1)} - \frac{4}{3}\times\frac{1}{(z+1)^2} + \frac{7}{9}\times\frac{1}{(z-2)}$

1. For $0<|z+1|<3$

$\frac{1}{(z-2)} = \frac{1}{(z+1)-3} = \frac{-1}{3}\times\left[\frac{1}{1-\left(\frac{z+1}{3}\right)}\right]$

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$\rightarrow$ $f(z) = -\frac{1}{3} \times \left[1 + \frac{(z+1)}{3} + \frac{(z+1)^2}{3^2} + \cdots \right]$

for $|z+1| < 3$

$\therefore f(z) = -\frac{7}{9} \times \frac{1}{(z+1)} - \frac{4}{3} \times \frac{1}{(z+1)^2} + \frac{7}{9} \times \frac{(-1)}{3} \left[1 + \frac{(z+1)}{3} + \frac{(z+1)^2}{3^2} + \cdots \right]$

for $0 < |z+1| < 3$

2. For $|z+1| > 3$

$\frac{1}{(z-2)} = \frac{1}{(z+1)-3} = \frac{1}{(z+1)} \times \frac{1}{\boxed{1 - \frac{3}{(z+1)}}}$

$\frac{1}{(z-2)} = \frac{1}{(z+1)} \times \left[1 + \frac{3}{(z+1)} + \frac{3^2}{(z+1)^2} + \cdots \right]$

for $\left| \frac{3}{(z+1)} \right| < 1 \Rightarrow |z+1| > 3$

$\therefore f(z) = -\frac{7}{9} \times \frac{1}{(z+1)} - \frac{4}{3} \times \frac{1}{(z+1)^2} + \frac{7}{9} \left[\frac{1}{(z+1)} + \frac{3}{(z+1)^2} + \frac{3^2}{(z+1)^2} + \cdots \right]$

for $|z+1| > 3$

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Q.6

$\to$ 3. $1<|z|<2$

$$\frac{1}{(z+1)} = \frac{1}{(1+z)} = 1+z+z^2+\dots \quad \text{for } |z|<1$$

$$\frac{1}{(z+1)^2} = \frac{1}{(1+z)^2} = 1-2z+3z^2 \dots \quad \text{for } |z|<1$$

$$\frac{1}{(z-2)} = \frac{1}{-2} \times \frac{1}{\left(1-\frac{z}{2}\right)} = -\frac{1}{2} \times \left[1+\frac{z}{2}+\frac{z^2}{2^2}+\dots\right]$$

for $|z|<2$

$$\therefore f(z) = -\frac{7}{9}\left[1+z+z^2+\dots\right] - \frac{4}{3}\times\left[1-2z+3z^2-\dots\right]$$

$$-\frac{7}{18}\left[1+\frac{z}{2}+\frac{z^2}{2^2}+\dots\right] \quad \text{for } 1<|z|<2$$

4. $|z-2|>3$

$$\frac{1}{(z+1)} = \frac{1}{(z-2)+3} = \frac{1}{3} \times \frac{1}{\left(1+\frac{(z-2)}{3}\right)}$$

$$= \frac{1}{3} \times \left[1-\frac{(z-2)}{3}+\frac{(z-2)}{3^2}-\dots\right] \quad \text{for } |z-2|<3$$

$$\therefore f(z) = -\frac{7}{9}\times\left[\frac{1}{(z-2)} - \frac{3}{(z-2)^2} + \frac{3^2}{(z-2)^3} - \dots\right]$$

$$-\frac{4}{3}\left[\frac{1}{(z-2)} + \frac{2\times3}{(z-2)^2} + \frac{3\times3^2}{(z-2)^2} + \dots\right] + \frac{7}{9}\times\frac{1}{(z-2)} \quad \text{for}$$

$|z-2|>3$

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5 $1<|z-1|<2$

$\rightarrow$

$$\frac{1}{z+1} = \frac{1}{(z-1)+2} = \frac{1}{2}\frac{1}{\left[\frac{(z-1)}{2}+1\right]}$$

$$= \frac{1}{2} \times \left[1 - \frac{(z-1)}{2} + \frac{(z-1)^2}{2^2} - \cdots\right] \quad \text{for } |z-1|<2$$

$$\frac{1}{(z+1)^2} = \frac{1}{\left[2+(z-1)\right]^2} = \frac{1}{4} \times \frac{1}{\left[1+\frac{(z-1)}{2}\right]^2}$$

$$\frac{1}{(z-2)} = \frac{1}{(z-1)-1} = -1 \times \frac{1}{\left[1-(z-1)\right]} = \frac{1}{(z-1)} \times \frac{1}{\left[1-\frac{1}{(z-1)}\right]}$$

$$\frac{1}{(z-2)} = \frac{1}{(z-1)} \times \left[1 + \frac{1}{(z-1)} + \frac{1}{(z-1)^2} + \cdots\right] \cdot \text{for}$$

$|z-1|>1$

$$\therefore f(z) = \frac{-7}{9} \times \frac{1}{2} \times \left[1 - \frac{(z-1)}{2} + \frac{(z-1)^2}{2^2} - \cdots\right]$$

$$\frac{-4}{3} \times \frac{1}{4}\left[1 - \frac{(z-1)}{2} + \frac{(z-1)^2}{2^2} - \cdots\right]$$

$$+\frac{7}{9} \times \left[\frac{1}{(z-1)} + \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3} + \cdots\right] \quad \text{for } 1<|z-1|<2$$

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Q.7. Find the type of singularity of $\frac{1}{z-\sin z}$ at $z=0$. Find residue at this point

→ we have

$$f(z) = \frac{1}{z-\left[z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right]}$$

$$= \frac{1}{\frac{z^3}{3!}\left[1-\frac{3!}{5!}z^2+\cdots\right]}$$

Hence, $z=0$. is a pole of order 3

Now, $f(z) = \frac{1}{z^3/3!}\left[1-\frac{1}{20}z^2+\cdots\right]^{-1}$

$$= \frac{6}{z^3}\left[1+\frac{z^2}{20}+\cdots\right]$$

$$= \frac{6}{z^2}+\frac{3}{10}\cdot\frac{1}{z}+\cdots$$

∴ Residue at $z=0 = b_1$.

$b_1 =$ Coefficient of $\frac{1}{z} = \frac{3}{10}$

∴ Residue at $z=0$ will be $\frac{3}{10}$

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Q.15. Evaluate $\int_C \frac{1}{z^3} e^{1-\cos z}\, dz$, $|z|=1$.

→ $f(z)$ has pole of order 3 at $z=0$ and point lies inside C

∴ Residue (at $z=0$)

$= \lim_{z\to 0} \frac{1}{2!} \left[\frac{d^2}{dz^2}\left(z^3 \cdot \frac{1}{z^3} e^{1-\cos z} \cdot dz\right)\right]$

$= \lim_{z\to 0} \frac{1}{2} \left(\frac{d^2}{dz^2} (e^{1-\cos z}) \cdot dz\right)$

$= \lim_{z\to 0} \frac{1}{2} \left[\frac{d}{dz}\left(e^{1-\cos z} \cdot \frac{d}{dz}(1-\cos z)\right)\right]$

$= \lim_{z\to 0} \frac{1}{2}\left[\frac{d}{dz}(\sin z \cdot e^{1-\cos z})\right]$

$= \lim_{z\to 0} \frac{1}{2}\left[e^{1-\cos z}\left[\sin^2(z) + \cos(z)\right]\right]$

$= \frac{1}{2} \times 1.$

$\therefore \int_C f(z)\,dz = 2\pi i\left(\frac{1}{2}\right)$

$\therefore \int_C f(z).dz = \pi i$ //