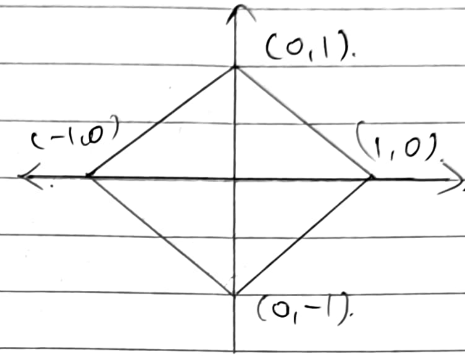
Maths Assignment - 1   manav - 17

Q. 3. $\int_C |z|^2 dz$, C: Square with vertices $(-1,0),(0,-1),(1,0),(0,1)$

→ $C = C_1 + C_2 + C_3 + C_4$

a. For C_1 :- $(-1,0)$ to $(0,-1)$



$\frac{y-0}{0+1} = \frac{x+1}{-1}$ $\therefore y = -x-1$.

$\therefore dy = -dx$

x from -1 to 0

$I_1 = \int_{C_1} |z|^2 dz = \int_{C_1} (x^2+y^2)(dx+i\,dy)$.

$= \int_{-1}^{0} [x^2 + (-x-1)^2](dx - i\,dx)$

$= \int_{-1}^{0} (x^2+x^2+2x+1)(dx - i\,dx)$

$= \int_{-1}^{0} (2x^2+2x+1)\,dx - i\int_{-1}^{0} (2x^2+2x+1)\,dx$

$= \left[\frac{2x^3}{3} + \frac{2x^2}{2} + x\right]_{-1}^{0} - i\left[\frac{2x^3}{3} + \frac{2x^2}{2} + x\right]_{-1}^{0}$

$= \frac{2}{3} - \frac{2i}{3}$

b. For C_2 :- $(0,-1)$ to $(1,0)$

$\frac{y+1}{-1-0} = \frac{x-0}{0-1}$ $\therefore y+1 = x$

i.e. $y = x-1$.

$dy = dx$

from $x=0$ to $x=1$.

Q.3.

-> $I_2 = \int_{C2} |z|^2 dz = \int_{C2} (x^2+y^2)(dx+idy)$.

$= \int_0^1 [x^2+(x-1)^2](dx+idy)$

$= \int_0^1 (x^2+x^2-2x+1)(dx+idx)$

$= \int_0^1 (2x^2-2x+1)\,dx + i\int_0^1 (2x^2-2x+1)\,dx$

$= \left[\frac{2x^3}{3}\right]_0^1 - \left[\frac{2x^2}{2}\right]_0^1 + [x]_0^1 + i\left\{\left[\frac{2x^3}{3}\right]_0^1 - \left[\frac{2x^2}{2}\right]_0^1 + [x]_0^1\right\}$

$= \frac{2}{3} + \frac{2i}{3}$

c. For C3:- (1,0) to (0,1)

$\frac{y-0}{0-1} = \frac{x-1}{1-0}$ $\therefore -y = x-1$.

$-dy = dx$

x from 1 to 0

$I_3 = \int_{C3} |z|^2 dz = \int_{C3} (x^2+y^2)(dx+idy)$

$= \int_1^0 [x^2+(-x+1)^2](dx-idx)$

$= \int_1^0 (x^2+x^2-2x+1)(dx-idx)$

$= \int_1^0 2x^2-2x+1 - \int_1^0 2x^2-2x+1 = -\frac{2}{3} + \frac{2i}{3}$

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Q.3

a.

C_4 :- $(0,1)$ to $(-1,0)$

$$\frac{y-1}{1-0} = \frac{x-0}{0+1} \quad : y-1=x$$

i.e. $y = x+1$.

$dy = dx$

x from 0 to -1.

$$I_4 = \int_{C_4} |z|^2 dz = \int_{C_4} (x^2+y^2)(dx+idy)$$

$$= \int_0^{-1} (x^2+2x+x^2+1)(dx+idx)$$

$$= \int_0^{-1} (2x^2+2x+1)\,dx + i\int_0^{-1} (2x^2+2x+1)\,dx$$

$$= -\frac{2}{3} - \frac{2}{3}i$$

$C = C_1 + C_2 + C_3 + C_4$

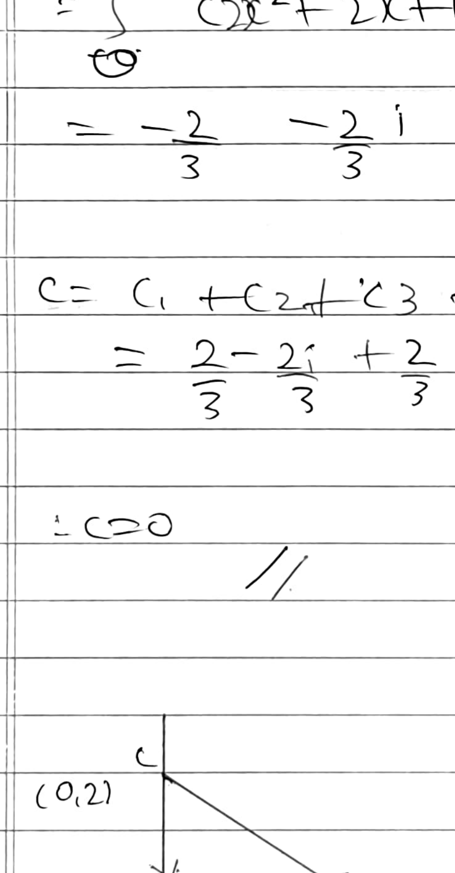
$$= \frac{2}{3} - \frac{2i}{3} + \frac{2}{3} + \frac{2i}{3} - \frac{2}{3} + \frac{2i}{3} - \frac{2}{3} - \frac{2i}{3}$$

$\therefore C = 0$

//

Q.7

→



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Q.7.

→ a. Now along line AB joining points (0,0) and (2,0)

Eqn of line $\frac{y-0}{0-0} = \frac{x-0}{0-2}$, ie $-2y = 0$

By differentiating, $dy = 0$

Now, $z = x + iy$

$z^2 = (x+iy)^2 = x^2 + 2xyi + y^2$

$\therefore dz = dx + idy$

$\therefore dz = dx \rightarrow dy = 0$

$C_1 = \int_0^2 x^2 + 2xyi - y^2 \cdot dx = \int_0^2 [x^2 + 0 - 0]\, dx$

$= \left(\frac{x^3}{3}\right)_0^2$

$\therefore C_1 = \frac{8}{3}$

b. Now, along line BC joining points (2,0), and (0,2).

Eqn. of line $\frac{y-0}{0-2} = \frac{x-2}{2-0}$ ie $x + y = 2$

By differentiating

$dx + dy = 0$ ie $dx = -dy$

$dz = dx + idy = dx + i(-dx) = (1-i)dx$

$C_2 = \int_2^0 x^2 + 2xyi - y^2 \cdot (1-i)dx$

$= \int_2^0 \{x^2 + 2x(2-x)i - (2-x)^2\} (1-i)dx$

$= (1-i) \int_2^0 (4xi - 2x^2 i - 4 + 4x)\, dx$

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Q.7.

b. $\therefore C_2 = (1-i)\left[-8i + \frac{16i}{3}\right]$

Now along line CA joining points (0,2) and (0,0)

Eqn of line $\frac{y-2}{2-0} = \frac{x-0}{0-0}$ i.e. $2x = 0$

By differentiating, $dx = 0$

Now, $C_3 = \int_0^2 (x^2 - y^2 + 2xyi)(i\,dy)$

$= \int_0^2 -y^2 i\,dy = -i\left[\frac{y^3}{3}\right]_0^2$

$\therefore C_3 = \frac{-8i}{3}$

Now for given triangle.

$C = C_1 + C_2 + C_3$

$= \frac{8}{3} - 8i + \frac{16i}{3} + 8i^2 - \frac{16i^2}{3} - \frac{8i}{3}$

$\therefore C = -\frac{16i}{3} - \frac{32}{3}$

//

Q.11. $I = \int_C \frac{1}{z(z-1)^2(z+3)}\,dz$ $|z| = 2$

$\rightarrow \frac{1}{z(z-1)^2(z+3)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{(z-1)^2} + \frac{D}{(z+3)}$

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Q.11

→ $1 = A(z-1)^2(z+3) + B(z)(z-1)(z+3) + C(z)(z+3) + D(z)(z-1)$

$\therefore 1 = A(z^2-2z+1)(z+3) + B(z^2-z)(z+3) + C(z^2+3z) + D(z^2-z)$

Coefficient of constant, $A = \frac{1}{3}$

Coefficient of z ; $0 = -5A - 3B + 3C - D$ --- (1)
Coefficient of z^2 ; $0 = 1A + 2B + C + D$ --- (2)
$\therefore B = \frac{-1}{3}$

Similarly, $C = \frac{-1}{12}$, $D = \frac{-11}{12}$

$$\therefore \frac{1}{z(z-1)^2(z+3)} = \frac{1/3}{z} + \frac{(-1/3)}{(z-1)} + \frac{(-1/12)}{(z-1)^2} + \frac{(-11/12)}{(z+3)}$$

$\therefore f(z)$ is not analytic at
$z(z-1)^2(z+3) = 0$
$\therefore z = 0,\ z = +1,\ z = -3$

$C: |z| = 2$
$z = +1$ and $z = 0$ both lies inside C.

$$I = \int_C f(z)\,dz = \frac{1}{3}\int_C \frac{1}{z}\,dz - \frac{1}{3}\int_C \frac{1}{(z-1)}\,dz - \frac{1}{12}\int_C \frac{1}{(z-1)^2}\,dz$$

$$-\frac{11}{12}\int_C \frac{1}{(z+3)}\,dz$$

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Q.11

→ $\therefore I = \frac{1}{3}2\pi i - \frac{1}{3}2\pi i - \frac{1}{12}2\pi i - 0 = 2\pi i\left[\frac{-1}{12}\right]$

$\therefore I = \frac{-\pi i}{6}$ //

Q.13. $I = \int_C \frac{2z^3+z^2+4}{z^4+4z^2}\,dz, \quad C: |z-2-2i| = 3.$

→ For $f(z)$ is not analytic at.

$z^4+4z^2 = 0$

$z^2(z^2+4) = 0$

$z = 0,\ z = +2i,\quad z = -2i$

for

$C: |z-(2+2i)| = 3$

$z = 0$, and $z = +2i$ lies inside the curve

$\frac{1}{z^4+4z^2} = \frac{1}{z^2(z-2i)(z+2i)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{(z-2i)} + \frac{D}{(z+2i)}$

$\therefore 1 = A(z)(z-2i)(z+2i) + B(z-2i)(z+2i) + C(z^2)(z+2i) + D(z^2)(z-2i)$

Put $z=0$ | Put $z=2i$ | Put $z=-2i$

$B = \frac{1}{4}$ | $C = \frac{1}{16}i$ | $D = \frac{-1}{16}i$

and $A = 0$

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Q.13

→

$$\therefore \frac{2z^3+z^2+4}{z^4+4z^2} = 0 + \frac{1}{4}\frac{(2z^3+z^2+4)}{z^2} + \frac{1}{16}\frac{i(2z^3+z^2+4)}{(z-2i)}$$

$$-\frac{1}{16}\frac{i(2z^3+z^2+4)}{(z+2i)}$$

$$I = \int_C f(z)\cdot dz$$

$$= \frac{1}{4}\times\frac{2\pi i}{1!}\frac{d}{dz}[(2z^3+z^2+4)]_{z=0} + \frac{i}{16}\times 2\pi i\,[2z^3+z^2+4]_{z=2i}$$

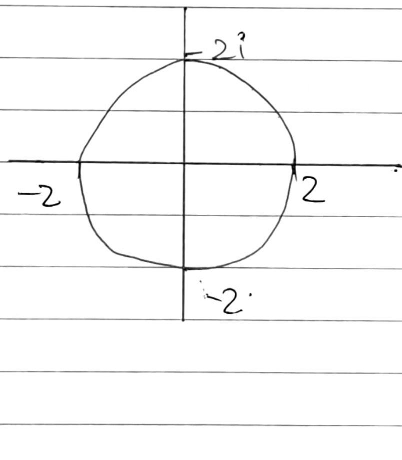
$$-0$$

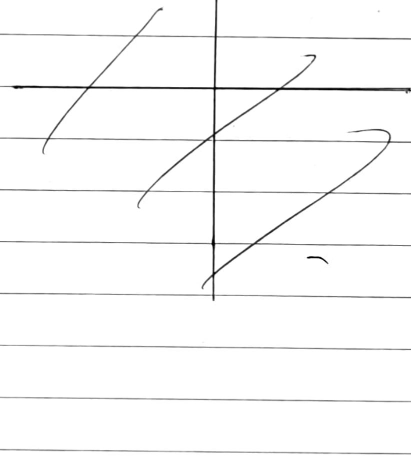
$$\therefore I = \frac{\pi i}{2}\times 0 + 2\pi i + \frac{4\pi}{8} - \frac{4\pi}{8}$$

$$\therefore I = 2\pi i$$ //.

Q.14

→





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S(4)

→ For f(1) the point $z=1$, lies inside the circle.
$f(z) = z^2+z+1$ is analytic in and on C and $z=1$ lies inside it. Hence by Cauchy's formula,

$$f(1) = \int_C \frac{z^2+z+1}{z-a}\, dz$$

the formula-

$$\int_C f(z)\, dz = \int_C \frac{g(z)}{(z-z_0)^n}\, dz$$

$$\int_C f(z)\, dz = \frac{2\pi i}{(n-1)!} \left[\frac{d^{n-1}}{dz^{n-1}} g(z)\right]_{z=z_0}$$

$$\therefore f(1) = \frac{2\pi i}{(1-1)!} \left[(z^2+z+1)\right]_{z=1}$$

→ $\therefore \cancel{f}(1) = 6\pi i$ //.

$$\therefore f(3i) = \int_C \frac{z^2+z+1}{z-a}\, dz$$

→ $f(3i) = 0$ //.

$\therefore$ For $f'(a)$, $f'(a) = \frac{d}{dz} \int_C \frac{z^2+z+1}{z-a}\, dz$

$$\therefore f'(a) = \int_C \frac{z^2+z+1}{(z-a)^2}\, dz$$

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Q14.

→ for $f'(i)$, $f'(i) = \frac{2\pi i}{1!} \left[\frac{d}{dz}(z^2+z+1)\right]_{z=i}$

→ $\therefore f'(i) = -4\pi + 2\pi i$ //

$f''(2.5) = \int_C \frac{z^2+z+1}{z-a}\, dz$

→ $\therefore f''(2.5) = 0$ //

for $f''(-1)$

$f''(a) = \frac{d}{da} \int_C \frac{z^2+z+1}{(z-a)^2}\, dz$

$f''(a) = \int_C \frac{z^2+z+1}{(z-a)^3}\, dz$

$\therefore f''(-1) = \frac{2\pi i}{(3-1)!} \left[\frac{d^{3-1}}{dz^{3-1}}(z^2+z+1)\right]_{z=-1}$

→ $\therefore f''(-1) = 2\pi i$ //