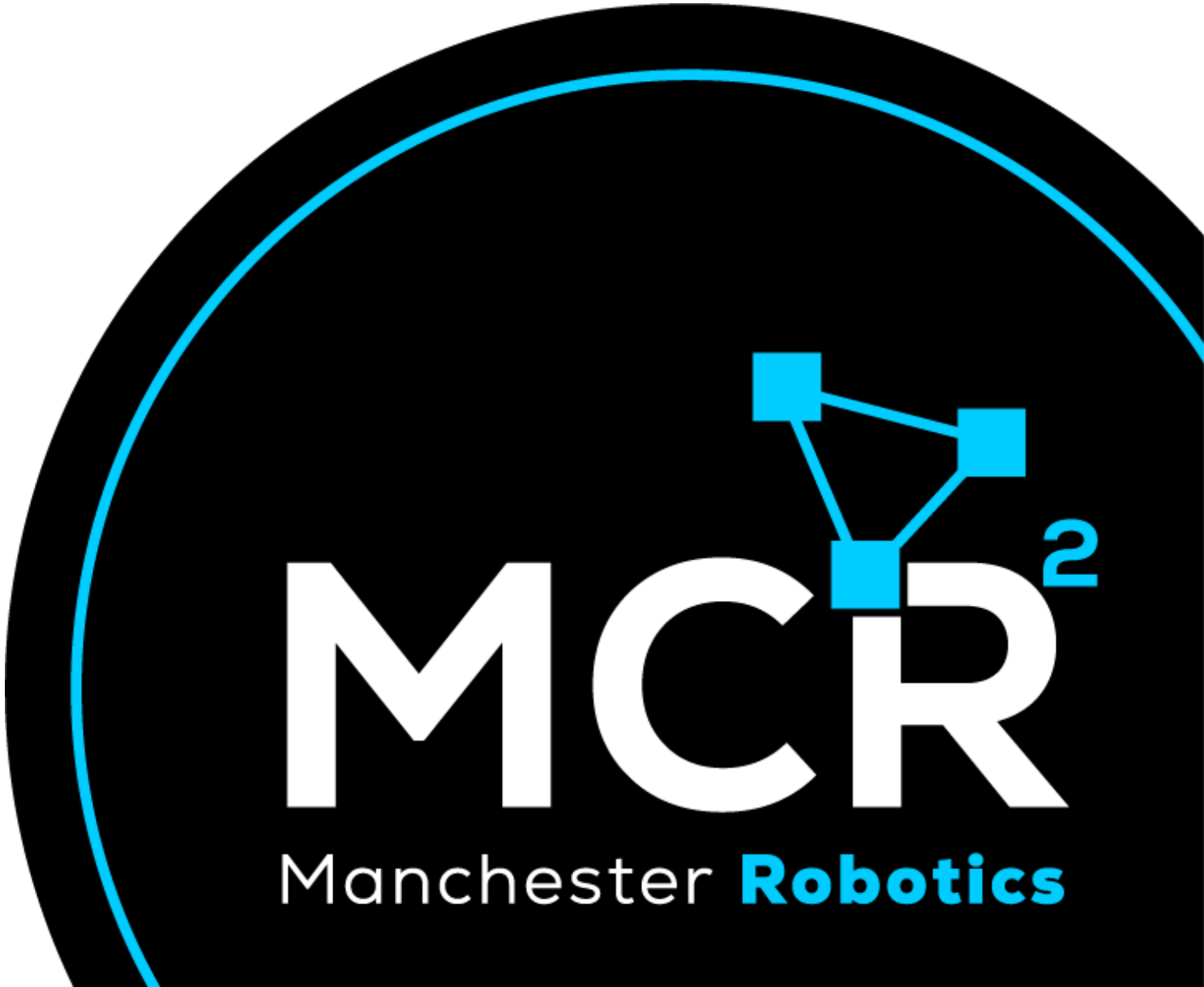
{Learn, Create, Innovate};

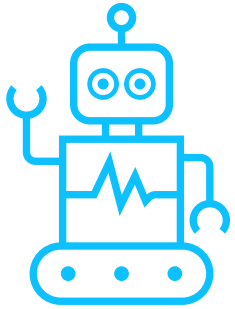
Dynamical Systems

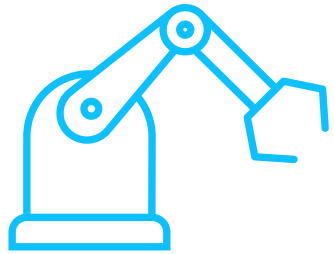
State Space Representation



Systems: general aspects

- In general, the notion of a **system** is used in many fields of activity. With this notion, we want to delimitate a form of existence in a well-defined space.
- Some examples of dynamic systems: the democratic system, the education system of a country, the nervous system, the automatic temperature regulation system, a mobile robot, a robotic manipulator, etc.

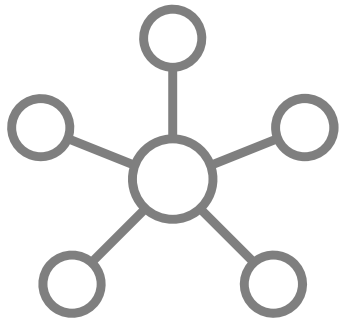


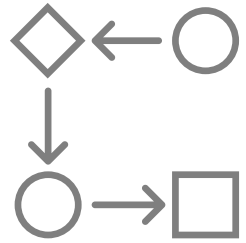


Property of MCR2

Systems: general aspects

- The notion of system helps us, in a first instance, to delimitate, for example:
  - A state management mechanism
  - A way of education at the national level
  - Part of the components that contribute to the integration of the human body into the environment
  - The elements necessary to obtain a constant temperature in an enclosure
  - The elements of a mobile robot, the elements of a robotic manipulator, etc





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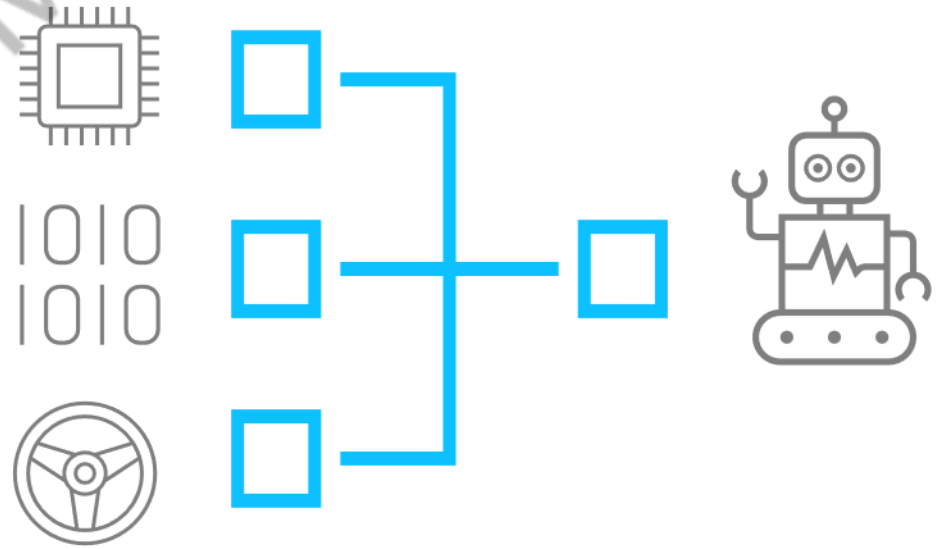
Systems: general aspects

- Generally speaking, a dynamic system is a structure that has multiple connections between its component parts.
- Another characteristic is that its elements are structured according to the same criteria or to achieve the same goal.
- In many situations a system can contain subsystems that can be regarded as independent systems.



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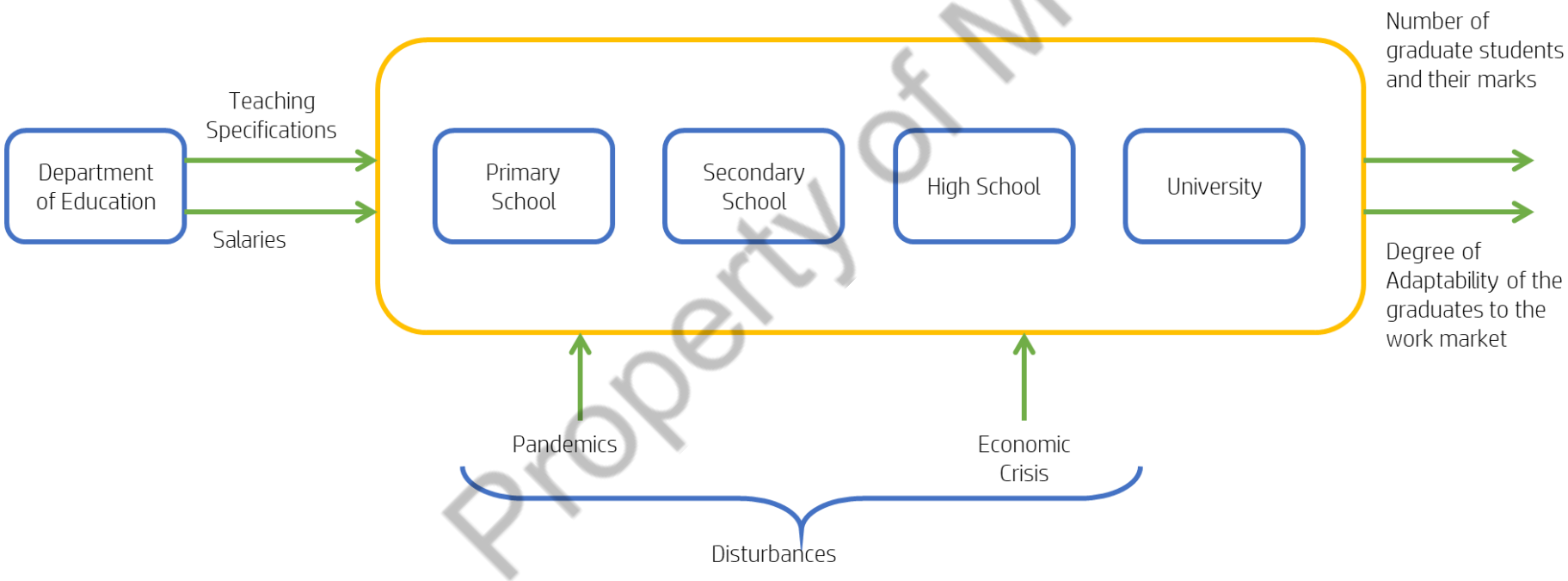
Systems general aspects

For a better understanding, consider one of the examples stated above, namely the education system. The block diagram of the mentioned system is represented in the following figure:



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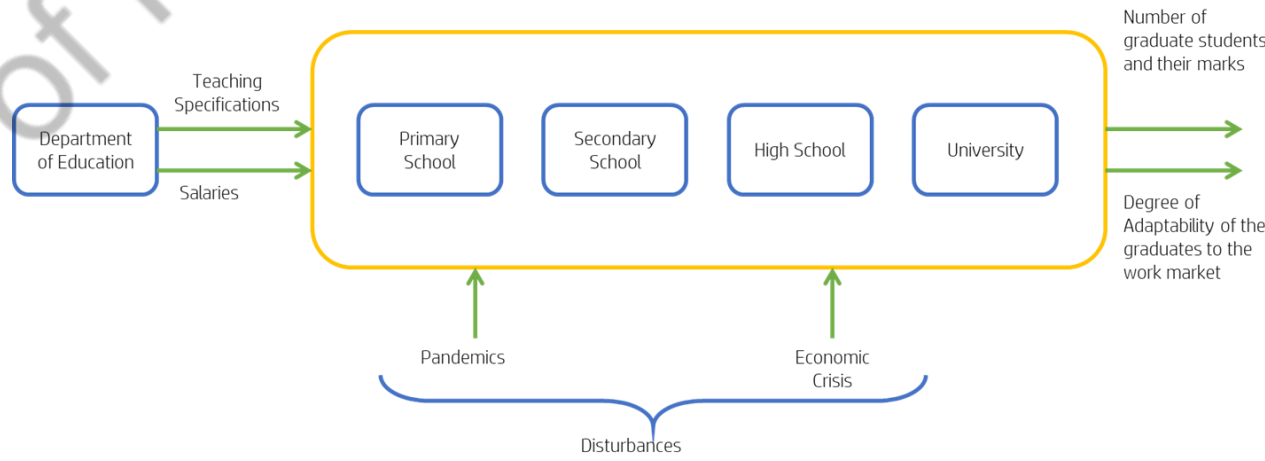
This is just a representation of a fictitious education system any similarity with reality is purely coincidental. In reality, the education system is more complex and presents other inputs, outputs and disturbances.

Systems general aspects

Aspects regarding the dynamic nature of the education system:

- The output performance depends on the structure of the teaching specifications over a period of time that includes the current year and also a number of previous years
- The salary has an important contribution to the quality of the teaching. Good salary attracts good teachers
- The relations between schools and universities have a decisive role in terms of the quality and continuity of the educational process
- The professionalism of the academics
- The facilities and the labs of each education institution contribute to student formation
- Another aspect regarding the output performance is related to the presence of disturbances, which can have negative consequences if a rejection mechanism is not applied.



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Dynamical systems

A dynamical system consists of two elements:

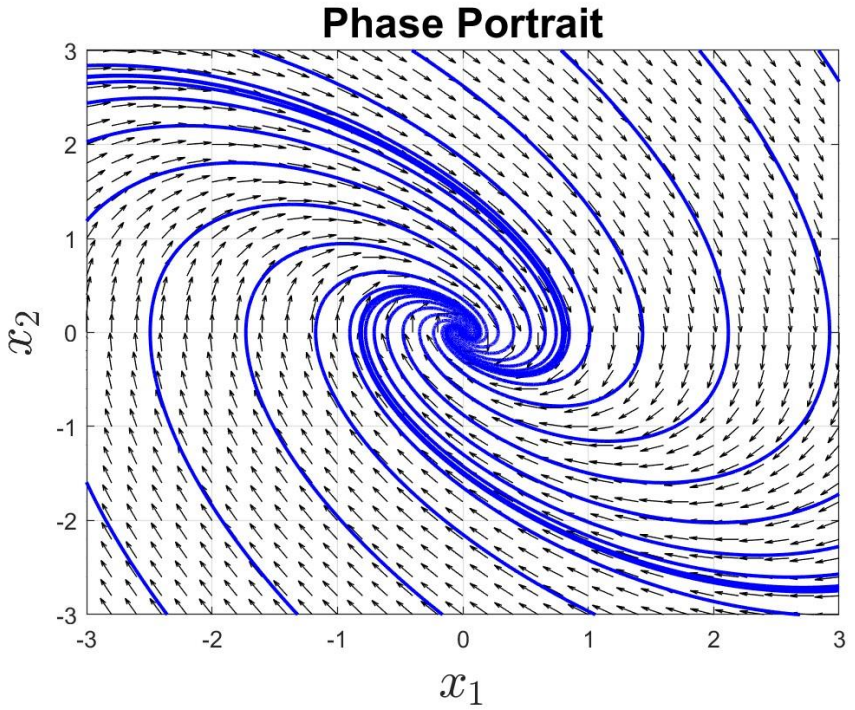
1. A non-empty space $\mathcal{D}$ (for instance $\mathbb{R}^2$)
2. A map from this space and the time into the same space $f : \mathcal{D} \times \mathbb{R} \to \mathcal{D}$

A dynamical system is described by the following differential equation (ODE – ordinary differential equation):

$$\dot{x}(t) = f(x(t), t) \tag{1}$$

This can be seen as a geometrical concept. In other words, for every point of the space $x \in \mathcal{D}$, the function $f(x, t)$ provides the information about the evolution of the system at the instant t.

Given an initial condition, the trajectory of the state follows the field of velocities $f = f(x, t)$. When the function f does not depend on time, i.e., $f = f(x)$, then the system is said to be a time-invariant system.



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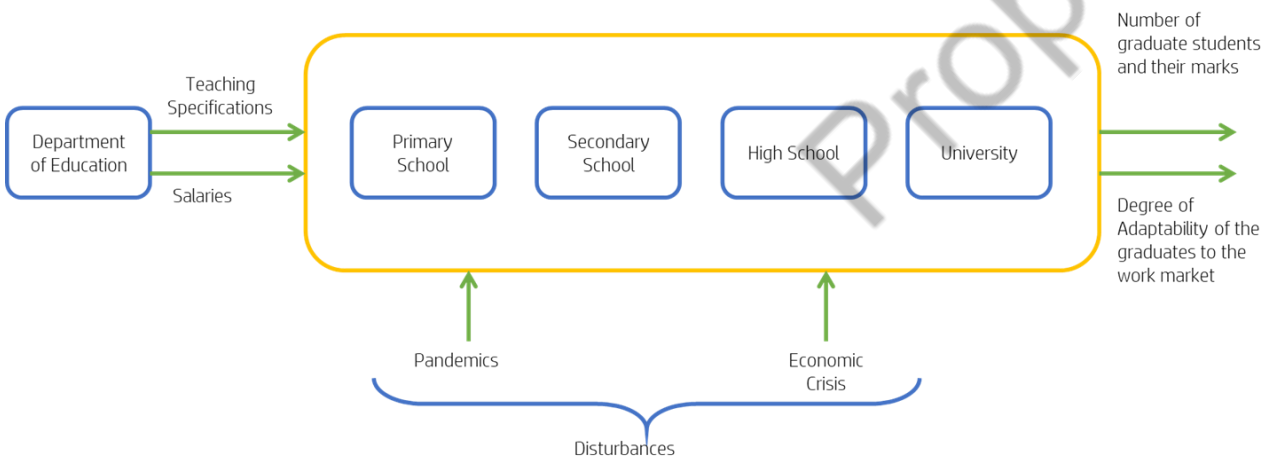
Dynamical systems

- A fundamental concept in dynamical systems is the **state** of the system.
- The **state** can be defined as the minimal information determining the system's future. In other words, the variables that describe the behaviour of the system.

Q: Which are the states for Educational systems illustrated previously?



- The dynamical systems can be classified in autonomous systems and non-autonomous systems.
- This classification is given by the source of energy which determine the future evolution of the system.
- For instance, the system (1) is isolated from the rest of the Universe, its evolution only depends on itself, and we say that the system (1) is an autonomous system.

Q: The Educational systems illustrated previously is autonomous or non-autonomous?

Property of MCR2

Dynamical systems

Worked example

- Let us consider the electrical circuit in Figure 1.

Q: Which are the states (the set of coordinates) which can describe the dynamics of this electrical circuit?

A: The dynamics of the circuit can be described using infinite set of coordinates, but two sets are straightforward:

- The changes at the capacitors $q = (q_1, q_2)$
- The current $i = (i_1, i_2)$.

In this example, we are going to model the same circuit using both sets of coordinates.

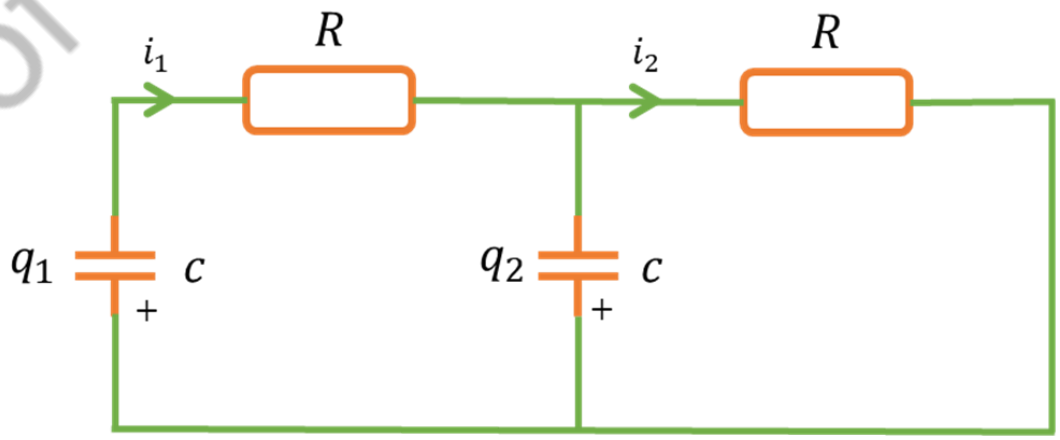


Figure 1: Electrical Circuit

Property of MCR2

Dynamical systems

Worked example

- The system can be analysed using two methodologies: using the charges $\boldsymbol{q}$ or the current $\boldsymbol{i}$.

1. Using the charge $\boldsymbol{q}$:
   - Applying Kirchhoff's Voltage Law (KVL) on the left mesh:

$$\sum_i V_i^{left} = \frac{1}{c}q_1 + i_1 R - \frac{1}{c}q_2 = 0 \Rightarrow$$
$$\Rightarrow i_1 = -\frac{1}{cR}q_1 + \frac{1}{cR}q_2 \quad (2)$$

   - And using KVL on the right:

$$\sum_i V_i^{right} = i_2 R + \frac{1}{c}q_2 = 0 \Rightarrow$$
$$\Rightarrow i_2 = -\frac{1}{cR}q_2 \quad (3)$$

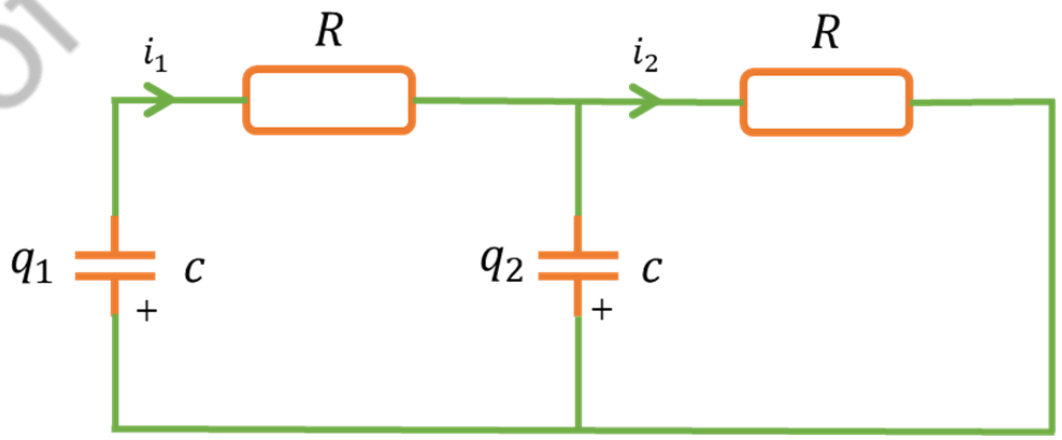


Figure 1: Electrical Circuit

Property of MCR2

Dynamical systems

Worked example

- Moreover, both charges and currents are related as follows:

$$i_x = i_2 - i_1 \tag{4}$$

- Therefore,

$$\dot{q_1} = i_1 = -\frac{1}{cR}q_1 + \frac{1}{cR}q_2 \tag{5}$$

$$\dot{q_2} = i_x = i_2 - i_1 = -\frac{1}{cR}q_2 + \frac{1}{cR}q_1 - \frac{1}{cR}q_2 \tag{6}$$

- Or equivalently, the matrix form:

$$\dot{\boldsymbol{q}} = \begin{bmatrix} -\frac{1}{Rc} & \frac{1}{Rc} \\ \frac{1}{Rc} & -\frac{2}{Rc} \end{bmatrix} \boldsymbol{q} \tag{7}$$

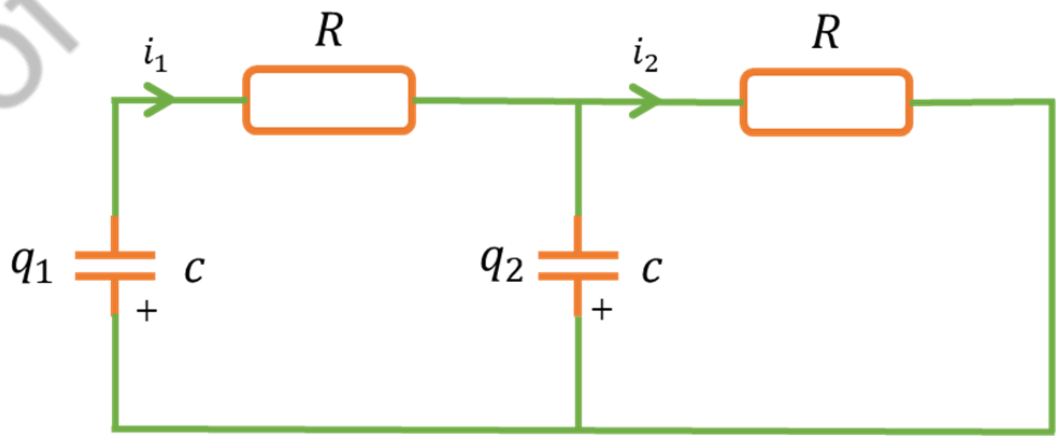


Figure 1: Electrical Circuit

Property of MCR2

Dynamical systems

Worked example

2. Using the current $\boldsymbol{i}$:

- The time derivatives of (3) and (2) are given by:

$$\dot{i_2}R + \frac{1}{c}\dot{q_2} = 0 \Rightarrow \dot{q_2} = -Rc\dot{i_2} \quad (8)$$

$$\frac{1}{c}\dot{q_1} + \dot{i_1}R - \frac{1}{c}\dot{q_2} = 0 \Rightarrow \dot{q_1} = i_1 = -Rc\dot{i_1} - Rc\dot{i_2} \quad (9)$$

- The dynamical equations in the capacitors can be written as:

$$i_1 = \dot{q_1} = -Rc\dot{i_1} - Rc\dot{i_2} \quad (10)$$

$$\dot{q_2} = i_2 - i_1 = -\frac{1}{cR}q_2 + \frac{1}{cR}q_1 - \frac{1}{cR}q_2 \quad (11)$$

- Reordering the above equation, we get the result in the matrix form:

$$\dot{\boldsymbol{i}} = \begin{bmatrix} -\frac{2}{Rc} & \frac{1}{Rc} \\ \frac{1}{Rc} & -\frac{1}{Rc} \end{bmatrix} \boldsymbol{i} \quad (12)$$

- **Q:** *This electrical circuit is an autonomous systems or a non-autonomous system?*

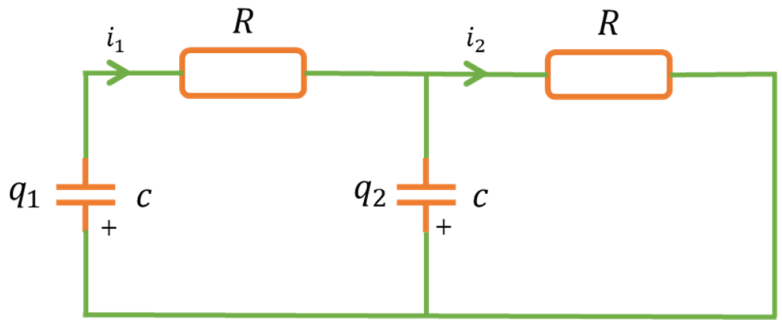


Figure 1: Electrical Circuit

Property of MCR2

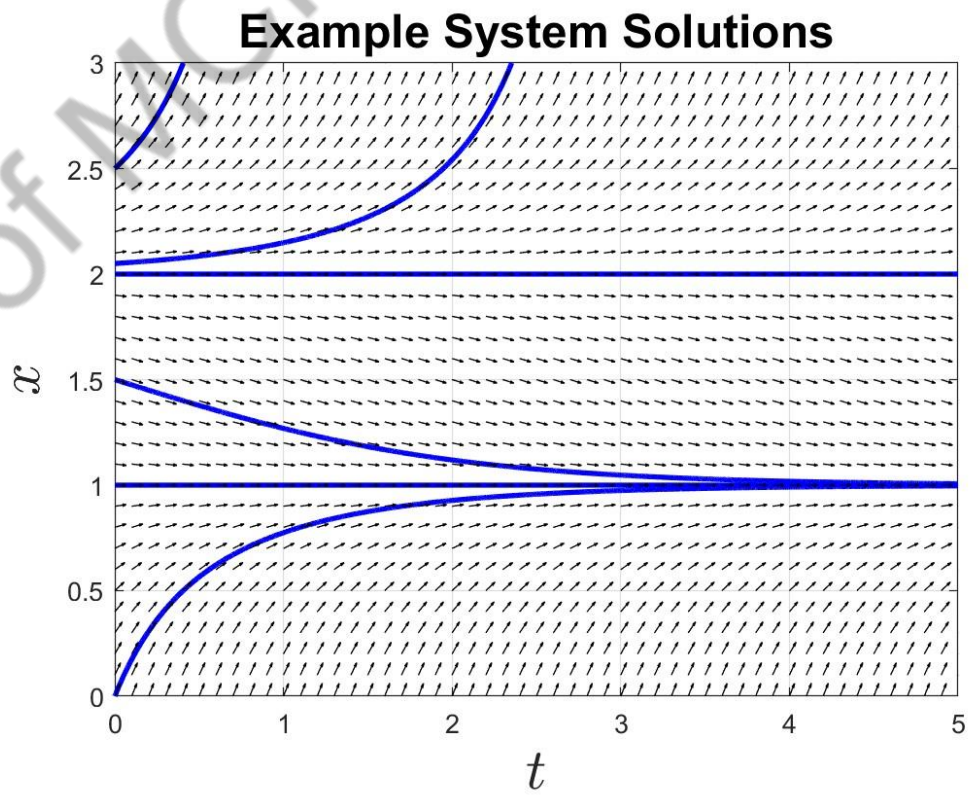
State-space representation of a linear system

Dynamical Systems Models

The key point in control engineering and systems theory is interaction.

We are interested in studying the dynamical evolution of interconnected systems.

In particular, feedback systems are the most important for us as robotics and control engineers.



Property of MCR2

State-space representation of a linear system

Dynamical Systems Models

- Therefore, we would like to model our system as a dynamical system, including explicitly input u and output y:

$$\dot{x} = f(x,u) \qquad x \in \mathbb{R}^{n_x}, \qquad u \in \mathbb{R}^{n_u} \tag{13}$$

$$y = h(x,u) \qquad y \in \mathbb{R}^{n_y} \tag{14}$$

where n_x is the number of state coordinates, n_u is the number of inputs, n_y is the number of outputs.

- This representation is called state-space representation.
- Is a very general, and most real systems can be modelled by (13) and (14).
- The equations (13) and (14) are referred to as the system equation and the output equation, respectively.
- In contrast with the transfer function representation of a system, the state-space representation is not limited to linear systems.

Property of MCR2

State-space representation of a linear system

Linear Dynamical Systems Models

- The general definition of a dynamical system can be used to describe the behaviour of a linear system as follows:

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^{n_x}, \qquad u \in \mathbb{R}^{n_u} \tag{15}$$

$$y = Cx + Du \qquad y \in \mathbb{R}^{n_y} \tag{16}$$

where $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times n_u}$, $C \in \mathbb{R}^{n_y \times n_x}$, and $D \in \mathbb{R}^{n_y \times n_u}$.

.

- Equations (15) and (16) are said to be the state-space representation of a linear system.
- In short, the four matrices (A, B, C, D) represent a time-invariant linear (LTI) system.
- For systems with single input and output, i.e., $n_u = n_y = 1$, B is a column vector, C is a row vector and D is a number.
- These systems are referred to as Single-Input Single-Output (SISO).
- Systems with several inputs and several outputs, i.e., $n_u > 1$, $n_y > 1$, are referred to as Multiple-Input Multiple-Output (MIMO).

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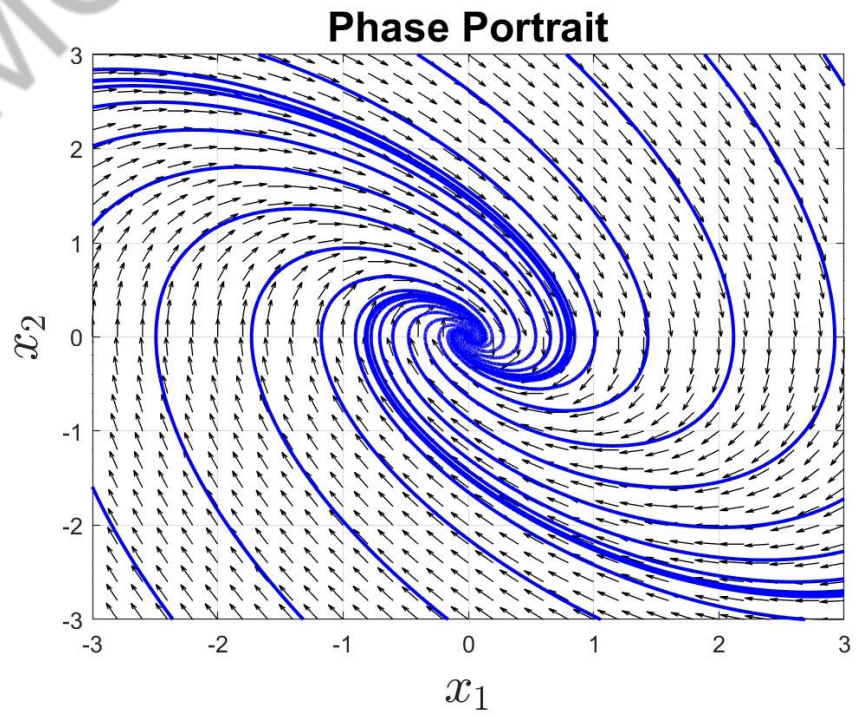
State-space representation of a linear system

- For today lecture we restrict our attention to SISO systems.
- Any ordinary differential equation in the form:

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_1 \frac{du}{dt} + b_0 u \quad (17)$$

with $m < n$ has an equivalent state-space representation.



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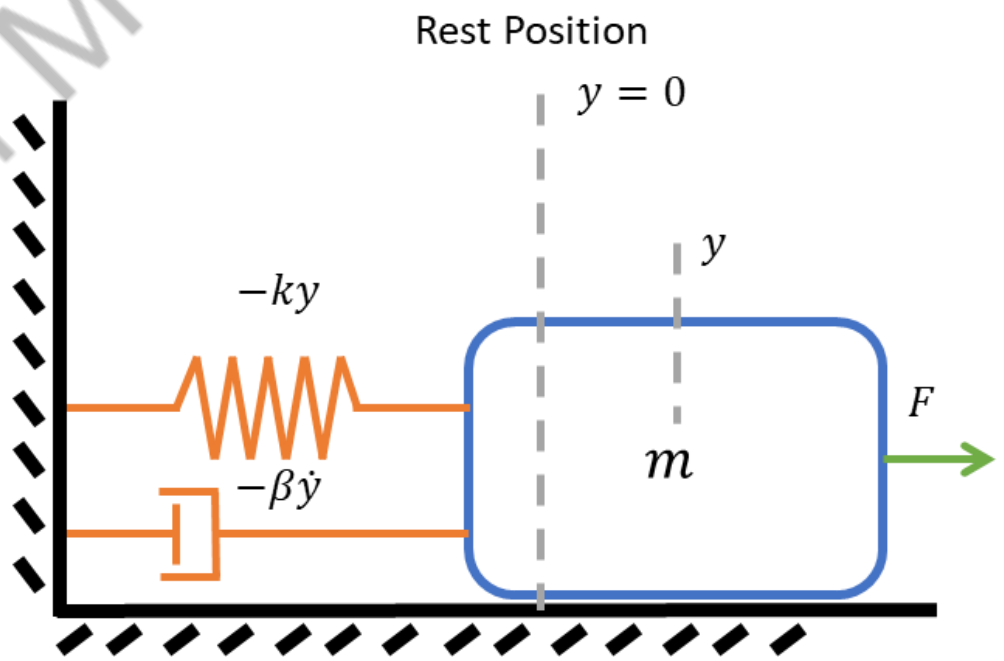
State-space representation of a linear system

Worked example

- Let us consider an Ideal Mass-Spring-Damper system where an external force F is applied on the mass. The output of the system is the position of the mass y.



Q: Which are the states (the set of coordinates) that describe the dynamics of this mechanical system?

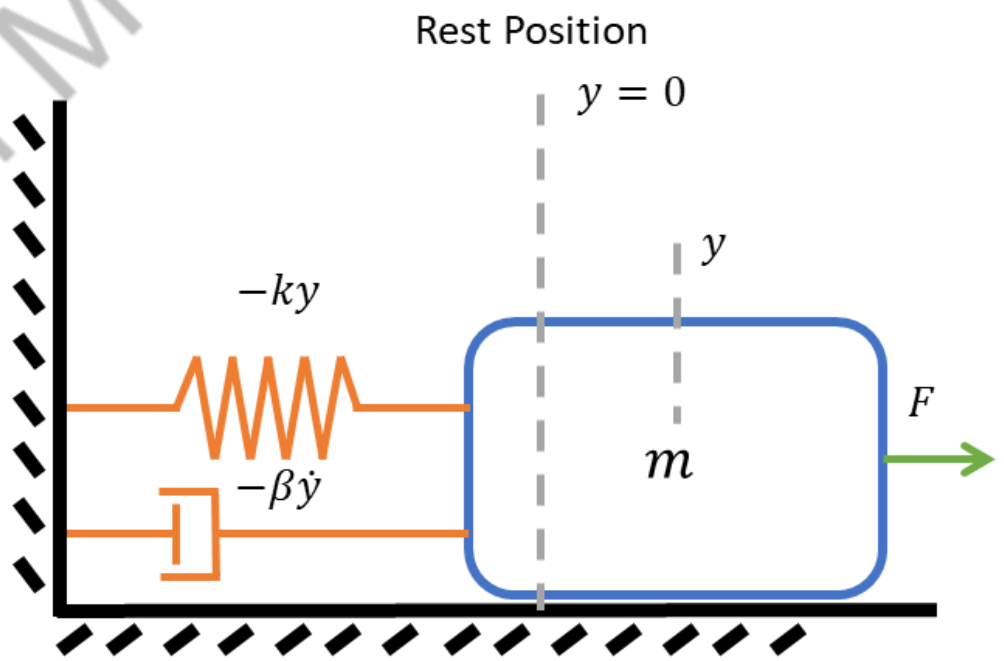
Property of MCR2

State-space representation of a linear system

Worked example

- Applying Newton's second law, the dynamics of the system are given by:

$$\sum_i F_i = ma = m\,\ddot{y} \tag{18}$$



Property of MCR2

State-space representation of a linear system

Worked example

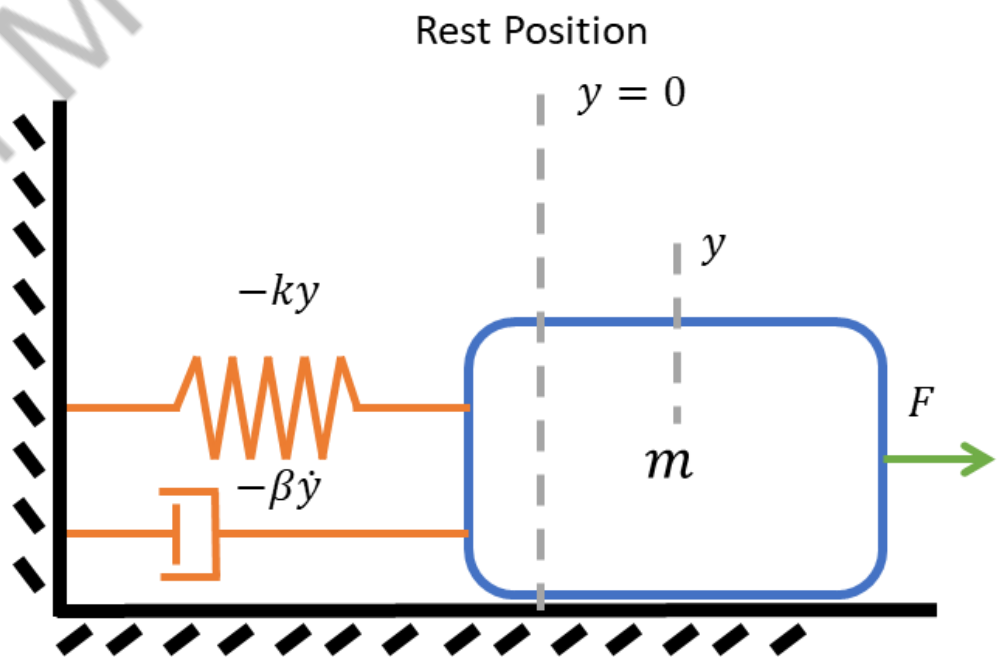
- There are three forces in the direction of y: the spring force ($-ky$), the damper force ($-\beta\,\dot{y}$), and the external force (F).

$$F + (-ky) + (-\beta\,\dot{y}) = m\,\ddot{y} \tag{19}$$

$$\ddot{y} + \frac{\beta}{m}\,\dot{y} + \frac{k}{m}y = \frac{F}{m} \tag{20}$$

- Let us define the set of states as:

$$\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \end{aligned} \tag{21}$$



Property of MCR2

State-space representation of a linear system

Worked example

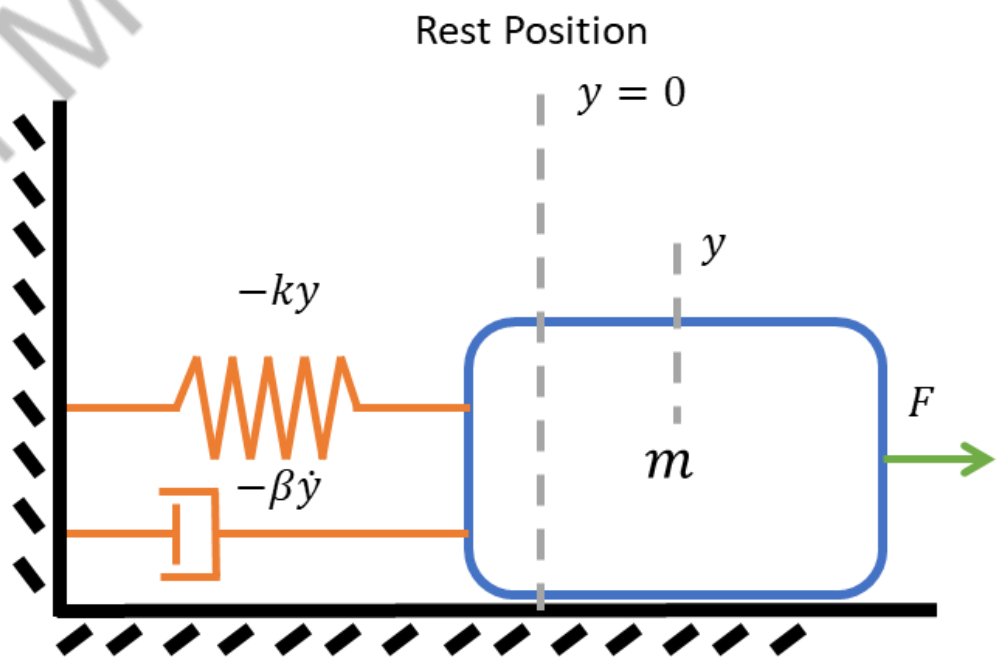
- Then we can find a state-space representation of this system. From the definition of both coordinates, it is trivial that $\dot{x}_1 = x_2$, then (18) can be rewritten in term of x_1, x_2, and $\dot{x}_2$.

$$\dot{x}_2 + \frac{\beta}{m} x_2 + \frac{k}{m} x_1 = \frac{F}{m} \quad (22)$$

$$\ddot{y} + \frac{\beta}{m} \dot{y} + \frac{k}{m} y = \frac{F}{m} \quad (23)$$

- As a result, the system is described by two first order differential equations:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{\beta}{m} x_2 - \frac{k}{m} x_1 + \frac{1}{m} F \end{cases} \quad (24)$$



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State-space representation of a linear system

Worked example

- We rewrite these two equations using matrices and the state $\boldsymbol{x} = (x_1, x_2)$.

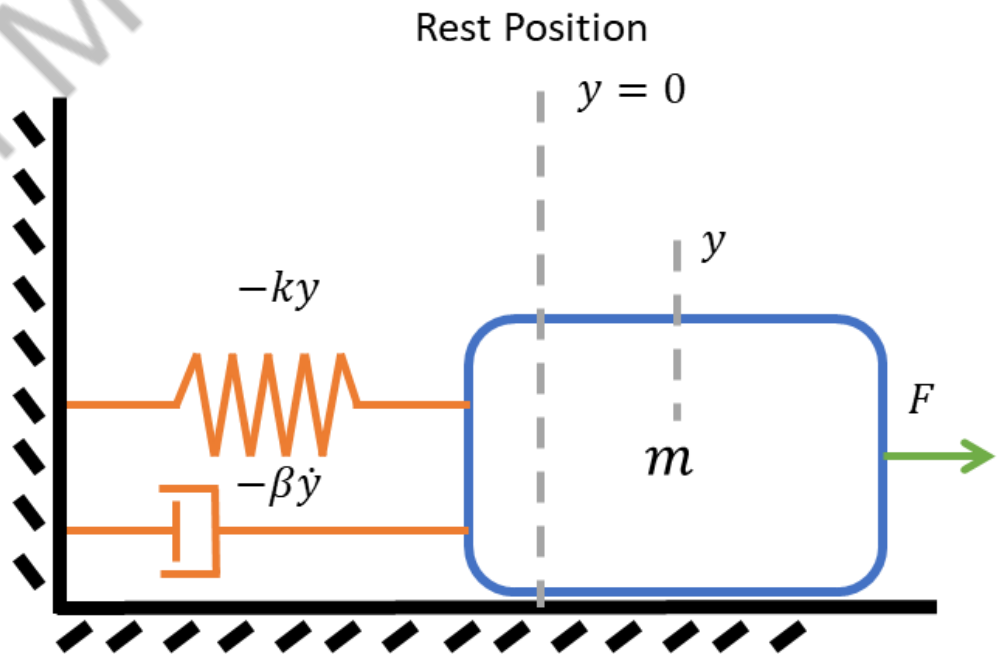
$$\dot{x}_1 = 0x_1 + x_2 + 0F \tag{25}$$

$$\dot{x}_2 = -\frac{\beta}{m}x_2 - \frac{k}{m}x_1 + \frac{1}{m}F \tag{26}$$

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\beta}{m} \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix}F \tag{27}$$

- Using (21), the output equation is given by:

$$y = x_1 + 0x_2 + 0F = \begin{bmatrix} 1 & 0 \end{bmatrix}\boldsymbol{x} + 0F \tag{28}$$



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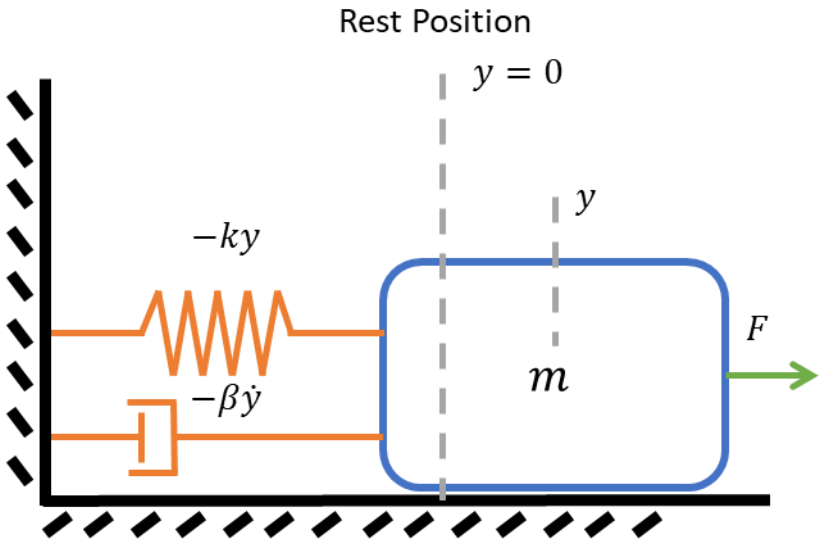
State-space representation of a linear system

Worked example

In summary, the state-space representation of an ideal mass-spring-damper is given by:

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\beta}{m} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1/m \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D = 0$$



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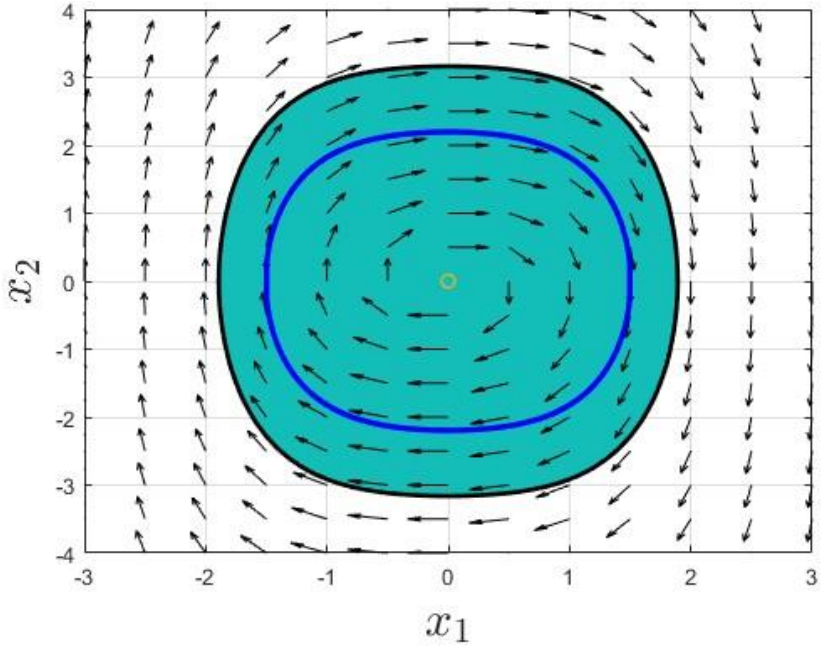
State-space representation of a linear system

- Observation: The state-space is a mathematical description of the system that could be different to the real space where the system performs is trajectory.
- In the example of the mechanical systems mass-spring-damper, the system moves along the y-axis, which is an one-dimensional space. However, the state of the system, as a mathematical concept, evolves on a two-dimensional space.
- As we have stated, there is a state-space representation of all ODEs. Usually, we prefer to express differential equations in the Laplace domain, and we speak about transfer functions.
- Any transfer function can be represented by infinite state-space representations. Three representations are very important: controller canonical form, observer canonical form, and modal form.



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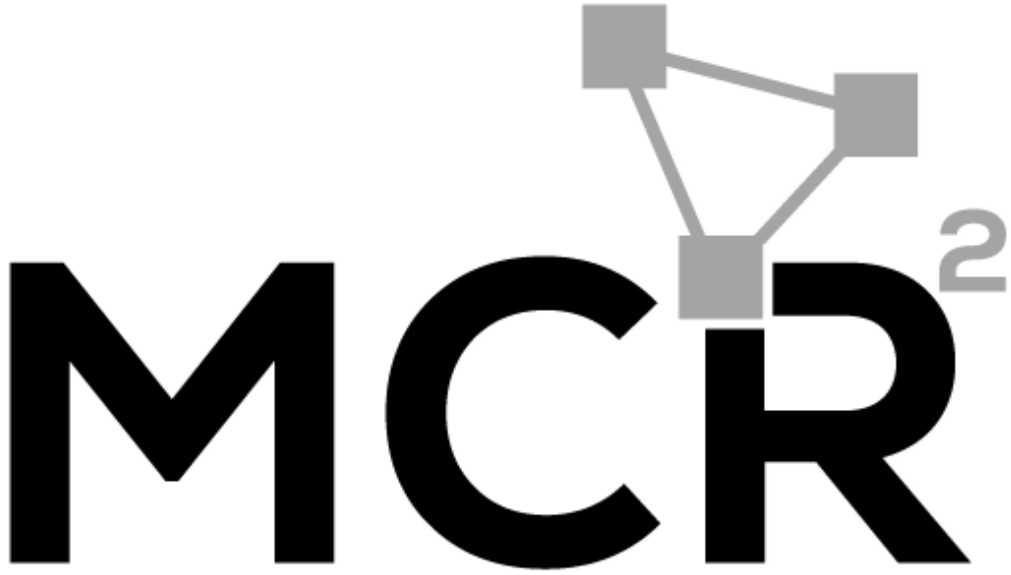
Nonlinear systems

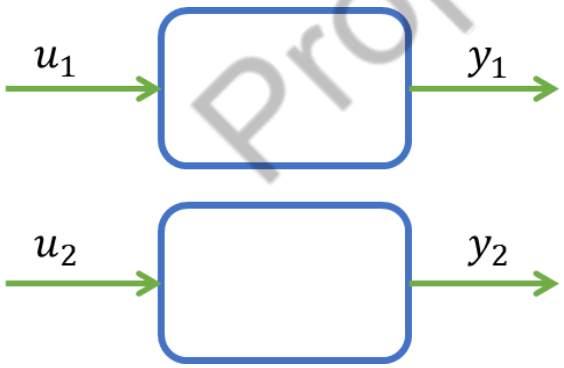
Introduction

Property of MCR2

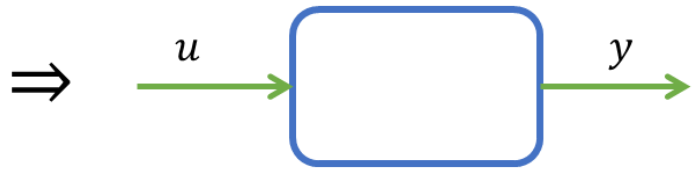


Nonlinear systems

- Compared to linear systems, nonlinear systems have a much rich behaviour.
- A nonlinear system can have a particular behaviour for a specific region of the space, but a completely different behaviour for other regions.



- The fundamental reason is that for linear systems (linear ODEs) the superposition principle holds. There are several formulations for the superposition principle.
- If input u_1 produces output y_1, and input u_2 produces output y_2, then input $\mathrm{u} = u_1 + u_2$ produces output $\mathrm{y} = y_1 + y_2$.



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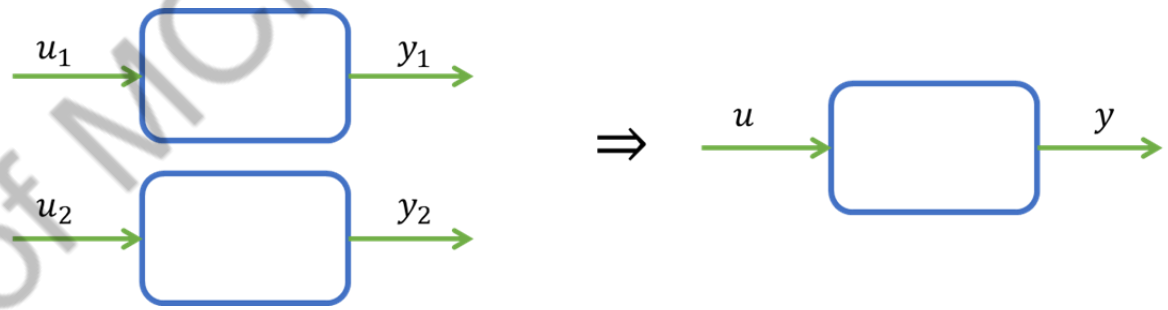
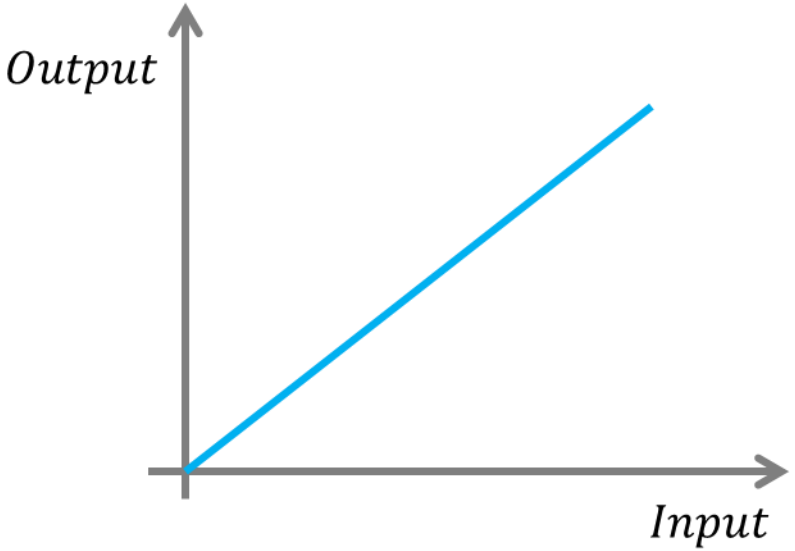
Nonlinear systems

- **Definition:** A system is said to be linear if it satisfies the superposition principle.
- Consider a system with inputs u_1 and u_2 and outputs $y_1 = f(u_1)\ and\ y_2 = f(u_2)$.

$$f(a_2u_1 + a_2u_2) = a_1f(u_1) + a_2f(u_2) \quad (29)$$





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Discrete-time dynamic models

Introduction

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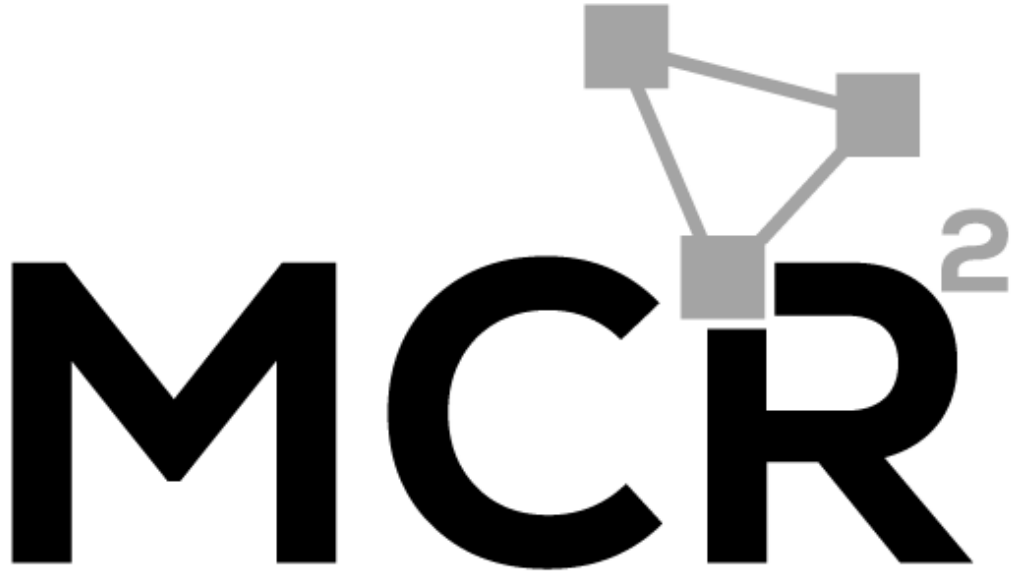

Manchester Robotics

Discrete-time dynamic models

- A digital computer by its very nature, deals internally with discrete-time data or numerical values of functions at equally spaced intervals determined by the sampling period.
- Thus, discrete-time models such as difference equations are widely used in computer control applications.
- One way a continuous-time dynamic model can be converted to discrete-time form is by employing a finite difference approximation.

- Consider a nonlinear differential equation

$$\frac{dy(t)}{dt} = f(y, u) \tag{30}$$

where y is the output variable and u is the input variable.

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Discrete-time dynamic models

- A This equation can be numerically integrated (for instance using Euler method) by introducing a finite difference approximation for the derivative.
- For example, the frst-order, backward difference approximation to the derivative at $t = k\Delta t$ is:

$$\frac{dy(t)}{dt} \cong \frac{y(k) - y(k-1)}{\Delta t} \quad (31)$$

where Δt is the integration interval (the control engineers name it sampling time) specified by the user and $y(k)$ denotes the values of $y(k)$ at $t = k\Delta t$.

- So,

$$\frac{y(k) - y(k-1)}{\Delta t} \cong f(y(k-1), u(k-1)) \quad (32)$$

- or:

$$y(k) = y(k-1) + \Delta t\, f(y(k-1), u(k-1)) \quad (33)$$

- This is a first-order difference equation that can be used to predict $y(k)$ based on information at the previous time step $(k-1)$. This type of expression is called a recurrence relation.

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Discrete-time dynamic models

- For higher-order ODEs, we can use a generalisation of the Euler method that we used for solving first-order ODEs. To illustrate the method, let us consider a 2nd order ODE:

$$\frac{d^2\,y(t)}{dt^2} = f(t, y, \frac{dy(t)}{dt}) \tag{34}$$

- Or:

$$\ddot{y} = f(t, y, \dot{y}) \tag{35}$$

- For discretization, the idea is to write the second order system (ODE) as a system of two first order systems (ODEs) and then apply Euler's method to the first order equations.
- So, as we did before, we'll define a new variable:

$$\begin{cases} y = x_1 \\ \dot{y} = x_2 = \dot{x}_1 \end{cases} \tag{36}$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f(t, x_1, x_2) \end{cases} \tag{37}$$

- We need initial conditions:

For Newton the initial conditions are the initial position and initial velocity.

$$\begin{cases} x_1(t_0) = 0 \\ x_2(t_0) = 0 \end{cases} \tag{38}$$

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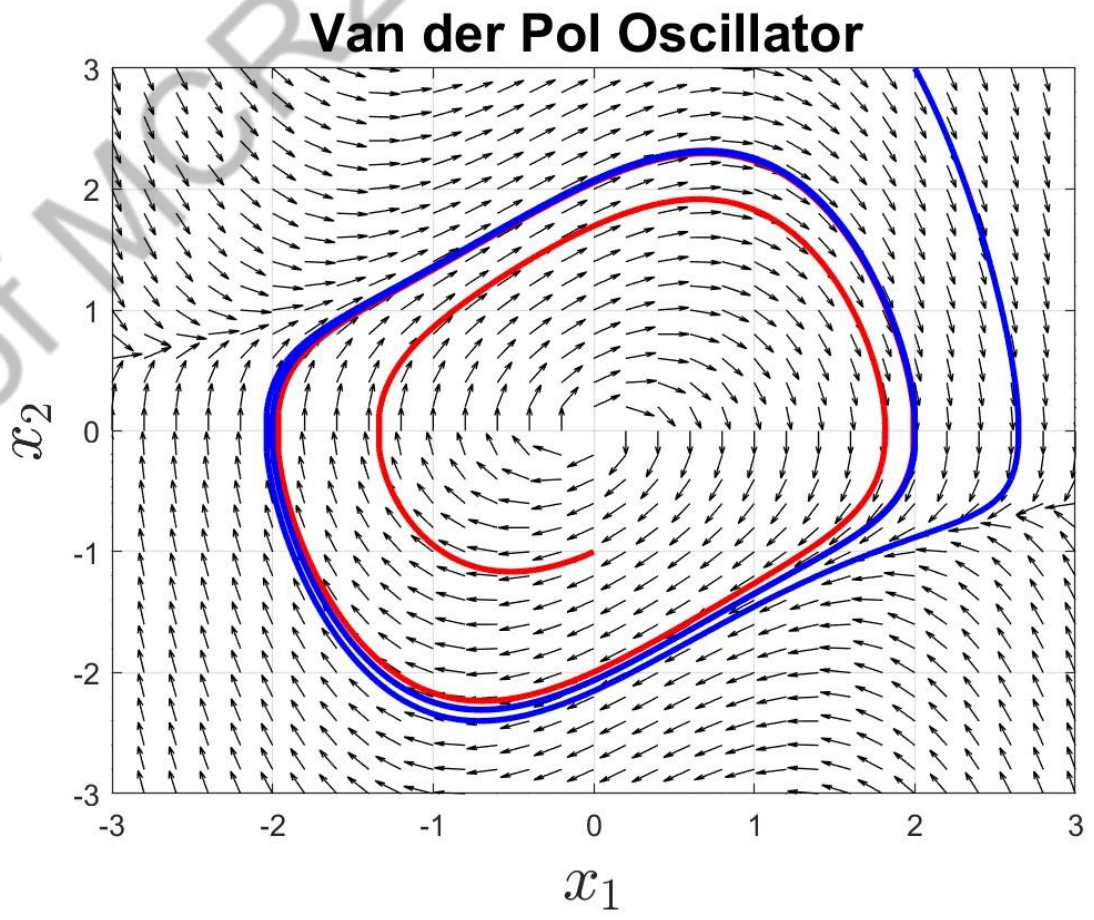
Discrete-time dynamic models

- Now, the idea is to solve both x_1 and x_2 simultaneously using Euler's method for both first order ODEs:

$$\begin{cases} \dfrac{x_1(k) - x_1(k-1)}{\Delta t} = x_2(k-1) \\ \dfrac{x_2(k) - x_2(k-1)}{\Delta t} = f(t, x_1, x_2) \end{cases} \quad (39)$$

$$\begin{cases} x_1(k) = x_1(k-1) + \Delta t\, x_2(k-1) \\ x_2(k) = x_2(k-1) + \Delta t\, f((k-1), x_1, x_2) \end{cases} \quad (40)$$

- This can be generalized to third order ODEs, or fourth order ODEs, as well as n order ODEs.



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Thank you

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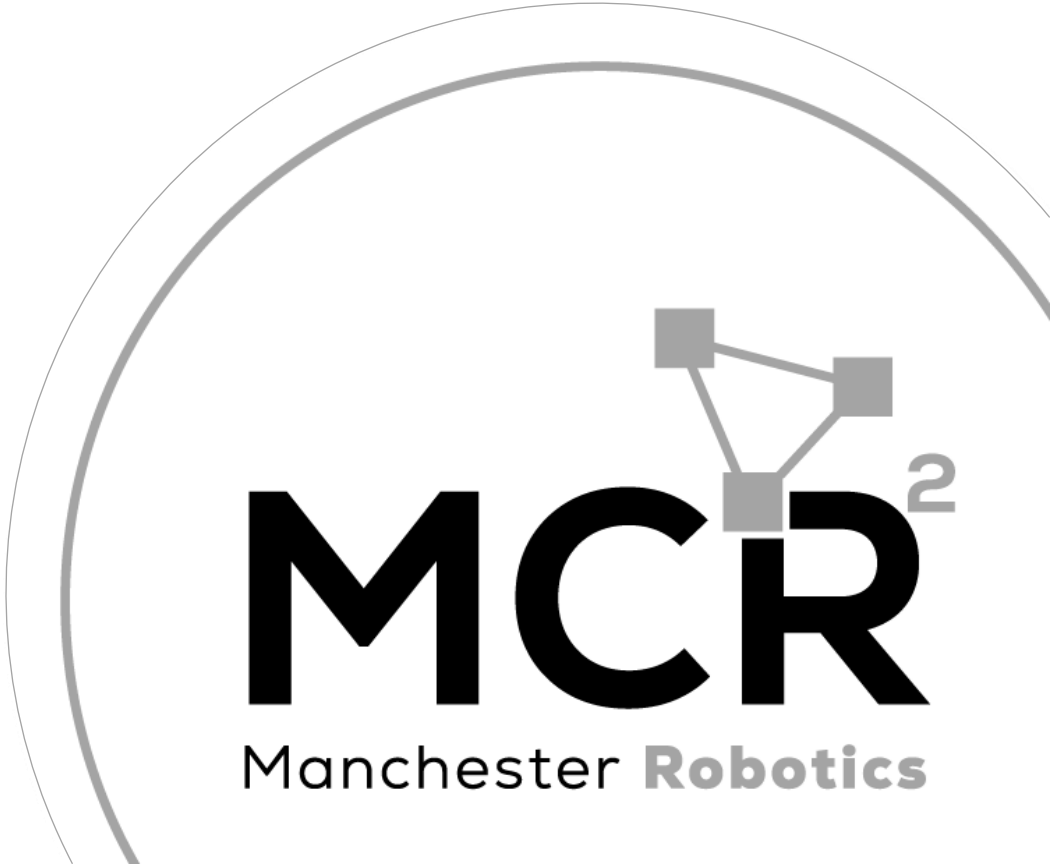

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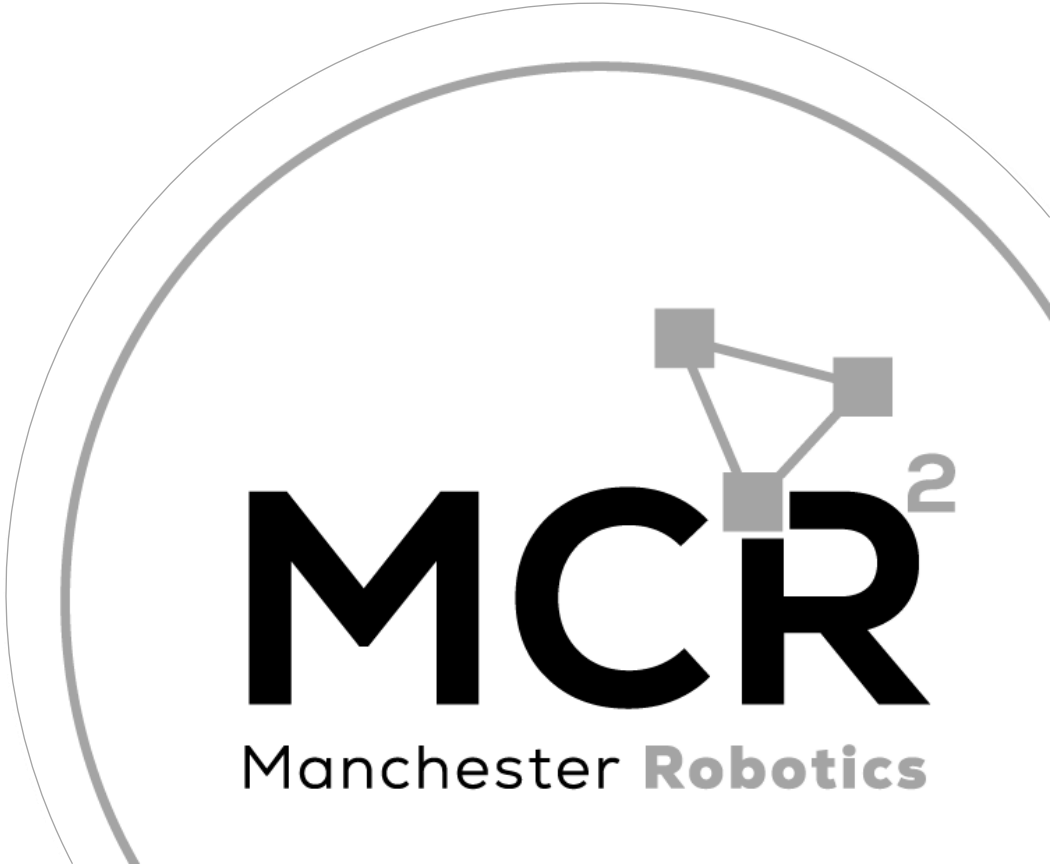

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