1. Let $A(t)$ denote the amount of radio active substance that is present in a sample at a time t. Then $A' = kA$ is a reasonable model for A. The rate of decay of radioactive isotopes is usually specified in terms of half-life. The half-life of an isotopes is the time required for one half of the initial amount to decay. If the half-life of an isotope is 12 years, what is k? What is the general relation between the half-life of the isotope and k?

2. Consider the differential equation $y'' = -k^2 y$ for some constant $k > 0$. Check that $y(x) = C_1 \cos(kx) + C_2 \sin(kx)$ is a general solution of this equation. Also consider $y'' = k^2 y$ for some constant $k > 0$. Check that $y(x) = C_1 \cosh(kx) + C_2 \sinh(kx)$ is a general solution of this equation.

3. For each of the following differential equations draw several isoclines and sketch some solution curves.
(i) $y' = 2x^2 - y$ (ii) $y' = \frac{x^2-y}{y}$.

4. Consider the differential equation $y' = \alpha y,\ x > 0$, where α is a constant. Show that
   (a) if $\phi(x)$ is any solution and $\psi(x) = \phi(x)e^{-\alpha x}$, then $\psi(x)$ is a constant.
   (b) if $\alpha < 0$, then every solutions tends to 0 as $x \to \infty$.

5. Find the orthogonal trajectories of the following families of curves:
(i) $e^x \sin y = c$ (ii) $y^2 = cx^3$

6. Reduce the differential equation $y' = F\left(\frac{ax+by+m}{cx+dy+n}\right)$, $ad - bc \neq 0$ to a separable form. Also discuss the case of $ad = bc$.

7. Show that the following equations are exact and hence find their general solution:

   (i) $(\cos x \cos y - \cot x) = (\sin x\, \sin y)y'$ (ii) $y' = 2x(ye^{-x^2} - y - 3x)/(x^2 + 3y^2 + e^{-x^2})$

8. Show that if the differential equation $M\,dx + N\,dy$ is of the form

$$x^a y^b (my\,dx + nx\,dy) + x^c y^d (py\,dx + qx\,dy) = 0,$$

   where a, b, c, d, m, n, p, q $(mq \neq np)$ are constants, then $x^h y^k$ is an integrating factor. Hence find a general solution of $(x^{1/2}y - xy^2) + (x^{3/2} + x^2 y)y' = 0$.

9. Solve the initial value problem $xy' = x + \sqrt{x^2 - y^2},\ y(x_0) = 0$ where $x_0 > 0$.

10. (a) Solve $y' + 2xy = e^{x-x^2}, y(0) = -1$.
(b) Add a constant of integration to the integral in the integrating factor and show that the solution you get in the end is the same.

11. Solve $2xe^{2y}y' = 3x^4 + e^{2y}$. (Observe that it is neither linear nor separable nor homogeneous nor Bernoulli)

12. Use Picard's method of successive approximation to solve the following initial value problems and compare these results with the exact solutions:
(i) $y' = 2\sqrt{x},\ y(0) = 1$ (ii) $y' + xy = x,\ y(0) = 0$ (iii) $y' = 2\sqrt{y}/3,\ y(0) = 0$

13. Reduce the following second order differential equation to first order differential equation and hence solve.
(i) $xy'' + y' = y'^2$ (iii) $yy'' + y'^2 + 1 = 0$ (iii) $y'' - 2y' \coth x = 0$

14. Find the differential equation satisfied by each of the following two-parameter families of plane curves:
(i) $y = \cos(ax + b)$ (ii) $y = ax + b/x$ (iii) $y = ae^x + bxe^x$

15. Find the values of m such that $y = e^{mx}$ is a solution of
(i) $y'' + 3y' + 2y = 0$ (ii) $y'' - 4y' + 4y = 0$ (iii) $y''' - 2y'' - y' + 2y = 0$

16. Find the values of m such that $y = x^m$ $(x > 0)$ is a solution of
(i) $x^2y'' - 4xy' + 4y = 0$ (ii) $x^2y'' - 3xy' - 5y = 0$

17. Are the following functions linearly dependent on the given intervals?
(i) $\sin 4x,\ \cos 4x$ $(-\infty, \infty)$ (ii) $\ln x,\ \ln x^3$ $(0, \infty)$
(iii) $\cos 2x,\ \sin^2 x$ $(0, \infty)$ (iv) $x^3,\ x^2|x|$ $[-1, 1]$

18. Let $y_1(x), y_2(x)$ be two twice continuously differentiable functions on an interval $\mathcal{I}$. Suppose that the Wronskian $W(y_1, y_2)$ does not vanish anywhere in $\mathcal{I}$. Show that there exists unique $p(x), q(x)$ on $\mathcal{I}$ such that (*) has y_1, y_2 as fundamental solutions.

19. Construct equations of the form (*) from the following pairs of solutions:
(i) $e^{-x},\ xe^{-x}$ (ii) $e^{-x}\sin 2x,\ e^{-x}\cos 2x$

20. Find general solution of the following differential equations given a known solution y_1:
(i) $x(1 - x)y'' + 2(1 - 2x)y' - 2y = 0$ $\quad y_1 = 1/x$
(ii) $(1 - x^2)y'' - 2xy' + 2y = 0$ $\quad y_1 = x$

21. Verify that $\sin x/\sqrt{x}$ is a solution of $x^2y'' + xy' + (x^2 - 1/4)y = 0$ over any interval on the positive x-axis and hence find its general solution.

22. Solve the following differential equations:
(i) $y'' - 4y' + 3y = 0$ (ii) $y'' + 2y' + (\omega^2 + 1)y = 0$, ω is real.

23. Solve the following initial value problems:
(i) $y'' + 4y' + 4y = 0$ $\quad y(0) = 1,\ y'(0) = -1$
(ii) $y'' - 2y' - 3y = 0$ $\quad y(0) = 1,\ y'(0) = 3$

24. The equation
$$x^2\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = 0,$$
where a, b are constants, is called the Euler-Cauchy equation. Show that under the transformation $x = e^t$ (when $x > 0$) for the independent variable, the above reduces to

$$\frac{d^2y}{dt^2} + (a-1)\frac{dy}{dt} + by = 0,$$

which is an equation with constant coefficients.
Hence solve: (i) $x^2y'' + 2xy' - 12y = 0$ (ii) $x^2y'' + xy' + y = 0$ (iii) $x^2y'' - xy' + y = 0$

25. (a) Show that the fundamental system of solutions of Legendre equation

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0$$

consists of $y_1(x) = \sum_{n=0}^{\infty} a_{2n}x^{2n}$ and $y_2(x) = \sum_{n=0}^{\infty} a_{2n+1}x^{2n+1}$, where $a_0 = a_1 = 1$ and

$$a_{2n+2} = -\frac{(p-2n)(p+2n+1)}{(2n+1)(2n+2)}a_{2n}, \qquad n = 0, 1, 2, \cdots$$

$$a_{2n+1} = -\frac{(p-2n+1)(p+2n)}{2n(2n+1)}a_{2n-1}, \qquad n = 1, 2, 3, \cdots .$$

(b) Verify that

$$y_1(x) = P_0(x) = 1,\ y_2(x) = \frac{1}{2}\ln\frac{1+x}{1-x} \quad \text{for } p = 0$$

$$y_2(x) = P_1(x) = x,\ y_1(x) = 1 - \frac{x}{2}\ln\frac{1-x}{1+x} \quad \text{for } p = 1.$$

(c) The expression, $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}[(x^2-1)^n]$, is called the Rodrigues' formula for Legendre polynomial P_n of degree n. Assuming this, find P_1, P_2, P_3.

26. Using Rodrigues' formula for $P_n(x)$, show that
(i) $P_n(-x) = (-1)^n P_n(x)$ (ii) $P_n'(-x) = (-1)^{n+1}P_n'(x)$
(iii) $\displaystyle\int_{-1}^{1} P_n(x)P_m(x)\,dx = \frac{2}{2n+1}\delta_{mn}$ (iv) $\displaystyle\int_{-1}^{1} x^m P_n(x)\,dx = 0$ if $m < n$

27. Suppose $m > n$. Show that $\displaystyle\int_{-1}^{1} x^m P_n(x)\,dx = 0$ if $m-n$ is odd. What happens if $m-n$ is even?

28. Expand the following functions in terms of Legendre polynomials over $[-1, 1]$:
(i) $f(x) = x^3 + x + 1$ (ii) $f(x) = \begin{cases} 0 & \text{if } -1 \le x < 0 \\ x & \text{if } 0 \le x \le 1 \end{cases}$ (first three nonzero terms)

29. Show that $2x^3y'' + (\cos 2x - 1)y' + 2xy = 0$ has only one Frobenius series solution.

30. Reduce $x^2y'' + xy' + (x^2 - 1/4)y = 0$ to normal form and hence find its general solution.

31. Find a solution bounded near $x = 0$ of the following ODE

$$x^2y'' + xy' + (\lambda^2x^2 - 1)y = 0$$

32. Using recurrence relations, show that
(i) $J_0''(x) = -J_0(x) + J_1(x)/x$ (ii) $xJ_{n+1}'(x) + (n+1)J_{n+1}(x) = xJ_n(x)$

33. Show that
    (i) $\int x^4 J_1(x)\,dx = (4x^3 - 16x)J_1(x) - (x^4 - 8x^2)J_0(x) + C$
    (ii) $\int J_5(x)\,dx = -2J_4(x) - 2J_2(x) - J_0(x) + C$

34. Express
    (i) $J_3(x)$ in terms of $J_1(x)$ and $J_0(x)$ (ii) $J_2'(x)$ in terms of $J_1(x)$ and $J_0(x)$
    (iii) $J_4(ax)$ in terms of $J_1(ax)$ and $J_0(ax)$

35. Prove that between each pair of consecutive positive zeros of $J_\nu(x)$, there is exactly one zero of $J_{\nu+1}(x)$ and vice versa.

36. Let $u(x)$ be any nontrivial solution of $u'' + [1 + q(x)]u = 0$, where $q(x) > 0$. Show that $u(x)$ has infinitely many zeros.

37. Let $F(s)$ be the Laplace transform of $f(t)$. Find the Laplace transform of $f(at)$ $(a > 0)$.

38. Find the Laplace transforms:
    (a) $[t]$ (greatest integer function), (b) $t^m \cosh bt$ ($m \in$ non-negative integers),
    (c) $e^t \sin at$, (d) $\dfrac{e^t \sin at}{t}$, (e) $\dfrac{\sin t \cosh t}{t}$, (f) $f(t) = \begin{cases} \sin 3t, & 0 < t < \pi, \\ 0, & t > \pi, \end{cases}$

39. Using convolution, find the inverse Laplace transforms:
    (a) $\dfrac{1}{s^2 - 5s + 6}$, (b) $\dfrac{2}{s^2 - 1}$, (c) $\dfrac{1}{s^2(s^2 + 4)}$, (d) $\dfrac{1}{(s-1)^2}$.

40. Use Laplace transform to solve the initial value problems:
    (a) $y'' + 4y = \cos 2t$, $y(0) = 0$, $y'(0) = 1$.
    (b) $y'' + 3y' + 2y = 4t$ if $0 < t < 1$ and 8 if $t > 1$; $y(0) = y'(0) = 0$
    (c) $y'' + 9y = 8\sin t$ if $0 < t < \pi$ and 0 if $t > \pi$; $y(0) = 0, y'(0) = 4$
    (d) $y_1' + 2y_1 + 6\displaystyle\int_0^t y_2(\tau)\,d\tau = 2u(t)$, $y_1' + y_2' = -y_2$, $y_1(0) = -5, y_2(0) = 6$