MTH101A: Mathematics-I

Problem Set 1: Real numbers and sequences

(To be discussed in the week starting on 5 August 2019)

The problems marked with an asterisk(*) will not be asked during any quiz or exam. The problems marked with a plus sign(+) are extra questions and will be discussed in the tutorial only if time permits.

1. Suppose a sequence (x_n) is not Cauchy. Express this using the language of ϵ and N.

   We have seen the following sequence in the last problem sheet: $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$ for $n \in \mathbb{N}$. Is this sequence Cauchy?

   (Hint: We have seen that $|x_{2n} - x_n| \geq \frac{1}{2}$.)

2. (+) Show that any subsequence of a convergent sequence is still convergent.

   Suppose $(x_n), (y_n)$ are two convergent sequences with limit α. Show that the spliced sequence (z_n) also converges to α where (z_n) is defined by $z_n := \begin{cases} x_{\frac{n}{2}} & \text{if n is even,} \\ y_{\frac{n+1}{2}} & \text{otherwise.} \end{cases}$.

3. Prove that for any bounded subset $A \subseteq \mathbb{R}$ we have $\sup A = -\inf(-A)$, where $(-A) := \{-a \mid a \in A\}$.

4. (+) Let (x_n) be a strict monotone increasing sequence converging to 0. For a subset $A \subseteq \mathbb{R}$ and $\alpha \in \mathbb{R}$, show that $\alpha = \sup A$ if and only if, for each $n \in \mathbb{N}$, $\alpha + x_n$ is not an upper bound of A and $\alpha - x_n$ is an upper bound of A.

5. Show that each of the following sequences is convergent and find its limit.

   (a) $\left(\frac{(-1)^n}{3^{2n}}\right)$

   (b) $\left(\frac{5n^{\frac{1}{4}}}{n^2+\sqrt{n}}\right)$

   (c) (c^n) for $0 < c < 1$

   (d) $(\frac{c^n}{n!})$ for $c > 0$

6. Suppose $A \subseteq \mathbb{R}$ is bounded above and $\sup A = \alpha$. If $\alpha \notin A$, then construct a strictly increasing sequence sequence (a_n) of elements of A such that $\lim_{n\to\infty} a_n = \alpha$.

7. (*) Show that a Cauchy sequence is bounded.

8. (*) Show that every sequence has a monotone subsequence.

9. (*)(+) Exponentials: Suppose α is any positive real number. While we know that α^3 means the product when α is multiplied by itself 3 times, what does an expression like α^π really mean?

   Using the completeness property of $\mathbb{R}$, we can assign meaning to the expression $\alpha^{\frac{m}{n}}$, for $m, n \in \mathbb{Z}, n \neq 0$, as the supremum of the set $\{x \mid x^n < \alpha^m\}$.

   If β is any real number, then there is a monotone increasing sequence (q_n) of rationals converging to β. We define α^β as the limit of the sequence α^{q_n}. (We are assuming that the limit is independent of the choice of the sequence (q_n), which can be proved.)

   Show that for $\alpha(> 0), \beta, \gamma \in \mathbb{R}$, we have

$$\alpha^{\beta+\gamma} = \alpha^\beta \alpha^\gamma,$$

$$(\alpha^\beta)^\gamma = \alpha^{\beta\cdot\gamma}.$$