Topological Sort (of an Acyclic Graph)

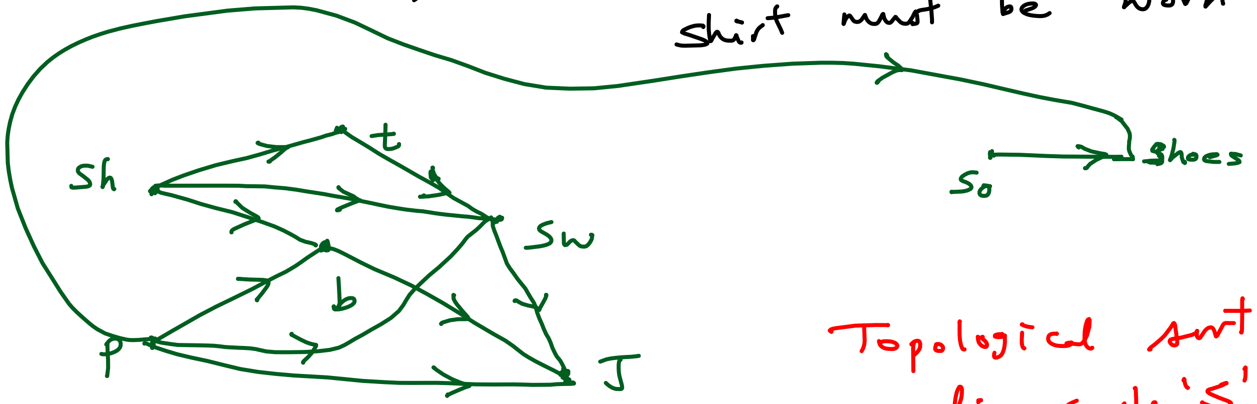
Professor : Dresses up by wearing
shirt, pant, socks, shoes, tie, sweater, belt, Jacket

Something must be worn before some other things,
for example, socks must be worn before shoes
shirt must be worn before belt.

We can show this 'precedence' relation by a graph

Vertices are articles to be donned by the prof.

Edge (x, y) indicates that x must be worn before y.



This graph is acyclic

Topological sort means a linear order '$\leq$' on the vertices s.t. if $(x, y) \in E$ then $x \leq y$

Topological Sort

$G = (V, E)$

Directed Acyclic Graph
(DAG)

Need to produce a linear order '$\leq$' on the set V s.t.

$$\forall_{xy} \left[(x, y) \in E \longrightarrow x \leq y \right]$$

(Can be done for acyclic graphs only)

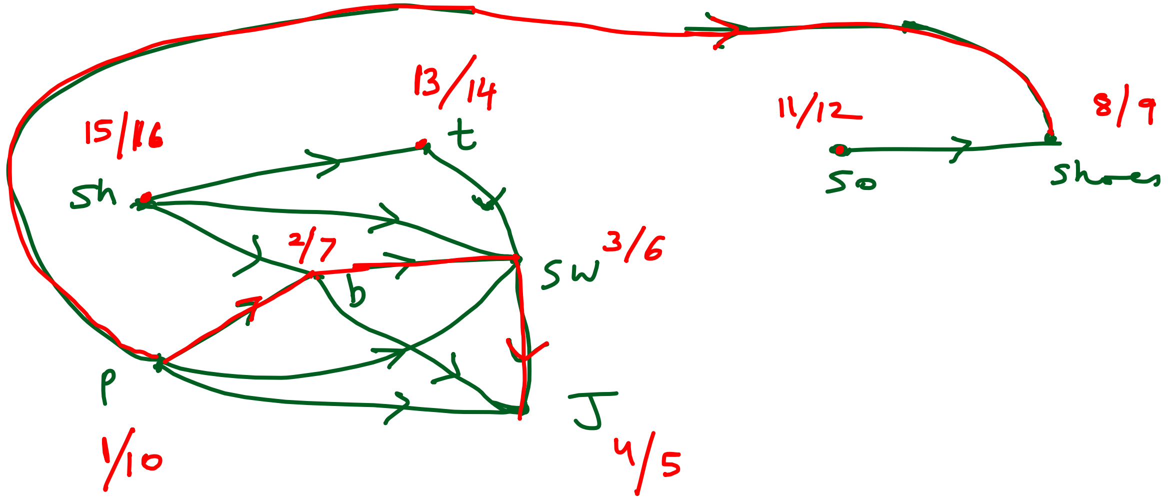
TopologicalSort (G)

1. Start DFS(G), L = nil
2. when a vertex finishes add it to the begining of L.
3. return L

// L is the list of vertices in the decreasing order of their finish time.

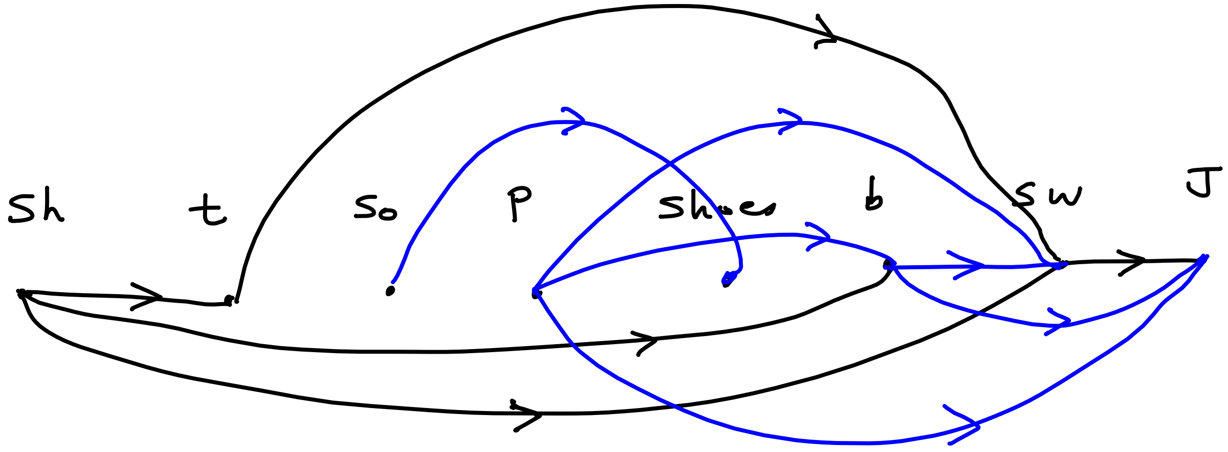
Complexity $O(|V| + |E|)$

linear time.



Graph G

DFS(G) produces a forest of 4 DFS trees.



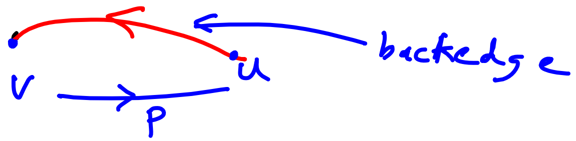
Topological sort of G

Arranging vertices of G in a horizontal line, s.t. all edges of G are left to right.

Correctness (of Topological sort)

Lemma: Let G be a (directed or undirected) graph and let F be its DFS forest. G is cyclic iff G has a backedge.

Proof: ($\Leftarrow$) If G has a back-edge (u,v) then by definition v is an ancestor of u in F. Therefore (by definition of ancestor relation) there is a path p in F from v to u.

Diagram 

shows a cycle in G from v to v

(P is in a DFS tree)

($\Longrightarrow$) Assume there is a simple cycle

$u_1 u_2 \dots u_m u_1$ in G.

Let u_i be first vertex in the cycle to be discovered during DFS(G). So all the other vertices on this cycle are white when u_i is discovered.

By white path theorem,

u_{i-1} is a descendent of u_i in F.

The edge (u_{i-1}, u_i) in G is by definition a backedge.

$\square$

Lemma: Let $G=(V,E)$ be a DAG, and let F be its DFS forest.
For all $(x,y) \in E$, $x.f > y.f$.

Proof: Any $(x,y) \in E$ is either a tree edge, forward edge or a cross edge (back edges are not possible by previous Lemma).

In the first two cases, y is a descendant of x. This by a DFS property implies that $y.f < x.f$.

If (x,y) is a cross-edge, $(x.d, x.f)$ and $(y.d, y.f)$ are disjoint

if $(y.d, y.f)$ precedes $(x.d, x.f)$ then clearly $y.f < x.f$.

if $(x.d,\ x.f) < (y.d,\ y.f)$

Edge (x,y) is visited when x is grey, that is before $x.f$.

when this edge is visited then

either y is white and then it becomes grey

or y is non-white.

In any case y is non_white before $x.f$.

$$\Longrightarrow \quad y.d < x.f$$

A contradiction

$\square$

<u>Corollary</u>: In a DAG $G=(V,E)$, topological sort produces a list of vertices s.t. if $(x,y) \in E$ then x occurs before y.

$\square$