Asymptotic Growth rate of functions

Insertion sort time complexity grows as n^2, n size of the input

In the analysis of algorithms several growth like

$\lceil \log n \rceil$, $\sqrt{n}$, n, $n\log n$, n^2

n^3, 2^n, $n!$

Commonly arise

Comparison of growth rates

$n \rightarrow$	10	100	1000	10^6
$\lceil \log n \rceil$	4	7	10	20
$\lceil n \log n \rceil$	34	665	9966	$\approx 20 \times 10^6$
n^2	100	10^4	10^6	10^{12}
n^3	10^3	10^6	10^9	10^{18}
2^n	1024	$\approx 10^{30}$	Too Big	Too Big

A processor 10^6 ops/sec.

Algorithm on a problem size 10^6

$\lceil n \log n \rceil$ 20 sec

n^2 277 hours

n^3 31 thousand years

These gaps are not bridged by technology easily.

Analysis of growth rates of time complexity is important.

Insertion Sort takes $\underline{\underline{O}}(n^2)$ time

Big-oh

To compare asymptotic growth rates of fns.

__Defn__ f, g are functions $N \to N$. f is $O(g)$ if there is a $c \in \mathbb{R}$, $c > 0$ and $n_0 \in N$ s.t. $\forall n \geq n_0 \left[f(n) \leq c \cdot g(n) \right]$.

Examples $\quad n^2$ is $O(n^2-n)$

$$n^2 \le 2(n^2-n) \quad \text{for all } n \ge 2$$

$$\boxed{\iff n^2 - 2n \ge 0 \iff n \ge 2}$$

2. n^2 is $O(n^3)$

3. $\log^2 n$ is $O(n)$

Lemma Any polynomial of degree k is $O(n^k)$.

Pf

$$p(n) = a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n + a_0$$

$$a_j \leq |a_j| \qquad \sum_{j=0}^{k} a_j n^j \leq \sum_{j=0}^{k} |a_j| n^k \qquad n \geq 1$$

$$p(n) \leq \left(\sum_{j=0}^{k} |a_j|\right) \cdot n^k \quad \text{for all } n \geq 1 \qquad \square$$

$\Omega(g)$ Lower bound

<u>Defn</u> f is $\Omega(g)$ if $\exists\, c, n_0$ s.t.

$$\forall n \geq n_0 \left[f(n) \geq c\, g(n) \right]$$

<u>Thm</u> f is $\Omega(g)$ iff g is $O(f)$.

<u>Pf</u> LHS $f(n) \geq c\, g(n) \iff g(n) \leq \frac{1}{c} f(n)$

$\iff g$ is $O(f)$

$\square$

Defn f is $\Theta(g)$ if f is $O(g)$ and g is $O(f)$.

f, g have same growth rate.

Ex: For any two polynomials p, q of degree k with leading coeff. positive,

$$p = \Theta(q).$$

n^k is $o(n^{k+1})$ n^{k+1} is a strict upper bound

Defn (Small o)

f is $o(g)$ if for all $c \in \mathbb{R}, c > 0$

$\exists n_0$ s.t.

$f(n) < c\,g(n)$ for all $n \geq n_0$

Ex n^k is $o(n^{k+1})$

Lemma For any $k \in \mathbb{N}$ and $a \in \mathbb{R}$, $a > 1$

n^k is $o(a^n)$

Pf

$$\lim_{n\to\infty} \frac{n^k}{a^n} = \lim_{n\to\infty} \frac{k n^{k-1}}{(\ln a)\, a^n} \quad \text{(L'Hopital's Rule)}$$

$$= \cdots \lim_{n\to\infty} \frac{k!}{(\ln a)^k} \frac{1}{a^n} = 0$$

$$\lim_{n\to\infty} \frac{n^k}{a^n} = 0$$

$$\lim_{n\to\infty} \frac{n^k}{a^n} = 0 \qquad \text{For any } \varepsilon > 0,\ \exists n_0 \text{ s.t.}$$

$$\frac{n^k}{a^n} < \varepsilon \qquad \forall n \geq n_0$$

$$n^k < \varepsilon a^n$$

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n^k is $o(a^n)$ $\square$

Corollary n^k is $O(a^n)$

Ex show that $\log^k n$ is $o(n^\varepsilon)$ for any $\varepsilon > 0$

Ex Give examples of f, g s.t. f is not $o(g)$ and g is not $o(f)$.

Factorial fn $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$

$$2^{n-1} \leq n! \leq n^n \quad \text{for } n \geq 1$$

Ex 2^{n-1} is $o(n!)$

$n!$ is $o(n^n)$

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\Theta\left(\frac{1}{n}\right)\right)$$

What does it mean?

(Stirling's approximation)

$\exists\, c_1, c_2 > 0$ s.t.

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\frac{c_1}{n}\right) \le n! \le \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\frac{c_2}{n}\right)$$

Elementary Mathematical Tools

$\lceil x \rceil$ = smallest integer $\geq x$

real

$\lfloor x \rfloor$ = largest integer $\leq x$

Ex: $\lceil 7.3 \rceil = 8$, $\lfloor 7.3 \rfloor = 7$, $\lceil 7 \rceil = 7$

Fact $x \leq \lceil x \rceil < x+1$, $x-1 < \lfloor x \rfloor \leq x$

Fact,

$$\left\lfloor \frac{x}{2} \right\rfloor + \left\lceil \frac{x}{2} \right\rceil = x$$

for any integer x

log fn $f(x) = \log_a x$

$a > 1$ Defined for $x > 0$

		$f(x)$	x
$f(x)$	is	$-ve$	$(0, 1)$
		0	1
		$(0, 1)$	$0 < x < a$
		1	a
		> 1	$> a$

Monotonically increasing fn.

$b, a > 1 \quad c > 0$

$$\log_a b \cdot \log_b c = \log_a c$$

(Exercise)

$\log_a x = n \iff a^n = x$

Change the base

$\log_2 n \longrightarrow \log_e n$

$$\log_2 n \cdot \underbrace{\log_e 2}_{\text{Constant}} = \log_e n$$

$O(n \log n)$

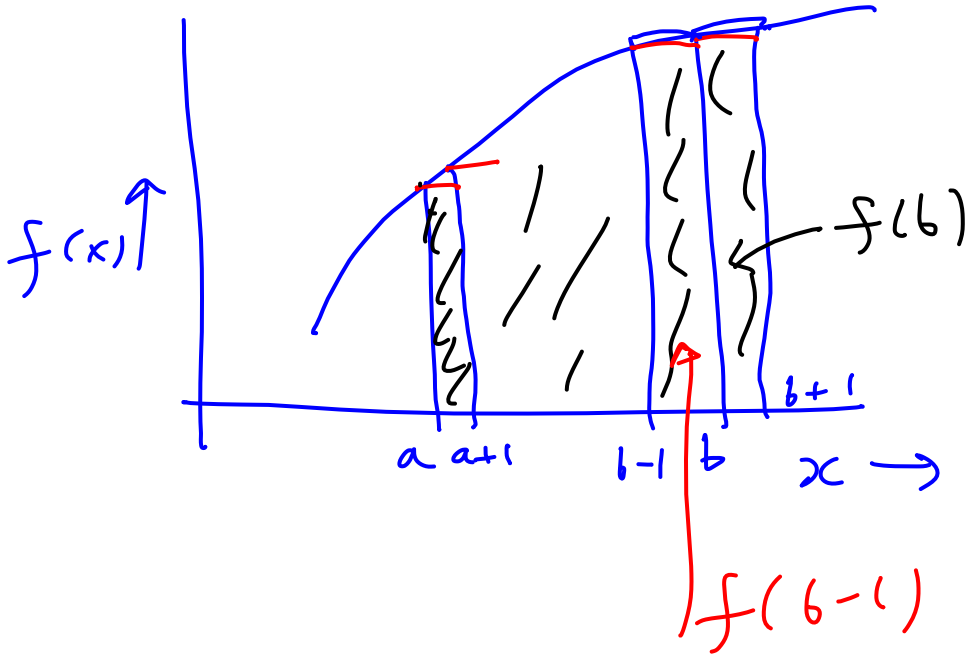
Base need not be specified as change of base only involves multiplication by a constant.

Approximating sums by integrals

f is monotonically increasing fn.

$$\sum_{x=a}^{b} f(x) = \text{Area of rectangular strips}$$

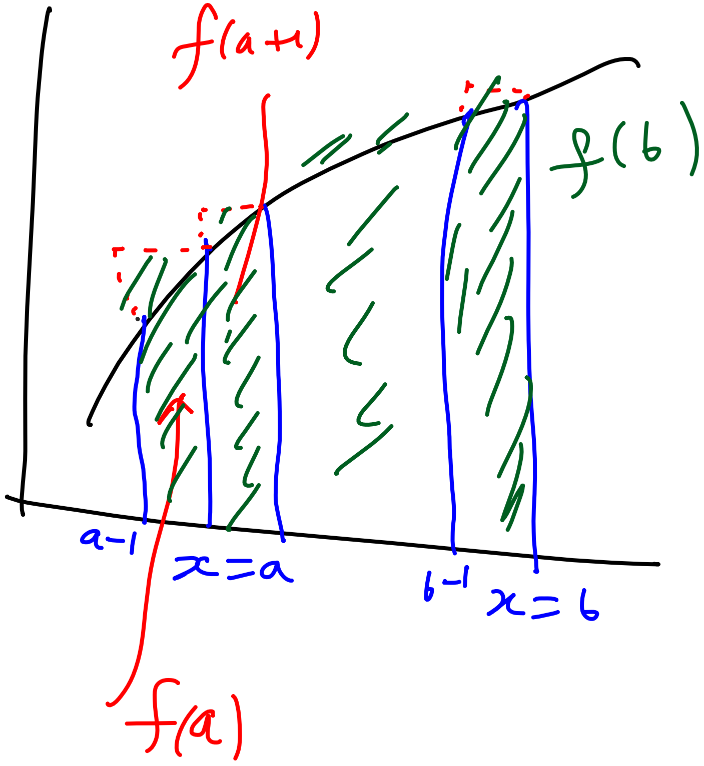
$$\leq \int_a^{b+1} f(x)\,dx$$



Lower bound

$$\sum_{x=a}^{b} f(x) = \text{Area shaded in green}$$

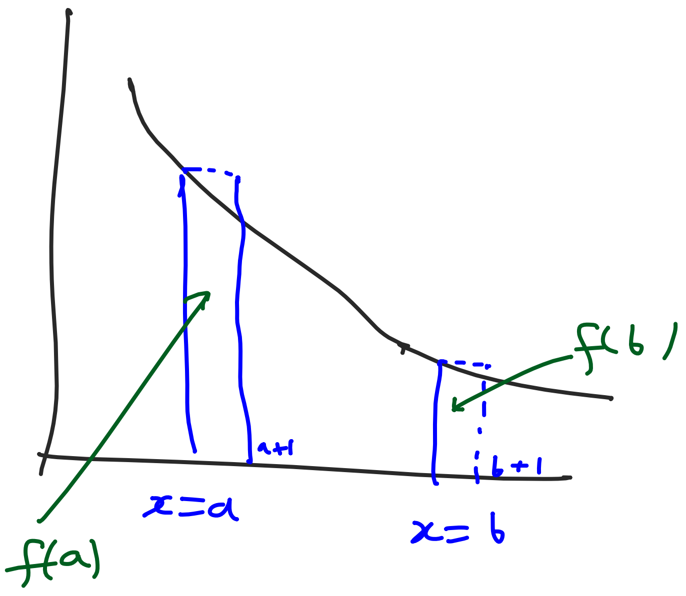
$$\geq \int_{a-1}^{b} f(x)\,dx$$



If f is monotonically decreasing

$$\sum_{x=a}^{b} f(x) = \text{Area shown by blue rectangles}$$

$$\geq \int_a^{b+1} f(x)\,dx$$



$$\int_a^{b+1} f(x)\,dx \leq \sum_{x=a}^{b} f(x) \leq \int_{a-1}^{b} f(x)\,dx$$

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

$$f(x) = \frac{1}{x} \qquad f \text{ is monotonically decreasing}$$

$$H_n = 1 + \sum_{x=2}^{n} \frac{1}{x} \leq 1 + \int_{x=1}^{n} \frac{1}{x}\,dx = 1 + \Big[\ln x\Big]_1^n$$

$$= 1 + \ln x$$

$$H_n = \sum_{x=1}^{n} \frac{1}{x} \geq \int_{1}^{n+1} \frac{1}{x}\,dx = \Big[\ln x\Big]_1^{n+1} = \ln(n+1)$$

$$\boxed{\ln(n+1) \leq H_n \leq 1 + \ln n}$$