Department of Mathematics and Statistics, I.I.T. Kanpur

MTH101A End-Semester Examination: Marking Scheme - 23.11.2011

Maximum Marks: 120 Time: 08:00 - 11:00 AM

1. (a) Investigate the convergence of the sequence defined as follows: $a_1 = 1$ and $a_{n+1} = \frac{1}{6}(a_n^2 + 8)$, for all $n \geq 1$. [5]

Ans:

- Consider $a_{n+2} - a_{n+1} = \frac{1}{6}(a_{n+1} - a_n)(a_{n+1} + a_n)$. [2]
- Since $a_2 = \frac{3}{2} > 1 = a_1$, it follows by induction that a_n is increasing sequence. [1]
- Again, $a_1 \leq 2$, and if $a_n \leq 2$, then $a_{n+1} \leq 2$. [1]
- Hence a_n is bounded above and therefore convergent. [1]
- **Alt:** $a_{n+2} - a_{n+1} = \frac{1}{6}(a_{n+1} - a_n)(a_{n+1} + a_n$. [2]
- $a_1 \leq 2$, and if $a_n \leq 2$, then $a_{n+1} \leq 2$. [1]
- Therefore, $|a_{n+2} - a_{n+1}| \leq \frac{4}{6}|a_{n+1} - a_n|$. Therefore $\alpha = \frac{4}{6} = \frac{2}{3} < 1$. [1]
- Hence, the sequence is Cauchy and therefore convergent. [1]

(b) Show that the polynomial $p(x) = x^4 + 2x^3 - 9$ has exactly two real roots. [5]

Ans:

- Consider $p(x) = x^4 + 2x^3 - 9$. The $p(0) = -9$, and $p(x) \to \infty$ as $x \to \pm\infty$. Therefore, by IVP, there exists two real roots say $a > 0, b < 0$ of p. [2]
- Note that $p'(x) = 4x^3 + 6x^2 = 2x^2(2x + 3)$. and $p(-\frac{3}{2}) \neq 0$, a, b are simple roots of p. Since complex roots occur in pair, if p has three real roots, it will have all four as real roots. Also none of them is a repeated root. Therefore, p' must vanish at three distinct points, which is not true. Hence p has exactly two real roots. [3]

(c) Let f be a continuous function on $[0, 1]$ and differentiable on $(0, 1)$. Suppose that $f(0) = 0 = f(1)$. Show that there exists a $c \in (0, 1)$ such that $f(c) + f'(c) = 0$. [5]

Ans:

- Consider $g(x) = f(x)e^x, x \in [0, 1]$. Then g is a continuous on $[0, 1]$ and differentiable on $(0, 1)$. Also $g(0) = g(1) = 0$. [2]
- By Mean Value theorem, $g'(c) = e^c(f(c) + f'(c)) = 0$, for some $c \in (0, 1)$. Therefore, $f(c) + f'(c) = 0$. [3]

(d) Is the series $\sum_{n=1}^{\infty} \frac{\cos n - 1}{n^2}$ convergent. Justify your answer. [5]

Ans:

- $a_n = \frac{\cos n - 1}{n^2}$ satisfies $|a_n| \leq \frac{2}{n^2}$. [2]
- $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, and absolute convergence implies convergence, the given series is convergent. [3]

2. (a) Determine all values of x for which the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n\,3^n}$ converges. Give reasons for your answer. [5]

Ans:

- Consider $|\frac{a_{n+1}}{a_n}| = \frac{n}{3(n+1)} \to \frac{1}{3}$ as $n \to \infty$. [1]
- Therefore, the series converges for $x : |x-3| < 3$. [2]
- For $x = 6$, $a_n = \frac{1}{n}$ which is divergent. [1]
- For $x = 0$, $a_n = \frac{(-1)^n}{n}$, which is convergent by the Leibniz test. [1]
- Therefore, the series converges for $x : |x-3| < 3$. and for $x = 0$.

(b) Sketch the graph of $f(x) = x^3 - 6x^2 + 9x + 1$ marking the local maxima/minima, intervals where f is concave up (i.e., convex) or concave down, points of inflection and asymptotes, if any. [10]

Ans:

- $f(x) = x^3 - 6x^2 + 9x + 1$.
- $f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$. $f'(x) = 0$, for $x = 1, 3$. $f''(x) = 3(x-3+3(x-1)$. $f''(1) = -6 < 0$ and $f''(3) = 6 > 0$. Therefore,$x = 1$ is local maximum and $x = 3$ is a local minimum. [3]
- $f''(x) = 6x - 12 = 6(x-2) > 0$ if $x > 2$ and less than 0 if $x < 2$. So f is convex in $(2, \infty)$, and concave in $(-\infty, 2)$. [2]
- $x = 2$ is an inflection point. [1]
- Award 4 marks for graph, this includes 1 mark for no asymptote. If graph has asymptote, deduct 1 mark.
- General observations: Do not deduct marks (a) if it is not explicitly mentioned that there are no asymptotes,as it would be clear from the graph, (b) if student does not plot the point where the graph cuts the axes, and (c) if observation $x \to \infty$, $f \to \infty$ and as $x \to -\infty$, $f \to -\infty$ is missing.

(c) Determine all values of p for which the improper integral $\int_0^{\infty} \frac{t^{2p}}{e^t}\,dt$ is convergent. [10]

Ans:

- Let $f(t) = \frac{t^{2p}}{e^t}$. $\int_0^{\infty} \frac{t^{2p}}{e^t}dt = \int_0^{1} \frac{t^{2p}}{e^t}dt + \int_1^{\infty} \frac{t^{2p}}{e^t}dt = I_1 + I_2$. [2]
- For I_1 : Let $g(t) = t^{2p}$. Then $\lim_{t\to 0} \frac{f(t)}{g(t)} = \lim_{t\to 0} \frac{1}{e^t} = 1$. Therefore, by limit comparison test, I_1 converges iff $\int_0^1 t^{2p}dt$ converges. [2]
- $\Rightarrow I_1 < \infty$ iff $-2p < 1$ or $p > -\frac{1}{2}$. [2]
- For I_2 : Let $h(t) = \frac{1}{t^2}$. Then $\lim_{t\to\infty} \frac{f(t)}{h(t)} = \lim_{t\to\infty} \frac{t^{2p+1}}{e^t} = 0$. Therefore, by limit comparison test I_2 converges for all p. [2]
- Hence $I < \infty$ if and only if $p > -\frac{1}{2}$. [2]

3. (a) Let (x_0, y_0) be the centroid of the parametric curve $x(t) = \cos^3 t$, $y(t) = \sin^3 t, 0 \le t \le \frac{\pi}{2}$. Using Pappus theorem find y_0. [10]

Ans:

- For tutors: Pappus Theorem: Let C be a plane curve. Suppose C is revolved about the line L which does not cut through the interior of C, then the area of the surface generated is

$$S = 2\pi\rho L$$

where ρ is the distance from the axis of revolution to the centroid and L is the length of the curve C.

- The curve is given by $\{(\cos^3 t, \sin^3 t) : t \in [0,].\}$ The length of this curve is

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = \int_0^{\frac{\pi}{2}} \sqrt{(-3\cos^2 t \sin t)^2 + (3\sin^2 t \cos t)^2\, dt}$$

[2]

-
$$\Rightarrow L = \int_0^{\frac{\pi}{2}} 3 \mid \cos t \sin t \mid dt = \frac{3}{2}\int_0^{\frac{\pi}{2}} \sin 2t dt = \frac{3}{2}.$$

[2]

-
$$S = \int_a^b 2\pi y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\, dt = \int_0^{\frac{\pi}{2}} 2\pi \sin^3 t\, 3 \sin t \cos t\, dt$$

[2]

-
$$\Rightarrow S = 6\pi \int_0^{\frac{\pi}{2}} \sin^4 t \cos t dt = 6\pi \frac{\sin^5 t}{t}|_0^{\frac{\pi}{2}} = \frac{6\pi}{5}.$$

[2]

- Therefore, $2\pi y_0 L = S \Rightarrow 2\pi y_0 \frac{3}{2} = \frac{6\pi}{5} \Rightarrow y_0 = \frac{2}{5}$. [2]

(b) Find the equation of the surface generated by the normals to the surface $xyz^2 + 2yz + 2x = 0$ at all points on the y-axis. [5]

Ans:

- Let $f(x, y, z) = xyz^2 + 2yz + 2x$. Then $\nabla f = (yz^2 + 2, xz^2 + 2z, 2xyz + 2y)|_{(0,t,0)} = (2, 0, 2t)$. [2]
- If (x, y, z) is any point on the given surface, then $\frac{x-0}{2} = \frac{y-t}{0} = \frac{z-0}{2t}$. Therefore, $xt = z, y = t$, i.e., the surface is $xy = z$. [3]

(c) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(0, 0) = 0$ and

$$f(x, y) = \frac{1}{\sqrt{x^2 + y^2}}\Big(\mid |x| - |y| \mid - |x| - |y|\Big),\ (x, y) \neq (0, 0).$$

Answer the following questions and give reasons for your answer:

- Is f continuous at $(0, 0)$?

- Which directional derivatives of f exist at $(0,0)$? [10]
- Is f differentiable at $(0,0)$?

Ans:

- Let $y = mx$. Then $f(x,y) = \dfrac{1}{\sqrt{1+m^2}}\Big(\big|\, 1 - |m|\,\big| - 1 - |m|\Big)$,
  which depends on m. [3]
  Therefore, f is discontinuous at $(0,0)$. [1]
- Take $u = (u_1, u_2)$, with $\|u\| = 1$. [2]
- Then $\nabla f_u(0,0) = \lim_{t\to 0}\dfrac{f(tu_1, tu_2)}{t} = \lim_{t\to 0}\dfrac{1}{t}(||u_1| - |u_2|| - |u_1 - |u_2||)$. [2]
  The above limit exists if and only if $u_1 = 0$ or $u_2 = 0$. [2]
- Since f is not continuous at $(0,0)$, it is not differentiable at $(0,0)$.

4. (a) A company produces steel boxes at three different plants in amounts x, y and z, respectively, producing an annual revenue of $f(x,y,z) = x^2yz - 2(x+y+z)$. The company is to produce 100 units annually. How should the production be distributed to maximize revenue? [10]

Ans:

- Here, $f(x,y,z) = x^2yz - 2(x+y+z)$ is to be maximized subject to the constraints that $g(x,y,z) := x+y+z-100 = 0$. [2]
- The Lagrange multiplier method implies that $\nabla f = \lambda \nabla g$, for some $\lambda \neq 0$. [2]
- i.e., $2xyz - 2 = \lambda, x^2z - 2 = \lambda, x^2y - 2 = \lambda$ and $x+y+z-100 = 0$. These imply that $z = y, x = 2y$. [4]
- Therefore $y = 25 \Rightarrow x = 50, z = 25$. [2]

(b) Compute $\iiint\limits_D e^{z^2} dV$ where D is the region in the space bounded by the planes $y = 2z, y = 0, z = 1, x = 0$ and $x = 1$. [5]

Ans:

- $I = \iiint\limits_D e^{z^2} dV = \int_0^1 \int_0^1 \int_0^{2z} e^{z^2} dydzdx$. [3]

  or $I = \int_0^1 \int_0^{2z} \int_0^1 e^{z^2} dxdydz$. [3]
- $\Rightarrow I = \int_0^1 e^{z^2} 2zdz = e - 1$. [2]

(c) Find the surface area of the portion of the sphere $x^2 + y^2 + z^2 = 4$ which lies inside the cylinder $y^2 + z^2 + 2z = 0$. [10]

Ans:

- Let T be the region enclosed by $y^2 + z^2 + 2z = 0$ in the yz plane. (This is shadow of the sphere in the yz-plane- see Figure). Then, because of the symmetry, the surface area is

$$a(S) = 2 \iint\limits_T \sqrt{1 + f_y^2 + f_z^2}\, dydz,$$

  where $x = f(y,z) = \sqrt{4 - y^2 - z^2}$. [2]
- $f_y = \frac{-y}{\sqrt{4-y^2-z^2}}, f_z = \frac{-z}{\sqrt{4-y^2-z^2}}$ and $\sqrt{1 + f_y^2 + f_z^2} = \sqrt{\frac{4}{4-y^2-z^2}}$. [2]
-

$$\Rightarrow a(S) = 2 \iint_T \sqrt{\frac{4}{4 - y^2 - z^2}} dydz = 2.2. \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} \frac{2rdrd\theta}{\sqrt{4 - r^2}}.$$

[4]

$$\Rightarrow a(S) = 8 \int_0^{\frac{\pi}{2}} -\sqrt{4 - r^2}|_0^{2\cos\theta} = 8 \int_0^{\frac{\pi}{2}} (-\sqrt{4 - 4\cos^2\theta} + \sqrt{4}) d\theta.$$

$$\Rightarrow a(S) = 16 \int_0^{\frac{\pi}{2}} 1 - \sin\theta d\theta = 16(\frac{\pi}{2} - 1) = 8(\pi - 2).$$

[2]

5. (a) Let D be the portion of the solid hemisphere $\{(x,y,z) : x^2+y^2+z^2 \le 4, z \ge 0\}$ which lies inside the cylinder $x^2+y^2=3$. By spherical and cylindrical coordinates,

$$\iiint_D dV = \int_\alpha^\beta \int_\gamma^\delta \int_\mu^\sigma \rho^2 \sin\phi d\rho\, d\theta\, d\phi + \int_a^b \int_c^d \int_e^f r\, dr\, d\theta dz$$

for some limits $\alpha, \beta, \gamma, \delta, \mu, \sigma$ and a, b, c, d, e, f. Find these limits. [10]

Ans:

- Spherical polar coordinates are taken on the portion of the sphere $x^2 + y^2 + z^2 \le 4$ and $z \ge 1$. Cylindrical coordinates are taken on the cylinder $x^2 + y^2 = 3, 0 \le z \le 1$.
Therefore,

$$\iiint_D dV = \int_0^{\frac{\pi}{3}} \int_0^{2\pi} \int_{\sec\phi}^{2} \rho^2 \sin\phi d\rho\, d\theta\, d\phi + \int_0^1 \int_0^{2\pi} \int_0^{\sqrt{3}} r\, dr\, d\theta dz = I_1 + I_2.$$

Award 3 marks each for limits of ρ and ϕ and 1 mark for limits in θ in I_1.
- Award 3 marks for limits in I_2, 1 mark for each limit.

(b) Let S be the part of the surface $z + x^2 - 1 = 0, 0 \le x \le 1$ and $-2 \le y \le 2$. Find:

- the unit normal to S at an arbitrary point $(x_0, y_0, z_0) \in S$.
- the line integral $\oint_C (2x\vec{i} + \vec{k}).dR$, where C is the simple closed curve given in the polar coordinates by $r = 2\sqrt{\cos\theta}, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.
- the surface integral $\iint_S \frac{1}{\sqrt{1+4x^2}} d\sigma$.

[10]

Ans:

- Unit normal: Surface is $f(x,y,z) := z + x^2 - 1 = 0$, therefore $\nabla f = (2x, 0, 1)$. Hence, $\vec{n} = \frac{2x\vec{i}+\vec{k}}{\sqrt{1+4x^2}}$. [2]
- Line integral: $2x\vec{i} + \vec{k} = \nabla f$, and C is a closed curve, therefore

$$\oint_C (2x\vec{i} + \vec{k}).dR = 0.$$

If a student just writes the value to be zero, without any explanation, as mention of fundamental theorem for line integrals, give no mark. [3]
- Surface integral: $S = \iint_S \frac{1}{\sqrt{1+4x^2}} d\sigma$.
$\Rightarrow S = \iint_T \frac{1}{\sqrt{1+4x^2}} \sqrt{1 + g_x^2 + g_y^2 dxdy}$,
where $z := 1 - x^2 = g(x,y)$.
Here $T = [0,1] \times [-2,2]$. [2]
Therefore, $S = \int_0^1 \int_{-2}^2 \frac{1}{\sqrt{1+4x^2}} \sqrt{1+4x^2} dydx$. [2]

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- Let T be the region enclosed by $y^2 + z^2 + 2z = 0$ in the yz plane. (This is shadow of the sphere in the yz-plane- see Figure). Then, because of the symmetry, the surface area is

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  where $x = f(y,z) = \sqrt{4 - y^2 - z^2}$. [2]
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$$\Rightarrow a(S) = 2 \iint_T \sqrt{\frac{4}{4-y^2-z^2}} dydz = 2.2. \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} \frac{2rdrd\theta}{\sqrt{4-r^2}}.$$

[4]

$$\Rightarrow a(S) = 8 \int_0^{\frac{\pi}{2}} -\sqrt{4-r^2}|_0^{2\cos\theta} = 8 \int_0^{\frac{\pi}{2}} (-\sqrt{4 - 4\cos^2\theta} + \sqrt{4}) d\theta.$$

$$\Rightarrow a(S) = 16 \int_0^{\frac{\pi}{2}} 1 - \sin\theta d\theta = 16(\frac{\pi}{2} - 1) = 8(\pi - 2).$$

[2]

5. (a) Let D be the portion of the solid hemisphere $\{(x, y, z) : x^2 + y^2 + z^2 \leq 4, z \geq 0\}$ which lies inside the cylinder $x^2 + y^2 = 3$. By spherical and cylindrical coordinates,

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for some limits $\alpha, \beta, \gamma, \delta, \mu, \sigma$ and a, b, c, d, e, f. Find these limits. [10]

Ans:

- Spherical polar coordinates are taken on the portion of the sphere $x^2 + y^2 + z^2 \leq 4$ and $z \geq 1$. Cylindrical coordinates are taken on the cylinder $x^2 + y^2 = 3, 0 \leq z \leq 1$.
  Therefore,

$$\iiint\limits_D dV = \int\limits_0^{\frac{\pi}{3}} \int\limits_0^{2\pi} \int\limits_{\sec\phi}^{2} \rho^2 \sin\phi d\rho\, d\theta\, d\phi + \int_0^1 \int_0^{2\pi} \int_0^{\sqrt{3}} r\, dr\, d\theta dz = I_1 + I_2.$$

  Award 3 marks each for limits of ρ and ϕ and 1 mark for limits in θ in I_1.
- Award 3 marks for limits in I_2, 1 mark for each limit.

(b) Let S be the part of the surface $z + x^2 - 1 = 0, 0 \leq x \leq 1$ and $-2 \leq y \leq 2$. Find:

- the unit normal to S at an arbitrary point $(x_0, y_0, z_0) \in S$.
- the line integral $\oint\limits_C (2x\vec{i} + \vec{k}).dR$, where C is the simple closed curve given in the polar coordinates by $r = 2\sqrt{\cos\theta}, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.
- the surface integral $\iint_S \frac{1}{\sqrt{1 + 4x^2}} d\sigma$.

[10]

Ans:

- Unit normal: Surface is $f(x, y, z) := z + x^2 - 1 = 0$, therefore $\nabla f = (2x, 0, 1)$. Hence, $\vec{n} = \frac{2x\vec{i}+\vec{k}}{\sqrt{1+4x^2}}$. [2]
- Line integral: $2x\vec{i} + \vec{k} = \nabla f$, and C is a closed curve, therefore

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  $\Rightarrow S = \iint_T \frac{1}{\sqrt{1 + 4x^2}} \sqrt{1 + g_x^2 + g_y^2 dx dy}$,
  where $z := 1 - x^2 = g(x, y)$.
  Here $T = [0, 1] \times [-2, 2]$. [2]
  Therefore, $S = \int_0^1 \int_{-2}^2 \frac{1}{\sqrt{1 + 4x^2}} \sqrt{1 + 4x^2} dy dx$. [2]

- $\Rightarrow S = 4.$ [1]

(c) Evaluate the line integral $\oint\limits_C (y^2 - tan^{-1}x)dx + (3x + \sin y)dy$, where C is the boundary of the region enclosed by $y = x^2$ and $y = 4$ oriented counterclockwise. [5]

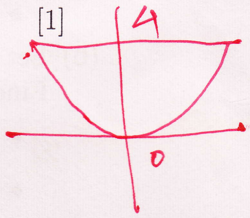
Ans:

- By Green's theorem,

$$I = \oint\limits_C (y^2 - tan^{-1}x)dx + (3x + \sin y)dy = \iint (3 - 2y)dA$$

[2]

- $\Rightarrow I = \int_{-2}^{2} \int_{x^2}^{4} (3 - 2y)dydx$ [2]
- $\Rightarrow I = \int_{-2}^{2} (x^4 - 3x^2 - 4)dx = \frac{-96}{5}.$ [1]



$\int_0^4 \int_{-\sqrt{y}}^{\sqrt{y}} (3-2y).$