MTH102A

Assignments Solution

March 13, 2016

1. Let $A(t)$ denote the amount of radio active substance that is present in a sample at a time t. Then $A' = kA$ is a reasonable model for A. The rate of decay of radioactive isotopes is usually specified in terms of half-life. The half-life of an isotopes is the time required for one half of the initial amount to decay. If the half-life of an isotope is 12 years, what is k? What is the general relation between the half-life of the isotope and k?

Solution: The general solution is $A(t) = Ce^{kt}$. Note that $C = A(0)$. If the half life is 12 years then $A(12) = A(0)/2$. Using the general form of the solution gives $A(0)/2 = A(0)e^{12k}$ from which $k = \frac{1}{12}\log\frac{1}{2}$. More generally the same argument shows that $k = \frac{1}{T}\log\frac{1}{2}$, where T is the half life.

2. Consider the differential equation $y'' = -k^2 y$ for some constant $k > 0$. Check that $y(x) = C_1\cos(kx) + C_2\sin(kx)$ is a general solution of this equation. Also consider $y'' = k^2 y$ for some constant $k > 0$. Check that $y(x) = C_1\cosh(kx) + C_2\sinh(kx)$ is a general solution of this equation.

Solution: Easy verification.

3. For each of the following differential equations draw several isoclines and sketch some solution curves.
(i) $y' = 2x^2 - y$ (ii) $y' = \frac{x^2-y}{y}$.

4. Consider the differential equation $y' - \alpha y,\ x > 0$, where α is a constant. Show that

   (a) if $\phi(x)$ is any solution and $\psi(x) = \phi(x)e^{-\alpha x}$, then $\psi(x)$ is a constant.

   (b) if $\alpha < 0$, then every solutions tends to 0 as $x \to \infty$.

Solution: From $\psi(x) = \phi(x)e^{-\alpha x}$ we get $\phi(x) = \psi(x)e^{\alpha x}$. Since $\phi(x)$ is a solution, substituting $\phi(x)$ into the ODE gives $\psi' = 0$. Thus $\psi = C$, a constant. Any solution of $y' = \alpha y,\ x > 0$ is therefore $y = Ce^{\alpha x},\ x > 0$. If $\alpha < 0$, then clearly $y(x) \to 0$ as $x \to \infty$.

5. Find the orthogonal trajectories of the following families of curves:
(i) $e^x \sin y = c$ (ii) $y^2 = cx^3$

Solution: (i) Differential equation for the family of curves $e^x\sin y = c$ is $\cos y y' + \sin y = 0$. Thus the orthogonal trajectories is governed by the ODE $-\cos y/y' + \sin y = 0$. Solving this we get $e^x\cos y = C$.

(ii) Orthogonal trajectories are $2x^2 + 3y^2 = C$.

6. Reduce the differential equation $y' = F\left(\frac{ax+by+m}{cx+dy+n}\right)$, $ad - bc \neq 0$ to a separable form. Also discuss the case of $ad = bc$.

Solution: Use transform $x = X + h$, $y = Y + k$ and find h, k so that $ah + bk + m = ch + dk + n = 0$ (possible since $ad - bc \neq 0$). Denoting $v = Y/X$, the ODE reduces to $\frac{dv}{f\left(\frac{a+bv}{c+dv}\right)-v} = \frac{dX}{X}$. If $ad = bc$, then $ax + by = \lambda(cx + dy)$. Use substitution $w = cx + dy$ to reduce the ODE to separable form.

7. Show that the following equations are exact and hence find their general solution:

(i) $(\cos x \cos y - \cot x) = (\sin x \sin y)y'$ (ii) $y' = 2x(ye^{-x^2} - y - 3x)/(x^2 + 3y^2 + e^{-x^2})$

Solution: (i) $M = (\cos x \cos y - \cot x)$, $N = -\sin x \sin y$. Clearly $\partial M/\partial y = \partial N/\partial x$. Now $u = \int M\, dx = \sin x \cos y - \log \sin x + f(y)$. Since $\partial u/\partial y = N$, we get $f' = 0$ and hence $f = C$. Thus the solution is $\sin x \cos y - \log \sin x = C$.

(ii) Similar to (i) above and the solution is $x^2 y + 2x^3 + ye^{-x^2} + y^3 = C$

8. Show that if the differential equation $M\, dx + N\, dy$ is of the form

$$x^a y^b (my\, dx + nx\, dy) + x^c y^d (py\, dx + qx\, dy) = 0,$$

where a, b, c, d, m, n, p, q $(mq \neq np)$ are constants, then $x^h y^k$ is an integrating factor. Hence find a general solution of $(x^{1/2}y - xy^2) + (x^{3/2} + x^2 y)y' = 0$.

Solution: Multiplying by $x^h y^k$ and using the condition for exactness, we get

$$\begin{aligned} nh - mk &= m(b+1) - n(a+1) \\ qh - pk &= p(d+1) - q(c+1) \end{aligned}$$

Since $np \neq mq$, we can solve for h, k.

Here $a = 1/2, b = 0$ and $c = d = 1$ and $m = n = q = 1, p = -1$. Thus $h = -9/4$, $k = -7/4$. Solution is $x^{-3/4}y^{-3/4} - 3x^{-1/4}y^{1/4}$=C

9. Solve the initial value problem $xy' = x + \sqrt{x^2 - y^2}$, $y(x_0) = 0$ where $x_0 > 0$.

Solution: By making substitution $v = \frac{y}{x}$ we get $v + xv' = v + \sqrt{1 - v^2}$. Thus, $\sin^{-1} v = \ln x + C$. Observe $v(x_0) = 0$. So, $y(x) = x \sin\left(\ln \frac{x}{x_0}\right)$.

10. (a) Solve $y' + 2xy = e^{x-x^2}, y(0) = -1$.
(b) Add a constant of integration to the integral in the integrating factor and show that the solution you get in the end is the same.

Solution: (a) Integrating factor is $\mu(x) = e^{x^2}$. So $y(x) = e - - x^2 + Ce^{-x^2}$. By putting the value at -1 we get $C = -2$.
(b) Let $\mu(x) = e^{\int 2x dx + c}$. Answer will be same.

11. Solve $2xe^{2y}y' = 3x^4 + e^{2y}$. (Observe that it is neither linear nor separable nor homogeneous nor Bernoulli)

Solution: Substitute $v = xe^y$. We get $v' - \frac{2v}{x} = 3x^4$.

12. Use Picard's method of successive approximation to solve the following initial value problems and compare these results with the exact solutions: (i) $y' = 2\sqrt{x},\ y(0) = 1$ (ii) $y' + xy = x,\ y(0) = 0$ (iii) $y' = 2\sqrt{y}/3,\ y(0) = 0$

Solution: Picard iteration is $y_{n+1}(x) = y_0 + \int_{x_0}^{x} f[t, y_n(t)]\, dt$.

(i) $y_0 = 0,\ y_n(x) = 1 + 2\int_0^x \sqrt{t}\, dt = 1 + (4/3)x^{3/2},\ n \geq 1$ (since f is independent of y). Here $y_n(x)$ $(n \geq 1)$ coincides with the exact solution.

(ii) Here $f(x, y) = x(1 - y)$ and $y_0 = 0$. Thus $y_1(x) = \int_0^x t(1 - 0)\, dt = x^2/2$. Using y_1, we get $y_2(x) = \int_0^x t(1 - t^2/2)\, dt = x^2/2 - (x^2/2)^2/2$. $y_3(x) = x^2/2 - (x^2/2)^2/2 + (x^2/2)^3/3!$. Proceeding this way we get $y_n(x) = \sum_{m=1}^{n}(-1)^{m-1}(x^2/2)^m/m!$. Thus $y_n(x) \to -\sum_{m=0}^{n}(-x^2/2)^m/m! + 1 = 1 - e^{-x^2/2}$, which is the exact solution.

(iii) Here $y_0 = 0$ and $f(x, y) = 2\sqrt{y}/3$. Thus $y_n(x) = 0,\ n \geq 1$. Now $y_n(x), \forall n$ coincides with the analytical solution $y(x) = 0$. The other solution $y(x) = (x/3)^2$ is not reachable from here.

13. Reduce the following second order differential equation to first order differential equation and hence solve.
(i) $xy'' + y' = y'^2$ (iii) $yy'' + y'^2 + 1 = 0$ (iii) $y'' - 2y' \coth x = 0$

Solution: (i) Dependent variable y absent. Substitute $y' = p \implies y'' = dp/dx$. Thus $xp' + p = p^2$. Solving $p = 1/(1 - ax)$ which on integrating again gives $y = b - \ln(1 - ax)/a$.

(ii) $yy'' + y'^2 + 1 = 0$. Substitute $y' = p \implies y'' = p\, dp/dy$. Thus $dp/dy + 2p/y = -1/y$. Solving $p^2 = a^2/y^2 - 1$, which on integrating again gives $(x + b)^2 + y^2 = a^2$.

(iii) $y'' - 2y' \coth x = 0$. Substitute $y' = p \implies y'' = dp/dx$. Thus $dp/dx = 2p \coth x$. Solving $p = a \sinh^2 x$, which on integrating again gives $y = a(\sinh 2x - 2x)/4 + b$.

14. Find the differential equation satisfied by each of the following two-parameter families of plane curves:
(i) $y = \cos(ax + b)$ (ii) $y = ax + b/x$ (iii) $y = ae^x + bxe^x$

Solution: Eliminate constants a and b by differentiating twice. Answers:
(i) $(1 - y^2)y'' + yy'^2 = 10$ (ii) $y = xy' + x^2y''$ (iii) $y'' - 2y' + y = 0$

15. (a) Find the values of m such that $y = e^{mx}$ is a solution of
(i) $y'' + 3y' + 2y = 0$ (ii) $y'' - 4y' + 4y = 0$ (iii) $y''' - 2y'' - y' + 2y = 0$
(b) Find the values of m such that $y = x^m$ $(x > 0)$ is a solution of
(i) $x^2y'' - 4xy' + 4y = 0$ (ii) $x^2y'' - 3xy' - 5y = 0$

Solution: (a) (i) $m = -2, -1$ (ii) $m = -2, -2$ (iii) $m = -1, 1, 2$
(b) Answers: (i) $m = 1, 4$ (ii) $m = -1, 5$

16. Are the following functions linearly dependent on the given intervals?

(i)	$\sin 4x,\ \cos 4x$	$(-\infty, \infty)$	(ii)	$\ln x,\ \ln x^3$	$(0, \infty)$
(iii)	$\cos 2x,\ \sin^2 x$	$(0, \infty)$	(iv)	$x^3,\ x^2\lvert x\rvert$	$[-1, 1]$

Solution: Answers: (i) No (ii) Yes (iii) No (iv) No

17. (a) Let $y_1(x), y_2(x)$ be two twice continuously differentiable functions on an interval $\mathcal{I}$. Suppose that the Wronskian $W(y_1, y_2)$ does not vanish anywhere in $\mathcal{I}$. Show that there exists unique $p(x), q(x)$ on $\mathcal{I}$ such that (*) has y_1, y_2 as fundamental solutions.

    (b) Construct equations of the form (*) from the following pairs of solutions:
    (i) e^{-x}, xe^{-x} (ii) $e^{-x}\sin 2x$, $e^{-x}\cos 2x$

18. We want to find $p(x), q(x)$ such that

$$y_1'' + p(x)y_1' + q(x)y_1 = 0, \tag{1}$$
$$y_2'' + p(x)y_2' + q(x)y_2 = 0. \tag{2}$$

Since the Wronskian $W(y_1, y_2)$ is never never zero in the whole interval, we can solve $p(x), q(x)$ from these two equations and find $p(x) = -(y_1y_2'' - y_2y_1'')/W(y_1, y_2)$ and $q(x) = (y_1'y_2'' - y_2'y_1'')/W(y_1, y_2)$

19. Find general solution of the following differential equations given a known solution y_1:
(i) $x(1-x)y'' + 2(1-2x)y' - 2y = 0$ $\quad y_1 = 1/x$
(ii) $(1-x^2)y'' - 2xy' + 2y = 0$ $\quad y_1 = x$

Solution: (i) Here $y_1 = 1/x$. Substitute $y = u(x)/x$ to get $(1-x)u'' - 2u' = 0$. Thus, $u' = 1/(1-x)^2$ and $u = 1/(1-x)$. Hence, $y_2 = x/(1-x)$ and the general solution is $y = a/x + b/(x(1-x))$.

(ii) Here $y_1 = x$. Substitute $y = xu(x)$ to get $(1-x^2)u'' = 2(2x^2-1)u$. Thus, $u' = 1/(x^2(1-x^2))$ and $u = -1/x + \frac{1}{2}\ln[(1+x)/(1-x)]$. Hence, $y_2 = -1 + \frac{x}{2}\ln[(1+x)/(1-x)]$ and the general solution is $y = ax + b\left\{-1 + \frac{x}{2}\ln[(1+x)/(1-x)]\right\}$.

20. Verify that $\sin x/\sqrt{x}$ is a solution of $x^2y'' + xy' + (x^2 - 1/4)y = 0$ over any interval on the positive x-axis and hence find its general solution.

Solution: Verification is easy.
Substitute $y = u(x)\sin x/\sqrt{x}$ to give $\sin x\, u'' + 2\cos x\, u' = 0$. Thus, $u' = \operatorname{cosec}^2 x$ and $u = -\cot x$. Hence, $y_2 = -\cos x/\sqrt{x}$ and the general solution is $y = (a\sin x + b\cos x)/\sqrt{x}$.

21. Solve the following differential equations:
(i) $y'' - 4y' + 3y = 0$ (ii) $y'' + 2y' + (\omega^2 + 1)y = 0$, ω is real.

Solution: (i) Characteristic (or auxiliary) equation: $m^2 - 4m + 3 = 0 \implies m = 1, 3$.
General sol: $y = Ae^x + Be^{3x}$

(ii) Characteristic equation: $m^2 + 2m + (1+\omega^2) = 0 \implies m = -1 \pm \omega i$.

Case 1: $\omega = 0 \implies$ equal roots $m = -1, -1$ and general sol: $y = (A + Bx)e^{-x}$

Case 2: $\omega \neq 0 \implies$ complex conjugate roots $m = -1 \pm \omega i$ and general sol: $y = e^{-x}(A\sin\omega x + B\cos\omega x)$

22. Solve the following initial value problems:
(i) $y'' + 4y' + 4y = 0 \qquad y(0) = 1,\ y'(0) = -1$
(ii) $y'' - 2y' - 3y = 0 \qquad y(0) = 1,\ y'(0) = 3$

Solution: Solve the following initial value problems:

(i) General sol: $y = e^{-2x}(A + Bx)$. Using initial conditions: $y = (x + 1)e^{-2x}$

(ii) General sol: $y = (Ae^{3x} + Be^{-x})$. Using initial conditions: $y = e^{3x}$

23. The equation
$$x^2 \frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = 0,$$
where a, b are constants, is called the Euler-Cauchy equation. Show that under the transformation $x = e^t$ (when $x > 0$) for the independent variable, the above reduces to
$$\frac{d^2y}{dt^2} + (a-1)\frac{dy}{dt} + by = 0,$$
which is an equation with constant coefficients.
Hence solve: (i) $x^2y'' + 2xy' - 12y = 0$ (ii) $x^2y'' + xy' + y = 0$ (iii) $x^2y'' - xy' + y = 0$

Solution: $x = e^t \implies dy/dx = (1/x)dy/dt$ and $d^2y/dt^2 = (1/x^2)[d^2y/dt^2 - dy/dt]$ etc., $x > 0$.

(i) Using $x = e^t$ gives $d^2y/dt^2 + dy/dt - 12y = 0 \implies y = Ae^{-4t} + Be^{3t}$
$\implies y = Ax^{-4} + Bx^3$

(ii) Using $x = e^t$ gives $d^2y/dt^2 + y = 0 \implies y = A\cos t + B\sin t$
$\implies y = A\cos(\log x) + B\sin(\log x)$

(iii) Using $x = e^t$ gives $d^2y/dt^2 - 2dy/dt + y = 0 \implies y = (A + Bt)e^t$
$\implies y = x(A + B\log x)$

24. Show that the fundamental system of solutions of Legendre equation
$$(1 - x^2)y'' - 2xy' + p(p+1)y = 0$$
consists of $y_1(x) = \sum_{n=0}^{\infty} a_{2n}x^{2n}$ and $y_2(x) = \sum_{n=0}^{\infty} a_{2n+1}x^{2n+1}$, where $a_0 = a_1 = 1$ and
$$a_{2n+2} = -\frac{(p-2n)(p+2n+1)}{(2n+1)(2n+2)}a_{2n}, \qquad n = 0, 1, 2, \cdots$$
$$a_{2n+1} = -\frac{(p-2n+1)(p+2n)}{2n(2n+1)}a_{2n-1}, \qquad n = 1, 2, 3, \cdots .$$

25. Verify that
$$y_1(x) = P_0(x) = 1,\ y_2(x) = \frac{1}{2}\ln\frac{1+x}{1-x} \quad \text{for } p = 0$$
$$y_2(x) = P_1(x) = x,\ y_1(x) = 1 - \frac{x}{2}\ln\frac{1-x}{1+x} \quad \text{for } p = 1.$$

26. The expression, $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}[(x^2-1)^n]$, is called the Rodrigues' formula for

Legendre polynomial P_n of degree n. Assuming this, find P_1, P_2, P_3.

(b) For $p = 0$, $a_{2n+2} = [2n/(2n+2)]a_{2n}$, $n = 0, 1, 2, \cdots \implies a_{2n} = 0$, $n \geq 1$ and hence $y_1(x) = P_0(x) = 1$. On the other hand, $a_{2n+1} = [(2n-1)/(2n+1)]a_{2n-1}$, $n = 1, 2, 3, \cdots$.. Thus, $y_2(x) = x + x^3/3 + x^5/5 + x^7/7 + \cdots = (1/2)\ln[(1+x)/(1-x)]$

For $p = 1$, $a_{2n+2} = [(2n-1)/(2n+1)]a_{2n}$, $n = 0, 1, 2, \cdots$ and hence $y_1(x) = 1 - x^2 - x^4/3 - x^6/5 - \cdots = 1 - (x/2)\ln[(1+x)/(1-x)]$. On the other hand, $a_{2n+1} = [(2n-2)/(2n)]a_{2n-1}$, $n = 1, 2, 3, \cdots \implies a_{2n+1} = 0$, $n \geq 1$. Thus, $y_2(x) = P_1(x) = x$

(c) Straight forward calculations: $P_1(x) = x$, $P_2(x) = (3x^2-1)/2$, $P_3(x) = (5x^3 - 3x)/2$.

27. Using Rodrigues' formula for $P_n(x)$, show that
(i) $P_n(-x) = (-1)^n P_n(x)$ (ii) $P_n'(-x) = (-1)^{n+1}P_n'(x)$
(iii) $\displaystyle\int_{-1}^{1} P_n(x)P_m(x)\,dx = \frac{2}{2n+1}\delta_{mn}$ (iv) $\displaystyle\int_{-1}^{1} x^m P_n(x)\,dx = 0$ if $m < n$

Solution: (i) Replace x in $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}[(x^2-1)^n]$ by $-x$ to get $P_n(-x) = (-1)^n \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}[(x^2-1)^n] = (-1)^n P_n(x)$

(ii) By differentiating (i) w.r.t. x, we get $P_n'(-x) = (-1)^{n+1}P_n(x)$.

(iii) Let $f(x)$ be any function with at least n continuous derivatives in $[-1, 1]$. Consider the integral $I = \int_{-1}^{1} f(x)P_n(x)\,dx = (1/2^n n!)\int_{-1}^{1} f(x)(d^n/dx^n)(x^2 - 1)^n\,dx$. Repetition of integration by parts gives $I = (-1)^n/(2^n n!)\int_{-1}^{1} f^n(x)(x^2 - 1)^n\,dx$. If $m \neq n$, without any loss of generality we take $f = P_m$, $m < n$ and then $f^n(x) = 0$ and $I = 0$. If $f(x) = P_n(x)$, then $f^n(x) = (1/2^n n!)(d^{2n}/dx^{2n})(x^2-1)^n = (2n!)/(2^n n!)$. Thus,

$$I = (2n!)/(2^{2n}(n!)^2)\int_{-1}^{1}(1-x^2)^n\,dx = 2(2n!)/(2^{2n}(n!)^2)\int_{0}^{1}(1-x^2)^n\,dx.$$

Substitute $x = \sin\theta$ to get

$$I = \frac{2(2n!)}{2^{2n}(n!)^2}\int_0^{\pi/2}\cos^{2n+1}\theta\,d\theta = \frac{2(2n!)}{2^{2n}(n!)^2}I_n.$$

Since

$$\int \cos^{2n+1}\,d\theta = \frac{1}{2n+1}\cos^{2n}\theta\sin\theta + \frac{2n}{2n+1}\int\cos^{2n-1}\theta\,d\theta,$$

we find

$$I_n = \frac{2n}{2n+1}I_{n-1} = \frac{2n}{2n+1}\frac{2n-2}{2n-1}\cdots\frac{2}{3}I_0.$$

Now

$$I_0 = \int_0^{\pi/2}\cos\theta\,d\theta = 1.$$

Hence,

$$I_n = \frac{2^n n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)} = \frac{2^{2n}(n!)^2}{(2n!)(2n+1)}$$

Thus, we get the required result.

(iv) See (iii)

28. Suppose $m > n$. Show that $\int_{-1}^{1} x^m P_n(x)\, dx = 0$ if $m-n$ is odd. What happens if $m-n$ is even?

Solution: Proceeding as in 5(iii), we get

$$I = \int_{-1}^{1} x^m P_n(x)\, dx = \frac{m(m-1)\cdots(m-n+1)}{2^n n!} \int_{-1}^{1} x^{m-n}(1-x^2)^n\, dx$$

If $m-n$ is odd, then $I = 0$, since integrand is an odd function.

If $m - n = 2k$ is even, then

$$\begin{aligned} I &= \frac{2m(m-1)\cdots(m-n+1)}{2^n n!} \int_0^{\pi/2} \sin^{2k}\theta \cos^{2n+1}\theta\, d\theta \\ &= \frac{2m(m-1)\cdots(m-n+1)}{2^n n!} I_{k,n} \end{aligned}$$

where

$$I_{k,n} = \int_0^{\pi/2} \sin^{2k}\theta \cos^{2n+1}\theta\, d\theta = \frac{2n}{2k+1} I_{k+1,n-1}$$

By repeated application of this relation, the last subscript becomes zero. Then the resulting integral can be evaluated by substitution. Thus,

$$\begin{aligned} I_{k,n} &= \frac{2n \cdot 2(n-1)\cdots 2.1}{(2k+1)(2k+3)\cdots(2\{k+n-1\}+1)} I_{k+n,0} \\ &= \frac{2^n n!}{(2k+1)(2k+3)\cdots(2\{k+n-1\}+1)(2\{k+n\}+1)} \end{aligned}$$

29. Expand the following functions in terms of Legendre polynomials over $[-1,1]$:

(i) $f(x) = x^3 + x + 1$ (ii) $f(x) = \begin{cases} 0 & \text{if} \quad -1 \le x < 0 \\ x & \text{if} \quad 0 \le x \le 1 \end{cases}$ (first three nonzero terms)

Solution: For any piecewise continuous function $f(x)$, the Legendre expansion is

$$f(x) = \sum_{n=0} a_n P_n(x), \qquad a_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx; \quad x \in [-1,1]$$

(i) We can use the above formula. Alternately, using

$$1 = P_0, x = P_1, x^3 = (2P_3 + 3P_1)/5,$$

we get

$$f(x) = (5p_0 + 8p_1 + 2p_3)/5$$

(ii) Using the above formula,

$$a_0 = 1/4, a_1 = 1/2, a_2 = 5/16.$$

Thus,

$$f(x) = P_0/4 + P_1/2 + 5P_2/16 + \cdots$$

30. Show that $2x^3y'' + (\cos 2x - 1)y' + 2xy = 0$ has only one Frobenius series solution.

Solution: Here $p(x) = (\cos 2x - 1)/2x^2$ and $q(x) = 1$. Hence, the indicial equation is

$$r^2 + (p(0) - 1)r + q(0) = 0 \implies r^2 - 2r + 1 \implies r = 1, 1.$$

Since the indicial equation has double root, it has only one Frobenius series solution.

31. Reduce $x^2y'' + xy' + (x^2 - 1/4)y = 0$ to normal form and hence find its general solution.

Solution: Here $p = 1/x, q = 1 - 1/(4x^2)$ and $\exp\left(-\frac{1}{2}\int p dx\right) = 1/\sqrt{x}$. Using the transformation $y = u/\sqrt{x}$, we get $u'' + u = 0$. Thus general solution is $u = A\sin x + B\cos x$. For the original equation, the general solution of is $y = (A\sin x + B\cos x)/\sqrt{x}$.

32. Find a solution bounded near $x = 0$ of the following ODE

$$x^2y'' + xy' + (\lambda^2x^2 - 1)y = 0$$

Solution: Substitute $\lambda x = t$ and we get

$$t^2\ddot{y} + t\dot{y} + (t^2 - 1)y = 0,$$

where $\dot{}$ is w.r.t. t. It is Bessel eqn. of order $\nu = 1$. General solution is $y = AJ_1(t) + BY_1(t) = AJ_1(\lambda x) + BY_1(\lambda x)$. Note that Y_1 is unbounded (due to presence of $\log x$) at $x = 0$ and hence we must have $y = AJ_1(\lambda x)$.

$\Big\{$Useful formulae for problems with Bessels functions:
$\left(x^\nu J_\nu\right)' = x^\nu J_{\nu-1}$, $\left(x^{-\nu} J_\nu\right)' = -x^{-\nu} J_{\nu+1}$, $J_{\nu-1} + J_{\nu+1} = 2\nu J_\nu/x$, $J_{\nu-1} - J_{\nu+1} = 2J'_\nu$.$\Big\}$

33. Using recurrence relations, show that
(i) $J''_0(x) = -J_0(x) + J_1(x)/x$ (ii) $xJ'_{n+1}(x) + (n+1)J_{n+1}(x) = xJ_n(x)$

Solution: (i) $2J'_0 = J_{-1} - J_1 = -2J_1 \implies 2J''_0 = -2J'_1 = J_2 - J_0 = 2J_1/x - 2J_0$

(ii) $\left(x^{n+1}J_{n+1}\right)' = x^{n+1}J_n \implies xJ'_{n+1} + (n+1)J_{n+1} = xJ_n$

34. Show that
(i) $\int x^4 J_1(x)\, dx = (4x^3 - 16x)J_1(x) - (x^4 - 8x^2)J_0(x) + C$
(ii) $\int J_5(x)\, dx = -2J_4(x) - 2J_2(x) - J_0(x) + C$

Solution: (i) From $J_0' = -J_1$ we have $\int x^4 J_1 = -\int x^4 J_0' = -x^4 J_0 + 4\int x^3 J_0 = -x^4 J_0 + 4\int x^2 [xJ_1]' = -x^4 J_0 + 4x^3 J_1 - 8\int x^2 J_1 = -x^4 J_0 + 4x^3 J_1 + 8\int x^2 J_0' = -x^4 J_0 + 4x^3 J_1 + 8x^2 J_0 - 16\int xJ_0 = -x^4 J_0 + 4x^3 J_1 + 8x^2 J_0 - 16\int [xJ_1]' = (4x^3 - 16x)J_1 - (x^4 - 8x^2)J_0 + C$

(ii) $\int J_5 = \int [J_3 - 2J_4'] = -\int [J_1 - 2J_2' - 2J_4'] = -\int [J_0' + 2J_2' + 2J_4'] = -2J_4(x) - 2J_2(x) - J_0(x) + C$

35. Express
(i) $J_3(x)$ in terms of $J_1(x)$ and $J_0(x)$ (ii) $J_2'(x)$ in terms of $J_1(x)$ and $J_0(x)$
(iii) $J_4(ax)$ in terms of $J_1(ax)$ and $J_0(ax)$

Solution: (i) Using $J_{\nu+1} = 2\nu J_\nu/x - J_{\nu-1}$ we have

$$\begin{aligned} J_3(x) &= \frac{4}{x}J_2(x) - J_1(x) = \frac{4}{x}\left[\frac{2}{x}J_1(x) - J_0(x)\right] - J_1(x) \\ &= \left(\frac{8}{x^2} - 1\right)J_1(x) - \frac{4}{x}J_0(x) \end{aligned}$$

(ii)

$$2J_2' = J_1 - J_3 = J_1 - \left[\frac{4}{x}J_2 - J_1\right] = 2J_1 - \frac{4}{x}\left[\frac{2}{x}J_1 - J_0\right]$$

Thus $J_2' = 2J_0/x - (1 - 4/x^2)J_1$

(iii) Using $J_{\nu+1} = 2\nu J_\nu/x - J_{\nu-1}$ we have

$$\begin{aligned} J_4(ax) &= \frac{6}{ax}J_3(ax) - J_2(ax) = \frac{6}{ax}\left[\frac{4}{ax}J_2(ax) - J_1(ax)\right] - J_2(ax) \\ &= \left(\frac{24}{a^2x^2} - 1\right)J_2(ax) - \frac{6}{ax}J_1(ax) \\ &= \left(\frac{24}{a^2x^2} - 1\right)\left[\frac{2}{ax}J_1(ax) - J_0(ax)\right] - \frac{6}{ax}J_1(ax) \end{aligned}$$

36. Prove that between each pair of consecutive positive zeros of $J_\nu(x)$, there is exactly one zero of $J_{\nu+1}(x)$ and vice versa.

Solution: Let α and β be two consecutive positive zeros of $J_{\nu+1}$. Let $f(x) = x^{\nu+1}J_{\nu+1}$. Then $f(\alpha) = f(\beta) = 0$. Thus there exists $c \in (\alpha, \beta)$ such that $f'(c) = 0$. Taking $\gamma = \nu + 1$ in $[x^\gamma J_\gamma]' = x^\gamma J_{\gamma-1}$, we see that $J_\nu(c) = 0$. Thus there exists a zero of J_ν between consecutive zeros of $J_{\nu+1}$. Similarly taking $\gamma = \nu$ in $[x^{-\gamma}J_\gamma]' = -x^{-\gamma}J_{\gamma+1}$, we conclude that there exists a zero of $J_{\nu+1}$ between consecutive positive zeros of J_ν. To prove uniqueness, let there exist two zero of J_ν between consecutive zeros α and β of $J_{\nu+1}$. This implies that there exist a zero of $J_{\nu+1}$ between α and β, which contradicts the fact that α and β are consecutive zeroes.

37. Let $u(x)$ be any nontrivial solution of $u'' + [1 + q(x)]u = 0$, where $q(x) > 0$. Show that $u(x)$ has infinitely many zeros.

Solution: Consider

$$v'' + v = 0, \qquad u'' + [1 + q(x)]u = 0$$

Now $v = \sin x$ is a nontrivial solution of $v'' + v = 0$. Since $1 + q(x) > 1$, by Strum comparison theorem, u must vanish between two zeros of $\sin x$. Since, $\sin x$ has infinitely many zeros, u also has infinitely may zeros.

38. Let $F(s)$ be the Laplace transform of $f(t)$. Find the Laplace transform of $f(at) \quad (a > 0)$.

Soln:

$$f(at) = \int_0^\infty e^{-st} f(at)\, dt = \frac{1}{a}\int_0^\infty e^{-(s/a)\tau} f(\tau)\, d\tau = F(s/a)/a$$

39. Find the Laplace transforms:

(a) $[t]$ (greatest integer function), (b) $t^m \cosh bt \quad (m \in$ non-negative integers),

(c) $e^t \sin at$, (d) $\dfrac{e^t\ \sin at}{t}$, (e) $\dfrac{\sin t \cosh t}{t}$, (f) $f(t) = \begin{cases} \sin 3t, & 0 < t < \pi, \\ 0, & t > \pi, \end{cases}$

Soln: (a)

$$\int_1^2 e^{-st}\, dt + 2\int_2^3 e^{-st}\, dt + 3\int_3^4 e^{-st}\, dt + \cdots$$

$$= \frac{e^{-s}}{s}(1 + e^{-s} + e^{-2s} + e^{-3s} + \cdots) = \frac{1}{s(e^s - 1)}$$

(b)

$$\begin{aligned} \mathcal{L}(t^m) &= \frac{m!}{s^{m+1}} \implies \mathcal{L}(t^m \cosh bt) = \frac{1}{2}\mathcal{L}(e^{bt}t^m + e^{-bt}t^m) \\ &= \frac{(m!}{2}\left[\frac{1}{(s-b)^{m+1}} + \frac{1}{(s+b)^{m+1}}\right] \end{aligned}$$

(c)

$$\mathcal{L}(\sin at) = \frac{a}{s^2 + a^2} \implies \mathcal{L}(e^t \sin at) = \frac{a}{(s-1)^2 + a^2}$$

(d) Use $\mathcal{L}[f(t)/t] = \int_s^\infty F(s)\, ds$. Now

$$\begin{aligned} \mathcal{L}(\sin at) &= \frac{a}{s^2 + a^2} \implies \mathcal{L}(\sin at/t) = a\int_s^\infty \frac{1}{s^2 + a^2} = \tan^{-1}(a/s) \\ \implies \mathcal{L}(e^t \sin at/t) &= \tan^{-1}\left(\frac{a}{s-1}\right) \end{aligned}$$

(e)

$$\mathcal{L}(\sin t/t) = \tan^{-1}(1/s) \implies \mathcal{L}(\cosh t \sin t/t) = \frac{1}{2}\left[\tan^{-1}\left(\frac{1}{s-1}\right) + \tan^{-1}\left(\frac{1}{s+1}\right)\right]$$

(f)

$$\mathcal{L}[f(t)] = \int_0^\pi e^{-st} \sin 3t\, dt = \frac{3(1 + e^{-\pi s})}{s^2 + 9}$$

40. Using convolution, find the inverse Laplace transforms:

(a) $\dfrac{1}{s^2-5s+6}$, (b) $\dfrac{2}{s^2-1}$, (c) $\dfrac{1}{s^2(s^2+4)}$, (d) $\dfrac{1}{(s-1)^2}$.

Soln: (a)

$$F(s)=\frac{1}{s^2-5s+6}=\frac{1}{(s-3)(s-2)}$$

Now

$$\mathcal{L}(e^{3t})=\frac{1}{s-3},\quad \mathcal{L}(e^{2t})=\frac{1}{s-2}.$$

Hence,

$$f(t)=\int_0^t e^{3\tau}e^{2(t-\tau)}d\tau=e^{2t}\int_0^t e^{\tau}d\tau=e^{3t}-e^{2t}$$

(b)

$$F(s)=\frac{2}{s^2-1}=\frac{2}{(s+1)(s-1)}$$

Now

$$\mathcal{L}(e^{t})=\frac{1}{s-1},\quad \mathcal{L}(e^{-t})=\frac{1}{s+1}.$$

Hence,

$$f(t)=2\int_0^t e^{\tau}e^{-(t-\tau)}d\tau=2e^{-t}\int_0^t e^{2\tau}d\tau=e^{t}-e^{-t}$$

(c)

$$F(s)=\frac{1}{s^2(s^2+4)}=\frac{1}{2}\frac{1}{s^2}\frac{2}{s^2+4}$$

Now

$$\mathcal{L}(t)=\frac{1}{s^2},\quad \mathcal{L}\left(\sin 2t\right)=\frac{2}{s^2+4}.$$

Hence,

$$f(t)=\frac{1}{2}\int_0^t (t-\tau)\sin(2\tau)d\tau=\frac{2t-\sin 2t}{8}$$

(d)

$$F(s)=\frac{1}{(s-1)^2}=\frac{1}{s-1}\frac{1}{s-1}$$

Now

$$\mathcal{L}(e^{t})=\frac{1}{s-1}.$$

Hence,

$$f(t)=\int_0^t e^{\tau}e^{t-\tau}d\tau=e^{t}\int_0^t d\tau=te^{t}$$

41. Use Laplace transform to solve the initial value problems:

(a) $y''+4y=\cos 2t,\qquad y(0)=0,\ y'(0)=1.$

(b) $y''+3y'+2y=4t$ if $0<t<1$ and 8 if $t>1$; $y(0)=y'(0)=0$

(c) $y''+9y=8\sin t$ if $0<t<\pi$ and 0 if $t>\pi$; $y(0)=0, y'(0)=4$

(d) $y_1'+2y_1+6\displaystyle\int_0^t y_2(\tau)\,d\tau = 2u(t), \quad y_1'+y_2' = -y_2, \qquad y_1(0) = -5, y_2(0) = 6$

Soln: (a) Taking Laplace Transform on both sides and simplifying (Y(s)=$\mathcal{L}$[y(t)])

$$Y(s) = s/(s^2+4)^2 + 1/(s^2+4)$$

Using convolution [or any other technique]

$$\begin{aligned} y(t) &= \frac{1}{2}\int_0^t \sin(2\tau)\cos(2(t-\tau))d\tau + \frac{\sin 2t}{2} \\ &= \frac{t\sin 2t}{4} + \frac{\sin 2t}{2} \end{aligned}$$

(b) Let $r(t) = 4u(t-0)t + 4u(t-1)(1-(t-1))$. Taking Laplace Transform on both sides of the ODE, we get

$$(s^2+3s+2)Y(s) = R(s) \implies Y(s) = \frac{4}{s^2(s+1)(s+2)} + e^{-s}\frac{s-1}{s^2(s+1)(s+2)}$$

Using partial fraction and solving we get

$$y(t) = -3 - e^{-2t} + 4e^{-t} + 2t + u(t-1)\left(5 + 3e^{-2(t-1)} - 8e^{-(t-1)} - 2(t-1)\right)$$

(c) Let $r(t) = 8(u(t-0) - u(t-\pi))\sin t = 8u(t-0)\sin t + u(t-\pi)\sin(t-\pi)$. Taking Laplace Transform on both sides of the ODE, we get

$$(s^2+9)Y(s) = R(s) + 4 \implies Y(s) = \frac{4}{s^2+9} + \frac{R(s)}{s^2+9}$$

We can explicitly write $R(s)$ and then use partial fraction technique. Otherwise, use convolution as follows

$$y(t) = \frac{4}{3}\sin 3t + \frac{1}{3}\int_0^t r(\tau)\sin 3(t-\tau)\,d\tau$$

Thus for $0 < t < \pi$, we get

$$y(t) = \frac{4}{3}\sin 3t + \frac{8}{3}\int_0^t \sin\tau\sin 3(t-\tau)\,d\tau = \frac{4}{3}\sin 3t + \sin t - \frac{1}{3}\sin 3t = \sin 3t + \sin t$$

and for $t > \pi$, we get [since $r(t) = 0$]

$$y(t) = \frac{4}{3}\sin 3t + \frac{8}{3}\int_0^\pi \sin\tau\sin 3(t-\tau)\,d\tau + \frac{1}{3}\int_\pi^t 0\,\sin 3(t-\tau)\,d\tau = \frac{4}{3}\sin 3t$$

(d) Taking Laplace transform, we get

$$\begin{aligned} (s+2)Y_1 + \frac{6Y_2}{s} &= \frac{2}{s} - 5 \\ sY_1 + (s+1)Y_2 &= 1 \end{aligned}$$

Solving

$$\begin{aligned} Y_1(s) &= \frac{1}{s} - \frac{12}{5}\frac{1}{s-1} - \frac{18}{5}\frac{1}{s+4} \\ Y_2(s) &= \frac{6}{5}\frac{1}{s-1} + \frac{24}{5}\frac{1}{s+4} \end{aligned}$$

Thus,

$$\begin{aligned} y_1(t) &= 1 - 12e^t/5 - 18e^{-4t}/5 \\ y_2(t) &= 6e^t/5 + 24e^{-4t}/5 \end{aligned}$$