MTH101A: Mathematics-I

Problem Set 2: Series

(To be discussed in the week starting on 12 August 2019)

The problems marked with an asterisk(*) will not be asked during any quiz or exam. The problems marked with a plus sign(+) are extra questions and will be discussed in the tutorial only if time permits.

1. (+) Use the Cauchy condensation test to find the values of $p > 0$ for which the p-Harmonic series $\sum_{n\geq 1} \frac{1}{n^p}$ converges. For what values of p does it diverge?

2. Are the following series convergent or divergent? Give appropriate reasons.

   (a) $\sum_{n\geq 1} \frac{1}{n^2} sin(\frac{n\pi}{4})$

   (b) $\sum_{n\geq 1} \frac{n-1}{2n+1}$

   (c) $\sum_{n\geq 1} \frac{n^2}{2^n}$

   (d) $\sum_{n\geq 1} ne^{-n}$

   (e) $\sum_{n\geq 1}(n\ln(1+\frac{1}{n}))$

   (f) $\sum_{n\geq 1}(\tan^{-1} n)^n$

3. Show that the alternating series $\sum_{n\geq 1} \frac{(-1)^{n+1}}{n^p}$ converges iff $p > 0$.

4. (+) (The Bouncing Ball.) Suppose that a rubber ball is dropped from a height of 1 metre and that each time it bounces it rises to a height of (2/3) of the previous height. How far does it travel before it stops bouncing (and yes, it does stop)?

5. Let (a_n) be a sequence of non-negative terms. Show that $\sum_{n\geq 1}$ converges iff $\sum_{n\geq 1} \frac{a_n}{1+a_n}$ converges.

6. Find the radius of convergence for the following power series:

   (a) $\sum_{n\geq 0} \frac{x^n}{n}$

   (b) $\sum_{n\geq 0} \frac{x^n}{n!}$

   (c) $\sum_{n\geq 0} n!x^n$

   (d) $\sum_{n\geq 0} \frac{(3n)!}{(n!)^3}x^n$

7. We showed above that $\sum_{n\geq 0} \frac{x^n}{n!}$ is convergent for all $x \in \mathbb{R}$. Define $e^x := \sum_{n\geq 0} \frac{x^n}{n!}$. Show that $e^x \cdot e^y = e^{x+y}$ for all $x, y \in \mathbb{R}$.

   Can you guess individual terms of the product of two series?

8. (*) Let $\sum_{n\geq 1}(-1)^{n+1}a_n$ be an absolutely convergent series. Show that for any one-to-one correspondence $\phi : \mathbb{N} \to \mathbb{N}$ we have that $\sum_{n\geq 1}(-1)^{n+1}a_{\phi(n)}$ converges to $\sum_{n\geq 1}(-1)^{n+1}a_n$.

   (Hint: Use the sequences (a_n^+) and (a_n^-).)

9. (*) (Riemann rearrangement theorem) Let $\sum_{n\geq 1}(-1)^{n+1}a_n$ be a conditionally convergent series. Show that given any $\alpha \in \mathbb{R}$, there exists a one-to-one correspondence $\phi : \mathbb{N} \to \mathbb{N}$ such that $\sum_{n\geq 1}(-1)^{n+1}a_{\phi(n)}$ converges to α.

   (Hint: Use the sequences (a_n^+) and (a_n^-).)