MTH101A, Mid-Sem. Exam. 16.09.13 (15:00-17:00 hrs) Max. Marks: 72

1. (a) For $a \geq 0$, let $x_1 = a$ and $x_{n+1} = \frac{1}{5}(x_n^2 + 6)$ for all $n \in \mathbb{N}$. [8]

   i. Find **(all)** the values of a in $[0, \infty)$ for which (x_n) is decreasing/increasing.

   ii. Verify whether the sequence (x_n) converges if $3 < a < \frac{7}{2}$.

   (b) Suppose $\sum_{n=1}^{\infty} |a_n|$ converges. Show that for $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|\sum_{i=m}^{n} a_i| < \epsilon$ for all $m, n \in \mathbb{N}$ satisfying $n > m > N$. [4]

(Tentative) Marking scheme for Question 1:

(a)(i) Note that $x_{n+1} - x_n = \frac{1}{5}(x_n^2 - x_{n-1}^2)$, [1]

$x_2 - x_1 = \frac{1}{5}(a^2 + 6) - a = \frac{1}{5}(a-2)(a-3)$, [1]

and $x_{n+1} \geq x_n$ if $x_2 - x_1 \geq 0$ and $x_{n+1} \leq x_n$ if $x_2 - x_1 \leq 0$.

Now (x_n) is increasing on $[0, 2]$ and $[3, \infty)$ [2]

and is decreasing on $[2, 3]$. [1]

(a)(ii) If $x_n \to \ell$ then $\ell^2 - 5\ell + 6 = 0$, i.e., $\ell = 2$ or 3 [2]

Since (x_n) is strictly increasing, (x_n) cannot converge either to 2 or 3. [1]

(b) The sequence of partial sums (S_n) of $\sum_{n=1}^{\infty} |a_n|$ satisfies the Cauchy criterion. [1]

Therefore for $\epsilon > 0$, $\exists\ N$ s.t. $|S_n - S_{m-1}| < \epsilon \ \ \forall\ m, n \in \mathbb{N}$, $n > m > N$.

That is $\sum_{i=m}^{n} |a_i| < \epsilon\ \forall\ m, n \in \mathbb{N}$, $n > m > N$. [2]

Use the fact that $|\sum_{i=m}^{n} a_i| \ \leq \ \sum_{i=m}^{n} |a_i|$. [1]

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2. (a) Let $f : (-1, 1) \to \mathbb{R}$ be a twice differentiable function such that $f''(0) > 0$. Show that there exists $n \in \mathbb{N}$ such that $f(\frac{1}{n}) \neq 1$. [7]

   (b) Let $0 < x_0 < 1$. Using the fixed point iteration method generate a sequence of approximate solutions of the equation $x^3 - 7x + 2 = 0$ for the starting value x_0. [5]

(Tentative) Marking scheme for Question 2:

(a) Suppose $f(\frac{1}{n}) = 1$ for all $n \in \mathbb{N}$ [2]

Then $f(0) = 1$ by the continuity of f, [1]

and $f'(0) = \lim_{n\to\infty} \frac{f(\frac{1}{n}) - f(0)}{\frac{1}{n}} = 0$. [1]

By Rolle's theorem, $\exists\ c_n \in (0, \frac{1}{n})$ such that $f'(c_n) = 0$ for all n [2]

Now $f''(0) = \lim_{n\to\infty} \frac{f'(c_n) - f'(0)}{c_n} = 0$ which is a contradiction. [1]

(b) Consider $x_{n+1} = \frac{1}{7}(x_n^3 + 2)$ [2]

Now $|x_{n+2} - x_{n+1}| \leq \frac{1}{7}|x_{n+1}^3 - x_n^3|$

$= \frac{1}{7}|x_{n+1}^2 + x_{n+1}x_n + x_n^2||x_{n+1} - x_n| \leq \frac{3}{7}|x_{n+1} - x_n|$. [3]

Therefore (x_n) converges (to a solution of $x^3 - 7x + 2 = 0$).

<u>*Alternate Solution:*</u>

Consider $x_{n+1} = f(x_n)$ where $f(x) = \frac{1}{7}(x^3 + 2)$. [2]

Observe that $f : [0, 1] \to [0, 1]$ i.e., $f([0, 1]) \subseteq [0, 1]$. [1]

and $|f'(x)| \leq \frac{3}{7}$ on $[0, 1]$ [2]

Hence (x_n) converges.

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3. (a) Sketch the graph of the function $f(x) = \frac{2x^2+1}{x^2+1}$ after finding the intervals of decrease/increase, intervals of concavity/convexity, points of local minima/local maxima, points of inflection and asymptotes. [8]

(b) Consider the series $\frac{1}{4} + \frac{1}{5} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{4^3} + \frac{1}{5^3} + \frac{1}{4^4} + \frac{1}{5^4} + \ldots$. Using the ROOT TEST show that the series converges. [4]

Tentative Marking Scheme for Question 3:

(a) $f(x) = 2 - \frac{1}{x^2+1} \Rightarrow y = 2$ is an asymptote. [1]

$f'(x) = \frac{2x}{(x^2+1)^2} \Rightarrow f$ is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$ [1]

and f has a local minimum at $x = 0$. [1]

$f''(x) = \frac{2(1-3x^2)}{(x^2+1)^3} \Rightarrow f$ is concave on $(-\infty, -\frac{1}{\sqrt{3}})$ and $(\frac{1}{\sqrt{3}}, \infty)$ [1]

convex on $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ [1]

and $-\frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$ are the points of inflection. [1]

For the graph see Figure 1. [2]

(b) The terms can be either $5^{-\frac{n}{2}}$ or $4^{-\frac{n+1}{2}}$. [1]

Since $(5^{-\frac{n}{2}})^{\frac{1}{n}} \to \frac{1}{\sqrt{5}}$ and $(4^{-\frac{n+1}{2}})^{\frac{1}{n}} \to \frac{1}{\sqrt{4}}$, [2]

$a_n^{\frac{1}{n}} < L$ eventually for some L satisfying $\frac{1}{2} < L < 1$. [1]

Hence by the root test the series converges.

Alternate Solution:

The series can be written as a sum of two series and for each series the root test can be applied.

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4. (a) Let $f : [a, b] \to \mathbb{R}$ be continuous. Show that f is bounded. [6]

(b) For $x > 0$, show that the Maclaurin series of e^x converges to e^x. [6]

Tentative Marking Scheme for Question 4:

(a) Suppose f is not bounded. [1]

Then for every $n \in \mathbb{N}$, $\exists\, x_n \in [a, b]$ such that $|f(x_n)| > n$ or $|f(x_n)| \to \infty$. [1]

By B-W Theorem, $\exists\, x_{n_k} \to x_0$ for some $x_0 \in [a, b]$. [2]

By the continuity of f, $f(x_{n_k}) \to f(x_0)$. [1]

Hence $(f(x_{n_k}))$ is bounded which is a contradiction. [1]

(b) Let $f(x) = e^x$. Fix $x > 0$. By Taylor's Theorem $\exists\, c_n \in (0, x)$ such that

$f(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + \frac{f^{n+1}(c_n)}{(n+1)!}x^{n+1}$. [2]

Note that $\frac{f^{n+1}(c_n)}{(n+1)!}x^{n+1} = \frac{e^{c_n}}{(n+1)!}x^{n+1} \leq \frac{e^x}{(n+1)!}x^{n+1}$. [2]

Let $a_{n+1} = \frac{e^x}{(n+1)!}x^{n+1}$, then $\frac{a_{n+1}}{a_n} = \frac{x}{n+1} \to 0$. This implies that $a_n \to 0$. [2]

This shows that $\frac{f^{n+1}(c_n)}{(n+1)!}x^{n+1} \to 0$ and hence $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges to e^x.

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5. (a) Let f be differentiable on $[a, b]$. Show that there exist $c_1, c_2, c_3 \in (a, b)$ such that $c_1 \neq c_3$ and $f'(c_2) + f'(c_3) = 2f'(c_1)$. [6]

   (b) Suppose that $a_n > 0$ for all n and $\sum_{n=1}^{\infty} a_n$ converges. Discuss the convergence/ divergence of the series $\sum_{n=1}^{\infty} \sqrt{a_n} \sin(a_n)$. [6]

<u>Tentative Marking Scheme for Question 5:</u>

(a) By MVT, $\exists\ c_1 \in (a, b)$ such that $f(b) - f(a) = f'(c_1)(b - a)$ [1]

By MVT, $\exists\ c_2 \in (a, \frac{a+b}{2})$ such that $f(\frac{a+b}{2}) - f(a) = f'(c_2)(\frac{b-a}{2})$ [1]

and $\exists\ c_3 \in (\frac{a+b}{2}, b)$ such that $f(b) - f(\frac{a+b}{2}) = f'(c_3)(\frac{b-a}{2})$ [1]

This implies that $f(b) - f(a) = (f'(c_2) + f'(c_3))(\frac{b-a}{2})$. [2]

That is $f'(c_2) + f'(c_3) = 2f'(c_1)$ [1]

(b) $\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow\ a_n \to 0$ and hence $\sqrt{a_n} \to 0$. [1]

Since $a_n\sqrt{a_n} \leq a_n$, by comparison test, $\sum_{n=1}^{\infty} a_n\sqrt{a_n}$ converges. [2]

Note that $\lim_{n\to\infty} \frac{\sqrt{a_n}\sin(a_n)}{\sqrt{a_n}a_n} \to 1$. [2]

By LCT $\sum_{n=1}^{\infty} \sqrt{a_n} \sin(a_n)$ converges. [1]

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6. (a) Let f be a differentiable function on $\mathbb{R}$ such that $f(0) = f(1) = 0$, $f'(0) > 0$ and $f'(1) > 0$. [8]

   i. Show that there exists a $\delta > 0$ such that $f(x) < 0$ on $(1 - \delta, 1)$.

   ii. Show that there exist $c_1, c_2 \in (0, 1)$ such that $c_1 \neq c_2$ and $f'(c_1) = f'(c_2) = 0$.

   (b) Let $f : (a, b) \to \mathbb{R}$ be twice differentiable and convex. If $x_0 \in (a, b)$ show that the graph of f is above the tangent line to the graph at $(x_0, f(x_0))$. [4]

<u>Tentative Marking Scheme for Question 6:</u>

(a)(i) If $\forall\ n$, $\exists\ x_n \in (1 - \frac{1}{n}, 1)$ s.t. $f(x_n) \geq 0$, then $f'(1) = \lim_{x_n\to1^-} \frac{f(x_n)}{x_n-1} \leq 0$.

(OR) Since $f'(1) = \lim_{x\to1^-} \frac{f(x)}{x-1} > 0$, $\exists\ \delta > 0$ such that $f(x) < 0$ on $(1 - \delta, 1)$. [3]

(a)(ii) Observe, as done in (i), that $\exists\ \delta_1 > 0$ such that $f(x) > 0$ on $(0, \delta_1)$ [2]

Choose $\delta_2 = \min\{\delta, \delta_1, \frac{1}{4}\}$.

By IVP, $\exists\ c \in (\delta_2, 1 - \delta_2)$ such that $f(c) = 0$. [2]

By Rolle's theorem $\exists c_1 \in (0, c)$ and $c_2 \in (c, 1)$ s.t. $f'(c_1) = 0$ and $f'(c_2) = 0$. [1]

(b) Since f is convex, $f''(x) \geq 0$ on (a, b). [1]

The equation of the tangent line is $y = f(x_0) + f'(x_0)(x - x_0)$. [1]

By Taylor's theorem there exists c between x and x_0 such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{(x-x_0)^2}{2} f''(c) \qquad [2]$$

This implies that $f(x) \geq f(x_0) + f'(x_0)(x - x_0)$.