# End-Semester Examination

(1) (a) Consider the matrix

$$A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

Find an orthogonal matrix $P$ and a diagonal matrix $D$ such that $A = P^{-1}DP$. [8]

(b) Prove that $2 \times 2$ symmetric matrix is diagonalisable. [5]

(c) Let $A$ be an invertible square matrix with integer entries. Show that $A^{-1}$ has integer entries if and only if $\det(A) = \pm 1$. [4]

**Solution:**

(a) The Eigen values of $A$ are 2,2,-2. [2]

The Eigen Space corresponding to the Eigen Value 2 is spanned by the orthonormal basis $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$. [2]

The Eigen Space corresponding to the Eigen Value $-2$ is spanned by the orthonormal basis $\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$. [2]

Let $P = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$. [2]

(b) $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$. The characteristic polynomial of $A$ is

$\lambda^2 - (a+c)\lambda + (ac - b^2) = 0$. [1]

Hence $\lambda = \dfrac{(a+c) \pm \sqrt{(a-c)^2 + 4b^2}}{2}$. [1]

If $a = c$ and $b = 0$, then $A$ is a diagonal matrix. [1]

In all the other cases the eigen values are distinct and therefore $A$ is diagonalisable. [2]

(c) If $\det(A) = \pm 1$, then since $A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A)$, the entries of $A^{-1}$ are integers. [2]

If entries of $A^{-1}$ are integers then $AA^{-1} = I$ implies that $1/\det(A) = \det(A^{-1}) \in \mathbb{Z}$. As $\det(A) \in \mathbb{Z}$, the result follows. [2]

**NOTE: 1a. Eigenvalues [2] marks, each eigenspace [2] marks without orthonormality thrown in, and for P [2] marks. 1b. One mark deducted in case discriminant equal to zero case is missing. No marks if the symmetric**

**matrix is taken to be the special one with same diagonal entries. 1c. Most students fail to realise that they have proved the statement only one way; majority proved that if $detA = \pm 1$, then $A^{-1}$ has integral entries using the Adjoint matrix. So in all these cases only 2 marks have been awarded.**

**Further, in Q1a most students have forgotten to ornormalise the eigenvectors while writing the matrix P. So [2] marks meant for writing P have been deducted in all these cases.**

(2) (a) Let $A$ and $B$ be orthogonal matrices. Show that

(i) $B(A^T + B^T)A = A + B$.

(ii) Show that if $\det(A) + \det(B) = 0$, then $A + B$ is not invertible. [4+4]

(b) Let $A \in M_{4\times3}(\mathbb{R})$ and $\dim(C(A)) = 2$.

(i) Compute $\dim(N(A))$.

(ii) Let $A \in M_{4\times3}(\mathbb{R})$ and $N(A) = \{0\}$. Show that $N(AB) = N(A)$.

(iii) Let $A \in M_{3\times2}(\mathbb{R})$, $B \in M_{2\times4}(\mathbb{R})$, $\text{Rank}(A) = 2$ and $\text{Rank}(AB) = 1$. Compute $\dim(C(B))$. [3+3+3]

**Solution:**

(a) (i) $A$ and $B$ are orthogonal imply that $AA^T = A^TA = BB^T = B^TB = I$. [2]

$B(A^T + B^T)A = BA^TA + BB^TA = B + A$. [2]

(ii) $\det(A) = \pm1$ and $\det(B) = \pm1$.

Since $B(A^T+B^T)A = A+B$, we get $\det(B)\det(A^T+B^T)\det(A) = \det(A+B) = \det(B)\det(A+B)^T\det(A) = \det(B)\det(A+B)\det(A)$.

$\implies \det(A+B)(1 - \det(B)\det(A)) = 0$. [2]

Since $\det(A) + \det(B) = 0$, $\det(A) = -\det(B)$ and $\det(A)\det(B) = -1$.

Hence $2\det(A+B) = 0 \implies A + B$ is not invertible. [2]

(b) (i) $A : \mathbb{R}^3 \to \mathbb{R}^4$. By the Rank Nullity Theorem, $\dim N(A) = 1$. [3]

(ii) $Bx = 0 \implies ABx = 0 \implies N(B) \subset N(AB)$. [1]

$ABx = 0 \implies A(Bx) = 0 \implies Bx \in N(A) \implies Bx = 0$, since $N(A) = \{0\}$. [2]

$N(AB) = N(B)$.

(iii) $AB : \mathbb{R}^4 \to \mathbb{R}^3$.

$\text{Rank}(A) = 2 \implies \dim(C(A)) = 2 \implies \dim(N(A)) = 0$.

$\text{Rank}(AB) = 1 \implies \dim(C(AB)) = 1 \implies \dim(N(AB)) = 3$. [1]

Since $N(A) = \{0\}$, from $(ii)$, $AN(B) = N(AB)$ and therefore $\dim(N(B)) = 3$

Hence $\dim(C(B)) = 1$. [2]

**Regarding question no. 2(b)(i), some students wrote just only "By the Rank Nullity Theorem, dimN(A) = 1" without any other justification. No marks was given in this case**.

(3) (a) Find solution curve for $\frac{dy}{dx} = \frac{x+y+4}{x+y-6}$ which passes through the point $(1, 1)$. [6]

(b) By method of variation of parameters find general solution of $(x^2-1)y'' - 2xy' + 2y = (x^2-1)^2$. [6]

(c) Show that there is no equation $y'' + p(x)y' + q(x)y = 0, x \in [0, 2]$, $p$ and $q$ continuous, for which $x$ and $e^x$ are two solutions. [4]

**Solution:**

(a) Put $w = x + y$, Then $\frac{dy}{dx} + 1 = \frac{dw}{dx}$ [2]

Given ODE: $\frac{dw}{dx} - 1 = \frac{w+4}{w-6}$.

Solving $w - 5\log(w-1) = 2x + c$ or $y - x - 5\log(x + y - 1) = c$ [3]

Passes through $(1, 1)$ implies $c = 0$. Hence solution $y - x - 5\log(x + y - 1) = 0$. [1]

(b) The homogeneous part $(x^2-1)y'' - 2xy' + 2y = 0$ has two independent solutions $y_1 = x$ and $y_2 = x^2 + 1$. (What ever correct methods they apply - follows straightforwardly using power series method) [2]

$W(y_1, y_2) = x^2 - 1$.

For a particular solution of equation $y'' - \frac{2x}{x^2-1}y' + \frac{2}{(x^2-1)^2}y = x^2 - 1$ we try $y_p = v_1y_1 + v_2y_2$.

Then $v_1 = \int \frac{-y_2(x^2-1)dx}{W(y_1,y_2)} = \frac{-x^3}{3} - x$ [2]

$v_2 = \int \frac{y_1(x^2-1)dx}{W(y_1,y_2)} = \frac{x^2}{2}$. [2].

General solution $c_1y_1 + c_2y_2 + y_p$.

(c) Suppose $y_1(x) = x,\ y_2(x) = e^x$ are solutions of the given equation.

Note that $y_1$ and $y_2$ are independent in the given interval. [1]

Thus if $y_1$ and $y_2$ are solutions then $W(y_1, y_2) \neq 0$ for all $x \in [0, 2]$. [1]

However, $W(y_1, y_2) = (x-1)e^x$ is zero at $x = 1$. [1]

Hence $y_1$ and $y_2$ can not be solutions. [1]

**Note: For part $(b)$, after finding the solution for homogeneous part, if anybody did not apply method of variation of parameters, no marks was awarded. For the $(c)$ part, full marks was awarded to those who did it putting $x$ and $e^x$ as solutions in the given equation and then working out a contradiction.**

(4) (a) The equation $y'' + (p - x^2)y = 0$ has power series solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$. Find the recursive relation for coefficients $(a_n)$. [4]

(b) Find two independent solutions of $y'' + xy' + y = 0$ by power series method and show that power series, in both the case, converges. [8]

(c) Show that the equation $x^2y'' - 3xy' + (4x+4)y = 0$ has only one Frobenius series solution. [4]

**Solution:**

(a) $y = \sum_{n=0}^{\infty} a_n x^n$. Hence $y'' = \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n$. [1]

$py = \sum_{n=0}^{\infty} pa_n x^n$ and $x^2y = \sum_{n=0}^{\infty} a_{n-2}x^n$ where $a_{-1} = a_{-2} = 0$. [1]

Putting back in equation and equating coefficients we have

$$(n+1)(n+2)a_{n+2} + pa_n - a_{n-2} = 0.$$

[2]

(b) $y = \sum_{n=0}^{\infty} a_n x^n$. $xy' = \sum_{n=0}^{\infty} na_n x^n$ and $y'' = \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n$. [1]

This gives

$$(n+1)(n+2)a_{n+2} + (n+1)a_n = 0.$$

[2]

Thus $a_{n+2} = -\frac{a_n}{n+2}$. [1]

Two independent solutions are

$$y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2 \cdot 4 \cdots 2n}, \; y_2 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{1 \cdot 3 \cdots 2(n+1)}.$$

[2]

For $y_1$: $\frac{a_{2n+2}}{a_{2n}} = \frac{1}{2n+2} \longrightarrow 0$. [1]

For $y_2$: $\frac{a_{2n+3}}{a_{2n+1}} = \frac{1}{2n+3} \longrightarrow 0$. [1]

(c) Indicial Equation: $m(m-1) - 3m + 4 = 0$. [2]

Roots are $m_1 = m_2 = 2$. Hence there is only one Frobenius series solution. [2]

**Note: In part (b), 1 mark deducted if you have not written the final answers in summation from. Also marks were deducted if ratio test for convergence was applied wrongly. 2 marks deducted if in the solution powers of $x$ are missing. 1 mark deducted if the recursive relation is not stated clearly**.

(5) (a) Show that $J_{\frac{1}{2}} = \sqrt{\frac{2}{\pi x}} \sin x$ and $J_{-\frac{1}{2}} = \sqrt{\frac{2}{\pi x}} \cos x$. [6]

(b) Show that $J_{\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi x}}(\frac{3 \sin x}{x^2} - \frac{3 \cos x}{x} - \sin x)$. [4]

(c) Show that $\int_{-1}^{1} x^m P_n(x) dx = 0$ in the following
(i) $m < n$ (ii) $m > n$ and $m - n$ is odd integer. [6]

**Solution:**

(a) $J_{\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+\frac{1}{2}}}{n!(n+\frac{1}{2})!} = \sqrt{\frac{2}{x}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n+1} n!(n+\frac{1}{2})!}$ [1]

$(n + \frac{1}{2})! = \sqrt{\pi} \frac{(2n+1)!}{2^{2n+1} n!}$ [2].

Hence the result follows.

$J_{\frac{-1}{2}} = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n-\frac{1}{2}}}{n!(n-\frac{1}{2})!} = \sqrt{\frac{2}{x}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} n!(n-\frac{1}{2})!}$ [1]

$(n - \frac{1}{2})! = \sqrt{\pi} \frac{(2n)!}{2^{2n} n!}$ [2].

Hence the result follows.

(b) Note that $\frac{2p}{x} J_p(x) = J_{p-1}(x) + J_{p+1}(x)$ [1]

Hence $J_{\frac{3}{2}} = \frac{1}{x} J_{\frac{1}{2}} - J_{-\frac{1}{2}} = \sqrt{\frac{2}{\pi x}}(\frac{\sin x}{x} - \cos x)$ [1]

Thus $J_{\frac{5}{2}} = \frac{3}{x} J_{\frac{3}{2}} - J_{\frac{1}{2}} = \sqrt{\frac{2}{\pi x}}(\frac{3 \sin x}{x^2} - \frac{3 \cos x}{x} - \sin x)$. [2]

(c) For any function $f(x)$ we have $\int_{-1}^{1} f(x) P_n(x) dx = \frac{1}{2^n n!} \int_{-1}^{1} f^{(n)}(x)(x^2 - 1)^n dx$. [2]

(i) $m < n,\ f(x) = x^m$. Then $f^{(n)} = 0$. Thus the results. [2]

(ii) $m - n$ is odd implies the function under integral is odd. Hence the result. [2]

**Note: For all the parts anybody who applied correct but different formulas, methods, full marks was awarded. If you did not get marks, be sure, that you have applied wrong methods or your calculations are wrong. For part $(a)$ if one has found up to 3 terms of both the series but did not write general term, full marks was awarded. Less than 3 terms 1 mark was deducted. For part $(b)$ if anyone had just written the expression for $J_{\frac{5}{2}}$, no marks was awarded.**

(6) (a) Show that $L(\sin x)(p) = \frac{1}{1-e^{-2\pi p}} \int_0^{2\pi} e^{-px} \sin x dx$ and hence evaluate the integral. (Hint: $\sin x$ is $2\pi$-periodic). [5]

(b) Show that the solution of $y'' + 4y = f(x),\ y(0) = y'(0) = 0$ is $y(x) = \frac{1}{2} \int_0^x f(t) \sin 2(x-t) dt$. [5]

(c) Show that $L(x \cos x)(p) = \frac{p^2-1}{(p^2+1)^2}$. Hence find a function $f$ such that $L(f)(p) = \frac{1}{(p^2+1)^2}$. [6]

**Solution:**

(a) $L(\sin x)(p) = \int_0^\infty e^{-px} \sin x dx$.
Put $z = x - 2\pi$. Then $L(\sin x)(p) = e^{-2\pi p} \int_{-2\pi}^\infty e^{-pz} \sin z dz$ [1]
Hence $L(\sin x)(p) = e^{-2\pi p}[\int_0^\infty e^{-pz} \sin z dz + \int_{-2\pi}^0 e^{-pz} \sin z dz] = e^{-2\pi p}[L(\sin x)(p) + \int_{-2\pi}^0 e^{-pz} \sin z dz]$ [2]
Put $z + 2\pi = y$. Then $L(\sin x)(p) = e^{-2\pi p} L(\sin x)(p) + \int_0^{2\pi} e^{-py} \sin y dy$. [1]
Thus $L(\sin x)(p) = \frac{1}{1-e^{-2\pi p}} \int_0^{2\pi} e^{-px} \sin x dx$. [1]

(b) Taking Laplace transform: $L(y'') + 4L(y) = L(f)$. Hence $p^2 L(y) - y'(0) - py(0) + 4L(y) = L(f)$ [1]
That is $(p^2+4)L(y) = L(f)$ or $L(y) = \frac{L(f)}{p^2+4}$ [1]
Hence $L(y) = \frac{1}{2} L(f) L(\sin 2x)$. [1]
Hence $y = \frac{1}{2} \int_0^x f(t) \sin 2(x-t) dt$. [2]

(c) Let $F(p) = L(\cos x) = \frac{p}{p^2+1}$. Then $L(-x \cos x) = F'(p) = -\frac{p^2-1}{(p^2+1)^2}$ implying $L(x \cos x) = \frac{p^2-1}{(p^2+1)^2}$. [2]
This gives $L(x \cos x) = \frac{1}{p^2+1} - \frac{2}{(p^2+1)^2} = L(\sin x - 2L(f))$. [2]
Thus $L(f) = L(\frac{\sin x - x \cos x}{2})$. [2]

**Note: For part $(a)$ if anybody separately evaluated the L. H. S. and R. H. S. and showed that they are equal full marks was awarded. For part $(b)$ if anybody correctly solved the problem without using Laplace transform, full marks was awarded. If someone has verified the RHS is a solution and also used uniqueness full marks was awarded. Otherwise marks were deducted. For part $(c)$ correct alternative method was awarded full marks - if you did not get marks by trying some alternative methods, your calculation must be wrong**.