Assignment - 5

Q.1.] Correct answers are (a.) and (c.)

Q.2] Given : $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Laplace equation before applying rotation:

$$\Delta u = 0$$

$$\Rightarrow \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = 0 \longrightarrow (1)$$

Now, we apply rotation. So,

$$y_1 = x_1\cos\theta + x_2\sin\theta \longrightarrow (2)$$

$$y_2 = -x_1\sin\theta + x_2\cos\theta \longrightarrow (3)$$

Now, $\frac{\partial y_1}{\partial x_1} = \cos\theta$, $\frac{\partial y_1}{\partial x_2} = \sin\theta$

$$\frac{\partial y_2}{\partial x_1} = -\sin\theta \quad , \quad \frac{\partial y_2}{\partial x_2} = \cos\theta$$

Consider $\frac{\partial u}{\partial x_1} = \frac{\partial v}{\partial x_1}$

$$= \frac{\partial v}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial v}{\partial y_2}\frac{\partial y_2}{\partial x_1}$$

$$\frac{\partial u}{\partial x_1} = \cos\theta \frac{\partial v}{\partial y_1} + (-\sin\theta)\frac{\partial v}{\partial y_2}$$

So, $\frac{\partial^2 u}{\partial x_1^2} = \cos\theta \frac{\partial\left(\frac{\partial v}{\partial y_1}\right)}{\partial x_1} + (-\sin\theta)\frac{\partial\left(\frac{\partial v}{\partial y_2}\right)}{\partial x_1}$

$$\frac{\partial^2 u}{\partial x_1^2} = \cos^2\theta \frac{\partial^2 v}{\partial y_1^2} + \sin^2\theta \frac{\partial^2 v}{\partial y_2^2} \rightarrow (4.)$$

Now,

$$\frac{\partial u}{\partial x_2} = \frac{\partial v}{\partial x_2}$$

$$= \frac{\partial v}{\partial y_1}\frac{\partial y_1}{\partial x_2} + \frac{\partial v}{\partial y_2}\frac{\partial y_2}{\partial x_2}$$

$$= \sin\theta \frac{\partial v}{\partial y_1} + \cos\theta \frac{\partial v}{\partial y_2}$$

$$\frac{\partial^2 u}{\partial x_2^2} = \sin\theta \frac{\partial\left(\frac{\partial v}{\partial y_1}\right)}{\partial x_2} + \cos\theta \frac{\partial\left(\frac{\partial v}{\partial y_2}\right)}{\partial x_2}$$

$$\frac{\partial^2 u}{\partial x_2^2} = \sin^2\theta \frac{\partial^2 v}{\partial y_1^2} + \cos^2\theta \frac{\partial^2 v}{\partial y_2^2} \rightarrow (5.)$$

Adding (4.) and (5.) :

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = \frac{\partial^2 v}{\partial y_1^2} + \frac{\partial^2 v}{\partial y_2^2}$$

Hence $\boxed{\Delta v = 0}$ whenever $\Delta u = 0$

Hence, Laplace equation is rotational invariant.

Q.3] Given problem:
$$\begin{cases} \Delta u = 0 & \text{on } (0,a) \times (0,b) \\ u_x(a,y) = f(y), & u_x(0,y) = 0 \\ u_y(x,0) = 0, & u_y(x,b) = 0 \end{cases}$$

Now, by separation of variables, assume

$$u(x,y) = F(x)\,G(y)$$

$$\frac{\partial^2 u}{\partial x^2} = F''(x)\,G(y) \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = F(x)\,G''(y)$$

Hence, $\Delta u = 0$

$$\Rightarrow \frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)}$$

Clearly, since one ratio depends only on x and other only on y, the ratios must be constant.

So, let $\dfrac{F''(x)}{F(x)} = -\dfrac{G''(y)}{G(y)} = \lambda \longrightarrow (1.)$, $\lambda \in \mathbb{R}$

Now, $u_x(a,y) = f(y)$, $u_x(0,y) = 0$

$\Rightarrow F'(a)\,G(y) = f(y)$ and $F'(0)\,G(y) = 0$

Clearly $G(y) \neq 0$ if we neglect trivial solutions.

Hence, $F'(0) = 0$ and $G(y) = \dfrac{f(y)}{F'(a)} \longrightarrow (2.)$

Also, $u_y(x,0) = 0$ and $u_y(x,b) = 0$

$\Rightarrow F(x)\,G'(0) = 0$ and $F(x)\,G'(b) = 0$

$\Rightarrow G'(0) = G'(b) = 0 \longrightarrow (3.)$

Now, from (2'):

$$G(y) = \frac{f(y)}{F'(a)}$$

$$G''(y) = \frac{f''(y)}{F'(a)}$$

So, the problem is: $G''(y) + \lambda G(y) = 0$

with $G'(0) = G'(b) = 0$

Consider 3 cases:

Case 1: $\lambda < 0$

So, $G(y) = A e^{\sqrt{-\lambda}\, y} + B e^{-\sqrt{-\lambda}\, y}$

$G'(y) = A\sqrt{-\lambda}\, e^{\sqrt{-\lambda}\, y} - B\sqrt{-\lambda}\, e^{-\sqrt{-\lambda}\, y}$

$G'(0) = 0 \Rightarrow A = B$

$G'(b) = 0 \Rightarrow A\left(e^{2\sqrt{-\lambda}\, b} - 1\right) = 0$

$\Rightarrow A = B = 0$

Hence, $\lambda < 0$ eigenvalue is not possible.

Case 2: $\lambda = 0$

So, $G(y) = Ay + B$

$G'(y) = A$

$G'(0) = 0 \Rightarrow A = 0$

So, $G'(y) = 0$

So, $G(y) = B$

So, $\lambda = 0$ is a possible eigenvalue.

Case 3: $\lambda > 0$

So, $G(y) = A\cos(\sqrt{\lambda}\,y) + B\sin(\sqrt{\lambda}\,y)$

$G'(y) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}\,y) + B\sqrt{\lambda}\cos(\sqrt{\lambda}\,y)$

$G'(0) = 0 \Rightarrow B = 0$

$G'(b) = 0 \Rightarrow -A\sqrt{\lambda}\sin(\sqrt{\lambda}\,b) = 0$

$\Rightarrow \sqrt{\lambda}\,b = n\pi$, $n \in \mathbb{N}$ as $\sqrt{\lambda} > 0$

$\Rightarrow \sqrt{\lambda} = \dfrac{n\pi}{b}$

$\Rightarrow \lambda = \dfrac{n^2\pi^2}{b^2} \longrightarrow (4.)$

So, $\boxed{G_n(y) = A\cos\left(\dfrac{n\pi y}{b}\right)} \longrightarrow (5.)$

So, from (4.) :

$\dfrac{F''(x)}{F(x)} = \dfrac{n^2\pi^2}{b^2}$ with $F'(0) = 0$

$\Rightarrow F_n(x) = C e^{\frac{n\pi}{b}x} + D e^{-\frac{n\pi x}{b}}$

$\Rightarrow F_n'(x) = \dfrac{Cn\pi}{b} e^{\frac{n\pi x}{b}} - \dfrac{Dn\pi}{b} e^{-\frac{n\pi x}{b}}$

$F'(0) = 0 \Rightarrow \boxed{C = D}$

So, $F_n(x) = C\left(e^{\frac{n\pi x}{b}} + e^{-\frac{n\pi x}{b}}\right)$

$F_n(x) = 2C\cosh\left(\dfrac{n\pi x}{b}\right)$

Take $2C = E$. Hence

$\boxed{F_n(x) = E\cosh\left(\dfrac{n\pi x}{b}\right)}$

So, $u_n(x,y) = F_n(x)\, G_n(y)$

$$u_n(x,y) = AE \cosh\left(\frac{n\pi x}{b}\right) \cos\left(\frac{n\pi y}{b}\right), \quad n \in \mathbb{N}$$

Now, by principle of superposition:

$$u(x,y) = \sum_{n=1}^{\infty} \left(E \cosh\left(\frac{n\pi x}{b}\right) A \cos\left(\frac{n\pi y}{b}\right)\right)$$

$$u_x(a,y) = f(y).$$

Now,

$$u_x(x,y) = \sum_{n=1}^{\infty} \left(\left(\frac{E n\pi}{b} \sinh\left(\frac{n\pi x}{b}\right)\right) A \cos\left(\frac{n\pi y}{b}\right)\right)$$

$$u_x(a,y) = \sum_{n=1}^{\infty} \left(\left(\frac{E n\pi}{b} \sinh\left(\frac{n\pi a}{b}\right)\right) A \cos\left(\frac{n\pi y}{b}\right)\right)$$

$$= f(y)$$

This is like a fourier cosine series for $f(y)$ without the constant term.

Since A and E are arbitary, we define:

$$\boxed{\frac{AE n\pi}{b} \sinh\left(\frac{n\pi a}{b}\right) = \frac{2}{b} \int_0^b f(y) \cos\left(\frac{n\pi y}{b}\right) dy}$$

Note that, since constant term is 0, we also require:

$$\boxed{\int_0^b f(y)\, dy = 0}$$

Hence our final solution is:

$$u(x,y) = \sum_{n=1}^{\infty} E \cosh\left(\frac{n\pi x}{b}\right) A \cos\left(\frac{n\pi y}{b}\right)$$

with

$$\boxed{\frac{AEn\pi}{b} \sinh\left(\frac{n\pi a}{b}\right) = \frac{2}{b} \int_0^b f(y) \cos\left(\frac{n\pi y}{b}\right) dy}$$

and $\boxed{\int_0^b f(y)\, dy = 0}$

If $\int_0^b f(y)\, dy \neq 0$, then the problem will not have a solution.

Another solution can arise if $\lambda = 0$.

Then $G(y) = c$, where c = constant.

Now, $\because \lambda = 0$

$\Rightarrow F''(x) = 0$, $F'(0) = 0$

$\Rightarrow F(x) = A_1 x + B_1$, $F'(0) = 0$

$\Rightarrow F'(x) = A_1$, $F'(0) = 0$

$\Rightarrow A = 0$

Hence, $F(x) = B_1$, where B_1 = constant.

So, $u(x,y) = k$, where k = constant.

So, $u_x = u_y = 0$

Hence if $f(y) \equiv 0 \quad \forall \; y \in (0, b)$, then

$u(x,y) = k$ is a solution, where k = constant

Q.4) Given problem : $\begin{cases} \Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$

Suppose the problem has 2 solutions, say, u_1 and u_2.

Then $\begin{cases} \Delta u_1 = f & \text{in } \Omega \\ u_1 = g & \text{on } \partial\Omega \end{cases}$ and

$\begin{cases} \Delta u_2 = f & \text{in } \Omega \\ u_2 = g & \text{on } \partial\Omega \end{cases}$

Now, consider a function $u_3 = u_1 - u_2$.

$\because$ Laplace operator is a linear operator

$$\Delta u_3 = \Delta u_1 - \Delta u_2$$
$$= f - f$$
$$= 0 \quad \text{in } \Omega$$

Also, since $u_1 = u_2 = g$ on $\partial\Omega$

$\Rightarrow$ $u_3 = 0$ on $\partial\Omega$

So, this means that u_3 satisfies the problem : $\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$

By, maximum principle, we see that this is only possible if $u_3 \equiv 0$ on $\Omega \cup \partial\Omega$

$\Rightarrow$ $u_1 = u_2$ on $\Omega \cup \partial\Omega$

Hence there can be atmost one solution to the above problem.

The above proof proves that if there exists a solution, it has to be unique. So either there is no solution or there is only 1 solution.

Hence, the problem has atmost 1 solution.