PP 28 : Directional derivative, gradient and tangent plane

1. Let $f(x,y) = |x|+|y|$ for $(x,y) \in \mathbb{R}^2$. Show that f is continuous at $(0,0)$ and no directional derivative of f at $(0,0)$ exists.

2. Let $f(x,y) = \sqrt{|xy|}$ for all $(x,y) \in \mathbb{R}^2$ and $(u,v) \in \mathbb{R}$ be such that $\|(u,v)\| = 1$. Show that the directional derivative of f at $(0,0)$ in the direction (u,v) exists if only if $(u,v) = (1,0)$ or $(u,v) = (0,1)$.

3. Let $f(x,y) = \frac{x^2y}{x^2+y^2}$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Show that the directional derivative of f at $(0,0)$ in all directions exist but f is not differentiable at $(0,0)$.

4. Consider the function $f(x,y) = \frac{3x^2y-y^3}{x^2+y^2}$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Find the directional derivative of f at $(0,0)$ in the direction $\frac{1}{\sqrt{2}}(1,1)$. Discuss the differentiabilty of f at $(0,0)$.

5. (a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $(u,v) \in \mathbb{R}^2$ be such that $\|(u,v)\| = 1$. For $(x_0,y_0) \in \mathbb{R}^2$, show that $D_{(x_0,y_0)}f(u,v)$ is the derivative of $f(x_0+tu, y_0+tv)$ with respect to t at $t=0$.

   (b) If $f(x,y) = xy$, using (a), find $D_{(1,1)}f(\frac{\sqrt{3}}{2}, \frac{1}{2})$.

6. Let $f(x,y) = x^2e^y + \cos(xy)$. Find the directional derivative of f at $(1,2)$ in the direction $(\frac{3}{5}, \frac{4}{5})$.

7. Let $f(x,y) = 2x^2 + xy + y^2$ describe the temperature at (x,y). Suppose a bug is at $(1,1)$ and it decides to cool off. What is the best direction for it to move?

8. For $X \in \mathbb{R}^3$, define $f(X) = \|X\|$. Let $X_0 = (x_0,y_0,z_0) \in \mathbb{R}^3$ and $\|X_0\| = 1$,

   (a) Show that $\nabla f(X_0) = X_0$.

   (b) Find a unit normal to the sphere $f(x,y,z) = 1$ at X_0.

   (c) Find the equation of the tangent plane of the sphere $f(x,y,z) = 1$ at X_0.

9. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be differentiable and $R(t) = (x(t), y(t), z(t))$, $t \in \mathbb{R}$, be a differentiable curve. Suppose that $f(R(t))$ attains its minimum at some t_0. Show that $\nabla f(R(t_0))$ is perpendicular to $R'(t_0)$.

10. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable and $c \in \mathbb{R}$. Suppose that C is a curve (graph or parametric curve) described by $f(x,y) = c$. Assume that C has tangent at every point on the curve. For $(x_0,y_0) \in C$, let $\nabla f(x_0,y_0) \neq (0,0)$. Show that

    (a) $\nabla f(x_0,y_0)$ is normal to C at (x_0,y_0).

    (b) The equation of the tangent line to the curve at (x_0,y_0) is $f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) = 0$.

    (c) If T is a tangent vector for C at (x_0,y_0) then $D_{(x_0,y_0)}f(T) = 0$

11. Let $f(x,y) = 6 - x^2 - 4y^2$. Find a vector which is perpendicular to

    (a) the curve $f(x,y) = 1$, i.e., $x^2 + 4y^2 = 5$, at $(1,1)$.

    (b) the surface $z = f(x,y)$ at the point $(1,1,1)$.

12. Consider the cone $z^2 = x^2 + y^2$.

    (a) Find the equation of the tangent plane to the cone at $(1,1,\sqrt{2})$.

(b) Find an equation for the normal line to the cone at this point.

13. Consider the surface $z = f(x, y) = x^2 - 2xy + 2y$. Find a point on the surface at which the surface has a horizontal tangent plane.

Practice Problems 28: Hints/Solutions

1. Let $(u, v) \in \mathbb{R}^2$ be arbitrary such that $\|(u, v)\| = 1$. Then $\lim_{t\to 0} \frac{f(tu,tv)}{t} = \lim_{t\to 0} \frac{|t|(|u|+|v|)}{t}$ does not exist. Therefore no directional derivative of f exists at $(0, 0)$.

2. The limit $\lim_{t\to 0} \frac{f(tu,tv)}{t} = \lim_{t\to 0} \frac{|t|\sqrt{|uv|}}{t}$ exists if and only if either $u = 0$ or $v = 0$.

3. Let $(u, v) \in \mathbb{R}^2$ be such that $\|(u, v)\| = 1$. Then $D_{(u,v)}f(0, 0) = \lim_{t\to 0} \frac{f(tu,tv)}{t} = u^2v$ but $D_{(0,0)}f(u, v) \neq \nabla f(0, 0) \cdot (u, v)$ if u and v are non-zeros. Therefore f is not differentiable.

4. $D_{(0,0)}f(\frac{1}{\sqrt{2}}(1, 1)) = \lim_{t\to 0} \frac{f(\frac{t}{\sqrt{2}}(1,1))}{t} = \frac{1}{\sqrt{2}}$. If f is differentiable at $(0, 0)$, then $D_{(0,0)}f(\frac{1}{\sqrt{2}}(1, 1)) = (f_x, f_y)|_{(0,0)} \cdot \frac{1}{\sqrt{2}}(1, 1)$. But $(f_x, f_y)|_{(0,0)} \cdot \frac{1}{\sqrt{2}}(1, 1) = -\frac{1}{\sqrt{2}}$. Therefore f is not differentiable.

5. (a) This follows from the definition of $D_{(x_0,y_0)}f(u, v)$.
   (b) By (a), $D_{(1,1)}f(\frac{\sqrt{3}}{2}, \frac{1}{2}) = \frac{d}{dt}\left[f(1 + \frac{\sqrt{3}}{2}t, 1 + \frac{1}{2}t)\right]|_{t=0} = \frac{1}{2}(1 + \sqrt{3})$.

6. Since f_x and f_y are continuous, f is differentiable. Therefore $D_{(1,2)}f(\frac{3}{5}, \frac{4}{5}) = f_x(1, 2) \cdot \frac{3}{5} + f_y(1, 2) \cdot \frac{4}{5}$.

7. The direction of the fastest decrease in the temperature is $-\nabla f(1, 1) = -(5, 3)$.

8. (a) For $X = (x, y, z)$, $f(X) = \sqrt{x^2 + y^2 + z^2}$ and hence $\nabla f(X) = (f_x, f_y, f_z)|_X = \left(\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}\right)$. Therefore $\nabla f(X_0) = X_0$.
   (b) The unit normal to the level surface $f(x, y, z) = 1$ at X_0 is $\nabla f(X_0) = X_0$.
   (c) The equation of the tangent plane at $X_0 = (x_0, y_0, z_0)$ is $xx_0 + yy_0 + zz_0 = 1$.

9. Since $\frac{df}{dt}|_{R(t_0)} = 0$, the problem follows from the chain rule.

10. (a) Suppose that C is described by $R(t) = (x(t), y(t))$. Since $f(R(t)) = c$, by the chain rule $\nabla f(R(t)) \cdot R'(t) = 0$ which proves (a).
    (b) This follows from (a).
    (c) $D_{(x_0,y_0)}f(T) = \nabla f(x_0, y_0) \cdot T$ which is 0 by (a).

11. (a) The gradient $\nabla f(1, 1) = (-2, -8)$ is a normal to the curve at $(1, 1)$.
    (b) If $g(x, y, z) = f(x, y) - z = 6 - x^2 - 4y^2 - z$ then the given surface is the level surface $g(x, y, z) = 0$. The gradient $\nabla g(1, 1, 1) = (-2 - 8, -1)$ is a required normal.

12. Since the cone is the level surface $g(x, y, z) = x^2+y^2-z^2 = 0$, $\nabla g(1, 1, \sqrt{2}) = (2, 2, -2\sqrt{2})$ is a normal to the tangent plane. Therefore the equation of the tangent plane is $2(x-1)+2(y-1)-2\sqrt{2}(z-\sqrt{2}) = 0$. An equation of the normal line is $(x, y, z) = (1, 1, \sqrt{2})+t(2, 2, -2\sqrt{2})$.

13. A normal at a point (x, y, z) on the level surface $g(x, y, z) = z - f(x, y) = 0$ is $\nabla g(x, y, z) = (-2x + 2y, 2x - 2, 1)$. Since the horizontal tangent plane to the surface at a point has the normal $(0, 0, 1)$, the point required satisfy the equations $-2x + 2y = 0$ and $2x - 2 = 0$; i.e., $x = 1$ and $y = 1$. The required point on the surface is $(1, 1, 1)$.