MA 201: Partial Differential Equations
Lecture - 2

What type of equations we will be studying?

- Linear, quasi-linear, and nonlinear first-order PDEs involving two independent variables.
- Linear second-order PDEs in two independent variables.

▶What is the objective of the course?

- Finding Solution of a PDE along with Initial and/or Boundary Conditions

Basic facts about ODE and PDE:

- Consider $u_x = 0$. Integrating we have $u(x, y) = c(y)$, i.e., any arbitrary function of y solves this PDE. It has an infinite dimensional space of solution.
- Solution $u(x, y) = c(y)$ gives all possible solutions of the PDE. Such solution is called general solution/integral.
- In PDE general solution involves arbitrary functions, whereas in ODE general solution involves arbitrary constants.
- What is the general solution for $u_{xy} = 0, \;\; u_{xx} = 0, \;\; u_{yy} = 0$?

Example

Let us warm up with a simple example

$$u_x = u + c, \quad c \text{ is function of } \ x, \ y. \tag{1}$$

Observe

- Equation (1) contains no derivative with respect to the y variable, we can regard this variable as a parameter.
- Thus, for fixed y, we are actually dealing with an ODE, the solution is immediate:

$$u(x,y) = e^x \Big[\int_0^x e^{-\xi} c(\xi, y) d\xi + T(y) \Big]. \tag{2}$$

- Suppose, we supplement (1) with the initial condition $u(0,y) = y$.
- Then the unique solution is given by

$$u(x,y) = e^x \Big[\int_0^x e^{-\xi} c(\xi, y) d\xi + y \Big]. \tag{3}$$

Example

- Consider following IVP

$$u_x = u, \ \ \& \ \ u(x,0) = 2x. \tag{4}$$

- The solution of (4) now becomes $u(x,y) = e^x T(y)$ and with the condition $u(x,0) = 2x$, we must have $T(0) = 2xe^{-x}$, which is of course impossible.
- We have seen so far an example in which a problem had a unique solution, and an example where there was no solution at all. It turns out that an equation might have infinitely many solutions.
- Consider following IVP

$$u_x = u, \ \ \& \ \ u(x,0) = 2e^x. \tag{5}$$

- Now $T(y)$ should satisfy $T(0) = 2$. Thus every function $T(y)$ satisfying $T(0) = 2$ will provide a solution for the equation together with the initial condition. Hence, the IVP has infinitely many solutions.

- **Well-posed Problem**(In the sense of Hadamard)

A problem (PDE + side condition) is said to be well-posed if it satisfies the following criteria:

1. The solution must exist.
2. The solution should be unique.
3. The solution should depend continuously on the initial and/or boundary data.

If one or more of the conditions above does not hold, we say that the problem is ill-posed.

Linear First-Order PDEs

The most general first-order linear PDE has the form

$$a(x,y)u_x + b(x,y)u_y = c(x,y)u + d(x,y), \tag{6}$$

where a, b, c, and d are given functions of x and y. These functions are assumed to be continuously differentiable.

Definition (Solution of a PDE)

A function $u(x,y)$ is a solution to (6) if u and its partial derivatives appearing in (6) satisfy (6) identically for (x,y) in some region $\Omega \subset \mathbb{R}^2$.

- Thus, for a solution to (6), we are looking for a function u of x and y which satisfies equation (6). **In such case solution is given explicitly in terms of independent variables x and y.**
- In more general terms, we are looking for an expression $F(x,y,u) = 0$ involving x, y and u which leads to equation (6). **In such case, we get an implicit form of the solution.**
- Observe that $u = u(x,y)$ means u is function of x and y or $F(x,y,u) = 0$ gives a surface in $\mathbb{R}^3$. **This is known as integral surface or solution surface.**

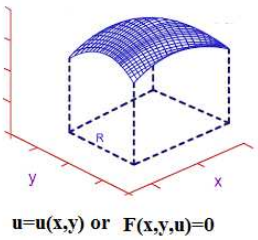

Figure : Image of an integral surface

Remarks: Thus, any point (x, y, u) on the integral surface will satisfy the equation

$$a(x,y)u_x + b(x,y)u_y = c(x,y)u + d(x,y).$$

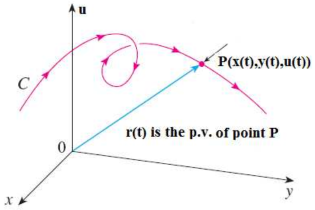

Figure : Parametric curve

- The curve C of all points (x, y, u) in space with

$$x = x(t), \quad y = y(t), \quad u = u(t), \tag{7}$$

  and t varying throughout the interval I, is called a space curve.
- The equations in (7) are called parametric equations of C and t is called a parameter.
- We can think of C as being traced out by a moving particle whose position at time t is $(x, y, u) = (x(t), y(t), u(t))$.

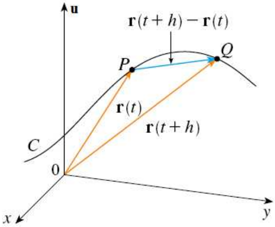

Figure : Tangent Vector

- $\mathbf{r}'(t)$ is called the tangent vector to the curve at $P(x(t), y(t), u(t))$ and is given by

$$\mathbf{r}'(t) = \langle x'(t), y'(t), u'(t) \rangle. \tag{8}$$

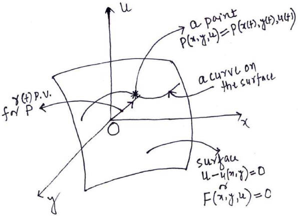

Figure : Integral surface corresponding to $au_x + bu_y = cu + d$.

▶Suppose $u - u(x, y) = 0$ or $F(x, y, u) = 0$ is the integral surface for

$$au_x + bu_y = cu + d.$$

▶Let C be a curve on the integral surface.
▶Let $P(x, y, u)$ be a point on the curve with parametric form

$$x = x(t),\ y = y(t),\ u = u(t).$$

Observe that

- $$u(t) = u = u(x, y) = u(x(t), y(t))$$

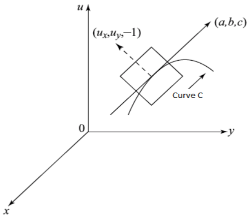

Figure : Tangent and normal vectors to curve C at $P(x, y, u)$.

Observe that

- $\langle u_x, u_y, -1\rangle$ is normal to the surface $u - u(x, y) = F(x, y, u) = 0$.
- $\langle a, b, cu + d\rangle \cdot \langle u_x, u_y, -1\rangle = 0$ along C.
- vector $\langle a(t), b(t), c(t)u(t) + d(t)\rangle$ is a tangent vector to C at $P(x(t), y(t), u(t))$.
- $\mathbf{r}'(t) = \langle x'(t), y'(t), u'(t)\rangle$ is also a tangent vector to C at $P(x(t), y(t), u(t))$.

Characteristic Curves

▶Hence, we have following system of ODEs

$$\frac{dx}{dt} = a(x(t), y(t)) = a(t), \tag{9}$$

$$\frac{dy}{dt} = b(x(t), y(t)) = b(t), \tag{10}$$

$$\begin{aligned}\frac{du}{dt} &= c(x(t), y(t))u(x(t), y(t)) + d(x(t), y(t)) \\ &= c(t)u(t) + d(t).\end{aligned} \tag{11}$$

Remarks.

- The ODEs (9)-(11) are known as the characteristic equations for the PDE

$$au_x + bu_y = cu + d. \tag{12}$$

  The solution curves of the characteristic equation are the characteristic curves for (12).
- The approach described above is called the method of characteristics.

Let $\mu(t) = \exp\left[-\int_0^t c(\tau)d\tau\right]$ be an integrating factor for (11). Then, the solution is given by

$$u(t) = \frac{1}{\mu(t)}\left[\int_0^t \mu(\tau)d(\tau)d\tau + u(0)\right]. \tag{13}$$

Remarks.

- The values $u(t)$ of the solution u along the entire characteristic curve are completely determined, once the value $u(0) = u(x(0), y(0))$ is prescribed.
- For a given initial point $P_n,\ n = 1, 2, 3, \ldots,$

  $$P_n(x(0) = x_{0n}, y(0) = y_{0n}, u(0) = u_{0n}),\ u(0) = u(x(0), y(0)),$$

  on the integral surface, we determine n-th characteristic curve C_n.
- Suppose Γ is a curve on the integral surface passing through all those initial points. We call Γ an initial curve on the integral surface.

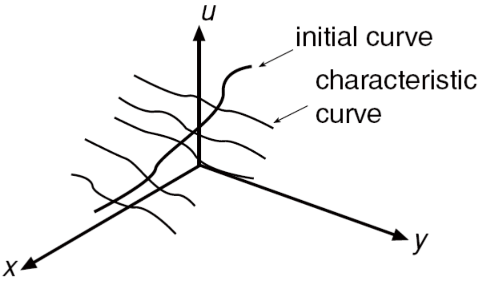

Figure : Characteristic curves and construction of the integral surface

- Considering each characteristic curve passing through the points on initial curve, we can construct the integral surface.
- Assume that initial curve intersects the characteristic curve at $t = 0$. That is $(x(0), y(0), u(0))$ is the point of intersection.
- Values of $x(0),\ y(0),\ u(0)$ will be changing according to the location of the initial point. For instance, $x(0) = x_{0n},\ y(0) = y_{0n},\ u(0) = u_{0n}$ at P_n.

Let the initial curve Γ be given parametrically as:

$$x(0) = x_0(s), \quad y(0) = y_0(s), \quad u(0) = u_0(s) \quad \text{for } s \in I. \tag{14}$$

- Every value of s fixes a point on Γ through which a unique characteristic curve passes.
- Using our notation, we have

$$u_0(s) = u(0) = u(x(0), y(0)) = u(x_0(s), y_0(s)).$$

- Thus, any point (x, y, u) on a characteristic curve passing through the initial point $(x_0(s), y_0(s), u_0(s))$ will depend on two parameters t and s. So, we shall explicitly write the points on characteristics curve as

$$x = x(t, s), \quad y = y(t, s), \quad u = u(t, s) = u(x(t, s), y(t, s))$$

with $t = 0$ corresponding to the initial curve Γ. That is, we have

$$x(0) = x(0, s) = x_0(s), \; y(0) = y(0, s) = y_0(s),$$
$$u(0) = u(0, s) = u_0(s) = u(x(0, s), y(0, s)).$$

Cauchy's problem or IVP for first-order linear PDEs: Find integral surface of the PDE

$$au_x + bu_y = cu + d$$

containing a initial curve

$$\Gamma = \{(x_0(s), y_0(s), u_0(s)) : \ s \in I\}.$$

- In other words, for each fixed s, we need to solve following system of equations

$$\begin{aligned}
\frac{d}{dt}x(t,s) &= a(x(t,s), y(t,s)) = a(t,s), \ \ x(0,s) = x_0(s), \\
\frac{d}{dt}y(t,s) &= b(x(t,s), y(t,s)) = b(t,s), \ \ y(0,s) = y_0(s), \\
\frac{d}{dt}u(t,s) &= c(t,s)u(t,s) + d(t,s), \ \ u(0,s) = u_0(s) = u(x_0(s), y_0(s))
\end{aligned}$$

- For $u(t,s)$, recall integrating factor

$$\mu(t,s) = \exp\left[-\int_0^t c(t,s)dt\right].$$

Analogous to formula (13), for each fixed s, we obtain

$$\boxed{u(t,s) = \frac{1}{\mu(t,s)}\left[\int_0^t \mu(t,s)d(t,s)dt + u_0(s)\right].} \tag{15}$$

$u(t,s)$ is the value of u at the point $(x(t,s), y(t,s))$.

Note: As s and t vary, the point (x, y, u), in xyu-space, given by

$$\boxed{x = x(t,s), \quad y = y(t,s), \quad u = u(t,s),} \tag{16}$$

traces out the surface of the graph of the solution u of PDE (6) which meets the initial curve Γ. The equations (16) constitute the parametric form of the solution of (6) satisfying the initial condition (i.e., a surface in (x, y, u)-space that contains the initial curve Γ)

Remarks.

- The parametric representation of the integral surface might hide further difficulties.
- The difficulty lies in the inversion of the transformation from the plane (t, s) to the plane (x, y).
- By implicit function theorem, if the Jacobian

$$\begin{aligned} J &= \frac{\partial(x, y)}{\partial(t, s)} \\ &= \frac{\partial x}{\partial t}\frac{\partial y}{\partial s} - \frac{\partial y}{\partial t}\frac{\partial x}{\partial s} \\ &= \begin{vmatrix} a & b \\ (x_0)_s & (y_0)_s \end{vmatrix} \neq 0 \end{aligned} \tag{17}$$

  on Γ, where $(x_0)_s = \frac{dx_0}{ds}$, $(y_0)_s = \frac{dy_0}{ds}$, then $x = x(t, s)$ and $y = y(t, s)$ can be inverted to give s and t as (smooth) functions of x and y, i.e., $s = s(x, y)$ and $t = t(x, y)$.
  The resulting function $u(x, y) = u(t(x, y), s(x, y))$ satisfies PDE (6) in a neighborhood of the curve Γ and is the unique solution of the IVP.
- The condition (17) is called transversality condition.

Example

Determine the solution the following IVP:

$$\frac{\partial u}{\partial y} + c\frac{\partial u}{\partial x} = 0, \quad u(x,0) = f(x),$$

where $f(x)$ is a given function and c is a constant.

Solution.

- **Step 1.**(Finding characteristic curves)
  To apply the method of characteristics, recall parametric form the initial curve Γ:

  $$\Gamma = \{(x_0(s), y_0(s), u_0(s)) : \ u_0(s) = u(x_0(s), y_0(s)), \ s \in I\}. \quad (18)$$

  ▸Observe that, for $x_0(s) = s, \ y_0(s) = 0$, we have

  $$u_0(s) = u(s,0) = f(s).$$

  The family of characteristic curves $x((t,s), y(t,s))$ are determined by solving the ODEs

  $$\frac{d}{dt}x(t,s) = c, \quad \frac{d}{dt}y(t,s) = 1$$

- **Step 2.** (Applying IC)
  Along with initial conditions

$$x(0,s) = s, \quad y(0,s) = 0.$$

  we find that

$$x(t,s) = ct + s \quad \text{and} \quad y(t,s) = t.$$

- **Step 3.** (Writing the parametric form of the solution)
  Comparing with (6), we have

$$c(x,y) = 0 \ \ \& \ \ d(x,y) = 0$$

  Therefore, using (15) and the fact $\mu(t,s) = 1$, we find the solution

$$u(t,s) = f(s).$$

  Thus, the parametric form of the solution of the problem is given by

$$x(t,s) = ct + s, \quad y(t,s) = t, \quad u(t,s) = f(s).$$

- **Step 4.** (Expressing $u(s,t)$ in terms of $u(x,y)$)
  Expressing s and t as $s = s(x,y)$ and $t = t(x,y)$, we have

$$s = x - cy, \quad t = y.$$

  We now write the solution in the explicit form as

$$u(x,y) = u(t(x,y), s(x,y)) = f(x - cy).$$

  Clearly, if $f(x)$ is differentiable, the solution $(x,y) = f(x - cy)$ satisfies given PDE as well as the initial condition.

- **Remarks.**
  The above example characterizes unidirectional wave motion with velocity c. If $c > 0$, the entire initial wave form $f(x)$ moves to the right without changing its shape with speed c (if $c < 0$, the direction of motion is reversed).