Assignment 4

Q.1] Correct option is (c).

Q.2] (a) Given equation is :

$$3u_{xx} + 10\,u_{xy} + 3u_{yy} = 0$$

So, $A(x,y) = 3$, $E(x,y) = 0$

$B(x,y) = 10$, $F(x,y) = 0$

$C(x,y) = 3$, $G(x,y) = 0$

$D(x,y) = 0$

$$B^2 - 4AC = 100 - 36$$
$$= 64$$
$$> 0$$

Hence, the equation is Hyperbolic.

Now, let $\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$

$$= \frac{10 \pm 8}{6}$$

$\frac{dy}{dx} = 3$ OR $\frac{dy}{dx} = \frac{1}{3}$

So, $y = 3x + c_1$ OR $y = \frac{x}{3} + c_2$

Let $p(x, y) = 3x - y$

$q(x, y) = x - 3y$

Now, $\bar{A}(p, q) = 3(3)^2 + 10(-3) + 3(-1)^2$

$= 27 + 3 - 30$

$= 0$

$\bar{B}(p, q) = 6(3) + 10(-9-1) + 6(3)$

$= 36 - 100$

$= -64$

$\bar{C}(p, q) = 3(1)^2 + 10(-3) + 3(-3)^2$

$= 3 - 30 + 27$

$= 0$

$\bar{D}(p, q) = 0$

$\bar{E}(p, q) = 0$

$\bar{F}(p, q) = 0$

$\bar{G}(p, q) = 0$

Hence the cannonical form is:

$$-64\, v_{pq} = 0$$

$$\Rightarrow \boxed{v_{pq} = 0}$$

(b.) Given equation is:

$$u_{xx} + 4u_{xy} + 4u_{yy} = 0$$

So, $A(x,y) = 1$, $E(x,y) = 0$

$B(x,y) = 4$, $F(x,y) = 0$

$C(x,y) = 4$, $G(x,y) = 0$

$D(x,y) = 0$

$$B^2 - 4AC = 16 - 16$$
$$= 0$$

Hence, it is Parabolic

let $\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$

$= \frac{4}{2} = 2$

So, $y = 2x + c_1$

let $p(x,y) = 2x - y$

$q(x,y) = x + 2y$

So, $\bar{A}(p,q) = 4 - 8 + 4$

$= 0$

$\bar{B}(p,q) = 4 + 12 - 16$

$= 0$

$\bar{C}(p,q) = 1 + 8 + 16$

$= 25$

$\bar{D}(p,q) = 0$

$\bar{E}(p,q) = 0$

$\bar{F}(p,q) = 0$ and $\bar{G}(p,q) = 0$

So, the cannonical form is:

$25\, V_{qq} = 0$

$\Rightarrow \boxed{V_{qq} = 0}$

(c.) Given equation is:

$$u_{xx} + x u_{yy} = 0 \quad , \quad x > 0$$

$A(x,y) = 1$, $E(x,y) = 0$

$B(x,y) = 0$, $F(x,y) = 0$

$C(x,y) = x$, $G(x,y) = 0$

$D(x,y) = 0$

$$B^2 - 4AC = -4x$$

$$< 0 \quad \text{as} \quad x > 0$$

So. it is Elliptical.

Let $$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$= \frac{\pm \sqrt{-4x}}{2}$$

$$\frac{dy}{dx} = \pm i\sqrt{x}$$

So, $y = \frac{2i}{3} x^{3/2} + c_1$ OR $y = \frac{-2i}{3} x^{3/2} + c_2$

Let $p(x,y) = x^{3/2}$ and
$q(x,y) = \frac{3y}{2}$

$\bar{A}(p,q) = \left(\frac{3}{2}\sqrt{x}\right)^2 = \frac{9x}{4}$

$\bar{B}(p,q) = 0$

$\bar{C}(p,q) = x\left(\frac{3}{2}\right)^2 = \frac{9x}{4}$

$\bar{D}(p,q) = \frac{3}{4\sqrt{x}}$

$\bar{E}(p,q) = 0$
$\bar{F}(p,q) = 0$ and $\bar{G}(p,q) = 0$

So, the canonical form of equation is:

$$\frac{9x}{4} v_{pp} + \frac{9x}{4} v_{qq} + \frac{3}{4\sqrt{x}} v_p = 0$$

$\because$ $x > 0$, we can multiply by $\sqrt{x}$ both sides:

$$3x^{3/2}(V_{pp} + V_{qq}) + V_p = 0$$

$$\boxed{3p(V_{pp} + V_{qq}) + V_p = 0}$$

Q.3]

$$\begin{cases} u_{tt} - 9u_{xx} = 0 & \text{in } (0,\pi) \times (0,\infty) \\ u(x,0) = 2 \\ u_t(x,0) = 1 \end{cases}$$

Now, assume $u(x,t) = F(x)\,G(t)$.

Hence, $\dfrac{F''(x)}{F(x)} = \dfrac{\ddot{G}(t)}{9G(t)}$ from the given PDE.

Clearly the ratios must be equal to a constant as if we vary one variable, say x, then one ratio will change while the second ratio will remain constant. But since both ratios are equal for all values of (x,t) in domain, the ratio is constant, say λ.

So, $\dfrac{F''(x)}{F(x)} = \dfrac{\ddot{G}(t)}{9G(t)} = \lambda$

Now, given $u(x,0) = 2$. Hence,

$$F(x)\, G(0) = 2$$

Clearly, $G(0) \neq 0$.

Hence, $F(x) = \dfrac{2}{G(0)}$ = constant

$$\Rightarrow \quad F''(x) = 0$$

Hence, $\dfrac{F''(x)}{F(x)} = \dfrac{\ddot{G}(t)}{9\, G(t)} = 0$

So, $\lambda = 0$

$$\Rightarrow \quad \ddot{G}(t) = 0$$

$\Rightarrow \quad G(t) = At + B$ for constants A and B

Now, $u(x,t) = F(x)\, G(t)$

$$= \frac{2}{G(0)} (At + B)$$

$$u(x,t) = \frac{2}{B} (At + B)$$

Now, $u_t(x,t) = \dfrac{2A}{B}$

But $\quad u_t(x,0) = 1$

So, $\quad 2A = B$

Hence, $\quad u(x,t) = \frac{2}{B}\left(\frac{tB}{2} + B\right)$

$$\boxed{u(x,t) = t+2}$$

Q.4.)

$$\begin{cases} u_{tt} - 4u_{xx} = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u(x,0) = \sin(x) \\ u_t(x,0) = 1 \end{cases}$$

By D' Almbert's formula:

$$u(x,t) = \frac{1}{2}\left[\sin(x+2t) + \sin(x-2t) \right] + \frac{1}{4} \int_{x-2t}^{x+2t} 1 \, ds$$

$$= \frac{1}{2}\left[\sin(x+2t) + \sin(x-2t) \right] + \frac{1}{4}(4t)$$

$$= \frac{1}{2}\left[\sin(x+2t) + \sin(x-2t) \right] + t$$

$$\boxed{u(x,t) = \sin(x)\cos(2t) + t}$$