Conjunctive Normal Form.

literal - $p \mid \neg p$

Clause - disjunction of literals

α is in CNF if it is a conjunction of clauses.

$$\alpha = C_1 \wedge C_2 \wedge \cdots \wedge C_m$$

clause

Semantic Equivalence.

α and β are semantically equivalent if $\alpha \equiv \beta$ is valid ($\vDash \alpha \equiv \beta$)

Question. Can every $\alpha \in \Phi$ be transformed into an equivalent α' in CNF.
(Semantically)

$$\Phi :: p \in P \mid \neg\alpha \mid \alpha \vee \beta \mid \alpha \wedge \beta$$

$$\Phi ::= p \in P \mid \neg p \in P \mid \beta_1 \vee \beta_2 \mid \beta_1 \wedge \beta_2$$

Example.

1. $\neg p \wedge (q \vee r)$

2. $\models \neg(p \vee q) \equiv \neg p \wedge \neg q$

3. $\models (p \wedge q) \vee r \equiv (p \vee r) \wedge (q \vee r)$

Negation Normal Form. (NNF).

α is in NNF if negation appears only with atomic propositions.

1. $\neg(\alpha \vee \beta) \equiv \neg\alpha \wedge \neg\beta$

2. $\neg(\alpha \wedge \beta) \equiv \neg\alpha \vee \neg\beta$

3. $\neg\neg\alpha \equiv \alpha$

NNF (α)

$\alpha = p / \neg p$; $NNF(\alpha) = p / \neg p$

$\alpha = \neg\neg\beta$; $NNF(\alpha) = NNF(\beta)$

$\alpha = \beta_1 \vee \beta_2$; $NNF(\alpha) = NNF(\beta_1) \vee NNF(\beta_2)$

$$\neg(\neg p \wedge (q \vee \neg(r \wedge s)))$$
$$\equiv \neg\neg p \vee \neg(q \vee \neg(r \wedge s)))$$
$$\equiv p \vee$$
$$\equiv p \vee (\neg q \wedge \neg\neg(r \wedge s))$$
$$\equiv p \vee \neg q \wedge r \wedge s.$$

$\alpha = \beta_1 \wedge \beta_2$; $NNF(\alpha) = NNF(\beta_1) \wedge NNF(\beta_2)$.

$\alpha = \neg(\beta_1 \vee \beta_2)$ $NNF(\alpha) = NNF(\neg\beta_1) \wedge NNF(\neg\beta_2)$

$\alpha = \neg(\beta_1 \wedge \beta_2)$ $NNF(\alpha) = NNF(\neg\beta_1) \vee NNF(\neg\beta_2)$

Distribution Rules.

1. $\alpha \vee (\beta_1 \wedge \cdots \wedge \beta_k) \equiv (\alpha \vee \beta_1) \wedge (\alpha \vee \beta_2) \wedge \cdots \wedge (\alpha \vee \beta_k)$

2. $(\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_k) \vee \alpha \equiv (\beta_1 \vee \alpha) \wedge (\beta_2 \vee \alpha) \wedge \cdots \wedge (\beta_k \vee \alpha)$

Assumption. α is in NNF

CNF(α)

$\alpha = p / \neg p$; $CNF(\alpha) = p / \neg p$

$\alpha = \beta_1 \wedge \beta_2$; $CNF(\alpha) = CNF(\beta_1) \wedge CNF(\beta_2)$

$\alpha = \beta_1 \vee \beta_2$; $CNF(\alpha) = DISTR\left(CNF(\beta_1), CNF(\beta_2)\right)$

$DISTR(\phi_1, \phi_2)$ $\qquad$ $\phi_1, \phi_2 \rightarrow CNF$.

$\phi_1 = \gamma_{11} \wedge \gamma_{12}$ $\quad DISTR(\phi_1, \phi_2) = DISTR(\gamma_{11}, \phi_2) \wedge DISTR(\gamma_{12}, \phi_2)$

$\phi_2 = \gamma_{21} \wedge \gamma_{22}$ $\quad DISTR(\phi_1, \phi_2) = DISTR(\phi_1, \gamma_{21}) \wedge DISTR(\phi_1, \gamma_{22})$.

Otherwise $DISTR(\phi_1, \phi_2) = \phi_1 \vee \phi_2$

$$(\neg p_0 \wedge p_1) \to (p_2 \wedge (p_3 \to p_4))$$

$$\equiv \neg(\neg p_0 \wedge p_1) \vee (p_2 \wedge (\neg p_3 \vee p_4))$$

$$\equiv (p_0 \vee \neg p_1) \vee (p_2 \wedge (\neg p_3 \vee p_4)) \quad - \text{ NNF.}$$

$$\equiv (p_0 \vee \neg p_1 \vee p_2) \wedge (p_0 \vee \neg p_1 \vee \neg p_3 \vee p_4) \quad - \text{ CNF}$$