Uniqueness Results for Solutions of (1) Wave equation and (2) Heat equation

(Reference - T. Amarnath. An Elementary Course in Partial Differential Equations.)

Part A: Uniqueness of solution for one dimensional wave equation with finite length

Theorem: The solution of the following problem, if it exists, is unique.

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= F(x,t), \;\; 0 < x < l, \;\; t > 0 \qquad (1) \\ u(x,0) &= f(x), \;\; 0 \le x \le l, \\ u_t(x,0) &= g(x), \;\; 0 \le x \le l, \\ u(0,t) &= u(l,t) = 0, \;\; t > 0 \end{aligned}$$

Proof The above uniqueness result for IBP of wave equation is equivalent to showing that the following IBP has only trivial solution,

$$\begin{aligned} v_{tt} &= c^2 v_{xx}, \;\; 0 < x < l, \;\; t > 0 \qquad (2) \\ v(x,0) &= 0, \;\; 0 \le x \le l, \\ v_t(x,0) &= 0, \;\; 0 \le x \le l, \\ v(0,t) &= v(l,t) = 0, \;\; t > 0 \end{aligned}$$

Let $v(x,t)$ be a solution of problem (2). Now consider,

$$E(t) = \frac{1}{2}\int_0^l (c^2 v_x^2 + v_t^2)dx.$$

Observe that $E(t)$ is a differentiable function of t, since $v(x,t)$ is twice differentiable. Therefore

$$\begin{aligned} \frac{dE}{dt} &= \int_0^l (c^2 v_x v_{xt} + v_t v_{tt})dx, \\ &= \int_0^l v_t v_{tt} dx + \left[c^2 v_x v_t\right]_0^l - \int_0^l c^2 v_t v_{xx} dx. \end{aligned}$$

$$v(0,t) = 0 \Rightarrow v_t(0,t) = 0 \;\; \forall\, t \ge 0, \text{ and } v(l,t) = 0 \Rightarrow v_t(l,t) = 0 \;\; \forall\, t \ge 0.$$

Therefore

$$\frac{dE}{dt} = \int_0^l v_t(v_{tt} - c^2 v_{xx})dx = 0 \Rightarrow E = \text{constant}.$$

Since $v(x,0) = 0$ implies $v_x(x,0) = 0$ and given that $v_t(x,0) = 0$, therefore

$$E(0) = 0 \Rightarrow E \equiv 0.$$

Hence $v_x \equiv 0, v_t \equiv 0 \ \ \forall\, t > 0,\ 0 < x < l$. This is possible only of $v(x,t)$ =constant, since $v(x,0) = 0,\ v \equiv 0$. Hence the theorem.

Part B: Uniqueness of solution for one dimensional heat equation with finite length

Theorem: The solution of the following problem, if it exists, is unique.

$$\begin{aligned} u_t - \kappa u_{xx} &= F(x,t),\ \ 0 < x < l,\ \ t > 0 \\ u(x,0) &= f(x),\ \ 0 \le x \le l, \\ u(0,t) &= u(l,t) = 0,\ \ t \ge 0 \end{aligned} \tag{3}$$

Proof The above uniqueness result for IBP of heat equation is equivalent to showing that the following IBP has only trivial solution,

$$\begin{aligned} v_t &= \kappa v_{xx},\ \ 0 < x < l,\ \ t > 0 \\ v(x,0) &= 0,\ \ 0 \le x \le l, \\ v(0,t) &= v(l,t) = 0,\ \ t \ge 0 \end{aligned} \tag{4}$$

Let $v(x,t)$ be a solution of problem (4). Now consider,

$$E(t) = \frac{1}{2\kappa}\int_0^1 v^2(x,t)dx.$$

Observe that $E(t)$ is a differentiable function of t, since $v(x,t)$ is twice differentiable.

Therefore

$$\begin{aligned} \frac{dE}{dt} &= \frac{1}{\kappa}\int_0^1 vv_t dx, \\ &= \int_0^1 vv_{xx}dx \\ &= vv_x|_0^l - \int_0^l v_x^2 dx \end{aligned}$$

Since $v(0,t) = v(l,t) = 0$, we have

$$\frac{dE}{dt} = -\int_0^l v_x^2 dx \le 0,$$

i.e. E is a decreasing function of t. Also, by definition, $E(t)$ is nonnegative and from the condition $v((x,0) = 0$ we have $E(0) = 0$. Therefore

$$E(t) \equiv 0 \ \ \forall t > 0 \Rightarrow v(x,t) \equiv 0 \ \text{ in } 0 \le x \le l,\ t \ge 0.$$

Hence the theorem.