MTH 102A - Linear Algebra - 2015-16-II Semester

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P. **Maximality**

• Let $S \subseteq T$. We say S is a maximal subset of T having a property (P) if
i) S has (P) and ii) no proper superset of S in T has (P).

• Let $T = \{2, 3, 4, 7, 8, 10, 12, 13, 14, 15\}$. Then a maximal subset of T of consecutive integers is $S = \{2, 3, 4\}$. Other maximal subsets are $\{7, 8\}, \{10\}, \{12, 13, 14, 15\}$. The subset $\{12, 13\}$ is not maximal. Why?

• $S \subseteq \mathbb{V}$ is called maximal lin.ind if
i) S is lin.ind and ii) no proper super set of S is lin.ind.

• In $\mathbb{R}^3$, the set $\{\mathbf{e}_1, \mathbf{e}_2\}$ is lin.ind but not maximal lin.ind. ‼

• In $\mathbb{R}^3$, the set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is maximal lin.ind. ‼

• Let $S \subseteq \mathbb{R}^n$ be lin.ind and $|S| = n$. Then S is maximal lin.ind. ‼

• Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subseteq \mathbb{R}^n$. Form the matrix A by taking $\mathbf{v}_i$'s as columns. Apply GJE: let $A = ER$, where $R = \text{RREF}(A)$. Let $R(:, i_1), \ldots, R(:, i_p)$ be the pivotal columns. Then $\{\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_p}\}$ is a maximal lin.ind subset of S. ‼

P. **Dimension**

• A set $S \subseteq \mathbb{V}$ is maximal lin.ind if and only if $\text{LS}(S) = \mathbb{V}$. ‼

• $\mathbb{V}$ is called finite dimensional if it has a finite subset S s.t. $\text{LS}(S) = \mathbb{V}$.

• $\mathbb{R}^n$ is finite dimensional as $\mathbb{R}^n = \text{LS}(\mathbf{e}_1, \ldots, \mathbf{e}_n)$.

• Let $\mathbb{V} = \text{LS}(S)$, $|S| = k$, $T \subseteq \mathbb{V}$ be lin.ind. We already know that $|T| \leq k$. Thus a maximal lin.ind subset of $\mathbb{V}$ has at most k vectors in it.

Th Let S, T be two finite maximal lin.ind subsets of $\mathbb{V}$. Then $|S| = |T|$.

Po. As $\text{LS}(S) = \mathbb{V}$ and T is lin.ind, we get $|T| \leq |S|$. Similarly, $|S| \leq |T|$. □

• Let $\mathbb{V} \neq \{\mathbf{0}\}$ be a VS and S be a maximal lin.ind subset. We call $|S|$ the (algebraic) dimension $\dim(\mathbb{V})$ of $\mathbb{V}$. Convention: dimension of $\{\mathbf{0}\}$ is 0.

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• Let $\mathbb{V} \neq \{\mathbf{0}\}$ be a VS. Then a maximal lin.ind subset of $\mathbb{V}$ is called the (Hamel) basis of $\mathbb{V}$. Note: basis of $\{\mathbf{0}\}$ is not defined.

P. **Minimal spanning set, Basis**

• Let $\mathbb{V} \neq \{\mathbf{0}\}$. A set $S \subseteq \mathbb{V}$ is called minimal spanning if $\text{LS}(S) = \mathbb{V}$ and no proper subset of S spans $\mathbb{V}$.

Th Let $\mathbb{V} \neq \{\mathbf{0}\}$ be a vector space. TFAE:

1) $\mathcal{B}$ is a basis (maximal lin.ind set) of $\mathbb{V}$.
2) $\mathcal{B}$ is lin.ind and it spans $\mathbb{V}$.
3) $\mathcal{B}$ is a minimal spanning set of $\mathbb{V}$.

Po. 1)$\Rightarrow$ 2): Basis $\Rightarrow$ Lin. ind and Maximal $\Rightarrow$ spans.

2)$\Rightarrow$3): Let S be a lin.ind set that spans $\mathbb{V}$. Then, for any $\mathbf{x} \in S$, $\mathbf{x} \notin \text{LS}\,(S - \{\mathbf{x}\})$. Hence $\text{LS}\,(S - \{\mathbf{x}\}) \neq \mathbb{V}$.

3)$\Rightarrow$1): Since $\mathcal{B}$ spans $\mathbb{V}$, for any $\mathbf{x} \in \mathbb{V} \setminus \mathcal{B}$ we have $\mathcal{B} \cup \{\mathbf{x}\}$ is lin.dep. Assume that $\mathcal{B}$ is lin.dep. Then there exists $\mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathcal{B}$ s.t. $\mathbf{v} = \sum \alpha_i \mathbf{v}_i$. In that case $\text{LS}(\mathcal{B} \setminus \{\mathbf{v}\}) = \text{LS}(\mathcal{B}) = \mathbb{V}$, means $\mathcal{B}$ is not minimal spanning. $\square$

Th Let $\mathbb{V}$ have dimension n and $S \subseteq \mathbb{V}$ be lin.ind. Then there exists a basis $T \supseteq S$.

Po. If $\text{LS}(S) = \mathbb{V}$, then S is a basis. Otherwise, choose $\mathbf{x}_1 \in \mathbb{V} \setminus \text{LS}(S)$.

By earlier result, $S_1 = S \cup \{\mathbf{x}_1\}$ is lin.ind. Repeat the process. Process must stop as dimension is n. When it stops $\text{LS}(S_k) = \mathbb{V}$. $\square$

Th Let F be invertible and $A_{m\times n} = FB$. Then $\text{ROW}(A) = \text{ROW}(B)$.

Po. As $A(i,:) = \sum_{j=1}^{m} f_{ij} B(j,:)$, $\text{ROW}(A) \subseteq \text{ROW}(B)$. As $A = F^{-1}R$, $\text{ROW}(B) \subseteq \text{ROW}(A)$. $\square$

Th $\text{RANK}(A) = \dim \text{ROW}(A)$.

Po. GJE: $R = FA$. Nonzero rows of R are lin.ind due to the positions of pivots. Thus $\text{RANK}(R) = \dim \text{ROW}(R) = \dim \text{ROW}(A)$. $\square$

Th Let $R = \text{RREF}(A)$ with the pivotal columns $i_1, \ldots, i_k$. Then columns $i_1, \ldots, , i_k$ of A form a basis for $\text{COL}(A)$. Thus $\text{RANK}\, A = \dim \text{COL}(A)$.

Po. Note: $R(:, i_1), \ldots, R(:, i_k)$ are lin.ind and other columns in R are lin.comb of them. So $A(:, i_1) \ldots, A(:, i_k)$ are lin.ind (as F is invertible) and other columns in A are lin.comb of them. So the columns $A(:, i_1) \ldots, A(:, i_k)$ form a basis for $\text{COL}(A)$. $\square$

Th $\text{RANK}(A) = \dim \text{ROW}(A) = \dim \text{COL}(A) = \text{RANK}(A^t)$.

Po. First two equality follow from earlier result. As $\text{COL}(A^t) = \text{ROW}(A)$, $\text{RANK}(A) = \dim \text{ROW}(A) = \dim \text{COL}(A^t) = \text{RANK}(A^t)$. $\square$

T/F Rows of A contain a basis of $\text{ROW}(A)$. T.

• In GJE we called the pivotal columns 'basic columns'. Notice that they give us a 'basis' for the column space.

• Let $S \subseteq \mathbb{R}^n$ be lin.ind. Can you use GJE to extend S to a basis?

• Yes. Form the matrix $A_{n\times m}$ $(m \le n)$ using the vectors in S as columns. Apply GJE: $A = ER$, where E is $n \times n$ and first m rows of R are nonzero. Extend R to $R' = [R \mid \mathbf{e}_{m+1} \quad \cdots \quad \mathbf{e}_n]$. Put $A' = ER'$. Columns of A' form the necessary extension.

Th Let $\textsc{rank}\, A_{m\times n} = k$. Then the maximum order of a nonsingular submatrix of A is k.

Po. Delete nonbasic columns of A. Call it A_1. Look at $B_{k\times m} = A_1^t$. It has rank k. So it has k basic columns. Delete the nonbasic ones. Call it B_1. Look at $C_{k\times k} = B_1^t$. It is a submatrix of A of rank k. So $\det C \neq 0$.

Take a submatrix $D_{k+1\times k+1}$ of A. Assume $\det D \neq 0$. Then columns of D are lin.ind. Extend each column of D (replace it with resp column of A). They still remain lin.ind. But this means $\textsc{rank}\, A > k$.

A contradiction. $\square$

Cor $\textsc{rank}(A) = \textsc{rank}(A^*)$. ‼

• Consider the VS $\mathbb{F}^n$ over $\mathbb{F}$. The basis $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ is called the standard basis. Notice that $\dim \mathbb{F}^n = n$.

• Let $\mathbb{V} = \mathbb{R}[x]$. Then $\{1, x, x^2, \cdots\}$ is a basis. ‼

• $\{1, x, x^2, \cdots, x^n\}$ is a basis for $\mathbb{R}[x; n]$. So $\dim \mathbb{R}[x; n] = n + 1$.

• Give a basis to show that $\dim \mathcal{M}_{m,n}(\mathbb{R})$ is mn. ‼

P. **Rank-Nullity**

Th[rank-nullity] $\dim \textsc{null}\, A_{m\times n} = n - \textsc{rank}\, A$.

Po. Imagine solving $A\mathbf{x} = \mathbf{0}$ by GJE. Let $\mathbf{x}_{i_1}, \ldots, \mathbf{x}_{i_k}$ be the free variables. Put $\mathbf{x}_{i_1} = 1$, $\mathbf{x}_{i_2} = \cdots = \mathbf{x}_{i_k} = 0$ to get a solution X_1. Put $\mathbf{x}_{i_k} = 1$, $\mathbf{x}_{i_1} = \cdots = \mathbf{x}_{i_{k-1}} = 0$ to get a solution X_k. We get k lin.ind solutions.

X_1	X_2	$\cdots$	X_k
$*$	$*$		$*$
1	0		0
0	1		0
		$\vdots$	
$*$	$*$		$*$
0	0		1

Let $\mathbf{y}$ be any solution of $A\mathbf{x} = \mathbf{0}$. Put $\mathbf{z} = \mathbf{y} - \mathbf{y}_{i_1} X_1 - \cdots - \mathbf{y}_{i_k} X_k$. So $A\mathbf{z} = A\mathbf{y} - \mathbf{y}_{i_1} A X_1 - \cdots - \mathbf{y}_{i_k} A X_k = \mathbf{0}$. So, $\mathbf{z}$ is a solution of $A\mathbf{x} = \mathbf{0}$ where each free variable is 0. Hence $\mathbf{z} = \mathbf{0}$. So y is a lin.comb of $X_1, \ldots, X_k$. So $\dim \textsc{null}\, A = k = n - \textsc{rank}\, A$. $\square$