Ordinary Differential Equations

MTH-102A

T. Muthukumar
tmk@iitk.ac.in

(In memory of Prof. Arbind Kumar Lal)

02 July 2021

Reference text for the Course

- *Ordinary Differential Equations* by Morris Tenenbaum and Harry Pollard.
- *Elementary Differential Equations and Boundary Value Problems* by William E. Boyce and Richard C. DiPrima.
- *Introduction to Ordinary Differential Equations* by Shepley L. Ross.
- *Differential Equations and Linear Algebra* by Gilbert Strang.
- *Differential Equations with Applications and Historical Notes* by George F. Simmons.

The above references are indicative, for further reading and not exhaustive. The contents of the course is *not* a linear adaptation of above references but intersect.

How Equations Arise?

The process of understanding natural phenomena may be viewed in three stages:

1. Modelling the phenomenon as a mathematical equation (algebraic, differential or integral equation) using physical laws such as Newton's law, momentum, conservation laws, balancing forces etc.
2. Solving the equation! This leads to the question of what constitutes as a solution to the equation?
3. Properties of the solution, especially in situations when exact solution is not within our reach.

In this course, we are mostly interested in differential equations in dimension one, i.e. one independent variable!

Algebraic Equations to Differential Equations

1. Algebraic equations are relations between (one or more) unknown quantities/numbers and known quantities.
2. The Linear algebra (earlier part of the course) may also be applied to solve system of linear (algebraic or differential) equations.
3. Differential equations are relations between (one or more) unknown functions along with its derivatives and known quantities.
4. Modelling into differential equations was facilitated by the invention of differential calculus in sixteenth century.
5. The first ordinary differential equation (ODE) probably comes from the Newton's second law: How much a body of mass m will be displaced when acted up on by a force F!
6. The first partial differential equation (PDE) was written down to study the vibrating strings, now known as the (one space dimension) wave equation.

Derivative as a Map

- Let $I := (a, b) \subset \mathbb{R}$ be an open interval not necessarily of finite length. Then the *derivative* of a function $y : I \to \mathbb{R}$, at $x \in I$, is defined as

$$\frac{dy}{dx}(x) = y'(x) := \lim_{h \to 0} \frac{y(x+h) - y(x)}{h}$$

  provided the limit exists.
- The ordinary differential operator of *order* k is denoted as $\frac{d^k}{dx^k}$. More concisely, the k-th order derivative of $y : I \to \mathbb{R}$ is denoted as $y^{(k)}$.
- A k-th order ordinary differential operator can be viewed as a map from $\frac{d^k}{dx^k} : C^k(\bar{I}) \to C(\bar{I})$ where $\bar{I} = [a, b]$ and $C^k(\bar{I})$ is the set of all k-times differentiable function in I with the k-the derivative being continuous and extended continuously to $\bar{I}$.
- (Exercise!) Verify that $C^k(\bar{I}) \subset C(\bar{I})$ for $k \in \mathbb{N}$. Also, seek examples of differentiable functions whose derivative is not continuous!

Ordinary Differential Equation

Definition

Let I be an open interval of $\mathbb{R}$. A k-th *order* ordinary differential equation of an *unknown* function $y : I \to \mathbb{R}$ is of the form

$$F\left(y^{(k)}, y^{(k-1)}, \dots y'(x), y(x), x\right) = 0, \tag{2.1}$$

for each $x \in I$, where $F : \mathbb{R}^{k+1} \times I \to \mathbb{R}$ is a given map such that F depends on the k-th order derivative y and is independent of $(k+j)$-th order derivative of y for all $j \in \mathbb{N}$.

The word *ordinary* refers to the situation of exactly one independent variable x. The unknown dependent variable y can be more than one, leading to a system of ODE.

Example

Example

- The DE $y' + y + x = 0$ comes from the $F(u, v, w) = u + v + w$.
- The DE $y'' + xy(y')^2 = 0$ comes from the $F(t, u, v, w) = t + wvu^2$.
- The DE $y^{(4)} + 5y'' + 3y = \sin x$ comes from the $F(r, s, t, u, v, w) = r + 5t + 3v - \sin w$.
- $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$ is a *partial* diferential equation of a two variable unknown function $u(x, y)$.
- $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ is a *partial* diferential equation of a three variable unknown function $u(x, y, z)$.

Classification of ODE in terms of Linearity

The level of difficulty in solving a ODE may depend on its order k and linearity of F.

Definition

A k-th order ODE is *linear* if F in (2.1) has the form

$$F := Ly - f(x)$$

where $Ly(x) := \sum_{i=0}^{k} a_i(x) y^{(i)}(x)$ for given functions f and a_i's such that a_k is not identically zero. Otherwise the ODE is *nonlinear*. If, in addition to being linear, one has $f \equiv 0$ then the ODE is *linear homogeneous*.

It is called linear because L is linear in y, i.e., $L(cy_1 + dy_2) = cL(y_1) + dL(y_2)$ for all $c, d \in \mathbb{R}$.

Examples

Example

1. $y'' + 5y' + 6y = 0$. Here $L := \frac{d^2}{dx^2} + 5\frac{d}{dx} + 6$. and is linear, homogeneous because $f \equiv 0$ and second order with constant coefficients.
2. $x^{(4)} + 5x^{(2)} + 3x = \sin t$ is linear and fourth order with constant coefficients.
3. $y^{(4)} + x^2 y^{(3)} + x^3 y' = xe^x$ is linear and fourth order with variable coefficients.
4. $y'' + xy(y')^2 = 0$ is nonlinear and second order.
5. $y'' + 5y' + 6y^2 = 0$ is nonlinear and second order.
6. $y'' + 5(y')^3 + 6y = 0$ is nonlinear and second order.
7. $y'' + 5yy' + 6y = 0$ is nonlinear and second order.

Explicit Solution of ODE

Definition

We say $u : I \to \mathbb{R}$ is an *explicit solution* to the ODE (2.1) on I, if $u^{(j)}(x)$ exists for all j explicitly present in (2.1), for all $x \in I$, and u satisfies the equation (2.1) in I.

Example

The function $y : \mathbb{R} \to \mathbb{R}$ defined as $y(x) := 2\sin x + 3\cos x$ is an explicit solution of the ODE $y'' + y = 0$ in $\mathbb{R}$.

Roughly speaking, an explicit solution curve is a graph of the solution in the plane!

Implicit Solution of ODE

Definition

We say a relation $v(x, y) = 0$ is an *implicit solution* to the ODE (2.1) on I, if it defines at least one real function $u : I \to \mathbb{R}$ of x variable which is a explicit solution of (2.1) on I.

Example

The relation $v(x, y) := x^2 + y^2 - 25 = 0$ is an implicit solution of the ODE $yy' + x = 0$ in $(-5, 5)$. The given relation v defines two real valued functions $u_1(x) := \sqrt{25 - x^2}$ and $u_2(x) := -\sqrt{25 - x^2}$ in $(-5, 5)$ and both are explicit solutions of the given ODE.

Roughly speaking, an implicit solution curve is not a graph of the solution but some part (locally) of the solution curve can be expressed as the graph of a function.

Formal solutions

- In the previous example, could we have just differentiated the relation $x^2 + y^2 - 25 = 0$ with respect to x to obtain $2x + 2yy' = 0$ and concluded that it is implicit solution?
- The answer is a 'no'!
- For instance, consider $x^2 + y^2 + 25 = 0$.
- It formally satisfies the ODE on implicit differentiation
- but is not an implicit solution because $y(x) := \pm\sqrt{-25 - x^2}$ cannot be expressed as (even locally) graph of a function.
- Complex valued functions can be solutions to ODE with real coefficients (as above) or $(y')^2 + 1 = 0$ has the family of solutions $y(x) = \imath x + c$.
- Henceforth, explicit, implicit and formal solution will be simply referred to as 'solution'.

Do solutions always exist?

- Do all ODE have solutions?
- If yes, how many?

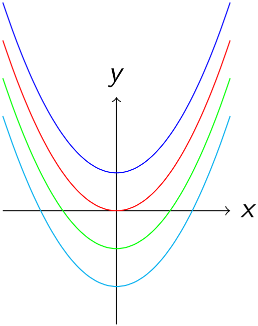
Example

The ODE $|y'| + |y| + 1 = 0$ admits *no* real (or complex) valued function y as a solution because sum of positive quantities cannot be a negative number.

Family of Solutions

Example

Consider the first order ODE $y' = 2x$. On integration we obtain a one parameter family of solutions $y_c(x) := x^2 + c$ indexed by $c \in \mathbb{R}$.



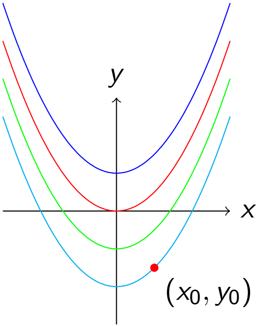
Solutions Outside Family of Solutions

Example

Consider the first order ODE $(y')^2 - 4y = 0$. The one parameter family $y_c(x) := (x+c)^2$ are solution to the ODE. In addition, $y \equiv 0$ is also a solution, not included in the above family of solutions.

IVP and BVP

- In practice, to choose a desirable solution from the family of solutions, some additional conditions are imposed along with the ODE.
- These additional conditions could be either an *initial condition* or *boundary condition* or their mixture.



Initial Value Problem

Definition

An *initial value problem* (IVP) corresponding to a k-th order ODE is given as

$$\begin{cases} F(y^{(k)}, \ldots, y, x) & = 0 \quad \text{on } I \\ y^{(j)}(x_0) & = y_j \quad \text{for all } 0 \leq j \leq k-1 \end{cases}$$

where k initial conditions are prescribed at some $x_0 \in \bar{I}$ and $\bar{I}$ is the union of I with its end-points.

Examples for IVP

Example

The first order IVP

$$\begin{cases} y' &= 2x \quad \text{in } \mathbb{R} \\ y(1) &= 4. \end{cases}$$

Example

The second order IVP

$$\begin{cases} y'' + y &= 0 \qquad \text{in } \mathbb{R} \\ y(1) &= 3 \\ y'(1) &= -4. \end{cases}$$

A first order boundary value problem is same as an initial value problem!

Definition

A *boundary value problem* corresponding to a second order ODE $F(y'', y', y, x) = 0$ on I is prescribed with one of the following additional conditions: For any $x_0, x_1 \in \overline{I}$

1. $y(x_0) = y_0$ and $y(x_1) = y_1$.
2. $y'(x_0) = y_0$ and $y'(x_1) = y_1$.
3. $c_1 y(x_0) + c_2 y'(x_0) = y_0$ and $d_1 y(x_1) + d_2 y'(x_1) = y_1$ for given $c_i, d_i \in \mathbb{R}$ and $i = 1, 2$.
4. $y(x_0) = y(x_1)$ and $y'(x_0) = y'(x_1)$.

Examples for BVP

Example

The second order BVP

$$\begin{cases} y'' + y &= 0 \quad \text{in } \mathbb{R} \\ y(0) &= 1 \\ y(\frac{\pi}{2}) &= 5. \end{cases}$$

Example

The second order BVP

$$\begin{cases} y'' + y &= 0 \quad \text{in } \mathbb{R} \\ y(0) &= 1 \\ y(\pi) &= 5. \end{cases} \tag{3.1}$$

Existence of Solutions

- Do all IVP and BVP have solutions?
- If yes, how many?
- It can be shown that the second example for BVP (3.1) admits no solution! (even in the simple case of linear homogeneous.)
- The situation is quite better for 'special' IVP! Special in the sense that the nonlinear ODE is necessarily linear in the highest order derivative.
- We state the existence and uniqueness result for first order 'special' IVP (without proof).

Existence for First Order of the form $y' = f(x, y)$

Theorem (Picard)

Let Ω be an open, connected subset (domain) of $\mathbb{R}^2$ and $(x_0, y_0) \in \Omega$. If $f : \Omega \to \mathbb{R}$ is continuous then the first order IVP

$$\begin{cases} y' &= f(x, y) \\ y(x_0) &= y_0 \end{cases} \tag{3.2}$$

admits at least one solution.

Non-uniqueness of Solution

In general the IVP (3.2) may admit more than one solution!

Example

Consider the IVP $y' = y^{1/3}$ in $\mathbb{R}$ with the initial conditions $y(0) = 0$. Note that $y \equiv 0$ is one solution. Also,

$$y_c(x) := \begin{cases} \left(\frac{2(x-c)}{3}\right)^{3/2} & x \geq c \\ 0 & x \leq c \end{cases}$$

is also a solution for all $c \geq 0$! Thus, this problem admits infinitely many solutions.

Picard's Uniqueness Theorem

Theorem (Uniqueness)

Let Ω be a domain of $\mathbb{R}^2$ and $(x_0, y_0) \in \Omega$. If both $f : \Omega \to \mathbb{R}$ and $\frac{\partial f}{\partial y} : \Omega \to \mathbb{R}$ are continuous then there exists a $h > 0$ such that the first order IVP (3.2) admits a unique solution in the interval $[x_0 - h, x_0 + h]$.

Examples for Picard's Theorem

Since we shall not be proving the theorem, let us verify the theorem with some examples.

Example

- Consider the IVP $y' = x^2 + y^2$ with $y(1) = 3$.
- Note that $f(x, y) = x^2 + y^2$ and $\frac{\partial f}{\partial y} = 2y$.
- Both are continuous in every domain of $\mathbb{R}^2$.
- By Picards' theorem, for every domain Ω that contains $(1, 3)$ there is a $h > 0$ such that the IVP has a unique solution in $[1 - h, 1 + h]$.

Examples for Picard's Theorem

Example

Consider the ODE $y' = yx^{-1/2}$. Note that $f(x,y) = yx^{-1/2}$ and $\frac{\partial f}{\partial y} = x^{-1/2}$. They are continuous in the open right half plane, i.e $\{(x,y) \mid x > 0\}$.

- If the initial condition imposed is $y(1) = 2$ then for every domain Ω that contains $(1,2)$ and *does* not contain the y-axis there is a $h > 0$ such that the IVP has a unique solution in $[1-h, 1+h]$.
- If the initial condition imposed is $y(0) = 2$ then both f and $\partial_y f$ are not continuous (undefined) at $(0,2)$. Thus, a comment on the existence and uniqueness of solution is inconclusive!

Topics to be covered in First order

In this course we shall introduce approximate solution and closed-form solution methods for first order ODE. The approximate methods:

- Picard's method of successive approximation;
- Method of Isoclines.

The following six types of first order ODE admit closed-form solutions, viz.

- Separable;
- Homogeneous;
- Exact;
- Linear;
- Bernoulli;
- One degree Polynomial coefficients.

The methods introduced are basically relying on two major ideas, viz. integrating factor and transformation.

Motivation for Picard's Successive Approximation

- Consider the IVP (3.2), i.e.

$$\begin{cases} y' &= f(x,y) \quad \text{in } I \\ y(x_0) &= y_0 \end{cases}$$

  such that $x_0 \in \bar{I}$.
- Integrating both sides of the differential equation between x_0 and x, we get the integral equation $y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt$.
- Solving for y in the ODE is same as solving for y in the integral equation.
- Define the map $T : C(\bar{I}) \to C(\bar{I})$ as $Ty(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt$ (ignore concerns related to the well-definedness of T for this course).
- Then the solution to the integral equation $y = Ty$ is a fixed point of T.
- Picard's approximation gives an algorithm to find the fixed point of T and, in turn the solution of the IVP (3.2).

Picard's Successive Approximation

- Consider the IVP (3.2), i.e.

$$\begin{cases} y' &= f(x, y) \\ y(x_0) &= y_0 \end{cases}$$

- Choose a function u_0 (zeroth approximation) such that $u_0(x_0) = y_0$.
- Next choose u_1 (first approximation) as a solution to

$$\begin{cases} u_1' &= f(x, u_0) \\ u_1(x_0) &= y_0 \end{cases}$$

or, equivalently, for 'nice' f

$$u_1(x) = y_0 + \int_{x_0}^{x} f[s, u_0(s)]\, ds.$$

Picard's Successive Approximation

- Similarly, the k-th approximation u_k is chosen such that
$$u_k(x) = y_0 + \int_{x_0}^{x} f[s, u_{k-1}(s)]\, ds.$$
- Under suitable conditions, it can be shown that $\{u_k\}$ will converge to some u which will be a solution of the IVP.

Example

Example

- Consider the IVP

$$\begin{cases} y' &= x^2 + y^2 \\ y(0) &= 1 \end{cases}$$

- Choose $u_0 \equiv 1$ and, for $k \geq 1$,

$$u_k(x) = 1 + \int_0^x \left[s^2 + u_{k-1}^2(s)\right] \, ds.$$

- On integration, we obtain, $u_1(x) = 1 + x + \frac{x^3}{3}$.
- Similarly,

$$u_2(x) = 1 + x + x^2 + \frac{2x^3}{3} + \frac{x^4}{6} + \frac{2x^5}{15} + \frac{x^7}{63}$$

Example

Example

- Consider the IVP
$$\begin{cases} y' &= 2\sqrt{x} \\ y(0) &= 1 \end{cases}$$
- Choose $u_0 \equiv 1$ and, for $k \geq 1$,
$$u_k(x) = 1 + 2\int_0^x \sqrt{s}\,ds = 1 + \frac{4}{3}x^{\frac{3}{2}}.$$
- Thus, we have a constant sequence $\{u_k\}$, for $k \geq 1$ converging to the exact solution obtained by integrating (separable ODE).

Total Differential

Definition

Let $F : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ be a two-variable function that admit continuous first order partial derivatives in Ω. The *total differential*, denoted as dF, is defined as

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy.$$

- The definition can be extended to n variable function in the obvious way.
- It is motivated from the chain rule structure! If we imagine x and y as functions of t the chain rule says that

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt}.$$

- The total differential signifies the changes in the function as linear combination of changes in each variable.

Gain or Loss of Solutions

Observe that any first order ODE of the form $y' = f(x, y)$ can be equivalently written in the differential form as

$$M(x, y)dx + N(x, y)dy = 0. \tag{3.3}$$

- The advantage of differential form version is that the roles of x and y can be interchanged.
- The equivalent forms are obtained by multiplication or division operation.
- It should be observed that solutions may be gained or lost while performing multiplication/division operations. We shall illustrate this phenomenon in the following example.

Example (loss of solution)

Example

Consider the ODE $y' = \frac{(x-4)y^4}{x^3(y^2-3)}$. Observe that $y \equiv 0$ is one possible solution to the ODE. Rewriting it in the differential form, we obtain

$$(x-4)y^4 dx - x^3(y^2-3)dy = 0.$$

Dividing the equation above by x^3y^4, we obtain

$$(x^{-2} - 4x^{-3})dx - (y^{-2} - 3y^{-4})dy = 0.$$

On integration, we have the family of solutions

$$-x^{-1} + 2x^{-2} + y^{-1} - y^{-3} = c.$$

Note that the solution $y = 0$ (a solution of $y^4 = 0$) is not a member of the above family which got eliminated with the restriction $y^4 \neq 0$.

Example (no loss of solution)

Example

Consider the ODE $y' = \frac{-x \sin y}{(x^2+1)\cos y}$. Observe that the roots of $\sin y = 0$, i.e. $y = k\pi$ for $k \in \mathbb{Z}$ are constant solutions to the ODE. Rewriting it in the differential form, we obtain

$$x \sin y \, dx + (x^2 + 1) \cos y \, dy = 0.$$

Dividing the equation above by $(x^2 + 1) \sin y$, we obtain

$$\frac{x}{x^2 + 1} dx + \frac{\cos y}{\sin y} dy = 0.$$

On integration, we have the family of solutions $\frac{1}{2} \ln(x^2 + 1) + \ln |\sin y| = \ln |c|$ or $(x^2 + 1) \sin^2 y = c^2$. Note that the solution $y = k\pi$, for all $k \in \mathbb{Z}$ (a solution of $\sin y = 0$) is *also* a member of the above family (for $c = 0$) in spite of imposing the restriction $\sin y \neq 0$.

Separable type ODE

Definition

The first order ODE is said to be of *separable* type if $f(x, y)$ is of the form $-\frac{M(x)}{N(y)}$. Equivalently, M and N are independent of the y and x variable, respectively, in the differential form $M(x)dx + N(y)dy = 0$.

The family of solutions for a separable equation is $F(x, y) := \int M(x)\, dx + \int N(y)\, dy = c$.

Example

Consider the ODE $(x^{-2} - 4x^{-3})dx - (y^{-2} - 3y^{-4})dy = 0$. Then, on integration, we obtain the one parameter family of solutions

$$-\frac{1}{x} + \frac{2}{x^2} + \frac{1}{y} - \frac{1}{y^3} = c.$$

Homogeneous ODE

Definition

A function $v : \mathbb{R}^2 \to \mathbb{R}$ is said to be *homogeneous of degree n* if $v(tx, ty) = t^n v(x, y)$.

For instance, $v(x, y) := x^2 + y^2$ is homogeneous of degree 2.

Definition

A first order ODE of the form (3.3) is *homogeneous* if both M and N are homogeneous of same degree. Equivalently, $y' = f(x, y)$ is *homogeneous* if $f(tx, ty) = f(x, y)$.

- For instance, $(y + \sqrt{x^2 + y^2})dx - xdy = 0$ is homogeneous.
- The reason for considering homogeneous ODEs is that it can be transformed to ODE with separated variables.

Homogeneous to Separable Form

Theorem

If $y' = f(x, y)$ is homogeneous then the change of variable $y = zx$ transforms the ODE into a separable type ODE in x and z variables.

Proof.

By homogeneity, since $f(tx, ty) = f(x, y)$ for any t, we choose in particular $t = 1/x$ and obtain $f(1, y/x) = f(x, y)$. Thus, the given ODE becomes $y'(x) = f(1, y/x)$. Using the change of variable $y(x) = xz(x)$ in the ODE, we get $z + xz' = f(1, z)$ or equivalently, $z' = \frac{f(1,z)-z}{x}$ is in the separable form. $\square$

Note that such a reduction is valid in region excluding $\{x = 0\}$.

Solution to Homogeneous ODE

- A homogeneous ODE $y' = f(x, y)$ can be reduced to the separable type $z' = \frac{f(1,z)-z}{x}$.
- The solution to the separable type form is given as

$$\int \frac{dz}{z - f(1, z)} + \int \frac{dx}{x} = c.$$

- Thus, in terms of x and y variable, the family of solutions is $F(y/x) + \ln|x| = c$ where

$$F(z) = \int \frac{dz}{z - f(1, z)}.$$

Example

Example

- Consider the ODE $(x^2 - 3y^2)dx + 2xydy = 0$ or, equivalently, $y' = \frac{3y^2 - x^2}{2xy}$.
- This ODE is homogeneous because M and N are homogeneous of degree 2, i.e. $M(tx, ty) = t^2(x^2 - 3y^2) = t^2 M(x, y)$ and $N(tx, ty) = t^2 2xy = t^2 N(x, y)$.
- Using $y = zx$ substitution, we get $z + xz' = \frac{3z^2 - 1}{2z}$ or, equivalently $xz' = \frac{z^2 - 1}{2z}$.
- Thus, the separable type ODE we obtain is

$$\frac{2z}{z^2 - 1} dz - \frac{dx}{x} = 0.$$

Example

Example

- On integration, the family of solutions is

$$\ln|z^2 - 1| - \ln|x| = \ln|c|$$

or, equivalently, $|z^2 - 1| = |cx|$. Finally, $|y^2 - x^2| - |cx|x^2 = 0$ is the family of solutions.

Example

- Consider the IVP

$$\begin{cases} (y + \sqrt{x^2 + y^2})dx - xdy & = 0 \\ y(1) & = 0. \end{cases}$$

The ODE can be rewritten as $y' = \frac{y + \sqrt{x^2+y^2}}{x}$. We have already checked that this ODE is homogeneous!

Homogeneous IVP

Example

- Using $y = zx$ substitution, we get $z + xz' = z + \sqrt{1+z^2}$ or, equivalently $xz' = \sqrt{1+z^2}$.
- Thus, the separable type ODE we obtain is

$$\frac{dz}{\sqrt{z^2+1}} - \frac{dx}{x} = 0.$$

- On integration, the family of solutions is $\ln|z + \sqrt{z^2+1}| - \ln|x| = \ln|c|$ or, equivalently, $|z + \sqrt{z^2+1}| = |cx|$. Finally, $|y + \sqrt{y^2+x^2}| - |c|x^2 = 0$ is the family of solutions.
- Using the initial conditions, we get $|c| = 1$.
- The particular integral (solution) is $|y + \sqrt{y^2+x^2}| - x^2 = 0$.

Definition

An ODE of the form (3.3) is said to be an *exact* differential equation if there exists a two variable function F such that $\partial_x F(x,y) = M(x,y)$ and $\partial_y F(x,y) = N(x,y)$.

The motivation for the above definition is that an exact differential equation can be rewritten as $dF(x,y) = 0$.

Examples

Example

The ODE $y^2dx + 2xydy = 0$ is an exact differential equation because $F(x, y) = xy^2$ satisfies the requirement.

Example

The ODE $ydx + 2xdy = 0$ is *not* exact!

Now, showing exactness of an ODE is not that easy! We need to either find a F or show non-existence of a F meeting the requirement. Thus, we need a test to determine exactness of ODE.

Necessary and Sufficient Condition for Exactness

Theorem

Let M and N admit continuous first order partial derivatives in a rectangular domain $\Omega \subset \mathbb{R}^2$. The ODE (3.3) is exact in Ω iff $\partial_y M(x,y) = \partial_x N(x,y)$ for all $(x,y) \in \Omega$.

Proof.

The necessary part is obvious! The sufficiency part involves finding suitable F. The arguments are shown by examples. Abstract proof follows the same line as you see in examples. □

Example

The ODE $y^2dx + 2xydy = 0$ is exact in every domain $\Omega \subset \mathbb{R}^2$ because $M(x,y) = y^2$ and $N(x,y) = 2xy$ with $\partial_y M(x,y) = 2y = \partial_x N(x,y)$.

Example

The ODE $ydx + 2xdy = 0$ is *not* exact because $M(x,y) = y$ and $N(x,y) = 2x$ with $\partial_y M(x,y) = 1 \neq 2 = \partial_x N(x,y)$.

Example

The ODE $(2x\sin y + y^3e^x)dx + (x^2\cos y + 3y^2e^x)dy = 0$ is exact because

$$\partial_y M(x,y) = 2x\cos y + 3y^2e^x = \partial_x N(x,y).$$

Theorem

If the ODE (3.3) is exact in Ω then there is a one parameter family of solutions to (3.3) given by $F(x,y) = c$ where c is arbitrary constant and F is such that $\partial_x F = M$ and $\partial_y F = N$ in Ω.

Proof.

By the definition of exact ODE there is a F satisfying $\partial_x F = M$ and $\partial_y F = N$. Thus, the ODE becomes $dF(x,y) = 0$. Hence, $F(x,y) = c$ for arbitrary c are solutions. $\square$

How to find the Solution F?

We shall present two methods to identify F

- The "standard" method
- Method of grouping

Example (Standard Method)

Consider the ODE $(3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0$. It is exact because $M(x, y) = 3x^2 + 4xy$ and $N(x, y) = 2x^2 + 2y$ with $\partial_y M(x, y) = 4x = \partial_x N(x, y)$. We seek F such that

$$\partial_x F(x, y) = M(x, y) = 3x^2 + 4xy \text{ and } \partial_y F(x, y) = N(x, y) = 2x^2 + 2y.$$

From the first identity,

$$F(x, y) = \int M(x, y)\, dx + g(y) = x^3 + 2x^2 y + g(y).$$

Standard Method to find F

Example

From the second identity, we must have

$$2x^2 + 2y = 2x^2 + g'(y).$$

Therefore, $g'(y) = 2y$ and $g(y) = y^2 + c_0$. Thus, $F(x, y) = x^3 + 2x^2y + y^2 + c_0$. Since $F(x, y) = c$ is the one paramater family of solutions, we get

$$x^3 + 2x^2y + y^2 + c_0 = c$$

or, equivalently, $x^3 + 2x^2y + y^2 = c$ is the one parameter family of solutions.

Method of grouping to find F

Example

Consider the same ODE $(3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0$. We have already checked it is exact! We rewrite the ODE as follows:

$$\begin{aligned} 3x^2dx + (4xydx + 2x^2dy) + 2ydy &= 0 \\ d(x^3) + d(2x^2y) + d(y^2) &= d(c) \\ d(x^3 + 2x^2y + y^2) &= d(c). \end{aligned}$$

Thus, $x^3 + 2x^2y + y^2 = c$ is a family of solutions.

Exact IVP

Example

Consider the IVP

$$\begin{cases} (2x\cos y + 3x^2 y)dx + (x^3 - x^2 \sin y - y)dy & = 0 \\ y(0) & = 2. \end{cases}$$

The equation is exact in $\mathbb{R}^2$ because $\partial_y M(x,y) = -2x\sin y + 3x^2 = \partial_x N(x,y)$ in $\mathbb{R}^2$. We seek F such that

$$\partial_x F(x,y) = M(x,y) = 2x\cos y + 3x^2 y \text{ and}$$

$$\partial_y F(x,y) = N(x,y) = x^3 - x^2 \sin y - y.$$

From the first identity,

$$F(x,y) = \int M(x,y)\,dx + f(y) = x^2 \cos y + x^3 y + g(y).$$

Example

From the second identity, we must have

$$x^3 - x^2 \sin y - y = -x^2 \sin y + x^3 + g'(y).$$

Therefore, $g'(y) = -y$ and $g(y) = -\frac{y^2}{2} + c_0$. Thus, $F(x, y) = x^2 \cos y + x^3 y - \frac{y^2}{2} + c_0$. Since $F(x, y) = c$ is the one paramater family of solutions, we get

$$x^2 \cos y + x^3 y - \frac{y^2}{2} + c_0 = c$$

or, equivalently, $x^2 \cos y + x^3 y - \frac{y^2}{2} = c$ is the one parameter family of solutions. Now, using the initial value we get $c = -2$ and $x^2 \cos y + x^3 y - \frac{y^2}{2} + 2 = 0$ is a solution.

Same Example via Grouping Method

Example

Consider the same IVP

$$\begin{cases} (2x\cos y + 3x^2y)dx + (x^3 - x^2\sin y - y)dy &= 0 \\ y(0) &= 2. \end{cases}$$

We have already checked it is exact! We rewrite the ODE as follows:

$$\begin{aligned} (2x\cos y dx - x^2\sin y dy) + (3x^2 y dx + x^3 dy) - y dy = 0 \\ d(x^2\cos y) + d(x^3 y) - d\left(\frac{y^2}{2}\right) = d(c). \end{aligned}$$

Thus, $x^2\cos y + x^3 y - \frac{y^2}{2} = c$ is a family of solutions and using the initial condition we get $c = -2$.

Integrating Factor

Some non-exact ODEs can be transformed into an exact ODE using integrating factor!

Definition

Let (3.3) be not exact in Ω. If there exists a function $\mu : \Omega \to \mathbb{R}$ such that

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

is exact in Ω then μ is said to be an *integrating factor* of (3.3).

- Please note that multiplication (by integrating factor) may lead to gain/loss of solutions!
- In general, finding the integrating factor μ is not easy. (It appears as a solution of partial differential equation involving M and N).
- However, for some special ODEs computing integrating factor is easier!

Integrating Factor for some Non-Exact ODE

Theorem

Consider the ODE $M(x,y)dx + N(x,y)dy = 0$ and let

$$d(x,y) := \frac{\partial M}{\partial y}(x,y) - \frac{\partial N}{\partial x}(x,y).$$

1. *If $\frac{d(x,y)}{N(x,y)}$ depends only on x then $\exp\left(\int \frac{d(x,y)}{N(x,y)}\,dx\right)$ is an I.F. of (3.3).*
2. *If $\frac{-d(x,y)}{M(x,y)}$ depends only on y then $\exp\left(\int \frac{-d(x,y)}{M(x,y)}\,dy\right)$ is an I.F. of (3.3).*

It is possible that the situations are different from above possibilities in which case we seek other means!

Example

Consider the ODE $(2x^2 + y)dx + (x^2y - x)dy = 0$. This ODE is not separable, not homogeneous, not exact. Then $d(x, y) := 2(1 - xy)$. Now,

$$\frac{d}{N} = \frac{2(1 - xy)}{x^2y - x} = \frac{-2}{x}$$

is dependent only on x. Thus, the I.F. is

$$\exp\left(-2\int \frac{1}{x}\, dx\right) = \exp(-2\ln|x|) = \frac{1}{x^2}.$$

Multiplying the I.F. in the equation we get the exact equation $(2 + \frac{y}{x^2})dx + (y - \frac{1}{x})dy = 0$.

Linear First Order ODE

Definition

A first order ODE is *linear* in the y-variable (unknown) if it is of the form $P(x)y' + Q(x)y(x) = R(x)$.

- The differential form is: $[Q(x)y - R(x)]dx + P(x)dy = 0$.
- A linear ODE is exact iff $Q(x) = P'(x)$. In that case the ODE becomes $(P(x)y)' = R(x)$ which is already in the separable form.

Family of Solution to Linear FODE

Theorem

The linear ODE $y' + Q(x)y(x) = R(x)$ *has the integrating factor* $\mu(x,y) := e^{\int Q(x)\,dx}$ *and its one parameter family of solutions is*

$$y(x) = e^{-\int Q(x)\,dx}\left[\int e^{\int Q(x)\,dx} R(x)\,dx + c\right].$$

Proof.

The differential form of the given linear ODE is $[Q(x)y - R(x)]\,dx + dy = 0$. This is exact iff $Q \equiv 0$. If Q is non-zero then $d(x,y) = Q(x)$ and the ratio $\frac{d}{N} = Q$ depends only on x. By the theorem on I.F. for non-exact ODE its I.F. is $\mu(x) := \exp(\int Q)$ which depends only on x. With this choice of μ the linear ODE becomes $(\mu y)' = R(x)\mu$. Thus, the family of solutions as given in the statement is obtained. $\square$

Example

Example

- Consider $y' + \left(\frac{2x+1}{x}\right) y = e^{-2x}$.
- Its integrating factor is

$$\begin{aligned} \mu(x) &:= \exp\left[\int \left(\frac{2x+1}{x}\right)\, dx\right] = \exp(2x + \ln|x|) \\ &= e^{2x}e^{\ln|x|} = xe^{2x}. \end{aligned}$$

- Using the I.F. in the ODE we get $xe^{2x}y' + e^{2x}(2x+1)y = x$ or, equivalently, $(xe^{2x}y)' = x$.
- Thus, $xe^{2x}y - \frac{x^2}{2} = c$ is a family of solutions.

IVP Example

Example

- Consider the IVP

$$\begin{cases} y' + \left(\frac{4x}{x^2+1}\right) y & = \frac{x}{x^2+1} \\ y(2) & = 1. \end{cases}$$

- The I.F. is $\mu(x) = \exp\left(\int \frac{4x}{x^2+1}\, dx\right) = \exp[\ln(x^2+1)^2] = (x^2+1)^2$.
- Using the integrating factor in the ODE we get $[(x^2+1)^2 y]' = x^3 + x$.
- Thus, its family of solutions is $(x^2+1)^2 y = \frac{x^4}{4} + \frac{x^2}{2} + c$.
- Using the initial condition, we get $c = 19$. Thus, the solution is $(x^2+1)^2 y = \frac{x^4}{4} + \frac{x^2}{2} + 19$.

Interchange of x and y variable in differential form

Example

- Consider the ODE $y' = \frac{y^2}{1-3xy}$. This example does not fall in any of the category we have seen so far: it is not separable, not homogeneous, not exact and not linear.
- However, treating the y as independent variable and x as dependent variable, the ODE $x' = \frac{1-3xy}{y^2}$ is linear in x. (The $'$ now denotes the derivative with respect to y variable).
- The justification of this interchange is made via the differential form of the ODE.
- The I.F. for the linear ODE is $\mu(y) = \exp\left(\int \frac{3}{y}\,dy\right) = y^3$.
- Multiplying the I.F in the linear ODE we get $(y^3x)' = y$.
- Thus, the family of solutions is $y^3x - \frac{y^2}{2} = c$.

Bernoulli Equations

Definition

An equation of the form $y' + Q(x)y = R(x)y^{\alpha}$, for a given fixed real number α, is called a *Bernoulli* differential equation.

- For α equal to zero or one, the Bernoulli equation is linear.
- For other values of α the Bernoulli equation can be transformed to a linear equation

Transforming to Linear

Theorem

Let $\alpha \neq 0, 1$. The transformation $z = y^{1-\alpha}$ reduces the Bernoulli equation to a linear equation in z.

Proof.

- We first divide y^α to the Bernoulli equation to obtain $y^{-\alpha}y' + Q(x)y^{1-\alpha} = R(x)$.
- Set $z = y^{1-\alpha}$ then $z' = (1-\alpha)y^{-\alpha}y'$.
- Using this transformation in the ODE above, we get the linear ODE

$$z' + (1-\alpha)Q(x)z = (1-\alpha)R(x).$$

$\square$

Example

Example

- Consider the Bernoulli equation $y' + y = xy^3$.
- We divide by y^3 to obtain $y^{-3}y' + y^{-2} = x$.
- Set $z = y^{-2}$. Then $z' = -2y^{-3}y'$ and the transformed linear ODE obtained is $z' - 2z = -2x$.
- Its I.F. is $\mu(x) = e^{-2x}$.
- Using the I.F. in the linear ODE we get $(e^{-2x}z)' = -2xe^{-2x}$.
- Integrating both sides, $e^{-2x}z = \frac{1}{2}e^{-2x}(2x+1) + c$. Thus, $e^{-2x}y^{-2} = \frac{1}{2}e^{-2x}(2x+1) + c$.

Bernoulli Equation as a Special case

- Bernoulli equation is a special case of $g'(y)y' + Q(x)g(y) = R(x)$ with $g(y) = y^{1-\alpha}$.
- Set $z = g(y)$. Then $z' = g'(y)y'$.
- Thus, the above ODE is transformed to a linear ODE in the z as $z' + Q(x)z = R(x)$ which can be solved using I.F.

Example

- Consider $\cos y y' + x^{-1}\sin y = 1$ where $g(y) = \sin y$.
- Use $z = \sin y$. Then we obtain the linear ODE in z, $z' + x^{-1}z = 1$.
- Its I.F. is $\mu(x) = |x|$.
- For $x > 0$, $x \sin y - \frac{x^2}{2} = c$.

One degree Polynomial Coefficients

Theorem

Consider the ODE $(a_{11}x + a_{12}y + b_1)dx + (a_{21}x + a_{22}y + b_2)dy = 0$ *for given constants* a_{ij} *and* b_i *with* $i, j = 1, 2$. *Let* $A = (a_{ij})$ *be the* 2×2 *matrix.*

- *If* A *is invertible and* $(h, k) = A^{-1}(b_1, b_2)$ *then the transformation* $w = x + h$ *and* $z = y + k$ *reduces the ODE to a homogeneous ODE in* w *and* z *variable of the form*

$$(a_{11}w + a_{12}z)dw + (a_{21}w + a_{22}z)dz = 0.$$

- *If* A *is not invertible then the transformation* $z = a_{11}x + a_{12}y$ *reduces the ODE to a separable ODE in* z *and* x *variables.*

Example

- Consider the ODE $(x - 2y + 1)dx + (4x - 3y - 6)dy = 0$.
- Determinant of the matrix A is $5 \neq 0$. Then its inverse is

$$A^{-1} = \frac{1}{5} \begin{pmatrix} -3 & 2 \\ -4 & 1 \end{pmatrix}.$$

- Therefore $(h, k) = A^{-1}(1, -6) = (-3, -2)$. Thus, $w = x - 3$ and $z = y - 2$.
- Using this transformation in the ODE we obtain $(w - 2z)dw + (4w - 3z)dz = 0$or $z' = \frac{2z-w}{4w-3z}$. This is homogeneous!
- Use the transformation $z = \zeta w$ to get $\zeta + w\zeta' = \frac{2\zeta-1}{4-3\zeta}$.

Example

Example

- or, equivalently,
$$\frac{4-3\zeta}{3\zeta^2-2\zeta-1}d\zeta = \frac{dw}{w}.$$
- On integration, we obtain
$$\frac{3}{4}\ln\left|\frac{3\zeta-3}{3\zeta+1}\right| - \frac{1}{2}\ln|3\zeta^2-2\zeta-1| = \ln|w| + \ln|c_0|$$

  or, equivalently, $\ln\left|\frac{3\zeta-3}{3\zeta+1}\right|^3 - \ln|3\zeta^2-2\zeta-1|^2 = \ln|wc_0|^4$ or $\ln\left|\frac{\zeta-1}{(3\zeta+1)^5}\right| = \ln|wc_0|^4$ or $\left|\frac{\zeta-1}{w^4(3\zeta+1)^5}\right| = c_0^4$.
- Replacing $\zeta = z/w$, we get $\left|\frac{z-w}{(3z+w)^5}\right| = c_0^4$.
- Going back to x and y variables, we get $\left|\frac{y-x+1}{(3y+x-9)^5}\right| = c_0^4$.

Example

Example

- Consider the ODE $(x + 2y + 3)dx + (2x + 4y - 1)dy = 0$.
- Determinant of the matrix A is 0.
- The matrix is not invertible.
- Using the transformation $z = x + 2y$ in the ODE we obtain $(z + 3)dx + (2z - 1)\left(\frac{dz - dx}{2}\right) = 0$ or $7dx + (2z - 1)dz = 0.$ This is separable!
- Integrating, we get $7x + z^2 - z = c$.
- Going back to x and y variables, we get $7x + (x + 2y)^2 - (x + 2y) = c$ or $x^2 + 4xy + 4y^2 + 6x - 2y = c$.

- Observe that the solution $y : I \to \mathbb{R}$ of an ODE can be geometrically viewed as the curve $(x, y(x))$ (graph of y, at least locally) in $\mathbb{R}^2$.
- Thus, a one parameter family of solutions for a first order ODE $y' = f(x, y)$ will correspond to a one parameter family of curves in $\mathbb{R}^2$.
- These curves have the property that the slope of their tangents at (x_0, y_0) is $f(x_0, y_0)$ and such curves are called the *integral curves* of the ODE $y' = f(x, y)$.
- For instance, the family of parabolae $x^2 + c$ are the *integral curves* of the ODE $y' = 2x$.

Graphical Method: An approximate Method

- The family of solutions $F(x, y, c) = 0$ of $y' = f(x, y)$ is not always computable exactly.
- Note that $y' = f(x, y)$ gives the slope $f(x, y)$ of the integral curves at each point (x, y). For instance, the solution curve of $y' = 2x + y$ has slope 4 at $(1, 2)$ because $f(1, 2) = 4$.
- Thus, at $(1, 2)$ one can draw a small *line element* of slope 4. Doing this at all points gives the *direction field* of the ODE.
- Smooth curves drawn tangent to the line elements provides the approximate 'graph' of the solution curve.
- This process is tedious, time-consuming and does not provide analytical expressions for solutions!

Method of Isoclines

Definition

An *isocline* of the ODE $y' = f(x, y)$ is a curve along which the slope $f(x, y)$ has a constant value c.

For different values of c, the curves $f(x, y) = c$ are the family of isoclines of the ODE.

- For each fixed c, draw the line elements with slope c (or angle θ such that $\tan\theta = c$) at each (x, y) in the curve $f(x, y) = c$.
- Repeat the procedure for different values of c.
- Smooth curves drawn tangent to the line elements provides the approximate 'graph' of the solution curve.

Example

Consider the ODE $y' = x$. The isoclines of the ODE are the vertical lines $\{x = c\}$.

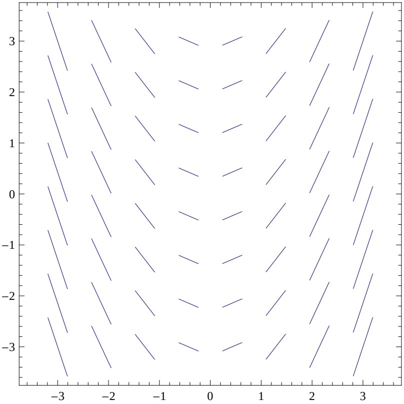


Figure: Line Elements (Image Courtesy: Prof. S. Ghorai)

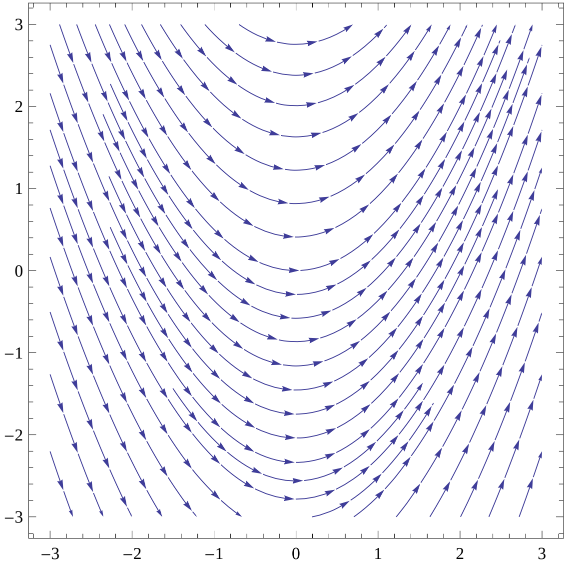


Figure: Approximate solution curves (Image Courtesy: Prof. S. Ghorai)

Application: Oblique and Orthogonal Trajectories

Definition

Let $F(x, y, c) = 0$ be a one parameter family of curves in $\mathbb{R}^2$. The *oblique trajectory* of the given family is any curve that intersects the curves in the given family at a constant angle θ. If $\theta = 90^\circ$ i.e. the intersection is at right angles, then the curve is said to be an *orthogonal trajectory* of the given family.

Example

Consider the family of circles $x^2 + y^2 = c^2$ with origin as centre and radius c. Then every straight line through the origin $y = mx$ is an orthogonal trajectory of the given family of circles. Conversly, each circle is an orthogonal trajectory of the family of straight lines passing through its centre.

Finding Oblique and Orthogonal Trajectories

- Given a family of curves $F(x,y,c) = 0$, differentiate w.r.t x and eliminate the constant c to obtain a first order ODE $y' = f(x,y)$.
- $y' = f(x,y)$ gives the slope $f(x,y)$ of the integral curves at (x,y). Thus, the angle of inclination of the tangent line is $\arctan[f(x,y)]$.
- The tangent line of the oblique trajectory intersecting the given family of curves at a constant angle θ will have the inclination $\arctan[f(x,y)] + \theta$ at (x,y).
- The slope of the tangent line of the oblique trajectory at (x,y) is given as

$$\tan\left[\arctan[f(x,y)] + \theta\right] = \frac{f(x,y) + \tan\theta}{1 - f(x,y)\tan\theta}.$$

- Solve the ODE $y' = \frac{f(x,y)+\tan\theta}{1-f(x,y)\tan\theta}$ to obtain the one parameter family of oblique curves, say $G(x,y,c) = 0$.
- Solve the ODE $y' = -\frac{1}{f(x,y)}$ to obtain the one parameter family of orthogonal curves, say $G(x,y,c) = 0$.

Example

Example

- Consider the family of circles $x^2 + y^2 = c^2$ centred at origin.
- Differentiating the family we get $x + yy' = 0$ or $y' = \frac{-x}{y}$.
- Then, the ODE for the orthogonal family is given by $y' = \frac{y}{x}$ which is separable.
- Solving, we get $y = mx$ a one parameter family of orthogonal curves.
- *Caution*: The orthogonal (vertical) line $\{x = 0\}$ is not obtained in the above family of curves and should be observed by inspection.

Example

Example

- Consider the family of parabolae $y = cx^2$.
- Differentiating the family we get $y' = 2cx$. Eliminating the parameter c between the ODE and the family of curves, we get $y' = \frac{2y}{x}$.
- Then, the ODE for the orthogonal family is given by $y' = -\frac{x}{2y}$ which is separable.
- Solving, we get $2y^2 + x^2 = b^2$ a one parameter family of ellipses centred at origin with x-axis as the major axis.

Example: Oblique Trajectory

Example

- Consider the family of straight lines $y = mx$. Let us find the oblique trajectory intersecting the given family at constant angle of 45°.
- Differentiating the family we get $y' = m$. Eliminating the parameter m between the ODE and the family of curves, we get $y' = \frac{y}{x}$.
- Now,
$$\frac{f(x,y) + \tan 45^\circ}{1 - f(x,y)\tan 45^\circ} = \frac{\frac{y}{x} + 1}{1 - \frac{y}{x}} = \frac{x+y}{x-y}.$$
- Then, the ODE for the oblique family with angle 45° is given by $y' = \frac{x+y}{x-y}$ which is homogeneous.
- Setting, $y = zx$, we get $z + xz' = \frac{1+z}{1-z}$.

Example: Oblique Trajectory

Example

- Thus, solving the separable ODE $\frac{(z-1)}{z^2+1}dz = -\frac{dx}{x}$, we get $\frac{1}{2}\ln(z^2+1) - \arctan z = -\ln|x| - \ln|c|$ or $\ln c^2(x^2+y^2) - 2\arctan(y/x) = 0$ is a one parameter family of oblique trajectories.

Topics to be covered in Higher order

- Linear Independence of Solutions.
- Reduction Order technique (Linearly independent solution from a given solution).
- Linear homogeneous with constant coefficients ODE
- Linear inhomogeneous with constant coefficients ODE (Method of Undetermined Coefficients)
- Linear ODE with variable coefficients (Method of Variation of Parameter)
- Power Series Solution

k-th order to System of first order ODE

- Consider a k-th order ODE of the form $y^{(k)} = f(x, y, y', \dots y^{(k-1)})$
- For $1 \leq i \leq k$, introduce k unknowns $y_i := y^{(i-1)}$ where $y = y_1$.
- We have the system of k first order ODEs

$$\begin{cases} y_i' &= y_{i+1} \, (1 \leq i \leq k-1) \\ y_k' &= f(x, y_1, y_2, \dots, y_{(k-1)}). \end{cases}$$

- Thus, the existence and uniqueness queries for above k-th order ODE is equivalent to similar queries posed for first order system of ODE.
- Picard's theorems on existence and uniqueness can be extended to first order system of ODEs — a topic beyond the purview of this course.

Existence and Uniqueness for k-th order Linear ODE

Theorem

Consider the linear k-th order IVP

$$\begin{cases} \sum_{i=0}^{k} a_i(x) y^{(i)}(x) &= f(x) \quad I \subset \mathbb{R} \\ y^{(j)}(x_0) &= y_j \quad \text{for } 0 \leq j \leq k-1 \end{cases}$$

for a given $x_0 \in \bar{I}$ and $y_j \in \mathbb{R}$ such that $a_k(x) \neq 0$ for all $x \in I$. If $a_i, f : \bar{I} \to \mathbb{R}$ are continuous, for all $0 \leq i \leq k$, then the IVP admits a unique solution in $\bar{I}$.

- If $f \equiv 0$ and $y_j = 0$, for all $0 \leq j \leq k-1$, then zero function is the unique solution.
- Denoting $Ly(x) := \sum_{i=0}^{k} a_i(x) y^{(i)}(x)$, one can view $L : C(\bar{I}) \to C(\bar{I})$ as a linear map where $C(\bar{I})$ is the set of all continuous real-valued functions on $\bar{I}$.
- Show that $C(\bar{I})$ forms a vector space over $\mathbb{R}$ (Exercise!).

Null Space of the Linear ODE

Definition

The null space of L, denoted as $N(L)$, is defined as the class of all functions $y \in C^k(\bar{I})$ such that $Ly = 0$ where $C^k(\bar{I})$ is the set of all k-times differentiable functions on I such that the k-th derivative can be continuously extended to the end-points of I.

Example

All constant functions in $C(\bar{I})$ is in the null space of the first order linear $L := \frac{d}{dx}$.

Example

All linear functions of the form $ax + b$ in $C(\bar{I})$ is in the null space of the second order linear $L := \frac{d^2}{dx^2}$.

Null Space is a subspace

Theorem

The null space, $N(L)$, is a subspace of $C(\bar{I})$.

Proof.

Obviously, the constant function zero is in $N(L)$. Further, if $u_1, \ldots, u_m$ are in $N(L)$,i.e. $Lu_i = 0$ for $1 \leq i \leq m$.Then

$$L(\alpha_1 u_1 + \ldots + \alpha_m u_m) = \alpha_1 Lu_1 + \ldots + \alpha_m Lu_m = 0.$$

Thus, the linear combination of any solutions in $N(L)$ is also in $N(L)$. □

Example

Example

Consider the second order differential operator $Ly := y'' + y$. Observe that $\sin x$ and $\cos x$ are in the null space of L. Also, any linear combination $\alpha \sin x + \beta \cos x$ is also in the null space of L, for any choice of $\alpha, \beta \in \mathbb{R}$.

Example

Consider the third order differential operator $Ly := y^{(3)} - 2y'' - y' + 2y$. Observe that e^x, e^{-x} and e^{2x} are in the null space of L. Also, any linear combination $\alpha_1 e^x + \alpha_2 e^{-x} + \alpha_3 e^{2x}$ is in the null space of L, for any choice of $\alpha_i \in \mathbb{R}$, for all $i = 1, 2, 3$.

Linear Dependence and Independence of Solutions

Definition

A collection of functions $u_1, \ldots, u_m$ are said to be *linearly independent* on $\bar{I}$ if

$$\alpha_1 u_1 + \ldots + \alpha_m u_m = 0$$

implies $\alpha_i = 0$ for all i. Otherwise, the collection is said to be *linearly dependent*.

Example

The two functions x and $2x$ are linearly dependent on $[0, 1]$ because $\alpha_1 x + 2\alpha_2 x = 0$ for the choice $\alpha_1 = 2$ and $\alpha_2 = -1$.

Example

Example

The three functions $\sin x$, $3\sin x$ and $-\sin x$ are linearly dependent on $[-1,2]$ because $\alpha_1 \sin x + 3\alpha_2 \sin x - \alpha_3 \sin x = 0$ for the choice $\alpha_1 = \alpha_2 = 1$ and $\alpha_3 = 4$.

Example

The two functions x and x^2 are linearly independent on $[0,1]$. For some α_1 and α_2, consider $\alpha_1 x + \alpha_2 x^2 = 0$. Differentiate w.r.t x to obtain $\alpha_1 + 2\alpha_2 x = 0$ for all $x \in [0,1]$. Then $\alpha_1 x + 2\alpha_2 x^2 = 0$ for all $x \in [0,1]$. Now solving for α_1 and α_2 in the two equations $\alpha_1 x + \alpha_2 x^2 = 0$ and $\alpha_1 x + 2\alpha_2 x^2 = 0$, we get $\alpha_2 = 0 = \alpha_1$.

Dimension of the Null Space $N(L)$

Theorem

Let $Ly(x) := \sum_{i=0}^{k} a_i(x)y^{(i)}(x)$ *be the* k*-th order linear differential map* $L : C(\bar{I}) \to C(\bar{I})$. *Then its null space* $N(L)$ *is a* k*-dimensional subspace of* $C(\bar{I})$*, i.e. there exists* k *linearly independent solutions to* $Ly = 0$ *and every other solution can be expressed as a linear combination of the* k *solutions with appropriate choice of constants.*

Definition

The general solution of the k-th order linear, homogeneous ODE $Ly = 0$ is the linear combination of k linearly independent solutions of $Ly = 0$.

Example

Example

For the second order differential operator $Ly := y'' + y$, $\sin x$ and $\cos x$ are two linearly independent solutions and its general soluton is given as the linear combination $y(x) := \alpha \sin x + \beta \cos x$.

Example

For the third order differential operator $Ly := y^{(3)} - 2y'' - y' + 2y$, e^x, e^{-x} and e^{2x} are three linearly independent solutions of $Ly = 0$. Thus, its general solution is the linear combination $y(x) := \alpha_1 e^x + \alpha_2 e^{-x} + \alpha_3 e^{2x}$.

Wronskian

Definition

For any collection of k functions $u_1, \ldots, u_k$ in $C^{(k-1)}(\bar{I})$, the *Wronskian* of the collection, denoted as $W(u_1, \ldots, u_k)$, is defined as the determinant of the $k \times k$ matrix

$$W(u_1, \ldots, u_k)(x) := \begin{vmatrix} u_1(x) & \ldots & u_k(x) \\ u_1'(x) & \ldots & u_k'(x) \\ \vdots & \ddots & \vdots \\ u_1^{(k-1)}(x) & \ldots & u_k^{(k-1)}(x). \end{vmatrix}$$

The Wronskian can be viewed as a map $W(u_1, \ldots, u_k) : I \to \mathbb{R}$.

Sufficiency Condition for Linear Independence

Theorem

If u_1 and u_2 are linearly dependent in I then $W(u_1, u_2)(x) = 0$ for all $x \in I$. Equivalently, if $W(u_1, u_2) \neq 0$ for some $x \in I$ then u_1 and u_2 are linearly independent.

Proof.

Suppose u_1 and u_2 are linearly dependent in I then there exists a non-zero pair (α, β) such that $\alpha u_1(x) + \beta u_2(x) = 0$. On differentiation, we have the matrix equation

$$\begin{pmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since we have a non-zero pair (α, β) as a solution, the determinant of the matrix should be zero for all $x \in I$. $\square$

Converse Not True

Example

- Consider $u_1 = x|x|$ and $u_2 = x^2$
- with derivative $u_1' = 2|x|$ and $u_2' = 2x$.
- Its Wronskian is zero everywhere.
- But u_1 and u_2 are linearly independent
- because $\alpha x|x| + \beta x^2 = 0$ implies that
- for $x < 0$, we get $\beta - \alpha = 0$
- and for $x > 0$, we get $\alpha + \beta = 0$
- implying that both $\alpha = \beta = 0$.

Converse is True for Linearly Independent Solutions

Theorem

The k solutions $u_1, \dots, u_k$ of the k-order linear, homogeneous ODE $Ly = 0$ in I is linearly dependent in I iff $W(u_1, \dots, u_k)(x) = 0$ for some $x \in I$.

second order proof.

One way is already proved in Theorem 90! Conversely, suppose $W(u_1, , u_2)(x_0) = 0$ for some $x_0 \in I$. Consider the equation $\alpha u_1(x_0) + \beta u_2(x_0) = 0$. We claim that at least either of α or β is non-zero. On differentiation we have the additional equation $\alpha u_1'(x_0) + \beta u_2'(x_0) = 0$. Since $W(u_1, , u_2)(x_0) = 0$, we have a non-zero pair (α, β). Set $y(x) = \alpha u_1(x) + \beta u_2(x)$. Then $Ly = 0$ and $y(x_0) = y'(x_0) = 0$. By the uniqueness of IVP, we have $y \equiv 0$ for non-zero pair (α, β) giving the linear dependence of u_1 and u_2. □

Note that for linearly independent solutions of the linear ODE the Wronskian is non-zero everywhere.

Vanishing Property of Wronskian

Theorem

If $u_1, \ldots, u_k$ are solutions of the k-order linear, homogeneous ODE $Ly = 0$ in I then the Wronskian $W(u_1, \ldots, u_k)$ is either identically zero on I or is never zero on I.

Proof.

If the Wronskian vanishes at some point $x_0 \in I$, then the proof of Theorem 92 implied that u_1 and u_2 are linearly dependent. But Theorem 90 implied that for linear dependent functions the Wronskian vanishes everywhere! $\square$

Example

The Wronskian of $u_1(x) := x$ and $u_2(x) := \sin x$ is $x \cos x - \sin x$. This Wronskian is non-zero, for instance at $x = \pi$, thus the functions are linearly independent by Theorem 90. However, the Wronskian is zero at $x = 0$.It is no contradiction to above theorem because x and $\sin x$ cannot span solutions of a second order ODE (see assignment problem)!

Example

For the second order differential operator $Ly := y'' + y$, $\sin x$ and $\cos x$ are two linearly independent solutions because its Wronskian

$$W(\sin x, \cos x)(x) := \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1 \neq 0$$

for all $x \in \mathbb{R}$.

Example

Example

For the third order differential operator $Ly := y^{(3)} - 2y'' - y' + 2y$, e^x, e^{-x} and e^{2x} are three linearly independent solutions because its Wronskian

$$\begin{aligned} W(e^x, e^{-x}, e^{2x})(x) &:= \begin{vmatrix} e^x & e^{-x} & e^{2x} \\ e^x & -e^{-x} & 2e^{2x} \\ e^x & e^{-x} & 4e^{2x} \end{vmatrix} = e^{2x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{vmatrix} \\ &= -6e^{2x} \neq 0 \end{aligned}$$

for all $x \in \mathbb{R}$.

Reduction of Order

This method is used to find a linearly independent solution corresponding to a given solution.

Theorem

Let u_1 be a non-trivial solution of the second order homogeneous linear ODE $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$. Then $u_2 := u_1(x)v(x)$ is a linearly independent solution where

$$v(x) := \int \frac{1}{u_1^2(x)} \exp\left[-\int \frac{a_1(x)}{a_2(x)}\,dx\right] dx. \tag{5.1}$$

Proof: Set $u_2(x) := u_1(x)v(x)$ where v shall be chosen appropriately. Then by product rule we have

$$u_2' = u_1 v' + u_1' v \text{ and } u_2'' = u_1 v'' + 2u_1' v' + u_1'' v.$$

Proof

Since u_2 is required to be a solution to the given ODE, we consider

$$\begin{aligned} 0 &= a_2(x)u_2'' + a_1(x)u_2' + a_0(x)u_2 \\ &= a_2 u_1 v'' + (2a_2 u_1' + a_1 u_1)v' + (a_2 u_1'' + a_1 u_1' + a_0 u_1)v. \end{aligned}$$

The coefficient of v is zero because u_1 is given to be a solution. Thus, if v is chosen as the solution to the ODE

$$a_2 u_1 v'' + (2a_2 u_1' + a_1 u_1)v' = 0 \text{ or } \frac{dv'}{v'} = -\left(2\frac{u_1'}{u_1} + \frac{a_1}{a_2}\right)$$

then u_2 is a solution to the given ODE. Then v is as given in (5.1). Now, $u_1$and u_2 are linearly independent because

$$W(u_1, u_2) = \begin{vmatrix} u_1 & u_1 v \\ u_1' & u_1 v' + u_1' v \end{vmatrix} = u_1^2(x)v'(x) = \exp\left[-\int \frac{a_1(x)}{a_2(x)}\,dx\right] \neq 0.$$

Example

Example

Consider the ODE $(x^2+1)y'' - 2xy' + 2y = 0$. Note that $u_1(x) = x$ is a solution to the ODE. Now

$$v(x) = \int \frac{1}{x^2} \exp\left[-\int \frac{-2x}{x^2+1}\,dx\right] dx = \int \frac{x^2+1}{x^2}\,dx = x - \frac{1}{x}.$$

Then $u_2(x) := x(x - x^{-1}) = x^2 - 1$ and the general solution is $y(x) = \alpha_1 x + \alpha_2(x^2-1)$.

Theorem

A linear, homogeneous k-th order ODE with constant coefficients is given by $Ly := \sum_{i=0}^{k} a_i y^{(i)} = 0$ *and its characteristic equation (CE) is the k-th degree polynomial of m,* $\sum_{i=0}^{k} a_i m^{(k)} = 0$.

- *If (CE) admits k distinct real roots* $\{m_i\}_1^k$ *then* $y(x) := \sum_{i=1}^{k} \alpha_i e^{m_i x}$ *is a general solution of the ODE* $Ly = 0$.
- *If (CE) admits* ℓ *repeated real roots m and the rest are distinct then* $y(x) := (\sum_{i=1}^{\ell} \alpha_i x^{i-1}) e^{mx} + \sum_{i=\ell+1}^{k} \alpha_i e^{m_i x}$.
- *If (CE) admits non-repeated pair of complex roots* $a \pm \imath b$ *the rest are distinct real roots then* $y(x) := e^{ax}(\alpha_1 \sin bx + \alpha_2 \cos bx) + \sum_{i=3}^{k} \alpha_i e^{m_i x}$.
- *If (CE) admits repeated pair of* ℓ *complex roots* $a \pm \imath b$ *then the corresponding part of general solution is written as* $e^{ax}[\sum_{i=1}^{\ell} \alpha_i x^{i-1} \sin bx + \sum_{i=1}^{\ell} \alpha_{\ell+i} x^{i-1} \cos bx]$.

Proof of Distinct Real Roots

Proof.

- Observe that the derivative of exponential is constant multiple of itself, i.e. $\frac{d^k}{dx^k}[e^{cx}] = c^k e^{cx}$.
- Thus, it is natural to expect the solution of constant coefficients linear ODE in the form of exponential.
- Let $y(x) = e^{mx}$ be a solution of the ODE where m is to be determined appropriately.
- Using it in the ODE, we get $(\sum_{i=0}^{k} a_i m^i)e^{mx} = 0$.
- Thus, the possible choices of m are the roots of the k degree polynomial and the general solution is the linear combination of the linearly independent solutions.

$\square$

Example

- Consider $y'' - 3y' + 2y = 0$.
- The CE is $m^2 - 3m + 2 = 0$ whose roots are $m_1 = 1$ and $m_2 = 2$.
- The roots are real and distinct.
- The corresponding solutions are e^x and e^{2x}.
- They are linearly independent because

$$W(e^x, e^{2x}) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x} \neq 0.$$

- Thus, the general solution is $y(x) := \alpha_1 e^x + \alpha_2 e^{2x}$.

Example

- Consider $y^{(3)} - 4y'' + y' + 6y = 0$.
- The CE is $m^3 - 4m^2 + m + 6 = 0$ whose roots are $m_1 = -1$, $m_2 = 2$ and $m_3 = 3$.
- The roots are real and distinct.
- The corresponding solutions are e^{-x}, e^{2x} and e^{3x}.
- They are linearly independent (exercise!).
- Thus, the general solution is $y(x) := \alpha_1 e^{-x} + \alpha_2 e^{2x} + \alpha_3 e^{3x}$.

Proof of Repeated Real Roots

Proof.

- If m is a repeated root, say twice, then e^{mx} is one solution.
- Now, using the existing solution $u_1 := e^{mx}$, we seek a linearly independent solution $e^{mx}v(x)$ for a suitable choice of v.
- It can be shown that $v(x) = x$ and xe^{mx} is a linearly independent solution.
- For three repeated roots we have e^{mx}, xe^{mx} and x^2e^{mx} are the linearly independent solutions.

$\square$

Example

Example

- Consider the ODE $y'' - 6y' + 9y = 0$.
- The CE is $m^2 - 6m + 9 = 0$ with two repeated roots $m_1 = m_2 = 3$.
- The corresponding solution is e^{3x}.
- A linear independent solution is xe^{3x} and the general solution is $y(x) := (\alpha_1 + \alpha_2 x)e^{3x}$.

Example

Example

- Consider $y^{(3)} - 4y'' - 3y' + 18y = 0$.
- The CE is $m^3 - 4m^2 - 3m + 18 = 0$ whose roots are $m_1 = m_2 = 3$ and $m_3 = -2$.
- The two of the roots are repeated.
- The corresponding solutions are e^{3x} and e^{-2x}.
- Thus, the general solution is $y(x) := (\alpha_1 + \alpha_2 x)e^{3x} + \alpha_3 e^{-2x}$.

Example

Example

- Consider $y^{(4)} - 5y^{(3)} + 6y'' + 4y' - 8y = 0$.
- The CE is $m^4 - 5m^3 + 6m^2 + 4m - 8 = 0$ whose roots are $m_1 = m_2 = m_3 = 2$ and $m_4 = -1$.
- Some roots are repeated!
- Thus, the general solution is $y(x) := (\alpha_1 + \alpha_2 x + \alpha_3 x^2)e^{2x} + \alpha_4 e^{-x}$.

Proof of Conjugate Complex Roots

Proof.

- If $a \pm \imath b$ are the conjugate pair of complex roots then $e^{(a+\imath b)x} = e^{ax}(\cos bx + \imath \sin bx)$ and $e^{(a-\imath b)x} = e^{ax}(\cos bx - \imath \sin bx)$ are complex solutions of the ODE.
- We seek real linearly independent solutions.
- We sum the above two complex solutions and divide by 2 to obtain $e^{ax} \cos bx$.
- Similarly, on subtraction and dividing by $2\imath$, we get $e^{ax} \sin bx$.
- Both are real linearly independent solutions and the general solution is $e^{ax} [\alpha_1 \sin bx + \alpha_2 \cos bx]$.

□

Example

- Consider the ODE $y'' + y = 0$.
- The CE is $m^2 + 1 = 0$ with pair of conjugate complex roots $m_1 = \imath$ and $m_2 = -\imath$.
- The general solution is $y(x) := \alpha_1 \sin x + \alpha_2 \cos x$.

Example

- Consider $y'' - 6y' + 25y = 0$.
- The CE is $m^2 - 6m + 25 = 0$ whose roots are $m_1 = 3 + \imath 4$ and $m_2 = 3 - \imath 4$.
- Thus, the general solution is $y(x) := e^{3x} (\alpha_1 \sin 4x + \alpha_2 \cos 4x)$.

Example

- Consider $y^{(4)} - 4y^{(3)} + 14y'' - 20y' + 25y = 0$.
- The CE is $m^4 - 4m^3 + 14m^2 - 20m + 25 = 0$ whose roots are $m_1 = m_2 = 1 + \imath 2$ and $m_3 = m_4 = 1 - \imath 2$.
- The complex pair of roots are repeated.
- Thus, the general solution is

$$y(x) := e^x \left[(\alpha_1 + \alpha_2 x) \sin 2x + (\alpha_3 + \alpha_4 x) \cos 2x \right].$$

Example IVP: Linear Homogeneous

Example

- Consider the IVP

$$\begin{cases} y'' - 6y' + 25y & = 0 \\ y(0) & = -3 \\ y'(0) & = -1. \end{cases}$$

- The CE is $m^2 - 6m + 25 = 0$ whose roots are $m_1 = 3 + \imath 4$ and $m_2 = 3 - \imath 4$.
- The general solution is $y(x) := e^{3x}(\alpha_1 \sin 4x + \alpha_2 \cos 4x)$.
- Then $y'(x) = e^{3x}\left[(3\alpha_1 - 4\alpha_2)\sin 4x + (4\alpha_1 + 3\alpha_2)\cos 4x\right]$.
- Using the initial condition $y(0) = -3$, we get $\alpha_2 = -3$.
- Similarly, using $y'(0) = -1$, we get $\alpha_1 = 2$.
- Thus the solution is $y(x) := e^{3x}(2\sin 4x - 3\cos 4x)$.

Linear, Non-Homogeneous ODE

Theorem

A linear, non-homogeneous k-th order ODE is given as $Ly := \sum_{i=0}^{k} a_i(x)y^{(k)} = f(x)$. If y_p is a given particular integral (not involving arbitrary constants) of the non-homogeneous ODE then $y_c + y_p$ is the general solution of the non-homogeneous ODE where y_c (called the complementary function) is the general solution of the corresponding homogeneous ODE ($f \equiv 0$).

Proof.

It is given as third problem in tenth assignment! Any element of $N(L)$ is denoted as y_c. Then the general solution of $Ly = f$ is $y_c + y_p$, translation of the null space $N(L)$ by y_p. □

Example

Example

Consider the non-homogeneous ODE $y'' + y = x$. Its complementary function is the solution corresponding to the homogeneous ODE $y'' + y = 0$, i.e. $y_c = \alpha_1 \sin x + \alpha_2 \cos x$. A particular integral to $y'' + y = x$ is $y_p(x) = x$. Then $y(x) = \alpha_1 \sin x + \alpha_2 \cos x + x$ is the general solution for the non-homogeneous ODE.

Linear, Non-homogeneous, Constant Coefficients

- We have already seen that the general solution of a k-th order ODE $Ly := \sum_{i=0}^{k} a_i y^{(i)} = f(x)$ is the sum of the general solution of its corresponding homogeneous equation (complementary function) and a particular integral of the non-homogeneous.
- Thus, to find the general solution of non-homogeneous it is enough to find a particular integral of the non-homogeneous.
- The **method of undetermined coefficients** facilitates in finding a particular integral for special cases of f.
- The special classes of f are x^n, e^{mx}, $\sin(ax+b)$, $\cos(ax+b)$, its point-wise product and linear combinations.

Annihilator

Definition

An *annihilator* of a function f is the differential map A such that $Af = 0$.

We introduce the symbol $D^k := \frac{d^k}{dx^k}$, for any positive integer k. The symbol x in the denominator denotes the independent variable.

Example

- For $f := e^{mx}$ the annihilator is $D - m$.
- For $f := x^n$ the annihilator D^{n+1}.
- For $f := \sin(ax + b)$ the annihilator is $D^2 + a^2$. Same for $f := \cos(ax + b)$.
- For the product $f := x^n e^{mx}$ the annihilator is $(D - m)^{n+1}$. For product of functions annihilators composes!

Example

- For $f := x^n \sin(ax+b)$ the annihilator is $(D^2+a^2)^{n+1}$. Same for $f := x^n \cos(ax+b)$.
- For $f := e^{mx} \sin(ax+b)$ the annihilator is $(D-m)^2+a^2$. Same for $f := e^{mx} \cos(ax+b)$. Product by exponential gives translation in differential operator D.
- For $f := x^n e^{mx} \sin(ax+b)$ the annihilator is $[(D-m)^2+a^2]^{n+1}$.

Method of Undetermined Coefficients

- Given an inhomogeneous ODE $Ly = f$, if f admits an annihilator A, i.e. $Af = 0$ then we have the 'new' homogeneous ODE $(A \circ L)y = 0$, possibly of higher order than L. We illustrate this for second order linear ODE.
- Consider the linear, constant coefficient, second order, inhomogeneous ODE $(a_2 D^2 + a_1 D + a_0)y = f$.
- If A is an annihilator of f then we get 'new' homogeneous the $A(a_2 D^2 + a_1 D + a_0)y = 0$.
- Thus, y is obtained as a linear combination of linearly independent solutions of $A(a_2 D^2 + a_1 D + a_0)$ subject to its solvability!

Exponentials in RHS

- Consider the ODE $a_2y'' + a_1y' + a_0y = e^{mx}$ where $f(x) = e^{mx}$.
- Applying the annihilator of f on both sides of the ODE, we get a homogeneous, constant coefficient ODE, i.e. $(D - m)(a_2D^2 + a_1D + a_0)y = 0$.
- The CE of the 'new' homogeneous equation is $(\mu - m)(a_2\mu^2 + a_1\mu + a_0) = 0$ with three roots, i.e. the roots m_1 and m_2 of the original CE and the additional root m.
- If m is distinct from m_1, m_2 then $y = y_c + y_p$ where y_c (which depends on m_1 and m_2) is the solution of homogeneous ODE and $y_p = Ae^{mx}$ where A is to be determined and called the *undetermined coefficient*.
- If m is equal to one of the m_1 and m_2 then $y = y_c + y_p$ where $y_p = Axe^{mx}$ where A is to be determined.
- Similarly, for each case!
- To obtain A in y_p use the y in the given ODE and equate like variables both sides.

Example

Example

- Consider the ODE $y'' - 2y' - 3y = 2e^{4x}$ with $f(x) = 2e^{4x}$.
- The annihilator of $2e^{4x}$ is $2(D-4)$ and the 'new' homogeneous ODE is $2(D-4)(D^2-2D-3)y = 0$.
- The roots are $3, -1$ and 4 and $y(x) := y_c + \alpha_3 e^{4x}$ where $y_c := \alpha_1 e^{3x} + \alpha_2 e^{-x}$ is the general solution of the original CE.
- Using the y in the given ODE, we get $(D^2 - 2D - 3)(y_c + \alpha_3 e^{4x}) = 2e^{4x}$ or $(D^2 - 2D - 3)\alpha_3 e^{4x} = 2e^{4x}$.
- We get $(16 - 8 - 3)\alpha_3 e^{4x} = 2e^{4x}$ or $\alpha_3 = \frac{2}{5}$.
- Thus, $y_p(x) = \frac{2}{5}e^{4x}$ and the general solution is $y(x) := \alpha_1 e^{3x} + \alpha_2 e^{-x} + \frac{2}{5}e^{4x}$.

Example

- Consider the ODE $y'' - 2y' - 3y = 2e^{3x}$ with $f(x) = 2e^{3x}$.
- The annihilator of $2e^{3x}$ is $2(D-3)$ and the 'new' homogeneous ODE is $2(D-3)(D^2-2D-3)y = 0$.
- The roots are $3, -1$ and 3 with the 'new' root same as one of the roots of the original CE.
- Thus, $y(x) := y_c + \alpha_3 x e^{3x}$ where $y_c := \alpha_1 e^{3x} + \alpha_2 e^{-x}$ is the general solution of the original CE.
- Using the y in the given ODE, we get $(D^2-2D-3)(y_c + \alpha_3 x e^{3x}) = 2e^{3x}$ or $(D^2-2D-3)\alpha_3 x e^{3x} = 2e^{3x}$.
- We get $[3(3x+2) - 6x - 2 - 3x]\,\alpha_3 e^{3x} = 2e^{3x}$ or $\alpha_3 = \frac{1}{2}$.
- Thus, $y_p(x) = \frac{x}{2}e^{3x}$ and the general solution is $y(x) := \alpha_1 e^{3x} + \alpha_2 e^{-x} + \frac{x}{2}e^{3x}$.

Example

- Consider the ODE $y'' - 3y' + 2y = x^2e^x$ with $f(x) = x^2e^x$.
- The annihilator of x^2e^x is $(D-1)^3$ and the 'new' homogeneous ODE is $(D-1)^3(D^2 - 3D + 2)y = 0$.
- The roots are $1, 2$ and three repeated roots 1 with the 'new' root same as one of the roots of the original CE.
- Thus, $y(x) := y_c + (\alpha_3 x + \alpha_4 x^2 + \alpha_5 x^3)e^x$ where $y_c := \alpha_1 e^x + \alpha_2 e^{2x}$ is the general solution of the original CE.
- Using the y in the given ODE, we get $(D^2 - 3D + 2)(\alpha_3 x + \alpha_4 x^2 + \alpha_5 x^3)e^x = x^2e^x$.
- We get $-3\alpha_5 = 1$, $6\alpha_5 - 2\alpha_4 = 0$ and $2\alpha_4 - \alpha_3 = 0$ or $\alpha_5 = \frac{-1}{3}$, $\alpha_4 = -1$ and $\alpha_3 = -2$.
- Thus, $y_p(x) = (-2x - x^2 - \frac{1}{3}x^3)e^x$ and the general solution is $y(x) := \alpha_1 e^x + \alpha_2 e^{2x} + y_p(x)$.

Example

Example

- Consider the ODE $y'' - 2y' + y = x^2e^x$ with again $f(x) = x^2e^x$.
- The annihilator of x^2e^x is $(D-1)^3$ and the 'new' homogeneous ODE is $(D-1)^3(D^2-2D+1)y = 0$.
- The roots are five 1 with the 'new' roots same as the roots of the original CE.
- Thus, $y(x) := y_c + (\alpha_3x^2 + \alpha_4x^3 + \alpha_5x^4)e^x$ where $y_c := (\alpha_1 + \alpha_2x)e^x$ is the general solution of the original CE.
- Using the y in the given ODE, we get $(D^2-2D+1)(\alpha_3x^2 + \alpha_4x^3 + \alpha_5x^4)e^x = x^2e^x$.
- We get $2\alpha_3 = 0$, $6\alpha_4 = 0$ and $12\alpha_5 = 1$ or $\alpha_5 = \frac{1}{12}$, $\alpha_4 = 0 = \alpha_3$.
- Thus, $y_p(x) = \frac{1}{12}x^4e^x$ and the general solution is $y(x) := (\alpha_1 + \alpha_2x)e^x + \frac{1}{12}x^4e^x$.

Sine/Cosine in RHS

Example

- Consider the ODE $a_2y'' + a_1y' + a_0y = \sin mx$ where $f(x) = \sin mx$.
- Using the annihilator of f, we get a homogeneous, constant coefficient ODE $(D^2 + m^2)(a_2D^2 + a_1D + a_0)y = 0$.
- The CE of the 'new' homogeneous equation is $(\mu^2 + m^2)(a_2\mu^2 + a_1\mu + a_0) = 0$ with four roots, i.e. the roots m_1 and m_2 of the original CE and the additional complex roots $\pm\imath m$.
- If the complex roots are distinct from m_i then $y = y_c + y_p$ where $y_p(x) := A\sin mx + B\cos mx$ where A and B are to be determined.
- If the complex roots are equal to m_i's then $y = y_c + y_p$ where $y_p(x) := x(A\sin mx + B\cos mx)$ where A and B are to be determined.
- Similarly, for each case!
- To obtain A and B in y_p, use the y in the given ODE and equate like variables both sides.

Example

Example

- Consider the ODE $y'' - 2y' - 3y = 2e^x - 10\sin x$ with $f(x) = 2e^x - 10\sin x$, a linear combination of exponential and sine.
- The corresponding CE is $m^2 - 2m - 3 = 0$ with roots $m_1 = 3$ and $m_2 = -1$ and the complementary function is $y_c(x) := \alpha_1 e^{3x} + \alpha_2 e^{-x}$.
- The roots of the CE corresponding to the annihilators $2(D-1)$ and $-10(D^2+1)$ are $m_3 = 1$ and $m_4 = \pm\imath$ which are distinct from m_1 and m_2. That is, the functions appearing in f, both e^x and $\sin x$ are linearly independent from the solutions of the homogeneous equation.
- Thus, $y_p(x) := \alpha_3 e^x + \alpha_4 \sin x + \alpha_5 \cos x$.
- Using it in the ODE, we get

$$-4\alpha_3 e^x + (-4\alpha_4 + 2\alpha_5)\sin x + (-4\alpha_5 - 2\alpha_4)\cos x = 2e^x - 10\sin x.$$

Example

Example

- Thus, $-4\alpha_3 = 2$, $-4\alpha_4 + 2\alpha_5 = -10$ and $-4\alpha_5 - 2\alpha_4 = 0$ and $\alpha_3 = -\frac{1}{2}$, $\alpha_4 = 2$ and $\alpha_5 = -1$.
- Hence, the general solution is

$$y(x) := \alpha_1 e^{3x} + \alpha_2 e^{-x} - \frac{1}{2}e^x + 2\sin x - \cos x.$$

Example IVP

Example

- Consider the ODE

$$\begin{cases} y'' - 2y' - 3y &= 2e^x - 10\sin x \\ y(0) &= 2 \\ y'(0) &= 4. \end{cases}$$

- Its general solution is $y(x) := \alpha_1 e^{3x} + \alpha_2 e^{-x} - \frac{1}{2}e^x + 2\sin x - \cos x$ (done earlier).
- Using the first initial condition, we have $2 = \alpha_1 + \alpha_2 - \frac{1}{2} - 1$.
- Differentiating the G.S. w.r.t x, we get

$$y'(x) := 3\alpha_1 e^{3x} - \alpha_2 e^{-x} - \frac{1}{2}e^x + 2\cos x + \sin x.$$

Example IVP

Example

- Using the second initial condition, we have $4 = 3\alpha_1 - \alpha_2 - \frac{1}{2} + 2$.
- Thus, $\alpha_1 + \alpha_2 = \frac{7}{2}$ and $3\alpha_1 - \alpha_2 = \frac{5}{2}$.
- Hence, $\alpha_1 = \frac{3}{2}$ and $\alpha_2 = 2$ and the unique solution is $y(x) := \frac{3}{2}e^{3x} + 2e^{-x} - \frac{1}{2}e^{x} + 2\sin x - \cos x$.

Polynomial in RHS

Example

- Consider the ODE $a_2y'' + a_1y' + a_0y = \sum_{i=0}^{k} b_i x^i$ with $f(x) = \sum_{i=0}^{k} b_i x^i$.
- Using the annihilator of f, we get a homogeneous, constant coefficient ODE $D^{k+1}(a_2D^2 + a_1D + a_0)y = 0$.
- The CE of the 'new' homogeneous equation is $\mu^{k+1}(a_2\mu^2 + a_1\mu + a_0) = 0$ with $k+3$ roots, i.e. the roots m_1 and m_2 of the original CE and the additional $k+1$ repeated roots $m = 0$.
- If both m_i's are non-zero then $y = y_c + y_p$ where $y_p(x) := \sum_{i=0}^{k} A_i x^i$ where A_i are to be determined. To obtain A_i's in y_p, use the y in the given ODE and equate like variables both sides.
- If at least one of the m_i is zero ($a_0 = 0$) then $y_p(x) = x\sum_{i=0}^{k} A_i x^i = \sum_{i=0}^{k} A_i x^{i+1}$. If both $m_1 = m_2 = 0$ ($a_1 = a_0 = 0$) then the ODE can be solved from direct integration.

Example

Example

- Consider the ODE $y^{(4)} + y'' = 3x^2 + 4\sin x - 2\cos x$.
- The corresponding CE is $m^4 + m^2 = 0$ with two repeated roots 0 and pair of conjugate complex roots $\pm\imath$ and the complementary function is $y_c(x) := \alpha_1 + \alpha_2 x + \alpha_3 \sin x + \alpha_4 \cos x$.
- The CE of the annihilator of $3x^2$ in f, D^3, also has zero as roots. Thus, its corresponding particular intergal is $Ax^4 + Bx^3 + cx^2$.
- Similarly, the CE of the annihilator of $4\sin x$ and $-2\cos x$ in f, $D^2 + 1$, also has $\pm\imath$ as its roots. Thus, its corresponding particular intergal is $Dx\sin x + Ex\cos x$.
- Thus, we seek $y_p(x) := Ax^4 + Bx^3 + Cx^2 + Dx\sin x + Ex\cos x$.

Example

- Using it in the ODE, we get

$$\begin{aligned} 24A + Dx\sin x - 4D\cos x & \\ +Ex\cos x + 4E\sin x + 12Ax^2 & \\ +6Bx + 2C - Dx\sin x & \\ +2D\cos x - Ex\cos x - 2E\sin x &= 3x^2 + 4\sin x - 2\cos x. \end{aligned}$$

- Thus, $A = \frac{1}{4}$, $B = 0$, $C = -3$, $D = 1$ and $E = 2$.
- Hence, the general solution is
  $y(x) := \alpha_1 + \alpha_2 x + \alpha_3 \sin x + \alpha_4 \cos x + \frac{1}{4}x^4 - 3x^2 + x\sin x + 2x\cos x.$

Example

Example

- Consider the ODE $y'' - 3y' + 2y = 2x^2 + e^x + 2xe^x + 4e^{3x}$.
- The corresponding CE is $m^2 - 3m + 2 = 0$ with roots 1 and 2 and the complementary function is $y_c(x) := \alpha_1 e^x + \alpha_2 e^{2x}$.
- The roots of CE corresponding to the annihilators of e^x and $2xe^x$ have roots same as 1, i.e. e^x and xe^x appearing in f is a solution of the homogeneous equation.
- Thus, we seek $y_p(x) := Ax^2 + Bx + C + Dxe^x + Ex^2e^x + Fe^{3x}$.
- Using it in the ODE, we get

$$
\begin{aligned}
(2A - 3B + 2C) + (2B - 6A)x + 2Ax^2 & \\
+2Fe^{3x} + (-2E)xe^x + (2E - D)e^x &= 2x^2 + e^x + 2xe^x + 4e^{3x}.
\end{aligned}
$$

Thus, $A = 1$, $B = 3$, $C = \frac{7}{2}$, $D = -3$, $E = -1$ and $F = 2$.

Example

Hence, the general solution is

$$y(x) := \alpha_1 e^x + \alpha_2 e^{2x} + x^2 + 3x + \frac{7}{2} - 3xe^x - x^2 e^x + 2e^{3x}.$$

Linear ODE with Variable Coefficients

- The method of undetermined coefficients to find a particular integral is valid only for a restricted class of constant coefficients and linear ODE.
- Even among with constant coefficients ODE it is not applicable to all. For instance, it cannot be applied for the ODE $y'' + y = \tan x$.
- We now introduce the method of *variation of parameters* for variable coefficients linear ODE.
- This method works for ODE whose complementary function is known!

Method of Variation of Parameters

- Given a ODE $a_2(x)y'' + a_1(x)y' + a_0y = f(x)$ such that $y_c(x) := \alpha_1 y_1(x) + \alpha_2 y_2(x)$ is the complementary function of the ODE.
- We seek a particular integral of the form $y_p(x) := v_1(x)y_1(x) + v_2(x)y_2(x)$ where we have *varied* the parameters α_1, α_2 with unknown functions v_1 and v_2 that has to be determined suitably.
- Then $y_p'(x) = v_1(x)y_1'(x) + v_2(x)y_2'(x) + v_1'(x)y_1(x) + v_2'(x)y_2(x)$.
- Impose the condition $v_1'(x)y_1(x) + v_2'(x)y_2(x) = 0$.
- Then $y_p''(x) = v_1(x)y_1''(x) + v_2(x)y_2''(x) + v_1'(x)y_1'(x) + v_2'(x)y_2'(x)$.
- We need to choose v_1 and v_2 such that y_p is a solution to the ODE.

Method of Variation of Parameters

- Thus, we get

$$\begin{aligned} f(x) &= a_2(x)y_p'' + a_1(x)y_p' + a_0 y_p \\ &= a_2(x)[v_1'y_1' + v_2'y_2'] + v_1(x)[a_2(x)y_1'' + a_1(x)y_1' + a_0 y_1] \\ &\quad + v_2(x)[a_2(x)y_2'' + a_1(x)y_2' + a_0 y_2] \end{aligned}$$

- The coefficients of v_1 and v_2 vanish because y_1 and y_2 are solutions of the corresponding homogeneous ODE.
- Thus, for y_p to satisfy the ODE v_1 and v_2 should be chosen such that

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{f(x)}{a_2(x)}. \end{pmatrix}$$

- Since y_1 and y_2 are linearly independent solutions, the determinant (Wronskian) $W(y_1, y_2) \neq 0$.
- Thus, $v_1' = \frac{-fy_2}{a_2 W}$ and $v_2' = \frac{fy_1}{a_2 W}$ and, on integration, we obtain v_1 and v_2.

Examples

Example

- Consider the ODE $y'' + y = \tan x$. Its complementary function is $y_c(x) = \alpha_1 \sin x + \alpha_2 \cos x$.
- The Wronskian of $\sin x$ and $\cos x$ is $W(\sin x, \cos x) = -1$. Then $v_1' = \tan x \cos x = \sin x$ and $v_2' = -\tan x \sin x = \frac{-\sin^2 x}{\cos x} = \cos x - \sec x$.
- Thus, $v_1(x) = -\cos x + c_1$ and $v_2(x) = \sin x - \ln|\sec x + \tan x| + c_2$.
- Hence, for specific choice of c_1 and c_2

$$y_p(x) := c_1 \sin x + c_2 \cos x - \cos x \ln|\sec x + \tan x|$$

and the general solution is

$$y(x) := (\alpha_1 + c_1)\sin x + (\alpha_2 + c_2)\cos x - \cos x \ln|\sec x + \tan x|.$$

Example: Variable Coefficient

Example

- Consider the ODE $(x^2+1)y'' - 2xy' + 2y = 6(x^2+1)^2$.
- We have already seen earlier (Reduced Order Method) that its complementary function is $y_c(x) = \alpha_1 x + \alpha_2(x^2-1)$.
- The Wronskian $W(x, x^2-1) = x^2+1 \neq 0$. Then $v_1' = -6(x^2-1)$ and $v_2' = 6x$.
- Thus, $v_1(x) = -2x^3 + 6x + c_1$ and $v_2(x) = 3x^2 + c_2$.
- We choose $c_1 = c_2 = 0$. Then

$$y_p(x) := (-2x^3+6x)x + 3x^2(x^2-1) = x^4 + 3x^2$$

and the general solution is

$$y(x) := \alpha_1 x + \alpha_2(x^2-1) + x^4 + 3x^2.$$

Example

- Consider the ODE $y^{(3)} - 6y'' - 11y' - 6y = e^x$. The complementary function is $y_c(x) = \alpha_1 e^x + \alpha_2 e^{2x} + \alpha_3 e^{3x}$.
- We seek a particular integral of the form $y_p(x) := v_1(x)e^x + v_2(x)e^{2x} + v_3(x)e^{3x}$ where v_1, v_2 and v_3 are to be determined suitably. Then $y_p'(x) = v_1(x)e^x + 2v_2(x)e^{2x} + 3v_3(x)e^{3x}$ with the additional condition that $v_1'(x)e^x + v_2'(x)e^{2x} + v_3'(x)e^{3x} = 0$.
- Similarly, $y_p''(x) = v_1(x)e^x + 4v_2(x)e^{2x} + 9v_3(x)e^{3x}$ with the additional condition that $v_1'(x)e^x + 2v_2'(x)e^{2x} + 3v_3'(x)e^{3x} = 0$.
- Then $y_p^{(3)}(x) =$ $v_1(x)e^x + 8v_2(x)e^{2x} + 27v_3(x)e^{3x} + v_1'(x)e^x + 4v_2'(x)e^{2x} + 9v_3'e^{3x}$.
- Since we need to choose v_1 and v_2 such that y_p is a solution to the ODE, we get the third identity $v_1'(x)e^x + 4v_2'e^{2x} + 9v_3'e^{3x} = e^x$.

Example: Higher Order

Example

- Thus, for y_p to satisfy the ODE, v_1 and v_2 should be chosen such that

$$\begin{pmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{pmatrix} \begin{pmatrix} v_1' \\ v_2' \\ v_3' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ e^x. \end{pmatrix}$$

- The determinant of the matrix is $2e^{6x} \neq 0$, hence, is invertible. Thus, $v_1' = \frac{1}{2}$, $v_2' = -e^{-x}$ and $v_3' = \frac{1}{2}e^{-2x}$.
- Thus, $v_1(x) = \frac{x}{2} + c_1$, $v_2(x) = e^{-x} + c_2$ and $v_3(x) = -\frac{1}{4}e^{-2x} + c_3$.
- We choose $c_1 = c_2 = c_3 = 0$ and obtain the general solution

$$y(x) := (\alpha_1 + \frac{3}{4})e^x + \alpha_2 e^{2x} + \alpha_3 e^{3x} + \frac{1}{2}xe^x.$$

Cauchy-Euler Equation

Cauchy-Euler equation is a special case of variable coefficient ODE where one can obtain the complementary function explicitly in closed form.

Definition

An ODE of the form

$$\sum_{i=0}^{k} a_i x^i y^{(i)} = f(x) \tag{7.1}$$

where a_i's are all constants is said to be the *Cauchy-Euler* equation.

Theorem

The transformation $x = e^s$ $(x > 0)$ and $x = -e^s$ $(x < 0)$ transforms a Cauchy-Euler ODE to a linear ODE with constant coefficients in the s variable.

Proof.

Set $x = e^s$ then $s = \ln x$. Then $\frac{dy}{ds} = \frac{dy}{dx}\frac{dx}{ds} = xy'$. Similarly,

$$\frac{d^2y}{ds^2} = \frac{d}{dx}\left(xy'\right)x = x^2y'' + xy') = x^2y'' + \frac{dy}{ds}$$

or $x^2y'' = \frac{d^2y}{ds^2} - \frac{dy}{ds}$. More generally,

$$x^k y^{(k)} = \left[\frac{d}{ds}(\frac{d}{ds} - 1)(\frac{d}{ds} - 2)\dots(\frac{d}{ds} - (k-1))\right]y.$$

For the second order, we get

$$a_2\frac{d^2y}{ds^2} + (a_1 - a_2)\frac{dy}{ds} + a_0 y = f(e^s).$$

Higher order can be similarly computed! $\square$

Example

Example

- Consider the ODE $x^2y'' - 2xy' + 2y = x^3$.
- Assuming $x > 0$, we set $s := \ln x$. Then the transformed ODE is

$$\frac{d^2y}{ds^2} - 3\frac{dy}{ds} + 2y = e^{3s}.$$

- The CE of this ODE is $m^2 - 3m + 2 = 0$ with roots 1 and 2. Thus, the complementary function of the ODE in s variable is $y_c := \alpha_1 e^s + \alpha_2 e^{2s}$.
- We now find the particular integral by the method of undetermined coefficients. The f in RHS is linearly independent of complementary function. Thus, we seek $y_p := Ae^{3s}$.

Example

Example

- Then $\frac{dy_p}{ds} = 3Ae^{3s}$ and $\frac{d^2y_p}{ds^2} = 9Ae^{3s}$ and using in the ODE of s variable, we get $2Ae^{3s} = e^{3s}$.
- Thus $A = \frac{1}{2}$ and $y_p = \frac{1}{2}e^{3s}$.
- Hence, the general solution for the s variable ODE is $y = \alpha_1 e^s + \alpha_2 e^{2s} + \frac{1}{2}e^{3s}$
- and the general solution in x variable is $y = \alpha_1 x + \alpha_2 x^2 + \frac{1}{2}x^3$.

Example: Alternate Approach

Example

- Consider the ODE $x^2y'' - 2xy' + 2y = x^3$.
- Assuming $x > 0$, we set $s := \ln x$ and the transformed ODE is

$$\frac{d^2y}{ds^2} - 3\frac{dy}{ds} + 2y = e^{3s}.$$

- The CE of this ODE is $m^2 - 3m + 2 = 0$ with roots 1 and 2 and the complementary function of the ODE in s-variable is $y_c := \alpha_1 e^s + \alpha_2 e^{2s}$ and the ODE in x variable is $y_c := \alpha_1 x + \alpha_2 x^2$.
- We now find the particular integral by the method of variation of parameters by seeking $y_p(x) := v_1(x)x + v_2(x)x^2$.
- Complete the solution!

Example

- Consider the ODE $x^3y^{(3)} - 4x^2y'' + 8xy' - 8y = 4\ln x$.
- Assuming $x > 0$, set $s = \ln x$. Then $x^3y^{(3)} = \frac{d^3y}{ds^3} - 3\frac{d^2y}{ds^2} + 2\frac{dy}{ds}$
- and the ODE in s variable becomes $\frac{d^3y}{ds^3} - 7\frac{d^2y}{ds^2} + 14\frac{dy}{ds} - 8y = 4s$.
- The complementary function is $y_c = \alpha_1 e^s + \alpha_2 e^{2s} + \alpha_3 e^{4s}$.
- We seek $y_p = As + B$, using method of undetermined coefficients.
- Then $y_p' = A$, $y_p'' = y_p^{(3)} = 0$ and $14A - 8As - 8B = 4s$.
- Thus, $-8A = 4$ and $14A - 8B = 0$ which gives $A = -\frac{1}{2}$ and $B = -\frac{7}{8}$.
- Thus, the general solution in s variable is $y(s) := \alpha_1 e^s + \alpha_2 e^{2s} + \alpha_3 e^{4s} - \frac{1}{2}s - \frac{7}{8}$ and in x variable is $y(x) := \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^4 - \frac{1}{2}\ln x - \frac{7}{8}$.

Analytic Functions

Definition

A function f is said to be *analytic* at x_0 if its Taylor series about x_0, i.e.

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

exists and converges to $f(x)$ for all x in some open interval containing x_0.

Example

- All polynomials, exponential and sine and cosine are all analytic everywhere.
- The reciprocal of these are also analytic except on its roots (zeroes).

Ordinary and Singular Point

Definition

A point x_0 is said to be an *ordinary* point of the ODE $y'' + P(x)y' + Q(x)y = 0$ (normalized form) if both P and Q are analytic at x_0.

If either P or Q (or both) are not analytic at x_0 then x_0 is a *singular* point of the ODE.

Example

- All points in $\mathbb{R}$ are ordinary point for the ODE $y'' + xy' + (x^2 + 2)y = 0$.
- The points $x = 0$ and $x = 1$ are singular points for the ODE $(x - 1)y'' + xy' + x^{-1}y = 0$ because $P(x) = x(x - 1)^{-1}$ and $Q(x) = [x(x - 1)]^{-1}$ which are not analytic at $x = 1$ and $x = 0, 1$, respectively. All other points are ordinary points.

Existence of Power series Solution at Ordinary Point

Theorem

If x_0 is an ordinary point of the ODE $y'' + P(x)y' + Q(x)y = 0$ then the ODE admits two linearly independent power series solution of the form $\sum_{k=0}^{\infty} c_k(x - x_0)^k$ such that the power series converge in some interval containing x_0.

Example

Consider the ODE $y'' + xy' + (x^2 + 2)y = 0$. Observe that all points are ordinary points. Let us, for instance, find power series solution about $x_0 = 0$. We seek

$$y(x) = \sum_{k=0}^{\infty} c_k x^k.$$

Thus, $y'(x) = \sum_{k=1}^{\infty} kc_k x^{k-1}$ and $y''(x) = \sum_{k=2}^{\infty} k(k-1)c_k x^{k-2}$.

Example

Example

- Using these form in the ODE, we get

$$\sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} + \sum_{k=1}^{\infty} k c_k x^k + \sum_{k=0}^{\infty} c_k x^{k+2} + 2\sum_{k=0}^{\infty} c_k x^k = 0$$

- or

$$\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2} x^k + \sum_{k=1}^{\infty} k c_k x^k + \sum_{k=2}^{\infty} c_{k-2} x^k + 2\sum_{k=0}^{\infty} c_k x^k = 0$$

Example

Example

- or

$$0 = (2c_0 + 2c_2) + (3c_1 + 6c_3)x + \sum_{k=2}^{\infty} \left[(k+2)(k+1)c_{k+2} + (k+2)c_k + c_{k-2}\right] x^k.$$

- Equating like powers of x both sides we get $c_2 = -c_0$, $c_3 = -\frac{c_1}{2}$ and, for $k \geq 2$,

$$c_{k+2} = -\frac{(k+2)c_k + c_{k-2}}{(k+2)(k+1)},$$

a recursive relation for power series coefficients.

Example

Example

- Thus, $c_4 = \frac{c_0}{4}$ and $c_5 = \frac{3c_1}{40}$.
- Hence,

$$y(x) = c_0\left(1 - x^2 + \frac{x^4}{4} + \dots\right) + c_1\left(x - \frac{x^3}{2} + \frac{3x^5}{40} + \dots\right)$$

a linear combination of two linearly independent power series.

Example IVP

Example

Consider the IVP

$$\begin{cases} (x^2-1)y''+3xy'+xy & = 0 \\ y(0) & = 4 \\ y'(0) & = 6. \end{cases}$$

Observe that $x_0 = \pm 1$ are singular points. However, the initial values are prescribed at $x = 0$ which is an ordinary point. We seek a power series solution about $x_0 = 0$. We seek

$$y(x) = \sum_{k=0}^{\infty} c_k x^k$$

and substitute in the ODE.

Example IVP

Example

Using these form in the ODE, we get

$$\sum_{k=2}^{\infty} k(k-1)c_k x^k - \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + 3\sum_{k=1}^{\infty} kc_k x^k + \sum_{k=1}^{\infty} c_{k-1}x^k = 0.$$

or

$$\begin{aligned} 0 &= -2c_2 + (c_0 + 3c_1 - 6c_3)x \\ &\quad + \sum_{k=2}^{\infty}\left[-(k+2)(k+1)c_{k-2} + k(k+2)c_k + c_{k-1}\right]x^k. \end{aligned}$$

Equating like powers of x both sides we get $-2c_2 = 0$, $c_0 + 3c_1 - 6c_3 = 0$

Example IVP

Example

- and, for $k \geq 2$,
$$c_{k+2} = \frac{k(k+2)c_k + c_{k-1}}{(k+2)(k+1)},$$
a recursive relation for power series coefficients.
- Thus, $c_2 = 0, c_3 = \frac{c_0}{6} + \frac{c_1}{2}$, $c_4 = \frac{c_1}{12}$ and $c_5 = \frac{c_0}{8} + \frac{3c_1}{8}$.
- Hence,
$$y(x) = c_0\left(1 + \frac{x^3}{6} + \frac{x^5}{8} + \dots\right) + c_1\left(x + \frac{x^3}{2} + \frac{x^4}{12} + \frac{3x^5}{8} + \dots\right)$$
a linear combination of two linearly independent power series.
- Applying the first initial condition we get $c_0 = 4$.

Example IVP

Example

- Differentiating the general solution, we get

$$y'(x) = c_0\left(\frac{x^2}{2} + \frac{5x^4}{8} + \dots\right) + c_1\left(1 + \frac{3x^2}{2} + \frac{x^3}{3} + \frac{15x^4}{8} + \dots\right)$$

- and using the second initial condition we get $c_1 = 6$.
- Thus,

$$y(x) = 4 + 6x + \frac{11x^3}{3} + \frac{x^4}{2} + \frac{11x^5}{4} + \dots$$

Power Series about the initial point

- In the above example, instead of initial point $x = 0$, if the initial conditions were prescribed on $x = 2$ (also an ordinary point) then the power series solution sought should be around $x = 2$, i.e.

$$y(x) = \sum_{k=0}^{\infty} c_k (x-2)^k.$$

- In this case, either do directly or choose the change of variable $s = x - 2$ and rewrite the ODE in s variable and proceed as earlier to obtain a power series solution around $s = 0$.

Convergence of Power Series Solution

- Observe that we have not checked the convergence of the power series solution obtained and its radius of convergence.
- In general if P and Q has radius of convergence R_1 and R_2 then the power series solution has at least the radius of convergence of $\min\{R_1, R_2\}$.

Regular Singular Point

- Recall that the power series solution existence was given for an ordinary point.
- In general, the result is not true for singular points. However, regular singular point behaves better!

Definition

Let x_0 be a singular point of $y'' + P(x)y' + Q(x)y = 0$. If x_0 is such that $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ are both analytic at x_0 then x_0 is called a *regular singular* point of the ODE.

Example

Example

Note that $x_0 = 0$ is a singular point of the ODE $2x^2y'' - xy' + (x-5)y = 0$.However, $x_0 = 0$ is a regular singular point because $xP(x) = -\frac{1}{2}$ and $x^2Q(x) = \frac{(x-5)}{2}$ are analytic at $x_0 = 0$.

Example

Consider the ODE $x^2(x-2)^2y'' + 2(x-2)y' + (x+1)y = 0$. The points $x_0 = 0, 2$ are singular points of the ODE.Now, $xP(x) = \frac{2}{x(x-2)}$ is not analytic at $x_0 = 0$ but $x^2Q(x) = \frac{(x+1)}{(x-2)^2}$ is analytic at $x_0 = 0$. Thus, $x_0 = 0$ is *not* a regular singular point. However, $x_0 = 2$ is a regular singular point because $(x-2)P(x) = \frac{2}{x^2}$ and $(x-2)^2Q(x) = \frac{(x+1)}{x^2}$ are both analytic $x_0 = 2$.

Existence of Power series Solution at Regular Singular Point

Theorem

If x_0 is a regular singular point of the ODE $y'' + P(x)y' + Q(x)y = 0$ then the ODE admits at least one nontrivial solution of the form $|x - x_0|^r \sum_{k=0}^{\infty} c_k (x - x_0)^k$ where r in $\mathbb{R}$ or $\mathbb{C}$ may be determined and the power series converge in some interval containing x_0 but x_0 removed.

Example (Frobenius Method)

Consider the ODE $2x^2y'' - xy' + (x - 5)y = 0$. We have already seen that $x_0 = 0$ is a regular singular point. We seek a solution of the form
$y(x) = \sum_{k=0}^{\infty} c_k x^{k+r}$ such that $c_0 \neq 0$. Then
$y'(x) = \sum_{k=0}^{\infty}(k + r)c_k x^{k+r-1}$ and
$y''(x) = \sum_{k=0}^{\infty}(k + r)(k + r - 1)c_k x^{k+r-2}$.

Example

Example

- Using these form in the ODE, we get

$$\sum_{k=0}^{\infty}[2(k+r)(k+r-1)-(k+r)-5]c_k x^{k+r}+\sum_{k=1}^{\infty}c_{k-1}x^{k+r}=0$$

- or

$$\begin{aligned}&[2r(r-1)-r-5]c_0x^r\\+\sum_{k=1}^{\infty}\left[\{2(k+r)(k+r-1)-(k+r)-5\}c_k+c_{k-1}\right]x^{k+r}&=0.\end{aligned}$$

- Equating like powers of x both sides we get the *indicial equation* $2r(r-1)-r-5=0$ and

Example

Example

- the recurrence relation, for $k \geq 1$,

$$c_k = \frac{-c_{k-1}}{2(k+r)(k+r-1)-(k+r)-5}.$$

- Solving for r in $2r^2 - 3r - 5 = 0$ we get the roots $r_1 = \frac{5}{2}$ and $r_2 = -1$.
- For $r_1 = \frac{5}{2}$, the recurrence relation is, for $k \geq 1$, $c_k = \frac{-c_{k-1}}{k(2k+7)}$.
- Then

$$y_1(x) = c_0 x^{\frac{5}{2}} \left(1 - \frac{x}{9} + \frac{x^2}{198} - \frac{x^3}{7722} + \dots\right).$$

Example

Example

- For $r_2 = -1$, the recurrence relation is, for $k \geq 1$, $c_k = \frac{-c_{k-1}}{k(2k-7)}$.
- Then
$$y_2(x) = c_0 x^{-1}\left(1 + \frac{x}{5} + \frac{x^2}{30} + \frac{x^3}{90} + \ldots\right).$$
- The solutions obtained are linearly independent and the general solution is a linear combination of y_1 and y_2.

Theorem

Let x_0 be a regular singular point and r_1 and r_2 (indexed such that $Re(r_1) \geq Re(r_2)$) are roots of the indicial equation.

- *If $r_1 - r_2 \notin \mathbb{Z}^+$ then there exist two nontrivial linearly independent solutions, for $i = 1, 2$,*

$$y_i(x) = |x - x_0|^{r_i} \sum_{k=0}^{\infty} c_{i,k}(x - x_0)^k$$

  with $c_{i,0} \neq 0$.

- *If $r_1 - r_2 \in \mathbb{N}$ then there exist two nontrivial linearly independent solutions*

$$y_1(x) = |x - x_0|^{r_1} \sum_{k=0}^{\infty} c_{1,k}(x - x_0)^k$$

Theorem

and

$$y_2(x) = |x - x_0|^{r_2} \sum_{k=0}^{\infty} c_{2,k}(x - x_0)^k + Cy_1(x) \ln |x - x_0|$$

with $c_{1,0} \neq 0$, $c_{2,0} \neq 0$, C *may be zero.*

- *If* $r_1 - r_2 = 0$ *then there exist two nontrivial linearly independent solutions*

$$y_1(x) = |x - x_0|^{r_1} \sum_{k=0}^{\infty} c_{1,k}(x - x_0)^k$$

with $c_{1,0} \neq 0$ *and*

$$y_2(x) = |x - x_0|^{r_1+1} \sum_{k=0}^{\infty} c_{2,k}(x - x_0)^k + y_1(x) \ln |x - x_0|.$$

Example: First Case

Example

- Consider the ODE $2x^2y'' + xy' + (x^2 - 3)y = 0$. Observe that $x_0 = 0$ is a regular singular point. We seek a solution of the form $y(x) = \sum_{k=0}^{\infty} c_k x^{k+r}$ such that $c_0 \neq 0$.
- Then $y'(x) = \sum_{k=0}^{\infty}(k+r)c_k x^{k+r-1}$ and $y''(x) = \sum_{k=0}^{\infty}(k+r)(k+r-1)c_k x^{k+r-2}$.
- Using these form in the ODE, we get

$$\begin{aligned} & [2r(r-1) + r - 3]c_0 x^r + [2(r+1)r + (r+1) - 3]c_1 x^{r+1} \\ & + \sum_{k=2}^{\infty} \left[\{2(k+r)(k+r-1) + (k+r) - 3\} c_k + c_{k-2}\right] x^{k+r} = 0. \end{aligned}$$

- Equating like powers of x, we get $2r(r-1) + r - 3 = 0$ and, for $k \geq 2$, $c_k = \frac{-c_{k-2}}{2(k+r)(k+r-1)+(k+r)-3}$.

Example: First Case

Example

Solving for r in $2r^2 - r - 3 = 0$ we get the roots $r_1 = \frac{3}{2}$ and $r_2 = -1$ with $r_1 - r_2 = \frac{5}{2} \notin \mathbb{Z}^+$. For $r_1 = \frac{3}{2}$, the recurrence relation is, for $k \geq 2$, $c_k = \frac{-c_{k-2}}{k(2k+5)}$. Then

$$y_1(x) = c_0 x^{\frac{3}{2}} \left(1 - \frac{x^2}{18} + \frac{x^4}{936} - \dots\right).$$

For $r_2 = -1$, the recurrence relation is, for $k \geq 2$, $c_k = \frac{-c_{k-2}}{k(2k-5)}$. Then

$$y_2(x) = c_0 x^{-1} \left(1 + \frac{x^2}{2} - \frac{x^4}{24} + \dots\right).$$

The solutions obtained are linearly independent and the general solution is a linear combination of y_1 and y_2.

Example: Second Case

Example

Consider the ODE $x^2y'' - xy' - \left(x^2 + \frac{5}{4}\right) y = 0$. Observe that $x_0 = 0$ is a regular singular point. We seek a solution of the form $y(x) = \sum_{k=0}^{\infty} c_k x^{k+r}$ such that $c_0 \neq 0$. Using this form in the ODE, we get

$$\left[r(r-1) - r - \frac{5}{4}\right] c_0 x^r + \left[(r+1)r - (r+1) - \frac{5}{4}\right] c_1 x^{r+1}$$
$$+\sum_{k=2}^{\infty}\left[\left\{(k+r)(k+r-1) - (k+r) - \frac{5}{4}\right\} c_k - c_{k-2}\right] x^{k+r} \quad = \quad 0.$$

Equating like powers of x both sides we get $r^2 - 2r - \frac{5}{4} = 0$ and the recurrence relation, for $k \geq 2$, $c_k = \frac{c_{k-2}}{(k+r)(k+r-1)-(k+r)-\frac{5}{4}}$.

Example: Second Case

Example

- We get the roots $r_1 = \frac{5}{2}$ and $r_2 = -\frac{1}{2}$ with $r_1 - r_2 = 3 \in \mathbb{N}$.
- For $r_1 = \frac{5}{2}$, $c_1 = 0$ and the recurrence relation is, for $k \geq 2$, $c_k = \frac{c_{k-2}}{k(k+3)}$. Then

$$y_1(x) = c_0 x^{\frac{5}{2}} \left(1 + \sum_{k=1}^{\infty} \frac{x^{2k}}{[2 \cdot 4 \cdot 6 \ldots (2k)][5 \cdot 7 \cdot 9 \ldots (2k+3)]} \right).$$

- For $r_2 = -\frac{1}{2}$, $c_1 = 0$ and the recurrence relation is, for $k \geq 2$ and $k \neq 3$, $c_k = \frac{c_{k-2}}{k(k-3)}$.
- For $k = 3$, $(3 \times 0)c_3 = c_1$ making the choice of c_3 generic.

Example: Second Case

Example

- Then the solution $y_2(x)$ is given as

$$c_0 x^{-\frac{1}{2}} \left(1 - \frac{x^2}{2} - \frac{x^4}{2 \cdot 4} - \sum_{k=3}^{\infty} \frac{x^{2k}}{[2 \cdot 4 \cdot 6 \ldots (2k)][3 \cdot 5 \cdot 7 \cdot \ldots (2k-3)]} \right)$$
$$+ c_3 x^{-\frac{1}{2}} \left(x^3 + \sum_{k=2}^{\infty} \frac{x^{2k+1}}{[2 \cdot 4 \cdot 6 \ldots (2k-2)][5 \cdot 7 \cdot \ldots (2k+1)]} \right)$$

- Observe that with $c_0 = 0$ and $c_3 = 1$ we obtain y_1 with $c_0 = 1$. Thus, y_2 is the general solution of the ODE.

As in the above example, in the case when $r_1 - r_2 \in \mathbb{N}$, it is possible to obtain the general solution from the smaller root r_2.

Example: Second Case Again

Example

- Consider the ODE $x^2y'' + (x^2 - 3x)y' + 3y = 0$. Observe that $x_0 = 0$ is a regular singular point. We seek a solution of the form $y(x) = \sum_{k=0}^{\infty} c_k x^{k+r}$ such that $c_0 \neq 0$.
- Using this form in the ODE, we get 0 is equal to

$$\sum_{k=1}^{\infty} \left[\{(k+r)(k+r-1) - 3(k+r) + 3\} c_k + (k+r-1)c_{k-1} \right] x^{k+r} + \left[r(r-1) - 3r + 3 \right] c_0 x^r.$$

- Equating like powers of x both sides we get $r^2 - 4r + 3 = 0$ and the recurrence relation, for $k \geq 1$, $c_k = \frac{-(k+r-1)c_{k-1}}{(k+r)(k+r-1)-3(k+r)+3}$.

Example: Second Case Again

Example

- We get the roots $r_1 = 3$ and $r_2 = 1$ with $r_1 - r_2 = 2 \in \mathbb{N}$. For $r_1 = 3$, the recurrence relation is, for $k \geq 1$, $c_k = \frac{-c_{k-1}}{k}$. Then $y_1(x) = c_0 x^3 e^{-x}$.
- For $r_2 = 1$, the recurrence relation is, for $k \geq 1$ and $k \neq 2$, $c_k = \frac{-c_{k-1}}{(k-2)}$.
- We get $c_1 = c_0$ and $0 \times c_2 + c_1 = 0$ implies $c_1 = 0$ which implies $c_0 = 0$, a contradiction. Thus, no corresponding solution.
- Using the recursive for $k \geq 3$, we get the same solution $y_2(x) = c_2 x^3 e^{-x} = y_1(x)$. Thus, we are yet to find a linearly independent solution.

Example: Second Case Again

Example

- We employ the reduction order technique. We seek $y_2(x) = v(x)x^3e^{-x}$. Using it in the ODE, we obtain $xv'' + (3-x)v' = 0$. Then $v(x) = \int x^{-3}e^x\,dx$.
- Thus,

$$
\begin{aligned}
y_2(x) &= x^3e^{-x}\int x^{-3}\left(1 + x + \frac{x^2}{2!} + \dots\right)dx \\
&= \left(x^3 - x^4 + \frac{x^5}{2!} - \dots\right)\left(-\frac{1}{2x^2} - \frac{1}{x} + \frac{\ln x}{2} + \frac{x}{6} + \dots\right) \\
&= \left(-\frac{x}{2} - \frac{x^2}{2} + \frac{3x^3}{4} - \frac{x^4}{4} + \dots\right) + \frac{1}{2}x^3e^{-x}\ln x.
\end{aligned}
$$

Thus, the general solution is linear combination of y_1 and y_2.

Bessel's Equation

Definition

For a given $p \in \mathbb{R}$, the ODE $x^2y'' + xy' + (x^2 - p^2)y = 0$ is called the *Bessel's* equation of order p.

Example (Third Case: Zero Order Bessel)

The zero order Bessel's equation is $xy'' + y' + xy = 0$. Observe that $x_0 = 0$ is a regular singular point. We seek a solution of the form $y(x) = \sum_{k=0}^{\infty} c_k x^{k+r}$ such that $c_0 \neq 0$. Using this form in the ODE, we get

$$r^2c_0x^{r-1} + (1+r)^2c_1x^r + \sum_{k=2}^{\infty} \left[(k+r)^2c_k + c - k - 2\right] x^{k+r-1} = 0.$$

Equating like powers of x both sides we get $r^2 = 0$, $(1+r)^2c_1 = 0$ and

General Solution of Bessel's of Zero Order

Example

the recurrence relation, for $k \geq 2$, $c_k = \frac{-c_{k-2}}{(k+r)^2}$. We get the roots $r_1 = r_2 = 0 = r$. For $r = 0$, we have $c_1 = 0$ and, for $k \geq 2$, $c_k = \frac{-c_{k-2}}{k^2}$. Then

$$y_1(x) = c_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{x}{2}\right)^{2k}.$$

Definition

The case $c_0 = 1$ is called the Bessel's function of *first kind* of order zero, denoted as J_0.

The linearly independent solution can be obtained by the reduction order technique. We seek $y_2(x) = v(x)J_0(x)$. Then, by Theorem 97, we get $y_2(x) = J_0(x) \int \frac{dx}{xJ_0^2(x)}$.

Laplace Transform

Definition

Let $f:(0,\infty)\to\mathbb{R}$. The *Laplace Transform* of f, denoted as $\mathcal{L}(f)$ is defined as

$$\mathcal{L}(f)(p) := \int_0^{\infty} e^{-px} f(x)\, dx$$

whenever the integral is finite. The *inverse Laplace* transform of F, denoted by $\mathcal{L}^{-1}(F)$ is defined as $\mathcal{L}^{-1}(F)(x) := f(x)$ where $\mathcal{L}(f)(p) = F(p)$.

Example

Example

Let $f : (0,\infty) \to \mathbb{R}$ be defined as $f(x) = 1$, for all $x \in (0,\infty)$. Then its Laplace transform is

$$\begin{aligned}
\mathcal{L}(1)(p) &= \int_0^\infty e^{-px}\,dx = \lim_{R\to\infty}\int_0^R e^{-px}\,dx = \lim_{R\to\infty}\left[-\frac{e^{-px}}{p}\right]_0^R \\
&= \lim_{R\to\infty}\left[-\frac{e^{-pR}}{p} + \frac{1}{p}\right] = \frac{1}{p}, \text{ valid for all } p > 0.
\end{aligned}$$

Example

Let $f:(0,\infty)\to\mathbb{R}$ be defined as $f(x)=x$, for all $x\in(0,\infty)$. Then its Laplace transform is

$$\begin{aligned}\mathcal{L}(x)(p) &= \int_0^\infty x\,e^{-px}\,dx = \lim_{R\to\infty}\int_0^R x\,d\left(\frac{e^{-px}}{-p}\right)\\ &= \lim_{R\to\infty}\left[-x\frac{e^{-px}}{p}-\frac{e^{-px}}{p^2}\right]_0^R = \frac{1}{p^2},\ \text{valid for all } p>0.\end{aligned}$$

More generally, for any given $n\in\mathbb{N}$, the Laplace transform of

$$\begin{aligned}\mathcal{L}(x^n)(p) &= \int_0^\infty x^n\,d\left(\frac{e^{-px}}{-p}\right) = \left[-x^n\frac{e^{-px}}{p}\right]_0^\infty + \frac{n}{p}\int_0^\infty x^{n-1}e^{-px}\,dx\\ &= \frac{n}{p}L(x^{n-1}) = \frac{n(n-1)}{p^2}L(x^{n-2}) = \ldots = \frac{n!}{p^n}L(1) = \frac{n!}{p^{n+1}}.\end{aligned}$$

Example

Example

Let $f : (0, \infty) \to \mathbb{R}$ be defined as $f(x) = e^{ax}$, for all $x \in (0, \infty)$. Then its Laplace transform is

$$
\begin{aligned}
\mathcal{L}(e^{ax})(p) &= \int_0^\infty e^{ax} e^{-px}\, dx = \lim_{R\to\infty} \int_0^R e^{(a-p)x}\, dx \\
&= \lim_{R\to\infty} \left[\frac{e^{(a-p)x}}{a-p} \right]_0^R \\
&= \lim_{R\to\infty} \left[\frac{e^{(a-p)R}}{a-p} - \frac{1}{(a-p)} \right] \\
&= \frac{1}{p-a}, \text{ valid for all } p > a.
\end{aligned}
$$

Example

Example

Let $f : (0, \infty) \to \mathbb{R}$ be defined as $f(x) = \sin ax$, for all $x \in (0, \infty)$. Then its Laplace transform is

$$
\begin{aligned}
\mathcal{L}(\sin ax)(p) &= \int_0^\infty \sin ax \, e^{-px} \, dx = \lim_{R\to\infty} \int_0^R \sin ax \, e^{-px} \, dx \\
&= \lim_{R\to\infty} \int_0^R \frac{1}{2\imath} \left(e^{\imath ax} - e^{-\imath ax} \right) e^{-px} \, dx \\
&= \lim_{R\to\infty} \int_0^R \frac{1}{2\imath} \left[e^{-(p-\imath a)x} - e^{-(p+\imath a)x} \right] dx \\
&= \lim_{R\to\infty} \frac{1}{2\imath} \left[\frac{e^{-(p-\imath a)x}}{-p+\imath a} + \frac{e^{-(p+\imath a)x}}{p+\imath a} \right]_0^R \\
&= \frac{1}{2\imath} \left[\frac{1}{-p+\imath a} + \frac{1}{p+\imath a} \right] = \frac{a}{p^2+a^2}, \text{ valid for all } p > 0.
\end{aligned}
$$

Example

Example

Let $f : (0,\infty) \to \mathbb{R}$ be defined as $f(x) = \cos ax$, for all $x \in (0,\infty)$. Then its Laplace transform is

$$
\begin{aligned}
\mathcal{L}(\cos ax)(p) &= \int_0^\infty \cos ax \, e^{-px}\, dx = \lim_{R\to\infty} \int_0^R \cos ax \, e^{-px}\, dx \\
&= \lim_{R\to\infty} \int_0^R \frac{1}{2}\left(e^{\imath ax} + e^{-\imath ax}\right) e^{-px}\, dx \\
&= \lim_{R\to\infty} \int_0^R \frac{1}{2}\left[e^{-(p-\imath a)x} + e^{-(p+\imath a)x}\right] dx \\
&= \lim_{R\to\infty} \frac{1}{2}\left[\frac{e^{-(p-\imath a)x}}{-p+\imath a} - \frac{e^{-(p+\imath a)x}}{p+\imath a}\right]_0^R \\
&= \frac{1}{2}\left[\frac{1}{-p+\imath a} - \frac{1}{p+\imath a}\right] = \frac{p}{p^2+a^2}, \text{ valid for all } p > 0.
\end{aligned}
$$

Piecewise Continuous

Definition

A function is *piecewise continuous* if it is continuous except at finite number of points and the left and right limit of the function at the points of discontinuity are finite.

Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is of *exponential order* if there exists a $\alpha \in \mathbb{R}$, $x_0 \in \mathbb{R}$ and $M > 0$ such that, for all $x > x_0$, $|f(x)| \leq Me^{\alpha x}$.

Example

- All bounded functions are of exponential order with $\alpha = 0$.
- x^n is of exponential order for any $\alpha > 0$ because $\lim_{x \to \infty} e^{-\alpha x} x^n = 0$.
- e^{x^2} is *not* of exponential order because $e^{x^2 - \alpha x}$ is unbounded for all any α.

Existence Theorem

Theorem

Let $f : [0, \infty) \to \mathbb{R}$ be piecewise continuous in every closed interval $[0, a]$ and is of exponential order, say α, then the Laplace transform exists for all $p > \alpha$.

Proof.

There exists a $\alpha \in \mathbb{R}$, x_0 and $M > 0$ such that, for all $x > x_0$, $|f(x)| \leq Me^{\alpha x}$. Now, note that

$$\mathcal{L}(f)(p) := \int_0^\infty e^{-px} f(x)\, dx = \int_0^{x_0} e^{-px} f(x)\, dx + \int_{x_0}^\infty e^{-px} f(x)\, dx.$$

We need to show that the last sum above is finite! The first integral in the sum is finite due to piecewise continuity! □

Proof Continued...

Proof.

Now, consider

$$
\begin{aligned}
\int_{x_0}^{\infty} e^{-px} f(x)\, dx &\leq \int_{x_0}^{\infty} e^{-px} |f(x)|\, dx \leq \int_{x_0}^{\infty} e^{-(p-\alpha)x} M\, dx \\
&\leq \lim_{R\to\infty} \left[-\frac{Me^{-(p-\alpha)x}}{p-\alpha} \right]_{x_0}^{R} = \left[\frac{Me^{-(p-\alpha)x_0}}{p-\alpha} \right]
\end{aligned}
$$

where the last equality is valid for $p > \alpha$. Thus, the Laplace transform exists. □

Heaviside Function (Piecewise Continuous)

Example

Let us compute the Laplace transform of $H_a : (0, \infty) \to \mathbb{R}$ defined as

$$H_a(x) = \begin{cases} 0 & x < a \\ 1 & x > a \end{cases}$$

where $a \geq 0$. Observe that if $a = 0$ then $H_a \equiv 1$ and its Laplace transform is $\frac{1}{p}$. Consider

$$\mathcal{L}(H_a)(p) = \int_0^a e^{-px} \cdot 0 \, dx + \int_a^\infty e^{-px} \, dx = \frac{e^{-ap}}{p} \text{ for } p > 0.$$

Laplace Transfom of Piecewise Continuous

Example

Let us compute the Laplace transform of

$$f(x) = \begin{cases} 1 & 0 < x < 3 \\ 2 & x = 3 \\ 1 & x > 3. \end{cases}$$

Then

$$\mathcal{L}(f)(p) = \int_0^3 e^{-px}\,dx + \int_3^\infty e^{-px}\,dx = \frac{1}{p} \text{ for } p > 0.$$

- Compare the above example with Laplace transform of the constant function 1.
- Thus, the inverse transform of $1/p$ is not necessarily unique. One of them is continuous and the other is not continuous!

Uniqueness of Continuous Inverse Laplace Transform

Theorem

If a continuous inverse transform exists for F then it has unique continuous inverse transform.

Linearity

Theorem

The Laplace transform is linear, i.e. $\mathcal{L}(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 \mathcal{L}(f_1) + \alpha_2 \mathcal{L}(f_2)$. *The inverse transform is also linear!*

Proof.

Easy to verify! $\square$

Example (Using Linearity to compute LT)

Let $f(x) = \sin^2 ax$. Then

$$\begin{aligned}
\mathcal{L}(\sin^2 ax) &= \mathcal{L}\left(\frac{1}{2} - \frac{\cos 2ax}{2}\right) = \frac{1}{2}\left[\mathcal{L}(1) - \mathcal{L}(\cos 2ax)\right] \\
&= \frac{1}{2}\left(\frac{1}{p} - \frac{p}{p^2 + 4a^2}\right) = \frac{2a^2}{p(p^2 + 4a^2)}.
\end{aligned}$$

First Shifting Theorem

Theorem

If Laplace transform of f exists for $p > \alpha$ then, for any given $a \in \mathbb{R}$,
$\mathcal{L}(e^{ax}f)(p) = \mathcal{L}(f)(p-a)$ for all $p > \alpha + a$.

Proof.

$\mathcal{L}(e^{ax}f)(p) = \int_0^\infty e^{-(p-a)x} f(x)\,dx = \mathcal{L}(f)(p-a)$. $\square$

Example

1. Let $f(x) = x$. Then, for all $p > a$, $\mathcal{L}(e^{ax}x) = \mathcal{L}(x)(p-a) = \frac{1}{(p-a)^2}$.
2. Let $f(x) = \sin bx$. Then, for all $p > a$,
   $\mathcal{L}(e^{ax}\sin bx) = \mathcal{L}(\sin bx)(p-a) = \frac{b}{(p-a)^2+b^2}$.

Second Shifting Theorem

Definition

Let $f : (0, \infty) \to \mathbb{R}$ be a given function. The *right translate* of f by $a > 0$, denoted as $\tau_a f$, is defined as

$$\tau_a f(x) := \begin{cases} 0 & 0 < x < a \\ f(x-a) & x > a. \end{cases}$$

Theorem

If the Laplace transform of f exists then $\mathcal{L}(\tau_a f)(p) = e^{-ap}\mathcal{L}(f)(p)$.

Proof: Consider

$$\mathcal{L}(\tau_a f)(p) = \int_a^\infty e^{-px} f(x-a)\,dx = \int_0^\infty e^{-p(s+a)} f(s)\,ds = e^{-pa}\mathcal{L}(f)(p).$$

Derivative of Laplace Transform

Theorem

If Laplace transform of f exists then, for any given $n \in \mathbb{N}$,
$(-1)^n \frac{d^n}{dp^n}\mathcal{L}(f)(p) = \mathcal{L}(x^n f)(p)$.

Proof.

Consider $\frac{d}{dp}\mathcal{L}(f)(p) = \frac{d}{dp}\int_0^\infty e^{-px} f(x)\,dx = -\int_0^\infty e^{-px} x\, f(x)\,dx = -\mathcal{L}(xf)(p)$. $\square$

Example

$$\begin{aligned}\mathcal{L}(x^2 \sin ax) &= (-1)^2 \frac{d^2}{dp^2}\mathcal{L}(\sin ax)(p) = \frac{d^2}{dp^2}\left(\frac{a}{p^2+a^2}\right)\\ &= \frac{6ap^2 - 2a^3}{(p^2+a^2)^3}.\end{aligned}$$

Laplace Transform of Derivatives

Theorem

Let $f : [0, \infty) \to \mathbb{R}$ *be continuous and is of exponential order with* $e^{\alpha x}$*. If* f' *is piecewise continuous in every closed interval of* $[0, a]$ *then* $\mathcal{L}(f')(p) = p\mathcal{L}(f)(p) - f(0)$ *for all* $p > \alpha$*. More generally, with similar hypotheses on* $f^{(k-1)}$ *and* $f^{(k)}$*, we have* $\mathcal{L}(f^{(k)})(p) = p^k\mathcal{L}(f)(p) - p^{k-1}f(0) - p^{k-2}f'(0) - \ldots - f^{(k-1)}(0)$*.*

Proof.

For continuous f', a simple integration by parts yields

$$\mathcal{L}(f')(p) = \left[e^{-px}f(x)\right]_0^\infty + p\int_0^\infty e^{-px}f(x)\,dx.$$

The argument for piecewise continuous is similar after splitting in to finite number of integrals along the jumps of f' and use the continuity of f on the jump points of f'. $\square$

Example

Example

Let $f(x) = \sin ax \cos ax$. Then $f(x) = g'(x)$ where $g(x) = \frac{1}{2a}\sin^2 ax$. Then

$$\begin{aligned}\mathcal{L}(\sin ax \cos ax) &= \frac{p}{2a}\mathcal{L}(\sin^2 ax) - f(0) = \frac{p}{2a}\frac{2a^2}{p(p^2+4a^2)} \\ &= \frac{a}{(p^2+4a^2)}.\end{aligned}$$

Alternately, $\sin ax \cos ax = \frac{\sin 2ax}{2}$. Thus, $\mathcal{L}(\sin ax \cos ax) = \frac{1}{2}\mathcal{L}(\sin 2ax) = \frac{a}{(p^2+4a^2)}$.

Laplace Transform of Integrals

Theorem

Let $f : [0,\infty) \to \mathbb{R}$ *be continuous and is of exponential order with* $e^{\alpha x}$*. Then*

$$\mathcal{L}\left(\int_0^x f(y)\,dy\right)(p) = \frac{\mathcal{L}(f)(p)}{p}.$$

Proof.

Set $g(x) := \int_0^x f(y)\,dy$. Then g is continuous and of exponential order because

$$|g(x)| \leq \int_0^x |f(y)|\,dy \leq M\int_0^x e^{\alpha y}\,dy \leq \frac{M}{\alpha}(e^{\alpha x}-1) \leq \frac{M}{\alpha}e^{\alpha x}.$$

Further $g' = f$ and $g(0) = 0$. Thus, $\mathcal{L}(g')(p) = p\mathcal{L}(g)(p) - g(0)$ and $\mathcal{L}(g)(p) = \mathcal{L}(f)(p)/p$. □

Convolution

Definition

If f and g are two piecewise continuous functions on $[a, b]$ then the *convolution* of f and g, denoted as $f * g$, is defined as

$$f * g(x) := \int_a^x f(t)g(x-t)\,dt.$$

By the change of variable $s = x - t$, it is easy to observe that $f * g = g * f$ because

$$\begin{aligned}\int_a^x f(t)g(x-t)\,dt &= -\int_{x-a}^0 f(x-s)g(s)\,ds = -\int_x^a f(x-s)g(s)\,ds \\ &= \int_a^x f(x-s)g(s)\,ds.\end{aligned}$$

Convolution Theorem

Theorem

Let f and g be piecewise continuous function on every closed interval of the form $[0, a]$ and is of exponential order, say α. Then $\mathcal{L}(f * g)(p) = \mathcal{L}(f)(p)\mathcal{L}(g)(p)$, for all $p > \alpha$.

Proof:

$$\begin{aligned}
\mathcal{L}[(f * g)(x)](p) &= \int_0^\infty e^{-px} \int_0^x f(t)g(x-t)\,dt\,dx \\
&= \int_0^\infty f(t) \left[\int_t^\infty e^{-px} g(x-t)\,dx\right] dt \\
&= \int_0^\infty f(t) \left[\int_0^\infty e^{-p(s+t)} g(s)\,ds\right] dt \\
&= \mathcal{L}(f)(p) \cdot \mathcal{L}(g)(p).
\end{aligned}$$

Consider the IVP

$$\begin{cases} y' - 3y &= 4e^{5x} \\ y(0) &= 6. \end{cases}$$

Applying Laplace transform to the ODE, we get

$$\begin{aligned} 0 &= \mathcal{L}(y') - 3\mathcal{L}(y) - 4\mathcal{L}(e^{5x}) = p\mathcal{L}(y) - y(0) - 3\mathcal{L}(y) - 4\mathcal{L}(e^{5x}) \\ &= (p-3)\mathcal{L}(y) - 6 - \frac{4}{p-5}. \end{aligned}$$

Thus, we seek y such that

$$\begin{aligned} \mathcal{L}(y)(p) &= \frac{6}{p-3} + \frac{4}{(p-3)(p-5)} = \frac{6}{p-3} + \frac{2}{p-5} - \frac{2}{p-3} \\ &= \frac{4}{p-3} + \frac{2}{p-5} = \mathcal{L}(4e^{3x}) + \mathcal{L}(2e^{5x}) \\ &= \mathcal{L}(4e^{3x} + 2e^{5x}). \end{aligned}$$

Hence, $y(x) = 4e^{3x} + 2e^{5x}$.

Applications to ODE

Given $y : [0, \infty)$ is a solution of the IVP

$$\begin{cases} y'' - 6y' + 5y &= 0 \\ y(0) &= 3 \\ y'(0) &= 7. \end{cases}$$

Applying Laplace transform to the ODE, we get

$$\begin{aligned} 0 &= \mathcal{L}(y'') - 6\mathcal{L}(y') + 5\mathcal{L}(y) \\ &= p^2\mathcal{L}(y) - py(0) - y'(0) - 6[p\mathcal{L}(y) - y(0)] + 5\mathcal{L}(y) \\ &= [p^2 - 6p + 5]\mathcal{L}(y) - 3p + 11. \end{aligned}$$

Thus, the Laplace transform of y is

$$\mathcal{L}(y)(p) = \frac{3p - 11}{p^2 - 6p + 5}.$$

Complete the computation to find y!