LECTURE - 5,6

Differentiability,
CR - equations,
Analytic functions
Power series.

Lecture 5 : Differentiability and CR-eqns

Let f be a fn defined in a neighbourhood of z_0. We say f is differentiable at z_0 if the function

$$\frac{f(z)-f(z_0)}{z-z_0}$$ defined in the deleted neighbourhood of z_0

has a limit.

In other words, there exists $l \in \mathbb{C}$ such that given any $\varepsilon > 0$ $\exists\ \delta > 0$

$$\ni \left| \frac{f(z)-f(z_0)}{z-z_0} - l \right| < \varepsilon \quad \forall\, 0 < |z-z_0| < \delta.$$

The limit is denoted as $f'(z_0)$.

Eg: $f(z) = z^n$

$$\lim_{z \to z_0} \frac{z^n - z_0^n}{z - z_0} = ??$$

$$\frac{z^n - z_0^n}{z - z_0} = z^{n-1} + z^{n-2} z_0 + \cdots + z_0^{n-1}$$

$$\therefore \lim_{z \to z_0} z^{n-1} + z^{n-2} z_0 + \cdots + z_0^{n-1} = n z_0^{n-1}$$

<u>Useful Remark</u>: If f is differentiable at z_0 then the function η defined as follows

$$\eta(z) = \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \quad \text{for } z \neq z_0$$

$$\eta(z_0) = 0$$

is continuous at z_0.

$$\left(\because |\eta(z) - \eta(z_0)| = \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| \right)$$

Proposition: If f is differentiable at z_0 then f is continuous at z_0, as well.

<u>Pf</u>:

We use the above remark, to get

$$f(z) = f(z_0) + \eta(z)(z - z_0) + f'(z_0)(z - z_0)$$

each of this is continuous hence $f(z)$ is continuous at z_0.

Arithmetic of differentiability:

Let f and g be differentiable at z_0. Then $f \pm g$, fg, f/g (if $g(z_0) \neq 0$) and cf are differentiable.

Further,
$$(f \pm g)'(z_0) = f'(z_0) + g'(z_0)$$
$$(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$$
$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}$$
$$(cf)'(z_0) = cf'(z_0).$$

Chain rule:

If f is differentiable at $g(z_0)$ and g is differentiable at z_0 then $(f \circ g)'(z_0) = f'(g(z_0))\, g'(z_0)$.

(Proofs of all the above is exactly as in the real case).

Eg: (1) $f(z) = |z|$: $\lim_{z \to 0} \frac{|z|}{z}$ does not exist.

So f is not differentiable at $z = 0$ ∎

(2) $f(z) = |z|^2$: $\lim_{z \to 0} \frac{|z|^2}{z} = \lim_{z \to 0} \bar{z} = 0$

So f is differentiable at 0.

$$\lim_{h\to 0} \frac{|z+h|^2 - |z|^2}{h} = \lim_{h\to 0} \frac{(z+h)(\bar{z}+\bar{h}) - z\bar{z}}{h}$$

$$= \lim_{h\to 0} \frac{z\bar{h} + h\bar{z}}{h} = \bar{z} + \underbrace{\lim_{h\to 0} z\frac{\bar{h}}{h}}_{\text{does not exist}}$$

$\therefore$ does not exist. ∎

Remark: 1) By example ① it follows that even if u, v are differentiable, f need not be differentiable

2) By Example ②, it is seen that even if f is differentiable at z_0 it need not be differentiable anywhere around z_0.

Remark 1, in particular, means that differentiability of u and v is not sufficient to conclude differentiability of $u+iv$.

So, we try to investigate a sufficiency criteria for f to be differentiable.

For this we work backwards:

If f is differentiable then $f'(z)$ exists

So, $f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$

Take $h = h_1 + i \cdot 0$. Then

$$f'(z) = \lim_{h_1 \to 0} \frac{u(x+h_1, y) + iv(x+h_1, y) - (u(x,y) + iv(x,y))}{h_1}$$

$$= \lim_{h_1 \to 0} \frac{u(x+h_1, y) - u(x,y)}{h_1} + i \lim_{h_1 \to 0} \frac{v(x+h_1, y) - v(x,y)}{h_1}$$

$$= u_x(x,y) + i v_x(x,y)$$

Take $h = 0 + ih_2$. Then,

$$f'(z) = \lim_{ih_2 \to 0} \frac{u(x, y+h_2) + iv(x, y+h_2) - (u(x,y) + iv(x,y))}{ih_2}$$

$$= \lim_{h_2 \to 0} \frac{u(x, y+h_2) - u(x,y) + i(v(x, y+h_2) - v(x,y))}{ih_2}$$

Thus, we get

$$f'(z) = u_x(x,y) + i v_x(x,y) \quad = u_x + i v_x.$$

$$= \frac{u_y(x,y) + i v_y(x,y)}{i} = v_y - i u_y$$

Cauchy-Riemann equations : $\boxed{\begin{aligned} u_x &= v_y \\ v_x &= -u_y \end{aligned}}$

We have just seen that : (Necessary condition for differentiability)

If f is differentiable at z_0 then

$$u_x(x_0,y_0) = v_y(x_0,y_0) \ \& \ v_x(x_0,y_0) = -u_y(x_0,y_0).$$

Does this give a sufficiency condition as well?? No! Example (see exercise sheet 2).

Let $z_0 = x_0 + i y_0$

THEOREM : Let the partial derivatives u_x, u_y, v_x, v_y exist in a nbhd of (x_0, y_0) and be continuous in that nbhd of (x_0,y_0). Then, f is differentiable at $z_0 = x_0 + i y_0$ whenever f satisfies the CR equations in a nbhd of (x_0,y_0). ▨

(Proof skipped : Ref : Shakarchi).

Applications of CR-equations

① Let f be differentiable on an open set U. Let $f'(z) = 0 \ \forall z \in U$. Then f is constant on U.

Pf: $f'(z_0) = u_x(x_0, y_0) + i v_x(x_0, y_0) = 0 \quad \forall z_0 \in U$

$$\left.\begin{aligned} &\Rightarrow u_x(x,y) = 0 = v_x(x,y) \\ &\Rightarrow u = u(y), \ v = v(y) \end{aligned}\right\} \text{on } U$$

$$\left.\begin{aligned} \text{But, } f'(z_0) &= v_y(x_0,y_0) - i u_y(x_0,y_0) \\ \text{So, } \quad & u_y = 0 = v_y \end{aligned}\right\} \text{on } U$$

Thus, $u = \text{constant} = v$ ∎

② Let f be a differentiable function on U

If $|f|$, $\operatorname{Re} f$ or $\operatorname{Im} f$ is constant on U then so is f.

Pf: $|f|^2 = \text{constant} \quad (\neq 0 \text{ w.l.o.g})$

$\Rightarrow u^2 + v^2 = \text{constant}$

$\therefore \dfrac{\partial}{\partial x}(u^2+v^2) = 0 \Rightarrow 2uu_x + 2vv_x = 0$

By CR eqns $\quad 2uv_y - 2vu_y = 0$

$$\frac{\partial}{\partial y}(u^2+v^2)=0 \Rightarrow 2uu_y+2vv_y=0$$

from the above 2 equations, we get

$$(u^2+v^2)u_y=0$$

$$\Rightarrow u_y=0 \quad ||^{ly}\ v_y=0$$

$$\Rightarrow f'(z)=0 \quad \forall\ z\in U$$

hence f is constant on U.

Others, similarly argued.

(3) Let f be differentiable on U. If f is real valued (or purely imaginary) then f is constant.

Polar form of CR equations:

$$x(r,\theta)=r\cos\theta, \quad y(r,\theta)=r\sin\theta$$

$$\therefore \frac{\partial x}{\partial r}=\cos\theta, \quad \frac{\partial x}{\partial \theta}=-r\sin\theta$$

$$\frac{\partial y}{\partial r}=\sin\theta, \quad \frac{\partial y}{\partial \theta}=r\cos\theta$$

Consider, $u = u(x, y)$

$$u_r = \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

$$= u_x \cos\theta + u_y \sin\theta \quad -① $$

III^{ly} $$V_r = V_x \cos\theta + V_y \sin\theta \quad -②$$

$$u_\theta = -r u_x \sin\theta + r u_y \cos\theta \quad -③$$

$$V_\theta = -r V_x \sin\theta + r V_y \cos\theta \quad -④$$

Applying CR equation to ④, we get

$$\left.\begin{aligned} V_\theta &= r u_x \cos\theta + r u_y \sin\theta \\ &= r u_r \\ \text{III}^{ly},\quad u_\theta &= -r V_r \end{aligned}\right\} \text{CR-equation in POLAR FORM.}$$

—x—

Lecture 6 : Analytic functions and power series

A function $f: U \to \mathbb{C}$ is said to be analytic at $z_0 \in U$ if it is differentiable in a neighbourhood of z_0.

Eg: ① $|z|^2$ is differentiable at '0' but not at any other point. So, it is not analytic at '0'.

② z^n $(n>0)$ is differentiable everywhere in $\mathbb{C}$ hence it is analytic at any pt of $\mathbb{C}$.

§§ Are there examples of analytic functions other than z^n ??

The answer is YES !! And the recipe is via POWER SERIES

A series of the form $\sum_{n=0}^{\infty} a_n (z-z_0)^n$, where $a_n \in \mathbb{C}$ $\forall n$, $z_0 \in \mathbb{C}$ and z is an indeterminate is called a power series around z_0

Eg: $\sum_{n=0}^{\infty} z^n$ is a power series around '0'.

Recall that $\sum_{n=0}^{\infty} z^n$ is convergent if $|z|<1$

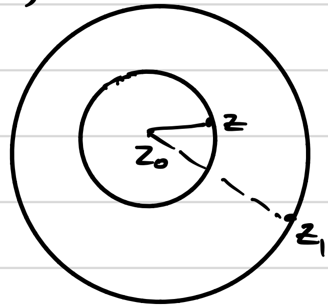
and it is divergent if $|z| \geq 1$

We say that 1 is the radius of convergence of $\sum_{n=0}^{\infty} z^n$.

Defn: Radius of convergence of a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n := \sup\{|z_1-z_0| \,/\, \sum_{n=0}^{\infty} a_n(z_1-z_0)^n \text{ converges}\}$

Why call it "radius of convergence"??

Proposition: Let the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ convergent at z_1. Then $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is convergent $\forall\, z \ni |z-z_0| < |z_1-z_0|$.



Pf: Let $\dfrac{|z-z_0|}{|z_1-z_0|} = \rho < 1$

$$\Rightarrow \frac{|z-z_0|^n}{|z_1-z_0|^n} = \rho^n$$

Further, $\sum_{n=0}^{\infty} a_n(z_1-z_0)^n$ is cgt $\Rightarrow a_n(z_1-z_0)^n \to 0$ as $n\to\infty$

$\therefore |a_n(z_1-z_0)^n| < K$ for some $K>0$

$$|a_n(z-z_0)^n| = |a_n| \frac{|z-z_0|^n}{|z_1-z_0|^n} |z_1-z_0|^n$$

$$= \rho^n |a_n||z_1-z_0|^n < K\rho^n \quad \forall n.$$

Since $\sum_{n=0}^{\infty} K\rho^n$ is cgt, by comparison test

$\sum a_n(z-z_0)^n$ is cgt, (absolutely) as required. ▨

Thus, we get an equivalent definition of Radius of convergence (which also justifies its name).

Radius of convergence of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is $R>0$ (including ∞) such that

$\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is cgt (absolutely) $\forall z \ni |z-z_0|<R$

and $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is dgt $\forall z \ni |z-z_0|>R$.

Hadamard's formula for Radius of convergence of a power series.

Recall, $\sum_{n=0}^{\infty} a_n$ be a series of complex number then if $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = l$

then $\sum_{n=0}^{\infty} a_n$ is cgt if $l < 1$

.... dgt if $l > 1$.

} Ratio Test

Consider the power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$.

Let $\lim_{n\to\infty} \frac{|a_{n+1}(z-z_0)^{n+1}|}{|a_n(z-z_0)^n|} = l$ (ie limit exists)

Then $l = |z-z_0| \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$

Applying Ratio test, we get $\sum a_n (z-z_0)^n$ that the series converges (absolutely) if $l < 1$

$$\Leftrightarrow |z-z_0| \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$$

$$\Leftrightarrow |z-z_0| < \frac{1}{\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}}$$

(with the convention $1/0 = \infty$ & $1/\infty = 0$)

the series diverges if $l > 1$

i.e. $|z - z_0| \lim \frac{|a_{n+1}|}{|a_n|} > 1$

i.e. $|z - z_0| > \frac{1}{\lim \frac{|a_{n+1}|}{|a_n|}}$ (again with the same convention as above: $1/0 = \infty$, $1/\infty = 0$)

Thus, by defn of Radius of convergence, the radius of convergence of $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ is

$$\boxed{1 \Big/ \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}}$$

whenever the limit exists (Convention: $1/\infty = 0$, $1/0 = \infty$).

On the other hand, if we apply Root test to $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ then the series converges (absolutely) if $\limsup \sqrt[n]{|a_n (z-z_0)^n|} < 1$ and it diverges if $\limsup \sqrt[n]{|a_n (z-z_0)^n|} > 1$.

Since $\limsup \sqrt[n]{|a_n||z-z_0|^n}$

$$= \left(\limsup \sqrt[n]{|a_n|}\right)|z-z_0|$$

We get that $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges

if $\quad |z-z_0| < \dfrac{1}{\limsup \sqrt[n]{|a_n|}}$

diverges if $|z-z_0| > \dfrac{1}{\limsup \sqrt[n]{|a_n|}}$

Convention: $\quad 1/0 = \infty$

$1/\infty = 0$

So, radius of convergence of $\sum_{n=0}^{\infty} a_n (z-z_0)^n$

is $\quad \dfrac{1}{\limsup \sqrt[n]{|a_n|}}$, with the convention

that $1/0 = \infty$ & $1/\infty = 0$.

Eg: $\sum_{n=0}^{\infty} z^n$, $\quad \lim_{n\to\infty} \dfrac{|a_{n+1}|}{|a_n|} = 1$

$$\limsup_n \sqrt[n]{|a_n|} = \limsup_n 1 = 1$$

$\therefore \sum_{n=0}^{\infty} z^n$ has radius of convergence 1.

§ Power series are analytic functions

THEOREM: Consider the function given by

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \forall\, z \ni |z-z_0| < R$$

where R = radius of convergence of the power series

Then f is analytic in $B_R(z_0)$ and

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}.$$

Lemma: Let $\sum_{n=0}^{\infty} a_n z^n$ be cgt for $|z|<R$. Then $\sum_{n=1}^{\infty} n a_n z^{n-1}$ is also cgt for $|z|<R$.

Pf: The radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$ is

$$R = \frac{1}{\limsup \sqrt[n]{|a_n|}}.$$

By Lecture 2 (Appendix), $\limsup \sqrt[n]{|a_n|}$

$$= \limsup \sqrt[n]{n|a_n|}$$

$$\left(\because \lim_{n\to\infty} \sqrt[n]{n} = 1\right)$$

$\therefore$ Radius of convergence of $\sum_{n=1}^{\infty} n a_n z^{n-1}$ is

$$\frac{1}{\limsup \sqrt[n]{n|a_n|}} = \frac{1}{\limsup \sqrt[n]{|a_n|}} = R$$

$\boxtimes$

This one is really better:

Pf of theorem:

Consider

$$\left| \frac{f(z)-f(w)}{z-w} - g(w) \right|.$$

For $N \in \mathbb{N}$, $\displaystyle\sum_{n=0}^{\infty} a_n z^n = \underbrace{\sum_{n=0}^{N} a_n z^n}_{S_N(z)} + \underbrace{\sum_{n=N+1}^{\infty} a_n z^n}_{E_N(z)}$

$S_N(z)$ is a polynomial whose derivative is

$$S_N'(z) = \sum_{n=0}^{N} n a_n z^{n-1} \left(= N\text{-th partial sum of } \sum_{n=0}^{\infty} n a_n z^{n-1} \right)$$

$$\frac{f(z)-f(w)}{z-w} - g(w) = \frac{\sum_{n=0} a_n z^n - \sum a_n w^n}{z-w} - g(w)$$

$$= \frac{S_N(z) + E_N(z) - (S_N(w) - E_N(w))}{z-w} - g(w)$$

$$= \frac{S_N(z) - S_N(w)}{z-w} + \frac{E_N(z) - E_N(w)}{z-w} - S_N'(w)$$

$$+ S_N'(w) - g(w)$$

$$\therefore \left| \frac{f(z)-f(\omega)}{z-\omega} - g(\omega) \right| \leq$$

$$\underbrace{\left| \frac{S_N(z)-S_N(\omega)}{z-\omega} - S_N'(\omega) \right|} + \left| S_N'(\omega) - g(\omega) \right|$$

$$+ \left| \frac{E_N(z) - E_N(\omega)}{z-\omega} \right|$$

$$\lim_{z\to\omega} \frac{S_N(z)-S_N(\omega)}{z-\omega} = S_N'(\omega)$$

Choose $\delta_1 > 0 \ni \underbrace{\left|\left(\qquad \right)\right|} < \varepsilon/3$

$S_N'(\omega) = N^{th}$ partial sum of $g(\omega)$

$\therefore$ as $N \to \infty$

$S_N'(\omega) \longrightarrow g(\omega)$

Choose $N_1 > 0$

$\left|\left(\qquad \right)\right| < \varepsilon/3$

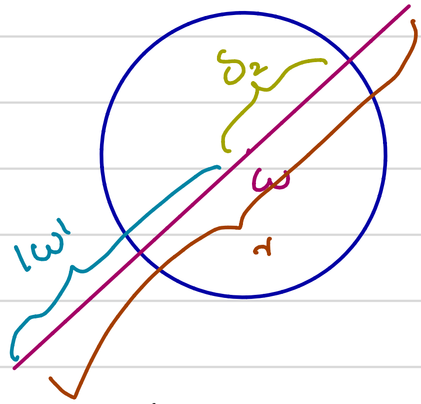
$$\left| \sum_{n=N+1}^{\infty} a_n \frac{(z^n - \omega^n)}{(z-\omega)} \right|$$

$$= \left| \sum_{n=N+1}^{\infty} a_n \left(z^{n-1} + z^{n-2}\omega + \cdots + \omega^{n-1} \right) \right|$$

$$\leq \sum_{n=N+1}^{\infty} |a_n| \left(|z|^{n-1} + |z|^{n-2}|\omega| + \cdots + |\omega|^{n-1} \right)$$

($\because$ partial sums satisfy this inequality).

Choose $\delta_2 \ni |\omega| + \delta_2 < r$



then for $z \in B_\omega(\delta)$

$$|z| \le |z-\omega| + |\omega| < r$$

$$\therefore \sum_{n=N+1}^{\infty} |a_n|\left(|z|^{n-1} + \cdots + |\omega|^{n-1}\right) \le \sum_{n=N+1}^{\infty} |a_n|\, n \cdot r^{n-1}$$

Since $r < R$, & $\sum_{n=0}^{\infty} n a_n z^{n-1}$ converges absolutely $\forall\ |z| < R$

we get a $N_2 >> 0 \ni \sum_{n=N_1+1}^{\infty} n|a_n| r^{n-1} < \varepsilon/3$

$$\left| \frac{E_{N_2}(z) - E_{N_2}(\omega)}{z-\omega} \right| < \varepsilon/3$$

Choose $N \geq \{N_1, N_2\}$

$\delta < \{\delta_1, \delta_2\}$

then,

$|\text{Term } 1| + |\text{Term } 2| + |\text{Term } 3| < \varepsilon$.