Practice problems 4 : Continuity and Limit

1. Find the value of α such that $\lim_{x\to -1} \frac{2x^2-\alpha x-14}{x^2-2x-3}$ exists. Find the limit.

2. Let $\lim_{x\to 0} \frac{f(x)}{x^2} = 5$. Show that $\lim_{x\to 0} \frac{f(x)}{x} = 0$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$. Suppose $\lim_{x\to x_0} f(x)$ exists. Show that $\lim_{x\to 0} f(x+x_0) = \lim_{x\to x_0} f(x)$.

4. Let $f(x) = |x|$ for every $x \in \mathbb{R}$. Show that f is continuous on $\mathbb{R}$.

5. Let $f : [0,\pi] \to \mathbb{R}$ be defined by $f(0) = 0$ and $f(x) = x sin\frac{1}{x} - \frac{1}{x} cos\frac{1}{x}$ for $x \neq 0$. Is f continuous ?

6. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous such that given any two points $x_1 < x_2$, there exists a point x_3 such that $x_1 < x_3 < x_2$ and $f(x_3) = g(x_3)$. Show that $f(x) = g(x)$ for all x.

7. Let $f(x) = 0$ when x is rational and 1 when x is irrational. Determine the points of continuity for the function f.

8. Let $[\cdot]$ denote the integer part function and $f : [0,\infty) \to \mathbb{R}$ be defined by $f(x) = [x^2] sin\pi x$. Show that f is continuous at each $x \neq \sqrt{n}$, $n = 1, 2, ...$ Further, show that f is discontinuous on $\{x \in [0,\infty) : x = \sqrt{n}$ where $n \neq k^2$, for some positive integer $k\}$.

9. Let $f : \mathbb{R} \to (0,\infty)$ satisfy $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. Suppose f is continuous at $x = 0$. Show that f is continuous at all $x \in \mathbb{R}$.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x) = f(x^2)$ for all $x \in \mathbb{R}$. Show that f is constant.

11. Suppose $f : [0,\infty) \to \mathbb{R}$ is continuous and $\lim_{x\to\infty} f(x)$ exists. Show that f is bounded on $[0,\infty)$.

12. **(*)** Let $f : [0,1] \to \mathbb{R}$ be one-one and onto function. Suppose f is continuous. Show that f^{-1} is also continuous.

13. **(*)** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Show that $f(x) = f(1)x$ for all $x \in \mathbb{R}$.

14. **(*)** Let $f : (0,1) \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ where } p, q \in \mathbb{N} \text{ and } p, q \text{ have no common factors} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

   (a) Let $x_n = \frac{p_n}{q_n} \in (0,1)$ where $p_n, q_n \in \mathbb{N}$ and have no common factors. Suppose $x_n \to x$ for some x with $x_n \neq x$ for all $n \in \mathbb{N}$. Show that $\lim_{n\to\infty} q_n = \infty$.

   (b) Show that f is continuous at every irrational.

   (c) Show that f is discontinuous at every rational.

Practice Problems 4: Hints/solutions

1. $\alpha = 12$ and the limit is 4.

2. Note that $\frac{f(x)}{x} = \frac{f(x)}{x^2}x$ for $x \neq 0$.

3. Let $\lim_{x\to x_0} f(x) = M$ for some $M \in \mathbb{R}$. Let $x_n \to 0, x_n \neq 0\ \forall\ n$. Then $x_n + x_0 \to x_0$. Since $\lim_{x\to x_0} f(x) = M$, $f(x_n + x_0) \to M$. This implies that $\lim_{x\to 0} f(x + x_0) = M$.

4. Let $x \in \mathbb{R}$ and $x_n \to x$. Then $|x_n| \to |x|$, because, $||x_n - |x|| \leq |x_n - x|$. Therefore f is continuous at x.

5. The function is not continuous at 0, because, $x_n = \frac{1}{2n\pi} \to 0$ but $f(\frac{1}{2n\pi}) \nrightarrow f(0)$.

6. Fix some $x_0 \in \mathbb{R}$. For every n, find x_n such that $x_0 - \frac{1}{n} < x_n < x_0$ and $(f - g)(x_n) = 0$. Allow $n \to \infty$ and apply the continuity.

7. Suppose x_0 is rational. Find an irrational sequence (x_n) such that $x_n \to x_0$. Then $f(x_n) = 1 \nrightarrow f(x_0) = 0$. Therefore f is not continuous at x_0. Let y_0 be rational. Show that f is not continuous at y_0.

8. <u>Case 1:</u> $x_0 \neq \sqrt{n},\ n = 1, 2, ...$ It is clear that f is continuous at x_0. <u>Case 2:</u> $x_0 = \sqrt{n}$ where $n = k^2$, for some positive integer k, i.e $x_0 = k$. In this case $\lim_{x\to k^+} f(x) = \lim_{x\to k^-} f(x) = 0$. <u>Case 3:</u> $x_0 = \sqrt{n}$ where $n \neq k^2$, for some positive integer k. In this case, $\lim_{x\to\sqrt{n}^+} f(x) = nsin(\pi\sqrt{n})$ and $\lim_{x\to\sqrt{n}^-} f(x) = (n - 1)sin(\pi\sqrt{n})$.

9. Since $f(0) = f(0)^2, f(0) = 1$ and since $f(x - x) = f(0), f(-x) = \frac{1}{f(x)}$. Let $x_0 \in \mathbb{R}$ and $x_n \to x_0$. By continuity at 0, $f(x_n - x_0) \to 1$ and hence $f(x_n) \to \frac{1}{f(-x_0)} = f(x_0)$.

10. Suppose $x > 0$. By the assumption, $f(x) = f(x^{\frac{1}{2}}) = f(x^{\frac{1}{2^2}}) = f(x^{\frac{1}{2^n}})$. Since $x^{\frac{1}{2^n}} \to 1, f(x^{\frac{1}{2^n}}) \to f(1)$, i.e. $f(x) = f(1)$. Now $f(-x) = f((-x)^2) = f(x^2) = f(x)$. At $x = 0$, by continuity, $\lim_{x\to 0} f(x) = f(0) = f(1)$. Therefore $f(x) = f(1)$ for all $x \in \mathbb{R}$.

11. Suppose $\lim_{x\to\infty} f(x) = \beta$ for some β. Then there exists a positive real number M such that $|f(x) - \beta| < 1$ for all x such that $x \geq M$. Then $|f(x)| \leq 1 + \beta$ for every x such that $x \geq M$. That is f is bounded on $\{x : x \geq M\}$. Also by continuity, f is bounded on $[0, M]$. Therefore f is bounded on $[0, \infty)$.

12. Let $f(x_n) \to f(x_0)$ for some $x_n, x_0 \in [0, 1]$. We show that $x_n \to x_0$ which proves that f^{-1} is continuous. If (x_{n_k}) is any subsequence, then by Bolzano-Weierstrass theorem, there exists a subsequence $(x_{n_{k_i}})$ such that $x_{n_{k_i}} \to \alpha$ for some $\alpha \in \mathbb{R}$. By continuity $f(x_{n_{k_i}}) \to f(\alpha)$. By assumption $f(\alpha) = f(x_0)$ and since f is one-one $x_0 = \alpha$. By Problem 8 of Practice problems 3, $x_n \to x_0$.

13. First observe that $f(0) = 0$ and $f(n) = nf(1)$ for all $n \in \mathbb{N}$. Next note that $f(-1) = -f(1)$ and $f(m) = f(1)m$ for all $m \in \mathbb{Z}$. By observing $f(\frac{1}{n}) = f(1)\frac{1}{n}$ for all $n \in \mathbb{N}$, show that $f(\frac{m}{n}) = f(1)\frac{m}{n}$ for all $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. Finally take any irrational number x and find $r_n \in \mathbb{Q}$ such that $r_n \to x$ and apply the continuity to conclude that $f(x) = f(1)x$.

14. (a) If for some $M \in \mathbb{N}$, $q_n < M$ for all $n \in \mathbb{N}$, then the set $\{x_n : n \in \mathbb{N}\}$ is finite which is not true. Similarly we can show that any subsequence of (q_n) cannot be bounded.

    (b) Suppose x_0 is rational in $(0, 1)$ and $x_n \to x_0$ where x_n can be rational or irrational. Apply (a) to show that $f(x_n) \to 0 = f(x_0)$.

    (c) Suppose x_0 is rational in $(0, 1)$. To show that f is discontinuous at x_0, choose an irrational sequence (x_n) such that $x_n \to x_0$.