CHAPTER 13

Fourier Series

Mathematicians of the eighteenth century, including Daniel Bernoulli and Leonard Euler, expressed the problem of the vibratory motion of a stretched string through partial differential equations that had no solutions in terms of "elementary functions." Their resolution of this difficulty was to introduce infinite series of sine and cosine functions that satisfied the equations. In the early nineteenth century, Joseph Fourier, while studying the problem of heat flow, developed a cohesive theory of such series. Consequently, they were named after him. Fourier series and Fourier integrals are investigated in this and the next chapter. As you explore the ideas, notice the similarities and differences with the chapters on infinite series and improper integrals.

PERIODIC FUNCTIONS

A function $f(x)$ is said to have a *period* T or to be *periodic* with period T if for all $x, f(x+T) = f(x)$, where T is a positive constant. The least value of $T > 0$ is called the *least period* or simply *the period* of $f(x)$.

EXAMPLE 1. The function $\sin x$ has periods $2\pi, 4\pi, 6\pi, \ldots,$ since $\sin(x+2\pi), \sin(x+4\pi), \sin(x+6\pi), \ldots$ all equal $\sin x$. However, 2π is the *least period* or *the period* of $\sin x$.

EXAMPLE 2. The period of $\sin nx$ or $\cos nx$, where n is a positive integer, is $2\pi/n$.

EXAMPLE 3. The period of $\tan x$ is π.

EXAMPLE 4. A constant has any positive number as period.

Other examples of periodic functions are shown in the graphs of Figures 13-1(a), (b), and (c) below.

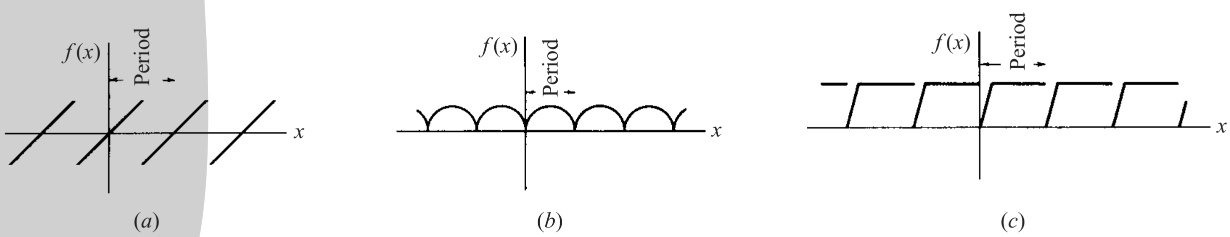

Fig. 13-1

FOURIER SERIES

Let $f(x)$ be defined in the interval $(-L, L)$ and outside of this interval by $f(x+2L) = f(x)$, i.e., $f(x)$ is $2L$-periodic. It is through this avenue that a new function on an infinite set of real numbers is created from the image on $(-L, L)$. The *Fourier series* or *Fourier expansion* corresponding to $f(x)$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \tag{1}$$

where the *Fourier coefficients* a_n and b_n are

$$\begin{cases} a_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \\ b_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \end{cases} \qquad n = 0, 1, 2, \ldots \tag{2}$$

ORTHOGONALITY CONDITIONS FOR THE SINE AND COSINE FUNCTIONS

Notice that the Fourier coefficients are integrals. These are obtained by starting with the series, (*1*), and employing the following properties called orthogonality conditions:

$$\begin{aligned} &(a)\quad \int_{-L}^{L}\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = 0 \text{ if } m \neq n \text{ and } L \text{ if } m = n \\ &(b)\quad \int_{-L}^{L}\sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = 0 \text{ if } m \neq n \text{ and } L \text{ if } m = n \\ &(c)\quad \int_{-L}^{L}\sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = 0. \text{ Where } m \text{ and } n \text{ can assume any positive integer values.} \end{aligned} \tag{3}$$

An explanation for calling these orthogonality conditions is given on Page 342. Their application in determining the Fourier coefficients is illustrated in the following pair of examples and then demonstrated in detail in Problem 13.4.

EXAMPLE 1. To determine the Fourier coefficient a_0, integrate both sides of the Fourier series (*1*), i.e.,

$$\int_{-L}^{L} f(x)\,dx = \int_{-L}^{L}\frac{a_0}{2}\,dx + \int_{-L}^{L}\sum_{n=1}^{\infty}\left\{a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right\}dx$$

Now $\displaystyle\int_{-L}^{L}\frac{a_0}{2}\,dx = a_0 L$, $\displaystyle\int_{-l}^{L}\sin\frac{n\pi x}{L}\,dx = 0$, $\displaystyle\int_{-L}^{L}\cos\frac{n\pi x}{L}\,dx = 0$, therefore, $a_0 = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\,dx$

EXAMPLE 2. To determine a_1, multiply both sides of (*1*) by $\cos\dfrac{\pi x}{L}$ and then integrate. Using the orthogonality conditions $(3)_a$ and $(3)_c$, we obtain $a_1 = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos\frac{\pi x}{L}\,dx$. Now see Problem 13.4.

If $L = \pi$, the series (*1*) and the coefficients (*2*) or (*3*) are particularly simple. The function in this case has the period 2π.

DIRICHLET CONDITIONS

Suppose that

(1) $f(x)$ is defined except possibly at a finite number of points in $(-L, L)$
(2) $f(x)$ is periodic outside $(-L, L)$ with period $2L$

(3) $f(x)$ and $f'(x)$ are piecewise continuous in $(-L, L)$.

Then the series *(1)* with Fourier coefficients converges to

(*a*) $f(x)$ if x is a point of continuity

(*b*) $\dfrac{f(x+0)+f(x-0)}{2}$ if x is a point of discontinuity

Here $f(x+0)$ and $f(x-0)$ are the right- and left-hand limits of $f(x)$ at x and represent $\lim\limits_{\epsilon\to 0+} f(x+\epsilon)$ and $\lim\limits_{\epsilon\to 0+} f(x-\epsilon)$, respectively. For a proof see Problems 13.18 through 13.23.

The conditions (1), (2), and (3) imposed on $f(x)$ are *sufficient* but not necessary, and are generally satisfied in practice. There are at present no known necessary and sufficient conditions for convergence of Fourier series. It is of interest that continuity of $f(x)$ does not *alone* ensure convergence of a Fourier series.

ODD AND EVEN FUNCTIONS

A function $f(x)$ is called *odd* if $f(-x) = -f(x)$. Thus, $x^3, x^5 - 3x^3 + 2x, \sin x, \tan 3x$ are odd functions.

A function $f(x)$ is called *even* if $f(-x) = f(x)$. Thus, $x^4, 2x^6 - 4x^2 + 5, \cos x, e^x + e^{-x}$ are even functions.

The functions portrayed graphically in Figures 13-1(*a*) and 13-1(*b*) are odd and even respectively, but that of Fig. 13-1(*c*) is neither odd nor even.

In the Fourier series corresponding to an odd function, only sine terms can be present. In the Fourier series corresponding to an even function, only cosine terms (and possibly a constant which we shall consider a cosine term) can be present.

HALF RANGE FOURIER SINE OR COSINE SERIES

A half range Fourier sine or cosine series is a series in which only sine terms or only cosine terms are present, respectively. When a half range series corresponding to a given function is desired, the function is generally defined in the interval $(0, L)$ [which is half of the interval $(-L, L)$, thus accounting for the name *half range*] and then the function is specified as odd or even, so that it is clearly defined in the other half of the interval, namely, $(-L, 0)$. In such case, we have

$$\begin{cases} a_n = 0, \quad b_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx & \text{for } \textit{half range sine series} \\ b_n = 0, \quad a_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx & \text{for } \textit{half range cosine series} \end{cases} \tag{4}$$

PARSEVAL'S IDENTITY

If a_n and b_n are the Fourier coefficients corresponding to $f(x)$ and if $f(x)$ satisfies the Dirichlet conditions.

Then
$$\frac{1}{L}\int_{-L}^{L}\{f(x)\}^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \tag{5}$$

(See Problem 13.13.)

DIFFERENTIATION AND INTEGRATION OF FOURIER SERIES

Differentiation and integration of Fourier series can be justified by using the theorems on Pages 271 and 272, which hold for series in general. It must be emphasized, however, that those theorems provide sufficient conditions and are not necessary. The following theorem for integration is especially useful.

Theorem. The Fourier series corresponding to $f(x)$ may be integrated term by term from a to x, and the resulting series will converge uniformly to $\int_a^x f(x)\,dx$ provided that $f(x)$ is piecewise continuous in $-L \leqq x \leqq L$ and both a and x are in this interval.

COMPLEX NOTATION FOR FOURIER SERIES

Using Euler's identities,

$$e^{i\theta} = \cos\theta + i\sin\theta, \qquad e^{-i\theta} = \cos\theta - i\sin\theta \tag{6}$$

where $i = \sqrt{-1}$ (see Problem 11.48, Chapter 11, Page 295), the Fourier series for $f(x)$ can be written as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{in\pi x/L} \tag{7}$$

where

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}\,dx \tag{8}$$

In writing the equality (*7*), we are supposing that the Dirichlet conditions are satisfied and further that $f(x)$ is continuous at x. If $f(x)$ is discontinuous at x, the left side of (*7*) should be replaced by $\dfrac{(f(x+0)+f(x-0)}{2}$.

BOUNDARY-VALUE PROBLEMS

Boundary-value problems seek to determine solutions of partial differential equations satisfying certain prescribed conditions called *boundary conditions*. Some of these problems can be solved by use of Fourier series (see Problem 13.24).

EXAMPLE. The classical problem of a vibrating string may be idealized in the following way. See Fig. 13-2.

Suppose a string is tautly stretched between points $(0, 0)$ and $(L, 0)$. Suppose the tension, **F**, is the same at every point of the string. The string is made to vibrate in the xy plane by pulling it to the parabolic position $g(x) = m(Lx - x^2)$ and releasing it. (m is a numerically small positive constant.) Its equation will be of the form $y = f(x, t)$. The problem of establishing this equation is idealized by (*a*) assuming that the constant tension, F, is so large as compared to the weight wL of the string that the gravitational force can be neglected, (*b*) the displacement at any point of the string is so small that the length of the string may be taken as L for any of its positions, and (*c*) the vibrations are purely transverse. The force acting on a segment PQ is $\frac{w}{g}\Delta x \frac{\partial^2 y}{\partial t^2}$, $x < x_1 < x + \Delta x$, $g \approx 32$ ft per sec.^{2}. If α and β are the angles that **F** makes with the horizontal, then the vertical

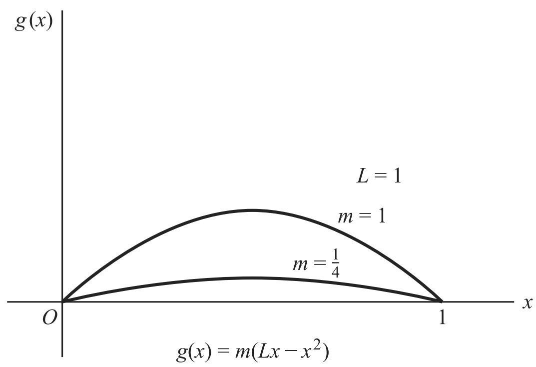

Fig. 13-2

difference in tensions is $F(\sin\alpha - \sin\beta)$. This is the force producing the acceleration that accounts for the vibratory motion.

Now $F\{\sin\alpha - \sin\beta\} = F\left\{\dfrac{\tan\alpha}{\sqrt{1+\tan^2\alpha}} - \dfrac{\tan\beta}{\sqrt{1+\tan^2\beta}}\right\} \approx F\{\tan\alpha - \tan\beta\} = F\left\{\dfrac{\partial y}{\partial x}(x+\Delta x, t) - \dfrac{\partial y}{\partial x}(x, t)\right\}$, where the squared terms in the denominator are neglected because the vibrations are small. Next, equate the two forms of the force, i.e.,

$$F\left\{\frac{\partial y}{\partial x}(x+\Delta x, t) - \frac{\partial y}{\partial x}(x, t)\right\} = \frac{w}{g}\Delta x \frac{\partial^2 y}{\partial t^2}$$

divide by Δx, and then let $\Delta x \to 0$. After letting $\alpha = \sqrt{\dfrac{Fg}{w}}$, the resulting equation is

$$\frac{\partial^2 y}{\partial t^2} = \alpha^2 \frac{\partial^2 y}{\partial x^2}$$

This homogeneous second partial derivative equation is the classical equation for the vibrating string. Associated boundary conditions are

$$y(0, t) = 0, y(L, t) = 0, t > 0$$

The initial conditions are

$$y(x, 0) = m(Lx - x^2), \frac{\partial y}{\partial t}(x, 0) = 0, 0 < x < L$$

The method of solution is to separate variables, i.e., assume

$$y(x, t) = G(x)H(t)$$

Then upon substituting

$$G(x)\,H''(t) = \alpha^2 G''(x)\,H(t)$$

Separating variables yields

$$\frac{G''}{G} = k, \frac{H''}{H} = \alpha^2 k, \text{ where } k \text{ is an arbitrary constant}$$

Since the solution must be periodic, trial solutions are

$$G(x) = c_1 \sin\sqrt{-k}\,x + c_2 \cos\sqrt{-k}\,x, < 0$$
$$H(t) = c_3 \sin\alpha\sqrt{-k}\,t + c_4 \cos\alpha\sqrt{-k}\,t$$

Therefore

$$y = GH = [c_1 \sin\sqrt{-k}\,x + c_2 \cos\sqrt{-k}\,x][c_3 \sin\alpha\sqrt{-k}\,t + c_4 \cos\alpha\sqrt{-k}\,t]$$

The initial condition $y = 0$ at $x = 0$ for all t leads to the evaluation $c_2 = 0$.

Thus

$$y = [c_1 \sin\sqrt{-k}\,x][c_3 \sin\alpha\sqrt{-k}\,t + c_4 \cos\alpha\sqrt{-k}\,t]$$

Now impose the boundary condition $y = 0$ at $x = L$, thus $0 = [c_1 \sin\sqrt{-k}\,L][c_3 \sin\alpha\sqrt{-k}\,t + c_4 \cos\alpha\sqrt{-k}\,t]$.

$c_1 \neq 0$ as that would imply $y = 0$ and a trivial solution. The next simplest solution results from the choice $\sqrt{-k} = \dfrac{n\pi}{L}$, since $y = \left[c_1 \sin\dfrac{n\pi}{L}x\right]\left[c_3 \sin\alpha\dfrac{n\pi}{l}\,t + c_4 \cos\alpha\dfrac{n\pi}{L}\,t\right]$ and the first factor is zero when $x = L$.

With this equation in place the boundary condition $\frac{\partial y}{\partial t}(x, 0) = 0,\ 0 < x < L$ can be considered.

$$\frac{\partial y}{\partial t} = \left[c_1 \sin\frac{n\pi}{L} x\right]\left[c_3\alpha\frac{n\pi}{L} \cos\alpha\frac{n\pi}{L} t - c_4\alpha\frac{n\pi}{L} \sin\alpha\frac{n\pi}{L} t\right]$$

At $t = 0$

$$0 = \left[c_1 \sin\frac{n\pi}{L} x\right] c_3\alpha\frac{n\pi}{L}$$

Since $c_1 \neq 0$ and $\sin\frac{n\pi}{L} x$ is not identically zero, it follows that $c_3 = 0$ and that

$$y = \left[c_1 \sin\frac{n\pi}{L} x\right]\left[c_4\alpha\frac{n\pi}{L} \cos\alpha\frac{n\pi}{L} t\right]$$

The remaining initial condition is

$$y(x, 0) = m(Lx - x^2),\ 0 < x < L$$

When it is imposed

$$m(Lx - x^2) = c_1 c_4 \alpha \frac{n\pi}{L} \sin\frac{n\pi}{L} x$$

However, this relation cannot be satisfied for all x on the interval $(0, L)$. Thus, the preceding extensive analysis of the problem of the vibrating string has led us to an inadequate form

$$y = c_1 c_4 \alpha \frac{n\pi}{L} \sin\frac{n\pi}{L} x \cos\alpha\frac{n\pi}{L} t$$

and an initial condition that is not satisfied. At this point the power of Fourier series is employed. In particular, a theorem of differential equations states that any finite sum of a particular solution also is a solution. Generalize this to infinite sum and consider

$$y = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L} x \cos\alpha\frac{n\pi}{L} t$$

with the initial condition expressed through a half range sine series, i.e.,

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L} x = m(Lx - x^2), \qquad t = 0$$

According to the formula of Page 338 for coefficient of a half range sine series

$$\frac{L}{2m} b_n = \int_0^L (Lx - x^2) \sin\frac{n\pi x}{L}\, dx$$

That is

$$\frac{L}{2m} b_n = \int_0^L Lx \sin\frac{n\pi x}{L}\, dx - \int_0^L x^2 \sin\frac{n\pi x}{L}\, dx$$

Application of integration by parts to the second integral yields

$$\frac{L}{2m} b_n = L\int_0^L x \sin\frac{n\pi x}{L}\, dx + \frac{L^3}{n\pi} \cos n\pi + \int_0^L \frac{L}{n\pi} \cos\frac{n\pi x}{L}\, 2x\, dx$$

When integration by parts is applied to the two integrals of this expression and a little algebra is employed the result is

$$b_n = \frac{4L^2}{(n\pi)^3}(1 - \cos n\pi)$$

Therefore,

$$y = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \cos \alpha \frac{n\pi}{L} t$$

with the coefficients b_n defined above.

ORTHOGONAL FUNCTIONS

Two vectors $\mathbf{A}$ and $\mathbf{B}$ are called *orthogonal* (perpendicular) if $\mathbf{A} \cdot \mathbf{B} = 0$ or $A_1B_1 + A_2B_2 + A_3B_3 = 0$, where $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$. Although not geometrically or physically evident, these ideas can be generalized to include vectors with more than three components. In particular, we can think of a function, say, $A(x)$, as being a vector with an *infinity of components* (i.e., an *infinite dimensional vector*), the value of each component being specified by substituting a particular value of x in some interval (a, b). It is natural in such case to define two functions, $A(x)$ and $B(x)$, as *orthogonal* in (a, b) if

$$\int_a^b A(x)\, B(x)\, dx = 0 \tag{9}$$

A vector $\mathbf{A}$ is called a *unit vector* or *normalized vector* if its magnitude is unity, i.e., if $\mathbf{A} \cdot \mathbf{A} = A^2 = 1$. Extending the concept, we say that the function $A(x)$ is *normal* or *normalized* in (a, b) if

$$\int_a^b \{A(x)\}^2\, dx = 1 \tag{10}$$

From the above it is clear that we can consider a set of functions $\{\phi_k(x)\}$, $k = 1, 2, 3, \ldots$, having the properties

$$\int_a^b \phi_m(x)\phi_n(x)\, dx = 0 \qquad m \neq n \tag{11}$$

$$\int_a^b \{\phi_m(x)\}^2\, dx = 1 \qquad m = 1, 2, 3, \ldots \tag{12}$$

In such case, each member of the set is orthogonal to every other member of the set and is also normalized. We call such a set of functions an *orthonormal set*.

The equations (*11*) and (*12*) can be summarized by writing

$$\int_a^b \phi_m(x)\phi_n(x)\, dx = \delta_{mn} \tag{13}$$

where δ_{mn}, called *Kronecker's symbol*, is defined as 0 if $m \neq n$ and 1 if $m = n$.

Just as any vector $\mathbf{r}$ in three dimensions can be expanded in a set of mutually orthogonal unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in the form $\mathbf{r} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, so we consider the possibility of expanding a function $f(x)$ in a set of orthonormal functions, i.e.,

$$f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x) \qquad a \leqq x \leqq b \tag{14}$$

As we have seen, Fourier series are constructed from orthogonal functions. Generalizations of Fourier series are of great interest and utility both from theoretical and applied viewpoints.

Solved Problems

FOURIER SERIES

13.1. Graph each of the following functions.

(*a*) $f(x) = \begin{cases} 3 & 0 < x < 5 \\ -3 & -5 < x < 0 \end{cases}$ Period = 10

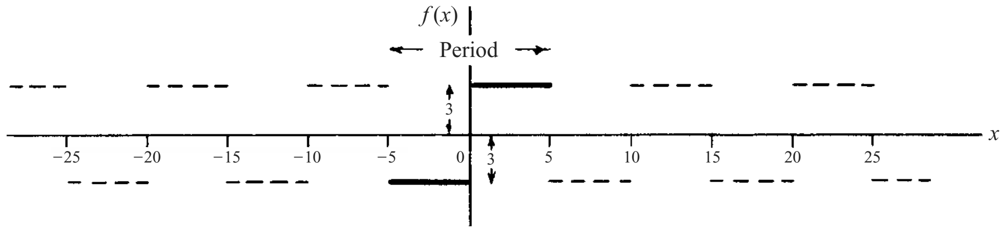

Fig. 13-3

Since the period is 10, that portion of the graph in $-5 < x < 5$ (indicated heavy in Fig. 13-3 above) is extended periodically outside this range (indicated dashed). Note that $f(x)$ is not defined at $x = 0, 5, -5, 10, -10, 15, -15$, and so on. These values are the *discontinuities* of $f(x)$.

(*b*) $f(x) = \begin{cases} \sin x & 0 \leqq x \leqq \pi \\ 0 & \pi < x < 2\pi \end{cases}$ Period $= 2\pi$

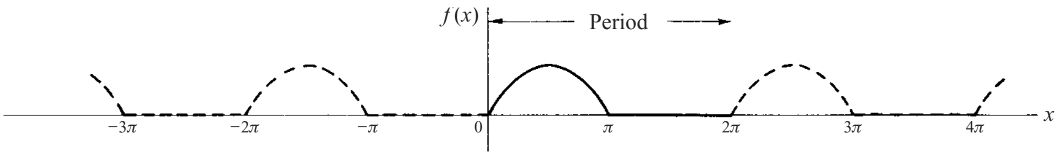

Fig. 13-4

Refer to Fig. 13-4 above. Note that $f(x)$ is defined for all x and is continuous everywhere.

(*c*) $f(x) = \begin{cases} 0 & 0 \leqq x < 2 \\ 1 & 2 \leqq x < 4 \\ 0 & 4 \leqq x < 6 \end{cases}$ Period = 6

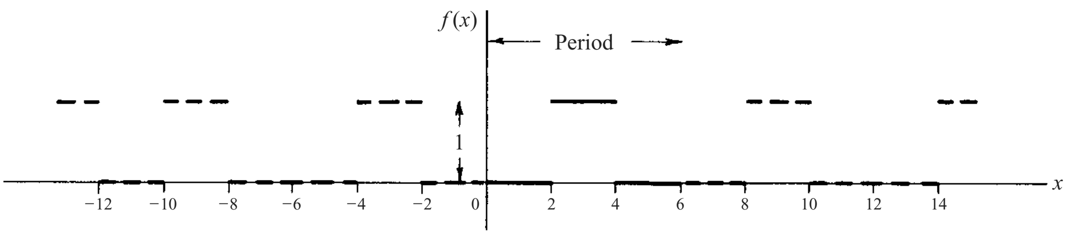

Fig. 13-5

Refer to Fig. 13-5 above. Note that $f(x)$ is defined for all x and is discontinuous at $x = \pm 2, \pm 4, \pm 8, \pm 10, \pm 14, \ldots$.

13.2. Prove $\displaystyle\int_{-L}^{L} \sin\frac{k\pi x}{L}\,dx = \int_{-L}^{L} \cos\frac{k\pi x}{L}\,dx = 0$ if $k = 1, 2, 3, \ldots$.

$$\int_{-L}^{L} \sin\frac{k\pi x}{L}\,dx = -\frac{L}{k\pi}\cos\frac{k\pi x}{L}\bigg|_{-L}^{L} = -\frac{L}{k\pi}\cos k\pi + \frac{L}{k\pi}\cos(-k\pi) = 0$$

$$\int_{-L}^{L} \cos\frac{k\pi x}{L}\,dx = \frac{L}{k\pi}\sin\frac{k\pi x}{L}\bigg|_{-L}^{L} = \frac{L}{k\pi}\sin k\pi - \frac{L}{k\pi}\sin(-k\pi) = 0$$

13.3. Prove (*a*) $\displaystyle\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = \int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$

(*b*) $\displaystyle\int_{-L}^{L} \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = 0$

where m and n can assume any of the values $1, 2, 3, \ldots$.

(*a*) From trigonometry: $\cos A\cos B = \frac{1}{2}\{\cos(A-B) + \cos(A+B)\}$, $\sin A\sin B = \frac{1}{2}\{\cos(A-B) - \cos(A+B)\}$.

Then, if $m \neq n$, by Problem 13.2,

$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_{-L}^{L}\left\{\cos\frac{(m-n)\pi x}{L} + \cos\frac{(m+n)\pi x}{L}\right\}dx = 0$$

Similarly, if $m \neq n$,

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_{-L}^{L}\left\{\cos\frac{(m-n)\pi x}{L} - \cos\frac{(m+n)\pi x}{L}\right\}dx = 0$$

If $m = n$, we have

$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_{-L}^{L}\left(1 + \cos\frac{2n\pi x}{L}\right)dx = L$$

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_{-L}^{L}\left(1 - \cos\frac{2n\pi x}{L}\right)dx = L$$

Note that if $m = n$ these integrals are equal to $2L$ and 0 respectively.

(*b*) We have $\sin A\cos B = \frac{1}{2}\{\sin(A-B) + \sin(A+B)\}$. Then by Problem 13.2, if $m \neq n$,

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_{-L}^{L}\left\{\sin\frac{(m-n)\pi x}{L} + \sin\frac{(m+n)\pi x}{L}\right\}dx = 0$$

If $m = n$,

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_{-L}^{L} \sin\frac{2n\pi x}{L}\,dx = 0$$

The results of parts (*a*) and (*b*) remain valid even when the limits of integration $-L, L$ are replaced by $c, c+2L$, respectively.

13.4. If the series $\displaystyle A + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)$ converges uniformly to $f(x)$ in $(-L, L)$, show that for $n = 1, 2, 3, \ldots$,

(*a*) $\displaystyle a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$, (*b*) $\displaystyle b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$, (*c*) $A = \dfrac{a_0}{2}$.

(*a*) Multiplying

$$f(x) = A + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \tag{1}$$

by $\cos\dfrac{m\pi x}{L}$ and integrating from $-L$ to L, using Problem 13.3, we have

$$\begin{aligned}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx &= A\int_{-L}^{L}\cos\frac{m\pi x}{L}\,dx \\ &\quad + \sum_{n=1}^{\infty}\left\{a_n\int_{-L}^{L}\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx + b_n\int_{-L}^{L}\cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx\right\} \\ &= a_m L \qquad \text{if } m \neq 0\end{aligned}$$

Thus
$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx \qquad \text{if } m = 1, 2, 3, \ldots$$

(*b*) Multiplying (*1*) by $\sin\dfrac{m\pi x}{L}$ and integrating from $-L$ to L, using Problem 13.3, we have

$$\begin{aligned}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx &= A\int_{-L}^{L}\sin\frac{m\pi x}{L}\,dx \\ &\quad + \sum_{n=1}^{\infty}\left\{a_n\int_{-L}^{L}\sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx + b_n\int_{-L}^{L}\sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx\right\} \\ &= b_m L\end{aligned}$$

Thus
$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx \qquad \text{if } m = 1, 2, 3, \ldots$$

(*c*) Integrating of (*1*) from $-L$ to L, using Problem 13.2, gives

$$\int_{-L}^{L} f(x)\,dx = 2AL \qquad \text{or} \qquad A = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

Putting $m = 0$ in the result of part (*a*), we find $a_0 = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\,dx$ and so $A = \dfrac{a_0}{2}$.

The above results also hold when the integration limits $-L, L$ are replaced by $c, c + 2L$.

Note that in all parts above, interchange of summation and integration is valid because the series is *assumed* to converge uniformly to $f(x)$ in $(-L, L)$. Even when this assumption is not warranted, the coefficients a_m and b_m as obtained above are called *Fourier coefficients* corresponding to $f(x)$, and the corresponding series with these values of a_m and b_m is called the *Fourier series* corresponding to $f(x)$. An important problem in this case is to investigate conditions under which this series actually converges to $f(x)$. Sufficient conditions for this convergence are the *Dirichlet conditions* established in Problems 13.18 through 13.23.

13.5. (*a*) Find the Fourier coefficients corresponding to the function

$$f(x) = \begin{cases} 0 & -5 < x < 0 \\ 3 & 0 < x < 5 \end{cases} \qquad \text{Period} = 10$$

(*b*) Write the corresponding Fourier series.

(*c*) How should $f(x)$ be defined at $x = -5$, $x = 0$, and $x = 5$ in order that the Fourier series will converge to $f(x)$ for $-5 \leqq x \leqq 5$?

The graph of $f(x)$ is shown in Fig. 13-6.

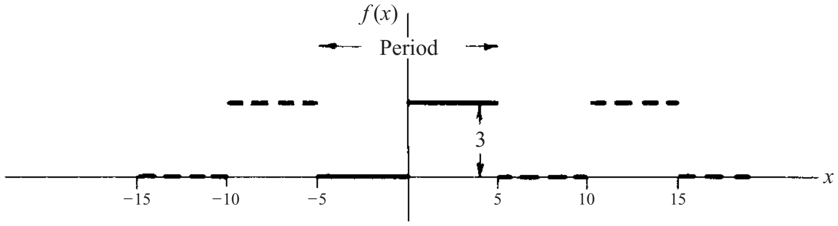

Fig. 13-6

(*a*) Period $= 2L = 10$ and $L = 5$. Choose the interval c to $c + 2L$ as -5 to 5, so that $c = -5$. Then

$$\begin{aligned} a_n &= \frac{1}{L}\int_c^{c+2L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{5}\int_{-5}^{5} f(x)\cos\frac{n\pi x}{5}\,dx \\ &= \frac{1}{5}\left\{\int_{-5}^{0}(0)\cos\frac{n\pi x}{5}\,dx + \int_0^5 (3)\cos\frac{n\pi x}{5}\,dx\right\} = \frac{3}{5}\int_0^5 \cos\frac{n\pi x}{5}\,dx \\ &= \frac{3}{5}\left(\frac{5}{n\pi}\sin\frac{n\pi x}{5}\right)\Bigg|_0^5 = 0 \qquad \text{if } n \neq 0 \end{aligned}$$

If $n = 0$, $a_n = a_0 = \dfrac{3}{5}\displaystyle\int_0^5 \cos\frac{0\pi x}{5}\,dx = \frac{3}{5}\int_0^5 dx = 3.$

$$\begin{aligned} b_n &= \frac{1}{L}\int_c^{c+2L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{5}\int_{-5}^{5} f(x)\sin\frac{n\pi x}{5}\,dx \\ &= \frac{1}{5}\left\{\int_{-5}^{0}(0)\sin\frac{n\pi x}{5}\,dx + \int_0^5 (3)\sin\frac{n\pi x}{5}\,dx\right\} = \frac{3}{5}\int_0^5 \sin\frac{n\pi x}{5}\,dx \\ &= \frac{3}{5}\left(-\frac{5}{n\pi}\cos\frac{n\pi x}{5}\right)\Bigg|_0^5 = \frac{3(1-\cos n\pi)}{n\pi} \end{aligned}$$

(*b*) The corresponding Fourier series is

$$\begin{aligned} \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right) &= \frac{3}{2} + \sum_{n=1}^{\infty}\frac{3(1-\cos n\pi)}{n\pi}\sin\frac{n\pi x}{5} \\ &= \frac{3}{2} + \frac{6}{\pi}\left(\sin\frac{\pi x}{5} + \frac{1}{3}\sin\frac{3\pi x}{5} + \frac{1}{5}\sin\frac{5\pi x}{5} + \cdots\right) \end{aligned}$$

(*c*) Since $f(x)$ satisfies the Dirichlet conditions, we can say that the series converges to $f(x)$ at all points of continuity and to $\dfrac{f(x+0)+f(x-0)}{2}$ at points of discontinuity. At $x = -5$, 0, and 5, which are points of discontinuity, the series converges to $(3+0)/2 = 3/2$ as seen from the graph. If we redefine $f(x)$ as follows,

$$f(x) = \begin{cases} 3/2 & x = -5 \\ 0 & -5 < x < 0 \\ 3/2 & x = 0 \\ 3 & 0 < x < 5 \\ 3/2 & x = 5 \end{cases} \qquad \text{Period} = 10$$

then the series will converge to $f(x)$ for $-5 \leqq x \leqq 5$.

13.6. Expand $f(x) = x^2$, $0 < x < 2\pi$ in a Fourier series if (*a*) the period is 2π, (*b*) the period is not specified.

(*a*) The graph of $f(x)$ with period 2π is shown in Fig. 13-7 below.

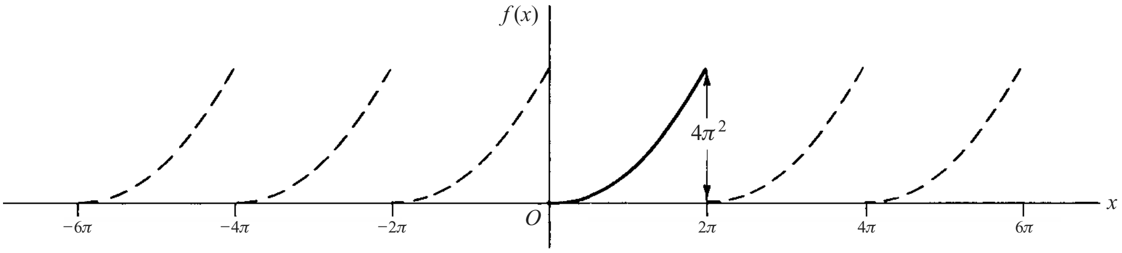

Fig. 13-7

Period $= 2L = 2\pi$ and $L = \pi$. Choosing $c = 0$, we have

$$a_n = \frac{1}{L}\int_c^{c+2L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{(x^2)\left(\frac{\sin nx}{n}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right\}\Bigg|_0^{2\pi} = \frac{4}{n^2}, \qquad n \neq 0$$

If $n = 0$, $a_0 = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} x^2\,dx = \dfrac{8\pi^2}{3}$.

$$b_n = \frac{1}{L}\int_c^{c+2L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{(x^2)\left(-\frac{\cos nx}{n}\right) - (2x)\left(-\frac{\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right\}\Bigg|_0^{2\pi} = \frac{-4\pi}{n}$$

Then $f(x) = x^2 = \dfrac{4\pi^2}{3} + \displaystyle\sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$.

This is valid for $0 < x < 2\pi$. At $x = 0$ and $x = 2\pi$ the series converges to $2\pi^2$.

(*b*) If the period is not specified, the Fourier series cannot be determined uniquely in general.

13.7. Using the results of Problem 13.6, prove that $\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \cdots = \dfrac{\pi^2}{6}$.

At $x = 0$ the Fourier series of Problem 13.6 reduces to $\dfrac{4\pi^2}{3} + \displaystyle\sum_{n=1}^{\infty}\frac{4}{n^2}$.

By the Dirichlet conditions, the series converges at $x = 0$ to $\frac{1}{2}(0 + 4\pi^2) = 2\pi^2$.

Then $\dfrac{4\pi^2}{3} + \displaystyle\sum_{n=1}^{\infty}\frac{4}{n^2} = 2\pi^2$, and so $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

ODD AND EVEN FUNCTIONS, HALF RANGE FOURIER SERIES

13.8. Classify each of the following functions according as they are even, odd, or neither even nor odd.

(*a*) $f(x) = \begin{cases} 2 & 0 < x < 3 \\ -2 & -3 < x < 0 \end{cases}$ Period $= 6$

From Fig. 13-8 below it is seen that $f(-x) = -f(x)$, so that the function is odd.

(*b*) $f(x) = \begin{cases} \cos x & 0 < x < \pi \\ 0 & \pi < x < 2\pi \end{cases}$ Period $= 2\pi$

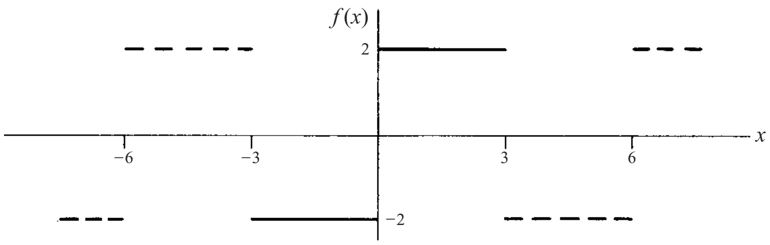

Fig. 13-8

From Fig. 13-9 below it is seen that the function is neither even nor odd.

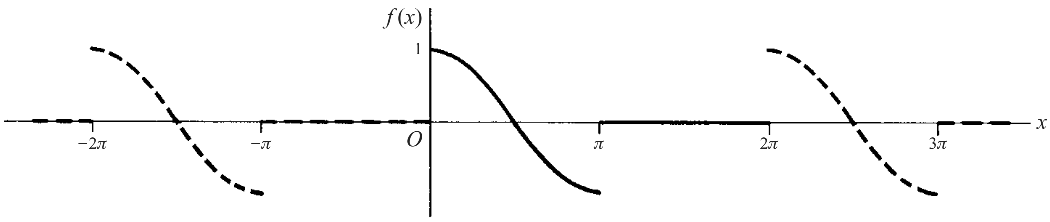

Fig. 13-9

(*c*) $f(x) = x(10 - x), 0 < x < 10, \text{Period} = 10.$

From Fig. 13-10 below the function is seen to be even.

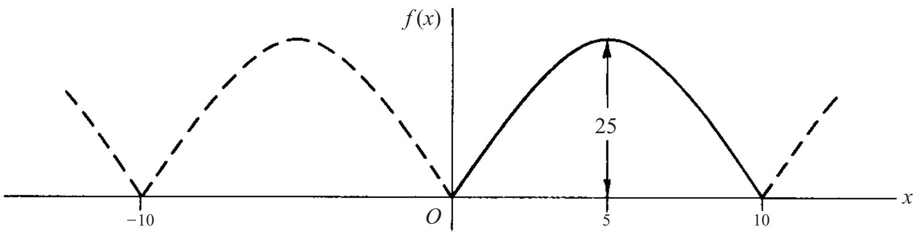

Fig. 13-10

13.9. Show that an even function can have no sine terms in its Fourier expansion.

Method 1: No sine terms appear if $b_n = 0, n = 1, 2, 3, \ldots$. To show this, let us write

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_{-L}^{0} f(x)\sin\frac{n\pi x}{L}\,dx + \frac{1}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \tag{1}$$

If we make the transformation $x = -u$ in the first integral on the right of (*1*), we obtain

$$\begin{aligned}\frac{1}{L}\int_{-L}^{0} f(x)\sin\frac{n\pi x}{L}\,dx &= \frac{1}{L}\int_{0}^{L} f(-u)\sin\left(-\frac{n\pi u}{L}\right)du = -\frac{1}{L}\int_{0}^{L} f(-u)\sin\frac{n\pi u}{L}\,du \\ &= -\frac{1}{L}\int_{0}^{L} f(u)\sin\frac{n\pi u}{L}\,du = -\frac{1}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx\end{aligned} \tag{2}$$

where we have used the fact that for an even function $f(-u) = f(u)$ and in the last step that the dummy variable of integration u can be replaced by any other symbol, in particular x. Thus, from (*1*), using (*2*), we have

$$b_n = -\frac{1}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx + \frac{1}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = 0$$

Method 2: Assume
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right).$$

Then
$$f(-x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} - b_N\sin\frac{n\pi x}{L}\right).$$

If $f(x)$ is even, $f(-x) = f(x)$. Hence,

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} - b_n\sin\frac{n\pi x}{L}\right)$$

and so
$$\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{L} = 0, \qquad \text{i.e.,}\ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{L}$$

and no sine terms appear.

In a similar manner we can show that an odd function has no cosine terms (or constant term) in its Fourier expansion.

13.10. If $f(x)$ is even, show that (*a*) $a_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$, (*b*) $b_n = 0$.

(*a*)
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_{-L}^{0} f(x)\cos\frac{n\pi x}{L}\,dx + \frac{1}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

Letting $x = -u$,

$$\frac{1}{L}\int_{-L}^{0} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_0^L f(-u)\cos\left(\frac{-n\pi u}{L}\right)du = \frac{1}{L}\int_0^L f(u)\cos\frac{n\pi u}{L}\,du$$

since by definition of an even function $f(-u) = f(u)$. Then

$$a_n = \frac{1}{L}\int_0^L f(u)\cos\frac{n\pi u}{L}\,du + \frac{1}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

(*b*) This follows by Method 1 of Problem 13.9.

13.11. Expand $f(x) = \sin x$, $0 < x < \pi$, in a Fourier cosine series.

A Fourier series consisting of cosine terms alone is obtained only for an even function. Hence, we extend the definition of $f(x)$ so that it becomes even (dashed part of Fig. 13-11 below). With this extension, $f(x)$ is then defined in an interval of length 2π. Taking the period as 2π, we have $2L = 2\pi$ so that $L = \pi$.

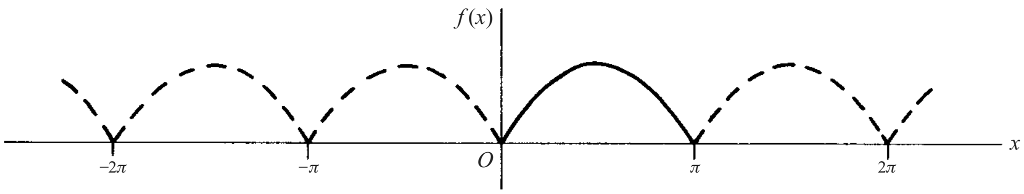

Fig. 13-11

By Problem 13.10, $b_n = 0$ and

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{\pi}\int_0^{\pi}\sin x\cos nx\,dx$$

$$
\begin{aligned}
&= \frac{1}{\pi}\int_0^\pi \{\sin(x+nx)+\sin(x-nx)\} = \frac{1}{\pi}\left\{-\frac{\cos(n+1)x}{n+1}+\frac{\cos(n-1)x}{n-1}\right\}\Bigg|_0^\pi \\
&= \frac{1}{\pi}\left\{\frac{1-\cos(n+1)\pi}{n+1}+\frac{\cos(n-1)\pi-1}{n-1}\right\} = \frac{1}{\pi}\left\{\frac{1+\cos n\pi}{n+1}-\frac{1+\cos n\pi}{n-1}\right\} \\
&= \frac{-2(1+\cos n\pi)}{\pi(n^2-1)} \qquad \text{if } n \neq 1.
\end{aligned}
$$

For $n = 1$, $\quad a_1 = \dfrac{2}{\pi}\displaystyle\int_0^\pi \sin x\cos x\,dx = \dfrac{2}{\pi}\dfrac{\sin^2 x}{2}\bigg|_0^\pi = 0.$

For $n = 0$, $\quad a_0 = \dfrac{2}{\pi}\displaystyle\int_0^\pi \sin x\,dx = \dfrac{2}{\pi}(-\cos x)\bigg|_0^\pi = \dfrac{4}{\pi}.$

Then

$$
\begin{aligned}
f(x) &= \frac{2}{\pi}-\frac{2}{\pi}\sum_{n=2}^\infty \frac{(1+\cos n\pi)}{n^2-1}\cos nx \\
&= \frac{2}{\pi}-\frac{4}{\pi}\left(\frac{\cos 2x}{2^2-1}+\frac{\cos 4x}{4^2-1}+\frac{\cos 6x}{6^2-1}+\cdots\right)
\end{aligned}
$$

13.12. Expand $f(x) = x, 0 < x < 2$, in a half range (*a*) sine series, (*b*) cosine series.

(*a*) Extend the definition of the given function to that of the odd function of period 4 shown in Fig. 13-12 below. This is sometimes called the *odd extension* of $f(x)$. Then $2L = 4, L = 2$.

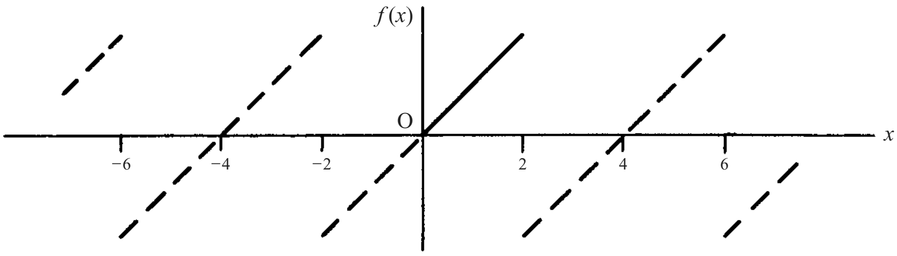

Fig. 13-12

Thus $a_n = 0$ and

$$
\begin{aligned}
b_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{2}\int_0^2 x\sin\frac{n\pi x}{2}\,dx \\
&= \left\{(x)\left(\frac{-2}{n\pi}\cos\frac{n\pi x}{2}\right)-(1)\left(\frac{-4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\right\}\Bigg|_0^2 = \frac{-4}{n\pi}\cos n\pi
\end{aligned}
$$

Then

$$
\begin{aligned}
f(x) &= \sum_{n=1}^\infty \frac{-4}{n\pi}\cos n\pi\sin\frac{n\pi x}{2} \\
&= \frac{4}{\pi}\left(\sin\frac{\pi x}{2}-\frac{1}{2}\sin\frac{2\pi x}{2}+\frac{1}{3}\sin\frac{3\pi x}{2}-\cdots\right)
\end{aligned}
$$

(*b*) Extend the definition of $f(x)$ to that of the even function of period 4 shown in Fig. 13-13 below. This is the *even extension* of $f(x)$. Then $2L = 4, L = 2$.

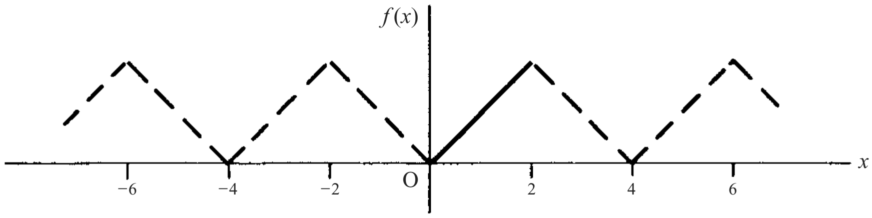

Fig. 13-13

Thus $b_n = 0$,

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,dx$$
$$= \left\{(x)\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (1)\left(\frac{-4}{n^2\pi^2}\,cos\,\frac{n\pi x}{2}\right)\right\}\Bigg|_0^2$$
$$= \frac{4}{n^2\pi^2}(\cos n\pi - 1) \qquad \text{If } n \neq 0$$

If $n = 0$, $a_0 = \int_0^2 x\,dx = 2$.

Then
$$f(x) = 1 + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}(\cos n\pi - 1)\cos\frac{n\pi x}{2}$$
$$= 1 - \frac{8}{\pi^2}\left(\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \cdots\right)$$

It should be noted that the given function $f(x) = x$, $0 < x < 2$, is represented *equally well* by the two *different* series in (*a*) and (*b*).

PARSEVAL'S IDENTITY

13.13. Assuming that the Fourier series corresponding to $f(x)$ converges uniformly to $f(x)$ in $(-L, L)$, prove Parseval's identity

$$\frac{1}{L}\int_{-L}^{L}\{f(x)\}^2\,dx = \frac{a_0^2}{2} + \Sigma(a_n^2 + b_n^2)$$

where the integral is assumed to exist.

If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)$, then multiplying by $f(x)$ and integrating term by term from $-L$ to L (which is justified since the series is uniformly convergent) we obtain

$$\int_{-L}^{L}\{f(x)\}^2\,dx = \frac{a_0}{2}\int_{-L}^{L}f(x)\,dx + \sum_{n=1}^{\infty}\left\{a_n\int_{-L}^{L}f(x)\cos\frac{n\pi x}{L}\,dx + b_n\int_{-L}^{L}f(x)\sin\frac{n\pi x}{L}\,dx\right\}$$
$$= \frac{a_0^2}{2}L + L\sum_{n=1}^{\infty}(a_n^2 + b_n^2) \tag{1}$$

where we have used the results

$$\int_{-L}^{L}f(x)\cos\frac{n\pi x}{L}\,dx = La_n, \qquad \int_{-L}^{L}f(x)\sin\frac{n\pi x}{L}\,dx = Lb_n, \qquad \int_{-L}^{L}f(x)\,dx = La_0 \tag{2}$$

obtained from the Fourier coefficients.

The required result follows on dividing both sides of (*1*) by L. Parseval's identity is valid under less restrictive conditions than that imposed here.

13.14. (*a*) Write Parseval's identity corresponding to the Fourier series of Problem 13.12(*b*).

(*b*) Determine from (*a*) the sum S of the series $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots + \frac{1}{n^4} + \cdots$.

(*a*) Here $L = 2, a_0 = 2, a_n = \frac{4}{n^2\pi^2}(\cos n\pi - 1), n \neq 0, b_n = 0$.

Then Parseval's identity becomes

$$\frac{1}{2}\int_{-2}^{2} \{f(x)\}^2\,dx = \frac{1}{2}\int_{-2}^{2} x^2\,dx = \frac{(2)^2}{2} + \sum_{n=1}^{\infty} \frac{16}{n^4\pi^4}(\cos n\pi - 1)^2$$

or $\frac{8}{3} = 2 + \frac{64}{\pi^4}\left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right)$, i.e., $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = \frac{\pi^4}{96.}$

(*b*)
$$\begin{aligned} S = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots &= \left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right) + \left(\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \cdots\right) \\ &= \left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right) + \frac{1}{2^4}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots\right) \\ &= \frac{\pi^4}{96} + \frac{S}{16}, \quad \text{from which } S = \frac{\pi^4}{90} \end{aligned}$$

13.15. Prove that for all positive integers M,

$$\frac{a_0^2}{2} + \sum_{n=1}^{M}(a_n^2 + b_n^2) \leqq \frac{1}{L}\int_{-L}^{L} \{f(x)\}^2\,dx$$

where a_n and b_n are the Fourier coefficients corresponding to $f(x)$, and $f(x)$ is assumed piecewise continuous in $(-L, L)$.

Let
$$S_M(x) = \frac{a_0}{2} + \sum_{n=1}^{M}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \tag{1}$$

For $M = 1, 2, 3, \ldots$ this is the sequence of partial sums of the Fourier series corresponding to $f(x)$.

We have

$$\int_{-L}^{L} \{f(x) - S_M(x)\}^2\,dx \geqq 0 \tag{2}$$

since the integrand is non-negative. Expanding the integrand, we obtain

$$2\int_{-L}^{L} f(x)\,S_M(x)\,dx - \int_{-L}^{L} S_M^2(x)\,dx \leqq \int_{-L}^{L} \{f(x)\}^2\,dx \tag{3}$$

Multiplying both sides of (*1*) by $2f(x)$ and integrating from $-L$ to L, using equations (*2*) of Problem 13.13, gives

$$2\int_{-L}^{L} f(x)\,S_M(x)\,dx = 2L\left\{\frac{a_0^2}{2} + \sum_{n=1}^{M}(a_n^2 + b_n^2)\right\} \tag{4}$$

Also, squaring (*1*) and integrating from $-L$ to L, using Problem 13.3, we find

$$\int_{-L}^{L} S_M^2(x)\,dx = L\left\{\frac{a_0^2}{2} + \sum_{n=1}^{M}(a_n^2 + b_n^2)\right\} \tag{5}$$

Substitution of (*4*) and (*5*) into (*3*) and dividing by L yields the required result.

Taking the limit as $M \to \infty$, we obtain *Bessel's inequality*

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \leqq \frac{1}{L}\int_{-L}^{L}\{f(x)\}^2\,dx \tag{6}$$

If the equality holds, we have Parseval's identity (Problem 13.13).

We can think of $S_M(x)$ as representing an *approximation* to $f(x)$, while the left-hand side of (*2*), divided by $2L$, represents the *mean square error* of the approximation. Parseval's identity indicates that as $M \to \infty$ the mean square error approaches zero, while Bessels' inequality indicates the possibility that this mean square error does not approach zero.

The results are connected with the idea of *completeness* of an orthonormal set. If, for example, we were to leave out one or more terms in a Fourier series (say $\cos 4\pi x/L$, for example), we could never get the mean square error to approach zero no matter how many terms we took. For an analogy with three-dimensional vectors, see Problem 13.60.

DIFFERENTIATION AND INTEGRATION OF FOURIER SERIES

13.16. (*a*) Find a Fourier series for $f(x) = x^2$, $0 < x < 2$, by integrating the series of Problem 13.12(*a*).
(*b*) Use (*a*) to evaluate the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}$.

(*a*) From Problem 13.12(*a*),

$$x = \frac{4}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} - \cdots\right) \tag{1}$$

Integrating both sides from 0 to x (applying the theorem of Page 339) and multiplying by 2, we find

$$x^2 = C - \frac{16}{\pi^2}\left(\cos\frac{\pi x}{2} - \frac{1}{2^2}\cos\frac{2\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} - \cdots\right) \tag{2}$$

where $C = \dfrac{16}{\pi^2}\left(1 - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \cdots\right)$.

(*b*) To determine C in another way, note that (*2*) represents the Fourier cosine series for x^2 in $0 < x < 2$. Then since $L = 2$ in this case,

$$C = \frac{a_0}{2} = \frac{1}{L}\int_0^L f(x) = \frac{1}{2}\int_0^2 x^2\,dx = \frac{4}{3}$$

Then from the value of C in (*a*), we have

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} = \frac{1}{4^2} + \cdots = \frac{\pi^2}{16}\cdot\frac{4}{3} = \frac{\pi^2}{12}$$

13.17. Show that term by term differentiation of the series in Problem 13.12(*a*) is not valid.

Term by term differentiation yields $2\left(\cos\dfrac{\pi x}{2} - \cos\dfrac{2\pi x}{2} + \cos\dfrac{3\pi x}{2} - \cdots\right)$.

Since the nth term of this series does not approach 0, the series does not converge for any value of x.

CONVERGENCE OF FOURIER SERIES

13.18. Prove that (*a*) $\frac{1}{2} + \cos t + \cos 2t + \cdots + \cos Mt = \dfrac{\sin(M+\frac{1}{2})t}{2\sin\frac{1}{2}t}$

$$(b)\ \frac{1}{\pi}\int_0^{\pi} \frac{\sin(M+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,dt = \frac{1}{2}, \qquad \frac{1}{\pi}\int_{-\pi}^{0} \frac{\sin(M+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,dt = \frac{1}{2}.$$

(*a*) We have $\cos nt \sin\frac{1}{2}t = \frac{1}{2}\{\sin(n+\frac{1}{2})t - \sin(n-\frac{1}{2})t\}$.
Then summing from $n = 1$ to M,

$$\begin{aligned}\sin\tfrac{1}{2}t\{\cos t + \cos 2t + \cdots + \cos Mt\} &= (\sin\tfrac{3}{2}t - \sin\tfrac{1}{2}t) + (\sin\tfrac{5}{2}t - \sin\tfrac{3}{2}t)\\ &\quad + \cdots + \left(\sin(M+\tfrac{1}{2})t - \sin(M-\tfrac{1}{2})t\right)\\ &= \tfrac{1}{2}\{\sin(M+\tfrac{1}{2})t - \sin\tfrac{1}{2}t\}\end{aligned}$$

On dividing by $\sin\frac{1}{2}t$ and adding $\frac{1}{2}$, the required result follows.

(*b*) Integrating the result in (*a*) from $-\pi$ to 0 and 0 to π, respectively. This gives the required results, since the integrals of all the cosine terms are zero.

13.19. Prove that $\lim\limits_{n\to\infty}\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \lim\limits_{n\to\infty}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0$ if $f(x)$ is piecewise continuous.

This follows at once from Problem 13.15, since if the series $\dfrac{a_0^2}{2} + \displaystyle\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$ is convergent, $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = 0$.

The result is sometimes called *Riemann's theorem*.

13.20. Prove that $\lim\limits_{M\to\infty}\displaystyle\int_{-\pi}^{\pi} f(x)\sin(M+\tfrac{1}{2})x\,dx = 0$ if $f(x)$ is piecewise continuous.

We have

$$\int_{-\pi}^{\pi} f(x)\sin(M+\tfrac{1}{2})x\,dx = \int_{-\pi}^{\pi}\{f(x)\sin\tfrac{1}{2}x\}\cos Mx\,dx + \int_{-\pi}^{\pi}\{f(x)\cos\tfrac{1}{2}x\}\sin Mx\,dx$$

Then the required result follows at once by using the result of Problem 13.19, with $f(x)$ replaced by $f(x)\sin\frac{1}{2}x$ and $f(x)\cos\frac{1}{2}x$ respectively, which are piecewise continuous if $f(x)$ is.

The result can also be proved when the integration limits are a and b instead of $-\pi$ and π.

13.21. Assuming that $L = \pi$, i.e., that the Fourier series corresponding to $f(x)$ has period $2L = 2\pi$, show that

$$S_M(x) = \frac{a_0}{2} + \sum_{n=1}^{M}(a_n\cos nx + b_n\sin nx) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t+x)\frac{\sin(M+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,dt$$

Using the formulas for the Fourier coefficients with $L = \pi$, we have

$$\begin{aligned}a_n\cos nx + b_n\sin nx &= \left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\cos nu\,du\right)\cos nx + \left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\sin nu\,du\right)\sin nx\\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)(\cos nu\cos nx + \sin nu\sin nx)\,du\\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\cos n(u-x)\,du\end{aligned}$$

Also,
$$\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)\,du$$

Then
$$\begin{aligned} S_M(x) &= \frac{a_0}{2} + \sum_{n=1}^{M}(a_n \cos nx + b_n \sin nx) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)\,du + \frac{1}{\pi}\sum_{n=1}^{M}\int_{-\pi}^{\pi} f(u)\cos n(u-x)\,du \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\left\{\frac{1}{2} + \sum_{n=1}^{M}\cos n(u-x)\right\}du \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\,\frac{\sin(M+\frac{1}{2})(u-x)}{2\sin\frac{1}{2}(u-x)}\,du \end{aligned}$$

using Problem 13.18. Letting $u - x = t$, we have

$$S_M(x) = \frac{1}{\pi}\int_{-\pi-x}^{\pi-x} f(t+x)\,\frac{\sin(M+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,dt$$

Since the integrand has period 2π, we can replace the interval $-\pi - x, \pi - x$ by any other interval of length 2π, in particular, $-\pi, \pi$. Thus, we obtain the required result.

13.22. Prove that

$$\begin{aligned} S_M(x) - \left(\frac{f(x+0)+f(x-0)}{2}\right) &= \frac{1}{\pi}\int_{-\pi}^{0}\left(\frac{f(t+x)-f(x-0)}{2\sin\frac{1}{2}t}\right)\sin(M+\tfrac{1}{2})t\,dt \\ &\quad + \frac{1}{\pi}\int_{0}^{\pi}\left(\frac{f(t+x)-f(x+0)}{2\sin\frac{1}{2}t}\right)\sin(M+\tfrac{1}{2})t\,dt \end{aligned}$$

From Problem 13.21,

$$S_M(x) = \frac{1}{\pi}\int_{-\pi}^{0} f(t+x)\frac{\sin(M+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,dt + \frac{1}{\pi}\int_{0}^{\pi} f(t+x)\frac{\sin(M+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,dt \tag{1}$$

Multiplying the integrals of Problem 13.18(*b*) by $f(x-0)$ and $f(x+0)$, respectively,

$$\frac{f(x+0)+f(x-0)}{2} = \frac{1}{\pi}\int_{-\pi}^{0} f(x-0)\frac{\sin(M+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,dt + \frac{1}{\pi}\int_{0}^{\pi} f(x+0)\frac{\sin(M+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,dt \tag{2}$$

Subtracting (*2*) from (*1*) yields the required result.

13.23. If $f(x)$ and $f'(x)$ are piecewise continuous in $(-\pi, \pi)$, prove that

$$\lim_{M\to\infty} S_M(x) = \frac{f(x+0)+f(x-0)}{2}$$

The function $\dfrac{f(t+x)-f(x+0)}{2\sin\frac{1}{2}t}$ is piecewise continuous in $0 < t \leqq \pi$ because $f(x)$ is piecewise contînous.

Also, $\displaystyle\lim_{t\to 0+}\frac{f(t+x)-f(x+0)}{2\sin\frac{1}{2}t} = \lim_{t\to 0+}\frac{f(t+x)-f(x+0)}{t}\cdot\frac{t}{2\sin\frac{1}{2}t} = \lim_{t\to 0+}\frac{f(t+x)-f(x+0)}{t}$ exists, since by hypothesis $f'(x)$ is piecewise continuous so that the right-hand derivative of $f(x)$ at each x exists.

Thus, $\dfrac{f(t+x)-f(x-0)}{2\sin\frac{1}{2}t}$ is piecewise continous in $0 \leqq t \leqq \pi$.

Similarly, $\dfrac{f(t+x)-f(x-0)}{2\sin\frac{1}{2}t}$ is piecewise continous in $-\pi \leqq t \leqq 0$.

Then from Problems 13.20 and 13.22, we have

$$\lim_{M\to\infty} S_M(x) - \left\{\frac{f(x+0)+f(x-0)}{2}\right\} = 0 \quad \text{or} \quad \lim_{M\to\infty} S_M(x) = \frac{f(x+0)+f(x-0)}{2}$$

BOUNDARY-VALUE PROBLEMS

13.24. Find a solution $U(x, t)$ of the boundary-value problem

$$\begin{array}{ll} \dfrac{\partial U}{\partial t} = 3\dfrac{\partial^2 U}{\partial x^2} & t > 0, 0 < x < 2 \\ U(0, t) = 0, U(2, t) = 0 & t > 0 \\ U(x, 0) = x & 0 < x < 2 \end{array}$$

A method commonly employed in practice is to assume the existence of a solution of the partial differential equation having the particular form $U(x, t) = X(x)\,T(t)$, where $X(x)$ and $T(t)$ are functions of x and t, respectively, which we shall try to determine. For this reason the method is often called the method of *separation of variables*.

Substitution in the differential equation yields

$$(1) \quad \frac{\partial}{\partial t}(XT) = 3\frac{\partial^2}{\partial x^2}(XT) \qquad \text{or} \qquad (2) \quad X\frac{dT}{dt} = 3T\frac{d^2X}{dx^2}$$

where we have written X and T in place of $X(x)$ and $T(t)$.

Equation (*2*) can be written as

$$\frac{1}{3T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} \tag{3}$$

Since one side depends only on t and the other only on x, and since x and t are independent variables, it is clear that each side must be a constant c.

In Problem 13.47 we see that if $c \geqq 0$, a solution satisfying the given boundary conditions cannot exist.

Let us thus assume that c is a negative constant which we write as $-\lambda^2$. Then from (*3*) we obtain two ordinary differentiation equations

$$\frac{dT}{dt} + 3\lambda^2 T = 0, \qquad \frac{d^2X}{dx^2} + \lambda^2 X = 0 \tag{4}$$

whose solutions are respectively

$$T = C_1 e^{-3\lambda^2 t}, \qquad X = A_1 \cos\lambda x + B_1 \sin\lambda x \tag{5}$$

A solution is given by the product of X and T which can be written

$$U(x, t) = e^{-3\lambda^2 t}(A\cos\lambda x + B\sin\lambda x) \tag{6}$$

where A and B are constants.

We now seek to determine A and B so that (*6*) satisfies the given boundary conditions. To satisfy the condition $U(0, t) = 0$, we must have

$$e^{-s\lambda^2 t}(A) = 0 \qquad \text{or} \qquad A = 0 \tag{7}$$

so that (*6*) becomes

$$U(x, t) = Be^{-s\lambda^2 t}\sin\lambda x \tag{8}$$

To satisfy the condition $U(2, t) = 0$, we must then have

$$Be^{-s\lambda^2 t}\sin 2\lambda = 0 \tag{9}$$

Since $B = 0$ makes the solution (*8*) identically zero, we avoid this choice and instead take

$$\sin 2\lambda = 0, \qquad \text{i.e.,} \quad 2\lambda = m\pi \quad \text{or} \quad \lambda = \frac{m\pi}{2} \tag{10}$$

where $m = 0, \pm 1, \pm 2, \ldots$.

Substitution in (*8*) now shows that a solution satisfying the first two boundary conditions is

$$U(x, t) = B_m e^{-3m^2\pi^2 t/4} \sin \frac{m\pi x}{2} \tag{11}$$

where we have replaced B by B_m, indicating that different constants can be used for different values of m.

If we now attempt to satisfy the last boundary condition $U(x, 0) = x, 0 < x < 2$, we find it to be impossible using (*11*). However, upon recognizing the fact that *sums* of solutions having the form (*11*) are also solutions (called the *principle of superposition*), we are led to the possible solution

$$U(x, t) = \sum_{m=1}^{\infty} B_m e^{-3m^2\pi^2 t/4} \sin \frac{m\pi x}{2} \tag{12}$$

From the condition $U(x, 0) = x, 0 < x < 2$, we see, on placing $t = 0$, that (*12*) becomes

$$x = \sum_{m=1}^{\infty} B_m \sin \frac{m\pi x}{2} \qquad 0 < x < 2 \tag{13}$$

This, however, is equivalent to the problem of expanding the function $f(x) = x$ for $0 < x < 2$ into a sine series. The solution to this is given in Problem 13.12(*a*), from which we see that $B_m = \dfrac{-4}{m\pi} \cos m\pi$ so that (*12*) becomes

$$U(x, t) = \sum_{m=1}^{\infty} \left(-\frac{4}{m\pi} \cos m\pi \right) e^{-3m^2\pi^2 t/4} \sin \frac{m\pi x}{2} \tag{14}$$

which is a *formal solution*. To check that (*14*) is actually a solution, we must show that it satisfies the partial differential equation and the boundary conditions. The proof consists in justification of term by term differentiation and use of limiting procedures for infinite series and may be accomplished by methods of Chapter 11.

The boundary value problem considered here has an interpretation in the theory of heat conduction. The equation $\dfrac{\partial U}{\partial t} = k \dfrac{\partial^2 U}{\partial x^2}$ is the equation for heat conduction in a thin rod or wire located on the x-axis between $x = 0$ and $x = L$ if the surface of the wire is insulated so that heat cannot enter or escape. $U(x, t)$ is the temperature at any place x in the rod at time t. The constant $k = K/s\rho$ (where K is the *thermal conductivity*, s is the *specific heat*, and ρ is the *density* of the conducting material) is called the *diffusivity*. The boundary conditions $U(0, t) = 0$ and $U(L, t) = 0$ indicate that the end temperatures of the rod are kept at zero units for all time $t > 0$, while $U(x, 0)$ indicates the initial temperature at any point x of the rod. In this problem the length of the rod is $L = 2$ units, while the diffusivity is $k = 3$ units.

ORTHOGONAL FUNCTIONS

13.25. (*a*) Show that the set of functions

$$1, \sin \frac{\pi x}{L}, \cos \frac{\pi x}{L}, \sin \frac{2\pi x}{L}, \cos \frac{2\pi x}{L}, \sin \frac{3\pi x}{L}, \cos \frac{3\pi x}{L}, \ldots$$

forms an orthogonal set in the interval $(-L, L)$.

(*b*) Determine the corresponding normalizing constants for the set in (*a*) so that the set is orthonormal in $(-L, L)$.

(*a*) This follows at once from the results of Problems 13.2 and 13.3.

(*b*) By Problem 13.3,

$$\int_{-L}^{L} \sin^2 \frac{m\pi x}{L}\, dx = L, \qquad \int_{-L}^{L} \cos^2 \frac{m\pi x}{L}\, dx = L$$

Then $$\int_{-L}^{L}\left(\sqrt{\frac{1}{L}}\sin\frac{m\pi x}{L}\right)^2 dx = 1, \qquad \int_{-L}^{L}\left(\sqrt{\frac{1}{L}}\cos\frac{m\pi x}{L}\right)^2 dx = 1$$

Also, $$\int_{-L}^{L}(1)^2\,dx = 2L \qquad \text{or} \qquad \int_{-L}^{L}\left(\frac{1}{\sqrt{2L}}\right)^2 dx = 1$$

Thus the required orthonormal set is given by

$$\frac{1}{\sqrt{2L}}, \frac{1}{\sqrt{L}}\sin\frac{\pi x}{L}, \frac{1}{\sqrt{L}}\cos\frac{\pi x}{L}, \frac{1}{\sqrt{L}}\sin\frac{2\pi x}{L}, \frac{1}{\sqrt{L}}\cos\frac{2\pi x}{L}, \ldots$$

MISCELLANEOUS PROBLEMS

13.26. Find a Fourier series for $f(x) = \cos\alpha x$, $-\pi \leqq x \leqq \pi$, where $\alpha \neq 0, \pm 1, \pm 2, \pm 3, \ldots$.

We shall take the period as 2π so that $2L = 2\pi$, $L = \pi$. Since the function is even, $b_n = 0$ and

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^\pi \cos\alpha x\cos nx\,dx \\ &= \frac{1}{\pi}\int_0^\pi \{\cos(\alpha - n)x + \cos(\alpha + n)x\}\,dx \\ &= \frac{1}{\pi}\left\{\frac{\sin(\alpha - n)\pi}{\alpha - n} + \frac{\sin(\alpha + n)\pi}{\alpha + n}\right\} = \frac{2\alpha\sin\alpha\pi\cos n\pi}{\pi(\alpha^2 - n^2)} \\ \alpha_0 &= \frac{2\sin\alpha\pi}{\alpha\pi} \end{aligned}$$

Then

$$\begin{aligned} \cos\alpha x &= \frac{\sin\alpha\pi}{\alpha\pi} + \frac{2\alpha\sin\alpha\pi}{\pi}\sum_{n=1}^{\infty}\frac{\cos n\pi}{\alpha^2 - n^2}\cos nx \\ &= \frac{\sin\alpha\pi}{\pi}\left(\frac{1}{\alpha} - \frac{2\alpha}{\alpha^2 - 1^2}\cos x + \frac{2\alpha}{\alpha^2 - 2^2}\cos 2x - \frac{2\alpha}{\alpha^2 - 3^2}\cos 3x + \cdots\right) \end{aligned}$$

13.27. Prove that $\sin x = x\left(1 - \dfrac{x^2}{\pi^2}\right)\left(1 - \dfrac{x^2}{(2\pi)^2}\right)\left(1 - \dfrac{x^2}{(3\pi)^2}\right)\cdots$.

Let $x = \pi$ in the Fourier series obtained in Problem 13.26. Then

$$\cos\alpha = \frac{\sin\alpha\pi}{\pi}\left(\frac{1}{\alpha} + \frac{2\alpha}{\alpha^2 - 1^2} + \frac{2\alpha}{\alpha^2 - 2^2} + \frac{2\alpha}{\alpha^2 - 3^2} + \cdots\right)$$

or

$$\pi\cot\alpha\pi - \frac{1}{\alpha} = \frac{2\alpha}{\alpha^2 - 1^2} + \frac{2\alpha}{\alpha^2 - 2^2} + \frac{2\alpha}{\alpha^2 - 3^2} + \cdots \qquad (1)$$

This result is of interest since it represents an expansion of the contangent into partial fractions.

By the Weierstrass M test, the series on the right of (1) converges uniformly for $0 \leqq |\alpha| \leqq |x| < 1$ and the left-hand side of (1) approaches zero as $\alpha \to 0$, as is seen by using L'Hospital's rule. Thus, we can integrate both sides of (1) from 0 to x to obtain

$$\int_0^x\left(\pi\cot\alpha\pi - \frac{1}{\alpha}\right)d\alpha = \int_0^x \frac{2\alpha}{\alpha^2 - 1}\,d\alpha + \int_0^x \frac{2\alpha}{\alpha^2 - 2^2}\,d\alpha + \cdots$$

or $$\ln\left(\frac{\sin\alpha\pi}{\alpha\pi}\right)\Bigg|_0^x = \ln\left(1 - \frac{x^2}{1^2}\right) + \ln\left(1 - \frac{x^2}{2^2}\right) + \cdots$$

i.e.,

$$\ln\left(\frac{\sin \pi x}{\pi x}\right) = \lim_{n\to\infty} \ln\left(1 - \frac{x^2}{1^2}\right) + \ln\left(1 - \frac{x^2}{2^2}\right) + \cdots + \ln\left(1 - \frac{x^2}{n^2}\right)$$
$$= \lim_{n\to\infty} \ln\left\{\left(1 - \frac{x^2}{1^2}\right)\left(1 - \frac{x^2}{2^2}\right)\cdots\left(1 - \frac{x^2}{n^2}\right)\right\}$$
$$= \ln\left\{\lim_{n\to\infty}\left(1 - \frac{x^2}{1^2}\right)\left(1 - \frac{x^2}{2^2}\right)\cdots\left(1 - \frac{x^2}{n^2}\right)\right\}$$

so that

$$\frac{\sin \pi x}{\pi x} = \lim_{n\to\infty}\left(1 - \frac{x^2}{1^2}\right)\left(1 - \frac{x^2}{2^2}\right)\cdots\left(1 - \frac{x^2}{n^2}\right) = \left(1 - \frac{x^2}{1^2}\right)\left(1 - \frac{x^2}{2^2}\right)\cdots \tag{2}$$

Replacing x by x/π, we obtain

$$\sin x = x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{(2\pi)^2}\right)\cdots \tag{3}$$

called the *infinite product for* $\sin x$, which can be shown valid for all x. The result is of interest since it corresponds to a factorization of $\sin x$ in a manner analogous to factorization of a polynomial.

13.28. Prove that $\dfrac{\pi}{2} = \dfrac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\ldots}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\ldots}$.

Let $x = 1/2$ in equation (*2*) of Problem 13.27. Then,

$$\frac{2}{\pi} = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{4^2}\right)\left(1 - \frac{1}{6^2}\right)\cdots = \left(\frac{1}{2}\cdot\frac{3}{2}\right)\left(\frac{3}{4}\cdot\frac{5}{4}\right)\left(\frac{5}{6}\cdot\frac{7}{6}\right)\cdots$$

Taking reciprocals of both sides, we obtain the required result, which is often called *Wallis' product*.

Supplementary Problems

FOURIER SERIES

13.29. Graph each of the following functions and find their corresponding Fourier series using properties of even and odd functions wherever applicable.

(*a*) $f(x) = \begin{cases} 8 & 0 < x < 2 \\ -8 & 2 < x < 4 \end{cases}$ Period 4 (*b*) $f(x) = \begin{cases} -x & -4 \leqq x \leqq 0 \\ x & 0 \leqq x \leqq 4 \end{cases}$ Period 8

(*c*) $f(x) = 4x, 0 < x < 10$, Period 10 (*d*) $f(x) = \begin{cases} 2x & 0 \leqq x < 3 \\ 0 & -3 < x < 0 \end{cases}$ Period 6

Ans. (*a*) $\dfrac{16}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{(1-\cos n\pi)}{n}\sin\frac{n\pi x}{2}$ (*b*) $2 - \dfrac{8}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{(1-\cos n\pi)}{n^2}\cos\frac{n\pi x}{4}$

(*c*) $20 - \dfrac{40}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{5}$ (*d*) $\dfrac{3}{2} + \displaystyle\sum_{n=1}^{\infty}\left\{\frac{6(\cos n\pi - 1)}{n^2\pi^2}\cos\frac{n\pi x}{3} - \frac{6\cos n\pi}{n\pi}\sin\frac{n\pi x}{3}\right\}$

13.30. In each part of Problem 13.29, tell where the discontinuities of $f(x)$ are located and to what value the series converges at the discontunities.

Ans. (*a*) $x = 0, \pm 2, \pm 4, \ldots; 0$ (*b*) no discontinuities (*c*) $x = 0, \pm 10, \pm 20, \ldots; 20$
(*d*) $x = \pm 3, \pm 9, \pm 15, \ldots; 3$

13.31. Expand $f(x) = \begin{cases} 2 - x & 0 < x < 4 \\ x - 6 & 4 < x < 8 \end{cases}$ in a Fourier series of period 8.

Ans. $\dfrac{16}{\pi^2}\left\{\cos\dfrac{\pi x}{4} + \dfrac{1}{3^2}\cos\dfrac{3\pi x}{4} + \dfrac{1}{5^2}\cos\dfrac{5\pi x}{4} + \cdots\right\}$

13.32. (*a*) Expand $f(x) = \cos x, 0 < x < \pi$, in a Fourier sine series.
(*b*) How should $f(x)$ be defined at $x = 0$ and $x = \pi$ so that the series will converge to $f(x)$ for $0 \leqq x \leqq \pi$?

Ans. (*a*) $\dfrac{8}{\pi}\displaystyle\sum_{n=1}^{\infty}\dfrac{n\sin 2nx}{4n^2 - 1}$ (*b*) $f(0) = f(\pi) = 0$

13.33. (*a*) Expand in a Fourier series $f(x) = \cos x, 0 < x < \pi$ if the period is π; and (*b*) compare with the result of Problem 13.32, explaining the similarities and differences if any.
Ans. Answer is the same as in Problem 13.32.

13.34. Expand $f(x) = \begin{cases} x & 0 < x < 4 \\ 8 - x & 4 < x < 8 \end{cases}$ in a series of (*a*) sines, (*b*) cosines.

Ans. (*a*) $\dfrac{32}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{n^2}\sin\dfrac{n\pi}{2}\sin\dfrac{n\pi x}{8}$ (*b*) $\dfrac{16}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\left(\dfrac{2\cos n\pi/2 - \cos n\pi - 1}{n^2}\right)\cos\dfrac{n\pi x}{8}$

13.35. Prove that for $0 \leqq x \leqq \pi$,

(*a*) $x(\pi - x) = \dfrac{\pi^2}{6} - \left(\dfrac{\cos 2x}{1^2} + \dfrac{\cos 4x}{2^2} + \dfrac{\cos 6x}{3^2} + \cdots\right)$

(*b*) $x(\pi - x) = \dfrac{8}{\pi}\left(\dfrac{\sin x}{1^3} + \dfrac{\sin 3x}{3^3} + \dfrac{\sin 5}{5^3} + \cdots\right)$

13.36. Use the preceding problem to show that

(*a*) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$, (*b*) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$, (*c*) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^3} = \frac{\pi^3}{32}$.

13.37. Show that $\dfrac{1}{1^3} + \dfrac{1}{3^3} - \dfrac{1}{5^3} - \dfrac{1}{7^3} + \dfrac{1}{9^3} + \dfrac{1}{11^3} - \cdots = \dfrac{3\pi^2\sqrt{2}}{16}$.

DIFFERENTIATION AND INTEGRATION OF FOURIER SERIES

13.38. (*a*) Show that for $-\pi < x < \pi$,

$$x = 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right)$$

(*b*) By integrating the result of (*a*), show that for $-\pi \leqq x \leqq \pi$,

$$x^2 = \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right)$$

(*c*) By integrating the result of (*b*), show that for $-\pi \leqq x \leqq \pi$,

$$x(\pi - x)(\pi + x) = 12\left(\frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \cdots\right)$$

13.39. (*a*) Show that for $-\pi < x < \pi$,

$$x\cos x = -\frac{1}{2}\sin x + 2\left(\frac{2}{1\cdot 3}\sin 2x - \frac{3}{2\cdot 4}\sin 3x + \frac{4}{3\cdot 5}\sin 4x - \cdots\right)$$

(*b*) Use (*a*) to show that for $-\pi \leqq x \leqq \pi$,

$$x\sin x = 1 - \frac{1}{2}\cos x - 2\left(\frac{\cos 2x}{1\cdot 3} - \frac{\cos 3x}{2\cdot 4} + \frac{\cos 4x}{3\cdot 5} - \cdots\right)$$

13.40. By differentiating the result of Problem 13.35(*b*), prove that for $0 \leqq x \leqq \pi$,

$$x = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right)$$

PARSEVAL'S IDENTITY

13.41. By using Problem 13.35 and Parseval's identity, show that

$$(a)\ \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \qquad (b)\ \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$$

13.42. Show that $\dfrac{1}{1^2 \cdot 3^2} + \dfrac{1}{3^2 \cdot 5^2} + \dfrac{1}{5^2 \cdot 7^2} + \cdots = \dfrac{\pi^2 - 8}{16}$. [Hint: Use Problem 13.11.]

13.43. Show that $(a)\ \displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$, $\quad (b)\ \displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} = \frac{\pi^6}{960}$.

13.44. Show that $\dfrac{1}{1^2 \cdot 2^2 \cdot 3^2} + \dfrac{1}{2^2 \cdot 3^2 \cdot 4^2} + \dfrac{1}{3^2 + 4^2 + 5^2} + \cdots = \dfrac{4\pi^2 - 39}{16}$.

BOUNDARY-VALUE PROBLEMS

13.45. (*a*) Solve $\dfrac{\partial U}{\partial t} = 2\dfrac{\partial^2 U}{\partial x^2}$ subject to the conditions $U(0, t) = 0$, $U(4, t) = 0$, $U(x, 0) = 3\sin \pi x - 2\sin 5\pi x$, where $0 < x < 4$, $t > 0$.

(*b*) Give a possible physical interpretation of the problem and solution.

Ans. (*a*) $U(x, t) = 3e^{-2\pi^2 t}\sin \pi x - 2e^{-50\pi^2 t}\sin 5\pi x$.

13.46. Solve $\dfrac{\partial U}{\partial t} = \dfrac{\partial^2 U}{\partial x^2}$ subject to the conditions $U(0, t) = 0$, $U(6, t) = 0$, $U(x, 0) = \begin{cases} 1 & 0 < x < 3 \\ 0 & 3 < x < 6 \end{cases}$ and interpret physically.

Ans. $\displaystyle U(x, t) = \sum_{m=1}^{\infty} 2\left[\frac{1 - \cos(m\pi/3)}{m\pi}\right] e^{-m^2\pi^2 t/36} \sin \frac{m\pi x}{6}$

13.47. Show that if each side of equation (*3*), Page 356, is a constant c where $c \geqq 0$, then there is no solution satisfying the boundary-value problem.

13.48. A flexible string of length π is tightly stretched between points $x = 0$ and $x = \pi$ on the x-axis, its ends are fixed at these points. When set into small transverse vibration, the displacement $Y(x, t)$ from the x-axis of any point x at time t is given by $\dfrac{\partial^2 Y}{\partial t^2} = a^2 \dfrac{\partial^2 Y}{\partial x^2}$, where $a^2 = T/\rho$, $T =$ tension, $\rho =$ mass per unit length.

(*a*) Find a solution of this equation (sometimes called the *wave equation*) with $a^2 = 4$ which satisfies the conditions $Y(0, t) = 0$, $Y(\pi, t) = 0$, $Y(x, 0) = 0.1\sin x + 0.01\sin 4x$, $Y_t(x, 0) = 0$ for $0 < x < \pi$, $t > 0$.

(*b*) Interpret physically the boundary conditions in (*a*) and the solution.

Ans. (*a*) $Y(x, t) = 0.1\sin x\cos 2t + 0.01\sin 4x\cos 8t$

13.49. (*a*) Solve the boundary-value problem $\dfrac{\partial^2 Y}{\partial t^2} = 9\dfrac{\partial^2 Y}{\partial x^2}$ subject to the conditions $Y(0, t) = 0$, $Y(2, t) = 0$, $Y(x, 0) = 0.05x(2 - x)$, $Y_t(x, 0) = 0$, where $0 < x < 2$, $t > 0$. (*b*) Interpret physically.

Ans. (*a*) $\displaystyle Y(x, t) = \frac{1.6}{\pi^3}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin \frac{(2n-1)\pi x}{2} \cos \frac{3(2n-1)\pi t}{2}$

13.50. Solve the boundary-value problem $\dfrac{\partial U}{\partial t} = \dfrac{\partial^2 U}{\partial x^2}$, $U(0, t) = 1$, $U(\pi, t) = 3$, $U(x, 0) = 2$.

[Hint: Let $U(x, t) = V(x, t) + F(x)$ and choose $F(x)$ so as to simplify the differential equation and boundary conditions for $V(x, t)$.]

Ans. $U(x, t) = 1 + \dfrac{2x}{\pi} + \displaystyle\sum_{m=1}^{\infty} \dfrac{4\cos m\pi}{m\pi} e^{-m^2 t} \sin mx$

13.51. Give a physical interpretation to Problem 13.50.

13.52. Solve Problem 13.49 with the boundary conditions for $Y(x, 0)$ and $Y_t(x, 0)$ interchanged, i.e., $Y(x,) = 0$, $Y_t(x, 0) = 0.05x(2 - x)$, and give a physical interpretation.

Ans. $Y(x, t) = \dfrac{3.2}{3\pi^4} \displaystyle\sum_{n=1}^{\infty} \dfrac{1}{(2n-1)^4} \sin \dfrac{(2n-1)\pi x}{2} \sin \dfrac{3(2n-1)\pi t}{2}$

13.53. Verify that the boundary-value problem of Problem 13.24 actually has the solution (*14*), Page 357.

MISCELLANEOUS PROBLEMS

13.54. If $-\pi < x < \pi$ and $\alpha \neq 0, \pm 1, \pm 2, \ldots$, prove that

$$\frac{\pi}{2} \frac{\sin \alpha x}{\sin \alpha\pi} = \frac{\sin x}{1^2 - \alpha^2} - \frac{2\sin 2x}{2^2 - \alpha^2} + \frac{3\sin 3x}{3^2 - \alpha^2} - \cdots$$

13.55. If $-\pi < x < \pi$, prove that

(*a*) $\dfrac{\pi}{2} \dfrac{\sinh \alpha x}{\sinh \alpha\pi} = \dfrac{\sin x}{\alpha^2 + 1^2} - \dfrac{2\sin 2x}{\alpha^2 + 2^3} + \dfrac{3\sin 3x}{\alpha^2 + 3^2} - \cdots$

(*b*) $\dfrac{\pi}{2} \dfrac{\cosh \alpha x}{\sinh \alpha\pi} = \dfrac{1}{2\alpha} - \dfrac{\alpha \cos x}{\alpha^2 + 1^2} + \dfrac{\alpha \cos 2x}{\alpha^2 + 2^2} - \cdots$

13.56. Prove that $\sinh x = x\left(1 + \dfrac{x^2}{\pi^2}\right)\left(1 + \dfrac{x^2}{(2\pi)^2}\right)\left(1 + \dfrac{x^2}{(3\pi)^2}\right)\cdots$

13.57. Prove that $\cos x = \left(1 - \dfrac{4x^2}{\pi^2}\right)\left(1 - \dfrac{4x^2}{(3\pi)^2}\right)\left(1 - \dfrac{4x^2}{(5\pi)^2}\right)\cdots$

[Hint: $\cos x = (\sin 2x)/(2\sin x)$.]

13.58. Show that (*a*) $\dfrac{\sqrt{2}}{2} = \dfrac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 22 \cdot 13 \cdot 15 \ldots}{2 \cdot 2 \cdot 6 \cdot 6 \cdot 10 \cdot 10 \cdot 14 \cdot 14 \ldots}$

(*b*) $\pi\sqrt{2} = 4\left(\dfrac{4 \cdot 4 \cdot 8 \cdot 8 \cdot 12 \cdot 12 \cdot 16 \cdot 16 \ldots}{3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot 15 \cdot 17 \ldots}\right)$

13.59. Let $\mathbf{r}$ be any three dimensional vector. Show that

(*a*) $(\mathbf{r} \cdot \mathbf{i})^2 + (\mathbf{r} \cdot \mathbf{j})^2 \leqq (\mathbf{r})^2$, (*b*) $(\mathbf{r} \cdot \mathbf{i})^2 + (\mathbf{r} \cdot \mathbf{j})^2 + (\mathbf{r} \cdot \mathbf{k})^2 = \mathbf{r}^2$

and discuss these with reference to Parseval's identity.

13.60. If $\{\phi_n(x)\}$, $n = 1, 2, 3, \ldots$ is orthonormal in (a, b), prove that $\displaystyle\int_a^b \left\{ f(x) - \sum_{n=1}^{\infty} c_n \phi_n(x) \right\}^2 dx$ is a minimum when

$$c_n = \int_a^b f(x)\, \phi_n(x)\, dx$$

Discuss the relevance of this result to Fourier series.