MTH 102A

Linear Algebra and Ordinary Differential Equations

2015-16-II Semester

Arbind Kumar Lal[*]

December 31, 2015

Course Number: MTH102A
Course Name: Linear Algebra & Differential Equations.

Instructors

Arbind K Lal Off-Phone: 7662. Email: arlal@iitk.ac.in
Kaushik Bal Off-Phone: 6217. Email: kaushik@iitk.ac.in

Course Materials:
Home Page: *http : //home.iitk.ac.in/ arlal/mth102a.htm*
Linear Algebra Notes: *http : //home.iitk.ac.in/ arlal/book/nptel/pdf/booklinear.pdf*
Reference Books:
Advanced Engineering Mathematics - Erwin Kreyszig
Linear Algebra - Kenneth M Hoffman and Ray Kunze

[*]Indian Institute of Technology Kanpur

- Linear Algebra Part: 50 Marks

Chapters	Classes
Matrices and Linear Equations	4
Vector Spaces	4
Inner Product Spaces	4
Eigenvalues and Eigenvectors	6
Linear Transformations	3

Exam	Due	About	Marks
Quiz1	$\sim$ week 4	LA	10
MidSem		LA	30
Quiz2	$\sim$ week 11	DE	10
Endsem		LA,DE	10,40

Quiz - I : Date: February 02, 2016

A rectangular array of numbers is called a matrix. Horizontal arrays are called its ROWS; Vertical arrays are called its COLUMNS.

- An $m \times n$ matrix $A = [a_{ij}]$ is an array of m rows and n columns.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

- Here $m \times n$ is called the size or the order of A.

We call a_{ij} the (i,j)-th entry of A. Also denoted by $A(i,j)$. a_{ij} is the entry at the intersection of the i^{th} row and j^{th} column. This is the 1st row of A. Notation : $A(1,:)$.

This is the 2nd column of A. Notation : $A(:,2)$.

$\mathbb{R}$: Real numbers; $\mathbb{C}$: Complex numbers

- Column vector : a matrix of single column.
- Elements of $\mathbb{R}^n$ (or $\mathbb{C}^n$) will be seen as column vectors.

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if they have the same order $m \times n$ and $a_{ij} = b_{ij}$ for each $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Can think of the linear equations $\begin{aligned} 2x + 3y &= 5 \\ 3x + 2y &= 5 \end{aligned}$

as

$$\begin{bmatrix} 2 & 3 & : & 5 \\ 3 & 2 & : & 5 \end{bmatrix} \text{ or } \left[\begin{array}{cc|c} 2 & 3 & 5 \\ 3 & 2 & 5 \end{array}\right].$$

Zero Matrix: Notation :- $\mathbf{0}$ For example,

$$\mathbf{0}_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \mathbf{0}_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Square Matrix: Notation :- $A_{n\times n}$ or A_n

Diagonal /Principal Diagonal entries: The entries $a_{11}, a_{22}, \ldots, a_{nn}$ of $A_n = [a_{ij}]$

Diagonal Matrix: Notation - $\mathrm{diag}(a_{11}, \ldots, a_{nn})$

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}.$$

Identity Matrix: Notation :- $I_n = \text{diag}(1, \ldots, 1)$.

Triangular Matrix: $A_{n\times n} = [a_{ij}]$. Then, A is

upper triangular if $a_{ij} = 0$, for all $1 \le j < i \le n$.

lower triangular if $a_{ij} = 0$, for all $1 \le i < j \le n$.

$$\begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}, \qquad \begin{bmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{bmatrix}$$

P. **Matrices - Operations**

- Transpose of $A_{m\times n} = [a_{ij}]$: Notation :- A^t.

is a matrix $B_{n\times m} = [b_{ij}]$, with $b_{ij} = a_{ji}$.

- Example: If $A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix}$ then, $A^t = \begin{bmatrix} 1 & 0 \\ 4 & 1 \\ 5 & 2 \end{bmatrix}$.

Thus, transpose of a row vector is a column vector and vice-versa.

Th For any matrix A, $(A^t)^t = A$.

- Addition of $A_{m\times n} = [a_{ij}]$ and $B_{m\times n} = [b_{ij}]$: Notation :- $A + B$.

is a matrix $C_{m\times n} = [c_{ij}]$ with $c_{ij} = a_{ij} + b_{ij}$.

Example: If $A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & -6 \end{bmatrix}$ then, $A + B = \begin{bmatrix} 3 & 3 & 8 \\ 4 & 3 & -4 \end{bmatrix}$.

- Scalar multiplication of $A = [a_{ij}]$ with $k \in \mathbb{R}$: Notation:- $kA = [ka_{ij}]$.

Example, if $A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix}$ and $k = 5$,

then $5A = \begin{bmatrix} 5 & 20 & 25 \\ 0 & 5 & 10 \end{bmatrix}$.

Th Let A, B and C be $m \times n$ matrices and $k \in \mathbb{R}$. Then,

- $A + B = B + A$ (commutativity).
- $(A + B) + C = A + (B + C)$ (associativity).
- $k(\ell A) = (k\ell)A$.
- $(k + \ell)A = kA + \ell A$.

Let A be an $m \times n$ matrix.

- Then there exists a matrix $B_{m\times n}$ with $A + B = \mathbf{0}_{m\times n}$.
  This matrix B is called the additive inverse of A, denoted $-A = (-1)A$.

- $A + \mathbf{0}_{m\times n} = \mathbf{0}_{m\times n} + A = A$.
  Hence, the matrix $\mathbf{0}_{m\times n}$ is called the additive identity.

Matrix Multiplication / Product : $A_{m\times n}B_{n\times p} = C_{m\times p} = [c_{ij}]$ with

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

Let $\mathbf{e}_1 = (1, 0, \ldots, 0)^t \in \mathbb{R}^n$, $\mathbf{e}_2 = (0, 1, 0, \ldots, 0)^t \in \mathbb{R}^n$ and $\mathbf{e}_i \in \mathbb{R}^n$ be the vector with 1 at the i-th place and 0, elsewhere.

Then, $\mathbf{e}_1^t\mathbf{e}_1 = \begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} = 1$, $\mathbf{e}_1^t\mathbf{e}_2 = \begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix} = 0$ and in general,

$$\mathbf{e}_i^t\mathbf{e}_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

What is $\mathbf{e}_i\mathbf{e}_j^t$? $\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\begin{bmatrix}0 & 1 & 0\end{bmatrix} =$ use $\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\\0&0&0\end{bmatrix} \rightarrow$ sometimes called e_{32}.

- What is $\begin{bmatrix}1&0&0\\0&2&0\\0&0&1\end{bmatrix}A_{3\times k}$?

Ans: $\begin{bmatrix}1&0&0\\0&2&0\\0&0&1\end{bmatrix}\begin{bmatrix}a&b&\cdots&d\\\alpha&\beta&\cdots&\delta\\u&v&\cdots&z\end{bmatrix} = \begin{bmatrix}a&b&\cdots&d\\2\alpha&2\beta&\cdots&2\delta\\u&v&\cdots&z\end{bmatrix}$.

The same matrix A, except that the second row is multiplied with 2.

- What is $A_{m\times 3}\begin{bmatrix}1&0&0\\0&2&0\\0&0&1\end{bmatrix}$?

Ans: The same matrix A, except that the second <u>column</u> is multiplied with 2.

- Example $A = \begin{bmatrix} \boxed{1 \quad 2 \quad 3} \\ 0 \quad 1 \quad -1 \end{bmatrix}$ and $B = \begin{bmatrix} a & b & c \\ e & f & g \\ u & v & w \end{bmatrix}$ then,

$$AB = \begin{bmatrix} \boxed{a+2e+3u} & b+2f+3v & c+2g+3w \\ e-u & f-v & g-w \end{bmatrix}.$$

$$\boxed{AB = \begin{bmatrix} a+2e+3u & b+2f+3v & c+2g+3w \end{bmatrix}} \\ \begin{bmatrix} e-u & f-v & g-w \end{bmatrix}.$$

$$\begin{aligned} AB(1,:) &= 1[a,b,c] + 2[e,f,g] + 3[u,v,w] \\ &= \boxed{[a+2e+3u, b+2f+3v, c+2g+3w]} \end{aligned}$$

$AB(2,:) = 0[a,b,c] + 1[e,f,g] + (-1)[u,v,w] = [e-u, f-v, g-w]$

$AB(:,1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} a + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e + \begin{bmatrix} 3 \\ -1 \end{bmatrix} u = \begin{bmatrix} a+2e+3u \\ e-u \end{bmatrix}.$

$AB(:,2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} b + \begin{bmatrix} 2 \\ 1 \end{bmatrix} f + \begin{bmatrix} 3 \\ -1 \end{bmatrix} v = \begin{bmatrix} b+2f+3v \\ f-v \end{bmatrix}.$

$AB(:,3) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} c + \begin{bmatrix} 2 \\ 1 \end{bmatrix} g + \begin{bmatrix} 3 \\ -1 \end{bmatrix} w = \begin{bmatrix} c+2g+3w \\ g-w \end{bmatrix}.$

In general,

$$AB(i,:) = a_{i1}B(1,:) + a_{i2}B(2,:) + \cdots + a_{in}B(n,:).$$

$$AB(:,j) = A(:,1)b_{1j} + A(:,2)b_{2j} + \cdots + A(:,n)b_{nj}.$$

- The product AB corresponds to operating on the *rows of the matrix* B using the entries of A.
- The product AB also corresponds to operating on the *columns of the matrix* A using the entries of B.
- Let $A_{m\times n}$ and $B_{n\times p}$. Then, $AB = [AB(:,1),\ AB(:,2),\ \ldots,\ AB(:,p)]$. Also, $AB = \begin{bmatrix} A(1,:)B \\ A(2,:)B \\ \vdots \\ A(n,:)B \end{bmatrix}.$

- The product $A_{m\times n}B_{n\times p}$ is defined but, the product BA is not defined, unless $p=m$.
- If $A_{m\times n}$ and $B_{n\times m}$ then, AB and BA are defined but their sizes are different.
- Even if A and B are $n\times n$, AB need not be equal to BA. Example,

$$\begin{bmatrix}1&1\\1&1\end{bmatrix}\begin{bmatrix}1&0\\-1&0\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}\neq\begin{bmatrix}1&1\\-1&-1\end{bmatrix}=\begin{bmatrix}1&0\\-1&0\end{bmatrix}\begin{bmatrix}1&1\\1&1\end{bmatrix}.$$

P. **Matrix Multiplication - Properties**

- Let $A_{m\times n}$, $B_{n\times p}$ and $C_{p\times r}$ and $k\in\mathbb{R}$.
  - $(AB)C=A(BC)$. Matrix multiplication is associative.
  - For any $k\in\mathbb{R}$, $(kA)B=k(AB)=A(kB)$.
  - $A(B+C)=AB+AC$. Multiplication distributes over addition.
  - If $m=n$ and $D=\text{diag}(d_1,d_2,\ldots,d_n)$ then,
    - $AI_n=I_nA=A$. $\quad I_n$ is the Multiplicative Identity
    - the first row of DA is d_1 times the first row of A
    - for $1\le i\le n$, the i^{th} row of DA is d_i times the i^{th} row of A.

  A similar statement holds for the columns of A when A is multiplied on the right by D.

P. **Invertible matrices**

Inverse: Let A be a square matrix. Then, B is said to be an inverse of A if $AB=BA=I$.

- We denote the inverse of A by A^{-1}. Note that $(A^{-1})^{-1}=A$ and $(A^{-1})^t=(A^t)^{-1}$.
  - Let $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$. If $ad-bc\neq 0$ then $A^{-1}=\frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$.

Q Let A and B be invertible. Is AB invertible? Yes. $(AB)^{-1}=B^{-1}A^{-1}$.

Q I have a matrix with a zero row. Is it invertible? No. If $A(i,:)$ is zero, then $(AB)(i,:)=0$ for any B. So, no AB will be I, as I does not have a zero row.

$$\begin{bmatrix}\\0&\cdots&0\\\\\end{bmatrix}\begin{bmatrix}*&*&*\\*&*&*\\*&*&*\end{bmatrix}=\begin{bmatrix}*&*&*\\0&\cdots&0\\*&*&*\end{bmatrix}$$

Q I have a matrix with a zero column. Is it invertible? No. If $A(:,i)$ is zero, then $(BA)(:,i)=0$ for any B. So, no BA will be I, as I does not have a zero column.

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} * & 0 & * \\ & \vdots & * \\ * & 0 & * \end{bmatrix}$$

P. Matrices - System of Linear Equations

• The system $\begin{matrix} 2x & +3y & +z & =6 \\ x & +2y & +z & =4 \\ x & & -z & =0 \end{matrix}$ is same as $\begin{bmatrix}2\\1\\1\end{bmatrix}x + \begin{bmatrix}3\\2\\0\end{bmatrix}y + \begin{bmatrix}1\\1\\-1\end{bmatrix}z = \begin{bmatrix}6\\4\\0\end{bmatrix}$

or equivalently, if $A = \begin{bmatrix}2 & 3 & 1\\1 & 2 & 1\\1 & 0 & -1\end{bmatrix}$ then $A\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}6\\4\\0\end{bmatrix}$.

Define $f : \mathbb{R}^3 \to \mathbb{R}^3$ by $f(\mathbf{x}) = A\mathbf{x}$. Then, $f(\mathbf{e}_1) = A\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}2\\1\\1\end{bmatrix}$, $f(\mathbf{e}_2) = A\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}3\\2\\0\end{bmatrix}$ and $f(\mathbf{e}_3) = A\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\1\\-1\end{bmatrix}$.

Q The above system is same as asking "is $\begin{bmatrix}6\\4\\0\end{bmatrix}$ an element of Image (f)"?

P. Matrices - Submatrix

Submatrix: A matrix obtained by deleting some of the rows and/or columns of a matrix is called a submatrix of the given matrix.

For example, if $A = \begin{bmatrix}1 & 4 & 5\\0 & 1 & 2\end{bmatrix}$, a few submatrices of A are $[1]$, $[2]$, $\begin{bmatrix}1\\0\end{bmatrix}$, $[1\ \ 5]$, $\begin{bmatrix}1 & 5\\0 & 2\end{bmatrix}$, A. Not a submatrix of A: $\begin{bmatrix}1\\1\end{bmatrix}$, $\begin{bmatrix}4\\0\end{bmatrix}$, $\begin{bmatrix}4\\2\end{bmatrix}$, $\begin{bmatrix}1 & 4\\1 & 0\end{bmatrix}$ and $\begin{bmatrix}1 & 4\\0 & 2\end{bmatrix}$.

Th Let $A = [a_{ij}] = [P_{m\times n}\ Q_{m\times r}]$ and $B = [b_{ij}] = \begin{bmatrix}H_{n\times t}\\K_{r\times t}\end{bmatrix}$. Then, the size/order of AB is $m \times t$ and

$$AB = PH + QK.$$

• Example: Let $A = \begin{bmatrix}1 & 2 & 0\\2 & 5 & 0\end{bmatrix}$ and $B = \begin{bmatrix}a & b\\c & d\\e & f\end{bmatrix}$. Then,

$$AB = \begin{bmatrix}1 & 2\\2 & 5\end{bmatrix}\begin{bmatrix}a & b\\c & d\end{bmatrix} + \begin{bmatrix}0\\0\end{bmatrix}[e\ f] = \begin{bmatrix}a+2c & b+2d\\2a+5c & 2b+5d\end{bmatrix}.$$