Conditional Distribution & Expectation

- We could restrict the scope of random variable X to some event B.

- Defn: • $X|_B := X : B \to \mathbb{R}$, $\omega \mapsto X(\omega)$ is called conditional distribution.

  • Probability mass fn. $P_{X|_B} : x \mapsto P(X = x \mid B) := P(B \cap X^{-1}(x)) / P(B)$

  • Conditional expectation $E[X|B] := \sum_{\omega \in B} P(\omega|B) \cdot X(\omega)$
  $= \frac{1}{P(B)} \cdot \sum_{\omega \in B} P(\omega) \cdot X(\omega)$.

- This gives a useful 'partition' formula (over disjoint events $B_1, \ldots, B_k$):

▷ $E[X] = \sum_{i \in [k]} E[X|B_i] \cdot P(B_i)$. (Why?)

- Eg: A miner is stuck in a mine with 3 doors.
  One door brings him back to the current place & wastes 7 hrs.
  Other door... wastes 5 hrs.
  Another door takes him out in 3 hrs.
  When could he expect to be out?

Analyse: • Let $X :=$ #hrs to get out. $P(\text{pick door-}i) = 1/3$.

$$\Rightarrow \quad E[X] = \sum_{i \in [3]} \underbrace{P(\text{pick door-}i)}_{\overset{!!}{} B_i} \cdot E[X \mid B_i]$$

$$= \frac{1}{3} \cdot \sum_i E[X \mid B_i]$$

$$\Rightarrow 3 \cdot E[X] = (7 + E[X]) + (5 + E[X]) + (3)$$

$$\Rightarrow \quad E[X] = 15. \qquad \square$$

- Say, X, Y are random variables on Ω. Then, for any $\alpha \in \mathbb{R}$, we can define $\underline{\alpha \cdot X} : \omega \mapsto \alpha \cdot X(\omega)$; which is again a random variable.
- Also, $\underline{X+Y} : \omega \mapsto X(\omega) + Y(\omega)$; is a random variable. What's its expectation?

Linearity of expectation

$\triangleright\ E[\alpha \cdot X] = \alpha \cdot E[X]\ ;\ \forall \alpha \in \mathbb{R}.$

Theorem: $E[X+Y] = E[X] + E[Y].$

Pf: $\cdot\ E[X+Y] =: \sum_{x,y} P(X=x \wedge Y=y)\cdot(x+y)$

$$= \sum_{x} x \cdot \left(\sum_{y} P(X=x \wedge Y=y)\right) + \sum_{y} y \cdot \left(\sum_{x} P(X=x \wedge Y=y)\right)$$

$$= \sum_{x} x \cdot P(X=x) + \sum_{y} y \cdot P(Y=y)$$

$$= E[X] + E[Y]. \qquad \square$$

Corollary: $E\left[\sum_{i} X_i\right] = \sum_{i} E[X_i].$

Corollary: $E\left[\underbrace{\sum_i \alpha_i \cdot X_i}_{\text{linear combination}}\right] = \sum_i \alpha_i \cdot E[X_i]$. $(\alpha_i \in \mathbb{R}, \forall i)$

- Despite being easy to prove, the property is very useful!

- Eg. 1: Recall the qn. of putting n letters into n (addressed) envelopes.

  Let $X :=$ #(letters correctly posted). What's $E[X]$?

• By defn, $E[X] = \sum_{0 \le k \le n} P(X=k) \cdot k$. Recall that even $P(X=0)$ was complicated!

• Better way: $X_i := \begin{cases} 1, & \text{if letter-}i \text{ is correctly posted,} \\ 0, & \text{else} \end{cases}$

▷ $X = \sum_{i=1}^{n} X_i$.

▷ $E[X] = \sum_{i=1}^{n} E[X_i]$.

$\Rightarrow E[X] = \sum_{i=1}^{n} \left(1 \cdot \frac{1}{n} + 0 \cdot \frac{n-1}{n}\right) = \boxed{1}$.

became very easy!