Steps to design Greedy Algorithms

1. Express solution of the given problem as a choice + solution to one subproblem

2. Come up with a greedy choice. Show it is safe. That is, there is an optimal solution with greedy choice

3. Optimal solution to the problem can be written as greedy choice + optimal solution to the subproblem.

$\Rightarrow$ recursive algorithm, making greedy choice at every stage.
Can be converted to iterative algorithm.

[Important to remember that a Greedy strategy often has relation to dynamic Programming formulation of the given problem]

Example (Making change)

(Unlimited) Coins of denominations 25 ¢, 10 ¢, 5 ¢, 1 ¢

Given a positive integer n, find minimum no. of coins (from above denominations) that add upto n.

Example (Concrete n) $n = 9$

$\overbrace{1 + 1 + - - 1}^{\text{nine times}}$ $\overbrace{10 \times 1 ¢}^{\text{no. of coins } 9}$

$5 + 1 + 1 + 1 + 1$ $\underbrace{1 \times 5 ¢ + 4 \times 1 ¢}_{\text{no. of coins } 5}$

(optimal Solution)

Input $(n, [25, 10, 5, 1])$

Step 1

Choice: how many coins of 25 ¢ are picked

Subproblem: remaining amount to be changed using coins of denominations [10, 5, 1].

n_{25} no. of coins of 25 ¢ chosen,

$(n - n_{25} \times 25, [10, 5, 1])$

$d_k > d_{k-1} \ldots > d_2 > d_1 = 1$

$\underset{\text{II}}{}$

Step 2

Greedy choice: Problem instance $(m, [d_k, d_{k-1}, \ldots, d_2, d_1])$

$1 \leq k \leq 4$

$\downarrow$

Pick as many coins of denomination d_k as possible. $\left\lfloor \frac{m}{d_k} \right\rfloor$

To show that Greedy choice is safe.

This needs to be shown for all subproblems

Subsubproblems

(i) $(m, [25, 10, 5, 1])$ $\longrightarrow$ $(m', [10, 5, 1])$

(ii) $(m, [10, 5, 1])$ $\longrightarrow$ $(m', [5, 1])$

(iii) $(m, [5, 1])$ $\longrightarrow$ $(m', \cancel{5}[1])$

(iv) $(m, [1])$ $\longrightarrow$ no subproblem

I show safety of greedy choice for (i). Other cases [(ii), (iii), (iv)] are easier and are left as exercise.

Claim: Greedy choice is safe for $(m, [25, 10, 5, 1])$

pf: Greedy choice is to pick $\lfloor \frac{m}{25} \rfloor$ many coins of 25 ¢.

Let e_{25}, e_{10}, e_5, e_1 be an optimal solution for this problem

(e_i : non-negative integers, denotes no. of coins picked of denomination i)

$$\Rightarrow \quad 25 \cdot e_{25} + 10 \cdot e_{10} + 5 \cdot e_5 + e_1 = m$$

if $e_{25} = \lfloor \frac{m}{25} \rfloor$ then we are done. (e_{25} can't be $> \lfloor \frac{m}{25} \rfloor$)

Otherwise let $e_{25} < \lfloor \frac{m}{25} \rfloor$

$$\Rightarrow \quad 10 e_{10} + 5 e_5 + e_1 \geq 25$$

If there are some coins from multiset $\{ \underbrace{10, \ldots 10}_{e_{10}}, \underbrace{5, \ldots 5}_{e_5}, \underbrace{1, \ldots, 1}_{e_1} \}$ that add upto 25 ¢ then we can replace them by a single 25 ¢ coin.

This results in lesser no. of coins realizing m. This contradicts optimality of solution $(e_{25}, e_{10}, e_5, e_1)$.

Let $x > 25$ be the least no. s.t. some coins from multiset $\{e_{10}, e_5, e_1\}$ add upto x.

Let E be a subset of the given multiset $\{e_{10}, e_5, e_1\}$ s.t. valuation of E is x.

<u>Case I</u> Suppose E has 1¢ coin.
Then $E - \{1¢\}$ gives a set realizing $x-1 \geq 25$
This is a contradiction

<u>Case II</u> E has coins of only 10¢ and 5¢.
$\Longrightarrow$ x is a multiple of 5.

Case (a) If E has a $5¢$ coin

then $x < 30$ $\left[\because\right.$ if $x \geq 30$ then $E - \{5¢\}$ adds up to $\left.\geq 25¢\right]$ a contradiction

$\Rightarrow x \leq 25$ ($\because x$ is multiple of 5)

a contradiction ($\because x$ is assumed be > 25)

Case (b)

E has only $10¢$ coins

$\Rightarrow x$ is a multiple of 10

If $x > 30$ then $x = 40$ $E - \{10¢\}$ adds upto $\geq 25¢$

Contradiction

$\Rightarrow x = 30$

$\Rightarrow E$ consists of 3 coins of $10¢$ each

Then E can be replaced by a 25¢ and a 5¢ coin.
leading to strictly lesser coins than in the optimal solution
$\Longrightarrow$ contradiction.

$$\Rightarrow \quad e_{25} = \left\lfloor \frac{m}{25} \right\rfloor$$

$\Rightarrow$ We have a optimal solution with a greedy choice. $\square$

We need to property ~~stated~~ in step 3.

Claim There is an optimal solution of $(m, [25, 10, 5, 1])$
consisting of greedy choice and optimal solution of
the subproblem $\left(m - \left\lfloor \frac{m}{25} \right\rfloor \cdot 25, \; [10, 5, 1]\right)$

Pf: We have seen that for subproblem $(m, [25, 10, 5, 1])$ there is an optimal solution with greedy choice

So. the optimal solution is

$$\left\{ \left\lfloor \frac{m}{25} \right\rfloor \text{ coins of } 25¢ \right\}$$

$$\cup \left\{ \underbrace{\text{solution of } \overbrace{\left(m - \left\lfloor \frac{m}{25} \right\rfloor \cdot 25 \,,\, [10, 5, 1] \right)}^{①}} \right\}$$

This has to be optimal solution to the subproblem ①

because if this is not an optimal solution than substituting an optimal solution for it will result in better solution for $(m, [25, 10, 5, 1])$ than the optimal solution. This is a contradiction.

$\square$

Pseudo code

make—change-greedy $(n, [25, 10, 5, 1])$

$i = 4$

$T = \phi$

While $n > 0$ and $i \geq 1$ do

greed st. $\quad x = \left\lfloor \frac{n}{\ell(i)} \right\rfloor$

$\quad T = T \cup \{(\ell(i), x)\}$

Sub problem $\begin{cases} n = n - x \cdot \ell(i) \\ i = i - 1 \end{cases}$

// end of while loop

return T

$\ell(4) = 25$

$\ell(3) = 10$

$\ell(2) = 5$

$\ell(1) = 1$

Example run : $\quad n = 89$

$i = 4, \quad T = \phi \qquad x = 3$

$T = \{(25, 3)\}, \quad n = 89 - 75 = 14, \; i = 3$

$x = \left\lfloor \frac{14}{10} \right\rfloor = 1, \quad T = \{(25, 3), (10, 1)\}$

$n = 14 - 1 \cdot 10 = 4, \quad i = 2$

$x = \left\lfloor \frac{4}{5} \right\rfloor = 0, \quad T = \{(25, 3), (10, 1), (5, 0)\}$

$n = 4, \quad i = 1$

$x = \left\lfloor \frac{4}{1} \right\rfloor = 4. \qquad \boxed{T = \{(25, 3), (10, 1), (5, 0), (1, 4)\}}$

$n = 4 - 4 \cdot 1 = 0, \quad i = 0 \rightarrow$ exit while

return T

Time complexity

$O(1)$ [In general no. of coins]

[Assuming, arithmetic ops have unit cost]

Does greedy strategy work for every set of denomination of coins.

NO.

Example

Coins of denomination [10, 8, 1]

$n = 16$

optimal solution has two coins (each of 8¢).

Greedy strategy 10¢ + 6 × 1¢ = 7 coins