Logical consequence.

For $X \subseteq \Phi$ and $\vartheta \in V$, $\vartheta \models X$ denotes $\vartheta \models \beta$ for all $\beta \in X$.

$X \models \alpha$ if for all $\vartheta \in V$ s.t $\vartheta \models X$, it also holds that $\vartheta \models \alpha$

α is a logical consequence of $X \subseteq \Phi$

Theorem (Compactness). Let $X \subseteq \Phi$ and $\alpha \in \Phi$, then

$X \models \alpha$ iff $\exists\, Y \subseteq_{FIN} X$ s.t $Y \models \alpha$.

Finite Satisfiability. Let $X \subseteq \Phi$, X is satisfiable iff every $Y \subseteq_{FIN} X$ is satisfiable.

($\Rightarrow$) Trivial

Suppose X is satisfiable. Then $\exists v \in V$ s.t $v \models X$
Then $v \models Y$ for every $Y \subseteq_{FIN} X$ as well.

($\Leftarrow$) Suppose X is not satisfiable.

We show: $\exists Y \subseteq_{FIN} X$ s.t Y is not satisfiable.

Let $P = \{p_1, p_2, \ldots\}$. Let $P_0 = \phi$ and $P_i = \{p_1, p_2, \ldots p_i\}$ for $i \geq 1$

Let Φ_i - Set of formulas generated using only the atomic propositions from P_i for $i \geq 1$.

Define $X_i = X \cap \Phi_i$

Construct a tree T: nodes are valuations over the set P_i, $i \geq 0$

Set of nodes: $\{ v \mid \exists i \in \{0,1,2,\ldots\}, v : P_i \to \{\top, \bot\} \}$

Consider any $v : P_i \to \{\top, \bot\}$. v has 2 children

v' and v'' both functions $P_{i+1} \to \{\top, \bot\}$.

v' extends v to P_{i+1} by setting p_{i+1} to $\top$ and
v'' " p_{i+1} to $\bot$.

That is, $\forall p \in P_i$, $v'(p) = v''(p) = v(p)$
$v'(p_{i+1}) = \top$, $v''(p_{i+1}) = \bot$

Observation. T is a complete binary tree.

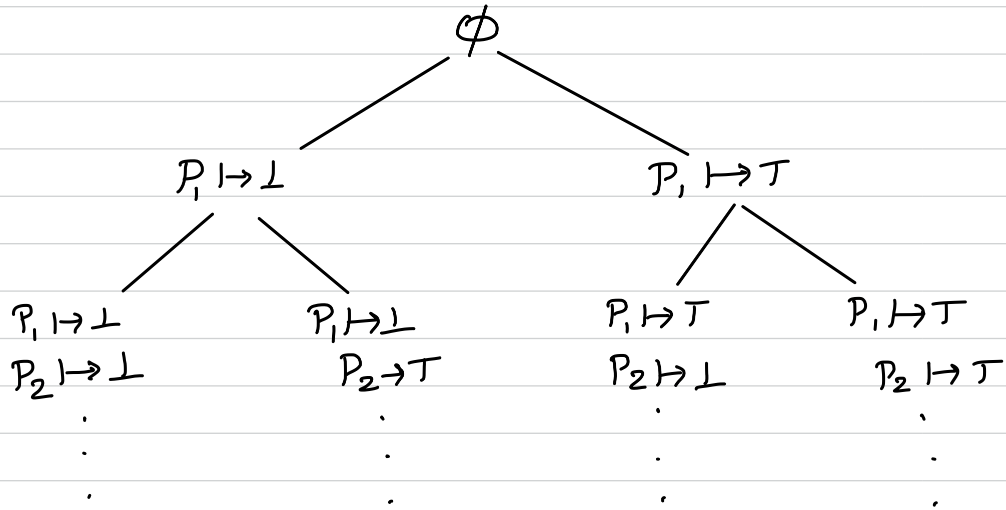
Level i in T consists of all possible valuations over P_i.

Infinite paths in T are in 1-1 correspondence with valuations over P.

Let $\pi = v_0\, v_1\, v_2 \cdots$ then $v_\pi : P \to \{\top, \bot\}$, $p_i \mapsto v_i(p_i)$

Given a valuation, we can find a unique path π_v in T

The complete binary tree T



A node v in T is bad if $v(\beta) = \perp$ for some $\beta \in X$

prune T by deleting all bad nodes which also have bad ancestors.

That is, on any path in T, retain only the nodes upto and including the first bad node.

In subtree T' of T, – all leaf nodes are bad
– all non-leaf nodes are not bad.

Claim 1. T' is finite.

Suppose Claim 1 is true. Let set of leaf nodes be

$\{v_1, v_2, \ldots, v_m\}$.

Every v_i is bad $\Rightarrow \exists \beta_i \in X$ s.t $v_i(\beta_i) = \perp$.

Claim 2. $\{\beta_1, \beta_2, \ldots, \beta_m\} \subseteq_{FIN} X$ is not satisfiable.

Consider any valuation v, the path π_v should pass through some node $v_j \in \{v_1, v_2, \ldots v_m\}$.

By definition, $v_{\pi_v}(\beta_j) = v_j(\beta_j) = \perp$

Therefore $v \not\models \{\beta_1, \beta_2, \ldots, \beta_m\}$.

Claim 1. T' is finite.

Proof. Suppose T' is not finite.

By König's Lemma, it contains an infinite path $\pi = v_0 v_1 \cdots$ s.t none of the nodes on π is bad.

By definition, π is also an infinite path in T.

Consider v_π and a $\beta \in X$. We have $\beta \in X_j$ for some $j \geq 1$ $\rightarrow X \cap \Phi_j$

So $v_\pi(\beta) = v_j(\beta) = T$. Thus $v_\pi \models X$ - a contradiction.

König's Lemma. Let T be a finitely branching tree*

if T has infinitely many nodes, then T has an infinite path.

* - (every node has a finite number of children).

Finite Satisfiability. Let $X \subseteq \Phi$, X is satisfiable iff every $Y \subseteq_{FIN} X$ is satisfiable.

Theorem (Compactness). Let $X \subseteq \Phi$ and $\alpha \in \Phi$, then

$X \models \alpha$ iff $\exists\, Y \subseteq_{FIN} X$ s.t $Y \models \alpha$.

Proof.

($\Leftarrow$) if $Y \subseteq_{FIN} X$ and $Y \models \alpha$ then $X \models \alpha$

if $v \models X$ then $v \models Y$. By assumption $Y \models \alpha$ so $v \models \alpha$.

($\Rightarrow$) For all $Z \subseteq \Phi$, $\forall \beta \in \Phi$, $Z \models \beta$ iff $Z \cup \{\neg\beta\}$ is not satisfiable.

Suppose $X \models \alpha$. Then $X \cup \{\neg\alpha\}$ is not satisfiable.

By finite satisfiability Lemma,

$\exists\, Y \subseteq_{FIN} X \cup \{\neg\alpha\}$ s.t Y is not satisfiable.

$\therefore$ $\underbrace{(Y \setminus \{\neg\alpha\})}_{\subseteq_{FIN} X} \cup \{\neg\alpha\}$ is not satisfiable.

Therefore, $Y \setminus \{\neg\alpha\} \models \alpha$