18.03 Practice Problems on Fourier Series – Solutions

Graphs appear at the end.

1. *What is the Fourier series for* $1+\sin^2 t$*?*

This function is periodic (of period 2π), so it has a unique expression as a Fourier series. It's easy to find using a trig identity. By the double angle formula, $\cos(2t) = 1 - 2\sin^2 t$, so

$$1+\sin^2 t = \frac{3}{2} - \frac{1}{2}\cos(2t).$$

The right hand side is a Fourier series; it happens to have only finitely many terms.

2. *Graph the function* $f(t)$ *which is even, periodic of period* 2π*, and such that* $f(t) = 2$ *for* $0 < t < \frac{\pi}{2}$ *and* $f(t) = 0$ *for* $\frac{\pi}{2} < t < \pi$*. Is the function even, odd, or neither?*

Here is the graph of $f(t)$. Note that there is only one way to extend the definition of f over all real t since f is specified to be even and periodic.

The function $f(t)$ is even.

Find its Fourier series in two ways:

(a) *Use parity if possible to see that some coefficients are zero. Then use the integral expressions for the remaining Fourier coefficients.*

The function $f(t)$ is even, so $b_n = 0$ for all $n > 0$. The only possibly nonzero coefficients are the a_n's. Compute a_0 first.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 2\,dt = 2.$$

Now compute a_n for $n > 0$. Since the function is even,

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(t)\cos(nt)\,dt = \frac{2}{\pi}\int_0^{\pi/2} 2\cos(nt)\,dt = \frac{4}{n\pi}\sin(nt)\Big|_0^{\pi/2} = \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

If n is even, this is always zero. If n is odd, then this alternates between $+\frac{4}{n\pi}$ when n of the form $4k+1$ and $-\frac{4}{n\pi}$ when n is of the form $4k+3$.

The Fourier series is then

$$f(t) = 1 + \frac{4}{\pi}\cos t - \frac{4}{3\pi}\cos(3t) + \frac{4}{5\pi}\cos(5t) - \frac{4}{7\pi}\cos(7t) + \cdots$$

(b) *Express* $f(t)$ *in terms of* $\mathrm{sq}(t)$*, substitute the Fourier series for* $\mathrm{sq}(t)$ *and use some trig identities.*

First we see that f can be expressed in terms of the standard square wave as

$$f(t) = 1 + \mathrm{sq}\left(t + \frac{\pi}{2}\right).$$

Now (see overleaf) the Fourier series for $\mathrm{sq}(t)$ is

$$\mathrm{sq}(t) = \frac{4}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \cdots\right),$$

so we can substitute this in to get the Fourier series for $f(t)$:

$$f(t) = 1 + \frac{4}{\pi}\left(\sin\left(t + \frac{\pi}{2}\right) + \frac{\sin\left(3t + \frac{3\pi}{2}\right)}{3} + \frac{\sin\left(5t + \frac{5\pi}{2}\right)}{5} + \cdots\right)$$
$$= 1 + \frac{4}{\pi}\cos t - \frac{4}{3\pi}\cos(3t) + \frac{4}{5\pi}\cos(5t) - \cdots.$$

Here we have used the angle addition formula for sine, and the same values of sine that we used in **(a)**. This coincides with the answer we got for Part **(a)**.

3. *Graph the function $h(t)$ which is odd and periodic of period 2π and such that $h(t) = t$ for $0 < t < \frac{\pi}{2}$ and $h(t) = \pi - t$ for $\frac{\pi}{2} < t < \pi$. What is its average value? Observe that $h'(t) = f(t) - 1$, where $f(t)$ is the function studied in **Problem 2**.*

The graph of $h(t)$ is a zigzag wave.

The function is odd, so its average is zero.

Use these observations to find its Fourier series.

We observe that the function $h(t)$ has derivative $f(t)-1$, where $f(t)$ is the function described in **Problem 1**. The Fourier series for $f(t)-1$ has zero constant term, so we can integrate it term by term to get the Fourier series for $h(t)$, up to a constant term given by the average of $h(t)$. Since $h(t)$ is odd, its average is 0. The rest of the series is computed below.

$$h(t) + c = \int (f(t) - 1)dt = \frac{4}{\pi}\int\left(\cos t - \frac{\cos(3t)}{3} + \frac{\cos(5t)}{5} - \cdots\right)dt$$
$$= \frac{4}{\pi}\left(\sin t - \frac{\sin(3t)}{9} + \frac{\sin(5t)}{25} - \cdots\right)$$

Since the right hand side has average value zero, $c = 0$.

4. *Explain why any function $F(x)$ is a sum of an even function and an odd function in just one way. Hint: $F_+(x) = \dfrac{F(x) + F(-x)}{2}$ is even. What is the even part of e^x? What is the odd part?*

First, $F_+(x)$ is even: $F_+(-x) = \dfrac{F(-x) + F(x)}{2} = F_+(x)$.

Now notice that $F(x) = F_+(x) + F_-(x)$ where $F_-(x) = \dfrac{F(x) - F(-x)}{2}$. I claim that $F_-(x)$ is odd: $F_-(-x) = \dfrac{F(-x) - F(x)}{2} = -F_-(x)$.

So $F(x)$ *is* the sum of an even function and an odd function.

To show that this decomposition is unique, we suppose we have another decomposition $\tilde{F}_+(x) + \tilde{F}_-(x) = F(x)$, where $\tilde{F}_+(x)$ is even and $\tilde{F}_-(x)$ is odd.

Then $F_+(x) + F_-(x) = F(x) = \tilde{F}_+(x) + \tilde{F}_-(x)$, so $F_+(x) - \tilde{F}_+(x) = \tilde{F}_-(x) - F_=(x)$. But the left hand side is even and the right hand side is odd, so they both must be zero, which says that $F_+(x) = \tilde{F}_+(x)$ and $F_-(x) = \tilde{F}_-(x)$.

This decomposition might seem familiar from hyperbolic trig function formulas: The even part of e^x is $\dfrac{e^x + e^{-x}}{2} = \cosh x$, and the odd part of e^x is $\dfrac{e^x - e^{-x}}{2} = \sinh x$.

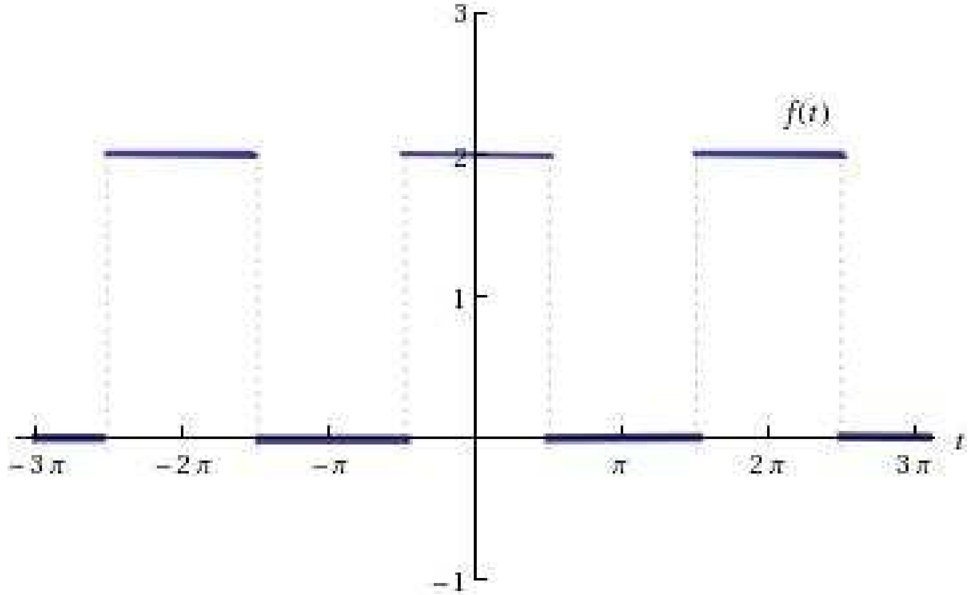

Figure 1: Graph of $f(t)$ over three periods.

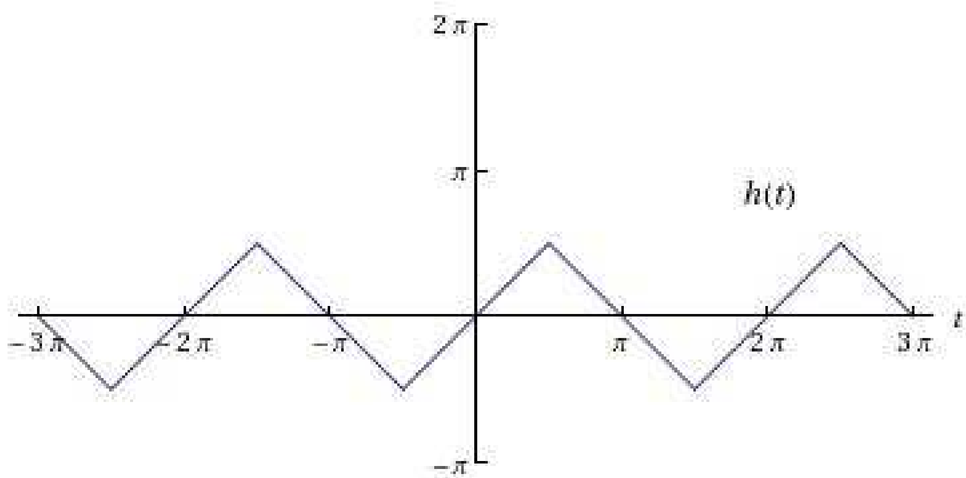

Figure 2: Graph of $h(t)$ over three periods.