⟹ You should switch your door-1 !

Random Variable & Expectation

- We defined $P(\cdot)$ as a map on subsets Σ of Ω.

  sigma-algebra (for Σ); sample space (for Ω)

- But, many a times we need numerical values associated to the outcome of the experiment. Eg. #H in n tosses; pay out in casino.

- Defn: • Random variable (real-valued) is a function $X : \Omega \to \mathbb{R}$, $\omega \mapsto X(\omega)$

- We now talk about events $[X = x]$, for $x \in \mathbb{R}$.
  ↳ is the subset $\{\omega \in \Omega \mid X(\omega) = x\} =: X^{-1}(x)$.
- $P(X=x) := \sum_{\omega \in X^{-1}(x)} P(\omega)$. [Probability mass function of X]
- Consider only those X that have a countable range. Then, X is discrete random variable.

— Just like there are many $P(\cdot)$ possible for $\mathcal{E}$,
〃 〃 〃 X 〃 〃 $(\mathcal{E}, P)$.

- Eg. 1: Say, a coin is tossed 10 times. What is $P(k \text{ Heads}) = ?$

- $\Omega = \{H, T\}^{10}$; $P(H) = P(T) = 0.5$;
- $X : \omega \mapsto (\#H\text{'s in } \omega)$.
- $P(X = k) = \sum_{S \in \binom{[10]}{k}} P(H \text{ in locations } S) \cdot P(X=k \mid H \text{ in locations } S)$

$$= \sum_S \left(\frac{1}{2}\right)^{|S|} \cdot \left(\frac{1}{2}\right)^{10-|S|} = \binom{10}{k} \cdot \frac{1}{2^{10}}.$$

Eg.2: In a class we ask the students their birthdates, one-by-one. Continue till a date repeats. (Assume that there are 365 dates.)

- Let X := #students asked.
- What's $P(X=k) = ?$, for $k \in \mathbb{N}$.

Analyse: • $P(X=k) = P(k\text{-}1 \text{ dates distinct} \wedge k\text{-th repeats})$

$$= \frac{365}{365} \cdot \frac{364}{365} \cdots \frac{(365-k+2)}{365} \cdot \frac{k-1}{365}$$

$$= \left(1-\frac{1}{365}\right) \cdots \left(1-\frac{k-2}{365}\right) \cdot \left(\frac{k-1}{365}\right).$$

▷ $P(X=k)$ falls as k grows. In fact, for $k \approx \sqrt{365 \times 2}$ the prob. of k dates having a repeat is quite high!

Independent random variables

- Defn: Random variables X, Y are independent if $\forall x, y \in \mathbb{R}$, $P(X=x \wedge Y=y) = P(X=x) \cdot P(Y=y)$.

- Eg. Define X: 1 if a dice-throw is even; else 0.
  Y: " " " " prime; " .
  - $X=1 \wedge Y=1 \iff$ throw value is 2 $\Rightarrow P(\cdot) = 1/6$.
  - $X=1 \iff$ value is $\{2,4,6\}$. $\Rightarrow P(\cdot) = 3/6 = 1/2$.
  - $Y=1 \iff$ " " $\{1,2,3,5\}$. $\Rightarrow P(\cdot) = 4/6 = 2/3$

$\Rightarrow P(X=1) \cdot P(Y=1) = 1/3 > 1/6 = P(X=1 \wedge Y=1)$.

▷ X, Y are dependent (& negative correlated!)

Expectation

- Random variable was introduced to associate numerical values to outcomes in an experiment.
- Could we talk about their average?

Defn: • Expected value of X is $E[X] := \sum_{x \in range(X)} P(X=x) \cdot x$

[Usually, range(X) is a finite subset of $\mathbb{R}$. Else convergence of the sum is needed.]

Note: $E[X] \notin \text{range}(X)$. So, it's abstract!

↳ $E[X]$ is just an average of possibilities weighted by their likelihood.

– Expectation is very useful to understand payoffs:

– Eg.1: Your friend gives Rs. 100/- if a prime turns up in a dice throw. How much should you pay her in the opposite case? Say, Rs. x/-

Analyse: • Define $\Omega = [6]$, $A := \{1,2,3,5\} \in \mathcal{E} := 2^{\Omega}$.

• $X : \omega \mapsto \begin{cases} +100, & \text{if } \omega \in A \\ -x, & \text{else} \end{cases}$. Fair game: $E[X] = 0$.

$$\Rightarrow \quad E[X] =: P(X=100)\cdot 100 - P(X\neq 100)\cdot x$$
$$= \frac{4}{6}\cdot 100 - \frac{2}{6}\cdot x$$
$$= \frac{200-x}{3} = 0$$

$\Rightarrow \quad x = 200.$ $\square$

[A casino would want $E[X] > 0$, to make profits!]

Eg. 2: Toss a coin n times. What's $\underbrace{\#H \text{ expected}}_{=:X}$?

$$\cdot\ E[X] = \sum_{i=0}^{n} P(X=i)\cdot i$$
$$= \sum_{i} \binom{n}{i}\cdot\frac{1}{2^i}\cdot\frac{1}{2^{n-i}}\cdot i = \frac{n}{2^n}\cdot\sum_{i=0}^{n}\binom{n-1}{i-1}$$
$$= \underline{n/2}. \quad \square$$

- As expected, expectation could be confusing when Ω is infinite:

- Eg. 3: Keep tossing a coin till H appears. The payoff doubles with every toss. What's your expected payoff?

Analyse:
- $X :=$ payoff.
- $E[X] = \sum_{k \geq 1} 2^k \cdot P(H \text{ at } k\text{-th toss})$

$$= \sum_{k \geq 1} 2^k \cdot \left(\frac{1}{2^k}\right) = \sum_{k=1}^{\infty} 1 \longrightarrow \infty$$ diverges!