MTH101A: Mathematics-I

Problem Set 0: Warm-up

(To be discussed in the week starting on 29 July, 2019)

The problems marked with an asterisk(*) will not be asked during any quiz or exam. The problems marked with a plus sign(+) are extra questions and will be discussed in the tutorial only if time permits.

1. (+) Familiarize yourself with the axioms for the arithmetic of real numbers and the terms used to describe those properties. You can read them from

   https://sites.math.washington.edu/ hart/m524/realprop.pdf

   The completeness axiom will be discussed in the class.

2. Show that 0 is the greatest lower bound of the set $\{\frac{1}{n} \mid n \in \mathbb{N}\}$.

3. Prove Bernoulli's inequality:

$$(1+x)^n \geq 1 + nx, \qquad \text{for } n \in \mathbb{N}, x \geq -1.$$

4. Probably you understand the meaning of the following expression:

$$1 + \frac{1}{2} + \frac{1}{4} + \ldots = 2.$$

   Can you guess the name of the term used to describe the infinite sum on the right hand side? Write the general formula in the variable x corresponding to the above identity. For what values of x does the identity work? Why does it fail for the other values?

5. (*) Read about the "Achilles and the Tortoise paradox" or watch a YouTube video explaining it. Understand the mathematical formulation of the problem so that you appreciate why it is not a paradox in the first place.

6. Let $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$ for $n \in \mathbb{N}$. Show that $|x_{2n} - x_n| \geq \frac{1}{2}$.

7. (*)(+) Continued fractions: Consider a real number $r_0 \in \mathbb{R}$. Suppose $a_0 := \lfloor r_0 \rfloor$ is the integer part of r. Recall that $r_1 := r_0 - a_0 \in [0, 1)$. If $r_1 \neq 0$, continue as $a_i := \lfloor \frac{1}{r_i} \rfloor$ and $r_{i+1} := \frac{1}{r_i} - a_i$ for $i \geq 1$ until $r_{i+1} = 0$. The sequence $[a_0; a_1, a_2, \ldots]$ is known as the 'continued fraction' of r_0. In the other direction, given the continued fraction $[a_0; a_1, a_2, \ldots]$, we can recover r as the expression $a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\cdots}}}$.

   Calculate the continued fraction of $\frac{49}{13}$.

   Convince yourself that this sequence is finite for r rational. Can you prove this?

   What happens when r_0 is irrational?