MTH101A: Mathematics-I

Problem Set 3: Limits and Continuity

(To be discussed in the week starting on 19 August 2019)

The problems marked with an asterisk(*) will not be asked during any quiz or exam. The problems marked with a plus sign(+) are extra questions and will be discussed in the tutorial only if time permits.

1. For a function $f : \mathbb{R} \to \mathbb{R}$, show that if $\lim_{x\to 0} f(x)$ exists, then it is unique.

2. (+) Show for a function $f : \mathbb{R} \to \mathbb{R}$ and $L \in \mathbb{R}$ that $\lim_{x\to x_0} f(x) = L$ iff for any sequence (x_n) of real numbers which converges to x_0, we have $f(x_n) \to L$.

3. Are the following functions continuous at given points? Give appropriate reasons.
   (a) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) := \sqrt{x^2 + 5}$ at $x_0 = \pi$.
   (b) (+) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) := |x|$ at $x = 0$.
   (c) (+) $f : [0, 1] \to \mathbb{R}$ given by $f(x) := 5x^3$ at $x_0 = 0.26$.
   (d) The floor function $\lfloor - \rfloor : \mathbb{R} \to \mathbb{R}$ at $x_0 = 5$.

4. A function $f : [a, b] \to \mathbb{R}$ is said to be Lipschitz continuous if there exists $K \in \mathbb{R}$ such that $|f(x) - f(y)| \le K|x - y|$ for all $x, y \in [a, b]$. Show that a Lipschitz continuous function is continuous at each point of the domain. Which of the functions in the list above are Lipschitz continuous?

5. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be additive if $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Show that if an additive function f is continuous at 0, then f is continuous at any x_0.

6. (+) For a function $f : \mathbb{R} \to \mathbb{R}$ and $L \in \mathbb{R}$, we say that $\lim_{x\to\infty} f(x) = L$ if there exists some $N \in \mathbb{N}$ such that $|f(x) - L| < \epsilon$ whenever $x > N$. Show that $\lim_{x\to\infty} e^{-x} = 0$.

7. This problem provides you with examples of functions that are continuous only on a special set. Show each of the following:
   - The Dirichlet function $1_{\mathbb{Q}} : \mathbb{R} \to \mathbb{R}$ defined by $1_{\mathbb{Q}}(x) := \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ otherwise.} \end{cases}$ is not continuous at any real number.
   - The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := \begin{cases} 0 \text{ if } x \in \mathbb{Q} \\ x \text{ otherwise.} \end{cases}$ is continuous only at $x_0 = 0$.
   - (*) The raindrop function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := \begin{cases} \frac{1}{q} \text{ if } x = \frac{p}{q} \in \mathbb{Q},\ gcd(p, q) = 1,\ q > 0 \\ 0 \text{ otherwise.} \end{cases}$ is continuous only at $x_0 \notin \mathbb{Q}$.

8. (*) Let $\sum_{n\ge 1} a_n x^n$ denote a formal power series with radius of convergence $R > 0$. Define a function $f : (-R, R) \to \mathbb{R}$ by $f(x) := \sum_{n\ge 1} a_n x^n$. Complete the proof that f is continuous everywhere in the domain using following steps:

   Let $\epsilon > 0$ be given.
   - First recall that any polynomial function $p : \mathbb{R} \to \mathbb{R}$ is a continuous function, since it can be obtained using sums and products of the constant function and the identity function.
   - Let $s_n(x)$ denote the sequence of partial sums of the power series. Show that $s_n(x)$ is a continuous function in the variable x.

- Using $s_n(x) \to f(x)$ for each $x \in (-R, R)$, conclude that the tail piece of the power series, $f(x) - s_n(x)$, can be controlled by $\frac{\epsilon}{3}$ for each value of x.
- Conclude the proof.