

Lectures 11 - 13 : Infinite Series, Convergence tests, Leibniz's theorem

Series : Let (a_n) be a sequence of real numbers. Then an expression of the form $a_1 + a_2 + a_3 + \dots$ denoted by $\sum_{n=1}^{\infty} a_n$, is called a series.

Examples : 1. $1 + \frac{1}{2} + \frac{1}{3} + \dots$ or $\sum_{n=1}^{\infty} \frac{1}{n}$ 2. $1 + \frac{1}{4} + \frac{1}{9} + \dots$ or $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Partial sums : $S_n = a_1 + a_2 + a_3 + \dots + a_n$ is called the n th partial sum of the series $\sum_{n=1}^{\infty} a_n$,

Convergence or Divergence of $\sum_{n=1}^{\infty} a_n$

If $S_n \rightarrow S$ for some S then we say that the series $\sum_{n=1}^{\infty} a_n$ converges to S . If (S_n) does not converge then we say that the series $\sum_{n=1}^{\infty} a_n$ diverges.

Examples :

1. $\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$ diverges because $S_n = \log(n+1)$.
2. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges because $S_n = 1 - \frac{1}{n+1} \rightarrow 1$.
3. If $0 < x < 1$, then the geometric series $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$ because $S_n = \frac{1-x^{n+1}}{1-x}$.

Necessary condition for convergence

Theorem 1 : If $\sum_{n=1}^{\infty} a_n$ converges then $a_n \rightarrow 0$.

Proof : $S_{n+1} - S_n = a_{n+1} \rightarrow S - S = 0$. □

The condition given in the above result is necessary but not sufficient i.e., it is possible that $a_n \rightarrow 0$ and $\sum_{n=1}^{\infty} a_n$ diverges.

Examples :

1. If $|x| \geq 1$, then $\sum_{n=1}^{\infty} x^n$ diverges because $a_n \not\rightarrow 0$.
2. $\sum_{n=1}^{\infty} \sin n$ diverges because $a_n \not\rightarrow 0$.
3. $\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$ diverges, however, $\log\left(\frac{n+1}{n}\right) \rightarrow 0$.

Necessary and sufficient condition for convergence

Theorem 2: Suppose $a_n \geq 0 \forall n$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if (S_n) is bounded above.

Proof : Note that under the hypothesis, (S_n) is an increasing sequence. □

Example : The Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges because

$$S_{2^k} \geq 1 + \frac{1}{2} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} + \dots + 2^{k-1} \cdot \frac{1}{2^k} = 1 + \frac{k}{2}$$

for all k .

Theorem 3: If $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

Proof : Since $\sum_{n=1}^{\infty} |a_n|$ converges the sequence of partial sums of $\sum_{n=1}^{\infty} |a_n|$ satisfies the Cauchy criterion. Therefore, the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$ satisfies the Cauchy criterion. □

Remark : Note that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=p}^{\infty} a_n$ converges for any $p \geq 1$.

Tests for Convergence

Let us determine the convergence or the divergence of a series by comparing it to one whose behavior is already known.

Theorem 4 : (Comparison test) Suppose $0 \leq a_n \leq b_n$ for $n \geq k$ for some k . Then

- (1) The convergence of $\sum_{n=1}^{\infty} b_n$ implies the convergence of $\sum_{n=1}^{\infty} a_n$.
- (2) The divergence of $\sum_{n=1}^{\infty} a_n$ implies the divergence of $\sum_{n=1}^{\infty} b_n$.

Proof : (1) Note that the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$ is bounded. Apply Theorem 2.

(2) This statement is the contrapositive of (1). \square

Examples:

1. $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ converges because $\frac{1}{(n+1)(n+1)} \leq \frac{1}{n(n+1)}$. This implies that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.
2. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges because $\frac{1}{n} \leq \frac{1}{\sqrt{n}}$.
3. $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges because $n^2 < n!$ for $n \geq 4$.

Problem 1 : Let $a_n \geq 0$. Then show that both the series $\sum_{n \geq 1} a_n$ and $\sum_{n \geq 1} \frac{a_n}{1+a_n}$ converge or diverge together.

Solution : Suppose $\sum_{n \geq 1} a_n$ converges. Since $0 \leq \frac{a_n}{1+a_n} \leq a_n$ by comparison test $\sum_{n \geq 1} \frac{a_n}{1+a_n}$ converges.

Suppose $\sum_{n \geq 1} \frac{a_n}{1+a_n}$ converges. By the Theorem 1, $\frac{a_n}{1+a_n} \rightarrow 0$. Hence $a_n \rightarrow 0$ and therefore $1 \leq 1 + a_n < 2$ eventually. Hence $0 \leq \frac{1}{2}a_n \leq \frac{a_n}{1+a_n}$. Apply the comparison test.

Theorem 5 : (Limit Comparison Test) Suppose $a_n, b_n \geq 0$ eventually. Suppose $\frac{a_n}{b_n} \rightarrow L$.

1. If $L \in \mathbb{R}, L > 0$, then both $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} a_n$ converge or diverge together.
2. If $L \in \mathbb{R}, L = 0$, and $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.
3. If $L = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof: 1. Since $L > 0$, choose $\epsilon > 0$, such that $L - \epsilon > 0$. There exists n_0 such that $0 \leq L - \epsilon < \frac{a_n}{b_n} < L + \epsilon$. Use the comparison test.

2. For each $\epsilon > 0$, there exists n_0 such that $0 < \frac{a_n}{b_n} < \epsilon, \forall n > n_0$. Use the comparison test.

3. Given $\alpha > 0$, there exists n_0 such that $\frac{a_n}{b_n} > \alpha \forall n > n_0$. Use the comparison test. \square

Examples :

1. $\sum_{n=1}^{\infty} (1 - n \sin \frac{1}{n})$ converges. Take $b_n = \frac{1}{n^2}$ in the previous result.
2. $\sum_{n=1}^{\infty} \frac{1}{n} \log(1 + \frac{1}{n})$ converges. Take $b_n = \frac{1}{n^2}$ in the previous result.

Theorem 6 (Cauchy Test or Cauchy condensation test) If $a_n \geq 0$ and $a_{n+1} \leq a_n \forall n$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges.

Proof : Let $S_n = a_1 + a_2 + \dots + a_n$ and $T_k = a_1 + 2a_2 + \dots + 2^k a_{2^k}$.

Suppose (T_k) converges. For a fixed n , choose k such that $2^k \geq n$. Then

$$\begin{aligned} S_n &= a_1 + a_2 + \dots + a_n \\ &\leq a_1 + (a_2 + a_3) + \dots + (a_{2^k} + \dots + a_{2^{k+1}-1}) \\ &\leq a_1 + 2a_2 + \dots + 2^k a_{2^k} \\ &= T_k. \end{aligned}$$

This shows that (S_n) is bounded above; hence (S_n) converges.

Suppose (S_n) converges. For a fixed k , choose n such that $n \geq 2^k$. Then

$$\begin{aligned} S_n &= a_1 + a_2 + \dots + a_n \\ &\geq a_1 + a_2 + (a_3 + a_4) + \dots + (a_{2^{k-1}+1} + \dots + a_{2^k}) \\ &\geq \frac{1}{2}a_1 + a_2 + 2a_4 + \dots + 2^{k-1}a_{2^k} \\ &= \frac{1}{2}T_k. \end{aligned}$$

This shows that (T_k) is bounded above; hence (T_k) converges. □

Examples:

1. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.



2. $\sum_{n=1}^{\infty} \frac{1}{n(\log n)^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Problem 2 : Let $a_n \geq 0, a_{n+1} \leq a_n \forall n$ and suppose $\sum a_n$ converges. Show that $na_n \rightarrow 0$ as $n \rightarrow \infty$.

Solution : By Cauchy condensation test $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges. Therefore $2^k a_{2^k} \rightarrow 0$ and hence $2^{k+1} a_{2^k} \rightarrow 0$ as $k \rightarrow \infty$. Let $2^k \leq n \leq 2^{k+1}$. Then $na_n \leq na_{2^k} \leq 2^{k+1} a_{2^k} \rightarrow 0$. This implies that $na_n \rightarrow 0$ as $n \rightarrow \infty$.

Theorem 7 (Ratio test) Consider the series $\sum_{n=1}^{\infty} a_n, a_n \neq 0 \forall n$.

1. If $|\frac{a_{n+1}}{a_n}| \leq q$ eventually for some $0 < q < 1$, then $\sum_{n=1}^{\infty} |a_n|$ converges.
2. If $|\frac{a_{n+1}}{a_n}| \geq 1$ eventually then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof: 1. Note that for some $N, |a_{n+1}| \leq q |a_n| \forall n \geq N$. Therefore, $|a_{N+p}| \leq q^p |a_N| \forall p > 0$. Apply the comparison test.

2. In this case $|a_n| \not\rightarrow 0$.

Corollary 1: Suppose $a_n \neq 0 \forall n$, and $|\frac{a_{n+1}}{a_n}| \rightarrow L$ for some L .

1. If $L < 1$ then $\sum_{n=1}^{\infty} |a_n|$ converges.
2. If $L > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $L = 1$ we cannot make any conclusion.

Proof :

1. Note that $|\frac{a_{n+1}}{a_n}| < L + \frac{(1-L)}{2}$ eventually. Apply the previous theorem.

2. Note that $|\frac{a_{n+1}}{a_n}| > L - \frac{(L-1)}{2}$ eventually. Apply the previous theorem.

Examples :

1. $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges because $\frac{a_{n+1}}{a_n} \rightarrow 0$.
2. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverges because $\frac{a_{n+1}}{a_n} = (1 + \frac{1}{n})^n \rightarrow e > 1$.
3. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, however, in both these cases $\frac{a_{n+1}}{a_n} \rightarrow 1$.

Theorem 8 : (Root Test) If $0 \leq a_n \leq x^n$ or $0 \leq a_n^{1/n} \leq x$ eventually for some $0 < x < 1$ then $\sum_{n=1}^{\infty} |a_n|$ converges.

Proof : Immediate from the comparison test. □

Corollary 2: Suppose $|a_n|^{1/n} \rightarrow L$ for some L . Then

1. If $L < 1$ then $\sum_{n=1}^{\infty} |a_n|$ converges.
2. If $L > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $L = 1$ we cannot make any conclusion.

Examples :

1. $\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$ converges because $a_n^{1/n} = \frac{1}{\log n} \rightarrow 0$.
2. $\sum_{n=1}^{\infty} (\frac{n}{n+1})^{n^2}$ converges because $a_n^{1/n} = \frac{1}{(1+\frac{1}{n})^n} \rightarrow \frac{1}{e} < 1$.
3. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, however, in both these cases $a_n^{1/n} \rightarrow 1$.

Theorem 9 : (Leibniz test) If (a_n) is decreasing and $a_n \rightarrow 0$, then $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Proof : Note that (S_{2n}) is increasing and bounded above by S_1 . Similarly, (S_{2n+1}) is decreasing and bounded below by S_2 . Therefore both converge. Since $S_{2n+1} - S_{2n} = a_{2n+1} \rightarrow 0$, both (S_{2n+1}) and (S_{2n}) converge to the same limit and therefore (S_n) converges. □

Examples : $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$, $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ and $\sum_{n=2}^{\infty} (-1)^n \frac{1}{\log n}$ converge.

Problem 3: Let $\{a_n\}$ be a decreasing sequence, $a_n \geq 0$ and $\lim_{n \rightarrow \infty} a_n = 0$. For each $n \in \mathbb{N}$, let $b_n = \frac{a_1 + a_2 + \dots + a_n}{n}$. Show that $\sum_{n \geq 1} (-1)^n b_n$ converges.

Solution : Note that $b_{n+1} - b_n = \frac{1}{n+1}(a_1 + a_2 + \dots + a_{n+1}) - \frac{1}{n}(a_1 + \dots + a_n) = \frac{a_{n+1}}{n+1} - \frac{(a_1 + \dots + a_n)}{n(n+1)}$. Since (a_n) is decreasing, $a_1 + \dots + a_n \geq na_n$. Therefore, $b_{n+1} - b_n \leq \frac{a_{n+1} - a_n}{n+1} \leq 0$. Hence (b_n) is decreasing.

We now need to show that $b_n \rightarrow 0$. For a given $\epsilon > 0$, since $a_n \rightarrow 0$, there exists n_0 such that $a_n < \frac{\epsilon}{2}$ for all $n \geq n_0$.

Therefore, $|\frac{a_1 + \dots + a_n}{n}| = |\frac{a_1 + \dots + a_{n_0}}{n} + \frac{a_{n_0+1} + \dots + a_n}{n}| \leq |\frac{a_1 + \dots + a_{n_0}}{n}| + \frac{n - n_0}{n} \frac{\epsilon}{2}$. Choose $N \geq n_0$ large enough so that $\frac{a_1 + \dots + a_{n_0}}{N} < \frac{\epsilon}{2}$. Then, for all $n \geq N$, $\frac{a_1 + \dots + a_n}{n} < \epsilon$. Hence, $b_n \rightarrow 0$. Use the Leibniz test for convergence.