MTH 102A - Linear Algebra - 2015-16-II Semester

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P. **Field**

- A field F is a set from which we choose our coefficients and scalars.

- Expected properties are

1) $a + b$ and $a \times b$ should be defined in it.

2) $a + b$ and $a \times b$ must be inside the field.

3) Both operations are commutative: $a + b = b + a$; $a \times b = b \times a$.

4) There should be identity elements for both operations. Identity element for $+$ is called 0 and that for $\times$ is called 1.

5) Inverse for "a w.r.t. $+$: $\forall a \in F$, $\exists b \in F$ s.t. $a + b = 0$"

"$a \neq 0$ w.r.t. $\times$: $\forall a \in F \setminus \{0\}$, $\exists b \in F$ s.t. $a \times b = 1$."

6) Value of $(a + b) + c$ and $a + (b + c)$ are 'equal'.

Value of $(a \times b) \times c$ and $a \times (b \times c)$ are 'equal'.

7) $\times$ distributes itself over $+$: $a \times (b + c) = (a \times b) + (a \times c)$.

Examples: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$.

Also $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ with

$a + b := (a + b) \pmod 5$ and $a \times b := (a \times b) \pmod 5$. Here, $3 + 4 = 2$, $4 \times 2 = 3$ and $4 \times 4 = 1$.

$\mathbb{Q}[\sqrt{2}] := \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$.

$(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2}$

$(a + b\sqrt{2}) \cdot (c + d\sqrt{2}) = (ac + 2bd) + (ad + bc)\sqrt{2}$

Also, whenever $(a + b\sqrt{2}) \neq 0$, we have $(a + b\sqrt{2})^{-1} = \dfrac{a}{a^2 - 2b^2} - \dfrac{b}{a^2 - 2b^2}\sqrt{2}$ as $\sqrt{2}$ is irrational.

P. **Linear Equations**

- A linear equation (over some field $\mathbb{F}$) is an expression of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b_1$$

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• Example: $2x_1 + 3x_2 + 7x_3 = 6$ is a linear equation. Over what?
Over any field. What does 2 mean in $\mathbb{F}$? $1+1$!
Anyway we never mention that!! Come on, isn't it obvious?

• A system of linear equations is a collection of m linear equations in the 'same' n variables. It has the form:

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

• $\left.\begin{aligned} x+y+z&=1 \\ 2y+3z&=7 \end{aligned}\right\}$ is a system of 2 linear equations in 3 variables.

• Matrix form of a system (of linear equations):

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \vdots & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}.$$

• The product AB corresponds to operating on the *columns of the matrix* A, using entries of B. Thus,

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \vdots & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}.$$

is same as

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} x_2 + \cdots + \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} x_n = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

(Similar to 'does there exist integers x, y such that $13x + 5y = 1$???")

P. The old story: A new way

Q Solve $\begin{aligned} 2x+y&=1 \\ x+2y&=-1 \end{aligned}$.

A $\begin{aligned} 2x+y&=1 \\ x+2y&=-1 \end{aligned} \xrightarrow{2(2)} \begin{aligned} 2x+y&=1 \\ 2x+4y&=-2 \end{aligned} \xrightarrow{(2)-(1)} \begin{aligned} 2x+y&=1 \\ 3y&=-3 \end{aligned}$. So...

• Write the Augmented coefficient matrix of the system with a matrix: $\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 1 & 2 & -1 \end{array}\right]$

- Above solution procedure is nothing but

$$\left[\begin{array}{cc|c}2&1&1\\1&2&-1\end{array}\right]\xrightarrow{2(2)}\left[\begin{array}{cc|c}2&1&1\\2&4&-2\end{array}\right]\xrightarrow{(2)-(1)}\left[\begin{array}{cc|c}2&1&1\\0&3&-3\end{array}\right].\ \text{Continue?}$$

$$\xrightarrow{\frac{1}{3}(2)}\left[\begin{array}{cc|c}2&1&1\\0&1&-1\end{array}\right]\xrightarrow{(1)-(2)}\left[\begin{array}{cc|c}2&0&2\\0&1&-1\end{array}\right]\xrightarrow{\frac{1}{2}(1)}\left[\begin{array}{cc|c}1&0&1\\0&1&-1\end{array}\right]$$

- System: $\left.\begin{array}{r}\boxed{x+y+z=1}\\2y+3z=7\end{array}\right\}$. That is, $\begin{bmatrix}\boxed{1\ \ 1\ \ 1}\\0\ \ 2\ \ 3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}\boxed{1}\\7\end{bmatrix}$.

- **Augmented Coefficient Matrix:** $\left[\begin{array}{ccc|c}1&1&1&1\\0&2&3&7\end{array}\right]$.

Apply elimination.

$$\left[\begin{array}{ccc|c}1&1&1&1\\0&2&3&7\end{array}\right]\xrightarrow{\frac{1}{2}(2)}\left[\begin{array}{ccc|c}1&1&1&1\\0&1&\frac{3}{2}&\frac{7}{2}\end{array}\right]\xrightarrow{(1)-(2)}\left[\begin{array}{ccc|c}\boxed{1}&0&-\frac{1}{2}&-\frac{5}{2}\\0&\boxed{1}&\frac{3}{2}&\frac{7}{2}\end{array}\right]$$

- Highlighted positions are called pivots/leading terms. Corresponding variables are basic variables. Others are free variables.

  - Here z is free. For solution: put $z=t$; then $\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-\frac{5}{2}+\frac{1}{2}t\\\frac{7}{2}-\frac{3}{2}t\\t\end{bmatrix}$.

  - For example, $t=0$ gives $\begin{bmatrix}-\frac{5}{2}\\\frac{7}{2}\\0\end{bmatrix}$ and $t=1$ gives $\begin{bmatrix}-2\\2\\1\end{bmatrix}$.

P. **We don't want to solve it. Let the computer do it!**

  - In that case we have to provide an algorithm. What is an algorithm?

A 'step by step' instruction to carry out the task.

  - Actually, we do not have to!! Gauss-Jordan have already done it!!

- To describe that we need 3 elementary row operations.
- Given the Augmented Coefficient Matrix (ACM) of a system: $\left[\begin{array}{ccc|c}a_{11}&\cdots&a_{1n}&b_1\\&\vdots&&\vdots\\a_{m1}&\cdots&a_{mn}&b_m\end{array}\right]$.

  1) $E_i(\alpha)$ Multiply ith row by $\alpha\neq 0$.

  For example, $\left[\begin{array}{ccc|c}0&1&1&1\\0&2&3&1\\1&0&2&-3\end{array}\right]\xrightarrow{E_2(\frac{1}{2})}\left[\begin{array}{ccc|c}0&1&1&1\\0&1&\frac{3}{2}&\frac{1}{2}\\1&0&2&-3\end{array}\right]$.

Result is the same as: $\begin{bmatrix}1 & 0 & 0\\ 0 & \frac{1}{2} & 0\\ 0 & 0 & 1\end{bmatrix}\left[\begin{array}{ccc|c}0 & 1 & 1 & 1\\ 0 & 2 & 3 & 1\\ 1 & 0 & 2 & -3\end{array}\right]$

- Elementary Matrix $E_2(\frac{1}{2})$

P. **Elementary row operations**

E_{ij} Interchange rows i and j.

$$\text{For example, } \left[\begin{array}{ccc|c}0 & 1 & 1 & 1\\ 0 & 2 & 3 & 1\\ 1 & 0 & 2 & -3\\ 1 & 2 & 3 & 0\\ 5 & 2 & -1 & 2\end{array}\right] \xrightarrow{E_{14}} \left[\begin{array}{ccc|c}1 & 2 & 3 & 0\\ 0 & 2 & 3 & 1\\ 1 & 0 & 2 & -3\\ 0 & 1 & 1 & 1\\ 5 & 2 & -1 & 2\end{array}\right].$$

$$\text{Result is the same as: } \begin{bmatrix}0 & & & 1 & \\ & 1 & & & \\ & & 1 & & \\ 1 & & & 0 & \\ & & & & 1\end{bmatrix}\left[\begin{array}{ccc|c}0 & 1 & 1 & 1\\ 0 & 2 & 3 & 1\\ 1 & 0 & 2 & -3\\ 1 & 2 & 3 & 0\\ 5 & 2 & -1 & 2\end{array}\right]$$

- Elementary Matrix E_{14}

$E_{ij}(\alpha)$ Replace ith row R_i by $R_i + \alpha R_j$

$$\text{For example, } \left[\begin{array}{ccc|c}0 & 1 & 1 & 1\\ 0 & 2 & 3 & 1\\ 1 & 0 & 2 & -3\\ 1 & 2 & 3 & 0\\ 5 & 2 & -1 & 2\end{array}\right] \xrightarrow{E_{35}(-\frac{1}{5})} \left[\begin{array}{ccc|c}1 & 2 & 3 & 0\\ 0 & 2 & 3 & 1\\ 0 & -\frac{2}{5} & \frac{11}{5} & -\frac{17}{5}\\ 0 & 1 & 1 & 1\\ 5 & 2 & -1 & 2\end{array}\right].$$

$$\text{Result is the same as: } \begin{bmatrix}1 & & & & \\ & 1 & & & \\ & & 1 & & -\frac{1}{5}\\ & & & 1 & \\ & & & & 1\end{bmatrix}\left[\begin{array}{ccc|c}0 & 1 & 1 & 1\\ 0 & 2 & 3 & 1\\ 1 & 0 & 2 & -3\\ 1 & 2 & 3 & 0\\ 5 & 2 & -1 & 2\end{array}\right]$$

- Elementary Matrix $E_{35}(-\frac{1}{5})$

P. **Echelon-form(EF)**

- Let A be a matrix. A pivot/leading term is the first (from left) nonzero element of a nonzero row in A. We use $\boxed{a_{ij}}$ to denote it.

- A matrix A is in echelon form (EF) (ladder like) if

1) Pivot of the $i+1$th row comes to the right of the ith.

2) Entries below the pivot in a 'pivotal column' are 0.

3) The zero rows are at the bottom.

- In EF: $\begin{bmatrix} \boxed{3} & 1 & 3 & 0 \\ 0 & \boxed{2} & 0 & 1 \\ 0 & 0 & 0 & \boxed{3} \end{bmatrix}$, $\begin{bmatrix} \boxed{1} & 2 & 0 & 5 \\ 0 & \boxed{2} & 0 & 6 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}$.
- Not in EF: $\begin{bmatrix} 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \end{bmatrix}$ (rule 1,2 fail); $\begin{bmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (rule 3 fails).

P. **Row-reduced-echelon-form(RREF)**

- A matrix A is in RREF if

  1) It is in EF.

2) Pivot of each nonzero row is 1.

3) Other entries in a 'pivotal column' are 0.

- In RREF: $\begin{bmatrix} 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} \boxed{1} & 0 & 0 & 5 \\ 0 & \boxed{1} & 0 & 6 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}$.
- Not in RREF: $\begin{bmatrix} 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

- A is row equivalent to B if B is the result of k elementary row operations on A. That is, if there exists some elementary matrices $E_1, \ldots, E_k$ s.t.

$$E_k E_{k-1} \cdots E_1 A = B.$$

P. **Gauss-Jordan elimination(GJE)**

GJE An algorithm. Uses row operations only. Input: A.

Output: a matrix B in RREF s.t. B is row equivalent to A.

0) Put 'region' $= A$.

1) If all entries in the region are 0, STOP.

Else, in the region, find the leftmost nonzero column and find its topmost nonzero entry.

Suppose it is a_{ij}. Box it. This is a pivot.

Take it to the top row of the region. Make it 1.

Make other entries of the whole matrix in it's column 0.

2) Put region = the submatrix below and to the right of the current pivot. Go to step 1).

- The process will stop, as we can get at most $\min\{m, n\}$ pivots.
- Apply GJE: $\begin{bmatrix} 0 & 2 & 3 & 7 \\ \boxed{1} & 1 & 1 & 1 \\ 1 & 3 & 4 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_{12}} \begin{bmatrix} \boxed{1} & 1 & 1 & 1 \\ 0 & 2 & 3 & 7 \\ 1 & 3 & 4 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_{31}(-1)} \begin{bmatrix} \boxed{1} & 1 & 1 & 1 \\ 0 & \boxed{2} & 3 & 7 \\ 0 & 2 & 3 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_2(\frac{1}{2})}$

$\begin{bmatrix} \boxed{1} & 1 & 1 & 1 \\ 0 & \boxed{1} & \frac{3}{2} & \frac{7}{2} \\ 0 & 2 & 3 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_{12}(-1), E_{32}(-2)} \begin{bmatrix} \boxed{1} & 0 & -\frac{1}{2} & -\frac{5}{2} \\ 0 & \boxed{1} & \frac{3}{2} & \frac{7}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix} \xrightarrow{E_{34}} \begin{bmatrix} \boxed{1} & 0 & -\frac{1}{2} & -\frac{5}{2} \\ 0 & \boxed{1} & \frac{3}{2} & \frac{7}{2} \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\xrightarrow{E_{13}(?), E_{23}(?)} \begin{bmatrix} \boxed{1} & 0 & -\frac{1}{2} & 0 \\ 0 & \boxed{1} & \frac{3}{2} & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Th Each matrix $A_{m\times n}$ is row equivalent to some matrix in RREF. ‼

P. **Gauss elimination**

GE It is the same as GJE except that

1) pivots need not be made 1 and

2) entries above the pivots need not be made 0.

- $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \xrightarrow{E_{21}(-1), E_{31}(-1)} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{bmatrix} \xrightarrow{E_{32}(-3)} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$.
- GJE may be viewed as an extension of GE.

P. **Inverses of elementary matrices**

Sometimes it helps to imagine an elementary matrix as an elementary row operation.

Recall Invertibility: $A_{n\times n}$ is invertible if there is a B s.t. $AB = BA = I$.

What is the inverse of $E_i(\alpha)$? That is, 'if I have multiplied ith row by α, how do i get back the original'? Must be $E_i(\frac{1}{\alpha})$.

What is the inverse of E_{ij}? That is, 'if I have interchanged R_i and R_j, how do i get back the original'? Must be $E_{ji} = E_{ij}$.

What is the inverse of $E_{ij}(\alpha)$? That is, 'if I have added αR_j to R_i, how do i get back the original'? Must be $E_{ij}(-\alpha)$.

Ex Inverse of an elementary matrix is an elementary matrix.

Ex If A is invertible, then it has a unique inverse.

- Recall that for any invertible matrix $A(i, :) \neq \mathbf{0}$.

Th Let $A_{n\times n}$ be invertible. Then, the RREF of A is I.

Po. A RREF of A is nothing but $E_k \cdots E_1 A = EA$ (say), where E_i's are elementary.

As EA is invertible, it has no zero row. How many pivots should it have? As EA is in RREF, it must be I. ∎

P. RREF

Th If A is row equivalent to B then, the systems $Ax = 0$ and $Bx = 0$ have the same solution set. ‼

Q Is $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ row equivalent to $B = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 0 \end{bmatrix}$?

No, $\begin{bmatrix} a \\ b \\ -1 \end{bmatrix}$ is a solution of $Bx = 0$, not of $Ax = 0$.

Th Let A and B be two row equivalent matrices in RREF. Then $A = B$.

P. Rank of a Matrix

Cor Each matrix A is row equivalent to a unique matrix in RREF. ‼

- We use RREF A to denote this matrix.
- The rank of a matrix A is the number of pivots in RREF A. Notation: $\mathsf{rank}\, A$.
- Thus, $\mathsf{rank}\, A_{m \times n} \le m, n$ and $\mathsf{rank}(\mathbf{0}) = 0$.

P. Gauss-Jordan say it

Th Take a system $Ax = b$ and an invertible matrix B. Then, y is a solution of $Ax = b$ if and only if y is a solution of $BAx = Bb$. ‼

Gauss-Jordan idea Take the ACM $[A|b]$ of a system. Keep on applying elementary row operations. <u>Solution space stays same!!!</u>

STOP at the RREF$[A'|b']$.

If $A'(i, :) = 0$ and $b'_i \neq 0$, then conclude that the system has no solution (inconsistent). Note that, here, $\mathsf{rank}(A) < \mathsf{rank}([A|b])$.

Otherwise, a general solution is obtained by assigning the free-variables arbitrary values and by evaluating the values of the basic variables. Note that, here, $\mathsf{rank}(A) = \mathsf{rank}([A|b])$.

P. Very Important Ideas

Th Let A be a matrix of RANK r. Then, there exist(s)

- Invertible P s.t. $PA = \begin{bmatrix} B \\ \mathbf{0} \end{bmatrix}$.

- Invertible P and Q such that

$$P_{m\times m}A_{m\times n}Q_{n\times n} = PAQ == \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Ex[Rank Factorization] Let $\mathsf{rank}\, A_{m\times n} = r$. Then there exist $B_{m\times r}, C_{r\times n}$ s.t. $\mathsf{rank}\, B = \mathsf{rank}\, C = r$ and $A = BC$ as

$$A = P^{-1}\begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} Q^{-1} = [P_1 \quad P_2]\begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = P_1 Q_1.$$

Th Consider the homogeneous linear system $A\mathbf{x} = \mathbf{0}$. Then,

- The zero vector, $\mathbf{0} = (0, \ldots, 0)^t$, is always a solution, called the TRIVIAL solution.

- Suppose $\mathbf{x}_1, \mathbf{x}_2 \neq \mathbf{0}$ are two solutions of $A\mathbf{x} = \mathbf{0}$. Then, $k_1\mathbf{x}_1 + k_2\mathbf{x}_2$ is also a solution of $A\mathbf{x} = \mathbf{0}$ for any $k_1, k_2 \in \mathbb{R}$.

Th Consider the linear system $A\mathbf{x} = \mathbf{b}$, where A is an $m\times n$ matrix and $\mathbf{x}^t = (x_1, \ldots, x_n)$. If $[C \,|\mathbf{d}] = \mathbf{rref}([A\,|\mathbf{b}])$ then,

- $A\mathbf{x} = \mathbf{b}$ is inconsistent (has no solution) if $[C \;\; \mathbf{d}]$ has a row of the form $[\mathbf{0}^t|1]$, where $\mathbf{0}^t = (0, \ldots, 0)$. That is,

$$\mathbf{Rank}(A) < \mathbf{Rank}([A\,|\mathbf{b}]) = \mathbf{Rank}([C\,|\mathbf{d}]).$$

- $A\mathbf{x} = \mathbf{b}$ is consistent (has a solution) if $[C\,|\mathbf{d}]$ has **NO row of the form** $[\mathbf{0}^t|1]$. That is,

$$\mathbf{Rank}(A) = \mathbf{Rank}([A\,|\mathbf{b}]) = \mathbf{Rank}([C\,|\mathbf{d}]).$$

  Furthermore, (recall n = Number of unknowns implies)

  - if $\mathbf{Rank}(A) = n$ then, $A\mathbf{x} = \mathbf{b}$ has A UNIQUE SOLUTION.
  - if $\mathbf{Rank}(A) < n$ then, $A\mathbf{x} = \mathbf{b}$ has INFINITE NUMBER OF SOLUTIONS.

P. **Example**

- System $\begin{aligned} x + y + z &= 1 \\ y + 3z &= 2 \\ -x + 2z &= 2 \end{aligned}$
- ACM: $A = \left[\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ -1 & 0 & 2 & 2 \end{array}\right]$ RREF$(A) = \left[\begin{array}{rrr|r} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$ $\rightarrow$ Matlab command rref(a)

• System is inconsistent (has no solution).

• System $\begin{aligned} x+y+z&=1\\ y+3z&=2\\ -x+2z&=1 \end{aligned}$

• ACM: $A = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1\\ 0 & 1 & 3 & 2\\ -1 & 0 & 2 & 1 \end{array}\right]$ $\text{RREF}(A) = \left[\begin{array}{ccc|c} 1 & 0 & -2 & -1\\ 0 & 1 & 3 & 2\\ 0 & 0 & 0 & 0 \end{array}\right]$

• x, y are basic, z is free. General solution: $\left\{ \begin{bmatrix} -1+2t\\ 2-3t\\ t \end{bmatrix} : t \in \mathbb{R} \right\}$.

• Let $[C|\mathbf{d}] = \text{RREF}([A|\mathbf{b}])$. Assume $C(i,:) \neq \mathbf{0}$ and the system is consistent. Notice that in $C(i,:)$ all entries are zero, except the pivot and entries corresponding to free variables. Thus, if we assign all free variables 0, then we get $x_i = d_i$.

• What happens if we put $z = t = 0$, in the above example?

• If $\mathsf{rank}(A_{m\times n}) < n$, then $A\mathbf{x} = \mathbf{0}$ has a nonzero solution. ‼

P. Invertibility and Gauss-Jordan

Let A be a square matrix of order n. Then the following statements are equivalent.

• A is invertible.

• The homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

• The (rref) row-reduced echelon form of A is I_n.

• A is a product of elementary matrices.

• If A is invertible then $rref(A) = I$. Thus, $E_k E_{k-1} \cdots E_2 E_1 A = I$ for some elementary matrices. Hence, $A^{-1} = E_k E_{k-1} \cdots E_2 E_1$.

Implication. Given a matrix $A_{n\times n}$, apply GJE to $[A|I_n]$. Then, we get elementary matrices $E_1, E_2, \ldots, E_k$ such that

$E_k E_{k-1} \cdots E_1 \, [A|I] = [E_k E_{k-1} \cdots E_1 A | E_k E_{k-1} \cdots E_1 I] = [I|A^{-1}]$.

P. Inverse - Example

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0\\ 2 & 1 & 0 & 1 \end{array}\right] \xrightarrow{E_{21}(-2)} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0\\ 0 & -3 & -2 & 1 \end{array}\right] \xrightarrow{E_2(-1/3)} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0\\ 0 & 1 & \frac{2}{3} & \frac{-1}{3} \end{array}\right]$$

$$\xrightarrow{E_{12}(-2)} \left[\begin{array}{cc|cc} 1 & 0 & \frac{-1}{3} & \frac{2}{3}\\ 0 & 1 & \frac{2}{3} & \frac{-1}{3} \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0\\ 2 & 4 & 7 & 0 & 1 & 0\\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[E_{31}(-1)]{E_{21}(-2)} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0\\ 0 & 0 & 1 & -2 & 1 & 0\\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}\right]$$

$[2,4,7,0,1,0] - 2[1,2,3,1,0,0] = [0,0,1,-2,1,0]$

$[1,1,1,0,0,1] - [1,2,3,1,0,0] = [0,-1,-2,-1,0,1]$.

$$\xrightarrow{E_{32}} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array}\right] \xrightarrow[E_{12}(-2)]{E_2(-1)} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 0 & 2 \\ 0 & 1 & 2 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[E_{13}(1)]{E_{23}(-2)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 1 & 2 \\ 0 & 1 & 0 & 5 & -2 & -1 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array}\right]$$

Thus, we have solved three linear systems simultaneously, namely

$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $A\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $A\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 1 & 1 & 1 \end{bmatrix}$.

$(x,y,z) = (-3,5,-2)$, $(\alpha,\beta,\gamma) = (1,-2,1)$ and $(u,v,w) = (2,-1,0)$.

P. Determinant

Notation:- Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 2 & 4 & 7 \end{bmatrix}$.

Then, $A(1|2) = \begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 2 & 4 & 7 \end{bmatrix}$,

$A(1|3) = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 2 & 4 & 7 \end{bmatrix}$,

and $A(1,2|1,3) = [4]$.

Determinant: Determinant of a square matrix $A = [a_{ij}]$, denoted $\det(A)$ (or $|A|$) is defined by

$$\det(A) = \begin{cases} a, & \text{if } A = [a] \ (n=1), \\ \sum\limits_{j=1}^{n} (-1)^{1+j} a_{1j} \det\big(A(1|j)\big), & \text{otherwise.} \end{cases}$$

- Let $A = [-2]$. Then $\det(A) = |A| = -2$.
- Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then, $\det(A) = |A| = a\,\big|A(1|1)\big| - b\,\big|A(1|2)\big| = ad - bc$.

  For $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$, $\det(A) = \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} = 1\cdot 5 - 2\cdot 3 = -1$.

Singular, Non-Singular: A matrix A is said to be a SINGULAR if $\det(A) = 0$. It is called NON-SINGULAR if $\det(A) \neq 0$.

Th Let A be an $n \times n$ matrix. If

- $B = E_{ij}A$ then, $\det(B) = -\det(A)$,
- $B = E_i(c), c \neq 0$ then, $\det(B) = c\det(A)$,
- $B = E_{ij}(c), c \neq 0$ then, $\det(B) = \det(A)$,
- all the elements of one row of A are 0 then, $\det(A) = 0$,
- two rows of A are equal then $\det(A) = 0$.
- A is a triangular matrix then $\det(A)$ is product of diagonal entries.

Th As $\det(I_n) = 1$, $\det(E_{ij}) = -1$ $\det(E_i(c)) = c$ whenever $c \neq 0$ $\det(E_{ij}(c)) = 1$, whenever $c \neq 0$

• Let $A_{n\times n} = [a_{ij}]$. Then, for any $k, 1 \leq k \leq n$

$$\det(A) = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} \det\big(A(k|j)\big).$$

• Let $A_{n\times n}$ matrix. Then, $|\det(A)|$ equals the volume of the n-dimensional parallelepiped formed by the rows of A.

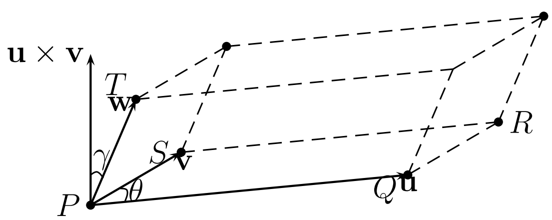

Figure 3: Parallelepiped with vertices P, Q, R and S as base

• Let $\mathbf{u}^t = (u_1, u_2, u_3), \mathbf{v}^t = (v_1, v_2, v_3)$ and $\mathbf{w}^t = (w_1, w_2, w_3)$.
Then, $\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$ and

$$\text{volume } (P) = \text{Area}(PQRS) \cdot \text{height} = |\mathbf{w} \bullet (\mathbf{u} \times \mathbf{v})| = \pm \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

P. **Cofactor**

$(i, j)^{\text{th}}$ minor of A Denoted A_{ij} equals $\det\left(A(i|j)\right)$.
$(i, j)^{\text{th}}$ cofactor of A: Denoted C_{ij} equals $(-1)^{i+j}A_{ij}$.
Adjoint of a Matrix: Denoted $Adj(A) = [C_{ji}]$

- Example $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 2 & 2 & 2 \end{bmatrix}$. Then, $Adj(A) = \begin{bmatrix} 4 & 2 & -7 \\ -2 & -4 & 5 \\ -2 & 2 & -1 \end{bmatrix}$ as

$C_{11} = (-1)^{1+1}A_{11} = 4, C_{21} = (-1)^{2+1}A_{21} = 2, \ldots, C_{33} = (-1)^{3+3}A_{33} = -1.$

$$\begin{aligned} A \cdot Adj(A) &= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 4 & 2 & -7 \\ -2 & -4 & 5 \\ -2 & 2 & -1 \end{bmatrix} \\ &= -6 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = Adj(A) \cdot A. \end{aligned}$$

Note that $\det(A) = -6$. So, Important Observation:

$$A \cdot Adj(A) = Adj(A) \cdot A = \det(A)I.$$

Is it always TRUE?

P. **Determinant-cofactor**

Th Let A be an $n \times n$ matrix. Then,

- for $1 \le i \le n$, $\sum\limits_{j=1}^{n} a_{ij}\, C_{ij} = \sum\limits_{j=1}^{n} a_{ij}(-1)^{i+j}\, A_{ij} = \det(A)$,
- for $i \neq \ell$, $\sum\limits_{j=1}^{n} a_{ij}\, C_{\ell j} = \sum\limits_{j=1}^{n} a_{ij}(-1)^{\ell+j}\, A_{\ell j} = 0$,
- $A(Adj(A)) = \det(A)I_n$. Thus,

$$\textbf{whenever } \det(A) \neq 0 \textbf{ one has } A^{-1} = \frac{1}{\det(A)}Adj(A).$$

Th A square matrix A is non-singular if and only if A is invertible.

Th Let A and B be square matrices of order n. Then

$$\det(AB) = \det(A)\det(B) = \det(BA).$$

Th Let A be a square matrix. Then $\det(A) = \det(A^t)$.

P. **Cramer's Rule**

Th The following statements are equivalent for a square matrix A:

- A is invertible.
- The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$.
- $\det(A) \neq 0$.

Cramer's Rule Let A be an $n \times n$ matrix. If $\det(A) \neq 0$ then, the unique solution of the linear system $A\mathbf{x} = \mathbf{b}$ is

$$x_j = \frac{\det(A_j)}{\det(A)}, \quad \text{for } j = 1, 2, \ldots, n,$$

where A_j is the matrix obtained from A by replacing the jth column of A by the column vector $\mathbf{b}$.