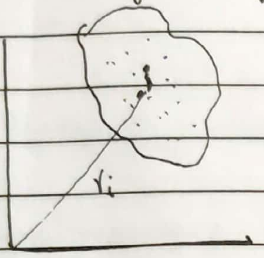
Newtonian Mechanics

Valid on Point particles [Bomb → not valid as it has chemical structures within it]

1^{st} law: Inertial frame exists
(non accelerating)

2^{nd} law: $F = ma$ assuming mass is constant

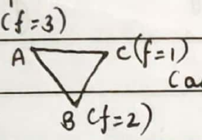
3^{rd} ~~law~~: System of particles: - [a closed system → no external forces]



$$m_i \ddot{\vec{r}}_i = \sum_{j \neq i} \vec{F}_{ij}$$

* Treat particles system by C.O.M

+ Rigid Body: - distance btwn any two particle remain constant



(any ~~three~~ together should have 6 degree of freedom

$3+2+1 = 6$

reduces 10^{23} d.o.f to 6

3^{rd} law: Conservation of linear and angular momentum for a system of particles.

Dimensional Analysis

$$\text{Dim}[x] = L^{\alpha} M^{\beta} T^{\gamma}$$

Limits of Newtonian Mechanics

① $v/c \ll 1$; Non realistic limit

② System:

$\lambda = \frac{h}{p}$, a:- interatomic distance

$\lambda \ll a$: $\lambda \simeq a$ → Quantum Mechanics

③. System at a finite temperature

Thermal De-Broglie Wavelength

$$\frac{p^2}{2m} \simeq K_B T \quad \rightarrow \quad \lambda_T = \frac{h}{\sqrt{2mK_B T}}$$

$\boxed{\lambda_T << a}$ high Temp, low density

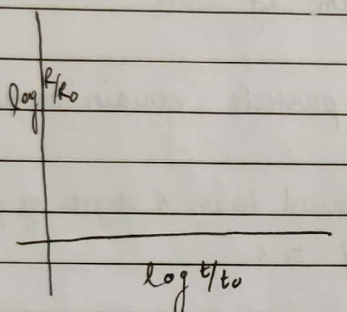
classical mechanics

Kepler's law

$$T = \frac{R^{3/2}}{\sqrt{GM_s}} f\left(\frac{M_p}{M_s}\right)$$

Energy released in bomb

$$\boxed{E = \frac{C' R^5 \rho}{t^2}}$$



$$\log R = 0.4 \log t + 0.2 \log (E/\rho)$$

Buckingham Pi Theorem

π. K = relevant dimensionfull quantities

r - fundamental dimensions

$r = 3$, M, L, T or $r = 4$ when q

$(K-r)$ independent dimensionless combination

call them $\pi_1, \pi_2 \ldots \ldots \pi_{K-r}$

The behaviour of the system

$$\Phi(\pi_1, \ldots \pi_2, \ldots \pi_{K-r}) = 0$$

eg: If we have π_1, π_2 : $\Phi(\pi_1, \pi_2) = 0$

$\Rightarrow \boxed{\pi_1 = g(\pi_2)}$ choice is not unique

Repeating variables ~~known~~ r

Non repeating Variables : K-r

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~~Restriction~~
Repeating variables must include all fundamental dimension

eg-1 $x = ut + \frac{1}{2}at^2$ Find out x as a function of t

given:- x, u, a, t

$K = 4$

$r = 2$ [M is not coming]

$K - r = 2 \rightarrow$ dimensionless combination

Repeating variables:- a and t [can choose any two]
LT^{-2} T

Non repeating variables:- x and u

$aT^{-2}(T)(L)$
$= L^0T^0$

[multiply by one non repeating variables]

$\pi_1 = a^{\alpha} t^{\beta} . x = L^0 T^0$

$L^0(T)$
LT^{-1}

$\pi_2 = a^{\alpha_1} t^{\beta_1} . u = L^0 T^0$

$\pi_1 = \frac{x}{at^2}$, $\pi_2 = \frac{u}{at}$

By the Buckingham Pi theorem

$\frac{x}{at^2} = g\left(\frac{u}{at}\right)$ $[x = ut + \frac{1}{2}at^2]$ $F = \frac{f}{A}$

$(L^0)(M)(L)$

$x = at^2 g(u/at)$

eg-2 Discharge through a pipe $\rightarrow$ with diameter d, density, pressure.

Q, d, p, ρ, η

$Q = \frac{d^2 p^{1/2}}{\rho^{1/2}}$ $\left(\frac{d \rho^{1/2} p^{1/2}}{\eta}\right)$ [whatever asked to calculate $\rightarrow$ choose it as non repeating variable]

L^3T^{-1}

$L^2(M)(L)(T^{-2})$

$K = 5$

$r = 3$

$K - r = 2$

Repeating variables:- Q, d, ρ

$Q \rightarrow L^3T^{-1}$, $d \rightarrow L$, $\rho \rightarrow ML^{-3}$

$F = 6\pi\eta r v$

Non repeating variables: η, P

$\eta \rightarrow ML^{-1}T^{-1}$, $P \rightarrow L^3MT^{-2}$

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$$\pi_1 = p \cdot Q^{\alpha_1} d^{\beta_1} \rho^{\gamma_1} = L^0 T^0 M^0 = \frac{dn}{\rho Q}$$

$$\pi_2 = \eta \cdot Q^{\alpha_2} d^{\beta_2} \cdot \rho^{\gamma_2} = L^0 T^0 M^0 = \frac{d^4 p}{\rho Q^2}$$

$\pi_1 = g(\pi_2)$

$$F\left(\frac{dn}{\rho Q}, \frac{d^4 p}{\rho Q^2}\right) = 0$$

$$\pi_2' = \frac{1}{\sqrt{\pi_2}} = \frac{\rho^{1/2} Q}{d^2 p^{1/2}} \qquad \pi_1' = \pi_1 \pi_2' = \frac{n}{d \rho^{1/2} p^{1/2}}$$

$$\pi_2' = g(\pi_1') \Rightarrow Q = \frac{d^2 p^{1/2}}{\rho^{1/2}} g\left(\frac{n}{d \rho^{1/2} p^{1/2}}\right)$$

Choose Q as a NRV.

$R_z(\phi) \Rightarrow RR^T = I$ check $\det R = \pm 1$

Vectors Why Vectors? $\vec{F} = m\vec{a}$

→ frame independent description

Defined w.r.t an operation:-

Rotation Matrix

axis, angle, sense

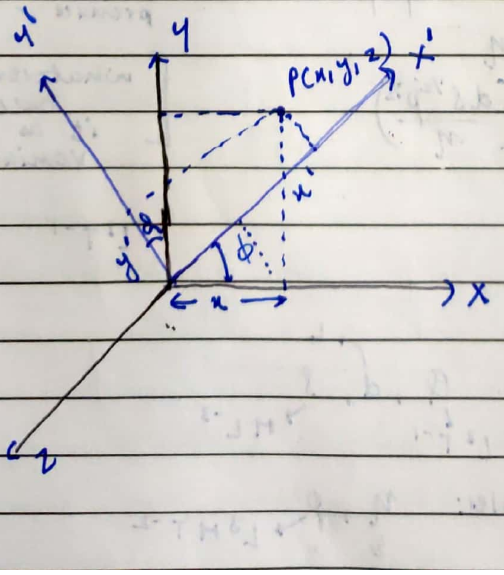
Classical Mechanics:-

Rotations of co-ordinates

axis: z-axis

angle: ϕ

sense: ACW



$$x' = x\cos\phi + y\sin\phi$$
$$y' = -x\sin\phi + y\cos\phi$$
$$z' = z$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Rotation Matrix: $R_z(\phi)$

HW: Find out $\boxed{R_x(\phi) \text{ and } R_y(\phi)}$

ii) Rotation matrix is orthogonal: $R^TR = I$

$\boxed{R^TR = I}$ Property of orthogonal matrix.

$\det(R^TR) = 1$

$\det R \det R^T = 1$

$\det R^T = \det R$

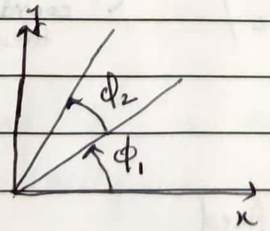
$\det R = \pm 1$

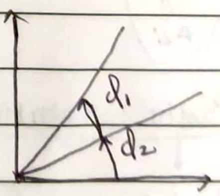
$\det R_z(\phi) = 1$ → {proper rotation}

In general: $R_{\hat{n}}(\phi)$

iii) Two dimensional rotations always commute.

about Z-axis:-





$$R_z(\phi_2)\, R_z(\phi_1) = R_z(\phi_1 + \phi_2) = R_z(\phi_2 + \phi_1)$$

iv) Two successive rotations constitute a single rotation.

$R_1 \quad R_2 \longrightarrow$ check orthogonality

$$(R_1R_2)^T R_1R_2 = R_2^T \underbrace{R_1^T R_1}_{I} R_2 = I$$

iv). Given an R :- axis and angle

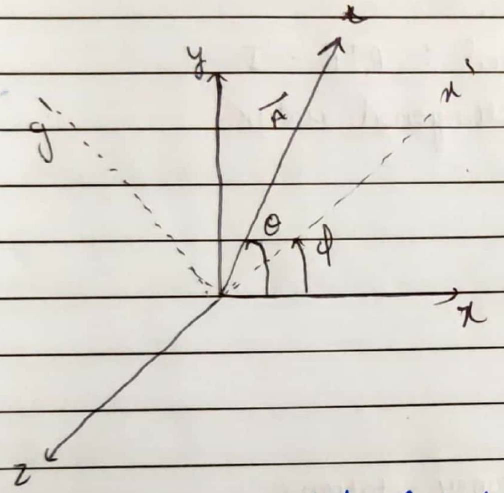
v) Rotation Matrix :- 3 angles : Euler angles
↳ why 3 ?

Inversion :-

$$R_I = \begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$$

$x \to -x$
$y \to -y$
$z \to -z$

$\det R_I = -1$



From S

$A_x = A\cos\theta$

$A_y = A\sin\theta$

From S'

$A_x = A\cos(\theta - \phi)$

$= A\cos\theta\cos\phi + A\sin\theta\sin\phi$

$A_x\cos\phi + A_y\sin\phi$

$A_y' = A\sin(\theta - \phi)$

$= -A_x\sin\phi + A_y\cos\phi$, $A_z' = A_z$

$$\begin{pmatrix} A_x' \\ A_y' \\ A_z' \end{pmatrix} = R_z(\phi)\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

{ Triplets of numbers that transform like coordinates under rotation themselves

Passive rotation
↓
Passive as Point P is fixed
coordinates

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Scalar Invariant under rotation { vectors: (components transform)

eg:- $\vec{A}\cdot\vec{B}$

check $A_xB_x + A_yB_y + A_zB_z = A_x'B_x' + A_y'B_y' + A_z'B_z'$

in frame S   in frame S'

$\vec{A}\cdot\vec{A}$ is scalar: Norm of vector is invariant under rotation

eg: $\vec{A}\times\vec{B} = \vec{C}$

Proof

$C_x = A_yB_z - A_zB_y$ $C_x' = A_y'B_z' - A_z'B_y'$

→ $R_x(\theta_1)R_z(\theta_2)$ can be seen as a single rotation but
$R_x(\theta_1)R_z(\theta_2) \neq R_z(\theta_2)R_x(\theta_1)$

Finite rotations about different axes do not commute

→ Infinitesimal rotations always commute.
$R_x(\delta\theta_1)R_z(\delta\theta_2) = R_z(\delta\theta_2)R_x(\delta\theta_1)$

→ $\delta\theta$ can be viewed as a vector
$\delta\theta_1 + \delta\theta_2 = \delta\theta_2 + \delta\theta_1$

→ $\vec{\delta\theta} = |\delta\theta|\hat{n}$ where $\hat{n}$ is the direction of rotation, defines angular velocity ω.

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R_z(\theta_i)R_x(\theta_2)\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Pseudo vector:- Inversion

$A_x \to -A_x$

$A_y \to -A_y$

$A_z \to -A_z$

$(\vec{A}\times\vec{B})$ what happens under inversion

$\to$ does not change sign under inversion if $\vec{A}$ and $\vec{B}$ do.

$\to$ pseudo vector

$\vec{L} = (\vec{r}\times\vec{p}) \to$ pseudo vector

Pseudo scalar:- changes sign under inversion.

Scalar: $\vec{r}\cdot\vec{L} \to$ Pseudo scalar

Levi - Civita Symbol

$\delta_{ij} = 1$ if $i=j$ $\qquad$ x, y, z as $1, 2, 3$

$\quad\; = 0$ if $i \neq j$

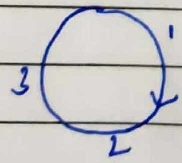
$\downarrow$

Kronecker delta $\qquad$ $\vec{A} = \{A_i\}$, $i = 1, 2, 3$

Levi Civita:-

ϵ_{ijk} completely anti-symmetric in indices

$\epsilon_{jik} = -\epsilon_{ijk}$



$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$

$\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1$

$\to$ cyclic order : $\epsilon_{ijk} = 1$

$\epsilon_{112} = -\epsilon_{112} = 0$

$\epsilon_{ijk} = 0$ if any two indices are equal

$\vec{A}\cdot\vec{B} = \sum_{i=1,2,3} A_i \cdot B_i$ Repeated indices = dummy = summed over

Cross product $(\vec{A}\times\vec{B})$: Consider the i^{th} component

$$(\vec{A}\times\vec{B})_i = \sum_{jk} \epsilon_{ijk} A_j B_k$$

i ↓ free index; i^{th} ↓ 1, 2, or 3

$i=1$ $= \sum_{jk} \epsilon_{1jk} A_j B_k$

$= \epsilon_{123} A_2 B_3 + \epsilon_{132} A_3 B_2$

possibility 1) $j=2$, $k=3$ $= A_2 B_3 - A_3 B_2$

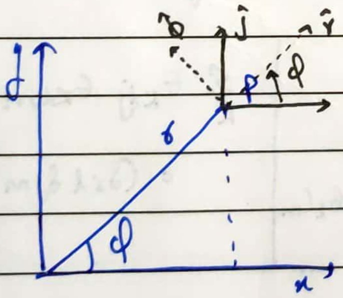
2) $j=3$, $k=2$ $= (\vec{A}\times\vec{B})_1$

$$\boxed{\sum_i \epsilon_{ijk}\epsilon_{ilm} = (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})}$$ only if $i,j,k: 1,2,3$

Plane-Polar Coordinates :- $(r,\theta) = (r,\phi)$

$\vec{F} = -\frac{GMm}{r^2}\hat{r}$ → central force → depends on r and in the $-\hat{r}$

* Motion is confined in a plane



$(r\cos\phi, r\sin\phi)$

$\hat{r} = \hat{i}\cos\phi + \hat{j}\sin\phi$

$\hat{\phi} = -\hat{i}\sin\phi + \hat{j}\cos\phi$

$\vec{r} = \hat{i}x + y\hat{j}$

$\frac{d\vec{r}}{dx} = \hat{i}$ → unit magnitude in the direction of increasing x

$\vec{r} = \hat{i}\cos\phi\, r + \hat{j} r\sin\phi$.

$\hat{r} = \frac{d\vec{r}}{dr} = \hat{i}\cos\phi + \hat{j}\sin\phi$.

$\hat{\phi} = \frac{d\vec{r}}{d\phi} = -\hat{i} r\sin\phi + \hat{j} r\cos\phi$ X (as it is not unit vector)

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$$\hat{\phi} = \frac{\frac{\partial \vec{r}}{\partial \phi}}{\left|\frac{\partial \vec{r}}{\partial \phi}\right|} = -\hat{i}\sin\phi + \hat{j}\cos\phi$$

→ Vectors and scalars are defined w.r.t. proper rotation (det = 1)

→ Whether a vector is a pseudo vector or a scalar is a pseudo scalar is determined w.r.t. inversion operation (det = −1)

Thumb rules:-

Never use a free index as a dummy index

Question

$(\vec{A}\times\vec{B})_i = \sum \epsilon_{ijk} A_j B_k$

$\vec{A}\times(\vec{B}\times\vec{C}) = ?$

start from i^{th} component.

$[\vec{A}\times(\vec{B}\times\vec{C})]_i = \sum_{jk} \epsilon_{ijk} A_j (\vec{B}\times\vec{C})_k$

$= \sum_{jklm} \epsilon_{ijk} A_j \epsilon_{klm} B_l C_m$

[cyclic order so no sign change]

$= \sum_{jklm} \epsilon_{kij} \epsilon_{klm} A_j B_l C_m$

$\sum_k \epsilon_{kij} \epsilon_{klm} = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})$

$\Rightarrow \sum_{jlm} (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) A_j B_l C_m$

$i=l \quad j=m$

$= B_i \sum_m A_m C_m - C_i \sum_l A_l B_l = B_i(\vec{A}\cdot\vec{C}) - C_i(\vec{A}\cdot\vec{B})$

$= \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$

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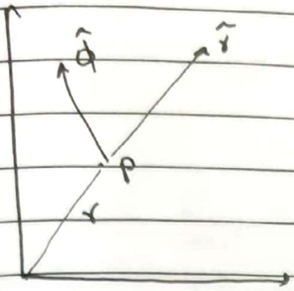
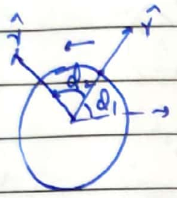
Polar Coordinates



$\hat{r} = \hat{i}\cos\phi + \hat{j}\sin\phi$

$\hat{\phi} = -\hat{i}\sin\phi + \hat{j}\cos\phi$

Position vector $\Rightarrow \vec{r} = r\hat{r} \rightarrow \hat{r} = \hat{r}(\phi)$



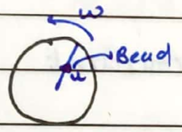
Circular situation (Motion)

$$\frac{d\vec{r}}{dt} = \hat{r}\frac{dr}{dt} + r\frac{d\hat{r}}{dt}$$

$$= \underbrace{\hat{r}\dot{r}}_{\text{Radial}} + \underbrace{r\dot{\phi}\hat{\phi}}_{\text{transverse}}$$

$\frac{d\hat{r}}{dt} = -\hat{i}\sin\phi\,\dot{\phi} + \hat{j}\cos\phi\,\dot{\phi} = \dot{\phi}\hat{\phi}$

Bead on spike



$V_r = u$ (Radial)

$r = ut$

$V_\theta = \omega r$ (transverse)

Acceleration

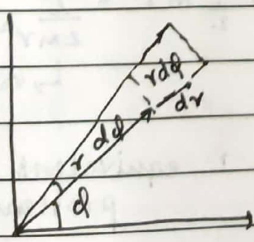
$$\vec{a} = \frac{d}{dt}(\hat{r}\dot{r} + r\dot{\phi}\hat{\phi})$$

$$\underbrace{(\ddot{r} - r\dot{\phi}^2)}_{\substack{\text{radial} \\ a_r}}\hat{r} + \underbrace{(r\ddot{\phi} + 2\dot{r}\dot{\phi})}_{\substack{\text{transverse} \\ a_\theta}}\hat{\phi}$$

$\frac{d\hat{r}}{dt} = \dot{\phi}\hat{\phi}$

$\frac{d\hat{\phi}}{dt} = -\dot{\phi}\hat{r}$

Note $\ddot{r} = 0$ even then a_r may not be a constant



elementary area $r\,dr\,d\phi$

$\phi \Rightarrow 0$ to 2π

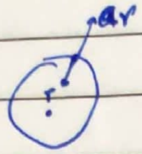
$r = \sqrt{x^2+y^2} \rightarrow 0$ to ∞

$r\,dr\,d\phi$

$\frac{a^2}{2}(2\pi)$

Integrate $\rightarrow 0$ to 2π and 0 to a

$\Rightarrow \pi a^2$

eg:-  $r = a \quad \dot{r} = 0$
$\dot{\phi} = \omega \quad v_\phi = \omega a$

Central field Problem

$$m(\ddot{r} - r\dot{\phi}^2) = f(r) \quad \text{— (1)}$$

$$m(r\ddot{\phi} + 2\dot{r}\dot{\phi}) = 0$$

↓

Implication

$$m[(r^2\ddot{\phi}) + (2r\dot{r}\dot{\phi})] = 0$$

$$\frac{d}{dt}(mr^2\dot{\phi}) = 0$$

$$\underset{\text{conserved}}{mr^2\dot{\phi} = l} = \text{constant}$$

$\frac{1}{2} I\omega^2$

$\frac{1}{2} m r_1^2 \omega^2$

$l = m v r$

$m\omega r^2$

$l^2 = \frac{m^2\omega^2 r^4}{2mr^2}$

Angular momentum vector is constant

$$\frac{d\vec{l}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \cancel{\frac{d\vec{r}}{dt}}$$

$$= \frac{dr}{dt} \times \vec{p} + \vec{r} \times \frac{dP}{dt}$$

$$= 0 + \vec{r} \times f(r)\hat{r} = 0$$

⇒ Motion is confined in a plane.

In general: $T = \frac{1}{2} m(v_r^2 + v_\phi^2)$

$$= \frac{1}{2} m(\dot{r}^2 + \dot{\phi}^2) = \underbrace{\frac{1}{2} m\dot{r}^2}_{\text{translational}} + \underbrace{\frac{l^2}{2mr^2}}_{\text{rotational}}$$

for central field l is constant

Eq: (1) as:- $\boxed{m\ddot{r} - \frac{l^2}{mr^3} = f(r)}$ → equivalent 1 dim problem

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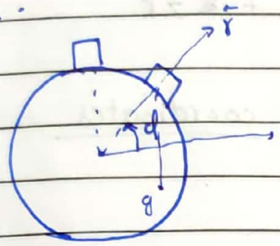
$\dot{\theta} = \frac{l}{mr^2}$

Equivalent 1 dim problem:-

$$E = \frac{1}{2} m\dot{r}^2 + \underbrace{\frac{l^2}{2mr^2} + v(r)}_{\text{effective potential}} ; \quad l = \text{constant}$$

$$\boxed{V_{eff} = \frac{l^2}{2mr^2} + v(r)}$$

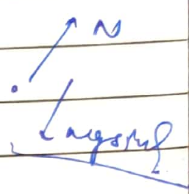
Problem:-



$$\cancel{N} - mg\sin\phi + N = F_r$$

$$F_\phi = -mg\cos\phi$$

$$m(\ddot{r} - r\dot{\phi}^2) = N - mg\sin\phi$$



$$\cancel{m(\ddot{r}}$$

$$m(r\ddot{\phi} + 2\dot{r}\dot{\phi}) = -mg\cos\phi$$

$l \to$ not constant

$r = R$

$$-MR^2\dot{\phi}^2 = N - mg\sin\phi \quad (1), \qquad MR\ddot{\phi} = -mg\cos\phi$$

$$\vec{l} = MR^2\dot{\phi}\,(-\hat{k}) \quad \text{into the paper}$$

$= -\frac{1}{2} m R^2 (mg\cos\phi)$

$\frac{g\cos\phi}{R}$

ϕ eqn -

$$\ddot{\phi} = \frac{1}{2}\frac{d}{d\phi}(\dot{\phi}^2) \quad (h.w)\ (\cancel{ccgt})$$

$$\frac{d\dot{\phi}^2}{d\phi} = \frac{-2g}{R}\cos\phi$$

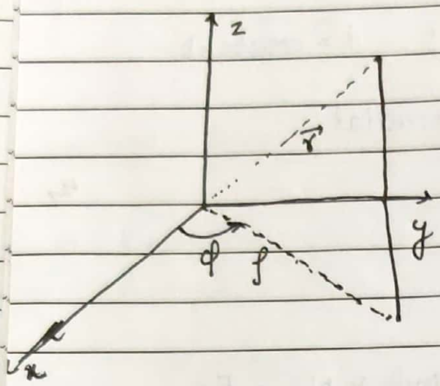
$R\dot{\phi}^2 = g(1-\sin\phi)$

$$\int_0^{\dot{\phi}} d\dot{\phi}^2 = -\frac{2g}{R}\int_{\pi/2}^{\phi}\cos\phi \quad \Rightarrow \quad R\dot{\phi}^2 = 2g(1-\sin\phi)$$

put in (1)

$$\boxed{N = 0; \quad \sin\phi = 2/3}$$

Cylindrical polar co-ordinates



$\rho = 0$ to ∞ $\quad \phi = 0$ to 2π

$z \to -\infty$ to ∞

$z = z, \quad x = \rho\cos\phi, \quad y = \rho\sin\phi$

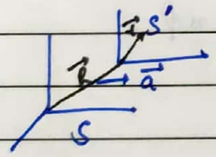
$\vec{r} = \rho\hat{\rho} + z\hat{k}$

Spherical coordinates

Non inertial frames:- Newton's law are strictly valid in an inertial frame.

S → inertial

S' → accelerating with acc. $\vec{a}$ with respect to S



pretending to write Newton's law in S'.

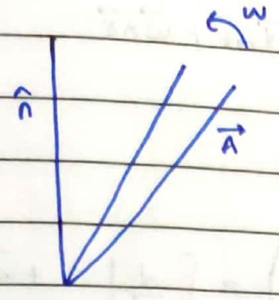
$\vec{r} = \vec{r}' + R \Rightarrow \ddot{\vec{r}} = \ddot{\vec{r}}' + \vec{a}$

$\ddot{\vec{R}} = \vec{a}$

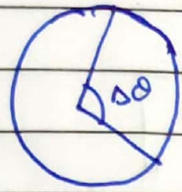
Newton's law in S: $\quad m\ddot{\vec{r}} = \vec{F}_{ext} \Rightarrow m(\ddot{\vec{r}}' + \vec{a}) = \vec{F}_{ext}$

$\underbrace{m\ddot{\vec{r}}'}_{\text{resembles N-2nd law}} = \vec{F}_{ext} \underbrace{- m\vec{a}}_{\text{Pseudo force}}$

Precessing Vector:- $\vec{A}$ about $\hat{n}$: Tip of the vector traces out a circle



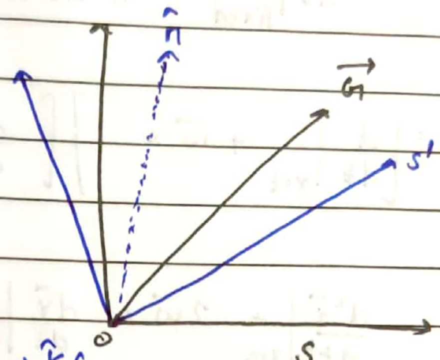
$$\frac{d\vec{A}}{dt} = \vec{\omega} \times \vec{A}$$



in time Δt

Two frames:- S : inertial

S' : rotating w.r.t. $\hat{n}$ by $\vec{\omega}$



(in s frame) $\vec{G} = \hat{i} G_1 + \hat{j} G_2 + \hat{k} G_3$

(in s' frame)

$\vec{G} = \hat{i}' G_1' + \hat{j}' G_2' + \hat{k}' G_3'$

$$\left.\frac{d\vec{G}}{dt}\right|_{s=\text{fixed}} = \hat{i}\frac{dG_1}{dt} + \hat{j}\frac{dG_2}{dt} + \hat{k}\frac{dG_3}{dt}$$

$$\left.\frac{d\vec{G}}{dt}\right|_{\text{fixed}=s'} \Rightarrow \underbrace{\hat{i}'\frac{dG_1'}{dt} + \hat{j}'\frac{dG_2'}{dt} + \hat{k}'\frac{dG_3'}{dt}}_{\text{observed by observer in } s'}$$

$$+ G_1'\frac{d\hat{i}'}{dt} + G_2'\frac{d\hat{j}'}{dt} + G_3'\frac{d\hat{k}'}{dt}.$$

$$\left.\frac{d\vec{G}}{dt}\right|_{\text{fixed}} = \left.\frac{d\vec{G}}{dt}\right|_{\text{rotating}} + \vec{\omega} \times (\hat{i}'G_1' + \hat{j}'G_2' + \hat{k}'G_3')$$

$$\left\{ \begin{aligned} \frac{d\hat{i}'}{dt} &= \vec{\omega} \times \hat{i}' \\ \frac{d\hat{j}'}{dt} &= \vec{\omega} \times \hat{j}' \\ &\vdots \end{aligned} \right.$$

$$= \left.\frac{d\vec{G}}{dt}\right|_{\text{rotating}} + (\vec{\omega} \times \vec{G})$$

True for any vector

$$\boxed{\left.\frac{d}{dt}\right|_{\text{fixed}} = \left.\frac{d}{dt}\right|_{\text{rot}} + \vec{\omega} \times \ldots}$$

$\vec{G} = \vec{r}$

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$$\left.\frac{d\vec{r}}{dt}\right|_{fixed} = \left.\frac{d\vec{r}}{dt}\right|_{rot} + \vec{\omega}\times\vec{r} \qquad \vec{v}_{fixed} = \vec{v}_{rot} + \vec{\omega}\times\vec{r}$$

for acceleration

$$\left.\frac{d^2\vec{r}}{dt^2}\right|_{fixed} = \left.\frac{d}{dt}\right|_{fixed}\left(\left.\frac{d\vec{r}}{dt}\right|_{rot} + \vec{\omega}\times\vec{r}\right) = \left[\left.\frac{d}{dt}\right|_{fixed} + \vec{\omega}\right]\times\vec{v}_{fixed}$$

$$= \left[\left.\frac{d}{dt}\right|_{rot} + \vec{\omega}\times\right]\left[\left.\frac{d\vec{r}}{dt}\right|_{rot} + \vec{\omega}\times\vec{r}\right]$$

$$\left.\frac{d^2\vec{r}}{dt^2}\right|_{rot} + 2\vec{\omega}\times\left.\frac{d\vec{r}}{dt}\right|_{rot} + \vec{\omega}\times(\vec{\omega}\times\vec{r})$$

$\vec{r} \longrightarrow$ in S ~~[illegible]~~

$$M\ddot{\vec{r}}\Big|_{fixed} = F_{ext} \qquad S\text{ frame}$$

2 pseudo forces

$$M\ddot{\vec{r}}\Big|_{rot} = \vec{F}_{ext} - \underbrace{2M\,\vec{\omega}\times\vec{v}_{rot}}_{\text{coriolis}} - \underbrace{M\vec{\omega}\times(\vec{\omega}\times\vec{r})}_{\text{centrifugal}}$$

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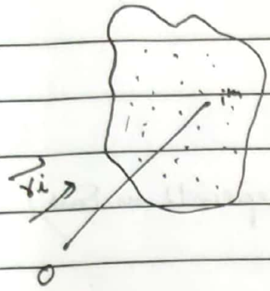
→ Newton's law are strictly valid in an inertial frame.

System of particles:-



$$m_i\ddot{\vec{r_i}} = F_i^{ext} + F_i^{int} \qquad P_i = m_i\dot{\vec{r_i}}$$

$$\sum \dot{\vec{P_i}} = \sum_i \vec{F_i}^{ext} + \sum_i F_i^{int}$$

$$= \vec{F}^{ext} + \sum_{j \neq i}\sum_i \vec{F}_{ij}^{int}$$

$$\Rightarrow \sum_i \dot{\vec{P_i}} = \vec{F}^{ext} + \underbrace{\frac{1}{2}\sum_{ij}(\vec{F_{ij}} + \vec{F_{ji}})}_{=0}$$

Newton's 3rd law:-

$$\vec{F_{ij}} = -F_{ji}$$

Weak form → implies conservation of linear momentum for a system of particles.

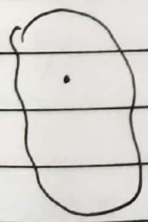
$$\sum \dot{\vec{P_i}} = \vec{F}^{ext}$$

If $\vec{F}^{ext} = 0$; $\sum \dot{\vec{P_i}} = 0 \Rightarrow \sum_i \vec{P_i} = \text{constant}$

Homogeneity of space

$m\vec{a} = \vec{F}$: Symmetry under spatial translation

Conservation of Angular Momentum about o



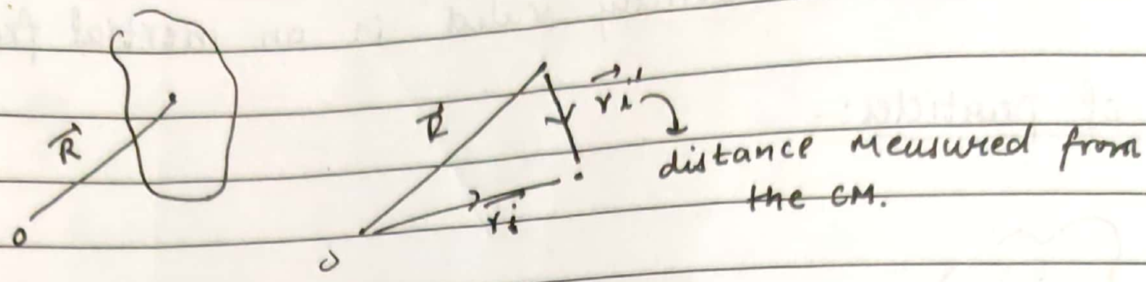
$$\vec{R}_{CM} = \vec{R} = \frac{\sum m_i\vec{r_i}}{\sum m_i} = \frac{\sum_i m_i\vec{r_i}}{M}$$

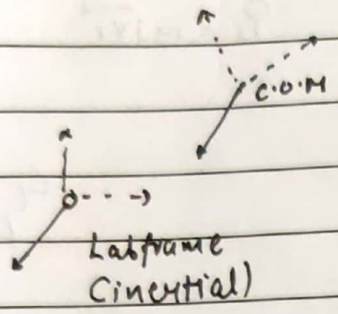
$$M\ddot{\vec{R}} = \vec{F}^{ext}$$

→ CM as a point particle of mass m on which $\vec{F}^{ext}$ acts

No external force

$$M\ddot{\vec{R}} = 0$$





C.O.M $\longrightarrow$ COM frame (may or may not be depends on Fext)

$$\vec{r_i} = \vec{r_i}' + \vec{R}$$

$$\Rightarrow \vec{r_i}' = \vec{r_i} - \vec{R}$$

$$\sum m_i \vec{r_i}' - M\vec{R} = 0$$

L about O:-

$$\vec{L} = \sum_i \vec{r_i} \times \vec{p_i} \Rightarrow \frac{d\vec{L}}{dt} = ?$$

$$\frac{d\vec{L}}{dt} = \sum_i \vec{r_i} \times \dot{\vec{p_i}} = \sum_i \vec{r_i} \times (\vec{F_i}^{ext} + \vec{F_i}^{int})$$

$$\Rightarrow \underbrace{\sum_i \vec{r_i} \times \vec{F_i}^{ext}}_{\substack{N \\ \downarrow \\ \text{Net torque}}} + \sum \vec{r_i} \times \vec{F_i}^{int} \to \sum_{\substack{ij \\ i \neq j}}' \vec{r_i} \times F_{ij}^{int}$$

$$\Rightarrow \frac{1}{2} \sum_{ij} (\vec{r_i} \times \vec{F_{ij}}^{int} + \vec{r_j} \times \underbrace{\vec{F_{ji}}^{int}}_{-F_{ij}^{int}})$$

$$\Rightarrow \frac{1}{2} \sum_{ij} (\vec{r_i} - \vec{r_j}) \times F_{ij}^{int}$$

$$\frac{d\vec{L}}{dt} = \vec{N} + \frac{1}{2} \sum_{ij} (\vec{r_i} - \vec{r_j}) \times \vec{F_{ij}}^{int}$$

$\Rightarrow$

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3^{rd} law in strong form

$$F_{ij} = f(|\vec{r_i} - \vec{r_j}|)(\vec{r_i} - \vec{r_j})$$

→ equal, opp. and acts along the line joining i and j

$$\Rightarrow \frac{1}{2}\sum_{ij} (\vec{r_i} - \vec{r_j}) \times F_{ij}$$
$$= \frac{1}{2}\sum (\vec{r_i} - \vec{r_j}) \times (\vec{r_i} - \vec{r_j}) f$$
$$= 0$$

$$\frac{d\vec{L}}{dt} = \vec{N} \qquad \text{if } \vec{N} = 0$$
$$\frac{d\vec{L}}{dt} = 0$$

Ang. momentum is conserved. Recall central field problem
↳ 3^{rd} law in strong form.

Calculate

$$\vec{L}_0 = \sum_i \vec{r_i} \times \vec{P_i} = \vec{L}_{cm,0} + \vec{L}_{ab/cm}$$

Two point particles: isolated
↳ reduced to a one body problem

If the force is radial, the angular momentum is conserved. If the force is explicitly time dependent, the energy is not conserved.

Isolated (closed) Two body problem.
↳ proton + electron / Earth - Sun.

No external force.

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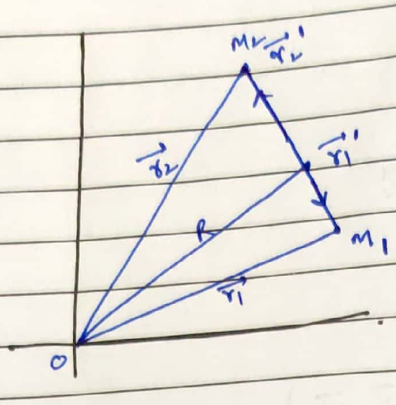

$$R_{CM} = \frac{M_1\vec{r}_1 + M_2\vec{r}_2}{M_1+M_2} \Rightarrow M_1\vec{r}_1{}' + M_2\vec{r}_2{}' = 0$$

$$\underset{\times M_2}{M_1\ddot{\vec{r}}_1} = \vec{F}_{12} \to ① \quad \text{and} \quad \underset{\times M_1}{M_2\ddot{\vec{r}}_2} = \vec{F}_{21} = -\vec{F}_{12} \to ②$$

$$① - ② \quad M_1 M_2 (\ddot{\vec{r}}_1 - \ddot{\vec{r}}_2) = (M_1+M_2)\vec{F}_{12}$$

$$\underbrace{\frac{M_1M_2}{M_1+M_2}}_{\mu}\ddot{\vec{r}} = \vec{F}_{12} \quad , \quad M\ddot{\vec{R}}_{COM} = 0$$

$\vec{r} = \vec{r}_1 - \vec{r}_2 \equiv$ Relative coordinate

$$\mu\ddot{\vec{r}} = \vec{F}_{12} \qquad \mu = \text{reduced mass}$$

Explicit time dependence : $f(x,t)$.

$$\frac{df}{dt} = \underbrace{\frac{\partial f}{\partial t}}_{\text{explicit}} + \underbrace{\frac{\partial f}{\partial x}\frac{dx}{dt}}_{\text{implicit}} \qquad \underbrace{\vec{F} = -\frac{GMm}{r^2}\hat{r}}_{\text{No explicit dependence}}$$

Forced oscillator:-

$$-Kx + A\cos(\omega_0 t)$$

$\to$ explicit (even if x is kept fixed. Force is time dependent)

$\to$ x depends on time

$\to$ Implicit

Implicit → Force depends on a parameter which depends on time.

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Newton's Second law:-

$$M \frac{d^2x}{dt^2} = F$$

$\underbrace{t \to t + t_0 = t'}_{\text{time translation}}$

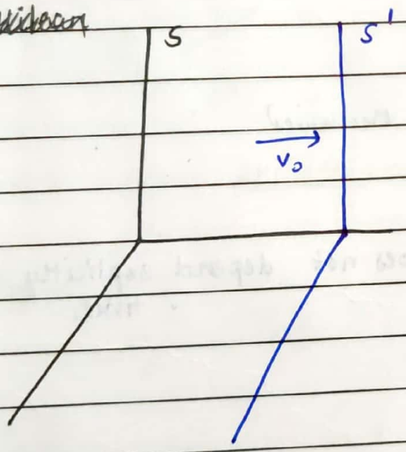
$$m \frac{d^2x}{dt'^2} = F$$

Newton's law are invariant under time translation True only iff force does not explicitly depend on time.

$\boxed{\text{Time translational invariance} \equiv \text{Energy conservation}}$

Galilean invariance:- (Relativity)

~~Gravitation~~



$P(x, y, z)$ as measured in S

$P(x', y', z')$ as measured in S'

$x = x' + v_0 t$

$y = y'$

$z = z'$

$t = t'$

↳ time is universal absolute

An event: fire a bullet

↳ characterised by co-ordinates and time

A: (x, y, z, t) as in S

: (x', y', z', t') as in S'

Newton's law are invariant under galilean Transformation.

2nd law:- Acceleration is invariant: 3rd law?

Maxwell's equations are not invariant under G.T as if it so then speed of light cannot be constant.

Lorentz transformation:- time is not the same in all frame

$\boxed{V_0 << c}$ Gallilean Invariance.

Two events:- A and B

$x_A = x_A' + v_0 t_A$
$x_B = x_B' + v_0 t_B$

$x_A - x_B = x_A' - x_B' + v_0 (t_A - t_B)$

i). Time ~~in general~~ interval are the same
ii). length is the same in all inertial frames. } for G.T

Time reversal invariance (Quantum Mechanics)

$m\frac{d^2x}{dt^2} = F$

→ Assumption - Force does not depend explicitly on time,

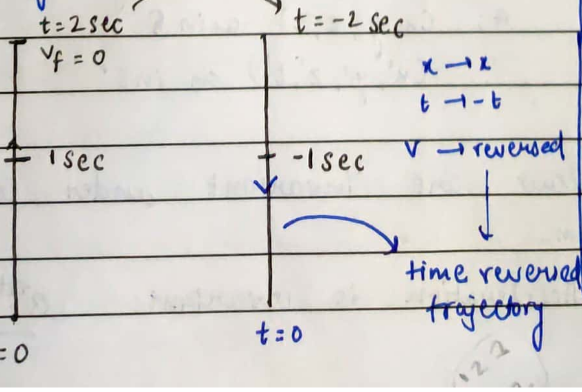
$x \longrightarrow x$ and $t \longrightarrow -t$

$t' = t$

$\frac{d^2x}{dt'^2} = F$

Vel. reverses sign ⇒ but acceleration does not.

No air drag
constant g



$x \rightarrow x$
$t \rightarrow -t$
$v \rightarrow$ reversed
time reversed trajectory

H.W.

$\sum_{ij} \epsilon_{ijk} S_{ij} = ?$

anti symm. symmetric

Drag Force / Damping force

$F_D = -mkv$ (or $-mk_1v^2$)

→ no energy cons

a particle falling under Gra

$m\frac{dv}{dt} = mg - mkv$

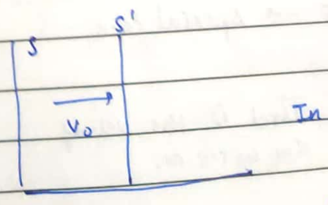
$\frac{dv}{g - kv} = dt \Rightarrow$

Initial condition:
at $t=0, v=0$
starts from rest

Variable mass problem:

recall:-

$\frac{d\vec{p}}{dt} = \vec{F}_{ext}$



In

Two d

① → special

NBS

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Drag Force / Damping force

$$F_D = -mkv \quad (\text{or } -mk_1 v^2)$$

} Not time reversal invariant

↳ no energy conservation

Lorentz force

$$\vec{F} = q(\vec{v} \times \vec{B})$$

$$\vec{F} \cdot \vec{v} = 0$$

↳ not a conservative force though magnetic field does not do any work.

a particle falling under Gravity : (constant g)

$$m\frac{dv}{dt} = mg - mkv$$

$$\frac{dv}{g-kv} = dt \Rightarrow \frac{d(g-kv)}{g-kv} = -k\,dt$$

Initial condition:
at $t=0, v=0$
starts from rest

$$v = \frac{g}{k}(1 - e^{-kt})$$

say $t \to \infty$

$$v = g/k$$

$$t \gg \frac{1}{k} : \quad v = \underset{\substack{\downarrow \\ v_{terminal}}}{v_t} = g/k$$

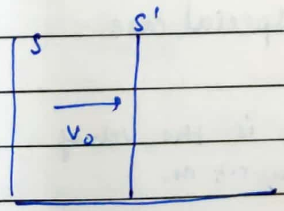
↳ if $mg - mkv_t = 0; \; v_t = g/k$

Variable mass problem:- i) Raindrop falling
ii) Rocket motion

recall:-

$$\frac{d\vec{p}}{dt} = \vec{F}_{ext}, \quad \frac{d}{dt}(m\vec{v}) = \vec{F}_{ext}$$

$$= m\frac{d\vec{v}}{dt} + \vec{v}\frac{dm}{dt} = \vec{F}_{ext} \quad \text{--- (1)}$$



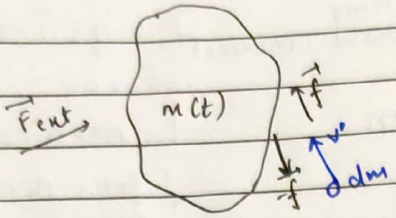
In fram S'

$$\frac{d\vec{p}}{dt} = m\frac{d\vec{v}'}{dt} + (\vec{v} - \vec{v}_0)\frac{dm}{dt} = \vec{F}_{ext}$$

Two different equations in different frames

Not galilean invariance

(1) → special case ---- ----



dm is absorbed in time dt to the bigger body m(t) which was moving with a vel. $\vec{v}(t)$ under $\vec{F}_{ext}$. Combined body moves with a vel $\vec{v}+d\vec{v}$ after collision

"dm" provides an impulsive force $= \vec{f}$
m also provides $-\vec{f}$ on dm $\Big\}$ 3^{rd} law

Inelastic collision $\Rightarrow$ Momentum conservation

for smaller mass dm

$$-\vec{f}dt = dm(\vec{v}+d\vec{v}-\vec{v}') \quad \text{- - - - (1)}$$

for Bigger Mass:-

$$\vec{F}_{ext}\,dt + \vec{f}dt = m(\vec{v}+d\vec{v}) - m\vec{v} \quad \text{- - - (2)}$$

add 1 and 2

$$\vec{F}_{ext}\,dt = m\,d\vec{v} + dm(\vec{v}-\vec{v}')$$

$$\underset{\downarrow\, m(t)}{m}\frac{d\vec{v}}{dt} = \vec{F}_{ext} + \underbrace{(\vec{v}'-\vec{v})\frac{dm}{dt}}$$

$\hookrightarrow$ variable mass equation.

If: $v'=0$, $m\dfrac{d\vec{v}}{dt} + \vec{v}\dfrac{dm}{dt} = \vec{F}_{ext}$ $\rightarrow$ special case

Variable mass equation

$$m\frac{d\vec{v}}{dt} = \vec{F}_{ext} + \vec{u}_{rel}\frac{dm}{dt}$$

$\vec{u}_{rel}$ is the vel. of dm w.r.t m.

$\hookrightarrow$ check the Galilean invariance (H.w.)

$\vec{u}_{rel}$ is the same in all inertial frames.

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Falling rain drop:-

- starts from $z=0, v_0$ | Rain drop gathers moisture.

↳ water particles are at rest w.r.t. ground ↓ $\vec{v}' = 0$

$x = 0$, x, $+ve\ x$

$$\frac{dm}{dx} = am$$

$$m\frac{d\vec{v}}{dt} + \vec{v}\frac{dm}{dt} = \vec{F}_{ext}$$

$$\rightarrow \frac{dm}{dt} = \frac{dm}{dx}\frac{dx}{dt} = amv$$

$$m(t)\frac{d\vec{v}}{dt} + \vec{v}\frac{dm}{dt} = mg$$

$$m(t)\frac{d\vec{v}}{dt} + \vec{v}am\vec{v} = mg$$

$$\cancel{m}\cdot\frac{dv}{dt} + a\cancel{m}v^2 = \cancel{m}g$$

$$\frac{dv}{dt} + av^2 = g$$

$$\frac{dm}{dx} = am; \quad m = m_0 e^{ax}$$

$$v\frac{dv}{dx} = g - av^2 \rightarrow \text{solve to find the terminal velocity.}$$

Rocket Motion:-

$\uparrow \vec{v}$, $\downarrow u_{rel}$

Assume Free space

Drop $-mg$

$$m(t)\frac{d\vec{v}}{dt} - \vec{u}_{rel}\frac{dm}{dt} = -mg$$

$$m(t)\frac{dv}{dt} = \vec{u}_{rel}\frac{dm}{dt}$$

$$m(t)\frac{dv}{dt} = -u_{rel}\frac{dm}{dt} \quad \Big| \quad dv = -u_{rel}\frac{dm}{m}$$

$$v_f - v_0 = \Delta v = -u_{rel}\int_{m_0}^{m_1}\frac{dm}{m}$$

$$m_1 = m(t) = m_0 e^{-\frac{\Delta v}{u_{rel}}}$$

Payload

higher, keeping Δv and m_0

for this u_{rel} → should be large

H.w. | $\frac{dm}{dt} = -\gamma$. Calculate Δv

(2) Include g and solve the equation of rocket motion.

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Conservative force (assume mass to be constant)

Total Mechanical energy is conserved

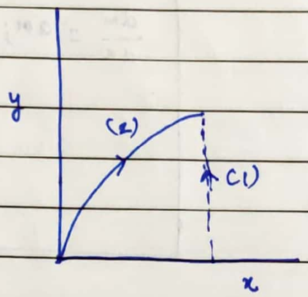
Work - Energy Theorem

$$W = \int_1^2 \vec{F} \cdot d\vec{r} = \int m \frac{d\vec{v}}{dt} \cdot d\vec{r}$$

$$= \int_1^2 m\vec{v} \cdot d\vec{v} = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2$$

Workdone is the change in kinetic energy

Work done depends on path.



Air drag is there

Work is not state function:
Path dependent.

Conservative: Work done does not depend on path; only on initial and final coordinates.
↳ potential
$\Psi(x, y, z)$

→ Work done in a closed loop is Zero
↳ Conservative force field

$\vec{\nabla}\psi$ = Mapping from a scalar field Ψ to a vector field $\vec{\nabla}\psi$

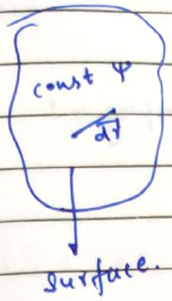
$\vec{\nabla} \times \vec{F} = 0$

$\vec{F} = -\vec{\nabla}\psi$

H.w. Show $\vec{\nabla}$ is a vector operator
S.D. Joglekar

$$\vec{\nabla} = \left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right)$$

$$\vec{\nabla}\psi = \hat{i}\frac{d\psi}{dx} + \hat{j}\frac{d\psi}{dy} + \hat{k}\frac{d\psi}{dz}$$



For displacement $d\vec{r}$; $d\varphi = 0$

$\vec{\nabla}\varphi \cdot d\vec{r}$

$= \left(\hat{i}\frac{d\varphi}{dx} + \hat{j}\frac{d\varphi}{dy} + \hat{k}\frac{d\varphi}{dz}\right)\left(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz\right)$

$= \frac{\partial\varphi}{\partial x}dx + \frac{d\varphi}{dy}dy + \frac{d\varphi}{\partial z}dz$

$= \cancel{2d\Psi} = d\Psi$

$d\Psi = 0$, for $d\vec{r}$

$\because \vec{\nabla}\varphi \cdot d\vec{r} = 0$

$\vec{\nabla}\varphi$ is perpendicular to constant φ surface (in the direction of max change)

$\Psi(r) = -\frac{Gm}{r}$

$-\vec{\nabla}\Psi = -\frac{GM}{r^2}\hat{r}$

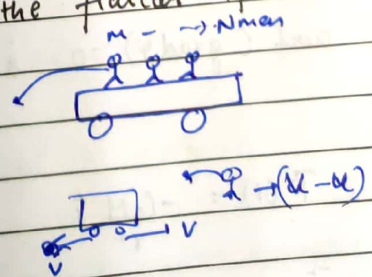
31/01/20

Kleppner 3.14

~~Flat car on a track~~

N men, each with mass m, stand on a railway flatcar of mass M. They jump off one end of the flatcar with velocity u relative to the car. The car rolls in the opp direction without friction

a) What is the final velocity of the flatcar if all the men jump at the same time



Momentum conservation

$\vec{P_i} = 0$

$\vec{P_f} = MV + mN(v-u)$

$V = \frac{Nmu}{Nm+M}$

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ii) jump out one by one

Suppose j^{th} fellow has jumped out & $(j+1)^{th}$ is jumping out

$$\vec{P_i} = [\,(N-j)m + M\,]\,v_j$$
$$\vec{P_f} = [\,(N-j-1)m + M\,]\,v_{j+1} + m(v_{j+1} - u)$$
$$\Delta\vec{P} = 0$$
$$v_{j+1} = \left[\frac{m}{(N-j)m + M}\right]u + v_j$$

$(v_0 = 0)$

$$v_{j+1} - v_j = \left[\frac{m}{(N-j)m+M}\right]u$$

$$\Sigma \Rightarrow v_N = mu \sum_{j=1}^{N-1} \frac{1}{Nm + M - jm}$$

Stokes Theorem :-

$$\oint \vec{F}\cdot d\vec{l} = \int_\sigma (\vec{\nabla}\times\vec{F})\cdot d\vec{\sigma}$$

always valid if loop does not include any singularity of $\vec{F}$.

anticlockwise sense

($\vec{F}$ does not have any divergence in the loop)

3) Conservative force: $\oint \vec{F}\cdot d\vec{l} = 0$ (for any arbitary closed loop)

$\Rightarrow \vec{\nabla}\times\vec{F} = 0$ (irrotational vectors)

4) curl (grad ψ) $= 0$ $\Rightarrow$ $\vec{F} = -\vec{\nabla}\psi$ → scalar product

→ $\vec{\nabla}\psi$ is $\perp$ to constant ψ surface

$\Rightarrow \vec{F}$ is $\perp$ to equipotential surface

5) $\vec{F}(r) = -\frac{GM}{r^2}\hat{r}$ $\qquad \psi(r) = -\frac{GM}{r}$

$$\vec{F} = -mg\,\hat{k}$$
$$\psi = mgz$$

Prove $\vec{\nabla} \times (\vec{\nabla}\psi) = 0$

$$\vec{\nabla} \times \vec{A} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{d}{dx_1} & \frac{d}{dy_1} & \frac{d}{dz_1} \\ A_1 & A_2 & A_3 \end{pmatrix}$$

$$[\vec{\nabla} \times (\vec{\nabla}\psi)]_i = \sum_{jk} \epsilon_{ijk}\, d_j\, (\vec{\nabla}\psi)_k = \sum_{jk} \underset{\text{anti-symmetric in } jk}{\epsilon_{ijk}}\, \underbrace{d_j d_k \psi}_{\text{symm in } j \,\&\, k}$$

$\therefore$ Pairwise cancellation:- $[\vec{\nabla} \times (\vec{\nabla}\psi)] = 0$

Work done:-

$\vec{F} = -\vec{\nabla}\psi$ (-ve sign ensures force is from higher potential to lower potential)

$$\int_1^2 \vec{F}\cdot d\vec{r} = -\int \vec{\nabla}\psi \cdot d\vec{r} = -\int_1^2 d\psi = \psi_1 - \psi_2$$

Work energy Theorem : $W = T_2 - T_1$

$T_2 - T_1 = \psi_1 - \psi_2 \Rightarrow \boxed{T_1 + \psi_1 = T_2 + \psi_2} \rightarrow$ Total mech. energy is conserved

SHM : curl is not defined for 1 Dimension.

I can define $\psi(x) = \frac{1}{2}Kx^2$ s.t. $\frac{-d\psi}{dx} = -Kx$

(Scalar potential is the hall mark of conservative force field)

H.w. ① $\vec{v} = r\omega\hat{\phi}$, calculate $\vec{\nabla} \times \vec{v} = ?$

② $\vec{F} = 3x^2y\,\hat{i} + x^3\hat{j} \Rightarrow \vec{\nabla} \times \vec{F} = 0$

what is $\psi(x,y) \Rightarrow -x^3y + C$

③ Central field : $\vec{F} = F(r)\vec{r}$
$\vec{\nabla} \times f(r)\vec{r} = 0$ } energy and angular momentum conserved.

(4). $F = -Kx \Rightarrow$ curl is not defined in one dimension

Define: $\psi(x) = \frac{1}{2}Kx^2$ s.t. $-\frac{d\psi}{dx} = -Kx$

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Homework: $\vec{V} = \omega r \hat{\phi}$; Calculate $\vec{\nabla} \times \vec{V}$

Conservative force field:-

$\vec{F} = -\vec{\nabla}\psi$ → Stoke's theorem
ψ → Scalar potential
$\vec{\nabla}$: Vector operator

so that $\vec{\nabla} \times \vec{F} = 0$
↳ irrotational vectors

One dimensional problem:- $\vec{F} = -Kx\hat{i}$: $\psi = \frac{1}{2}Kx^2$
↓
Curl is not defined

Total energy (mechanical) conserved
$T_1 + \psi_1 = T_2 + \psi_2$

potential Ψ is a state function: depends on coordinates

$W = \Psi_1 - \Psi_2 \Rightarrow$ path independent

Explicit time dependence:- Not there

$\boxed{\vec{F} = 3x^2y\,\hat{i} + x^3\hat{j} \quad \psi = ?}$

Central field:- Energy and ang. momentum both conserved

Prove:- $[\vec{\nabla} \times \vec{F}(r)]_i = [\vec{\nabla} \times f(r)\vec{r}]_i = \sum_{jK} \epsilon_{ijK}\, \partial_j (f(r)\vec{r})_K$

$\Rightarrow \sum_{jK} \epsilon_{ijK}\, \partial_j x_K + \sum_{jK} \epsilon_{ijK}\, x_K\, \partial_j [f(r)]$

$= 0 + \sum \epsilon_{ijK}\, x_K\, \partial_r f(r)\, \partial_j r$
(as $j \neq K$)

$\Rightarrow \sum \epsilon_{ijK}\, x_K \{\partial_r f(r)\} \frac{x_j}{r}$

$\Rightarrow \sum \epsilon_{ijK}\, x_j x_K \left[\frac{1}{r} \{\partial_r f(r)\} \right]$

$= 0$

$\frac{\partial}{\partial x_j} f(r)$

$= \frac{d}{dr} f(r) \frac{dr}{dx_j}$

$\partial_j = \frac{\partial}{\partial x_j}$

$\frac{\partial}{\partial x_1}$ or $\frac{\partial}{\partial x_2}$ or $\frac{\partial}{\partial x_3}$

$r = \sqrt{x_1^2 + x_2^2 + x_3^2}$
$= \sqrt{\sum x_i x_i}$

$\vec{r} = \hat{i}x_1 + \hat{j}x_2 + \hat{k}x_3$

Scanned by CamScanner

Example: 5.9 Kleppner chap 4 and 5 Klepner

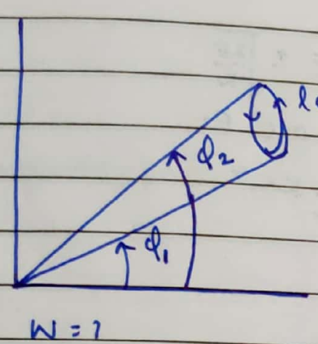
$\vec{F}(r) = \frac{A}{r}\hat{\phi}$: Conservative ?

$A \to$ const

Work done:-

$$W = \int \vec{F}\cdot d\vec{r} = \int \vec{F}\cdot(dr\hat{r} + r\,d\phi\,\hat{\phi})$$
$$= A\int d\phi$$

Closed loop:-

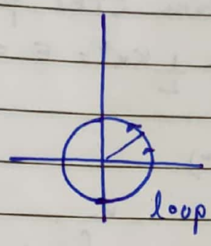


W = ?

for loop 1

$W = A(\phi_2 - \phi_1) + A(\phi_1 - \phi_2) = 0$

$\hookrightarrow$ Is it conservative?



loop 2:-

work done is $2\pi A$

For any closed loop enclosed the origin: $W = 2\pi A$

Not conservative

$\to$ $\vec{\nabla}\times\vec{F}$ is singular at the origin.

Origin:-

$$\vec{\nabla}\times\left(\frac{A}{r}\hat{\phi}\right) = \frac{1}{r}\frac{d}{dr}\left(r\frac{A}{r}\right) = 0$$

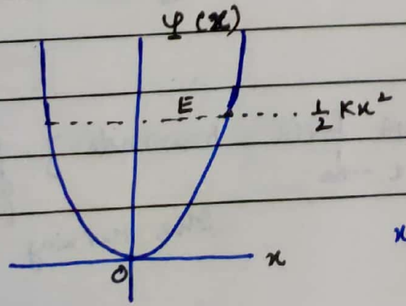
$\downarrow$

everywhere except for $r = 0$.

Hence, not a conservative force

One dimensional problem:-

Harmonic Oscillator:- $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}Kx^2$



$\frac{d\varphi}{dx} > 0$ for $x > 0$

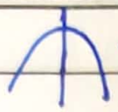
$\frac{d\varphi}{dx} < 0$ for $x < 0$

force is always restoring

always towards $x = 0$

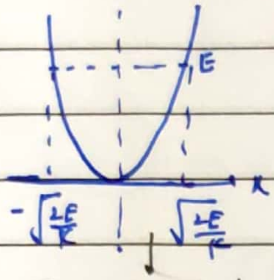
$x = 0$ stable eq^m

$\hookrightarrow \frac{d^2\varphi}{dx^2} > 0$

 → unstable

for all E motion is bounded between classical turning points.

boundstate



Two classical turning points

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}Kx^2 = E$$

Classical turning points: K.E. Vanishes (x_0) ⇒

$$\frac{1}{2}Kx_0^2 = E \Rightarrow x_0 = \pm\sqrt{\frac{2E}{K}}$$

Motion is bounded betwn CTPs for all E:

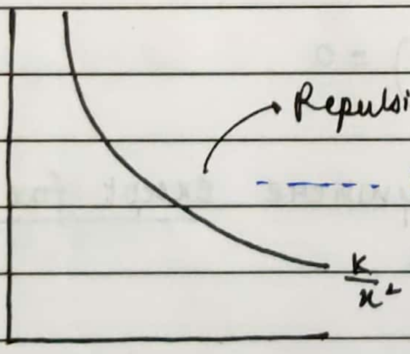
Time Period:- $\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}Kx^2 = E = \frac{1}{2}Kx_0^2$

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 = \frac{K}{m}(x_0^2 - x^2)$$

$$T = 2\times\int_{-x_0}^{x_0}\frac{dx}{\sqrt{\frac{K}{m}}(x_0^2-x^2)^{1/2}} = 2\times 2\int_0^{x_0}\frac{dx}{\sqrt{\frac{K}{m}}(x_0^2-x^2)^{1/2}} = \frac{2\pi}{\omega_0}$$

$$\omega_0 = \sqrt{\frac{K}{m}}$$

Q.



Scattering state

v_0 $+\infty$

mass m projected with $\frac{1}{2}mv_0^2$ from $x = +\infty$

$F = -\frac{\partial\varphi}{\partial x}$

$\varphi(x) = \frac{K}{x^2}$

$\varphi(x)$ diverges as $x \to 0$

E is conserved

what is the closed distance: (x_0)

Sol^n: $\frac{1}{2}mv_0^2 = \frac{K}{x_0^2}$

No bound state:

∵ Repulsive force towards $x \to \infty$ } particle goes back

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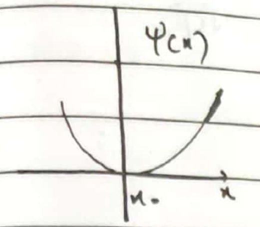
Harmonic Approximation:- (stable eq^n, minimum φ at $x = x_0$)



Not a parabola

"~~Small~~ motion" close to x_0 so that $(x-x_0)$ is small

Extremum condition = 0

$$\varphi(x) = \varphi(x_0) + \left.\frac{d\varphi}{dx}\right|_{x_0}(x-x_0) + \frac{1}{2}\left.\frac{d^2\varphi}{dx^2}\right|_{x_0}(x-x_0)^2 + \dots$$

Expand φ around $x = x_0$

$\varphi(x_0)$ → constant, set it to zero.

$\left.\frac{d\varphi}{dx}\right|_{x_0} = 0$: No Force acts

$$\varphi(x) = \frac{1}{2}\left.\frac{d^2\varphi}{dx^2}\right|_{x_0}(x-x_0)^2 > 0$$

+ve because φ is minimum at x_0

Compare with $\varphi(x) = \frac{1}{2}kx^2$

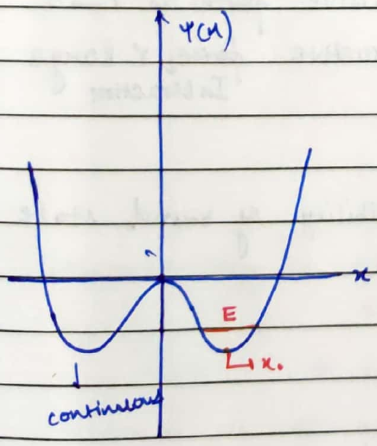
$\Rightarrow$ Harmonic oscillator :

$$K_{eff} = \left.\frac{d^2\varphi}{dx^2}\right|_{x_0} \rightarrow +ve$$

→ Motion close to a stable eq^n is like Harmonic oscillator with $K_{eff} = \frac{1}{2}\left.\frac{d^2\varphi}{dx^2}\right|_{x_0}$

Take a potential:-

$$\varphi(x) = -\frac{\alpha}{2}x^2 + \beta/_2\, x^4$$ quadratic term



$x = 0 \quad \varphi(x) = 0$

$x \to -x$ symmetric

$F = -\frac{dv}{dx}$

-ve

$\frac{d\varphi}{dx} = 0$, $-\alpha x + 2\beta x^3 = 0$

∴ Three extremas :

$x = 0, \sqrt{\frac{\alpha}{2\beta}}, -\sqrt{\frac{\alpha}{2\beta}}$

$\{x = 0, \text{ unstable eq}^n.\}$

$2\beta x^2$

~~Amplitude of small oscillation~~

~~$K_{eff} = \left.\frac{d^2\varphi}{dx^2}\right|_{x_0} = 2\alpha$~~

Amplitude of small oscillation: Expand $\Psi(x)$ around $x = x_0 = \sqrt{\frac{\alpha}{2\beta}}$.

$$K_{eff} = \left.\frac{d^2\Psi}{dx^2}\right|_{x_0} = 2\alpha$$

Lenard - Jones potential:

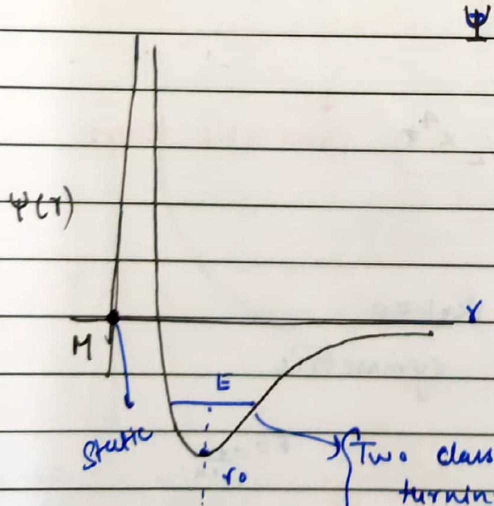
Isolated 2-body problem: can be reduced to 1-body problem in the C.O.M frame in terms of relative co-ordinates $\vec{r} = \vec{r_1} - \vec{r_2}$ & $\mu = \frac{m_1 m_2}{m_1 + m_2}$

Simplification: Assume $M \gg m$
$\mu = m_1$

C.O.M falls on the mass M which is static
m is moving in field of M:

$$\Psi(r) = \left[\left(\frac{r_0}{r}\right)^{12} - 2\left(\frac{r_0}{r}\right)^6 \right]$$

repulsive attractive



strong repulsive ~~force~~ interaction as $r \to 0$
weekly attractive ~~force~~ Interaction, r large

{Two classical turning points} → possibility of bound state

Date
Page

Central field:-

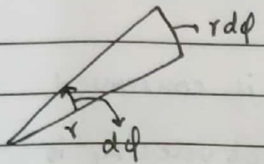
Energy and angular momentum are conserved.

Assume : $M >> m$
$\downarrow$
static
$\{\mu \simeq m\}$

Energy of mass(m): $= \frac{1}{2}m\dot{r}^2 + \boxed{\frac{l^2}{2mr^2} + \psi(r)}$

equivalent one dim problem

Recall: Kepler's problem: $\psi(r) = -K/r$



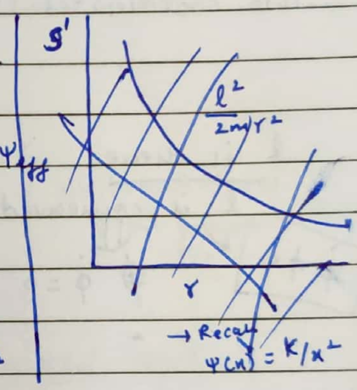
Area covered in time dt: $\frac{1}{2}r^2 d\phi$

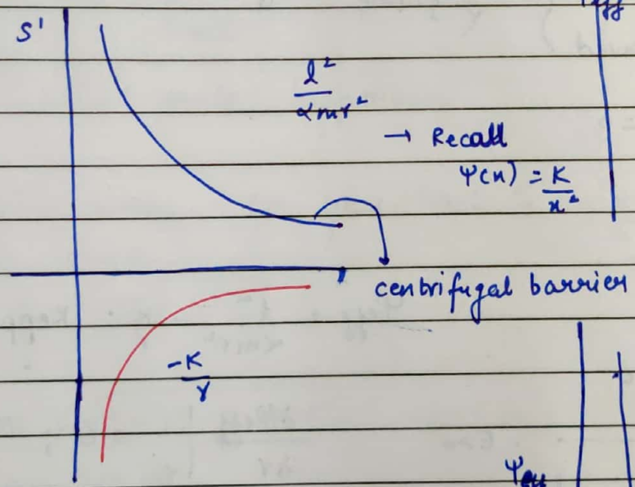
Areal velocity $= \frac{1}{2}r^2\dot{\phi} \to$ constant

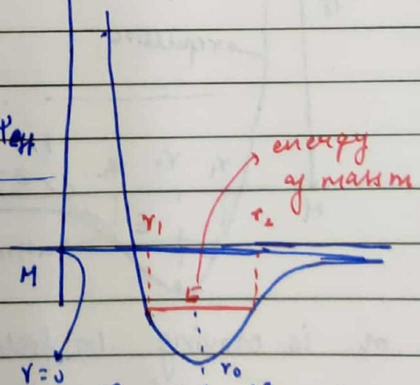
equivalent one dim problem: l is constant

$$\psi_{eff} = \frac{l^2}{2mr^2} - \frac{K}{r}$$

S': rotating with const. angular mom l.







Minimise $\psi_{eff} \Rightarrow \left.\frac{d\psi_{eff}}{dr}\right|_{r_0} = 0,$

$\Rightarrow r_0 = \frac{l^2}{mK} \to$ circular orbit.

mass m is always equidistant from M

$\left\{ \begin{array}{ll} r \to 0, & \frac{l^2}{2mr^2} \\ r \to \infty & -\frac{K}{r} \end{array} \right\}$

$$E = \left[\frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} - \frac{K}{r}\right] = \frac{l^2}{2mr_0^2} - \frac{K}{r_0} = \frac{-K}{2r_0}$$

In this case:- CTPs as $\frac{1}{2}m\dot{r}^2 = 0$

so, $\boxed{\frac{l^2}{2mr_0^2} - \frac{k}{r_0} = E}$ → Two real sol^n: an elliptical orbit

Recap

In polar coordinates

$$T = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2}$$

Klepner chap-9

In a central field problem:

$$E = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} - \varphi(r)$$

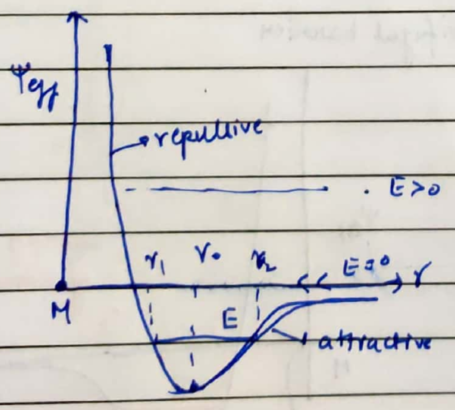
→ l is conserved
⇓
- Areal velocity is constant, Keppler's law
- Motion is confined in a plane.

$$\varphi_{eff} = \frac{l^2}{2mr^2} - \frac{k}{r}$$

Two coordinates (r, ϕ) ⇒ reduces only to r-equation
→ Equivalent one dim

l is there:
l is conserved } { frame S

$\boxed{\dot{\phi} \neq 0}$ ⇓ $\not\Rightarrow \dot{\phi} = 0$



$\varphi_{eff} = \frac{l^2}{2mr^2} - \frac{k}{r}$: Keppler problem

$$\left.\frac{\partial \varphi_{eff}}{\partial r}\right|_{r_0} = 0 \; ; \quad r_0 = \frac{l^2}{mk}$$

$r_0 = \frac{l^2}{mk}$: Circular motion

m is moving in field of M
m << M (simplifying assumption)
m is always equidistant from M
but moving with a constant ang. mom. l
⇓
Not static

$$r_0 = \frac{(mr_0^2\omega_0)^2}{mk}$$
↓
$$T^2 \propto r_0^3$$

at $r = r_0$; $E = V_{eff}\Big]_{r_0} = \frac{-K}{2r_0}$

circular orbit: $\frac{1}{2}m\dot{r}^2 = 0$

simple Harmonic oscillator

$$E = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} - \frac{K}{r}$$

Turning pts: S·H·O $\frac{1}{2}m\dot{x}^2 = 0$

Turning points: $\frac{1}{2}m\dot{r}^2 = 0$; denoted by '$r_0$' $E = \frac{l^2}{2mr_0^2} - \frac{K}{r_0}$ --- (1)

$$r_0 = \frac{-2mK \pm \sqrt{4m^2K^2 + 8mEl^2}}{4mE} \quad \text{--- (2)}$$

If $\sqrt{4m^2K^2 + 8mEl^2} = 0$

$r_0 = -\frac{K}{2E}$; $E = -\frac{K}{2r_0}$ (E ~~for~~ same as that of circular orbit)

For elliptical orbit, r_0 must have two real values

1. $E = 0$, ~~$r_0 = \frac{-2mK \pm 2mK}{4mE}$~~

from (1)

$r_0 = \frac{l^2}{2mK}$ --- ~~for~~ parabola

2) $E > 0$:

No bound state; hyperbola

No bound states from graph.

3. From (2), when do you have a closed orbit.

r_0 must have two real values.

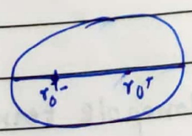
Ans: $E < 0$

$E = -|E|$

$$r_0 = \frac{2mK \mp \sqrt{4m^2K^2 - 8m|E|l^2}}{4m|E|} \quad \text{.... (3)}$$

Two solutions: r_0^+ and r_0^-

$r_0^+ + r_0^- = 2a$



$r_0^+ + r_0^- = \frac{4mK}{4m|E|} = 2a$

$\Rightarrow E = -|E| = -\frac{K}{2a}$

To calculate:-

$$\nabla\cdot\left(\frac{\vec r}{r^3}\right) = ? = 4\pi\delta^3(\vec r) \longrightarrow \text{Revise}$$

$$\vec\nabla\cdot\left(-\frac{GM\vec r}{r^3}\right) = -4\pi G\delta^3(\vec r).$$

Gauss's Theorem.

$$\int \vec E\cdot d\vec r = -4\pi GM.$$

↳ Mass enclosed.

Divergence theorem

$$\int \vec\nabla\cdot\vec F\, d\tau = -4\pi G\int \rho\, d\tau$$

↳ elementary volume

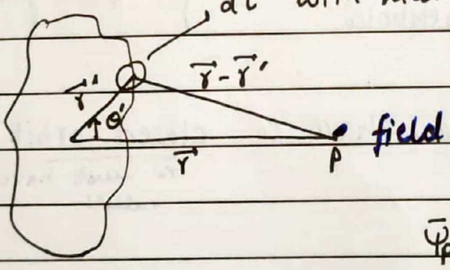
$$\vec F = -\vec\nabla\Psi$$

$$\boxed{\nabla^2\Psi = 4\pi G\rho}$$

$$\vec\nabla\cdot\vec F = -4\pi GM$$

An arbitary mass distribution

$d\tau'$ with mass $\rho\, d\tau'$



$$d\Psi_P = -\frac{G\rho\, d\tau}{|\vec r-\vec r'|}$$

$$\Psi_P = -G\int \frac{\rho(\vec r')\,d\tau'}{|\vec r-\vec r'|}$$

$$= -G\int \frac{\rho(\vec r')\,d\tau'}{\sqrt{r^2+r'^2-2rr'\cos\theta'}}$$

$$\Psi = -G\int \frac{\rho(\vec r')\,d\tau'}{r\sqrt{1+\left(\frac{r'}{r}\right)^2-2\frac{r'}{r}\cos\theta'}}$$

Approx: $\frac{r'}{r} << 1$:

$$\Psi_0 = -\frac{G}{r}\int \rho(r')\,d\tau' = -\frac{GM}{r}: \text{ Monopole term}$$

Show that :

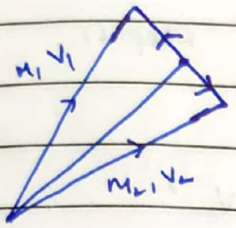
$$= -\frac{GM}{r} - \frac{G}{r^2}\int \rho(\vec{r}')\,d\tau'(r'\cos\theta)$$

↳ Prove that for a spherically symmetric $\rho(\vec{r}')$: This term vanishes.

~~Chap 4, Kleppner~~

Scattering: Elastic collision: Energy and momentum both conserved.

Lab frame and COM frame



$$\vec{V}_{com} = \vec{V} = \frac{m_1\vec{v_1} + m_2\vec{v_2}}{m_1+m_2}$$

$$\vec{V}_{1c} = v_1 - \vec{V} = \frac{m_2(\vec{v_1}-\vec{v_2})}{m_1+m_2} = \frac{m_2\vec{u}}{m_1+m_2}$$

↳ relative velocity

$$m_1\vec{V}_{1c} = \frac{m_1 m_2}{m_1+m_2}(\vec{u}) = \mu\vec{u} = \vec{p}_{1c}$$

llly $\qquad m_2\vec{V}_{2c} = -\mu\vec{u} = \vec{p}_{2c}$

$$\boxed{\vec{p}_{1c} + \vec{p}_{2c} = 0}$$

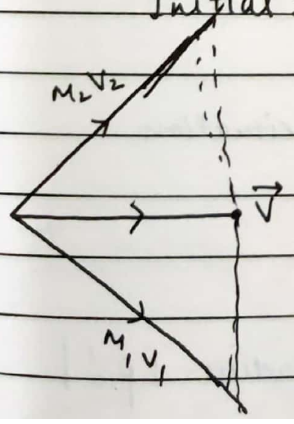
In Lab frame

$$m_1\vec{v_1} + m_2\vec{v_2} = (m_1+m_2)\vec{V}$$

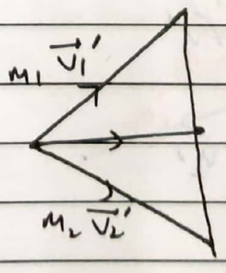
10th Feb

From the lab frame:

Initial:



Final:



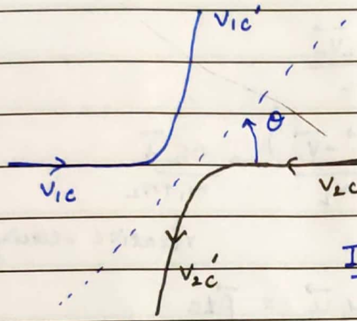
• Not in the same plane

Azimuthal symmetry: ϕ does not appear

In the CM frame:

$$\vec{p}_{1c} + \vec{p}_{2c} = \vec{p}\,'_{1c} + \vec{p}\,'_{2c} = 0$$

define plane of scattering



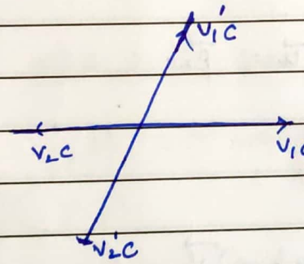
Kleppner

Energy conservation:

$v_{1c} = v_{1c}'$ & $v_{2c} = v_{2c}'$

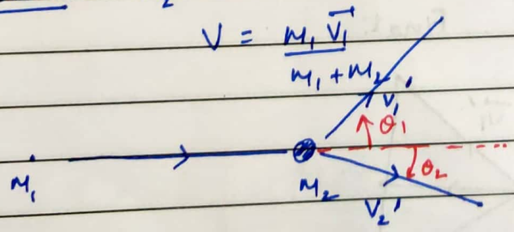
In the CM frame:

Velocity vectors simply rotate.



Assume: m_2 is at rest:

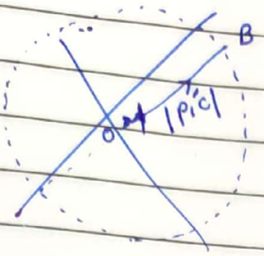
$$V = \frac{m_1 \vec{v}_1}{m_1 + m_2}$$

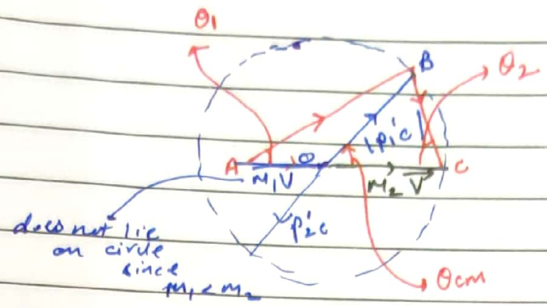


Experimental situation

Lab frame

Momentum diagram:- Draw a circle of radius $|p_{1c}'|$





m_2 is static, $v_2 = 0$

$$m_1\vec{V} = \frac{m_1^2 v_1}{m_1+m_2} \qquad V = \frac{m_1\vec{v_1}}{m_1+m_2}$$

<u>$m_1 < m_2$</u>

$$m_2\vec{V} = \frac{m_1 m_2 \vec{v_1}}{m_1+m_2} = \mu \vec{v_1} \quad \text{— (1)}$$

$$\vec{p_{1c}} = \mu \vec{u}_{rel} = \mu\vec{v_1}, \quad \boxed{|\vec{v}_{1c}| = |\vec{v}_{1c}'|}$$

↳ energy conservation

$$|\vec{p_{1c}'}| = \mu\vec{u} = \mu\vec{v_1} \quad \text{— (2)}$$

Compare (1) and (2): $m_2\vec{V}$ lies on circle

$p_{1c} + p_{2c} = p_{1c}' + p_{2c}' = 0$

$$\vec{AB} \Rightarrow m_1\vec{V} + |\vec{p_1'c}| = \vec{p_1}'$$

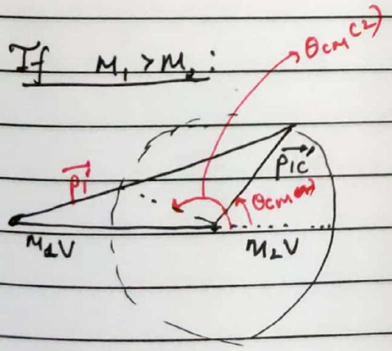
$$\vec{BC} = m_2\vec{V} - \vec{p_{1c}'} = m_2\vec{V} + \vec{p_{2c}'} = \vec{p_2'}$$

<u>Homework:-</u>

From this diagram: show:

$$\tan\theta_1 = \frac{p_1c' \sin\theta_{cm}}{m_1\vec{V} + p_{1c}' \cos\theta_{cm}} = \frac{\sin\theta_{cm}}{\frac{m_1}{m_2} + \cos\theta_{cm}}$$

<u>If $m_1 > m_2$:</u>



$m_1 \longrightarrow \quad m$

I have $\theta_{cm}^{(1)} < \pi/2$: forward scattering

$\theta_{cm}^{(2)} > \pi/2$: backward scattering

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