Semantics.

FO language $L = (R, F, C)$

Need to define the following.

- Fix an underlying set
- Specify the interpretation for each relation symbol $r \in R$.
- Specify the interpretation for each function symbol $f \in F$.
- Specify the interpretation for each constant symbol $c \in C$.

Done using a first order structure

$\mathcal{L} = (R, F, C)$ - FO language.

FO structure for $\mathcal{L}$ is a pair $\mathcal{M} = (S, \tau)$

- S is a non-empty set
- τ is a function over $R \cup F \cup C$ such that

• For each $r \in R$ with $\#(r) = n$, $\tau(r) \subseteq S^n$.

• For each $f \in F$ with $\#(f) = n$, $\tau(f) = S^n \to S$

• For each $c \in C$, $\tau(c) \in S$.

Notation. Denote $\tau(r) : r^{\mathcal{M}}$, $\tau(f) : f^{\mathcal{M}}$, $\tau(c) : c^{\mathcal{M}}$.

FO structure for $\mathcal{L}$ is called an $\mathcal{L}$-structure

Interpretation. Let $\mathcal{L} = (R, F, C)$. An interpretation is $I = (\mathcal{M}, \sigma)$ where $\mathcal{M} = (S, \tau)$ and $\sigma : Var \to S$.
(assignment)

Given $\sigma : Var \to S$, $\sigma[x_1 \mapsto s_1, x_2 \mapsto s_2, \ldots, x_n \mapsto s_n]$ the assignment σ' where $\sigma'(x_i) = s_i$, $\forall i \in \{1, \ldots, n\}$ and
$\sigma'(z) = \sigma(z)$ $\forall z \notin \{x_1, \ldots x_n\}$.

For $I = (\mathcal{M}, \sigma)$ denote by $I[x_1 \mapsto s_1, \ldots, x_n \mapsto s_n]$ the modified interpretation $(\mathcal{M}, \sigma[x_1 \mapsto s_1, \ldots, x_n \mapsto s_n])$.

Given $I = (M, \sigma)$ where $M = (S, \tau)$ each term t over L maps to a unique element in S – t^I

- if t is a constant $c \in C$, $t^I = c^M$.
- if t is a variable $x \in Var$, $t^I = \sigma(x)$
- if t is of the form $f(t_1, t_2, \ldots t_n)$, $t^I = f^M(t_1^I, \ldots t_n^I)$

Satisfaction relation for FO formula.

Let $L = (R, F, C)$: FO language and $I = (M, \sigma)$: interpretation

$I \models \varphi$ ($\varphi \in \Phi_L$ is satisfied under I) defined as:

- $I \models t_1 \equiv t_2$ if $t_1^I = t_2^I$
- $I \models r(t_1, \ldots t_n)$ if $(t_1^I, \ldots t_n^I) \in r^M$.
- $I \models \neg\varphi$ if $I \not\models \varphi$.
- $I \models \varphi \vee \psi$ if $I \models \varphi$ or $I \models \psi$.
- $I \models \exists x \varphi$ if $\exists s \in S$ s.t $I[x \mapsto s] \models \varphi$.

[- $I \models \forall x \varphi$ if $\forall s \in S$, $I[x \mapsto s] \models \varphi$]

$\varphi \in \Phi_L$ is satisfiable if there is an interpretation I based on an L-structure $\mathcal{M}$ s.t $I \models \varphi$

$\varphi \in \Phi_L$ is valid if for every L-structure $\mathcal{M}$ and every interpretation I based on $\mathcal{M}$, $I \models \varphi$.

A model of φ is an interpretation I s.t $I \models \varphi$.

Bound and Free variables.

An assignment σ fixes the value of all variables.

- $\exists x \psi$, $\forall x \psi$. value assigned by σ to x is irrelevant.

Scope of a quantifier : $\exists x \psi$: scope of $\exists x$ is ψ.
Variable x is free in φ if it is not in the scope of a quantifier $\exists x$ $(\forall x)$

Inductive definition of $FV(\varphi)$.

- if φ is an atomic formula : $r(t_1 \ldots t_n)$
  $FV(\varphi)$ is the set of variables in $t_1, \ldots t_n$
- if φ is an atomic formula $t_1 \equiv t_2$,
  $FV(\varphi)$ is the set of variables in t_1 and t_2.
- $FV(\neg\varphi) = FV(\varphi)$
- $FV(\varphi \vee \psi) = FV(\varphi) \cup FV(\psi)$
- $FV(\exists x \varphi) = FV(\varphi) \setminus \{x\}$.

Notation. $\varphi(x_1, x_2, \ldots x_n)$ denotes that
$FV(\varphi) \subseteq \{x_1, x_2, \ldots x_n\}$.

Proposition. Let L be a FO Language and $\varphi \in \Phi_L$

Let $\mathcal{M}$ be an L-structure and σ, σ' be assignments that agree on $FV(\varphi)$ then

$$(\mathcal{M}, \sigma) \models \varphi \text{ iff } (\mathcal{M}, \sigma') \models \varphi.$$

Sentence : A first order formula with no free variable.

Corollary. Let L be a FO language and $\varphi \in \Phi_L$ be a sentence. Let $\mathcal{M}$ be an L-structure and σ, σ' be assignments.

- $(\mathcal{M}, \sigma) \models \varphi$ iff $(\mathcal{M}, \sigma') \models \varphi$.

Example. How to characterise equivalence relation.

Let $r \in R$ be a binary relation symbol

$\#(r) = 2$

How to ensure that r is interpreted as an equivalence relation?

φ_1 - $\forall x \, r(x,x)$

φ_2 - $\forall x \forall y \, (r(x,y) \leftrightarrow r(y,x))$

φ_3 - $\forall x \forall y \forall z \, ((r(x,y) \wedge r(y,z)) \rightarrow r(x,z))$

Example. Strict linear order $<$: Binary relation over a set S that is irreflexive, transitive and satisfies the property

- Any two distinct elements in S are related by $<$.

Example: $<$ on natural numbers.

How to ensure that $< \in R$ is interpreted as a strict linear order?

- $\forall x (\neg < (x,x))$: $\forall x (\neg (x < x))$
- $\forall x \forall y \forall z ((x < y \wedge y < z) \rightarrow x < z)$
- $\forall x \forall y \, (x < y \vee x \equiv y \vee y < x)$

Example.

$$\varphi_{\geq 2} : \exists x \exists y \neg (x \equiv y)$$

Any structure that models $\varphi_{\geq 2}$ should have atleast two distinct elements in the underlying set S.

$$\varphi_{\geq n} : \exists x_1 \exists x_2 \cdots \exists x_n \bigwedge_{i \neq j} \neg (x_i \equiv x_j)$$

$$\neg \varphi_{\geq 2} : \forall x \forall y (x \equiv y)$$

Atmost one element in the underlying set S.
(Exactly one element since S is assumed to be non-empty).