MTH 102A - Linear Algebra - 2015-16-II Semester

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P. **Eigenvalues and eigenvectors**

- Matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ behaves like the scalar 3 when multiplied with $\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. That is, $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
- Physically: The LT $f(\boldsymbol{x}) = A\boldsymbol{x}$ magnifies the nonzero vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \mathbb{C}^2$ three (3) times.
- Similarly, $A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. So, behaves by changing the direction of the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$
- Take $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$. Do I have a nonzero $\boldsymbol{x} \in \mathbb{C}^2$ which gets magnified by A?
- So I am looking for $\boldsymbol{x} \neq 0$ and α s.t. $A\boldsymbol{x} = \alpha\boldsymbol{x}$. Using $\boldsymbol{x} \neq 0$, we have $A\boldsymbol{x} = \alpha\boldsymbol{x}$ if and only if $[\alpha I - A]\boldsymbol{x} = \mathbf{0}$ if and only if $\text{DET}[\alpha I - A] = 0$.
- $\text{DET}[\alpha I - A] = \text{DET} \begin{bmatrix} \alpha - 1 & -2 \\ -1 & \alpha - 3 \end{bmatrix} = \alpha^2 - 4\alpha + 1$. So $\alpha = 2 \pm \sqrt{3}$.
- Take $\alpha = 2 + \sqrt{3}$. To find $\boldsymbol{x}$, solve $\begin{bmatrix} 1+\sqrt{3} & -2 \\ -1 & \sqrt{3}-1 \end{bmatrix} \boldsymbol{x} = \mathbf{0}$: using GJE, for instance. We get $\boldsymbol{x} = \begin{bmatrix} \sqrt{3}-1 \\ 1 \end{bmatrix}$. Moreover

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \sqrt{3}-1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3}+1 \\ 2+\sqrt{3} \end{bmatrix} = (2+\sqrt{3}) \begin{bmatrix} \sqrt{3}-1 \\ 1 \end{bmatrix}.$$

- We call $\lambda \in \mathbb{C}$ an eigenvalue of $A_{n\times n}$ if there exists $\boldsymbol{x} \in \mathbb{C}^n$, $\boldsymbol{x} \neq 0$ s.t. $A\boldsymbol{x} = \lambda\boldsymbol{x}$. We call $\boldsymbol{x}$ an eigenvector of A for the eigenvalue λ. We call $(\lambda, \boldsymbol{x})$ an eigenpair.
- If $(\lambda, \boldsymbol{x})$ is an eigenpair of A, then so is $(\lambda, c\boldsymbol{x})$, for each $c \neq 0$, $c \in \mathbb{C}$.

Th. The number λ is an eigenvalue of $A_{n\times n}$ if and only if $\text{DET}(\lambda I - A) = 0$.

Pr. Follows from: there exists a nontrivial solution of $[\lambda I - A]\boldsymbol{x} = \mathbf{0}$ if and only if $\text{DET}[\lambda I - A] = 0$. ∎

- Find eigenpairs of $A = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$.
- To find the eigenvalues, solve $\text{DET}(\lambda I - A) = 0$. So $\lambda = 3 \pm \sqrt{2}i$.
- To find an eigenvector $\begin{bmatrix} x \\ y \end{bmatrix}$ for the eigenvalue $3 + \sqrt{2}i$:

$$\left[\begin{array}{cc|c} 2+\sqrt{2}i & -3 & 0 \\ 2 & \sqrt{2}i-2 & 0 \end{array}\right] \xrightarrow{E_{21}(\frac{2}{2+\sqrt{2}i})} \left[\begin{array}{cc|c} 2+\sqrt{2}i & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]. \text{ So } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3}{2+\sqrt{2}i} \\ 1 \end{bmatrix}.$$

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$\left(3+\sqrt{2}i, \begin{bmatrix} \frac{3}{2+\sqrt{2}i} \\ 1 \end{bmatrix}\right)$.

- Let D be a diagonal matrix. Then eigenpairs of D are $(d_{ii}, \boldsymbol{e}_i)$.

Q. Should $A_{n\times n}$ have n eigenvalues (including multiplicities)?. Yes.

- While discussing eigenvalues etc, we consider square matrices only.

Q. What are the eigenvalues of an upper triangular matrix? Diagonal entries.

- The multiset of eigenvalues of A is called the spectrum of A. Notation: $\sigma(A)$.

Q. Let $\lambda \in \sigma(A)$. Must we have at least one eigenvector for λ? Yes, by GJE.

- The multiplicity of λ in $\sigma(A)$ is called the algebraic multiplicity of A.

Q. Suppose that algebraic multiplicity of λ in $\sigma(A)$ is 2. Must we have 2 lin.ind eigenvectors for λ? No. For example, take $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $\sigma(A) = \{0, 0\}$ and $\boldsymbol{e}_1$ is an eigenvector. That is, $A\boldsymbol{e}_1 = \boldsymbol{0}$. If $\boldsymbol{x}$ is another eigenvector, then $A\boldsymbol{x} = \boldsymbol{0}$. That is, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. So $x_2 = 0$ and $\boldsymbol{x} = x_1\boldsymbol{e}_1$.

[If $(\lambda_1, \boldsymbol{x}_1), \ldots, (\lambda_k, \boldsymbol{x}_k)$ are eigenpairs, then $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$ may not be linearly independent unless λ_i's are distinct!!]

- If $\lambda \in \sigma(A)$, then the maximum number of linearly independent eigenvectors for λ is the geometric multiplicity of λ. [Geometry: maximum k such that the LT magnifies a k-dimensional subspace by λ. Take LT: $\boldsymbol{e}_1 \to \boldsymbol{e}_1$, $\boldsymbol{e}_2 \to \boldsymbol{e}_2$, $\boldsymbol{e}_3 \to \boldsymbol{e}_1 + \boldsymbol{e}_3$.

Th. Let $(\lambda_i, \boldsymbol{v}_i)$ be some eigenpairs of $A_{n\times n}$, λ_i's are distinct. Then $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ are linearly independent. [Contrast it with the earlier comment.]

Pr. Suppose that $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ are lin.dep. Then there exists ℓ smallest, and $\boldsymbol{\beta}$ s.t. $\boldsymbol{v}_{\ell+1} = \beta_1\boldsymbol{v}_1 + \cdots + \beta_\ell\boldsymbol{v}_\ell$. Note that $\boldsymbol{\beta} \neq 0$. So

$$\begin{array}{lcr}
A\boldsymbol{v}_{\ell+1} = A[\beta_1\boldsymbol{v}_1 + \cdots + \beta_\ell\boldsymbol{v}_\ell] & = & \lambda_1\beta_1\boldsymbol{v}_1 + \cdots + \lambda_\ell\beta_\ell\boldsymbol{v}_\ell \\
\lambda_{\ell+1}\boldsymbol{v}_{\ell+1} & = & \lambda_{\ell+1}\beta_1\boldsymbol{v}_1 + \cdots + \lambda_{\ell+1}\beta_\ell\boldsymbol{v}_\ell \\
\hline
0 & = & [\lambda_{\ell+1} - \lambda_1]\beta_1\boldsymbol{v}_1 + \cdots + [\lambda_{\ell+1} - \lambda_l]\beta_\ell\boldsymbol{v}_\ell.
\end{array}$$

So $\boldsymbol{v}_\ell \in \text{LS}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{\ell-1})$, a contradiction. ■

- The converse of the above is not true. Think of I.
- $0 \in \sigma(A)$ if and only if $\text{NULL}\, A \neq \{\boldsymbol{0}\}$ if and only if A is singular.
- Let A be invertible. Then $\lambda \in \sigma(A)$ if and only if $\lambda^{-1} \in \sigma(A^{-1})$.
- Polynomial $p_A(x) = \text{DET}\,[xI - A]$ is called the characteristic polynomial of A.

The equation $p_A(x) = 0$ is called the characteristic equation of A. So eigenvalues are roots of the characteristic equation.

- Let $\lambda \in \sigma(A)$. The eigenspace for λ is the span of all eigenvectors for λ.

[The dimension of the eigenspace is the geometric multiplicity.]

Q. Is $\sigma(A) = \sigma(A^t)$? Yes, same characteristic polynomial: $p_A(x) = p_{A^t}(x)$.

- Let $A\in\mathcal{M}_n(\mathbb{R})$. Then $p_A(\lambda)=0$ if and only if $p_A(\overline{\lambda})=0$. So $\lambda\in\sigma(A)$ if and only if $\overline{\lambda}\in\sigma(A)$.

[This is not true for complex matrices.]

Similar: A and B are said to be matrices if there exists an invertible matrix S s.t. $A=SBS^{-1}$.

Th. Similar matrices have the same characteristic polynomial.

Pr. Let A be similar to B. So there exists S s.t $B=S^{-1}AS$. So $\text{DET}\,[xI-B]=\text{DET}\,[xI-S^{-1}AS]=\text{DET}\,[S^{-1}(xI-A)S]=\text{DET}\,[xI-A]$. ∎

- Let B be invertible. Then AB is similar to BA, as $BA=B(AB)B^{-1}$. So $p_{AB}(x)=p_{BA}(x)$ and $\sigma(AB)=\sigma(BA)$.
- Let $B=S^{-1}AS$, $\lambda\in\sigma(A)$ and $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k$ be lin.ind eigenvectors of A for λ.

Are $S^{-1}\boldsymbol{v}_1,\ldots,S^{-1}\boldsymbol{v}_k$ lin.ind? Yes, as S is invertible. Is $S^{-1}\boldsymbol{v}_i$ an eigenvector of B? Yes, $B(S^{-1}v_i)=S^{-1}ASS^{-1}v_i=S^{-1}Av_i=\lambda(S^{-1}v_i)$. So the geometric multiplicity of λ in A and B are the same. [Geometry: if A has eigenvector v and $T\boldsymbol{x}=A\boldsymbol{x}$, then $[T]$ has eigenvector $[v]$.]

P. **Geometric and algebraic multiplicity**

Th. Let $\lambda\in\sigma(A)$. Then geom. multipli of $\lambda\le$ algeb multipli of λ.

Pr. Let geometric multiplicity of λ be k and $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k$ be lin.ind eigenvectors for λ. Apply GS: let $\boldsymbol{w}_1,\ldots,\boldsymbol{w}_k$ be the orthonormal eigenvectors for λ. Extend it to an orthonormal basis:

$\{\boldsymbol{w}_1,\ldots,\boldsymbol{w}_k,\boldsymbol{w}_{k+1},\ldots,\boldsymbol{w}_n\}$. Put $P(:,i)=\boldsymbol{w}_i$.

Then $P^*AP=P^*[A\boldsymbol{w}_1\ \cdots\ A\boldsymbol{w}_k\ \ A\boldsymbol{w}_{k+1}\ \cdots\ A\boldsymbol{w}_n]=$

$$\begin{bmatrix}\boldsymbol{w}_1^*\\ \vdots\\ \boldsymbol{w}_k^*\\ \boldsymbol{w}_{k+1}^*\\ \vdots\\ \boldsymbol{w}_n^*\end{bmatrix}[\lambda\boldsymbol{w}_1\ \cdots\ \lambda\boldsymbol{w}_k\ \ *\ \cdots\ *]=\left[\begin{array}{ccc|ccc}\lambda&\cdots&0&*&\cdots&*\\ 0&\ddots&0&*&\cdots&*\\ 0&\cdots&\lambda&*&\cdots&*\\ \hline 0&\cdots&0&*&\cdots&*\\ \vdots&&&&\boldsymbol{D}&\\ 0&\cdots&0&*&\cdots&*\end{array}\right].$$

Now $p_A(x)=p_{P^*AP}(x)=\text{DET}\,(xI-P^*AP)=(x-\lambda)^k\,\text{DET}\,(xI-D)$.

So algeb. multipli of λ in A = algeb. multipli of λ in $P^*AP\ge k$. ∎

P. **Schur unitary triangularization**

Th. Let $A\in\mathcal{M}_n(\mathbb{C})$. Then there exists U unitary, s.t. $T=U^*AU$ is upper triangular. Further, if A and $\sigma(A)$ are real, then U can be real orthogonal.

Pr. Use induction on n. Case $n=1$ is trivial. Let $n>1$. Let $(\lambda_1,\boldsymbol{w}_1)$ be an eigenpair of A, $\|\boldsymbol{w}_1\| = 1$. Take an orthonormal basis $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n\}$ and form $W = [\boldsymbol{w}_1 \quad \cdots \quad \boldsymbol{w}_n]$. Then, $W^*AW = \begin{bmatrix} \lambda_1 & * \\ 0 & A' \end{bmatrix}$, $A' \in \mathcal{M}_{n-1}$.

$W^*AW = \begin{bmatrix} \lambda_1 & * \\ 0 & A' \end{bmatrix}$, $A' \in \mathcal{M}_{n-1}$. By induction hypothesis, there exists U' unitary, s.t. $T' = U'^*A'U'$ is upper triangular.

Put $U = W \begin{bmatrix} 1 & 0 \\ 0 & U' \end{bmatrix}$. Note: U is unitary. Now

$U^*AU = \begin{bmatrix} 1 & 0 \\ 0 & U'^* \end{bmatrix} W^*AW \begin{bmatrix} 1 & 0 \\ 0 & U' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & U'^* \end{bmatrix} \begin{bmatrix} \lambda_1 & * \\ 0 & A' \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & U' \end{bmatrix} = \begin{bmatrix} \lambda_1 & * \\ 0 & U'^*A'U' \end{bmatrix} = \begin{bmatrix} \lambda_1 & * \\ 0 & T' \end{bmatrix}$.

Further: 'if A, λ are real then there exists $\boldsymbol{x} \neq \boldsymbol{0}$ real, s.t. $A\boldsymbol{x} = \lambda\boldsymbol{x}$'. ■

P. Applications: Schur unitary triangularization (SUT)

Remark. In SUT, as $U^*AU = T$, we have

$$\{\lambda_1, \ldots, \lambda_n\} = \sigma(A) = \sigma(T) = \{t_{11}, \ldots, t_{nn}\}.$$

Further, we can get the λ_i's in the diagonal of T in any prescribed order.

Cor. Let $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$. Then $\operatorname{DET} A = \prod \lambda_i$ and $\operatorname{TR} A = \sum \lambda_i$.

Pr. By SUT, there exists U unitary, s.t. $U^*AU = T$ is upper triangular. So $\sigma(T) = \sigma(A)$. So $\sum \lambda_i = \sum t_{ii} = \operatorname{TR}(T) = \operatorname{TR}(U^*AU) = \operatorname{TR}(AUU^*) = \operatorname{TR}(A)$.

And $\prod \lambda_i = \prod t_{ii} = \operatorname{DET} T = \operatorname{DET} U^*AU = \operatorname{DET} A$. ■

- By SUT, $U^*AU = T$ and

$$\{\lambda_1, \ldots, \lambda_n\} = \sigma(A) = \sigma(T) = \{t_{11}, \ldots, t_{nn}\}.$$

- By $\operatorname{DIAG} A$ we denote the diagonal $\{a_{11}, \ldots, a_{nn}\}$ of A. By $\operatorname{DIAG} \boldsymbol{v}$ we denote the diagonal matrix A with $a_{11} = \boldsymbol{v}_1, \ldots, a_{nn} = \boldsymbol{v}_n$.

Th[spectral Theorem]. Let A be normal ($AA^* = A^*A$). Then there exists U unitary, s.t. $U^*AU = D$ is diagonal. Further, $\operatorname{DIAG} D = \sigma(A)$.

Pr. By SUT, there exists U unitary, s.t $U^*AU = T$ is upper triangular. As $A^*A = AA^*$, we get $T^*T = TT^*$. Note: $\sum |t_{1i}|^2 = (TT^*)_{11} = (T^*T)_{11} = |t_{11}|^2$. So $t_{12} = \cdots = t_{1n} = 0$. Repeating the process, T is diagonal. ■

P. Applications: Schur unitary triangularization

Cor. Let A be Hermitian. Then there exists U unitary, s.t. $U^*AU = D$ is <u>real</u> diagonal. Further, if A is real, then U can be chosen as real orthogonal.

Pr. By SUT, we can find U unitary, s.t. $U^*AU = T$ is upper triangular. As $A^* = A$, we get $T^* = T$. So T is real diagonal. ■

Multiply

$$\begin{bmatrix} 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix} = \begin{bmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So $\begin{bmatrix} 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{0}.$

Cor[Cayley-Hamilton]. Every matrix satisfies its characteristic equation.

Pr. Let $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$. By SUT, there exists U unitary, s.t. $U^*AU = T$ is upper triangular; $t_{ii} = \lambda_i$ for each i. Let $p(x)$ be the characteristic polynomial of A. So $p(x) = \prod(x - \lambda_i)$. So $p(A) = \prod(A - \lambda_i I) = \prod(UTU^* - \lambda_i UIU^*) = U\Big[(T - \lambda_1 I)(T - \lambda_2 I)\cdots(T - \lambda_n I)\Big]U^* = U\mathbf{0}U^* = \mathbf{0}$. ■

P. **Diagonalizability**

- We say 'A is diagonalizable' if there exists S invertible, s.t. $S^{-1}AS = D$ is diagonal. Obviously, $\lambda_i = d_{ii}$ are the eigenvalues of A.

Th. $A_{n\times n}$ is diagonalizable if and only if it has n linearly independent eigenvectors.

Pr. $\Rightarrow$: Let A be diagonalizable. So there exists S invertible, s.t. $S^{-1}AS = D$ is diagonal. Put $\boldsymbol{s}_i = S(:, i)$ and $\lambda_i = d_{ii}$. Then

$[A\boldsymbol{s}_1 \cdots A\boldsymbol{s}_n] = A[\boldsymbol{s}_1 \cdots \boldsymbol{s}_n] = AS = SD = [\boldsymbol{s}_1 \cdots \boldsymbol{s}_n]D = [\lambda_1\boldsymbol{s}_1 \cdots \lambda_n\boldsymbol{s}_n]$.

That is, $A\boldsymbol{s}_i = \lambda_i\boldsymbol{s}_i$. So $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n$ are n linearly independent eigenvectors.

$\Leftarrow$: let $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n$ be n linearly independent eigenvectors, $A\boldsymbol{s}_i = \lambda_i\boldsymbol{s}_i$.

Put $S = [\boldsymbol{s}_1 \cdots \boldsymbol{s}_n]$ and $D = \textsc{diag}(\lambda_1, \ldots, \lambda_n)$. Then
$AS = [A\boldsymbol{s}_1 \cdots A\boldsymbol{s}_n] = [\lambda_1\boldsymbol{s}_1 \cdots \lambda_n\boldsymbol{s}_n] = SD$. So $S^{-1}AS = D$. ■

Cor. Let $A_{n\times n}$ have n distinct eigenvalues. Then A is diagonalizable.

Pr. Follows as the corresponding eigenvectors are linearly independent. ■

P. **Non-diagonalizability-Example**

Q. Is $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ diagonalizable? No. Assume it is. Then there exists S invertible, s.t. $S^{-1}AS = D$ is diagonal. As $\sigma(A) = \sigma(S^{-1}AS) = \sigma(D)$, we must have $D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. But then $A = SDS^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, a contradiction.