# Assignment - 7

Q.1] Trignometric form of DTFS is given as:

$$\tilde{x}[n] = a_0 + \sum_{k=1}^{M}\left(a_k \cos\left(\frac{2kn}{N_0}\right) + b_k \sin\left(\frac{2kn}{N_0}\right)\right)$$

where $N_0 = 7$ and hence

$M = 3$

Exponential form of DTFS is given as:

$$\tilde{x}[n] = \sum_{k=0}^{N_0-1} C_k e^{j(2\pi kn/N_0)}$$

where $N_0 = 7$

and $$C_k = \frac{1}{N_0}\sum_{n=0}^{N_0-1} \tilde{x}(n) e^{-2j\pi kn/N_0}$$

Now, $\tilde{x}[0] = \tilde{x}[1] = \tilde{x}[2] = \tilde{x}[3] = 1$

$\tilde{x}[4] = \tilde{x}[5] = \tilde{x}[6] = \tilde{x}[7] = 0$

So, $$C_k = \frac{1}{7}\left(1 + e^{-2j\pi k/N_0} + e^{-4j\pi k/N_0} + e^{-6j\pi k/N_0}\right)$$

$$c_k = \frac{1}{7}\left(\frac{1-\left(e^{-2j\pi k/7}\right)^4}{1-e^{-2j\pi k/7}}\right)$$

$$c_k = \frac{1}{7}\left(\frac{1-e^{-8j\pi k/7}}{1-e^{-2j\pi k/7}}\right)$$

$$\tilde{x}[n] = \sum_{k=0}^{6} c_k\, e^{\frac{2j\pi k n}{7}}$$

$$\tilde{x}[n] = \frac{1}{7}\sum_{k=0}^{6}\left(\frac{1-e^{-8\pi j k/7}}{1-e^{-2\pi j k/7}}\right) e^{2j\pi k n/7}$$

$$\tilde{x}[n] = \frac{1}{7}\sum_{k=0}^{6}\left(1+e^{-4\pi j k/7}\right)\left(1+e^{-2\pi j k/7}\right) e^{2\pi j k n/7}$$

Q.2] (a) $X_1(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_1[n]\, e^{-j\omega n}$

$$= \sum_{n=1}^{\infty} \alpha^n e^{-j\omega n}$$

$$= \alpha e^{-j\omega} + \alpha^2 e^{-2j\omega} \dots$$

$$= \frac{\alpha e^{-j\omega}}{1 - \alpha e^{-j\omega}}$$

$$X_1(e^{j\omega}) = \frac{\alpha e^{-j\omega}}{1 - \alpha e^{-j\omega}}$$

(b) $X_2(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_2[n]\, e^{-j\omega n}$

$$= \sum_{n=-1}^{\infty} \alpha^n e^{-j\omega n}$$

$$= \frac{e^{j\omega}}{\alpha} + 1 + \alpha e^{-j\omega} + \dots$$

$$= 1 + \frac{e^{j\omega}}{\alpha} + \frac{\alpha e^{-j\omega}}{1 - \alpha e^{-j\omega}}$$

$$= \frac{\alpha(1-\alpha e^{-j\omega}) + (1-\alpha e^{-j\omega})e^{j\omega} + \alpha^2 e^{-j\omega}}{\alpha(1-\alpha e^{-j\omega})}$$

$$= \frac{\alpha - \alpha^2 e^{-j\omega} + e^{j\omega} - \alpha + \alpha^2 e^{-j\omega}}{\alpha(1-\alpha e^{-j\omega})}$$

$$X_2(e^{j\omega}) = \frac{e^{j\omega}}{\alpha(1-\alpha e^{-j\omega})}$$

(c) $$X_3(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_3[n]\, e^{-j\omega n}$$

$$= \sum_{n=1}^{\infty} n\alpha^n e^{-j\omega n}$$

$$= \alpha e^{-j\omega} + 2\alpha^2 e^{-2j\omega} \ldots$$

Let $$S = \alpha e^{-j\omega} + 2\alpha^2 e^{-2j\omega} + 3\alpha^3 e^{-3j\omega}$$

$$\alpha e^{-j\omega} S = \alpha^2 e^{-2j\omega} + 2\alpha^3 e^{-3j\omega} \ldots$$

$$(-) \qquad (-) \qquad (-)$$

$$S(1-\alpha e^{-j\omega}) = \alpha e^{-j\omega} + \alpha^2 e^{-2j\omega} + \alpha^3 e^{-3j\omega} \ldots$$

$$S(1-\alpha e^{-j\omega}) = \frac{\alpha e^{-j\omega}}{1-\alpha e^{-j\omega}}$$

$$S = \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2}$$

So, $X_3(e^{j\omega}) = \dfrac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2}$

(d) $X_4(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_4[n] e^{-j\omega n}$

$$= \sum_{n=-1}^{\infty} n\alpha^n e^{-j\omega n}$$

let $S = -\dfrac{e^{j\omega}}{\alpha} + \alpha e^{-j\omega} + 2\alpha^2 e^{-2j\omega} \dots$

So, $S = -\dfrac{e^{+j\omega}}{\alpha} + \dfrac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2}$

$$S = \frac{\alpha^2 e^{-j\omega} - (1+\alpha^2 e^{-2j\omega} - 2\alpha e^{-j\omega})(e^{j\omega})}{\alpha(1-\alpha e^{-j\omega})^2}$$

$$S = \frac{\alpha^2 e^{-j\omega} - (e^{j\omega} + \alpha^2 e^{-j\omega} - 2\alpha)}{\alpha(1-\alpha e^{-j\omega})^2}$$

$$S = \frac{2\alpha - e^{+j\omega}}{\alpha(1-\alpha e^{-j\omega})^2}$$

So, $$X_4(e^{j\omega}) = \frac{2\alpha - e^{j\omega}}{\alpha(1-\alpha e^{-j\omega})^2}$$

Q.3]  $x[n] = \mu[n+1] - \mu[n-1]$

$= \delta[n+1] + \delta[n]$

Now, $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

$X(e^{j\omega}) = e^{j\omega} + 1$

Q.4] (a)

$$x_a[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_a(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\cos(2\omega)\, e^{j\omega n}\, d\omega$$

$$= \frac{2}{2\pi}\int_{-\pi}^{\pi} \cos(2\omega)\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{\pi}\left[\frac{e^{j\omega n}}{(4-n^2)}\left[2\sin(2\omega) + jn\cos(2\omega)\right]\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(4-n^2)}\left(e^{j\pi n}(jn) - e^{-j\pi n}(jn)\right)$$

$$= \frac{jn}{\pi(4-n^2)}\left(e^{j\pi n} - e^{-j\pi n}\right)$$

$$= \frac{jn}{\pi(4-n^2)}\, 2j\sin(\pi n)$$

$$x_a[n] = \frac{2n\sin(\pi n)}{\pi(n^2-4)}$$

(b.) $X_b(e^{j\omega}) = 3\cos(3\omega) + 4\sin(2\omega)$

First we find inverse DTFTs of $\cos(3\omega)$ and $\sin(2\omega)$.

So, $$y[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(3\omega)\, e^{j\omega n}\, d\omega$$

$$y[n] = \frac{-2n\sin(\pi n)}{\pi(n^2-9)}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(2\omega)\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{(4-n^2)}\left[jn\sin(2\omega) - 2\cos(2\omega)\right]\right]_{-\pi}^{\pi}$$

$$= \frac{-2}{2\pi(4-n^2)}\left(e^{j\pi n} - e^{-j\pi n}\right)$$

$$x[n] = \frac{2j\sin(\pi n)}{\pi(n^2-4)}$$

So, $$x_b[n] = 3\,y[n] + 4\,x[n]$$

Q.5] Given: $g[n] \xleftrightarrow{DTFT} G(e^{j\omega})$

Now, $$G(e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n]\, e^{-j\omega n}$$

So, $$G(e^{4j\omega}) = \sum_{n=-\infty}^{\infty} g[n]\, e^{-j4\omega n}$$

$$= \sum_{n=-\infty}^{\infty} g[n]\, e^{-j\omega(4n)}$$

Let $4n = m$, $m \in \mathbb{Z}$

So, $n = \frac{m}{4}$

So, $$G(e^{4j\omega}) = \sum_{m=-\infty}^{\infty} g[m/4]\, e^{-jm\omega}$$

So, $h[n] = g[n/4]$ where $n = 4k$, $k \in \mathbb{Z}$

Q.6] (a) We know that:

$$\alpha^n \mu[n] \xleftrightarrow{z} \frac{1}{1-\alpha z^{-1}}, \quad |z|>|\alpha|$$

By time - shifting property:

$$\alpha^{n-1} \mu[n-1] \xleftrightarrow{z} z^{-1}\left(\frac{1}{1-\alpha z^{-1}}\right)$$

and by linearity property:

$$\alpha^n \mu[n-1] \xleftrightarrow{z} \alpha z^{-1}\left(\frac{1}{1-\alpha z^{-1}}\right)$$

So, $X_1(z) = \dfrac{\alpha}{(z-\alpha)}$

ROC: $|z| > |\alpha|$ ~~except~~ with $z=0$ and $z=\infty$

So, $X_1(e^{j\omega}) = \dfrac{\alpha e^{-j\omega}}{1-\alpha e^{-j\omega}}$ obtained

by replacing $z^{-1}$ by $e^{-j\omega}$

(b) By time-shifting and linearity properties:

$$\alpha^n \mu[n+1] \overset{z}{\longleftrightarrow} \frac{z}{\alpha}\left(\frac{1}{1-\alpha z^{-1}}\right)$$

for $|z| > |\alpha|$

Hence, $$X_2(z) = \frac{z}{\alpha(1-\alpha z^{-1})}$$

ROC: $|z| > |\alpha|$ and ~~except~~ at $z=0$

$$X_2(e^{j\omega}) = \frac{}{\alpha e^{-j\omega}(1-\alpha e^{-j\omega})}$$

$$X_2(e^{j\omega}) = \frac{e^{j\omega}}{\alpha(1-\alpha e^{-j\omega})}$$

(c:) We know:

$$n\alpha^n \mu[n] \overset{z}{\longleftrightarrow} \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}, \quad |z| > |\alpha|$$

$$X_3(e^{j\omega}) = \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2},$$

ROC: $|z| > |\alpha|$

(d:) We know:

$$n\alpha^n \mu[n] \overset{z}{\longleftrightarrow} \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}, \quad |z| > |\alpha|$$

$\cancel{(n+1)\alpha^{n+1}\mu[n+1]}$

$$(n+1)\alpha^{n+1}\mu[n+1] = n\alpha^{n+1}\mu[n+1] + \alpha^{n+1}\mu[n+1]$$

$$X_4(z) = \sum_{n=-\infty}^{\infty} x_4[n] z^{-n}$$

$$= \sum_{n=-1}^{\infty} n\alpha^n z^{-n}$$

$$= \sum_{n=-1}^{\infty} n\left(\frac{\alpha}{z}\right)^n$$

$$X_4(z) = \frac{2\alpha - z}{\alpha(1 - \alpha z^{-1})^2}$$

ROC: $|z| > |\alpha|$ and $z = 0$

So, $$X_4(e^{j\omega}) = \frac{2\alpha - e^{j\omega}}{\alpha(1 - \alpha e^{-j\omega})^2}$$

Q.7] (a) $y_a[n] \overset{z}{\longleftrightarrow} \alpha^3 z^{-3}\left(\frac{1}{1-\alpha z^{-1}}\right), \quad |z|>|\alpha|$

ROC: $|z| > |\alpha|$ and $z = \infty$

$$y_a[n] \overset{z}{\longleftrightarrow} \frac{(\alpha/z)^3}{(1-(\alpha/z))}$$

(b) Now, $\mu[n] \overset{z}{\longleftrightarrow} \frac{1}{1-z^{-1}}, \quad |z|>1$

$$\mu[-n] \overset{z}{\longleftrightarrow} \frac{1}{1-z}, \quad |z|<1$$

$$\alpha\mu[-n] \overset{z}{\longleftrightarrow} \frac{\alpha}{1-z}, \quad |z|<1$$

ROC: $|z| < 1$ and $z = \infty$

Q8] (a) $$X_1(z) = \frac{2+0.4z^{-1}}{1+0.5z^{-1}}$$

Let $$\frac{2+0.4z^{-1}}{1+0.5z^{-1}} = K_1 + \frac{K_2}{1+0.5z^{-1}}$$

$$\frac{2+0.4z^{-1}}{1+0.5z^{-1}} = \frac{(K_1+K_2) + 0.5K_1 z^{-1}}{(1+0.5z^{-1})}$$

$$\Rightarrow \quad K_1 + K_2 = 2$$

$$0.5K_1 = 0.4$$

$$\Rightarrow \quad K_1 = 0.8$$

$$\Rightarrow \quad K_2 = 1.2$$

So, $$X_1(z) = 0.8 + \frac{1.2}{1+0.5z^{-1}}$$

So, $$x_1[n] = 0.8\,\delta[n] + 1.2\,(-0.5)^n \mu[n]$$

$$x_1[n] = \frac{4}{5}\delta[n] + \frac{6}{5}\left(\frac{-1}{2}\right)^n \mu[n]$$

(b)

$$X_2(z) = \frac{3}{1+0.25z^{-2}}$$

$$= \frac{3}{(1)^2 - (0.5jz^{-1})^2}$$

$$= \frac{3}{(1-0.5jz^{-1})(1+0.5jz^{-1})}$$

$$= \frac{3}{2}\left(\frac{1}{1-0.5jz^{-1}} + \frac{1}{1+0.5jz^{-1}}\right)$$

$$x_2[n] = \frac{3}{2}\left((0.5j)^n \mu[n] + (-0.5j)^n \mu[n]\right)$$

$$x_2[n] = \frac{3}{2}(0.5j)^n \mu[n] + \frac{3}{2}(-0.5j)^n \mu[n]$$

(c) $X_3(z) = \dfrac{1}{1 - z^{-4}}$

$$= \frac{1}{(1-z^{-1})(1+z^{-1})(1+z^{-2})}$$

$$= \frac{1}{(1-z^{-1})(1+z^{-1})(1-jz^{-1})(1+jz^{-1})}$$

Let $1 = A(1+z^{-1})(1+z^{-2}) + B(1-z^{-1})(1+z^{-2})$
$+ C(1-z^{-2})(1+jz^{-1})$
$+ D(1-z^{-2})(1-jz^{-1})$

$A = \frac{1}{4}$, $B = \frac{1}{4}$

$C = \frac{1}{4}$, $D = \frac{1}{4}$

So, $x_3[n] = \frac{1}{4}\Big( \mu[n] + (-1)^n \mu[n] + (j)^n \mu[n]$
$+ (-j)^n \mu[n] \Big)$