Assignment - 2

Q.1] $P(t)\, y''(t) + Q(t)\, y'(t) + R(t)\, y(t) + \lambda y(t) = 0$

Multiplying above equation by $\mu(t) > 0$ function:

$$P(t)\,\mu(t)\, y''(t) + Q(t)\,\mu(t)\,y'(t) + R(t)\,\mu(t)\,y(t) + \lambda \mu(t)\, y(t) = 0 \quad \hookrightarrow (1)$$

Consider

$$(p(t)\, y'(t))' + s(t)\, y(t) + \lambda\, r(t)\, y(t) = 0$$

$$\Rightarrow p'(t)\, y'(t) + p(t)\, y''(t) + s(t)\, y(t) + \lambda\, r(t)\, y(t) = 0$$

$$\Rightarrow p(t)\, y''(t) + p'(t)\, y'(t) + s(t)\, y(t) + \lambda r(t)\, y(t) = 0 \quad \hookrightarrow (2)$$

Comparing (1) and (2) :

19 Friday

$$p(t) = P(t)\,\mu(t) \longrightarrow (3)$$

$$p'(t) = P'(t)\,\mu(t) + P(t)\,\mu'(t) \longrightarrow (4)$$

$$p'(t) = Q(t)\,\mu(t) \longrightarrow (5)$$

$$s(t) = R(t)\,\mu(t) \longrightarrow (6)$$

$$\lambda r(t) = \lambda\,\mu(t) \longrightarrow (7.)$$

From (4.) and (5):

$$\mu'(t)\,P(t) = \left(Q(t) - P'(t)\right)\mu(t)$$

20 Saturday

$$\int \frac{\mu'(t)}{\mu(t)} = \int \frac{Q(t) - P'(t)}{P(t)}$$

$$\int \frac{1}{\mu(t)} \frac{d(\mu(t))}{dt}\,dt = \int \frac{\left(Q(t) - P'(t)\right)}{P(t)}\,dt$$

$$\ln|\mu(t)| = \int \left(\frac{Q(t) - P'(t)}{P(t)}\right) dt + C$$

21 Sunday

$$\mu(t) = e^{\int \left(\frac{Q(t) - P'(t)}{P(t)}\right) dt}\, e^{c}$$

where $\mu(t) > 0$ and
$c =$ const. of integration

April 2010

S	M	T	W	T	F	S
				1	2	3
4	5	6	7	8	9	10
11	12	13	14	15	16	17
18	19	20	21	22	23	24
25	26	27	28	29	30	

Monday 22

$c = 0$ can be chosen w.l.o.g.

So, $$\mu(t) = e^{\int \left(\frac{Q(t) - P'(t)}{P(t)} \right) dt}$$

From eqn (3), (6) and (7):

$$p(t) = P(t)\, e^{\int \left(\frac{Q(t) - P'(t)}{P(t)} \right) dt}$$

$$s(t) = R(t)\, e^{\int \left(\frac{Q(t) - P'(t)}{P(t)} \right) dt}$$

$$r(t) = e^{\int \left(\frac{Q(t) - P'(t)}{P(t)} \right) dt} \qquad (\lambda \neq 0)$$

Tuesday 23

w.l.o.g = without loss of generality

February 2010	S	M	T	W	T	F	S
		1	2	3	4	5	6
	7	8	9	10	11	12	13
	14	15	16	17	18	19	20
	21	22	23	24	25	26	27
	28						

24 Wednesday

Q.2] (c.), (d.) are the correct options.

25 Thursday

Q.3] (a) From Q1, we have

$P(t) = t$, $Q(t) = 2$, $R(t) = 0$

So, $p(t) = P(t)\, e^{\int \frac{Q(t)-P'(t)}{P(t)} dt}$

Now, $\int \left(\frac{Q(t) - P'(t)}{P(t)} \right) dt = \int \frac{2-1}{t} dt$

$= \ln(|t|)$

So, $e^{\ln(|t|)} = |t|$

For $t \geq 0$, $|t| = t$

So, $p(t) = t \cdot t$

$= t^2$

$s(t) = 0$

$r(t) = t$

So, the SLEVP is :

$(t^2 y'(t))' + \lambda t y(t) = 0$

S	M	T	W	T	F	S
	1	2	3	4	5	6
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28						

29 Monday

For $t<0$, $|t| = -t$

$$p(t) = -t^2$$
$$s(t) = 0$$
$$r(t) = -t$$

So, the SLEVP is:

$$(-t^2 y'(t))' - \lambda t\, y(t) = 0$$

$$(t^2 y'(t))' + \lambda t\, y(t) = 0$$

30 Tuesday

Q.4.] (a.) Only possible eigenvalues are when $\lambda > 0$

General solution is :

$$y(t) = A\cos(\sqrt{\lambda}\,t) + B\sin(\sqrt{\lambda}\,t)$$

For $y(0) = 0 \Rightarrow A = 0$

$y(L) = 0 \Rightarrow B\sin(\sqrt{\lambda}\,L) = 0$

$\Rightarrow \sqrt{\lambda}\,L = n\pi$ where $n \in \mathbb{N}$

$\Rightarrow \sqrt{\lambda} = \dfrac{n\pi}{L}$ where $n \in \mathbb{N}$

So, $y_n(t) = B\sin\left(\dfrac{n\pi}{L}t\right)$, $n \in \mathbb{N}$

16 Friday

8.4] (c) $\frac{y''(t)}{t} - \frac{y'(t)}{t^2} + (\lambda+1)\frac{y(t)}{t^3} = 0, \quad t \in (1, e^{\pi})$

$$t^2 y''(t) - t y'(t) + (\lambda+1) y(t) = 0, \quad t \in (1, e^{\pi})$$

This is a Homogeneous Euler-Cauchy D.E. Hence, characteristic equation:

$$m^2 - 2m + (\lambda+1) = 0$$

$$m = \frac{2 \pm \sqrt{4 - 4(\lambda+1)}}{2}$$

$$= \frac{2 \pm \sqrt{-4\lambda}}{2}$$

17 Saturday

[Variable x is used just for calculations]

Since, this is a RSLEVP, $\lambda \in \mathbb{R}$.

Case I: $\lambda < 0$.

So, $m = \frac{2 \pm 2\sqrt{-\lambda}}{2}$

$$= 1 \pm \sqrt{-\lambda}$$

18 Sunday

So, $y(x) = A x^{1+\sqrt{-\lambda}} + B x^{1-\sqrt{-\lambda}}$

$y(1) = 0 \Rightarrow 0 = A + B$

$y(e^{\pi}) = 0 \Rightarrow 0 = A e^{\pi\sqrt{-\lambda}} + B e^{-\pi\sqrt{-\lambda}}$

$\Rightarrow A\left(e^{2\pi\sqrt{-\lambda}} - 1\right) = 0$

$\Rightarrow A = 0$

$\Rightarrow B = 0$

Hence, $y(x) = 0$

Hence, $\lambda < 0$ is not possible.

Case II: $\lambda = 0$

So, $m = 1$ (Repeated)

So, $y(x) = Ax + Bx\ln(x)$

$y(1) = 0 \Rightarrow A = 0$

$y(e^{\pi}) = 0 \Rightarrow B = 0$

$\Rightarrow y(x) = 0$

Hence, $\lambda = 0$ is not possible.

Case III : $\lambda > 0$

So, $m = 1 \pm i\sqrt{\lambda}$

So, $y(x) = Ax\cos(\sqrt{\lambda}\ln(x)) + Bx\sin(\sqrt{\lambda}\ln(x))$

$y(1) = 0 \implies A = 0$

$y(e^{\pi}) = 0 \implies 0 = B\sin(\sqrt{\lambda}\pi)$

$\implies \sqrt{\lambda}\pi = n\pi \quad , \quad n \in \mathbb{N}$

So, $\sqrt{\lambda} = n \quad , \quad n \in \mathbb{N}$

So, $y_n(t) = Bt\sin(n\ln(t))$, $n \in \mathbb{N}$
is an eigenfunction for
eigenvalue λ

So, $y_n(t) = Bt\sin(n\ln(t))$ is a
solution for $\lambda > 0$
eigenvalues.

Q.5.] Given: $\{\sin(nx), \cos(nx)\}_{n\in\mathbb{N}}$ is a orthogonal family on $(-\pi,\pi)$

Consider $\sin(nx)$, $\cos(mx)$ for any $n,m\in\mathbb{N}$

So, $\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,dx = 0$ is given.

let $\pi t = x$

$\pi dt = dx$

$$\int_{-1}^{1} \pi \sin(nt\pi)\cos(mt\pi)\,dt = 0$$

$$\Rightarrow \int_{-1}^{1} \sin(nt\pi)\cos(mt\pi)\,dt = 0 \quad \text{for any } n,m\in\mathbb{N}$$

$\hookrightarrow$ (1.)

6 Friday

Now, consider $\sin(nx)$, $\sin(mx)$ for any $n, m \in \mathbb{N}$ s.t $n \neq m$

So, $\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = 0$ is given.

Let $\pi t = x$

$\pi dt = dx$

$$\int_{-1}^{1} \pi \sin(nt\pi)\sin(mt\pi)\,dt = 0$$

7 Saturday

$$\Rightarrow \int_{-1}^{1} \sin(nt\pi)\sin(mt\pi)\,dt = 0$$ for any $n, m \in \mathbb{N}$ s.t $n \neq m$

$\hookrightarrow$ 2.

8 Sunday

Now, consider ~~dx~~

Now, consider $\cos(nx)$ and $\cos(mx)$ for any $n, m \in \mathbb{N}$ s.t $n \neq m$

So, $\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = 0$ is given.

Let $t\pi = x$

$\pi\,dt = dx$

$$\int_{-1}^{1} \pi \cos(nt\pi)\cos(mt\pi)\,dt = 0$$

$$\Rightarrow \int_{-1}^{1} \cos(nt\pi)\cos(mt\pi)\,dt = 0$$ for any $n, m \in \mathbb{N}$ s.t $n \neq m$

3

Combining 1, 2 and 3, we get that if $\{\sin(nx), \cos(nx)\}_{n\in\mathbb{N}}$ is orthogonal on $(-\pi, \pi)$ then $\{\sin(n\pi x), \cos(n\pi x)\}_{n\in\mathbb{N}}$ is orthogonal family on $(-1, 1)$

Q.6] Consider 2 DEs :

$$y''(t) + y(t) = 0 \rightarrow (1) \text{ and}$$
$$y''(t) + (e^{t^2}+1)\, y(t) = 0 \rightarrow (2)$$

Now, $e^{t^2}+1 > 1 \quad \forall t \in \mathbb{R}$

Solution of DE (1) is:

$$m^2+1=0$$
$$\Rightarrow m = \pm i$$

So, $y(t) = A\sin(t+\phi)$

W.l.o.g $\phi = 0$ and $A = 1$ can be assumed.

$$y(t) = \sin(t)$$

$\sin(t)$ has ∞ many zeros in $\mathbb{R}$.

Now, consider $y_1(t)$ as a solution of (2). By Strum Comparison Theorem:

Between any 2 zeros of $\sin(t)$ $\exists$ atleast 1 zero of $y_1(t)$.

$\because$ $\sin(t)$ has ∞ zeros, it follows that $y_1(t)$ has ∞ zeros in $\mathbb{R}$.

10 Monday

[Variable x is used for intermediate calculations]

Q.7.] Consider the DE:

$$t^2 y''(t) + t y'(t) + \lambda y(t) = 0 \quad \text{on } (1, e)$$
$$\text{with} \quad y(1) = y(e) = 0$$

This is a Homogeneous Euler Cauchy D. Equation. Hence, characteristic eqn:

$$m^2 + \lambda = 0$$

Case 1: $\lambda < 0$

Then, $m = \pm\sqrt{-\lambda}$

11 Tuesday

So, $y(x) = Ax^{-\sqrt{-\lambda}} + Bx^{\sqrt{-\lambda}}$

$y(1) = 0 \implies 0 = A + B$

$y(e) = 0 \implies 0 = Ae^{-\sqrt{-\lambda}} + Be^{\sqrt{-\lambda}}$

$\implies 0 = Be^{2\sqrt{-\lambda}} + A$

$\implies 0 = B\left(e^{2\sqrt{-\lambda}} - 1\right)$

$\implies B = 0$

$\implies A = 0 \implies y(x) = 0$ on $(1, e)$

Hence, $\lambda < 0$ is not possible for eigenvalue.

Case 2: $\lambda = 0$

$\Rightarrow \quad m = 0$

Hence, $y(x) = A + B\ln(x)$

$y(1) = 0 \Rightarrow A = 0$

$y(e) = 0 \Rightarrow B = 0$

Hence, $y(x) = 0$ on $(1, e)$

Hence, $\lambda = 0$ eigenvalue is not possible.

Case 3: $\lambda > 0$

$\Rightarrow \quad m = \pm i\sqrt{\lambda}$

So, $y(x) = A\cos(\sqrt{\lambda}\ln(x)) + B\sin(\sqrt{\lambda}\ln(x))$

$y(1) = 0 \Rightarrow A = 0$

$y(e) = 0 \Rightarrow 0 = B\sin(\sqrt{\lambda})$

So, $\sqrt{\lambda} = n\pi, \quad n \in \mathbb{N}$

So, $y_n(t) = B\sin(n\pi\ln(t))$ are solutions, for eigenvalues $\lambda = (n\pi)^2$

$\because$ distinct eigenvalues's eigenfunctions are orthogonal to each other.

Hence, the family $\{\sin(n\pi\log(t))\}_{n\in\mathbb{N}}$ is an orthogonal family.

14 Friday

Consider $\int_1^e \sin^2(n\pi \ln(t))\, dt$

let $\ln(t) = x$

$dt = t\,dx$

$= e^x dx$

So, $\int_1^e \sin^2(n\pi \ln(t))\, dt = \int_0^1 \sin^2(n\pi x)\, e^x dx$

$= \frac{1}{2}\int_0^1 e^x(1-\cos(2n\pi x))\, dx$

15 Saturday

$= \frac{1}{2}\left[\int_0^1 e^x dx - \int_0^1 e^x \cos(2n\pi x)\, dx\right]$

$= \frac{1}{2}\left[(e-1) - \frac{e^x}{1+(2n\pi)^2}\left(\cos(2n\pi x) + 2n\pi \sin(2n\pi x)\right)\Big|_0^1\right]$

16 Sunday

$= \frac{1}{2}\left[(e-1) - \left(\frac{1}{1+4n^2\pi^2}\right)\left[e-1\right]\right]$

$= \frac{1}{2}\left[\frac{(e-1) + 4n^2\pi^2(e-1) - (e-1)}{1+4n^2\pi^2}\right]$

$$= \frac{2n^2\pi^2(e-1)}{4n^2\pi^2+1}$$

No, it is not orthonormal. It can be made orthonormal by dividing $\sin(n\pi\ln(t))$ by $\sqrt{\dfrac{2n^2\pi^2(e-1)}{4n^2\pi^2+1}}$

Q.8] Consider :

$$(p(t)\,\phi'(t))' + q(t)\,\phi(t) + \mu\,\sigma(t)\,\phi(t) = 0$$

Multiply by $\phi(t)$ on both sides:

$$\phi(t)\,(p(t)\,(\phi'(t)))' + q(t)\,(\phi(t))^2 + \mu\,\sigma(t)\,(\phi(t))^2 = 0$$

Integrate from a to b on both sides:

$$\int_a^b \phi(t)\,(p(t)\,\phi'(t))'\,dt + \int_a^b q(t)\,(\phi(t))^2\,dt + \int_a^b \mu\,\sigma(t)\,(\phi(t))^2\,dt = 0$$

$$\phi(t)\,p(t)\,\phi'(t)\Big|_a^b - \int_a^b p(t)\,(\phi'(t))^2\,dt + \int_a^b q(t)(\phi(t))^2\,dt + \int_a^b \mu\,\sigma(t)(\phi(t))^2\,dt = 0$$

$$\mu \int_a^b \sigma(t)(\phi(t))^2\,dt = -p(t)\,\phi(t)\,\phi'(t)\Big|_a^b + \int_a^b \left(p(t)(\phi'(t))^2 - q(t)(\phi(t))^2\right)dt$$

$$\mu = \frac{-p(t)\,\phi(t)\,\phi'(t)\Big|_a^b + \int_a^b \left(p(t)(\phi'(t))^2 - q(t)(\phi(t))^2\right)dt}{\int_a^b \sigma(t)(\phi(t))^2\,dt}$$

28 Friday

With $q(t) = 0$ on (a, b) and $y(a) = 0$, $y'(b) = 0$:

$$\mu = \frac{-p(t)\phi(t)\phi'(t)\Big|_a^b + \int_a^b p(t)(\phi'(t))^2 dt}{\int_a^b \sigma(t)(\phi(t))^2 dt}$$

$\because$ $\phi(a) = 0$ and $\phi'(b) = 0$:

29 Saturday

$$\mu = \frac{\int_a^b p(t)(\phi'(t))^2 dt}{\int_a^b \sigma(t)(\phi(t))^2 dt}$$

$\because$ it is a RSLEVP:

$p(t) > 0$ and $\sigma(t) > 0$ on $[a, b]$

30 Sunday

Hence, $\mu \geq 0$

For $\mu = 0 \Rightarrow \int_a^b p(t)(\phi'(t))^2 dt = 0$

$\Rightarrow \quad p(t)(\phi'(t))^2 = 0 \quad$ on (a,b)

$\Rightarrow \quad (\phi'(t))^2 = 0 \quad$ on (a,b)

$\Rightarrow \quad \phi'(t) = 0 \quad$ on (a,b)

$\Rightarrow \quad \phi(t) =$ constant on (a,b)

$\because \quad \phi(a) = 0$

$\Rightarrow \quad \phi(t) = 0 \quad$ on (a,b)

But $\phi(t) = 0$ on (a,b) is a trivial solution.

Hence, $\mu = 0$ is not possible eigenvalue.

Hence, $\mu > 0$

$\Rightarrow \mu$ is strictly positive