Ordinary Differential Equation

Assignment 1

(1) Classify each of the following differential equations linear, nonlinear and specify the orders.
(i) $y'' + y \sin x = 0$ (ii) $y'' + x \sin y = 0$ (iii) $y' = \sqrt{1+y}$
(iv) $y'' + (y')^2 + y = x$ (v) $y'' + x(y') = \cos y'$ (vi) $(xy')' = xy$

(2) Verify that $y = -\frac{1}{x+c}$ is general solution of $y' = y^2$. Find particular solutions such that (i) $y(0) = 1$ and (ii) $y(0) = -1$. In both cases find the largest interval I on which y is defined.

(3) For each of the following differential equations draw several isoclines and sketch some solution curves.
(i) $y' = -\frac{x}{y}$ (ii) $y' = x^2 + y^2$.

(4) Show that the following families of curves are self-orthogonal:
(i) $y^2 = 4c(x + c)$ (ii) $x^2/c^2 + y^2/(c^2 - 1) = 1$

(5) Find the family of oblique trajectories which intersect the family of straight lines $y = cx$ at an angle of 45^o.

(6) Find general solution of the following differential equations:

$$(i)\ (x + 2y + 1) - (2x + y - 1)y' = 0 \quad (ii)\ y' = (8x - 2y + 1)^2/(4x - y - 1)^2$$

Assignment 2

(1) Show that the set of solutions of the homogeneous linear equation, $y' + P(x)y = 0$ on an interval $I = [a, b]$ form a vector subspace W of the real vector space of continuous functions on I. What is the dimension of W?

(2) Show that the equation $(3y^2 - x) + 2y(y^2 - 3x)y' = 0$ admits an integrating factor which is a function of $(x + y^2)$. Hence solve the differential equation.

(3) Solve $xy' + y(x + 1) + xy^5 = 0,\ \ y(1) = 1$.

(4) Reduce the following differential equations into linear form and solve:
(i) $y^2y' + y^3/x = \sin x$ (ii) $y' \sin y + x \cos y = x$ (iii) $y' = y(xy^3 - 1)$

(5) Let $f(x, y)$ be continuous on the closed rectangle $R : |x - x_0| \le a,\ |y - y_0| \le b$.
(i) Show that y is a solution of the initial value problem $y' = f(x, y),\ y(x_0) = y_0$ <u>iff</u>

$$y(x) = y_0 + \int_{x_0}^{x} f[t, y(t)]\, dt.$$

(ii) Let $|f(x, y)| \le M$ and $y_n(x) = y_0 + \displaystyle\int_{x_0}^{x} f[t, y_{n-1}(t)]\, dt$, with $y_0(x) = y_0$. Show by the method of induction that $|y_n(x) - y_0| \le b$ for $|x - x_0| \le h$, where $h = \min\{a, b/M\}$.

(6) Solve $y' = (y-x)^{2/3}+1$. Show that $y = x$ is also a solution. What can be said about the uniqueness of the initial value problem consisting of the above equation with $y(x_0) = y_0$, where (x_0, y_0) lies on the line $y = x$.

(7) Discuss the existence and uniqueness of the solution of the inital value problem

$$(x^2 - 2x)y' = 2(x - 1)y, \qquad y(x_0) = y_0.$$

Assignment 3

(1) Find the curve $y = y(x)$ passing through origin for which $y'' = y'$ and the line $y = x$ is tangent at the origin.

(2) If $p(x), q(x), r(x)$ are continuous functions on an interval $\mathcal{I}$, then show that the set of solutions of the following linear homogeneous equation is a real vector space:

$$y''+p(x)y'+q(x)y = 0, \qquad x \in \mathcal{I}. \qquad (*)$$

Also show that the set of solutions of the linear non-homogeneous equation

$$y''+p(x)y'+q(x)y = r(x), \qquad x \in \mathcal{I} \qquad (\#)$$

is not a real vector space. Further, suppose $y_1(x), y_2(x)$ are any two solutions of (#). Obtain conditions on the constants a and b so that $ay_1 + by_2$ is also its solution.

(3) (a) Show that a solution to (*) with x-axis as tangent at any point in $\mathcal{I}$ must be identically zero on $\mathcal{I}$.
(b) Let $y_1(x), y_2(x)$ be two solutions of (*) with a common zero at any point in $\mathcal{I}$. Show that y_1, y_2 are linearly dependent on $\mathcal{I}$.
(c) Show that $y = x$ and $y = \sin x$ are not a pair solutions of equation (*), where $p(x), q(x)$ are continuous functions on $\mathcal{I} = (-\infty, \infty)$.

(4) Let $y_1(x), y_2(x)$ are two linearly independent solutions of (*). Show that
(i) between consecutive zeros of y_1, there exists a unique zero of y_2;
(ii) $\phi(x) = \alpha y_1(x) + \beta y_2(x)$ and $\psi(x) = \gamma y_1(x) + \delta y_2(x)$ are two linearly independent solutions of (*) iff $\alpha\delta \neq \beta\gamma$.

Assignment 4

(1) Verify that $y = x^2 \sin x$ and $y = 0$ both are solutions of the initial value problem

$$x^2 y'' - 4xy' + (x^2 + 6)y = 0, \quad y(0) = y'(0) = 0.$$

Does it contradict the uniqueness?

(2) Find a particular solution of each of the following equations.
(i) $y'' - 4y' + 3y = 6e^{3x}$ (ii) $y'' - 2y' + 5y = 25x^2 + 12$
(iii) $y'' - y = e^{-x}(\sin x + \cos x)$ (iv) $y'' - y' + 3y = x^2 e^x$

(3) By using the method of variation of parameters, find the general solution of:
(i) $y'' + 4y = 2\cos^2 x + 10e^x$ (ii) $y'' + y = x \sin x$
(iii) $y'' + y = \cot^2 x$ (iv) $x^2 y'' - x(x+2)y' + (x+2)y = x^3, \quad x > 0.$

Assignment 5

(1) The equation $y'' + y' - xy = 0$ has a power series solution of the form $y = \sum a_n x^n$.
(i) Find the power series solutions $y_1(x)$ and $y_2(x)$ such that $y_1(0) = 1, y_1'(0) = 0$ and $y_2(0) = 0, y_2'(0) = 1$.
(ii) Find the radius of convergence for $y_1(x)$ and $y_2(x)$.

(2) Consider the equation $(1 + x^2)y'' - 4xy' + 6y = 0$.
(i) Find its general solution in the form $y = a_0 y_2(x) + a_1 y_1(x)$, where $y_1(x)$ and $y_2(x)$ are power series.
(ii) Find the radius of convergence for $y_1(x)$ and $y_2(x)$.

(3) The function on the left side of $\dfrac{1}{\sqrt{1 - 2xt + t^2}} = \displaystyle\sum_{n=0}^{\infty} P_n(x) t^n$ is called the generating function of the Legendre polynomial P_n. Using this relation, show that
(i) $(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$ (ii) $nP_n(x) = xP_n'(x) - P_{n-1}'(x)$
(iii) $P_{n+1}'(x) - xP_n'(x) = (n+1)P_n(x)$ (iv) $P_n(1) = 1, \ P_n(-1) = (-1)^n$
(v) $P_{2n+1}(0) = 0, \ P_{2n}(0) = (-1)^n \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2^n n!}$

Assignment 6

(1) Locate and classify the singular points in the following:
(i) $x^3(x-1)y'' - 2(x-1)y' + 3xy = 0$ (ii) $(3x+1)xy'' - xy' + 2y = 0$

(2) For each of the following, verify that the origin is a regular singular point and find two linearly independent solutions:
(a) $9x^2 y'' + (9x^2 + 2)y = 0$ (b) $x^2(x^2 - 1)y'' - x(1 + x^2)y' + (1 + x^2)y = 0$
(c) $xy'' + (1 - 2x)y' + (x - 1)y = 0$ (d) $x(x-1)y'' + 2(2x-1)y' + 2y = 0$

(3) Let $u(x)$ be any nontrivial solution of $u'' + q(x)u = 0$ on a closed interval $[a, b]$. Show that $u(x)$ has at most a finite number of zeros in $[a, b]$.
(4) Show that any nontrivial solution of $u'' + q(x)u = 0,\ q(x) < 0$ has at most one zero.
(5) Let y_ν be a nontrivial solution of Bessel's equation of order ν on the positive x-axis. Show that (i) if $0 \le \nu < 1/2$, then every interval of length π contains at least one zero of $y_\nu(x)$; (ii) if $\nu = 1/2$, then the distance between successive zeros of y_ν is exactly π; and (iii) if $\nu > 1/2$, then every interval of length π contains at most one zero of $y_\nu(x)$.
(6) Show that the Bessel functions J_ν $(\nu \ge 0)$ satisfy

$$\int_0^1 x J_\nu(\lambda_m x) J_\nu(\lambda_n x)\, dx = \frac{1}{2} J_{\nu+1}^2(\lambda_n)\delta_{mn},$$

where λ_i are the positive zeros of J_ν.

Assignment 7

(1) Find the Laplace transforms :

(a) $f(t) = \begin{cases} 1, & 0 < t < \pi, \\ 0, & \pi < t < 2\pi, \\ \cos t, & t > 2\pi, \end{cases}$ (b) $f(t) = \begin{cases} 0, & 0 < t < 1, \\ \cos(\pi t), & 1 < t < 2, \\ 0, & t > 2 \end{cases}$

(2) Find the inverse Laplace transforms of

(a) $\tan^{-1}(a/s)$, (b) $\ln \dfrac{s^2+1}{(s+1)^2}$, (c) $\dfrac{s+2}{(s^2+4s-5)^2}$, (d) $\dfrac{se^{-\pi s}}{s^2+4}$, (e) $\dfrac{(1-e^{-2s})(1-3e^{-2s})}{s^2}$.

(3) Solve the integral equations:

(a) $y(t) + \displaystyle\int_0^t y(\tau)\, d\tau = u(t-a) + u(t-b)$

(b) $e^{-t} = y(t) + 2\displaystyle\int_0^t \cos(t-\tau) y(\tau)\, d\tau$

(c) $3\sin 2t = y(t) + \displaystyle\int_0^t (t-\tau) y(\tau)\, d\tau$

(4) Sketch the following functions and find their Laplace transforms:

(a) $f(t) = \begin{cases} u(t) - 2u(t-1), & 0 \le t < 2, \\ f(t-2), & t > 2, \end{cases}$ (b) $f(t) = \begin{cases} t[u(t) - u(t-1)], & 0 \le t < 2, \\ f(t-2), & t > 2, \end{cases}$

(c) $f(t) = \begin{cases} tu(t) - 2(t-1)u(t-1), & 0 \le t < 2, \\ f(t-2), & t > 2. \end{cases}$