Axiom System (AX) – Propositional logic.

(A1) $\alpha \to (\beta \to \alpha)$

(A2) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$

(A3) $(\neg\beta \to \neg\alpha) \to ((\neg\beta \to \alpha) \to \beta)$

Inference Rule $\dfrac{\alpha \quad \alpha \to \beta}{\beta}$ MP.

Axiom scheme : $p \to (q \to p)$ is an instance of (A1)

A derivation of α using AX is a finite sequence $\beta_1, \beta_2, \ldots, \beta_n$ s.t

1. $\beta_n = \alpha$
2. $\forall i \in \{1, \ldots, n\}$ β_i is either an instance of one of the axioms A1 – A3 or β_i is deduced by MP to formulas β_j, β_k where $j, k < i$.

Notation: $\vdash_{AX} \alpha$ denotes - α is derivable in AX

Example : $\vdash p \to p$.

1 $(p \to ((p \to p) \to p)) \to ((p \to (p \to p)) \to (p \to p))$ (A2)
2. $p \to ((p \to p) \to p)$ (A1)
3. $(p \to (p \to p)) \to (p \to p)$ MP 1, 2
4. $p \to (p \to p)$ (A1)
5. $p \to p$ MP 3, 4

"Hilbert-style" axiomatisation

- Many axioms and very few inference rules

Natural deduction - "Gentzen style" axiomatisation

- Many inference rules and very few axioms.

Theorem. For all $\alpha \in \Phi$, $\vdash_{Ax} \alpha$ iff $\models \alpha$.

Lemma (Soundness): if $\vdash_{Ax} \alpha$ then $\models \alpha$.

Proof. Induction on the length of the derivation.

Suffices to show

- All axioms define valid formulas
- The inference rule (MP) preserves validity.

[Similar argument to what we saw earlier]

Lemma (Completeness). if $\models \alpha$ then $\vdash_{Ax} \alpha$.

α is consistent if $\nvdash \neg\alpha$

Lemma (Henkin). For all $\beta \in \Phi$, if β is consistent then β is satisfiable.

Claim. Completeness follows from Henkin's Lemma.

Proof. Suppose $\nvDash \alpha$. We can show that $\vdash \neg\neg\alpha \to \alpha$

if $\nvdash \alpha$ then $\nvdash \neg\neg\alpha$. Otherwise we can use MP to derive α from $\vdash \neg\neg\alpha \to \alpha$ and $\vdash \neg\neg\alpha$.

Since $\nvdash \neg(\neg\alpha)$, we have $\neg\alpha$ is consistent.

By Henkin's Lemma, $\neg\alpha$ is satisfiable.

This implies that α is not valid.

Proves the contrapositive of Lemma (Completeness)

Maximal Consistent Sets - extend consistency to sets.

A finite set $X = \{\alpha_1, \alpha_2, --, \alpha_n\}$ is Consistent if $\nvdash \neg(\alpha_1 \wedge \alpha_2 \wedge -- \wedge \alpha_n)$ (i.e $\alpha_1 \wedge \alpha_2 \wedge -- \wedge \alpha_n$ is Consistent)

$X \subseteq \Phi$ is Consistent if every finite subset of X is Consistent.

$X \subseteq \Phi$ is a maximal Consistent set (MCS) iff X is Consistent and $\forall \alpha \notin X$, $X \cup \{\alpha\}$ is inconsistent.

Lemma (Lindenbaum). Every consistent set can be extended to an MCS.

Proof. Let $\alpha_0, \alpha_1, \alpha_2, \ldots$ be an enumeration of Φ

Let $X \subseteq \Phi$ be a consistent set.

Define an infinite sequence of sets: $X_0, X_1, X_2, \ldots$ as

- $X_0 = X$
- For $i \geq 0$, $X_{i+1} = \begin{cases} X_i \cup \{\alpha_i\} & \text{if } X \cup \{\alpha_i\} \text{ is consistent} \\ X_i & \text{otherwise} \end{cases}$

Observation. $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$ and X_i is consistent for all $i \geq 0$.

Let $Y = \bigcup_{i \geq 0} X_i$

Claim. Y is an MCS extending X.

To prove:

- Y is consistent
- Y is maximal.

Claim 1. Y is Consistent.

Suppose not, then $\exists Z \subseteq_{FIN} Y$ s.t Z is inconsistent.

Let $Z = \{\beta_1, \beta_2, \ldots \beta_n\} = \underline{\{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_n}\}}$

where indices match the enumeration of Φ

Let $j = \max(i_1, i_2, \ldots, i_n)$ then

$Z \subseteq_{FIN} X_{j+1}$ (in the sequence $X_0 \subseteq X_1 \subseteq \cdots \subseteq Y$).

This implies that X_{j+1} is inconsistent – A contradiction

Claim 2. Y is maximal

Suppose $Y \cup \{\beta\}$ is consistent for some $\beta \notin Y$.

According to our enumeration of Φ, $\beta = \alpha_j$ for some j.

Since $\alpha_j \notin Y$, α_j was not added at step $j+1$ in the Construction of Y.

$\Rightarrow X_j \cup \{\alpha_j\}$ is inconsistent

$\therefore \exists Z \subseteq_{FIN} X_j$ s.t $Z \cup \{\alpha_j\}$ is inconsistent.

Since $X_j \subseteq Y$ we have $Z \subseteq_{FIN} Y$

Contradicts the assumption that $Y \cup \{\alpha_j\}$ is Consistent.

Properties of MCS.

Lemma 3. Let X be a MCS. Then:

1. for all α, $\alpha \in X$ iff $\neg\alpha \notin X$
2. for all α, β, $\alpha \vee \beta \in X$ iff $\alpha \in X$ or $\beta \in X$.

MCS and valuations.

With every MCS, X we can associate a canonical valuation ϑ_X.

$$\vartheta_X(p) = \begin{cases} \top & \text{if } p \in X \\ \bot & \text{otherwise} \end{cases} \quad \Big/ \quad \vartheta_X = \{p \in P \mid p \in X\}.$$

Lemma 4. Let X be a MCS. For all $\alpha \in \Phi$,
$\vartheta_X \models \alpha$ iff $\alpha \in X$.

Proof. Induction on structure of α.

Base case $\alpha = p$. $\vartheta_X \models p$ iff $p \in X$ (Defn of ϑ_X).

Induction step.

$\alpha = \neg\beta$ $\quad \vartheta_X \models \neg\beta$ iff $\vartheta_X \not\models \beta$ (Semantics)
iff $\beta \notin X$ (I H)
iff $\neg\beta \in X$ (property of MCS) Lemma 3.

$\alpha = \beta \vee \gamma$ $\quad \vartheta_X \models \beta \vee \gamma$ iff $\vartheta_X \models \beta$ or $V_X \models \gamma$ (Semantics)

iff $\beta \in X$ or $\gamma \in X$ (I H)

iff $\beta \vee \gamma \in X$ (Lemma 3)

Conclusion. Every MCS defines a valuation.
Conversly, every valuation ϑ also defines a canonical MCS $\quad X_\vartheta = \{\alpha \mid V \models \alpha\}$.

Lemma 4. Let X be a MCS. For all $\alpha \in \Phi$,
$\vartheta_X \models \alpha$ iff $\alpha \in X$.

Lemma (Henkin). For all $\beta \in \Phi$, if β is consistent then β is satisfiable.

Proof. Suppose β is consistent.

By Lemma (Lindenbaum) $\{\beta\}$ can be extended to an MCS X.

By Lemma 4 $\vartheta_X \models \beta$ since $\beta \in X$.

$\therefore$ β is satisfiable.