First order logic:

Syntax.

First order language.

$L = (R, F, C)$ where $R = \{r_1, r_2, --\}$, $F = \{f_1, f_2, ---\}$ and $C = \{c_1, c_2, ---\}$.

Countable sets.

Each symbol $r \in R$ and $f \in F$ has an associated arity - number of arguments the symbol takes.

Notation $\#(r)$, $\#(f)$ - arity of r / f.

Example. E has arity 2 since it is a binary relation

$Var = \{v_1, v_2, ---\}$ Countable set of variables.

FO formula over an FO language:

Atomic formulas + Connective $\neg, \vee, \exists$ - existential quantifier

Define Terms.

Terms

Let $L = (R, F, C)$ be a FO language.

Set of terms over L is the smallest set satisfying

- Every constant symbol $c \in C$ is a term
- Every variable $x \in Var$ is a term
- if $t_1, t_2, \ldots t_n$ are terms and $f \in F$ is a function symbol of arity n then $f(t_1, \ldots, t_n)$ is a term.

Closed term: A term that does not have any variables.

Atomic Formulas.

Let $L = (R, F, C)$ be a FOL. Atomic formulas over L are defined as:

- if $r \in R$ is of arity n and $t_1, t_2, \ldots t_n$ are terms over L then $r(t_1, t_2, \ldots t_n)$ is an atomic formula.
- if t_1, t_2 are terms over L then $t_1 \equiv t_2$ is an atomic formula.

FO Formulas over L.

Φ_L is the smallest set satisfying the following:

- Every atomic formula over L is in Φ_L.
- if $\varphi \in \Phi_L$ then $\neg\varphi \in \Phi_L$
- if $\varphi, \psi \in \Phi_L$ then $\varphi \vee \psi \in \Phi_L$
- if $\varphi \in \Phi_L$ and $x \in Var$ then $\exists x . \varphi \in \Phi_L$

There exists x

Dual of $\exists$ is $\forall$ (for all)

$$\forall x\, \varphi \overset{\text{def}}{=} \neg \exists x \neg \varphi .$$