```
BFS(G, s)        // G = (V, E), s ∈ V

  for all v ∈ V do
        v.col = white
        v.d   = ∞
        v.π   = nil

  s.col = Grey
  s.d   = 0
  s.π   = nil
  Enqueue(Q, s)
  While notempty(Q) do

     x = dequeue(Q)
     for all y in 'Adjacency List of x' do
         if ( y.col == white)
               y.col = Grey
               y.d   = x.d + 1
               y.π   = x
               Enqueue(Q, y)
      x.col = Black
```

Properties of BFS

At any stage of execution we define

$G_\pi = (V_\pi, E_\pi)$ as follows: $(E_\pi \subseteq V_\pi \times V_\pi)$

$V_\pi = \{ v \in V \mid v.col = \text{Grey or } v.col = \text{Black} \}$, $E_\pi = \{ (v_1, v_2) \mid v_1, v_2 \in V_\pi, \; v_2.\pi = v_1 \}$

<u>Lemma</u>: At any stage of execution of BFS(G, s), where $G = (V, E)$,

(a) $G_\pi = (V_\pi, E_\pi)$ is a tree rooted at s. $E_\pi \subseteq E$.

(b) For all $v \in V_\pi$, Unique path from s to v in G_π has length $v.d$

Proof: We prove the following by induction on i.

"Statements (a) and (b) hold at all stages upto iteration i of the while loop."

Base Case: $i = 0$. This is before any iteration of the while loop.

$$V_\pi = \{s\}, \quad E_\pi = \phi, \quad s.d = 0$$

There is a trivial path from s to s of length 0.

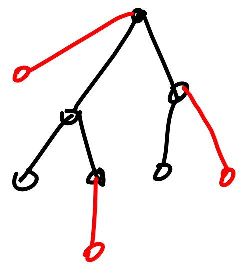
Induction Step: Assume the statement holds for $i = i_0$.

Consider $(i_0 + 1)^{th}$ iteration of while loop.

By I.H. $G_\pi = (V_\pi, E_\pi)$ is a tree

In $(i_0+1)^{th}$ iterations, we add only white vertices adjacent to some vertex x of the tree. white vertices $\notin V_\pi$, so we are adding some new vertices (not in the tree already) to the existing tree as children of x.

This opreation preserves the property that the graph is a tree.



If the added vertex is y, then there is path from s to y as follows:

$$\underset{\text{(By I.H.)}}{s \xrightarrow{\text{length } x.d} x} \longrightarrow y$$

length of this path $= x.d + 1 = y.d$

$(x, y) \in E$, since $y.\pi = x$, the edge added to E_π is (x, y).

$\Rightarrow E_\pi \subseteq E$.

$\square$

Observe that BFS tree is G_π when the procedure BFS(G, s) terminates.

Above claim about G_π implies that any vertex v in the BFS tree is reachable (via a path of length $\leq v.d$) from s.

Lemma: Any vertex v reachable from s is colored Black when procedure BFS(G, s) halts.

Proof: By induction on $\delta(s, v)$.

Base Case $\delta(s, v) = 0 \Rightarrow v = s$

s is colored grey in the initialization, and will become black when it leaves the queue.

Induction Step: Let $\delta(s, v) = r+1$

$\Rightarrow s, v_1, \dots, v_r, v$

$\Rightarrow \delta(s, v_r) = r$ (By property of shortest path)

$\Rightarrow v_r$ is colored black (By I.H.)

$\Rightarrow v_r$ was dequeued.

When v_r was dequeued, v would be put in the queue, if v is not already grey/black and would get grey color.

$\Rightarrow v$ gets grey/black color

$\Rightarrow v$ gets black color in the BFS tree.

$\square$

Corollary: BFS tree consists of exactly the set of reachable vertices from s.

Lemma: At any stage in the program execution if queue Q contains vertices $v_1 \cdots v_r$

front $\uparrow$ (at v_1) rear $\uparrow$ (at v_r)

then $v_1 \cdot d \leq v_2 \cdot d \leq \cdots \leq v_r \cdot d$

and $v_r \cdot d \leq v_1 \cdot d + 1$

Proof: We prove by induction on i that "For any configuration of Q arising upto the end of ith iteration of the while loop, property claimed in the lemma hold".

Base case: $i = 0$. Only queue configuration is singleton element.

$$v_1 = v_r = s$$

The claim clearly holds for it.

Induction step Suppose the claim holds at the end of k^{th} iteration.

In $(k+1)^{th}$ iteration, v_1 is dequeued and $v_{r+1} \cdots v_{r+t}$ are inserted in the queue. $v_{r+1}.d = v_{r+2}.d = \cdots = v_{r+t}.d = v_1.d + 1$

$v_{r+1}.d = v_1.d + 1 \le v_2.d + 1$ (because by I.H. $v_1.d \le v_2.d$)

(This shows second condition in the claim as v_2 is new front of the queue)

Similarly for $v_{r+2}, \cdots v_{r+t}$.

Also $\underbrace{v_r.d \le v_1.d + 1}_{\text{By I.H.}} = v_{r+1}.d$.

$\square$

Corollary: If vertex v enters the queue after vertex u then

$$v.d \geq u.d\,.$$

Proof: By induction on i we show that "For all vertices entered in Q upto i^{th} iteration of the while loop the claim holds".

Base case: $i = 0$. Trivially true (only one vertex entered Q)

Induction Step: claim holds upto r^{th} iteration. At the end of r^{th} iteration let Q be $v_1, \ldots, v_t$ $(t \geq 1)$.

now if u is entered in Q in this iteration then by monotonicity claim above $v_1.d \leq v_2.d \leq v_3.d \leq \ldots \leq v_t.d \leq v_{t+1}.d \leq \ldots \leq u.d$

$\square$

Theorem: When BFS procedure halts then for all $v \in V$,

$$v.d = \delta(s, v)$$

Proof: If v is white then $v.d = \infty$ (such a v never enters Q and its 'd' attribute is as in initialization)

By a result before, v is not reachable from s. That is $\delta(s, v) = \infty$

$$\Rightarrow v.d = \delta(s, v) = \infty$$

If v is black then there is a path from s to v of length $v.d$. $\Rightarrow$ $v.d \geq \delta(s, v)$

Towards contradiction assume that $v.d > \delta(s, v)$, for some v.

Without of generality let v be such a vertex with minimum $\delta(s, v)$.

$\Longrightarrow \quad \delta(s, v) \geq 1 \qquad$ (If $\delta(s, v) = 0$ then $v = s$ and $\delta(s, s) = s.d = 0$)

Let $\overbrace{\underbrace{s, v_1, \ldots, u}_{\text{length of this path } \delta(s,u)}, v}^{\delta(s,v)}$ be a path of length $\delta(s, v)$.

Let $v.\pi = w$

$v.d = w.d + 1 > \delta(s, v) = \delta(s, u) + 1$

$\Longrightarrow \quad w.d > \delta(s, u)$

$\Longrightarrow \quad w.d > \underbrace{\delta(s, u) = u.d}$ ($\because$ v is chosen to be min distance vertex violating the property)

$\Longrightarrow \quad w.d > u.d$

$\Longrightarrow$ w enters the queue after u (By previous corollary)

$\Longrightarrow$ When u is dequeued, v gets either grey/Black color (Before w is dequeued)

$\Rightarrow$ When w is dequeued, $v.col \neq$ white

This contradicts $v.\pi = w$

$\square$