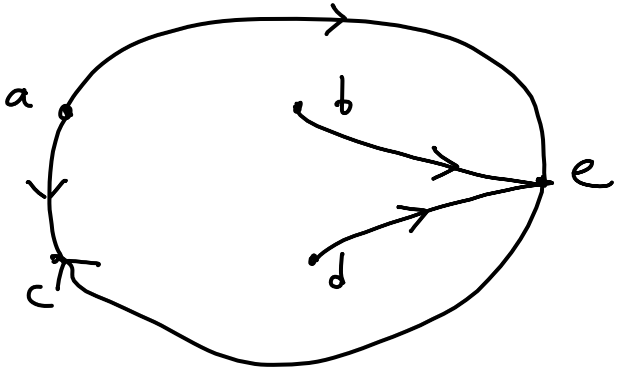
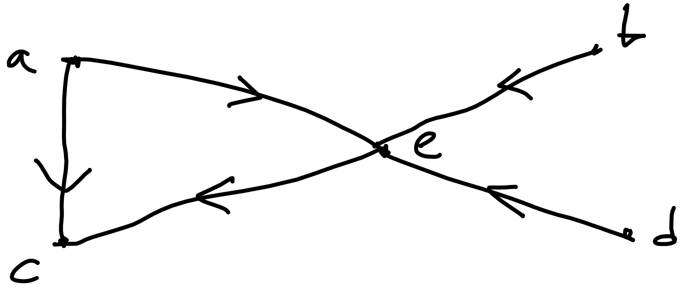
Graphs (Introduction)

Definition: A graph G is a pair (V, E), where V is a set and $E \subseteq V \times V$. V is called the set of vertices and E is called the set of edges of G.

Example: $G = (\{a, b, c, d, e\}, \{(a,c), (a,e), (b,e), (e,c), (d,e)\})$

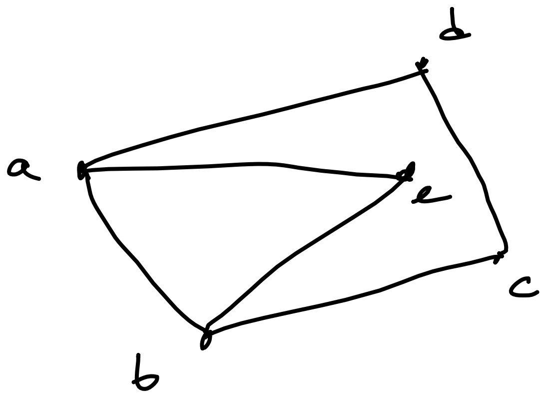




Two drawing of a graph G.

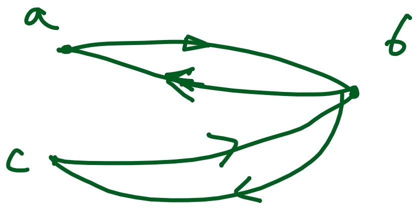
This was definition of a direction graph.

An undirected graph is one in which edges do not have any direction



Defn: G is undirected if E is a symmetric relation.

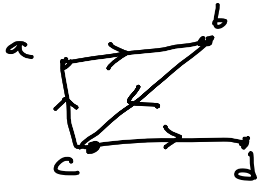
$$[(x, y) \in E \iff (y, x) \in E]$$



Since for each edge (x, y), we also have edge (y, x) in the other direction. We may think of whole $\{(x,y), (y,x)\}$ as an undirected edge.

Defn

A path $v_1, v_2, v_3 \dots v_i, v_{i+1}, \dots v_n$ is in a graph $G = (V, E)$ is a sequence of vertices s.t. for each $1 \le i < n$, $(v_i, v_{i+1}) \in E$.



A path is simple if no vertices repeat on it.

a b c a b c d is a path, c a b is a simple path.

A path $v_1 v_2 \dots v_n v_1$ is a simple cycle if vertices $v_2, \dots v_n$ are all distinct and $\neq v_1$.

Example c a b c is a simple cycle, a b c a b c a is not a simple cycle.
a b c d is not a cycle,

Length of a path is defined as number of edges in it.

Example: a b c a b c has length 5.

Exercise: If there is a path between u, v in $G(V, E)$ then there is such a path of length $< |V|$.

Distance between two vertices u, v, denoted $\delta(u, v)$, is the shortest path from u to v.

Example: In figure on the previous page,

$\delta(b, d) = 2$, $\delta(c, d) = 1$, $\delta(d, c) = \infty$

Degree of a vertex v in an undirected graph $G(V, E)$ is the no. of edges incident on v. $\left[\deg(v) = \left| \{ w \in V \mid (v, w) \in E \} \right| \right]$

Exercise: $\sum_{v \in V} \deg(v) = 2 \cdot |E|$ (prove it)

In a directed graph a vertex has two degrees, d_{in} and d_{out}.

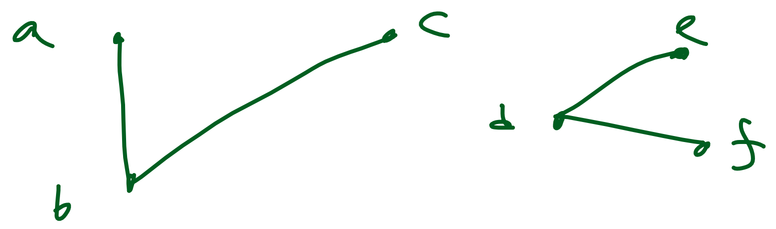
$$d_{in}(v) = \left| \{ w \in V \mid (w, v) \in E \} \right|$$

$$d_{out}(v) = \left| \{ w \in V \mid (v, w) \in E \} \right|$$

Exercise Formulate and prove a statement analogous to the one in the previous exercise, for directed graphs.

Definition A vertex v in G is reachable from a vertex u, if there is a path from u to v in G.

Example



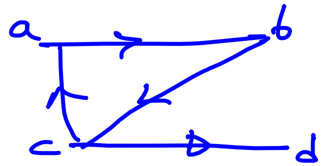
a is reachable from c but a is not reachable from e.

Definition: An undirected graph is said to be connected if for any $u_1, u_2 \in V$, u_2 is reachable from u_1.

The graph in the example above is not connected.

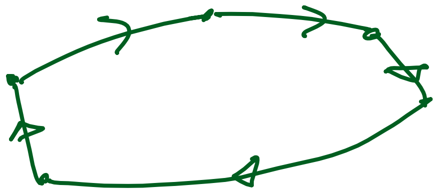
Definition: A directed graph $G(V, E)$ is strongly connected if for all $x, y \in V$, x is reachable from y and y is reachable from x.

In the earlier example



the directed graph is not strongly connected.

As another example:



is strongly connected.

Weighted graphs: $G = (V, E, W)$ $W : E \to \mathbb{R}$

Each edge has a real no. weight.

Why graphs? Many real life situations may be modeled as a graph.

Example: Travelling salesperson problem. (TSP) [Cities are modeled as vertices, edges have weight equal to distance between cities]

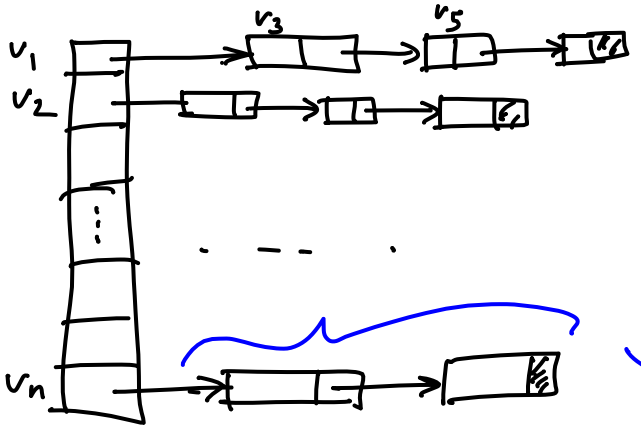
Example: A tree is also a graph

(it is a connected and acyclic graph)

Computer representation of graphs

$G = (V, E)$ both, V and E are finite and given explicitly.

First representation: Adjacency List



set of vertices adjacent to v_n

array of vertices

Size of this representation is

$$O(|V| + |E|)$$

vertex attributes may be stored in vertex array.

Edge attributes may be stored in linked list nodes.

For example, weight of an edge may be represented this way.

Second representation

Adjacency Matrix

$$A = \begin{array}{cc} & \begin{array}{ccccc} v_1 & \cdots & v_j & \cdots & v_n \end{array} \\ \begin{array}{c} v_1 \\ \vdots \\ v_i \\ \vdots \\ v_n \end{array} & \begin{bmatrix} & & \vdots & & \\ & & \vdots & & \\ \cdots & \cdots & 1/0 & & \\ & & & & \\ & & & & \end{bmatrix} \end{array}$$

$$A[i,j] = \begin{cases} 1 & \text{if there is an edge from } v_i \text{ to } v_j \\ 0 & \text{otherwise} \end{cases}$$

Disadvantage: $O(|V|^2)$ space

(not efficient if graph is sparse)

Advantage: presence or absence of any edge (v_i, v_j) can be tested in $O(1)$ time.

A graph with $|V|$ vertices may have at most $|V|^2$ edges.
But no. of edges may be much less in many applications.

Adjacency list is more commonly used representation for graphs.