Another Example of Divide & Conquer.

Matrix Multiplication $\qquad A_{m \times n} \times B_{n \times p} = C_{m \times p}$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

n multiplications

$n-1$ additions

$O(n)$ operations

$A_{n\times n}$ $B_{n\times n}$

Each entry requires
$O(n)$ time
(no. of arithmetic operations)

$(n^2 \text{ entries}) \cdot (n \text{ steps})$

$O(n^3)$ time

Can one do better?

Yes Surprising

Strassen 1969

$A_{n \times n} \quad B_{n \times n}$

Assume n is a power of 2

$n = 2^m$

If not, one can always add some 0-columns and 0-rows to a matrix to get to the nearest power of 2.

original product can be recovered from the multiplication of modified matrices by restricting ourselves to appropriate columns and rows.

<u>EX</u> verify the above statement.

$$\underbrace{\begin{array}{cc} & \overset{\longleftrightarrow}{n/2} \\ \updownarrow \frac{n}{2} & \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \end{array}}_{A} \qquad \underbrace{\begin{array}{cc} \overset{\longleftrightarrow}{n/2} & \\ \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right] & \updownarrow \frac{n}{2} \end{array}}_{B}$$

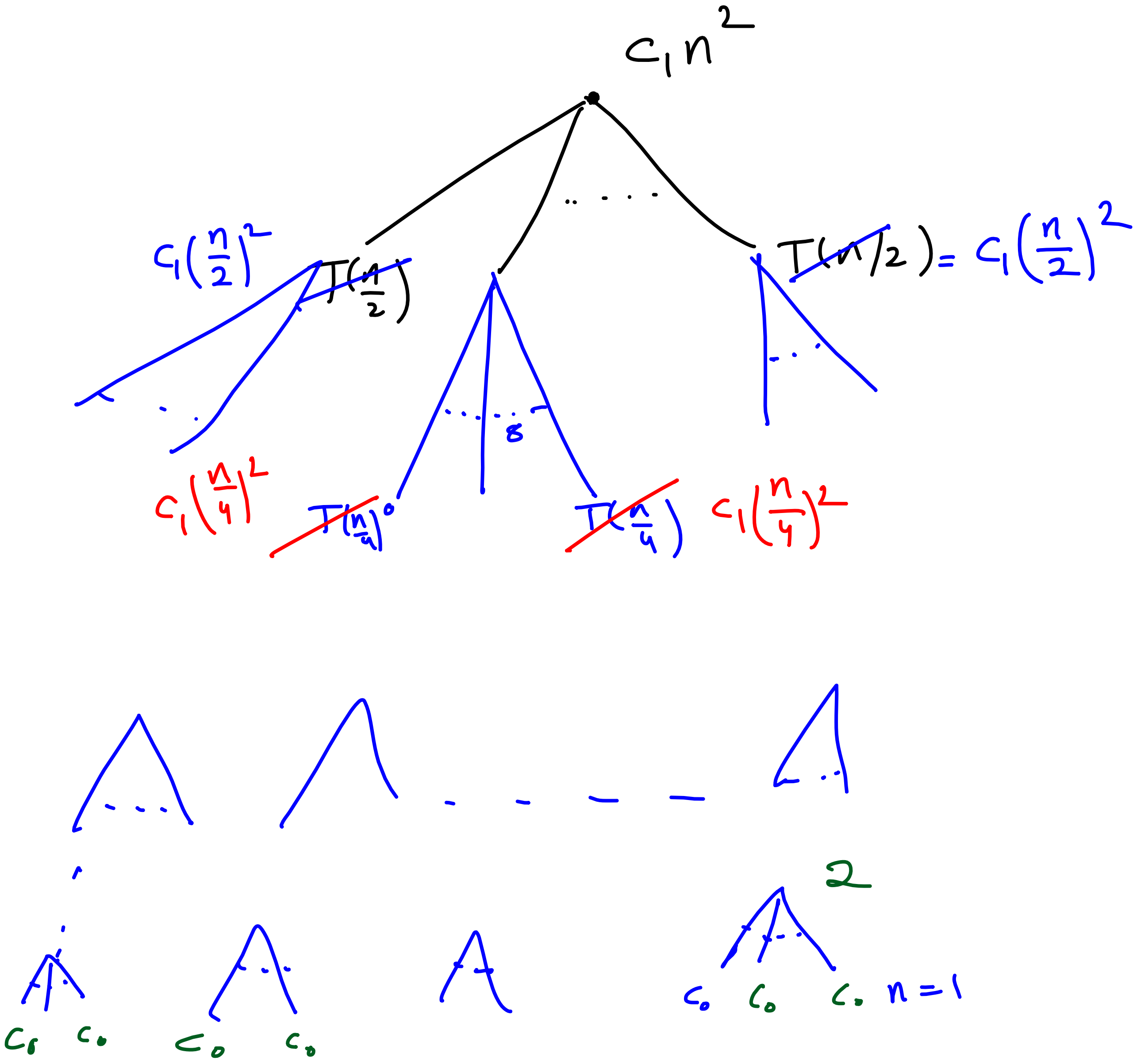
$$A \cdot B = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

<u>Exercise</u>: verify the above equation.

We reduced the problem of multiplying two $n \times n$ matrices to multiplying 8 $\frac{n}{2} \times \frac{n}{2}$ matrices and 4 addition of $\frac{n}{2} \times \frac{n}{2}$ matrices

$$T(n) = \begin{cases} 8T\left(\frac{n}{2}\right) + c_1 n^2 & n > 1 \\ c_0 & n = 1 \end{cases}$$

$$T_{new}(n) = \begin{cases} 7T_{new}\left(\frac{n}{2}\right) + d_1 n^2 & n > 1 \\ d_0 & n = 1 \end{cases}$$



$$n = 2^m$$

$$c_1 n^2$$

$$8\, c_1\left(\frac{n}{2}\right)^2$$

$$8^2 c_1\left(\frac{n}{4}\right)^2$$

$$8^k c_1\left(\frac{n}{2^k}\right)^2$$

$$8^{m-1} c_1\left(\frac{n}{2^{m-1}}\right)^2$$

$$8^m c_0$$

$$T(n) = \sum_{k=0}^{m-1} 8^k c_1 \left(\frac{n}{2^k}\right)^2 + 8^m c_0$$

$$= c_1 n^2 \sum_{k=0}^{m-1} \frac{2^{3k}}{2^{2k}} + (2^3)^m c_0$$

$$= c_1 n^2 \sum_{k=0}^{m-1} 2^k + (2^m)^3 c_0$$

$$= c_1 n^2 [2^m - 1] + n^3 c_0$$

$$= c_1 n^2 [n - 1] + c_0 n^3 \qquad O(n^3) \qquad \theta(n^3)$$

Not gained anything so far.

Time taken by Divide & Conquer Matrix-multiplication is the same as time taken by the straightforward algorithm derived from the definition.

Strassen: 7 multiplications instead of 8 suffice.

$$T_{new}(n) = \sum_{k=0}^{m-1} d_1 \cdot 7^k \left(\frac{n}{2^k}\right)^2 + d_0 7^m$$

$$= d_1 n^2 \sum_{k=0}^{m-1} \left(\frac{7}{4}\right)^k + d_0 \left(2^{\log_2 7}\right)^m$$

$$= d_1 n^2 \left[\frac{\left(\frac{7}{4}\right)^m - 1}{(7/4) - 1}\right] + d_0 \left(2^m\right)^{\log_2 7}$$

$$= d_1 n^2 \cdot \frac{4}{3} \left[\frac{7^m}{4^m} - 1\right] + d_0 n^{\log_2 7}$$

$$\leq d_1 \cancel{n^2} \frac{4}{3} \cdot \frac{n^{\log_2 7}}{\cancel{2^{2m}}} + d_0 n^{\log_2 7} = \frac{4}{3} d_1 n^{\log_2 7} + d_0 n^{\log_2 7}$$

$$\leq c\, n^{\log_2 7} \qquad O\left(n^{\log_2 7}\right)$$

$O(n^{\log_2 7})$

$O(n^{2.81})$

$\log_2 7 \approx 2.81$

Better than $O(n^3)$

Strassen's method

$S_1 = B_{12} - B_{22}$

$S_2 = A_{11} + A_{12}$

$S_3 = A_{21} + A_{22}$

$S_4 = B_{21} - B_{11}$

$S_5 = A_{11} + A_{22}$

$S_6 = B_{11} + B_{22}$

$S_7 = A_{12} - A_{22}$

$S_8 = B_{21} + B_{22}$

$S_9 = A_{11} - A_{21}$

$S_{10} = B_{11} + B_{12}$

Sums of $\frac{n}{2} \times \frac{n}{2}$ matrices

$10 \times \left(\frac{n}{2}\right)^2 = 2.5 n^2$ ops multiplication of

$\frac{n}{2} \times \frac{n}{2}$ matrices

$$P_1 = A_{11} \cdot S_1$$

$$P_2 = S_2 \cdot B_{22}$$

$$P_3 = S_3 \cdot B_{11}$$

$$P_4 = A_{22} \cdot S_4$$

$$P_5 = S_5 \cdot S_6$$

$$P_6 = S_7 \cdot S_8$$

$$P_7 = S_9 \cdot S_{10}$$

$$C_{11} = P_5 + P_4 - P_2 + P_6$$

$$C_{12} = P_1 + P_2$$

$$C_{21} = P_3 + P_4$$

$$C_{22} = P_5 + P_1 - P_3 - P_7$$

$$P_1 + P_2$$

$$= A_{11} \cdot S_1 + S_2 \cdot B_{22}$$

$$= A_{11} \cdot (B_{12} - B_{22}) + (A_{11} + A_{12}) \cdot B_{22}$$

$$= A_{11} B_{12} - \cancel{A_{11} B_{22}} + \cancel{A_{11} B_{22}} + A_{12} B_{22}$$

$$= A_{11} B_{12} + A_{12} B_{22}$$

$$= C_{12}$$

<u>Ex</u> verify all the other equations for C_{ij} $i = 1, 2$, $j = 1, 2$

18 sums of $\frac{n}{2} \times \frac{n}{2}$ matrices $\qquad O(n^2) \quad \underline{\Theta(n^2)}$

and 7 products of $\frac{n}{2} \times \frac{n}{2}$ matrices

$$\underline{T(n) = 7T\left(\frac{n}{2}\right) + \Theta(n^2)} \qquad n > 1$$

Verification is easy

But intuition (of Strassen's method)?

How to arrive at $S_1, \ldots S_{10}$ and at $P_1, \ldots P_7$ and be sure that $C_{11}, C_{12}, C_{21}, C_{22}$ are expressible in terms of them?

Matrix multiplication $\quad O\left(n^{2.37}\right) \quad$ improved bound