4) Discrepancy

Theorem 4: Given n unit vectors $v_i \in \mathbb{R}^n$, $i \in [n]$.
Then, $\exists$ "binary"-vector $b \in \{-1,1\}^n$ s.t.

$$\left\| \sum_{i \in [n]} b_i \cdot v_i \right\| \leq \sqrt{n} .$$

signed linear combination

Proof:

- Idea – Pick $b_i \in \{-1,1\}$ randomly (i.i.d).
  What's the exp. value of $\sum b_i v_i = ?$
- $X := \left\| \sum_{i \in [n]} b_i v_i \right\| \in \mathbb{R}_{\geq 0}$ is a rnd. variable.

$\triangleright$ $X^2 = \left(\sum b_i v_i\right)^T \cdot \left(\sum b_i v_i\right) = \left(\sum b_i v_i^T\right) \cdot \left(\sum b_i v_i\right)$

$= \sum_{i,j \in [n]} (b_i b_j) \cdot (v_i^T \cdot v_j)$

$\triangleright$ $E[X^2] = \sum_{i,j} E[b_i b_j] \cdot (v_i^T \cdot v_j) = \sum_{i=1}^{n} E[b_i^2] \cdot (v_i^T v_i) +$

$\sum_{i \neq j} E[b_i] \cdot E[b_j] \cdot (v_i^T v_j)$ $[\because b_i, b_j \text{ indep.}]$

$= n + 0 = n$ $[\because E[b_i] = 0]$

$\Rightarrow$

$\exists\, \bar{b} \in \{-1, 1\}^n, \quad X^2 \leq n \quad (\text{resp. } \geq n)$

$\Leftrightarrow$ " $, \quad X \leq \sqrt{n} \quad (\text{resp. } \geq \sqrt{n}).$

$\square$

5) Extremal Set Families

- **Defn:** Let $\mathcal{F} =: \{(A_i, B_i) \mid i \in [h]\}$ be a family of set-pairs. It is called (k,ℓ)-system if $|A_i| = k$, $|B_j| = \ell$ & $\begin{cases} A_i \cap B_i = \phi, \\ A_i \cap B_j \neq \phi, & \forall i \neq j \in [h]. \end{cases}$

- Eg. $U := [k+\ell]$. $\mathcal{F} := \{(A, A^c) \mid A \in \binom{U}{k}\}$ is a (k,ℓ)-system. (why?)

  $\triangleright$ $|\mathcal{F}| =: h = \binom{k+\ell}{k}$.

Qn: Is there a larger $\mathcal{F}$ (for the same k, ℓ)?

Thm 5: (Bollobás, 1965) $\mathcal{F}$ is (k,ℓ)-system

$$\Rightarrow \quad |\mathcal{F}| \le \binom{k+\ell}{\ell}.$$

Pf: Idea – Let $U := \bigcup_{i=1}^{n} (A_i \cup B_i)$ & consider random order π on U.

• E_i := event that elements of A_i precede B_i.

▷ For the given (A_i, B_i), π will keep A_i before B_i with prob.

$$P(E_i) = 1 \Big/ \binom{k+\ell}{k}$$

⇜ # ways π can pick first k elements!

Claim: $\forall i \neq j \in [h]$, E_i, E_j are disjoint.

Pf: • Suppose not. $\Rightarrow$ There's an order π wrt which: $\begin{cases} A_i < B_i, & \\ A_j < B_j \end{cases}$

• Wlog $\max(A_i) < \max(A_j)$.

$\Rightarrow A_i < B_j$ (wrt π).

$\Rightarrow A_i \cap B_j = \phi \Rightarrow$ ↯.

$\Rightarrow$ disjointness of E_i, E_j. $\square$

$\triangleright$ $1 \geq P\left(\bigcup_{i \in [h]} E_i\right) = \sum_{i \in [h]} P(E_i) = \sum_i \frac{1}{\binom{k+\ell}{k}} = h \Big/ \binom{k+\ell}{\ell}$.

$\Rightarrow h \leq \binom{k+\ell}{\ell}$. $\square$

6) Super-concentrator

- Defn: Super-concentrator is dag $G=(V,E)$ with n special input nodes $I \subset V$ & n output nodes $O \subset V$: $\forall k$, $\forall S \in \binom{I}{k}$, $\forall T \in \binom{O}{k}$,

vertices in S connects to T with k disjoint paths.

- Eg. This allows a robust telecommunication network:
  Any k people in I can talk to any k in O, simultaneously!

Exercise: A superconcentrator exists with $|V| = 2n$ & $|E| = n^2$.

Qn: Could we optimize to: $V, E = O(n)$?

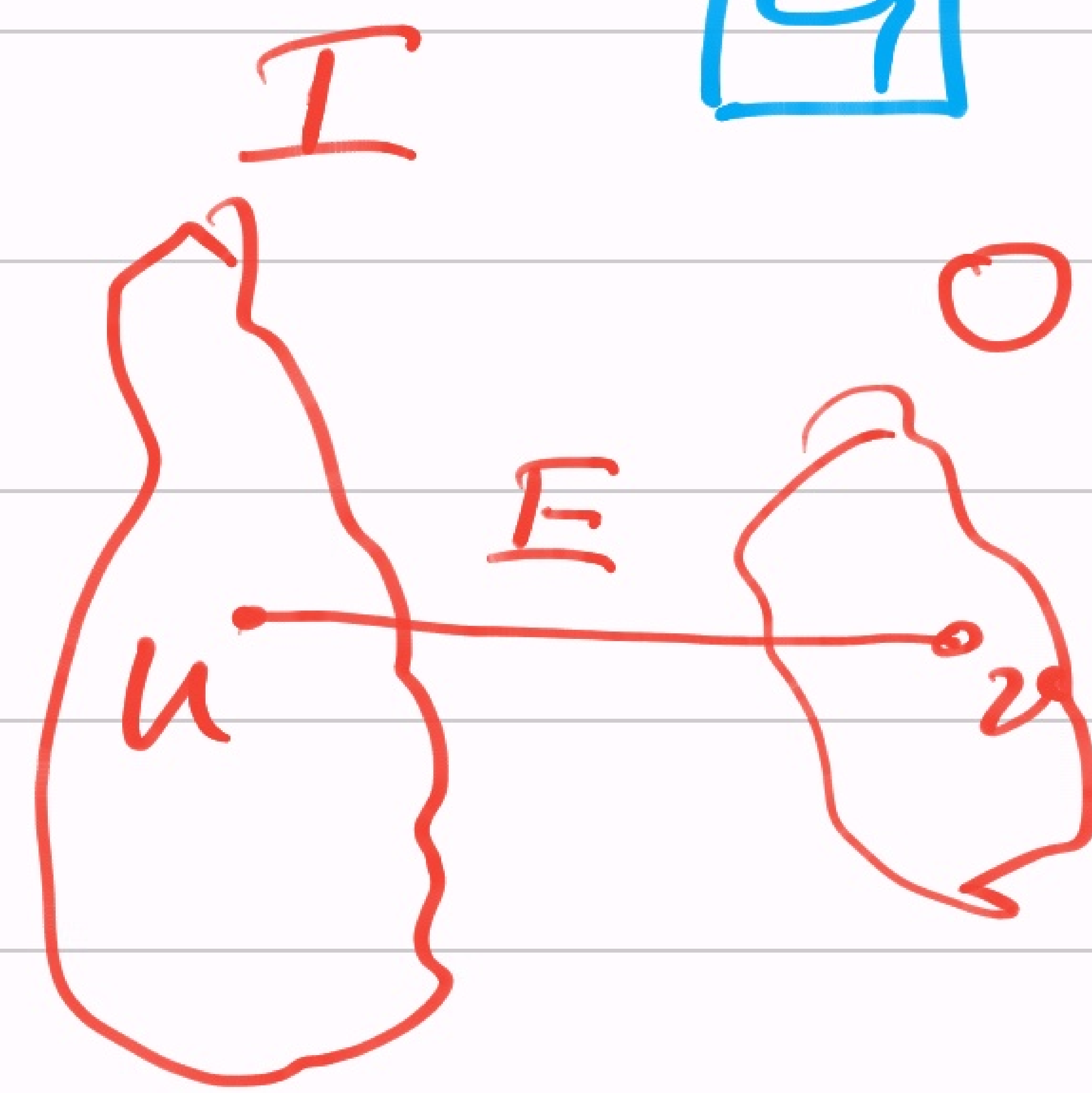
- Defn: (n_1, n_2, u)-concentrator is a bipartite graph that has n_1 input nodes I, n_2 output nodes O s.t. $\forall k \in [u]$, $\forall S \in \binom{I}{k}$, $\exists T \in \binom{O}{k}$, vertices in S connect to T by k disjoint paths.

- First, we show: Lemma: An efficient randomized algo. constructs $(6j, 4j, 3j)$-concentrator, with $|E| = O(j)$.

Pf: • Consider bipartite $G = (I \sqcup O, E)$ with $|I| = 6j$ & $|O| = 4j$. $\boxed{G}$

• Let E be <u>random</u> $36j$ edges.

Place them as:



• <u>Out-deg</u> of each $u \in I$ is 6, &

<u>In-deg</u> " " $v \in O$ is 9.

$$\Rightarrow \begin{cases} \#\text{ out-edges of } I \text{ is: } |I \times [6]| \\ \#\text{ in-edges of } O \text{ is: } |[9] \times O| \end{cases} = 36j.$$

▷ Any bijection $\phi : I \times [6] \longrightarrow [9] \times O$ defines E (& G), with an <u>ordered</u> adjacency list.

↪ 1-1 correspondence!

$\triangleright$ # such G is $(36j)!$.

- If G is a non-concentrator then: $\exists k \le 3j$, $\exists S \in \binom{I}{k}$, $\forall T \in \binom{O}{k}$, $S \not\rightsquigarrow T$ by disjoint paths.

$\Rightarrow$ $\exists S$, neighborhood $|N(S)| < |S|$. [Why? Hall's thm?]

- Let's use this S, say of size k to count non-concentrators:

$\Rightarrow$ $N(S)$ is a proper subset of some $T \in \binom{O}{k}$

$\Rightarrow$ $(6k)$-out-edges of S matched with $<(6k)$-in-edges of T.

$\Rightarrow$ # non-concentrators (for this S, T)

$$< (36j-6k)! \cdot \binom{9k}{6k} \cdot (6k)!$$

$\Rightarrow$ # non-concentrators (varying k, S & T)

$$< \sum_{k=1}^{3j} \binom{6j}{k} \cdot \binom{4j}{k} \cdot \binom{9k}{6k} \cdot (6k)! \cdot (36j-6k)!$$

$\triangleright$ $P(G \text{ is non-concentrator}) < \sum_{k=1}^{3j} \dfrac{\binom{6j}{k} \cdot \binom{4j}{k} \cdot \binom{9k}{6k}}{\binom{36j}{6k}}$

$$< (3j+1) \cdot \frac{1}{j^4} < 1.$$

$\Rightarrow$ A $(6j, 4j, 3j)$-concentrator will be found whp, having $|E| = 36j$.

$\square$

Thm 6: A randomized poly-time algo. designs a superconcentrator $G=(V,E)$ with $|V|=20j$ & $|E|=O(j)$.

Pf: • Use the Lemma to design: $\underline{G}$:

• This recurses from 6j to 4j superconcentrator.

- The recurrence is $E(6j) \leq E(4j) + 2 \times 36j + 6j$
$$= E(4j) + 78j$$

$\Rightarrow E(6j) \leq O(j)$.

- Let $S \in \binom{I}{k}$ & $T \in \binom{O}{k}$:

▷ $k \leq 3j \Rightarrow S \rightsquigarrow T' \rightsquigarrow S' \rightsquigarrow T$ gives k disjoint paths.

▷ $3j < k \leq 6j \Rightarrow$ # vertices in (S, T) that are matched by M are $\geq (k - 3j)$.

$\Rightarrow$ # unmatched vertices in S is $\leq 3j$.

$\Rightarrow$ Use the case: "$k \leq 3j$". □