Practice Problems 13 : Ratio and Root tests, Leibniz test

1. Determine the values of $\alpha \in \mathbb{R}$ for which $\sum_{n=1}^{\infty} \left(\frac{\alpha n}{n+1}\right)^n$ converges.

2. Consider $\sum_{n=1}^{\infty} a_n$ where $a_n > 0$ for all n. Prove or disprove the following statements.
   (a) If $\frac{a_{n+1}}{a_n} < 1$ for all n then the series converges.
   (b) If $\frac{a_{n+1}}{a_n} > 1$ for all n then the series diverges.

3. Show that the series $\frac{1}{1^2} + \frac{1}{2^3} + \frac{1}{3^2} + \frac{1}{4^3} + \frac{1}{5^2} + \frac{1}{6^3} + \ldots$ converges and that the root test and ratio test are not applicable.

4. Consider the rearranged geometric series $\frac{1}{2} + 1 + \frac{1}{8} + \frac{1}{4} + \frac{1}{32} + \frac{1}{16} + \frac{1}{128} + \frac{1}{64} \ldots$ Show that the series converges by the root test and that the ratio test is not applicable.

5. Consider the series $\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots$. Show that the ratio test is not applicable. Further show that $(a_n)^{\frac{1}{n}}$ does not converge and that the root test given in Theorem 8 is applicable.

6. (a) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converges absolutely, show that $\sum_{n=1}^{\infty} a_n b_n$ converges absolutely.
   (b) If $\sum_{n=1}^{\infty} a_n$ converges absolutely and (b_n) is a bounded sequence then $\sum_{n=1}^{\infty} a_n b_n$ converges absolutely.
   (c) Give an example of a convergent series $\sum_{n=1}^{\infty} a_n$ and a bounded sequence (b_n) such that $\sum_{n=1}^{\infty} a_n b_n$ diverges.

7. In each of the following cases, discuss the convergence/divergence of the series $\sum_{n=1}^{\infty} a_n$ where a_n equals:

   (a) $\frac{n!}{e^{n^2}}$ (b) $\frac{n^2 2^n}{(2n+1)!}$ (c) $\left(1 - \frac{1}{n}\right)^{n^2}$ (d) $\frac{n^2}{3^n}\left(1 + \frac{1}{n}\right)^{n^2}$

   (e) $\sin\left(\frac{(-1)^n}{n^p}\right), p > 0$ (f) $(-1)^n \frac{(\ln n)^3}{n}$ (g) $(-1)^n \left(n^{\frac{1}{n}} - 1\right)^n$ (h) $\frac{2^n + n^2 - \ln n}{n!}$

   (i) $\frac{\cos(\pi n) \ln n}{n}$ (j) $\left(1 + \frac{2}{n}\right)^{n^2 - \sqrt{n}}$ (k) $\frac{n^2(2\pi + (-1)^n)^n}{10^n}$

8. **(*)** Let $a_n \in \mathbb{R}$ and $a_n > 0$ for all n.

   (a) If $\frac{a_{n+1}}{a_n} \leq \lambda$ eventually for some $\lambda > 0$ then show that $a_n^{\frac{1}{n}} \leq \lambda + \epsilon$ eventually for every $\epsilon > 0$. Observe that if the ratio test (Theorem 7) gives the convergence of a series then the root test (Theorem 8) also gives the convergence, but the converse is not true (why?).

   (b) If $\lim_{n\to\infty} a_n^{\frac{1}{n}} = \alpha$ and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \beta$, show that $\alpha \leq \beta$.

9. (a) **(*) (Dirichlet test)** Let $\sum_{n=1}^{\infty} a_n$ be a series whose sequence of partial sums is bounded. Let (b_n) be a decreasing sequence which converges to 0. Show that $\sum_{n=1}^{\infty} a_n b_n$ converges. Observe that Leibniz test is a particular case of the Dirichlet test.
   (b) **(*) (Abel's test)** Let $\sum_{n=1}^{\infty} a_n$ be a convergent series and (b_n) be a monotonic convergent sequence. Show that $\sum_{n=1}^{\infty} a_n b_n$ converges.
   (c) Show that the series $1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \ldots$ converges whereas the series $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \ldots$ diverges.
   (d) Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}(1 + \frac{1}{n})^n$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \cos\frac{1}{n}$ and $\sum_{n=1}^{\infty} (-1)^n \frac{\tan^{-1} n}{\sqrt{n}}$ converge.

Practice Problems 13 : Hints/Solutions

1. Since $|\frac{\alpha n}{n+1}| \to \alpha$, by the root test, the series converges for $|\alpha| < 1$ and diverges for $|\alpha| > 1$. For $|\alpha| = 1$, the series diverges because $(\frac{n}{n+1})^n \to \frac{1}{e} \neq 0$.

2. (a) For $a_n = \frac{1}{n}$, $\frac{a_{n+1}}{a_n} < 1$ for all n but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

   (b) If $\frac{a_{n+1}}{a_n} > 1$ then $a_n \nrightarrow 0$. Hence $\sum_{n=1}^{\infty} a_n$ diverges.

3. By the comparison test (with $\frac{1}{n^2}$) the series converges.

4. The nth term a_n is $\frac{1}{2^n}$ if n is odd and $\frac{1}{2^{n-2}}$ if n is even. Since the consecutive ratio alternate in value between $\frac{1}{8}$ and 2, the ratio test is not applicable. However $a_n^{\frac{1}{n}} \to \frac{1}{2}$.

5. Observe that $((\frac{3}{2})^n)$ and $((\frac{2}{3})^n)$ are subsequences of $(\frac{a_{n+1}}{a_n})$; and $(\frac{3}{2})^n \to \infty$ $(\frac{2}{3})^n \to 0$. Therefore the ratio test (Theorem 7) is not applicable. But a_n can be either $3^{-\frac{n}{2}}$ or $2^{-\frac{n+1}{2}}$. Since $(3^{-\frac{n}{2}})^{\frac{1}{n}} \to \frac{1}{\sqrt{3}}$ and $(2^{-\frac{n+1}{2}})^{\frac{1}{n}} \to \frac{1}{\sqrt{2}}$, $a_n^{\frac{1}{n}} < L$ eventually for some L satisfying $\frac{1}{\sqrt{2}} < L < 1$. Hence the root test (Theorem 8) is applicable and the series converges.

6. (a) Since $b_n \to 0$, $|a_n b_n| \leq |a_n|$ eventually. Use the comparison test.

   (b) Let $|b_n| \leq M$ for some M. Then $|a_n b_n| \leq M|a_n|$. Use the comparison test.

   (c) Consider $a_n = \frac{(-1)^n}{n}$ and $b_n = (-1)^n$.

7. (a) Converges by the Ratio test.

   (b) Converges by the Ratio test.

   (c) Converges by the Root test: $(1 - \frac{1}{n})^n \to \frac{1}{e}$ (If $y = (1 - \frac{1}{n})^n$ then $\ln y = \frac{\ln(1-\frac{1}{n})}{\frac{1}{n}} \to -1$).

   (d) Converges by the Root test: $a_n^{\frac{1}{n}} \to \frac{e}{3} < 1$.

   (e) Converges by Leibniz test: $\sin(\frac{(-1)^n}{n^p}) = (-1)^n \sin(\frac{1}{n^p})$.

   (f) Converges by Leibniz test: If $f(x) = \frac{(\ln x)^3}{x}$ then $f'(x) < 0$ for all $x > e^3$.

   (g) Converges absolutely by the Root test.

   (h) Converges: By the LCT test with $\frac{2^n}{n!}$ and then the Ratio test for $\sum_{n-1}^{\infty} \frac{2^n}{n!}$.

   (i) Converges by Leibniz test: $\cos(\pi n) = (-1)^n$.

   (j) Diverges because $\left(1 + \frac{2}{n}\right)^{n^2 - \sqrt{n}} \nrightarrow 0$ as $(1 + \frac{2}{n}) > 1$.

   (k) Converges absolutely: Use $|a_n| \leq \frac{n^2(2\pi+1)^n}{10^n}$ and then the Ratio test.

8. (a) Suppose $\frac{a_{n+1}}{a_n} \leq \lambda$ for all $n \geq N$ for some N. Then for all $n \geq N$,

$$a_n = \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_{N+1}}{a_N} a_N \leq \lambda^{n-N} a_N.$$

   Therefore $a_n^{\frac{1}{n}} = (\lambda^{1-\frac{N}{n}}) a_N^{\frac{1}{n}} \leq \lambda + \epsilon$ eventually for any $\epsilon > 0$ as $a_N^{\frac{1}{n}} \to 1$.

   Suppose the Ratio test (Theorem 7) implies the convergence of a series $\sum_{n=1}^{\infty} a_n$. Then there exists a λ such that $0 < \lambda < 1$ and $\frac{a_{n+1}}{a_n} \leq \lambda$ eventually. Then, by the previous part, $a_n^{\frac{1}{n}} \leq \lambda + \frac{(1-\lambda)}{2} < 1$ eventually. Therefore the Root test (Theorem 8) implies the convergence of the series. The converse is not true (See Problem 5).

(b) Follows from (a).

9. (a) Compare the Dirichlet Test with Problem 15 of the Practice Problems 11. Repeat the steps (a)-(c) given in the problem mentioned above by taking b_n in place of $\frac{1}{n}$.

   (b) Compare Abel's Test with Problem 15 of the Practice Problems 11. Repeat the steps (a)-(c) given in the problem mentioned above by taking b_n in place of $\frac{1}{n}$. In Abel's test (b_n) could be increasing. However, the proofs of the steps (a)-(c) go through.

   (c) Apply the Dirichlet test or consider the sequence of partial sums.

   (d) Apply Abel's test.