MTH 102A - Linear Algebra - 2015-16-II Semester

Arbind Kumar Lal *

P. **Vector space**

- A set $\mathbb{V}$ over a field $\mathbb{F}$ is a vector space if

$x + y$ is defined in $\mathbb{V}$ for all $x, y \in \mathbb{V}$

and αx is defined in $\mathbb{V}$ for all $x \in \mathbb{V}, \alpha \in \mathbb{F}$ s.t.

1) $+$ is commutative, associative.

2) Each $x \in \mathbb{V}$ has a inverse w.r.t $+$.

3) Identity element 0 exists in $\mathbb{V}$ for $+$.

4) $1x = x$ holds $\forall x \in \mathbb{V}$, where 1 is the multiplicative identity of $\mathbb{F}$.

5) $(\alpha\beta)x = \alpha(\beta x), (\alpha + \beta)x = \alpha x + \beta x$ hold $\forall \alpha, \beta \in \mathbb{F}, x \in \mathbb{V}$.

6) $\alpha(x + y) = \alpha x + \alpha y$ holds $\forall \alpha \in \mathbb{F}, \forall x, y \in \mathbb{V}$.

- $\mathbb{R}^3$ is a vector space (VS) over $\mathbb{R}$. Here

$$(x_1, x_2, x_3) + \alpha(y_1, y_2, y_3) := (x_1 + \alpha y_1, x_2 + \alpha y_2, x_3 + \alpha y_3).$$

- $\mathcal{M}_{m,n} := \{A_{m\times n} : a_{ij} \in \mathbb{C}\}$ is a VS over $\mathbb{C}$. Here
$$[A + \alpha B]_{ij} := a_{ij} + \alpha b_{ij}.$$
- $\mathbb{R}^S := \{\text{functions from } S \text{ to } \mathbb{R}\}$ is a VS over $\mathbb{R}$. Here
$$(f + \alpha g)(s) := f(s) + \alpha g(s).$$
- $\mathbb{R}[x] = \{p(x) \mid p(x) \text{ is a real polynomial in } x\}$ is a VS over $\mathbb{R}$.
- $\mathbb{R}[x; 5] = \{p(x) \in \mathbb{R}[x] \mid \text{degree of } p(x) \le 5\}$ is a VS over $\mathbb{R}$.
- The set $\{0\}$ is the smallest VS over any field.
- The set of all real sequences (being $\mathbb{R}^{\mathbb{N}}$) is a VS over $\mathbb{R}$.
- The set of all bounded real sequences is a VS over $\mathbb{R}$.
- The set of all convergent real sequences is a VS over $\mathbb{R}$.
- $\{(a_n) \mid a_n \in \mathbb{R},\ a_n \to 0\}$ is a VS over $\mathbb{R}$. Change 0 to 1??? No.
- $\mathcal{C}((a, b), \mathbb{R}) := \{f : (a, b) \to \mathbb{R} \mid f \text{ is continuous}\}$ is a VS over $\mathbb{R}$.

*Indian Institute of Technology Kanpur

- $\mathcal{C}^2((a,b),\mathbb{R}) := \{f : (a,b) \to \mathbb{R} \mid f'' \text{ is continuous}\}$ is a VS over $\mathbb{R}$.
- $\mathcal{C}^\infty((a,b),\mathbb{R}) := \{f : (a,b) \to \mathbb{R} \mid f \text{ infinitely differentiable}\}$ is a VS.
- $\{f : (a,b) \to \mathbb{R} \mid f'' + 3f' + 5f = 0\}$ is a VS over $\mathbb{R}$.
- $\{A_{n\times n} \mid a_{ij} \in \mathbb{R},\ a_{11} = 0\}$ is a VS over $\mathbb{R}$.
- $\{A_{n\times n} \mid a_{ij} \in \mathbb{R},\ A \text{ upper triangular}\}$ is a VS over $\mathbb{R}$.
- $\{A \in \mathcal{M}_{n\times n}(\mathbb{R}) \mid A \text{ is symmetric}\}$ is a VS over $\mathbb{R}$. What about the skew symmetric ones? How about $A^t = 2A$???
- Let $\mathbb{W}$ and $\mathbb{V}$ be VSs over $\mathbb{F}$. We call $\mathbb{W}$ a subspace of $\mathbb{V}$ if $\mathbb{W} \subseteq \mathbb{V}$. Notation: $\mathbb{W} \preccurlyeq \mathbb{V}$.

Th Let $\mathbb{V}$ be a VS over $\mathbb{F}$ and $\mathbb{W} \subseteq \mathbb{V}$. Then $\mathbb{W} \preccurlyeq \mathbb{V}$ if and only if

'for all $\mathbf{x}, \mathbf{y} \in \mathbb{W}$ we have $\alpha\mathbf{x} + \mathbf{y} \in \mathbb{W}$ for all $\alpha \in \mathbb{F}$'. !!

- $\{\mathbf{0}\}$ and $\mathbb{V}$ are subspaces of any VS $\mathbb{V}$, called the trivial subspaces.
- The VS $\mathbb{R}$ over $\mathbb{R}$ does not have a nontrivial subspace.

Take a subspace $\mathbb{W} \neq \{0\}$. Then $\exists a \in \mathbb{W}$ s.t. $a \neq 0$. So $\alpha a \in \mathbb{W}$ for each $\alpha \in \mathbb{R}$. So $\mathbb{W} = \mathbb{R}$.

- The VS $\mathbb{R}$ over $\mathbb{Q}$ has many nontrivial subspaces. Two of them are $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ and $\mathbb{Q}[\sqrt{3}] = \{a + b\sqrt{3} : a, b \in \mathbb{Q}\}$.

P. **Subspaces**

- Fix a real $A_{m\times n}$. The set $\{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\}$ is a subspace of $\mathbb{R}^m$, called the range/column space of A. Our notation: $\textsc{col}(A)$.
- Note that $\textsc{col}(A)$ is basically the set of all vectors obtained by adding and subtracting real multiples of column vectors of A.
- Fix a real $A_{m\times n}$. The set $\{\mathbf{x}^t A : \mathbf{x} \in \mathbb{R}^m\}$ is a subspace of $\mathbb{R}^n$, called the row space of A. Our notation: $\textsc{row}(A)$.
- Fix a real $A_{m\times n}$. The set $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$ is a subspace of $\mathbb{R}^n$, called the null space of A. Our notation: $\textsc{null}(A)$.
- $\textsc{col}(A)$, $\textsc{row}(A)$, $\textsc{null}(A)$, $\textsc{null}(A^t)$ are the four fundamental subspaces related to a real/complex matrix A.
- Let $\mathbf{a} \in \mathbb{R}^2$, $\mathbf{a} \neq 0$. Then, $\{\mathbf{x} | a^t\mathbf{x} = 0\}$ is a nontrivial subspace of $\mathbb{R}^2$. Geometrically, it represents a straight line passing through the origin. In fact, these are the only nontrivial subspaces of $\mathbb{R}^2$.
- $\{\mathbf{x} \in \mathbb{R}^2 \mid x_1x_2 = 0\}$ is NOT a subspace of $\mathbb{R}^2$. IMPORTANT: union of axes; first time union of subspaces is not a subspace.
- Let $\mathbf{x}_i \in \mathbb{V}$, $\alpha_i \in \mathbb{F}$, $i = 1, \cdots, k$. We call $\sum\limits_{i=1}^{k} \alpha_i\mathbf{x}_i$ a linear combination (lin.comb) of

$\mathbf{x}_1, \cdots, \mathbf{x}_k$. The set $\{\sum_{i=1}^{k} \alpha_i \mathbf{x}_i \mid \alpha_i \in \mathbb{F}\}$ is a subspace of $\mathbb{V}$.

- Let $S \subseteq \mathbb{V}$. By span of S we mean $\{\sum\limits_{i=1}^{m} \alpha_i \mathbf{x}_i \mid \alpha_i \in \mathbb{F}, \mathbf{x}_i \in S\}$.

  Notation: $\mathbf{L}S(S)$. This is a subspace of $\mathbb{V}$. Convention: $\mathbf{L}S(\emptyset) = \{0\}$.

  Let $\mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)^t \in \mathbb{R}^n$ with 1 at the i-th coordinate. Then, $\mathbf{L}S(\mathbf{e}_, \mathbf{e}_2, \ldots, \mathbf{e}_n)$ is $\mathbb{R}^n$.

P. Linear Span

- Let $\mathbf{x}^t = [1 \quad 2 \quad 1]$ and $\mathbf{y}^t = [1 \quad -2 \quad 3]$. Then,

$$ls(\mathbf{x}, \mathbf{y}) = \left\{ a \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} : a, b \in \mathbb{R} \right\}$$
$$= \left\{ \begin{bmatrix} a+b \\ 2a-2b \\ a+3b \end{bmatrix} : a, b \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 4x - y - 2z = 0 \right\}$$ as $4(a+b) - (2a-2b) - 2(a+3b) = 0$.

Note that we have obtained conditions on x, y and $z$such that the system $a + b = x, 2a - 2b = y$ and $a + 3b = z$ has a solution for all choices of a, b and c.

- If $\mathbb{U}, \mathbb{W}$ are subspaces of $\mathbb{V}$ then, $\mathbb{U} + \mathbb{W} := \{\mathbf{u} + \mathbf{w} \mid \mathbf{u} \in \mathbb{U}, \mathbf{w} \in \mathbb{W}\}$ is a subspace of $\mathbb{V}$, called the sum of $\mathbb{U}$ and $\mathbb{W}$. When $\mathbb{U} \cap \mathbb{W} = \{0\}$, it is called the internal direct sum of $\mathbb{U}$ and $\mathbb{W}$. Notation: $\mathbb{U} \oplus \mathbb{W}$.

  Internal: they are inside $\mathbb{V}$; external: will come later;

x-axis + y-axis in $\mathbb{R}^2$, $\mathbb{R}^3$; x-axis + xy-plane in $\mathbb{R}^3$; xy-plane + xz-plane in $\mathbb{R}^3$.

Th Let $\mathbb{U}, \mathbb{W} \preccurlyeq \mathbb{V}$. Then, $\mathbb{U} + \mathbb{W}$ is the smallest subspace of $\mathbb{V}$ which contains both $\mathbb{U}$ and $\mathbb{W}$. ‼

IMPORTANT Is $\mathbb{U} \preccurlyeq \mathbb{U}+\mathbb{W}$? Is $\mathbb{W} \preccurlyeq \mathbb{U}+\mathbb{W}$? Let $\mathbb{X} \preccurlyeq \mathbb{V}$ s.t. $\mathbb{U}, \mathbb{W} \subseteq \mathbb{X}$. Is $\mathbb{U}+\mathbb{W} \subseteq \mathbb{X}$?

- If $\mathbb{U} \preccurlyeq \mathbb{W}$ and $\mathbb{W} \preccurlyeq \mathbb{V}$, then $\mathbb{U} \preccurlyeq \mathbb{V}$.
- Let $\{\mathbb{U}_t | \mathbb{U}_t \preccurlyeq \mathbb{V}\}$ be a nonempty class. Then $\bigcap\limits_t \mathbb{U}_t$ is a subspace. ‼
- Let $\mathbb{U}, \mathbb{W} \preccurlyeq \mathbb{V}$. Then $\mathbb{U} \cup \mathbb{W} \preccurlyeq \mathbb{V}$ if and only if either $\mathbb{U} \subseteq \mathbb{W}$ or $\mathbb{W} \subseteq \mathbb{U}$. ‼

Q If $\mathbb{U} \cup \mathbb{W} \preccurlyeq \mathbb{V}$, is it necessary that $\mathbb{U} \preccurlyeq \mathbb{V}$?

- Let $\mathbb{V}, \mathbb{W}$ be VSs over $\mathbb{F}$. Then $\mathbb{V} \times \mathbb{W}$ is a VS over $\mathbb{F}$ with

  $(v_1, w_1) + (v_2, w_2) := (v_1 + v_2, w_1 + w_2)$ and $\alpha(v, w) = (\alpha v, \alpha w)$.

  Notation: $\mathbb{V} \oplus \mathbb{W}$. Called external direct sum of $\mathbb{V}$ and $\mathbb{W}$.

Ex Give a bijection $f : \mathbb{R}^3 \to \mathbb{R}[x; 2]$ s.t. $f(\alpha u + v) = \alpha f(u) + f(v)$ for each $u, v \in \mathbb{R}^3$. Take 3 subspaces of $\mathbb{R}^3$. Observe their image in $\mathbb{R}[x; 2]$.

Ex Give vector spaces such that $\mathbb{U}_1 \succ \mathbb{U}_2 \succ \mathbb{U}_3 \succ \cdots$.

Ex Let $\mathbb{U}, \mathbb{W} \preccurlyeq \mathbb{V}$. Then $\mathbb{V} = \mathbb{U} \oplus \mathbb{W}$ (int/ext?) if and only if for each $v \in \mathbb{V}$, $\exists$ unique $u \in \mathbb{U}$ and unique $w \in \mathbb{W}$ s.t. $v = u + w$.

- Put $\mathbb{V}=\mathcal{M}_2(\mathbb{R})$, $\mathbb{W}=\left\{\begin{bmatrix}x_1 & x_2\\ x_3 & 0\end{bmatrix}: x_i\in\mathbb{R}\right\}$, $\mathbb{U}=\left\{\begin{bmatrix}x_1 & 0\\ 0 & x_2\end{bmatrix}: x_i\in\mathbb{R}\right\}$.
- Are $\mathbb{W}, \mathbb{U}$ subspaces of $\mathbb{V}$? Yes.
- Is $\mathbb{V}=\mathbb{W}+\mathbb{U}$? Yes.
- Is $\mathbb{V}=\mathbb{W}\oplus\mathbb{U}$? No.
- What is the subspace $\mathbb{W}\cap\mathbb{U}$? $\left\{\begin{bmatrix}x & 0\\ 0 & 0\end{bmatrix}: x\in\mathbb{R}\right\}$.

Th Let $\mathbb{V}$ be a VS and $S\subseteq\mathbb{V}$. Let $L=\{\mathbb{U}\mid \mathbb{U}\preccurlyeq\mathbb{V},\ S\subseteq\mathbb{U}\}$. Then $\textsc{ls}(S)=\bigcap\limits_{\mathbb{U}\in L}\mathbb{U}$. ‼

P. **Linear dependency**

Ex Let A be row-equivalent to B. Then $\textsc{row}(A)=\textsc{row}(B)$. Converse?

- Let $\mathbf{x}_1,\ldots,\mathbf{x}_k\in\mathbb{V}$. If a nonzero lin.comb of $\mathbf{x}_i$'s becomes $\mathbf{0}$, then we say $\mathbf{x}_1,\ldots,\mathbf{x}_k$ are linearly dependent (lin.dep).
- We say $\mathbf{x}_1,\ldots,\mathbf{x}_k$ are linearly independent, if they are not lin.dep.
- In $\mathbb{R}^2$, the vectors $\mathbf{x}=\begin{bmatrix}1\\2\end{bmatrix}, \mathbf{e}_1=\begin{bmatrix}1\\0\end{bmatrix}$ and $\mathbf{e}_2=\begin{bmatrix}0\\1\end{bmatrix}$ are lin.dep, as $\mathbf{x}-\mathbf{e}_1-2\mathbf{e}_2=\mathbf{0}$.
- $\mathbf{v}_1=\begin{bmatrix}3\\0\\-3\end{bmatrix}, \mathbf{v}_2=\begin{bmatrix}-1\\1\\2\end{bmatrix}, \mathbf{v}_3=\begin{bmatrix}4\\2\\-2\end{bmatrix}, \mathbf{v}_4=\begin{bmatrix}2\\1\\1\end{bmatrix}$ are lin.dep in $\mathbb{R}^3$,
  as $2\mathbf{v}_1+2\mathbf{v}_2-\mathbf{v}_3+0\mathbf{v}_4=\mathbf{0}$.
- $\mathbf{v}_1=\begin{bmatrix}1\\1\\1\end{bmatrix}, \mathbf{v}_2=\begin{bmatrix}-1\\1\\1\end{bmatrix}$ are lin.ind in $\mathbb{R}^3$. How?
- Answering the previous question: the vectors will be linearly dependent (lin.dep) if we can find
  $\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\neq\mathbf{0}$ s.t. $\alpha\begin{bmatrix}1\\1\\1\end{bmatrix}+\beta\begin{bmatrix}-1\\1\\1\end{bmatrix}=\mathbf{0}$. That is, if $\begin{bmatrix}1 & -1\\ 1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$.
- Apply GJE, if there are no free variables, then only solution is $\begin{bmatrix}\alpha\\ \beta\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$. In that case we conclude lin.ind.
- If there are free variables, then we have nonzero solutions for $\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$. In this case we conclude lin.dep.
- $\mathbf{v}_1=\begin{bmatrix}1\\1\\1\end{bmatrix}, \mathbf{v}_2=\begin{bmatrix}-1\\1\\1\end{bmatrix}, \mathbf{v}_3=\begin{bmatrix}0\\0\\0\end{bmatrix}$ are lin.dep. Why? As the variable for v_3 will be free.
  Or $0\cdot\mathbf{v}_1+0\cdot\mathbf{v}_2+\alpha\mathbf{v}_3=\mathbf{0}$ for all $\alpha\in\mathbb{R}$.

Th Vectors $\mathbf{v},\mathbf{w},\mathbf{z}$ in $\mathbb{R}^2$ are linearly dependent.

Po. Solving $\alpha\mathbf{v}+\beta\mathbf{w}+\gamma\mathbf{z}=\mathbf{0}$ means solving $\begin{bmatrix}\mathbf{v}_1 & \mathbf{w}_1 & \mathbf{z}_1\\ \mathbf{v}_2 & \mathbf{w}_2 & \mathbf{z}_2\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$.

That is, we have to find the RREF of $\left[\begin{array}{ccc|c}\mathbf{v}_1 & \mathbf{w}_1 & \mathbf{z}_1 & 0\\ \mathbf{v}_2 & \mathbf{w}_2 & \mathbf{z}_2 & 0\end{array}\right]$.

Applying GJE, as there are only two rows, the number of pivots is at most 2. As the number variables is 3, we will have a free variable. Hence we will have a nonzero solution.

$\square$

- We call a set $S \subseteq \mathbb{V}$ linearly dependent if contains a finite lin.dep set.
- Thus $\emptyset$ is lin.ind and any set containing $\mathbf{0}$ is lin.dep.
- So $S \subseteq \mathbb{V}$ is lin.ind if and only if each finite subset of S is lin.ind.
- Take $A_{m\times n}$, $m > n$. Then $R = \text{RREF}(A)$ has last row $\mathbf{0}$. That is, rows of A are lin.dep.

How? GJE: $R = FA$. Then $\sum_{i=1}^{n} f_{mi} A(i,:) = \mathbf{0}^t$.

P. **Linear dependency: crucial result**

Th Let $S = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\} \subseteq \mathbb{V}$ and $T = \{\mathbf{w}_1, \cdots, \mathbf{w}_m\} \subseteq \text{LS}(S)$ s.t. $m = |T| > |S|$. Then T is linearly dependent.

Po. Write $\mathbf{w}_i = a_{i1}\mathbf{v}_1 + a_{i2}\mathbf{v}_2 + \cdots + a_{in}\mathbf{v}_n$. So $\mathbf{w}_i = A(i,:)\begin{bmatrix}\mathbf{v}_1\\ \vdots\\ \mathbf{v}_n\end{bmatrix}$ and

$$\begin{bmatrix}\mathbf{w}_1\\ \vdots\\ \mathbf{w}_m\end{bmatrix} = \begin{bmatrix}a_{11}\mathbf{v}_1 + \cdots + a_{1n}\mathbf{v}_n\\ \vdots\\ a_{m1}\mathbf{v}_1 + \cdots + a_{mn}\mathbf{v}_n\end{bmatrix} = \begin{bmatrix}a_{11} & \cdots & a_{1n}\\ \vdots & & \vdots\\ a_{m1} & \cdots & a_{mn}\end{bmatrix}\begin{bmatrix}\mathbf{v}_1\\ \vdots\\ \mathbf{v}_n\end{bmatrix}.$$

As $m > n$, we see that rows of $A = (a_{ij})$ are lin.dep. That is, there exists α_i, not all zero, s.t. $\sum_{i=1}^{m} \alpha_i A(i,:) = \mathbf{0}^t$. Thus

$$\sum_{i=1}^{m} \alpha_i \mathbf{w}_i = \sum_{i=1}^{m} \alpha_i \left(A(i,:)\begin{bmatrix}\mathbf{v}_1\\ \vdots\\ \mathbf{v}_n\end{bmatrix}\right) = \left(\sum_{i=1}^{m} \alpha_i A(i,:)\right)\begin{bmatrix}\mathbf{v}_1\\ \vdots\\ \mathbf{v}_n\end{bmatrix} = \mathbf{0}^t \begin{bmatrix}\mathbf{v}_1\\ \vdots\\ \mathbf{v}_n\end{bmatrix} = \mathbf{0}.$$

$\square$

Cor Any $n+1$ vectors in $\mathbb{R}^n$ is lin.dep.

Po. Follows as $\mathbb{R}^n = \textbf{linear span}\,(\mathbf{e}_1, \ldots, \mathbf{e}_n)$. $\square$

Th $A_{n\times n}$ is invertible if and only if columns of A are linearly independent.

Po. It is a restatement of 'A is invertible if and only if $A\mathbf{x} = \mathbf{0}$ has a unique solution'. $\square$

Th $A_{n\times n}$ is invertible if and only if rows of A are linearly independent.

Po. Follows as 'A is invertible if and only if A^t is invertible'. $\square$

- Let $\mathbf{u}_1, \cdots, \mathbf{u}_m \in \mathbb{V}$ be lin.ind. Choose $\alpha_1, \ldots, \alpha_m \in \mathbb{F}$. Put $\mathbf{u} = \sum_1^m \alpha_i \mathbf{u}_i$. Can you find another set of $\beta_i \in \mathbb{F}$ s.t. $\mathbf{u} = \sum_1^m \beta_i \mathbf{u}_i$?
- No. If so, we have $\sum(\alpha_i - \beta_i)\mathbf{u}_i = \mathbf{0}$. Then $\mathbf{u}_i$'s are lin.dep., A contradiction.

Ex $\mathbf{u}_1, \cdots, \mathbf{u}_k \in \mathbb{R}^n$ are lin.ind if and only if $A\mathbf{u}_1, \cdots, A\mathbf{u}_k$ are lin.ind for any invertible A_n.

Ex Let $\mathbf{u}, \mathbf{v} \in \mathbb{V}$. Then $\mathbf{u}, \mathbf{v}$ are lin.ind if and only if $\mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}$ are lin.ind.

P. **Fundamental results: Linear dependence and linear span**

Th Let $S \subseteq \mathbb{V}$ be lin.ind, $\mathbf{x} \in \mathbb{V} \setminus S$. Then $S \cup \{\mathbf{x}\}$ is lin.dep if and only if $\mathbf{x} \in \text{LS}(S)$.

Po. Let $S \cup \{\mathbf{x}\}$ be lin.dep. Then some $\sum_{i=1}^{m} \alpha_i \mathbf{s}_i + \alpha_{m+1}\mathbf{x} = \mathbf{0}$, where α (the vector of α_is) is not $\mathbf{0}$. Note: α_{m+1} cannot be 0, otherwise S is lin.dep. So $\mathbf{x} = \sum_{i=1}^{m}(-\frac{\alpha_i}{\alpha_{m+1}})\mathbf{s}_i$. So $\mathbf{x} \in \text{LS}(S)$.

Conversely, let $\mathbf{x} \in \text{LS}(S)$. So $\mathbf{x} = \sum_{i=1}^{k} \alpha_i \mathbf{s}_i$, for some $\alpha_i \in \mathbb{F}, \mathbf{s}_i \in S$, not all zero. So $\mathbf{x} - \sum_{i=1}^{k} \alpha_i \mathbf{s}_i = \mathbf{0}$. So $S \cup \{\mathbf{x}\}$ is lin.dep. $\square$

Cor Let $S \subseteq \mathbb{V}$ be lin.ind, $\mathbf{x} \in \mathbb{V} \setminus S$. Then $S \cup \{\mathbf{x}\}$ is lin.ind if and only if $\mathbf{x} \notin \text{LS}(S)$. ‼

Cor Let $S \subseteq \mathbb{V}$ be lin.ind. Then $\text{LS}(S) = \mathbb{V}$ if and only if each proper superset of S is lin.dep. ‼

Th Let $A = ER$, where $R = \text{RREF}\, A$ and E is invertible. Then rows of A corresp. to the pivotal rows of R are lin.ind. Also columns of A corresp. to the pivotal columns of R are lin.ind.

Po. Pivotal rows of R are lin.ind due to the pivotal 1s. Let the pivotal rows of R be $R(1,:), \ldots, R(k,:)$ and R_1 be the submatrix formed by these rows. Let A_1 be the submatrix of A which gives R_1.

So $R_1 = \text{RREF}(A_1)$. That is, $A_1 = E_1 R_1$, for some invertible E_1.

Suppose there exists $\mathbf{x}_0 \neq \mathbf{0}$ s.t. $\mathbf{x}_0^t A_1 = \mathbf{0}$. So $\mathbf{x}_0^t E_1 R_1 = \mathbf{0}$. Note that $\mathbf{x}_0^t E_1 = \mathbf{y}_0^t \neq \mathbf{0}$. So $\mathbf{y}_0^t R_1 = \mathbf{0}$, a contradiction.

Other part: note that pivotal columns in R are lin.ind, due to the pivotal 1s. Let $R(:, j_1), \ldots, R(:, j_k)$ be the pivotal columns of R. Since E is invertible, $A(:, j_1) = ER(:, j_1), \ldots, A(:, j_k) = ER(:, j_k)$ are lin.ind.

Th Let $\mathbf{x}_1, \cdots, \mathbf{x}_n$ be lin.dep, $\mathbf{x}_1 \neq \mathbf{0}$. Then, there exists $k > 1$ s.t. $\mathbf{x}_k$ is a lin.comb of $\mathbf{x}_1, \cdots, \mathbf{x}_{k-1}$.

Po. Consider $\{\mathbf{x}_1\}, \{\mathbf{x}_1, \mathbf{x}_2\}, \cdots, \{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n\}$ one by one. Take the smallest $k > 1$ s.t. $S_k = \{\mathbf{x}_1, \cdots, \mathbf{x}_k\}$ is lin.dep. So S_{k-1} is lin.ind. In that case $\mathbf{x}_k \in \text{LS}(S_{k-1})$. $\square$

- Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subseteq \mathbb{R}^n$ be lin.dep, $\mathbf{v}_1 \neq \mathbf{0}$. One may take help of GJE to find out the first $\mathbf{v}_i$ which is a lin.comb of the earlier ones and to find a lin.ind subset $T \subseteq S$ s.t. $\text{LS}\, T = \text{LS}\, S$.
- $\mathbf{v}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix}1\\-1\\1\end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix}0\\1\\2\end{bmatrix}$. $A = \begin{bmatrix}1&1&1&0\\1&0&-1&1\\1&1&1&2\end{bmatrix}$.
- $\text{RREF}\, A = \begin{bmatrix}\boxed{\begin{matrix}1\\0\\0\end{matrix}}&\boxed{\begin{matrix}0\\1\\0\end{matrix}}&\begin{matrix}\boxed{\begin{matrix}-1\\2\end{matrix}}\\0\end{matrix}&\boxed{\begin{matrix}0\\0\\1\end{matrix}}\end{bmatrix}$. So $v_3 = 2v_2 - v_1$; $T = \{v_1, v_2, v_4\}$.