Natural Deduction.

Rules for implication.

Modus Ponens.

$$\frac{\phi \quad \phi \to \psi}{\psi} \to e$$

if today is Wednesday then there will be a quiz.
$P \to q$

Today is wednesday.

Conclusion: Therefore there will be a quiz.

Curry - Howard Correspondence.

if F is a function from $P \to Q$ and
x is of type P then $F x$ is of type Q.

Example.

$p \to (q \to r), \; p \to q, \; p \vdash r$

1.	$p \to (q \to r)$	premise
2.	$p \to q$	premise.
3.	p	premise.
4	$q \to r$	$\to e$ 1, 3
5	q	$\to e$ 2,3
6	r	$\to e$ 4,5

Modus Tollens.

Suppose $p \to q$ and $\neg q$

if p holds then use $\to_e$ to conclude that q holds.

Conclusion: $\neg p$ is true

$$\frac{\phi \to \psi \qquad \neg\psi}{\neg\phi}\ MT$$

Example.

$$p \to (q \to r),\ p,\ \neg r \vdash \neg q$$

1.	$p \to (q \to r)$	Premises.
2.	p.	Premises.
3.	$\neg r$	Premises.
4	$q \to r$	$\to_e$ 1, 2
5.	$\neg q$	MT 4, 3

Example.

$p \to \neg q, \quad q \vdash \neg p.$

1. $p \to \neg q$ } premises.
2. q
3. $\neg\neg q$ $\neg\neg i$ 2
4. $\neg p$ MT 1,3

$p \to q,\ \neg q \vdash \neg p$

$p \to q \vdash \neg q \to \neg p.$

Suppose $p \to q$ holds.

Assume that $\neg q$ holds, using MT we can infer $\neg p$

Given the premise $p \to q$, Assuming $\underbrace{\neg q \text{ we can infer } \neg p}_{\neg q \to \neg p.}$

1.	$p \to q$	Premise.
2	$\neg q$	Assumption.
3	$\neg p$	MT 1,2
4	$\neg q \to \neg p$	$\to i$ 2-3

Rule - implication introduction.

$$\frac{\boxed{\begin{matrix}\phi \\ \vdots \\ \psi\end{matrix}}}{\phi \to \psi}\ \to i.$$

Example. $\neg q \to \neg p \vdash p \to \neg\neg q$

1.	$\neg q \to \neg p$	premise.
2.	p	Assumption
3.	$\neg\neg p$	$\neg\neg i$ 2
4	$\neg\neg q$	MT 1,3
5	$p \to \neg\neg q$	$\to i$ 2-4

Theorems.

Exemple. Formulas ϕ with valid sequent $\vdash \phi$

$$\vdash (q \to r) \to ((\neg q \to \neg p) \to (p \to r))$$

1.	$q \to r$	Assumption
2.	$\neg q \to \neg p$	Assumption
3.	p	Assumption
4	$\neg\neg p$	$\neg\neg i$ 3
5	$\neg\neg q$	MT 2,4
6	q	$\neg\neg e$ 5
7	r	$\to e$ 1,6
8	$p \to r$	$\to i$ 3-7
9	$(\neg q \to \neg p) \to (p \to r)$	$\to i$ 2-8
10	$(q \to r) \to ((\neg q \to \neg p) \to (p \to r))$	$\to i$ 1-9

Example

$$p \wedge q \to r \vdash p \to (q \to r)$$

1.	$p \wedge q \to r$	premise
2.	p	Assumption
3	q	Assumption
4	$p \wedge q$	$\wedge i$ 2,3
5	r	$\to e$ 1,4
6	$q \to r$	$\to i$ 3-5
7	$p \to (q \to r)$	$\to i$ 2-6

Example

$$p \to (q \to r) \vdash p \wedge q \to r$$

1.	$p \to (q \to r)$	Premise
2.	$p \wedge q$	Assumption
3	p	$\wedge e_1$ 2
4	q	$\wedge e_2$ 2
5	$q \to r$	$\to e$ 1,3
6	r	$\to e$ 5,4
7	$p \wedge q \to r$	$\to i$ 2-6

Rules of disjunction.

$$\frac{\phi}{\phi \vee \psi} \vee i_1 \qquad \frac{\psi}{\phi \vee \psi} \vee i_2$$