Marking Scheme for the end semester examination of MTH101, 2013-14 (I)

1. (a) Let $x_1 = 0, x_2 = 1$ and $x_{n+2} = \frac{x_n + x_{n+1}}{2}$ for $n \in \mathbb{N}$. Show that (x_n) converges and find its limit. [5]

Marking Scheme:

Observe that $|x_{n+2} - x_{n+1}| = \frac{1}{2}|x_{n+1} - x_n|$ [2]

The sequence satisfies the Cauchy criterion and hence it converges.

Note that $x_{n+2} + \frac{x_{n+1}}{2} = x_{n+1} + \frac{x_n}{2}$. [2]

If $x_n \to \ell$ then $\ell + \frac{\ell}{2} = 1$ and hence $\ell = \frac{2}{3}$. [1]

(b) Suppose that $f : [-1, 1] \to \mathbb{R}$ is three times differentiable s.t. $f(-1) = 0, f(1) = 1$ and $f'(0) = 0$. Show that there exists c in $(-1, 1)$ such that $f'''(c) \geq 3$. [5]

Marking Scheme:

By Taylor's Theorem $f(1) = f(0) + f'(0) + \frac{f''(0)}{2} + \frac{f'''(c_1)}{6}$ for some $c_1 \in (0, 1)$ [2]

and $f(-1) = f(0) - f'(0) + \frac{f''(0)}{2} - \frac{f'''(c_2)}{6}$ for some $c_2 \in (-1, 0)$ [1]

Therefore $\frac{f'''(c_1)+f'''(c_2)}{6} = 1$. [1]

Hence either $f'''(c_1)$ or $f'''(c_2) \geq 3$. [1]

(c) Let $f : [0, 1] \to \mathbb{R}$ be continuous and $a_n = f(\sin \frac{\pi}{n}) - f(\sin \frac{\pi}{n+1})$ for $n = 1, 2, ...$

i. Show that $\sum_{n=1}^{\infty} a_n$ converges and find its sum/limit.

ii. If f is differentiable and $|f'(x)| < 1$ for all $x \in [0, 1]$, discuss the convergence/divergence of $\sum_{n=1}^{\infty} |a_n|$. [6]

Marking Scheme:

i. The partial sum $S_n = f(0) - f(\sin \frac{\pi}{n+1})$ which converges. [2]
   Since $f(\sin \frac{\pi}{n+1}) \to f(0)$, the sum of the series is $f(0) - f(0) = 0$. [1]

ii. By MVT, $|f(\sin \frac{\pi}{n}) - f(\sin \frac{\pi}{n+1})| \leq |\sin \frac{\pi}{n} - \sin \frac{\pi}{n+1}| \leq |\frac{\pi}{n} - \frac{\pi}{n+1}|$. [2]
   Since $|a_n| \leq \frac{\pi}{n(n+1)}$, by comparison test, $\sum |a_n|$ converges. [1]

2. (a) Let $a_n = \frac{n^2}{n^3+200}$, $n \in \mathbb{N}$. Find the largest term of the sequence (a_n). [5]

Marking Scheme:

If $f(x) = \frac{x^2}{x^3+200}, x > 0$ then $f'(x) = \frac{x(400-x^3)}{(x^3+200)^2}$. [2]

The point of maximum for f is $400^{\frac{1}{3}}$. [1]

Since $7 < 400^{\frac{1}{3}} < 8$ and $a_7 = \frac{49}{543} > a_8 = \frac{8}{89}$, a_7 is the largest term [2]

(b) Find $\lim_{n\to\infty} \frac{1}{n^{18}} \sum_{k=1}^{n} k^{16}$. [4]

Marking Scheme:

Note that $\frac{1}{n^{18}} \sum_{k=1}^{n} k^{16} = \frac{1}{n}\left[\frac{1}{n}\sum_{k=1}^{n}(\frac{k}{n})^{16}\right]$ [2]
and $\frac{1}{n}\sum_{k=1}^{n}(\frac{k}{n})^{16} \to \int_0^1 x^{16}dx$. [1]
Therefore $\lim_{n\to\infty} \frac{1}{n^{18}} \sum_{k=1}^{n} k^{16} = 0$. [1]

(c) Let $f : [0,1] \to \mathbb{R}$ be twice differentiable such that $\int_0^1 f(x)dx < f(\frac{1}{2})$. Show that there exists $x_0 \in [0,1]$ such that $f''(x_0) \leq 0$. [7]

Marking Scheme:

Suppose $f''(x) > 0$ for all $x \in [0,1]$. [2]
Then by Taylor's Theorem, $f(x) \geq f(\frac{1}{2}) + f'(\frac{1}{2})(x - \frac{1}{2})$ for all $x \in [0,1]$. [3]
This implies that $\int_0^1 f(x)dx \geq f(\frac{1}{2}) + f'(\frac{1}{2})\frac{1}{2} - f'(\frac{1}{2})\frac{1}{2} = f(\frac{1}{2})$. [2]
which is a contradiction.

3. (a) Using the Riemann criterion show that every increasing function on $[0,1]$ is integrable. [5]

Marking Scheme:

Let $f[0,1] \to \mathbb{R}$ be increasing.
For $n \in \mathbb{N}$, consider the partition $P_n = \{x_0, x_1, x_2, ..., x_n\}$, where $x_i = \frac{i}{n}$. [1]
Then $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\} = f(x_i)$ and
$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\} = f(x_{i-1})$. [2]
Then $U(P_n, f) - L(P_n, f) = \frac{1}{n}\sum_{i=1}^{n}[f(x_i) - f(x_{i-1})] = \frac{1}{n}[f(1) - f(0)] \to 0$. [2]
By the Riemann Criterion the function is integrable.

(b) Derive an equation for the surface generated by revolving the curve $4x^2 + 9y^2 = 36, z = 0$ around the y-axis. [5]

Marking Scheme:

Let $P(x, y, z)$ be any point on the surface.
Consider a point $Q = (x_0, y, 0)$ on the curve for some x_0. [2]
Note that the distance from Q to the y-axis and the distance from P to the y-axis are the same. [1]
Therefore $x_0^2 = x^2 + z^2$. [1]
An equation of the surface is $4(x^2 + z^2) + 9y^2 = 36$. [1]

(c) Find a point on the curve $y = e^x$ at which the curvature is maximum. [6]

Marking Scheme:

Observe that $\kappa(x) = \frac{|f''(x)|}{[1+(f'(x))^2]^{\frac{3}{2}}} = \frac{e^x}{(1+e^{2x})^{\frac{3}{2}}}$ [2]

and $\kappa'(x) = \frac{e^x(1+e^{2x})^{\frac{1}{2}}(1-2e^{2x})}{(1+e^{2x})^3}$. [2]

The curvature is maximum at $(\frac{1}{2}\ln\frac{1}{2}, \frac{1}{\sqrt{2}})$. [2]

4. Consider the function $f(x,y) = \frac{3x^2y-y^3}{x^2+y^2}$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$.

(a) Verify whether f is continuous at $(0,0)$. [3]

Marking Scheme:

$|f(x,y) - f(0,0)| \leq \frac{|y||3x^2-y^2|}{x^2+y^2} \leq \frac{|y||3x^2+3y^2|}{x^2+y^2} \leq 3|y| \to 0$ as $(x,y) \to (0,0)$. [3]

(b) Find the directional derivatives of f at $(0,0)$ in the directions $(0,1), (1,0)$ and $\frac{1}{\sqrt{2}}(1,1)$. [4]

Marking Scheme:

$f_x(0,0) = \lim_{t\to 0} \frac{f(t,0)}{t} = 0$ [1]

$f_y(0,0) = \lim_{t\to 0} \frac{f(0,t)}{t} = -1$ [1]

$Df_{(0,0)}(\frac{1}{\sqrt{2}}(1,1)) = \lim_{t\to 0} \frac{f(\frac{t}{\sqrt{2}}(1,1))}{t}$ [1]

Therefore $Df_{(0,0)}(\frac{1}{\sqrt{2}}(1,1)) = \frac{1}{\sqrt{2}}$ [1]

(c) Using (b) (NOT the definition of differentiability) verify whether f is differentiable at $(0,0)$. [4]

Marking Scheme:

If f is differentiable at $(0,0)$, then $Df_{(0,0)}(\frac{1}{\sqrt{2}}(1,1)) = (f_x, f_y)|_{(0,0)} \cdot \frac{1}{\sqrt{2}}(1,1)$ [3]

But $Df_{(0,0)}(\frac{1}{\sqrt{2}}(1,1)) = \frac{1}{\sqrt{2}}$ and $(f_x, f_y)|_{(0,0)} \cdot \frac{1}{\sqrt{2}}(1,1) = -\frac{1}{\sqrt{2}}$ [1]

(d) Evaluate $f_y(x,0)$ for $x \neq 0$. [3]

Marking Scheme:

Since $f_y(x,0) = \lim_{t\to 0} \frac{f(x,t)-f(x,0)}{t}$ [2]

$f_y(x,0) = 3$. [1]

(e) Verify whether f_y is continuous at $(0,0)$. [2]

Marking Scheme:

Note that $f_y(x,0) = 3 \nrightarrow f_y(0,0) = -1$ as $x \to 0$. [2]

5. (a) Let S be the sphere $x^2 + y^2 + z^2 = 1$. Evaluate the surface integral $I = \iint_S (2x^2 - y^2 + 2z^2 + 3e^{z^2}x - e^{x^2}y + z\cos^2 y)d\sigma$. [6]

Marking Scheme:

Note that $I = \iint_S \left[2x + 3e^{z^2}, -y - e^{x^2}, 2z + \cos^2 y\right] \cdot (x, y, z)d\sigma$ [2]

$n = (x, y, z)$ is the unit normal to S and let $F(x, y, z) = \left[2x + 3e^{z^2}, -y - e^{x^2}, 2z + \cos^2 y\right]$.

Hence $I = \iint_S F \cdot nd\sigma$. [2]

By divergence Theorem $I = \iiint_D divFdV$ where D is the solid sphere. [1]

Therefore $I = 4\pi$. [1]

(b) Evaluate the line integral $I = \oint_C zdx + (x + e^{y^2})dy + (y + e^{z^2})dz$ where C is the curve which is the intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. Orient C counterclockwise as viewed from above. [6]

Marking Scheme:

By Stokes' Theorem $I = \iint_S (curlF) \cdot nd\sigma$ [2]

where S is the portion of the plane lying inside the cylinder, $F(x, y, z) = (z, x + e^{y^2}, y + e^{z^2})$ and n is the unit outward normal to the said portion.

Note that $curlF = i + j + k$ [1]

and $n = \frac{1}{\sqrt{2}}(0, 1, 1)$ [1]

Hence $I = \sqrt{2} \iint_R \sqrt{1 + f_x^2 + f_y^2}dxdy$ where $R := x^2 + y^2 \leq 1$ and $f(x, y) = 2 - y$. [1]

Therefore $I = 2\pi$ [1]

(c) Consider the circle in the yz-plane with center $y = 5, z = 0$ and radius 3. Let S be the surface obtained by rotating this circle about the z-axis. [6]

i. Find a parametric equation/representation to describe this surface with one parameter θ, where θ is described below. If (x, y, z) is any point on the surface then θ is the angle between the x-axis and the line joining $(0, 0, 0)$ and $(x, y, 0)$.

Marking Scheme:

$x = (5 + 3\cos\phi)\cos\theta, y = (5 + 3\cos\phi)\sin\theta, z = 3\sin\phi$ [2]

where $0 \leq \theta \leq 2\pi, 0 \leq \phi \leq 2\pi$ [1]

and ϕ is the angle between the line joining (x, y, z) and the center of the moving circle (which contains (x, y, z)) with the xy-plane

ii. Set up a single integral (with one variable) to find $\iint_S zd\sigma$.

Marking Scheme:

$\iint_S zd\sigma = \int_0^{2\pi}\int_0^{2\pi}(3\sin\phi)\sqrt{EG - F^2}d\theta d\phi$. [1]

Since $\sqrt{EG - F^2} = 3(5 + 3\cos\phi)$ [1]

$\iint_S zd\sigma = 18\pi\int_0^{2\pi}(\sin\phi)(5 + 3\cos\phi)d\phi$. [1]

6. (a) Find the points of absolute maximum and absolute minimum of the function $f(x,y) = x^2+y^2-2x+2$ on the region $\{(x,y) : x^2+y^2 \leq 4 \text{ with } y \geq 0\}$. [6]

Marking Scheme:

$f_x = 0 \Rightarrow x = 1$ and $f_y = 0 \Rightarrow y = 0$ [1]

There is no critical point in the interior of the region.

On the curve $x^2+y^2 = 4, y \geq 0$, the function is $x^2+4-x^2-2x+2 = -2x+6$ [1]

The candidates for the points of maxima/minima are $(-2,0)$ and $(2,0)$. [1]

On the line segment joining $(-2,0)$ and $(2,0)$, the function is x^2-2x+2 and the critical point is $(1,0)$. [1]

Since $f(-2,0) = 10$ and $f(2,0) = 2$ and $f(1,0) = 1$, [1]

the point of maximum is $(-2,0)$ and the point of minimum is $(1,0)$. [1]

(b) Find the volume of the solid that lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = 1$. [6]

Marking Scheme:

Solving $z = \sqrt{x^2+y^2}$ and $x^2+y^2+z^2 = 1$ we get $x^2+y^2 = \frac{1}{2}$. [1]

The required volume is $I = \iint_R (\sqrt{1-x^2-y^2} - \sqrt{x^2+y^2})dxdy$ [2]

where $R = \{(x,y) : x^2+y^2 \leq \frac{1}{2}\}$.

Use polar co-ordinates to get $I = \int_0^{\frac{1}{\sqrt{2}}} \int_0^{2\pi} (\sqrt{1-r^2} - r) r d\theta dr$ [2]

The value of $I = \frac{2\pi}{3}[1 - \frac{1}{\sqrt{2}}]$ [1]

(c) Let D be the solid cone bounded below by $z = \sqrt{x^2+y^2}$ and above by $z = 2$. Convert the integral $\iiint_D z dV$ as iterated integrals of the form $\int_a^b \int_c^d \int_e^f g(\rho,\theta,\phi) d\rho d\phi d\theta$ for some a,b,c,d,e,f,g where ρ, ϕ and θ are spherical co-ordinates. [6]

Marking Scheme:

The required integral is $\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\frac{2}{\cos\phi}} \rho\cos\phi(\rho^2 \sin\phi) d\rho d\phi d\theta$

$a = 0, c = 0$ and $e = 0$ [1]

$b = 2\pi$ and $d = \frac{\pi}{4}$ [2]

$f = \frac{2}{\cos\phi}$ [2]

$g(\rho,\theta,\phi) = \rho^3 \cos\phi \sin\phi$ [1]