MTH 102A - Linear Algebra - 2015-16-II Semester

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P. Linear Transformations

- Let $\mathbb{V}, \mathbb{W}$ be VS's over $\mathbb{F}$. A function $T : \mathbb{V} \to \mathbb{W}$ is a linear map/ transformation if $T(\boldsymbol{u}+\alpha\boldsymbol{v}) = T(\boldsymbol{u})+\alpha T(\boldsymbol{v})$ for each $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{V}$, $\alpha \in \mathbb{F}$. The class of all linear transformations (LT) from $\mathbb{V}$ to $\mathbb{W}$ is denoted by $\mathcal{L}(\mathbb{V}, \mathbb{W})$.
- Define $T(\boldsymbol{v}) = \mathbf{0}$ for all $v \in \mathbb{V}$. Then T is a LT from $\mathbb{V}$ to any $\mathbb{W}$.
- Define $T(\boldsymbol{v}) = \boldsymbol{v}$ for all $\boldsymbol{v} \in \mathbb{V}$. Then T is a LT from $\mathbb{V}$ to $\mathbb{V}$, called the identity map.
- Fix $A_{m\times n}$. For $\boldsymbol{v} \in \mathbb{C}^n$ define $T(\boldsymbol{v}) = A\boldsymbol{v}$. Then $T \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^m)$.
- Let $\mathbb{V} = \mathbb{R}[x]$ and $T(p(x)) = p'(x)$. Then $T \in \mathcal{L}(\mathbb{V}, \mathbb{V})$.
- For $\boldsymbol{f} \in \mathbb{V} = \mathcal{C}(\mathbb{R}, \mathbb{R})$, define $\boldsymbol{g} = T(\boldsymbol{f})$ as $\boldsymbol{g}(a) = \int_0^a \boldsymbol{f}(t)dt$. So $[T(\sin t)](a) = \int_0^a \sin t dt = 1-\cos a$. So $T(\sin t) = 1-\cos t$. In general $T(\boldsymbol{f}+\alpha\boldsymbol{g}) = F+\alpha G-F(0)-\alpha G(0)$, where $F = \int \boldsymbol{f}$ and $G = \int \boldsymbol{g}$. So $T \in \mathcal{L}(\mathbb{V}, \mathbb{V})$.
- Fix $\boldsymbol{z} \in \mathbb{R}^n$. For $\boldsymbol{x} \in \mathbb{R}^n$ define $T(\boldsymbol{x}) = \langle \boldsymbol{x}, \boldsymbol{z}\rangle$. Then $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R})$.
- Sometimes we write $T\boldsymbol{v}$ to mean $T(\boldsymbol{v})$.
- For $\boldsymbol{x} \in \mathbb{R}^2$ define $T(\boldsymbol{x}) = \begin{bmatrix} x_1 + x_2 \\ 3x_1 \end{bmatrix}$. Then $T \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$.
- For $\boldsymbol{x} \in \mathbb{R}^2$ define $T(\boldsymbol{x}) = \begin{bmatrix} x_1 x_2 \\ 3x_1 \end{bmatrix}$. Then T is not a LT.
- Let $T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$. Then $T(\mathbf{0}) = T(\mathbf{0} + \mathbf{0}) = T(\mathbf{0}) + T(\mathbf{0})$. So $T(\mathbf{0}) = \mathbf{0}$.
- Let $T \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ s.t. $T(\boldsymbol{e}_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $T(\boldsymbol{e}_2) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. So
  $T\begin{bmatrix} x \\ y \end{bmatrix} = xT\begin{bmatrix} 1 \\ 0 \end{bmatrix} + yT\begin{bmatrix} 0 \\ 1 \end{bmatrix} = x\begin{bmatrix} 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} x + y \\ x + 3y \end{bmatrix}$.
- Let $T \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ s.t. $T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $T(\boldsymbol{e}_2) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. What is $T\begin{bmatrix} x \\ y \end{bmatrix}$?
  We have $T(\boldsymbol{e}_1) = T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) - T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$.
  So $T\begin{bmatrix} x \\ y \end{bmatrix} = xT(\boldsymbol{e}_1) + yT(\boldsymbol{e}_2) = x\begin{bmatrix} -2 \\ 2 \end{bmatrix} + y\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -2x + 3y \\ 2x + y \end{bmatrix}$.

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• Alternate: $T\left(\begin{bmatrix}1&0\\1&1\end{bmatrix}\begin{bmatrix}\alpha\\\beta\end{bmatrix}\right) = T\left(\alpha\begin{bmatrix}1\\1\end{bmatrix}+\beta\begin{bmatrix}0\\1\end{bmatrix}\right) = \alpha T\begin{bmatrix}1\\1\end{bmatrix}+\beta T\begin{bmatrix}0\\1\end{bmatrix} = \alpha\begin{bmatrix}1\\3\end{bmatrix}+\beta\begin{bmatrix}3\\1\end{bmatrix}$ $= \begin{bmatrix}1&3\\3&1\end{bmatrix}\begin{bmatrix}\alpha\\\beta\end{bmatrix} = \begin{bmatrix}T\begin{bmatrix}1\\1\end{bmatrix} & T\begin{bmatrix}0\\1\end{bmatrix}\end{bmatrix}\begin{bmatrix}\alpha\\\beta\end{bmatrix}$.

So, $T\begin{bmatrix}x\\y\end{bmatrix} = T\left(\begin{bmatrix}1&0\\1&1\end{bmatrix}\left(\begin{bmatrix}1&0\\1&1\end{bmatrix}^{-1}\begin{bmatrix}x\\y\end{bmatrix}\right)\right) = \begin{bmatrix}1&3\\3&1\end{bmatrix}\left(\begin{bmatrix}1&0\\1&1\end{bmatrix}^{-1}\begin{bmatrix}x\\y\end{bmatrix}\right)$

$= \begin{bmatrix}1&3\\3&1\end{bmatrix}\begin{bmatrix}1&0\\-1&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-2x+3y\\2x+y\end{bmatrix}$.

• Let $\{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n\}$ be a basis of $\mathbb{V}$ and $T\in\mathcal{L}(\mathbb{V},\mathbb{W})$. Then T is determined by $\{T\boldsymbol{v}_1,\ldots,T\boldsymbol{v}_n\}$. If we know $\{T\boldsymbol{v}_1,\ldots,T\boldsymbol{v}_n\}$, then we know $T\boldsymbol{v}$ for all $v\in\mathbb{V}$ as $\boldsymbol{v}=\sum\alpha_i\boldsymbol{v}_i \Rightarrow T(\boldsymbol{v}) = \sum\alpha_i T(\boldsymbol{v}_i)$.

Th. Let $T\in\mathcal{L}(\mathbb{V},\mathbb{W})$ and $S\subseteq\mathbb{V}$ be lin.dep. Then $T(S)$ is lin.dep.

Pr. As S is lin.dep, there exist $\boldsymbol{v}_1,\cdots,\boldsymbol{v}_k\in\mathbb{V}$, $\boldsymbol{\alpha_i}\neq 0$ s.t. $\sum\alpha_i\boldsymbol{v}_i=\boldsymbol{0}$. So $\sum\alpha_i T\boldsymbol{v}_i = T(\sum\alpha_i\boldsymbol{v}_i) = T(\boldsymbol{0}) = \boldsymbol{0}$. Hence $\{T\boldsymbol{v}_1,\cdots,T\boldsymbol{v}_k\}$ is lin.dep. ■

• So $T(S)$ is linearly independent $\Rightarrow$ S is linearly independent.

P. Linear Transformations- Kernel and Range

• Let $T\in\mathcal{L}(\mathbb{V},\mathbb{W})$. Then
the kernel/null space of T is $\text{KER}\,T = \{\boldsymbol{v}\in\mathbb{V} | T\boldsymbol{v}=\boldsymbol{0}\}$.
The range/image of T is $\text{RNG}\,T = \{T\boldsymbol{v} : \boldsymbol{v}\in\mathbb{V}\}$.

• If $\text{DIM}\,\mathbb{V}$ is finite, we define $\text{RANK}\,T := \text{DIM RNG}\,T$; $\text{NULLITY}\,T := \text{DIM KER}\,T$.

Ex. Let $T\in\mathcal{L}(\mathbb{V},\mathbb{W})$. Is $\text{KER}\,T$ a subspace of $\mathbb{V}$? Is $\text{RNG}\,T$ a subspace of $\mathbb{W}$?

• Let $\mathcal{B}$ be a basis for $\mathbb{V}$ and $T\in\mathcal{L}(\mathbb{V},\mathbb{W})$. Then $\text{RNG}\,T = \text{LS}\,T(\mathcal{B})$.‼
T is determined by $\{T\boldsymbol{v} : \boldsymbol{v}\in\mathcal{B}\}$

• An LT from $\mathbb{V}$ to $\mathbb{V}$ is also called a linear operator on $\mathbb{V}$.

P. Linear Transformations-Ordered basis

• Is $\{\boldsymbol{v}_1=\boldsymbol{e}_1-\boldsymbol{e}_2, \boldsymbol{v}_2=\boldsymbol{e}_1+\boldsymbol{e}_2\}$ a basis of $\mathbb{R}^2$? Ans: Yes.

• Write $\boldsymbol{x}=\begin{bmatrix}4\\2\end{bmatrix}$ as lin.comb of $\boldsymbol{v}_1,\boldsymbol{v}_2$: Ans: $\boldsymbol{x}=\boldsymbol{v}_1+3\boldsymbol{v}_2$. Coefficient vector: $\begin{bmatrix}1\\3\end{bmatrix}$.

• Which $\boldsymbol{x}$ has coefficient vector $\begin{bmatrix}1\\2\end{bmatrix}$? $\boldsymbol{x}=\begin{bmatrix}3\\1\end{bmatrix}$. Here we use the ith coefficient for the ith basis vector v_i. That is, $\{\boldsymbol{v}_1,\boldsymbol{v}_2\}$ is ordered.

• An ordered basis $\mathcal{B}$ in $\mathbb{V}$ is an 'ordered set' of vectors that form a basis of $\mathbb{V}$. For finite dimension, it means calling them 1st basis vector, 2nd basis vector, etc.

• Let $\text{DIM}\,\mathbb{V}$ be finite, $\mathcal{B}$ be an ordered basis and $\boldsymbol{x}\in\mathbb{V}$. The coefficient vector (coordinate matrix) of $\boldsymbol{x}$ w.r.t $\mathcal{B}$ is denoted by $[\boldsymbol{x}]_{\mathcal{B}}$.

• In $\mathbb{R}^2$, take $\mathcal{B}=\left\{\begin{bmatrix}1\\-1\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right\}$, $\mathcal{B}'=\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix}\right\}$. Let $\boldsymbol{x}=\begin{bmatrix}\boldsymbol{x}_1\\\boldsymbol{x}_2\end{bmatrix}\in\mathbb{R}^2$.

Then $[\boldsymbol{x}]_{\mathcal{B}} = \frac{1}{2}\begin{bmatrix}x_1-x_2\\x_1+x_2\end{bmatrix}$ and $[\boldsymbol{x}]_{\mathcal{B}'} = \frac{1}{2}\begin{bmatrix}x_1+x_2\\x_1-x_2\end{bmatrix}$. Note that

$\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}[\boldsymbol{x}]_{\mathcal{B}} = \begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\frac{1}{2}\begin{bmatrix}\boldsymbol{x}_1-\boldsymbol{x}_2\\ \boldsymbol{x}_1+\boldsymbol{x}_2\end{bmatrix} = \boldsymbol{x}$ and $\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}[\boldsymbol{x}]_{\mathcal{B}'} = \boldsymbol{x}$.

• Let $\boldsymbol{v} \in \mathbb{R}^n$ and $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ be an ordered basis of $\mathbb{R}^n$. Form a matrix $B = [\boldsymbol{v}_1 \ \cdots \ \boldsymbol{v}_n]$. Then by definition $B[\boldsymbol{v}]_{\mathcal{B}} = \boldsymbol{v}$. We are considering the augmented matrix $[B \mid v]$ to get the entries of $[\boldsymbol{v}]_{\mathcal{B}}$.

P. Linear Transformations-Matrix-$[\boldsymbol{v}]_{\mathcal{B}} = B^{-1}\boldsymbol{v}$

Th. Let $\boldsymbol{v} \in \mathbb{R}^n$ and $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ be an ordered basis of $\mathbb{R}^n$. Put $B = [\boldsymbol{v}_1 \ \cdots \ \boldsymbol{v}_n]$ (this is called the basis matrix). Then

$B[[\boldsymbol{e}_1]_{\mathcal{B}} \ \cdots \ [\boldsymbol{e}_n]_{\mathcal{B}}] = [B[\boldsymbol{e}_1]_{\mathcal{B}} \ \cdots \ B[\boldsymbol{e}_n]_{\mathcal{B}}] = [\boldsymbol{e}_1 \ \cdots \ \boldsymbol{e}_n] = I$. Hence, B is invertible and $[\boldsymbol{v}]_{\mathcal{B}} = B^{-1}\boldsymbol{v}$.

• Let $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ be an ordered basis of $\mathbb{V}$. Let $\mathcal{B}' = \{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m\}$ be an ordered basis of $\mathbb{W}$ and $T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$.

• Let A be a matrix s.t. $A[\boldsymbol{v}]_{\mathcal{B}} = [T\boldsymbol{v}]_{\mathcal{B}'}$ for each $\boldsymbol{v} \in \mathbb{V}$. Then 'in some sense' A takes $\boldsymbol{v}$ to $T\boldsymbol{v}$. So we call A a coordinate matrix of T.

• Note: $A(:, i) = A\boldsymbol{e}_i = A[\boldsymbol{v}_i]_{\mathcal{B}} = [T\boldsymbol{v}_i]_{\mathcal{B}'}$, by definition. So A is unique.

• We denote the coordinate matrix of T by $T[\mathcal{B}, \mathcal{B}']$ or simply by $[T]$.

• When there is no mention of basis, we take the standard basis.

Example. Take $T\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}x_1+x_2\\ x_1-x_2\end{bmatrix}$ on $\mathbb{R}^2$.

a) Then $[T] = [[Te_1] \ \ [Te_2]] = \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$.

b) On the image space take the ordered basis $\mathcal{B}' = \left\{\begin{bmatrix}1\\ 0\end{bmatrix}, \begin{bmatrix}1\\ 1\end{bmatrix}\right\}$.

Then $[T] = [[Te_1]_{\mathcal{B}'} \ \ [Te_2]_{\mathcal{B}'}] = \begin{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}_{\mathcal{B}'} & \begin{bmatrix}1\\ -1\end{bmatrix}_{\mathcal{B}'}\end{bmatrix} = \begin{bmatrix}0 & 2\\ 1 & -1\end{bmatrix}$.

c) In b) take $\mathcal{B} = \left\{\begin{bmatrix}-1\\ 1\end{bmatrix}, \begin{bmatrix}3\\ 1\end{bmatrix}\right\}$ as the ordered basis of domain.

Then $[T] = \begin{bmatrix}\left[T\begin{bmatrix}-1\\ 1\end{bmatrix}\right]_{\mathcal{B}'} & \left[T\begin{bmatrix}3\\ 1\end{bmatrix}\right]_{\mathcal{B}'}\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}0\\ -2\end{bmatrix}_{\mathcal{B}'} & \begin{bmatrix}4\\ 2\end{bmatrix}_{\mathcal{B}'}\end{bmatrix} = \begin{bmatrix}2 & 2\\ -2 & 2\end{bmatrix}$.

• Fix $A \in \mathcal{M}_n(\mathbb{C})$. Take $T \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^n)$ defined as $T\boldsymbol{x} = A\boldsymbol{x}$. Let $\mathcal{B}$ be the standard basis in $\mathbb{C}^n$. Put $B = [T]$. Then $B(:, i) = [T\boldsymbol{e}_i]_{\mathcal{B}} = [A\boldsymbol{e}_i]_{\mathcal{B}} = [A(:, i)]_{\mathcal{B}} = A(:, i)$. Is this a bijection?

• Fix $A \in \mathcal{M}_n(\mathbb{C})$. Define $T \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^n)$ as $T\boldsymbol{x} = A\boldsymbol{x}$. Take ordered bases $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ on the domain and $\mathcal{B}' = \{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n\}$ on the range. Let B and B' be the respective basis matrices. Then

$[T] = [[T\boldsymbol{v}_1]_{\mathcal{B}'} \ \cdots \ [T\boldsymbol{v}_n]_{\mathcal{B}'}] = [B'^{-1}T\boldsymbol{v}_1 \ \cdots \ B'^{-1}T\boldsymbol{v}_n] = [B'^{-1}A\boldsymbol{v}_1 \ \cdots \ B'^{-1}A\boldsymbol{v}_n] = B'^{-1}A[\boldsymbol{v}_1 \ \cdots \ \boldsymbol{v}_n]$
$= B'^{-1}AB$.

• In particular, if $\mathcal{B} = \mathcal{B}'$, then we have $[T] = B^{-1}AB$.

- Let $\mathbb{V} = \mathbb{R}[t;3]$ and $\mathbb{W} = \mathbb{R}[t;2]$. Take $\mathcal{B} = \{1, t, t^2, t^3\}$ on $\mathbb{V}$ and $\mathcal{B}' = \{1, t, t^2\}$ on $\mathbb{W}$.

Take $T : \mathbb{V} \to \mathbb{W}$ defined as $T(f) = f'$, differentiation transformation. Then

$[T] = [[T1]_{\mathcal{B}'} \quad [Tt]_{\mathcal{B}'} \quad [Tt^2]_{\mathcal{B}'} \quad [Tt^3]_{\mathcal{B}'}]$

$= [[0]_{\mathcal{B}'} \quad [1]_{\mathcal{B}'} \quad [2t]_{\mathcal{B}'} \quad [3t^2]_{\mathcal{B}'}] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$

- General: let $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$, $\mathcal{B}' = \{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m\}$ be ordered bases in $\mathbb{V}$ and $\mathbb{W}$ with basis matrices B and B', respectively. Let $T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$. Then $Tv = T\sum \alpha_i v_i = \sum \alpha_i T v_i$

$= [Tv_1 \quad \cdots \quad Tv_n] \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = [w_1 \quad \cdots \quad w_m][[Tv_1]_{\mathcal{B}'} \quad \cdots \quad [Tv_n]_{\mathcal{B}'}] \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = [w_1 \quad \cdots \quad w_m] T[\mathcal{B}, \mathcal{B}'][v]_{\mathcal{B}} =$

$B'[T][v]_{\mathcal{B}} = B'[T]B^{-1}v \qquad \text{(R1)}$

- Fix $A \in \mathcal{M}_{m\times n}(\mathbb{R})$, bases $\mathcal{B}$ in $\mathbb{R}^n$ and $\mathcal{B}'$ in $\mathbb{R}^m$. Then there exists $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ s.t. $[T] = A$. Let B be the matrix of $\mathcal{B}$ and B' be that of $\mathcal{B}'$. Then by (R1)

$$T\boldsymbol{v} = B'A[\boldsymbol{v}]_{\mathcal{B}} = B'AB^{-1}\boldsymbol{v}. \qquad \text{(R2)}$$

- On $\mathbb{R}^2$ take $\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$ and $\mathcal{B}' = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$. Take $A = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$.
- Then the associated linear transformation is $T\boldsymbol{x} = B'A[\boldsymbol{x}]_{\mathcal{B}} = B'AB^{-1}\boldsymbol{x} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} -\frac{1}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$

$\begin{bmatrix} 0 & 4 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} -\frac{1}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$

P. Composition of Linear Transformations-Matrix Product

Ex. Let $\mathbb{U}, \mathbb{V}, \mathbb{W}$ be VS's over $\mathbb{F}$. Let $T \in \mathcal{L}(\mathbb{U}, \mathbb{V})$ and $S \in \mathcal{L}(\mathbb{V}, \mathbb{W})$. Define composition $ST : \mathbb{U} \to \mathbb{W}$ as $(ST)(\boldsymbol{x}) = S(T(\boldsymbol{x}))$. Then, ST is a linear transformation.

Now, let $\mathcal{B}, \mathcal{B}_1$ and $\mathcal{B}_2$ be bases of $\mathbb{U}, \mathbb{V}$ and $\mathbb{W}$, respectively. Then, by definition $[ST(\boldsymbol{x})]_{\mathcal{B}_2} = ST[\mathcal{B}, \mathcal{B}_2][\boldsymbol{x}]_{\mathcal{B}}$.

Also,

$$[ST(\boldsymbol{x})]_{\mathcal{B}_2} = [S\ (T(\boldsymbol{x}))]_{\mathcal{B}_2} = S[\mathcal{B}_1, \mathcal{B}_2][T(\boldsymbol{x})]_{\mathcal{B}_1} = S[\mathcal{B}_1, \mathcal{B}_2]T[\mathcal{B}, \mathcal{B}_1][\boldsymbol{x}]_{\mathcal{B}}.$$

Hence, for all $\boldsymbol{x} \in \mathbb{U}$, we have

$$ST[\mathcal{B}, \mathcal{B}_2][\boldsymbol{x}]_{\mathcal{B}} = S[\mathcal{B}_1, \mathcal{B}_2]T[\mathcal{B}, \mathcal{B}_1][\boldsymbol{x}]_{\mathcal{B}}.$$

Thus,

$$ST[\mathcal{B}, \mathcal{B}_2] = S[\mathcal{B}_1, \mathcal{B}_2]T[\mathcal{B}, \mathcal{B}_1].$$