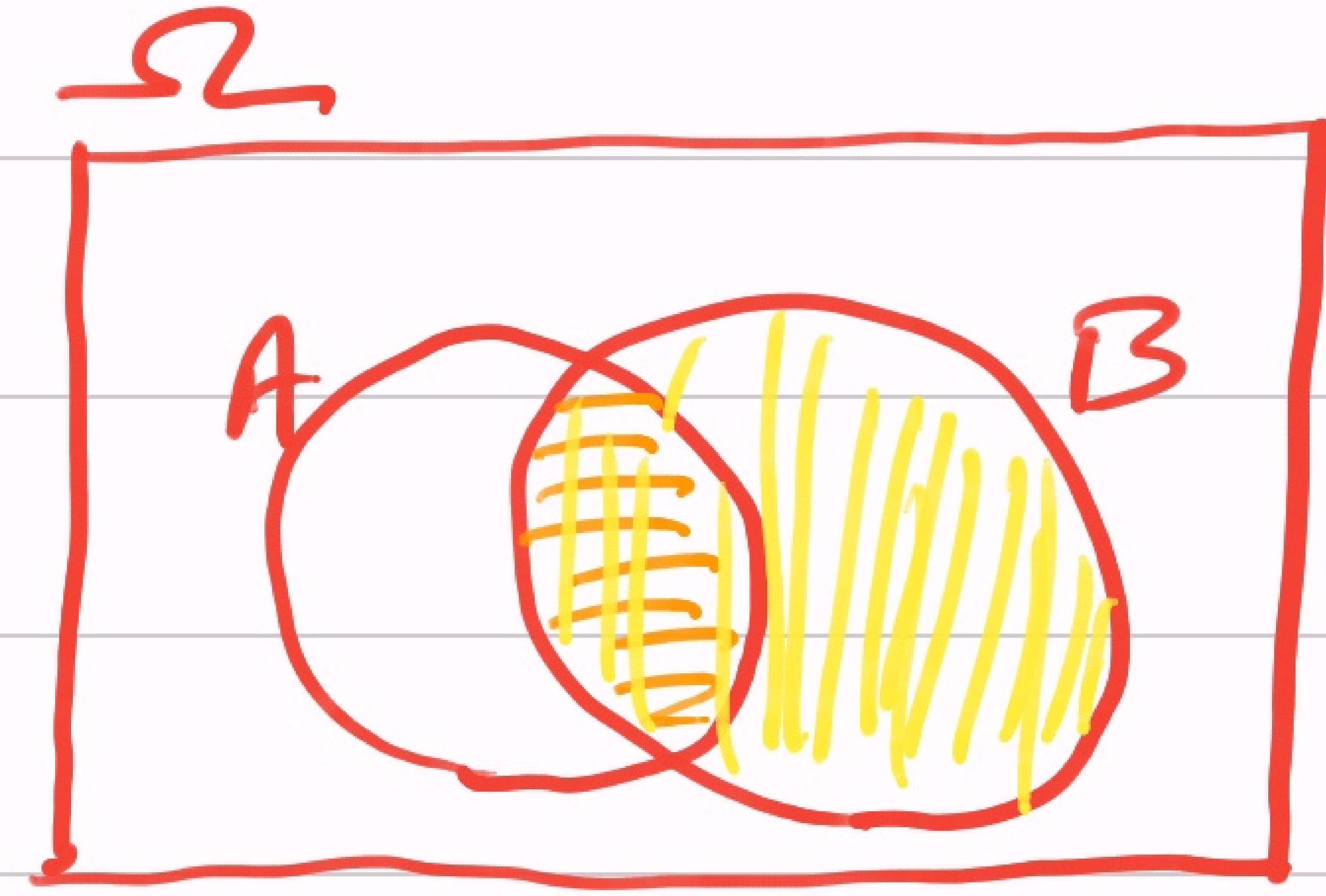
Conditional Probability

- Qn to ask: How does extra knowledge B affect the probability of an event A?

- This qn is different from complement, union, intersection, minus.
    - ↳ Relates somewhat to $A \cap B$:



- Without B, analysis was: A vs A^c.
- With B, analysis is: $B \cap A$ vs $B \cap A^c$.
- Defn: Conditional Probability of A given B is:

$$\underline{P(A|B)} := P(A \cap B) \,/\, P(B).$$

- It's easy to deduce:

▷ $$P(A|B) = \frac{P(A\cap B)}{\underbrace{P(A\cap B) + P(A^c\cap B)}_{= P(B)}}.$$

▷ $$P(A\cap B) = P(B)\cdot P(A|B).$$

▷ $$ = P(A)\cdot P(B|A).$$

- Eg. • Suppose there are two coins:

1st has H & T (normal coin N)

2nd has H & H (biased coin B)

• I randomly picked one & tossed, to get H.

Qn: What's the probability that I picked coin B?

- Is it $1/2$?
- No, it's wrong. Because B is biased towards H, so intuitively chance of B is $> 1/2$.
- How much more?

▷ $$P(B \text{ picked} \mid H \text{ appears}) = \frac{P(B \text{ pick} \cap H \text{ appears})}{P(H \text{ appears})}$$

$$= \frac{P(B \cap H)}{P(B \cap H) + P(N \cap H)} = \frac{\frac{1}{2} \cdot 1}{\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2}} = \frac{2}{3}.$$

these two P(·) are calculated by looking at the 4 possibilities & the favorable cases.

Partition Formula

- The above example inspires us to simplify $P(A)$ in terms of a given partition of Ω.

  I.e, $\Omega = \bigsqcup_{i=1}^{m} B_i$, where B_i's are mutually disjoint & cover Ω.

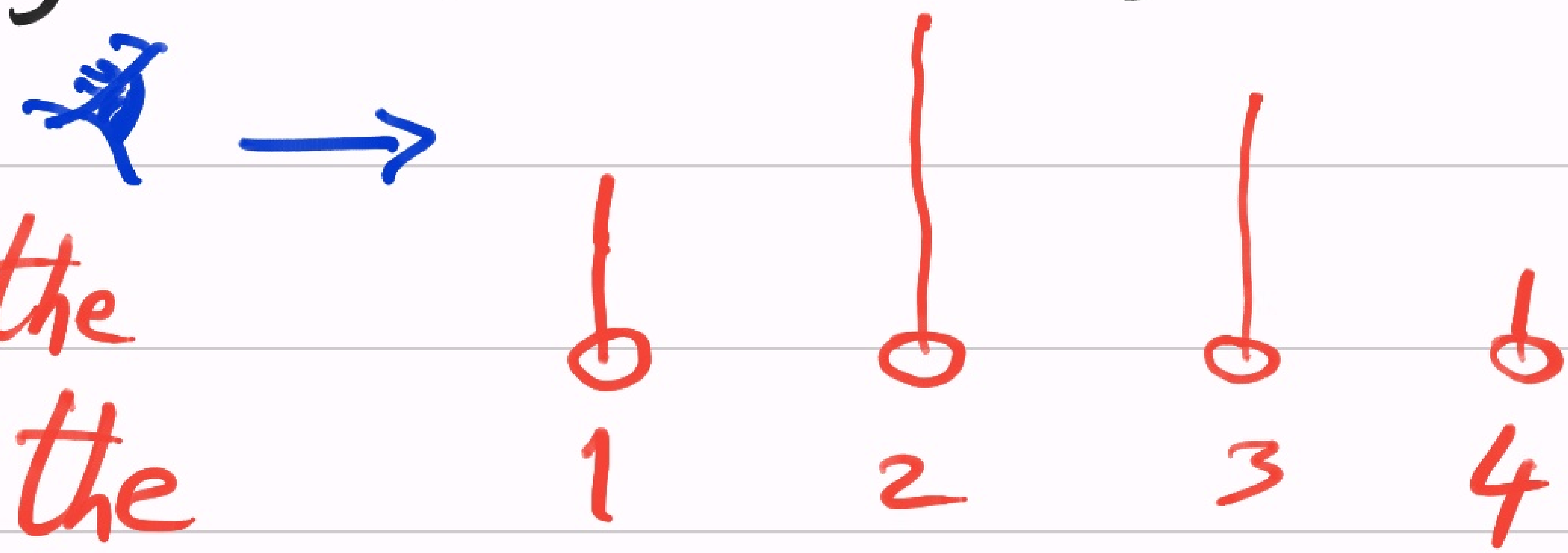
Lemma: $P(A) = \sum_{i=1}^{m} P(B_i) \cdot P(A|B_i)$ $. = \sum_i P(A \cap B_i)$

Pf:

- $A = \bigsqcup_{i=1}^{m} (A \cap B_i)$

$\Rightarrow P(A) = \sum_i P(A \cap B_i) = \sum_{i=1}^{m} P(B_i) \cdot P(A|B_i)$. $\square$

- You should try to find a good partition of Ω.
- e.g. Given n sticks of distinct lengths & n holes arranged in a line.
- Given $k \in [n]$: what's the probability that stick in the k-th hole is visible from the left? [Call the event A]



- $\Omega :=$ all permutations on sticks $\{1, \dots, n\}$.
- Partition 1: Arrange the sticks in increasing order of the length. $B_i :=$ permutations where i-th stick is in k-th hole.

▷ $\Omega = \bigsqcup_{i=1}^{n} B_i$ (is a partition).

▷ $P(A) = \sum_{i=1}^{n} \underbrace{P(B_i)}_{1/n} \cdot P(A|B_i)$

$= \dfrac{\binom{i-1}{k-1} \cdot (k-1)! \times (n-k)!}{(n-1)!}$

$$= \sum_{i=1}^{n} \frac{(i-1)! \times (n-k)!}{n! \times (i-k)!} = \frac{(n-k)!}{n!} \times \sum_{i=1}^{n} \frac{(i-1)!}{(i-k)!}$$

- Partition 2: Define event B_S: sticks from subset S of $[n]$ of size $=k$, go to holes $[k]$.

▷ $\Omega = \bigsqcup_{S \in \binom{[n]}{k}} B_S$. (Why?)

$$\Rightarrow P(A) = \sum_{S \in \binom{[n]}{k}} P(B_S) \cdot P(A \mid B_S)$$

prob. that the largest in S goes last!

$$= \sum_{S} P(B_S) \cdot \frac{1}{k} = \frac{1}{k} \cdot \sum_{S} P(B_S) = \frac{1}{k}.$$

- Qn: What do you make of $P(A \mid B) = P(A)$?