Q.1.] Correct options are (a.), (b.), (d.)

Q.2] (a.) It is a linear PDE.

Order = 2

(b.) It is a linear PDE

Order = 2

(c.) It is a linear PDE

Order = 2

(d.) It is a quasi-linear PDE.

Order = 2

(e.) It is a quasi-linear PDE.

Order = 2

(f.) It is a quasi-linear PDE.

Order = 1

(g.) It is a linear PDE

Order = 3

Q.3]

Problem 1: $\begin{cases} u_x - 2u_y = u & \text{in } \mathbb{R}^2 \\ u(x,0) = 1 \end{cases}$

From the usual notations:

$$a(x,y,u) = 1$$
$$b(x,y,u) = -2$$
$$c(x,y,u) = u$$

Curve Γ = x-axis

General point is : $(s,0,1)$.

Now, by method of characteristics:

$$x'(t) = 1 \Rightarrow x(t) = t + \lambda_1$$
$$y'(t) = -2 \Rightarrow y(t) = -2t + \lambda_2$$
$$u'(t) = u(t) \Rightarrow u(t) = \lambda_3 e^t$$

Now, at $t=0$, the characteristic curve passes through $(s,0,1)$.

So,
$$\lambda_1 = s$$
$$\lambda_2 = 0$$
$$\lambda_3 = 1$$

So,
$$x(t) = t+s$$
$$y(t) = -2t$$
$$u(t) = e^t$$

So, $u(x,y) = e^{-y/2}$

Q.4]
$$\begin{cases} u_x + u_y = u^{1/2} \\ u(x,0) = 0 \end{cases}$$

From the usual notations:

$a(x,y,u) = 1$

$b(x,y,u) = 1$

$c(x,y,u) = u^{1/2}$

Now, by observation, we can see that $u(x,y) = 0$ is a trivial solution to the above problem.

By method of characteristics:

$x'(t) = 1 \Rightarrow x(t) = t + \lambda_1$

$y'(t) = 1 \Rightarrow y(t) = t + \lambda_2$

$u'(t) = (u(t))^{1/2}$

Consider the 3^{rd} ODE:
$$\frac{d(u(t))}{dt} = (u(t))^{1/2}, \quad u(0) = 0.$$

By observation, $u(t) \equiv 0$ is a trivial solution. Also for $t \geq 0$
$$2\sqrt{u(t)} = t + \lambda_3, \quad u(0) = 0$$
$$\Rightarrow 2\sqrt{u(t)} = t$$
is also a valid solution.

$$\Rightarrow u(t) = \frac{t^2}{4}$$

Also, $\because x(0) = s, \ y(0) = 0$:

$x(t) = t + s$ and

$y(t) = t$

so, by eliminating t:

$u(x,y) = \dfrac{y^2}{4}$ is also a

solution.

No, the existence theorem is not violated. The theorem can be applied only when $c(x, y, u)$ is continuously differentiable in the required domain. But the derivative of $\sqrt{u}$ is not continuous at points where $u = 0$. Hence $u(x, y) = \frac{y^2}{4}$ is valid only for $y > 0$. Otherwise $u(x, y) = 0$ is a valid solution.

Q.5]

$$\begin{cases} uu_x + u_y = u^2 \\ u(x,0) = 1 \end{cases}$$

From the usual notations:

$a(x,y,u) = u$

$b(x,y,u) = 1$

$c(x,y,u) = u^2$

General point on the initial curve is $(s,0,1)$.

By, method of characteristics:

$x'(t) = u(t)$

$y'(t) = 1$

$u'(t) = (u(t))^2$

Now, $y'(t) = 1$, $y(0) = 0$

$\Rightarrow y(t) = t$

Now, $u'(t) = (u(t))^2$, $u(0) = 1$

$\Rightarrow \frac{-1}{u(t)} = t + \lambda_3$, $u(0) = 1$

$\Rightarrow \frac{-1}{u(t)} = t - 1$

$\Rightarrow u(t) = \frac{1}{1-t}$

So, $x'(t) = \frac{1}{1-t}$, $x(0) = s$

$\Rightarrow x(t) = -\ln(|1-t|) + \lambda_1$, $x(0) = s$

$\Rightarrow x(t) = -\ln(|t-1|) + s$

Eliminating t, we get :

$$u(x,y) = \frac{1}{1-y}$$

Date
Page

Q.6] $\begin{cases} u_x + u_y = 0 \\ u(x,x) = 1 \end{cases}$

From usual notations:

$a(x,y,u) = 1$

$b(x,y,u) = 1$

$c(x,y,u) = 0$

General point on initial curve is $(s,s,1)$.

By method of characteristics:

$x'(t) = 1 \Rightarrow x(t) = t + \lambda_1$

$y'(t) = 1 \Rightarrow y(t) = t + \lambda_2$

$u'(t) = 0 \Rightarrow u(t) = \lambda_3$

At $t=0$, characteristic curve passes through $(s,s,1)$.

So, $x(t) = t+s$

$y(t) = t+s$

$u(t) = 1$

So, $u(x,y) = 1$ and $x = y$

Hence, we can in general write a general function $u(x,y) = 1 + k(x-y)$ where $k \in \mathbb{R}$, which satisfies the above problem:

$\frac{\partial u}{\partial x} = k \quad \frac{\partial u}{\partial y} = -k \Rightarrow u_x + u_y = 0$

and $u(x,x) = 1$

Hence there are infinitely many solutions.

Correct option is (a).

Date
Page

Q.7) For problem 4:

Consider $x = y + s_1$
$x = y + s_2$
are 2 different PCs.

Clearly as the lines are parallel, the PCs don't intersect.

For problem 5:

Consider $x = -\ln|y-1| + s_1$
$x = -\ln|y-1| + s_2$
are 2 different PCs.

Now, $-(x - s_1) = \ln(|y-1|)$
$-(x - s_2) = \ln(|y-1|)$

So, $|y-1| = e^{-(x-s_1)}$
$|y-1| = e^{-(x-s_2)}$

These are just shifted copies of each other. They will only intersect when $s_1 = s_2$. Hence, since $s_1 \neq s_2$, the curves do not intersect.

Date
Page

Q.8] For the problem 4 :
let Γ be parametrized by $(s, 0)$.
So, $f(s) = s$, $h(s) = 0$
So, $f'(s) = 1$, $h'(s) = 0$

Now, $a(f(s), h(s), g(s)) = 1$ and
$b(f(s), h(s), g(s)) = 1$

Non - characteristics condition:
$$f'(s)\, a(f(s), h(s), g(s)) \neq h'(s)\, b(f(s), h(s), g(s))$$

Now, $f'(s)\, a(f(s), h(s), g(s)) = 1$
$h'(s)\, b(f(s), h(s), g(s)) = 0$

Hence, the Non- characteristics condition is satisfied.

For the toy problem 2 :
let Γ be parametrized by $(s, 0)$.
So, $f(s) = s$, $f'(s) = 1$
$h(s) = 0$, $h'(s) = 0$

Also, given: $a(f(s), h(s), g(s)) = 1$
$b(f(s), h(s), g(s)) = 0$

So, $f'(s)\, a(f(s), h(s), g(s)) = 1$
$h'(s)\, b(f(s), h(s), g(s)) = 0$

Hence, the Non- characteristic condition is satisfied.