MTH 102A - Linear Algebra - 2015-16-II Semester

Arbind Kumar Lal *

P. **Inner Product-Definition**

• Let $\mathbb{V}$ be a VS over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$). Take a function $f : \mathbb{V} \times \mathbb{V} \to \mathbb{F}$. Put $\langle \mathbf{u}, \mathbf{v} \rangle := f(\mathbf{u}, \mathbf{v})$. We call f an inner product if the following are satisfied.

1) $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ for all $\mathbf{v} \in \mathbb{V}$ and $\langle \mathbf{v}, \mathbf{v} \rangle = 0 \Rightarrow \mathbf{v} = \mathbf{0}$, for all $\mathbf{v} \in \mathbb{V}$.

2) $\langle \mathbf{v}, \mathbf{w} \rangle = \overline{\langle \mathbf{w}, \mathbf{v} \rangle}$ for all $\mathbf{v}, \mathbf{w} \in \mathbb{V}$.

3) $\langle \alpha\mathbf{v} + \mathbf{u}, \mathbf{w} \rangle = \alpha \langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$ for all $\alpha \in \mathbb{F}, \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{V}$.

• It is immediate that:

1) $\langle \mathbf{v}, \alpha\mathbf{w} \rangle = \overline{\alpha} \langle \mathbf{v}, \mathbf{w} \rangle$.

2) $\langle \mathbf{0}, \mathbf{0} \rangle = \langle 2 \times \mathbf{0}, \mathbf{0} \rangle = 2\langle \mathbf{0}, \mathbf{0} \rangle \Rightarrow \langle \mathbf{0}, \mathbf{0} \rangle = 0$.

3) If $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ for all $\mathbf{v} \in \mathbb{V}$, then in particular $\langle \mathbf{u}, \mathbf{u} \rangle = 0$. So, $\mathbf{u} = \mathbf{0}$.

VS: $\mathbb{R}^n$ over $\mathbb{R}$. IP: $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^t\mathbf{x} = \sum \mathbf{y}_i\mathbf{x}_i$. This is the usual IP.

VS: $\mathbb{C}^n$ over $\mathbb{C}$. IP: $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^*\mathbf{x} = \sum \overline{\mathbf{y}_i}\mathbf{x}_i$. This is the usual IP.

• VS: $\mathbb{R}^2$. IP: $\langle \mathbf{x}, \mathbf{y} \rangle = 2\mathbf{x}_1\mathbf{y}_1 + \mathbf{x}_1\mathbf{y}_2 + \mathbf{x}_2\mathbf{y}_1 + \mathbf{x}_2\mathbf{y}_2$.‼

$$= \begin{bmatrix} x_1 & y_1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}.$$

• Fix $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$, $a, c > 0$, $ac > b^2$. Then $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^t A\mathbf{x}$ is an IP on $\mathbb{R}^2$.‼ $\langle \mathrm{x}, \mathrm{x} \rangle = ax_1^2 + 2bx_1x_2 + cx_2^2 = a\left[x_1 + \frac{bx_2}{a}\right]^2 + \frac{1}{a}\left[ac - b^2\right]x_2^2$.

• VS: $\mathcal{M}_n(\mathbb{C})$. Then $\langle A, B \rangle = \text{TR}[AB^*] = \sum\limits_{i,j=1}^{n} a_{ij}\overline{b_{ij}}$ is an IP.‼

• VS: $\mathcal{C}([-1, 1], \mathbb{C})$. Then $\langle \mathbf{f}, \mathbf{g} \rangle := \int\limits_{-1}^{1} \mathbf{f}(x)\overline{\mathbf{g}(x)}dx$ is an IP.

Let us verify.

a) $\langle \mathbf{f}, \mathbf{f} \rangle = \int\limits_{-1}^{1} |\mathbf{f}(x)|^2 dx \geq 0$ as $|\mathbf{f}(x)|^2 \geq 0$ and this integral is 0 if and only if $\mathbf{f} \equiv \mathbf{0}$.

b) $\overline{\langle \mathbf{g}, \mathbf{f} \rangle} = \overline{\int\limits_{-1}^{1} \mathbf{g}(x)\overline{\mathbf{f}(x)}dx} = \int\limits_{-1}^{1} \overline{\mathbf{g}(x)\overline{\mathbf{f}(x)}}dx = \int\limits_{-1}^{1} \mathbf{f}(x)\overline{\mathbf{g}(x)}dx = \langle \mathbf{f}, \mathbf{g} \rangle$.

*Indian Institute of Technology Kanpur

c) $\langle \mathbf{f}+\mathbf{g}, \mathbf{h}\rangle = \int\limits_{-1}^{1} (\mathbf{f}+\mathbf{g})(x)\overline{\mathbf{h}(x)}dx = \int\limits_{-1}^{1} [\mathbf{f}(x)\overline{\mathbf{h}(x)} + \mathbf{g}(x)\overline{\mathbf{h}(x)}]dx = \int\limits_{-1}^{1} \mathbf{f}(x)\overline{\mathbf{h}(x)}dx + \int\limits_{-1}^{1} \mathbf{g}(x)\overline{\mathbf{h}(x)}dx = \langle \mathbf{f}, \mathbf{h}\rangle + \langle \mathbf{g}, \mathbf{h}\rangle$.

d) $\langle \alpha\mathbf{f}, \mathbf{g}\rangle = \int\limits_{-1}^{1} (\alpha\mathbf{f}(x))\overline{\mathbf{g}(x)}dx = \alpha \int\limits_{-1}^{1} \mathbf{f}(x)\overline{\mathbf{g}(x)}dx = \alpha\langle \mathbf{f}, \mathbf{g}\rangle$.

P. **Inner Product- A fundamental result**

- A VS with an IP specified on it is called an inner product space (IPS).

Th[Cauchy-Bunyakovskii-Schwartz inequality] Let $\mathbb{V}$ be an IPS and $\mathbf{x}, \mathbf{y} \in \mathbb{V}$. Then $|\langle \mathbf{x}, \mathbf{y}\rangle|^2 \le \langle \mathbf{x}, \mathbf{x}\rangle\langle \mathbf{y}, \mathbf{y}\rangle$. Equality holds if and only if x, y are linearly dependent.

Po. If $\mathbf{y} = \mathbf{0}$, the result holds with equality.

Let $y \neq \mathbf{0}$. Put $\alpha = \langle \mathbf{x}, \mathbf{y}\rangle / \langle \mathbf{y}, \mathbf{y}\rangle$. Then

$0 \le \langle \mathbf{x} - \alpha\mathbf{y}, \mathbf{x} - \alpha\mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{x}\rangle - \alpha\langle \mathbf{y}, \mathbf{x}\rangle - \overline{\alpha}\langle \mathbf{x}, \mathbf{y}\rangle + |\alpha|^2\langle \mathbf{y}, \mathbf{y}\rangle$

$= \langle \mathbf{x}, \mathbf{x}\rangle - \frac{\langle \mathbf{x},\mathbf{y}\rangle}{\langle \mathbf{y},\mathbf{y}\rangle}\langle \mathbf{y}, \mathbf{x}\rangle - \overline{\frac{\langle \mathbf{x},\mathbf{y}\rangle}{\langle \mathbf{y},\mathbf{y}\rangle}}\langle \mathbf{x}, \mathbf{y}\rangle + |\frac{\langle \mathbf{x},\mathbf{y}\rangle}{\langle \mathbf{y},\mathbf{y}\rangle}|^2\langle \mathbf{y}, \mathbf{y}\rangle$

$= \langle \mathbf{x}, \mathbf{x}\rangle - \frac{\langle \mathbf{x},\mathbf{y}\rangle}{\langle \mathbf{y},\mathbf{y}\rangle}\langle \mathbf{y}, \mathbf{x}\rangle = \langle \mathbf{x}, \mathbf{x}\rangle - |\langle \mathbf{x}, \mathbf{y}\rangle|^2 / \langle \mathbf{y}, \mathbf{y}\rangle$.

$0 \le \langle \mathbf{x} - \alpha\mathbf{y}, \mathbf{x} - \alpha\mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{x}\rangle - \alpha\langle \mathbf{y}, \mathbf{x}\rangle - \overline{\alpha}\langle \mathbf{x}, \mathbf{y}\rangle + |\alpha|^2\langle \mathbf{y}, \mathbf{y}\rangle$ Equality holds if and only if equality holds here.

If and only if $\langle \mathbf{x} - \alpha\mathbf{y}, \mathbf{x} - \alpha\mathbf{y}\rangle = 0$. Equivalently, $\mathbf{x} - \alpha\mathbf{y} = \mathbf{0}$. Or equivalently, $\mathbf{x}, \mathbf{y}$ are linearly dependent. □

Cor Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then $\left(\sum\limits_{i=1}^{n} \mathbf{x}_i\mathbf{y}_i\right)^2 \le \left(\sum\limits_{i=1}^{n} \mathbf{x}_i^2\right)\left(\sum\limits_{i=1}^{n} \mathbf{y}_i^2\right)$.**!!**

P. **Inner Product-Angle**

- Angle between $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n \setminus \{0\}$ is the $\theta \in [0, \pi]$ s.t. $\cos\theta = \dfrac{\langle \mathbf{x}, \mathbf{y}\rangle}{\sqrt{\langle \mathbf{x}, \mathbf{x}\rangle\langle \mathbf{y}, \mathbf{y}\rangle}}$.

- Take $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix} \in \mathbb{R}^2$; usual IP. Then $\cos\theta = \frac{1}{\sqrt{2}}$. So $\theta = \pi/4$.
- Take $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \in \mathbb{R}^3$; usual IP. Angle between them $\beta = \cos^{-1}\frac{2}{\sqrt{6}}$.
- Angle depends on the IP. Take $\langle \mathbf{x}, \mathbf{y}\rangle = 2\mathbf{x}_1\mathbf{y}_1 + \mathbf{x}_1\mathbf{y}_2 + \mathbf{x}_2\mathbf{y}_1 + \mathbf{x}_2\mathbf{y}_2$ on $\mathbb{R}^2$. Angle between $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$ is $\cos^{-1}\frac{3}{\sqrt{10}}$.
- Angle between $\mathbf{x}$ and $\mathbf{y}$ is the same as that between $\mathbf{y}$ and $\mathbf{x}$.

P. **Inner Product-Orthogonality**

- Euclidean space: a finite dimensional real IPS. Unitary space: a complex IPS.
- We say $\mathbf{x}$ is orthogonal to $\mathbf{y}$ ($\mathbf{x} \perp \mathbf{y}$) if $\langle \mathbf{x}, \mathbf{y}\rangle = 0$. Thus $\mathbf{0} \perp \mathbf{x}$ for all $\mathbf{x}$.

• Let $\emptyset \neq S \subseteq \mathbb{V}$, an IPS. The orthogonal complement $S^{\perp}$ of S is the set $\{\mathbf{y} \in \mathbb{V} \mid \mathbf{y} \perp \mathbf{x} \text{ for all } \mathbf{x} \in S\}$. We write $\mathbf{x}^{\perp}$ to denote $\{\mathbf{x}\}^{\perp}$.

• A set $E \subseteq \mathbb{V}$ is orthogonal if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ for all $\mathbf{x}, \mathbf{y} \in E$, $\mathbf{x} \neq \mathbf{y}$. Thus $\emptyset$ and $\{\mathbf{x}\}$ are orthogonal sets.

Q Let $a = [1 \quad 2]^t$. What is $a^{\perp}$ in $\mathbb{R}^2$? $\{[x_1 \quad x_2] \mid x_1 + 2x_2 = 0\}$. Is this a subspace? Of course yes. This is null space of $[1 \quad 2]$.

Q Let $S = \{\mathbf{x}, \mathbf{y}\} \subseteq \mathbb{R}^3$. Is $S^{\perp}$ a subspace of $\mathbb{R}^3$? Yes, as it is the intersection of the subspaces $\mathbf{x}^{\perp}$ and $\mathbf{y}^{\perp}$.

Th Let $\mathbb{V}$ be an IPS and $\emptyset \neq S \subseteq \mathbb{V}$. Then $S^{\perp}$ is a subspace. ‼

P. **Inner Product- Norm**

• A linear space is a vector space over $\mathbb{F}$ (that is, $\mathbb{R}$ or $\mathbb{C}$).

• A norm on a linear space $\mathbb{V}$ is a function $f(\mathbf{x}) = \|\mathbf{x}\|$ from $\mathbb{V}$ to $\mathbb{R}$ s.t.

a) $\|\mathbf{x}\| \geq 0$ for all $\mathbf{x}$ and $\|\mathbf{x}\| = 0 \Rightarrow \mathbf{x} = \mathbf{0}$.

b) $\|\alpha \mathbf{x}\| = |\alpha| \|\mathbf{x}\|$ for all $\alpha, \mathbf{x}$.

c) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y}$.

• A linear space with a norm on it is a normed linear space (NLS).

• Let $\mathbb{V}$ be a NLS and $\mathbf{x}, \mathbf{y} \in \mathbb{V}$. Then $\Big| \|\mathbf{x}\| - \|\mathbf{y}\| \Big| \leq \|\mathbf{x} - \mathbf{y}\|$. ‼

• On $\mathbb{R}^3$, $\|\mathbf{x}\| = \sqrt{\mathbf{x}_1^2 + \mathbf{x}_2^2 + \mathbf{x}_3^2}$ is a norm. It is nothing but $\sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$.

Q Let $\mathbb{V}$ be an IPS. Must $f(\mathbf{x}) = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ be a norm? Yes. Verifying a),b) is easy. Let us verify c)

$\|\mathbf{x} + \mathbf{y}\|^2 = \langle \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle \leq ?\|\mathbf{x}\|^2 + \|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{y}\|^2 = (\|\mathbf{x}\| + \|\mathbf{y}\|)^2$.

• $\|\mathbf{x}\| = \langle \mathbf{x}, \mathbf{x} \rangle^{\frac{1}{2}}$ is called the norm induced by the IP $\langle \cdot, \cdot \rangle$.

Ex [Polar Identity] The following identity holds in an IPS.

complex IPS $\quad 4\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x} - \mathbf{y}\|^2 + i\|\mathbf{x} + i\mathbf{y}\|^2 - i\|\mathbf{x} - i\mathbf{y}\|^2$

real IPS $\quad 4\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x} - \mathbf{y}\|^2$

Ex [IP-norm] Let $\|\cdot\|$ be a norm on $\mathbb{V}$. Then $\|\cdot\|$ is induced by some IP if and only if $\|\cdot\|$ satisfies the parallelogram law: $\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$.

• On $\mathbb{R}^2$ the function $\|\mathbf{x}\|_1 := |\mathbf{x}_1| + |\mathbf{x}_2|$ is a norm. ‼ Taking $\mathbf{x} = e_1$ and $\mathbf{y} = e_2$, we have

$$\|\mathbf{x} + y\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = \|(1,1)\|^2 + \|(1,-1)\|^2 = (|1| + |1|)^2 + (|1| + |-1|)^2 = 8$$

. Hence, $2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2) = 4$. So the parallelogram law fails. Thus $\|\mathbf{x}\|_1$ is not induced by any IP.

Th An orthogonal set S of nonzero vectors is linearly independent.

Po. Let S be lin.dep. So, $\exists$ nonzero $\mathbf{x}_1, \cdots, \mathbf{x}_n \in S$ and $\alpha_i \neq 0$ s.t. $\sum \alpha_i \mathbf{x}_i = \mathbf{0}$. As $\mathbf{x}_i \in S$, $\langle \mathbf{x}_i, \mathbf{x}_1 \rangle = 0$ for all $i \neq 1$. So $\alpha_i \langle \mathbf{x}_i, \mathbf{x}_1 \rangle = 0$ for all $i \neq 1$ and $\langle \sum\limits_{i=2}^{n} \alpha_i \mathbf{x}_i, \mathbf{x}_1 \rangle = 0$. So $\alpha_1 \|\mathbf{x}_1\|^2 = \langle \alpha_1 \mathbf{x}_1, \mathbf{x}_1 \rangle + \langle \sum\limits_{i=2}^{n} \alpha_i \mathbf{x}_i, \mathbf{x}_1 \rangle = \langle \sum\limits_{i=1}^{n} \alpha_i \mathbf{x}_i, \mathbf{x}_1 \rangle = \langle \mathbf{0}, \mathbf{x}_1 \rangle = 0$. So $\alpha_1 = 0$ (as $\mathbf{x}_1 \neq 0$), A contradiction.

P. **Inner Product-Advantage of orthonormal set**

- Let $\mathbb{V}$ be an IPS. An orthogonal set S in which $\|\mathbf{x}\| = 1$ for all $\mathbf{x}$ is an orthonormal set.

Ex An orthonormal set in an IPS is linearly independent.

- Consider the IPS $\mathbb{R}^3$.
- It is easy to check that $\left\{ \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \mathbf{z} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\}$ is a basis.
- Thus $\mathbf{w}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a lin.comb of the basis vectors in a unique way. Can we determine the coefficients of $\mathbf{x}, \mathbf{y}, \mathbf{z}$ in this representation?
- Yes, we can. All we need to do is to solve the system

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

for α, β, γ. This will take us some time.

- Suppose instead we have some orthonormal set as basis, say,

$$\left\{ \mathbf{u} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \mathbf{w} = \begin{bmatrix} -\frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix} \right\}.$$

- We ask the same question here. The answer is immediate.
- If $\mathbf{w}_0 = \alpha \mathbf{u} + \beta \mathbf{v} + \gamma \mathbf{w}$ then

$$\langle \mathbf{w}_0, \mathbf{u} \rangle = \langle \alpha \mathbf{u} + \beta \mathbf{v} + \gamma \mathbf{w}, \mathbf{u} \rangle = \alpha \langle \mathbf{u}, \mathbf{u} \rangle + \beta \langle \mathbf{v}, \mathbf{u} \rangle + \gamma \langle \mathbf{w}, \mathbf{u} \rangle = \alpha,$$

as $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is orthonormal. Similarly, $\beta = \langle \mathbf{w}_0, \mathbf{v} \rangle$ and $\gamma = \langle \mathbf{w}_0, \mathbf{w} \rangle$.

- Taking $\mathbf{w}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ we have $\alpha = \langle \mathbf{w}_0, \mathbf{u} \rangle = \frac{1}{\sqrt{3}}$, $\beta = \langle \mathbf{w}_0, \mathbf{v} \rangle = \sqrt{2}$ and $\gamma = \langle \mathbf{w}_0, \mathbf{w} \rangle = \frac{-2}{\sqrt{6}}$.

P. **Inner Product-Closest element-Feet of the perpendicular**

- If $\|\cdot\|$ is a norm, then $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$ defines a distance function.

Ex Let $\mathbf{x}, \mathbf{y} \in \mathbb{V}$ (IPS). Then $\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2\,\mathrm{RE}\langle \mathbf{x}, \mathbf{y}\rangle + \|\mathbf{y}\|^2$. Moreover if $\mathbf{x} \perp \mathbf{y}$ then, $\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$.

Q Determine the foot of the perpendicular from the point $(1,2,3)$ on the XY-plane.

Answer: $(1,2,0)$! Is this the point in the XY-plane that is closest to $(1,2,3)$? Yes. How did we find it? Equation of the XY-plane is $z=0$. So, the direction ratios of the normal vector of the XY-plane is $(0,0,1)$ as $z=0 \Rightarrow \langle (0,0,1),(x,y,z)\rangle = 0$. Note that $(1,2,3) - \langle (1,2,3),(0,0,1)\rangle (0,0,1) = (1,2,3) - 3(0,0,1) = (1,2,0)$.

Q Determine the foot of the perpendicular from the point $(1,2,3)$ on the plane generated by the vectors $(1,0,1)$ and $(0,1,1)$.

Answer: There is a unique plane containing the points $(0,0,0),(1,0,1)$ and $(0,1,1)$, namely $x+y-z=0$. The direction ratios of this plane is given by $(1,1,-1)$ as $x+y-z=0 \Leftrightarrow \langle (1,1,-1),(x,y,z)\rangle = 0$.

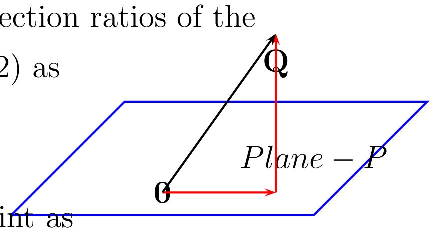
So, can we get the required point as $(1,2,3) - \langle (1,2,3),(1,1,-1)\rangle \frac{1}{3}(1,1,-1) = (1,2,3) - 0(1,1,-1) = (1,2,3)$. Why does it work?

Is this the point on the plane that is closest to $(1,2,3)$?

Q Determine the foot of the perpendicular from the point $Q=(1,2,3,4)$ on the plane generated by the vectors $(1,1,0,0),(1,0,1,0)$ and $(0,1,1,1)$.

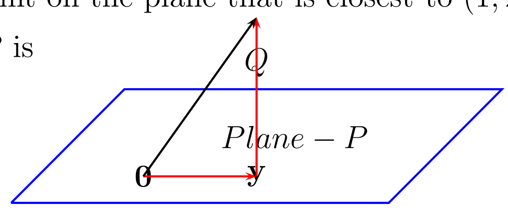
Answer: Note that equation of the required plane is $x-y-z+2w=0$ as $1-1+0+0=0$, $1-0-1+0=0$ and $0-1-1+2=0$.

Plane-P: $x-y-z+2w=0$. Direction ratios of the normal of the plane is $(1,-1,-1,2)$ as $x-y-z+2w=0$
$\Leftrightarrow \langle (1,-1,-1,2),(x,y,z,w)\rangle = 0$.



So, can we get the required point as $(1,2,3,4) - \langle (1,2,3,4),(1,-1,-1,2)\rangle \frac{1}{1+1+1+4}(1,-1,-1,2) = (1,2,3,4) - \frac{4}{7}(1,-1,-1,2)$ $= \frac{1}{7}(3,18,25,20)$. Is this the point on the plane that is closest to $(1,2,3,4)$?

Q Which point on the plane P is closest to the point, say Q?



- Answer: find the foot, say $\mathbf{y}$, of the $\perp$ from Q on P. Why?

  Use Pythagorus Theorem. Can there be more than one closest point? No.‼

How do we find $\mathbf{y}$? Note, $\vec{\mathbf{y}Q}$ gives a normal vector of the plane P. Hence, $\mathbf{y} = \vec{Q} - \vec{\mathbf{y}Q}$. So, need to find a way to compute $\vec{\mathbf{y}Q}$.

• Let $\mathbf{u}, \mathbf{v}$ be two non-zero vectors in an IPS $\mathbb{V}$.

Q Decompose $\mathbf{v}$ into two components, say $\mathbf{y}$ and $\mathbf{z}$, s.t. $\mathbf{y} \parallel \mathbf{u}$ and $\mathbf{z} \perp \mathbf{u}$. $\mathbf{y} = \mathbf{u}\cos(\theta)$ and $\mathbf{z} = \mathbf{u}\sin(\theta)$

P. **Inner Product-Orthogonal projection**

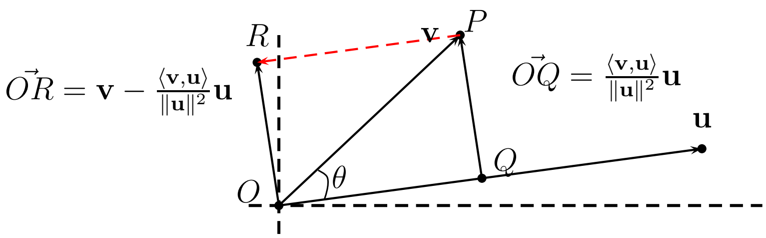


Figure 3: Decomposition of vector $\mathbf{v}$

• Note $\hat{\mathbf{u}} = \dfrac{\mathbf{u}}{\|\mathbf{u}\|}$ is a unit vector in the direction of $\mathbf{u}$. $\cos(\theta) = \frac{\|\vec{OQ}\|}{\|\vec{OP}\|}$. Hence $\|\vec{OQ}\| = \|\vec{OP}\| \ \cos(\theta) = \|\mathbf{v}\| \ \frac{\langle \mathbf{v},\mathbf{u}\rangle}{\|\mathbf{v}\| \ \|\mathbf{u}\|} = \frac{\langle \mathbf{v},\mathbf{u}\rangle}{\|\mathbf{u}\|} = \langle \mathbf{v}, \frac{\mathbf{u}}{\|\mathbf{u}\|}\rangle$. Thus, $\vec{OQ} = \|\vec{OQ}\| \ \hat{\mathbf{u}} = \langle \mathbf{v}, \frac{\mathbf{u}}{\|\mathbf{u}\|}\rangle \ \frac{\mathbf{u}}{\|\mathbf{u}\|}$.

Hence, $\mathbf{y} = \vec{OQ} = \langle \mathbf{v}, \frac{\mathbf{u}}{\|\mathbf{u}\|}\rangle \ \frac{\mathbf{u}}{\|\mathbf{u}\|}$ and $\mathbf{z} = \vec{OR} = \mathbf{v} - \langle \mathbf{v}, \frac{\mathbf{u}}{\|\mathbf{u}\|}\rangle \ \frac{\mathbf{u}}{\|\mathbf{u}\|}$. Verify: $\mathbf{v} = \mathbf{y} + \mathbf{z}$, $\mathbf{y} \parallel \mathbf{u}$ and $\mathbf{z} \perp \mathbf{u}$.

P. **Inner Product-Gram-Schmidt orthonormalization**

$\mathbf{y} = \vec{OQ}$ is the orthogonal projection of $\mathbf{v}$ on $\mathbf{u}$, denoted $\mathrm{Proj}_{\mathbf{u}}(\mathbf{v})$.
$\mathrm{Proj}_{\mathbf{u}}(\mathbf{v}) = \langle \mathbf{v}, \frac{\mathbf{u}}{\|\mathbf{u}\|}\rangle \ \frac{\mathbf{u}}{\|\mathbf{u}\|}$ and $\|\mathrm{Proj}_{\mathbf{u}}(\mathbf{v})\| = \|\vec{OQ}\| = \left|\frac{\langle \mathbf{v},\mathbf{u}\rangle}{\|\mathbf{u}\|}\right|$.

Also, distance of $\mathbf{u}$ from P equals $\|\vec{OR}\| = \|\vec{PQ}\| = \|\mathbf{z}\| = \|\mathbf{v} - \langle \mathbf{v}, \frac{\mathbf{u}}{\|\mathbf{u}\|}\rangle \ \frac{\mathbf{u}}{\|\mathbf{u}\|}\|$.

Th[Gram-Schmidt] Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a lin.ind subset of an IPS $\mathbb{V}$. Then there exists $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$, orthonormal s.t. $\mathrm{LS}(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \mathrm{LS}(\mathbf{w}_1, \ldots, \mathbf{w}_k)$, for all k.

Po. Put $\mathbf{w}_1 = \hat{\mathbf{v}}_1$. So $\mathrm{LS}(\mathbf{v}_1) = \mathrm{LS}(\mathbf{w}_1)$.

• Put $\mathbf{v}_2' = \mathbf{v}_2 - \langle \mathbf{v}_2, \mathbf{w}_1\rangle \mathbf{w}_1$. As $\mathbf{v}_2 \notin \mathrm{LS}(\mathbf{v}_1)$, we get $\mathbf{v}_2' \neq \mathbf{0}$. Put $\mathbf{w}_2 = \hat{\mathbf{v}}_2'$.

• Note $\{\mathbf{w}_1, \mathbf{w}_2\}$ is orthonormal and $\mathrm{LS}(\{\mathbf{v}_1, \mathbf{v}_2\}) = \mathrm{LS}(\{\mathbf{w}_1, \mathbf{w}_2\})$. ‼

• Assume that we have got $\{\mathbf{w}_1, \ldots, \mathbf{w}_{k-1}\}$ orthonormal s.t. $\mathrm{LS}(\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}) = \mathrm{LS}(\mathbf{w}_1, \ldots, \mathbf{w}_{k-1})$.

• Note: $\mathbf{v}_k' = \mathbf{v}_k - \sum\limits_{i=1}^{k-1} \langle \mathbf{v}_k, \mathbf{w}_i\rangle \mathbf{w}_i \neq \mathbf{0}$, as $\mathbf{v}_k \notin \mathrm{LS}(\mathbf{v}_1, \ldots, \mathbf{v}_{k-1})$.

Put $\mathbf{w}_k = \hat{\mathbf{v}}_k' = \mathbf{v}_k'/\|\mathbf{v}_k'\|$. Note $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is orthonormal as $\|w_k\| = 1$ & $\|v_k'\|\langle w_k, w_1\rangle = \langle \mathbf{v}_k', \mathbf{w}_1\rangle = \langle \mathbf{v}_k - \sum\limits_{i=1}^{k-1} \langle \mathbf{v}_k, \mathbf{w}_i\rangle \mathbf{w}_i, \ \mathbf{w}_1\rangle = \langle \mathbf{v}_k, \mathbf{w}_1\rangle - \langle \sum\limits_{i=1}^{k-1} \langle \mathbf{v}_k, \mathbf{w}_i\rangle \mathbf{w}_i, \ \mathbf{w}_1\rangle = \langle \mathbf{v}_k, \mathbf{w}_1\rangle - \sum\limits_{i=1}^{k-1} \langle \mathbf{v}_k, \mathbf{w}_i\rangle \langle \mathbf{w}_i, \ \mathbf{w}_1\rangle = \langle \mathbf{v}_k, \mathbf{w}_1\rangle - \langle \mathbf{v}_k, \mathbf{w}_1\rangle = 0$.

• By GS $\mathbf{w}_k = \mathbf{v}_k'/\|\mathbf{v}_k'\|$ is a lin.comb of $\mathbf{w}_1, \ldots, \mathbf{w}_{k-1}, \mathbf{v}_k$. So $\mathbf{w}_k \in \mathrm{LS}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$.

• As $\mathbf{v}_k = \|\mathbf{v}'_k\|\mathbf{w}_k + \sum_{i=1}^{k-1}\langle \mathbf{v}_k, \mathbf{w}_i\rangle \mathbf{w}_i$, we get $\mathbf{v}_k \in \text{LS}(\mathbf{w}_1, \ldots, \mathbf{w}_k)$.

• So $\text{LS}(\mathbf{w}_1, \ldots, \mathbf{w}_k) = \text{LS}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$. Proof finishes using induction on k. □

Ex Let $\mathcal{U} = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ be a basis in an IPS $\mathbb{V}$. Then $\mathcal{U}$ is orthonormal if and only if $\mathbf{x} \in \mathbb{V} \Rightarrow \mathbf{x} = \sum_{i=1}^{n}\langle \mathbf{x}, \mathbf{u}_i\rangle \mathbf{u}_i$.

Q Find an orthonormal set S s.t. $\text{LS}(S) = \text{LS}(T)$, where $T = \{\mathbf{v}_1 = [2 \quad 0 \quad 0]^t, \mathbf{v}_2 = [\frac{3}{2} \quad 2 \quad 0]^t, \mathbf{v}_3 = [\frac{1}{2} \quad \frac{3}{2} \quad 0]^t, \mathbf{v}_4 = [1 \quad 1 \quad 1]\}^t$.

• Take $\mathbf{w}_1 = \hat{\mathbf{v}}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = [1 \quad 0 \quad 0]^t = \mathbf{e}_1$. Note $\text{LS}(\mathbf{w}_1) = \text{LS}(\mathbf{v}_1)$.

This means: line (via $\mathbf{0}$) of $\mathbf{v}_1$ is line of $\mathbf{w}_1$.

• Next: does $\mathbf{v}_2 \in \text{LS}(\mathbf{w}_1)$? It is iff $\mathbf{v}_2$ equals $\sum\langle \mathbf{v}_2, \mathbf{w}_i\rangle \mathbf{w}_i = \frac{3}{2}\mathbf{w}_1$, not true. Then, find the $\perp$ from $\mathbf{v}_2$ to $\text{LS}(\mathbf{w}_1)$:

$\mathbf{v}'_2 = \mathbf{v}_2 - \frac{3}{2}\mathbf{w}_1 = [0 \quad 2 \quad 0]^t$. Put $\mathbf{w}_2 = \hat{\mathbf{v}}'_2 = \frac{\mathbf{v}'_2}{\|\mathbf{v}'_2\|} = [0 \quad 1 \quad 0]^t = \mathbf{e}_2$.

• Next: does $\mathbf{v}_3 \in \text{LS}(\mathbf{w}_1, \mathbf{w}_2)$? It is iff $\mathbf{v}_3 = \sum\langle \mathbf{v}_3, \mathbf{w}_i\rangle \mathbf{w}_i = \frac{1}{2}\mathbf{w}_1 + \frac{3}{2}\mathbf{w}_2$, true.

• Next: does $\mathbf{v}_4 \in \text{LS}(\mathbf{w}_1, \mathbf{w}_2)$? It is iff $\mathbf{v}_4 = \sum\langle \mathbf{v}_4, \mathbf{w}_i\rangle \mathbf{w}_i = \mathbf{w}_1 + \mathbf{w}_2$, not true. Find the $\perp$ from $\mathbf{v}_4$ to the plane $\text{LS}(\mathbf{w}_1, \mathbf{w}_2)$: $\mathbf{v}'_4 = \mathbf{v}_4 - \mathbf{w}_1 - \mathbf{w}_2 = \mathbf{e}_3$. So $\mathbf{w}_4 = \hat{v}'_4 = \mathbf{e}_3$. So $S = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

Q Find an orthonormal set S s.t. $\text{LS}(S) = \text{LS}(T)$, where $T = \{\mathbf{v}_1 = [2 \quad 0 \quad 0]^t, \mathbf{v}_2 = [\frac{3}{2} \quad 2 \quad 0]^t, \mathbf{v}_3 = [\frac{1}{2} \quad \frac{3}{2} \quad 0]^t, \mathbf{v}_4 = [1 \quad 1 \quad 1]\}^t$.

$\mathbf{w}_1 = [1 \quad 0 \quad 0]^t = \mathbf{e}_1$. $\mathbf{w}_2 = [0 \quad 1 \quad 0]^t = \mathbf{e}_2$. $\mathbf{w}_4 = [0 \quad 0 \quad 1]^t = \mathbf{e}_3$

So, $\begin{bmatrix} 2 & \frac{3}{2} & \frac{1}{2} & 1 \\ 0 & 2 & \frac{3}{2} & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & \frac{3}{2} & \frac{1}{2} & 1 \\ 0 & 2 & \frac{3}{2} & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Q Find an orthonormal set S s.t. $\text{LS}(S) = \text{LS}(T)$, where $T = \{(1, -1, 1, 1), (1, 0, 1, 0), (1, -2, 1, 2), (0, 1, 0, 1)\} \subseteq \mathbb{R}^4$.

As $\mathbf{u}_1 = (1, -1, 1, 1)$, $\mathbf{w}_1 = \frac{1}{2}\mathbf{u}_1$.

$\mathbf{w}_1 = \frac{1}{2}(1, -1, 1, 1)$.

Let $\mathbf{u}_2 = (1, 0, 1, 0)$. Then, $\mathbf{v}_2 = (1, 0, 1, 0) - \langle \mathbf{u}_2, \mathbf{w}_1\rangle \mathbf{w}_1 = (1, 0, 1, 0) - \mathbf{w}_1 = \frac{1}{2}(1, 1, 1, -1)$.

$\mathbf{w}_2 = \frac{1}{2}(1, 1, 1, -1)$.

Let $\mathbf{u}_3 = (1, -2, 1, 2)$. Then

$$\mathbf{v}_3 = \mathbf{u}_3 - \langle \mathbf{u}_3, \mathbf{w}_1\rangle \mathbf{w}_1 - \langle \mathbf{u}_3, \mathbf{w}_2\rangle \mathbf{w}_2 = \mathbf{u}_3 - 3\mathbf{w}_1 + \mathbf{w}_2 = \mathbf{0}.$$

So, we again take $\mathbf{u}_3 = (0, 1, 0, 1)$. Then $\mathbf{v}_3 = \mathbf{u}_3 - \langle \mathbf{u}_3, \mathbf{w}_1\rangle \mathbf{w}_1 - \langle \mathbf{u}_3, \mathbf{w}_2\rangle \mathbf{w}_2 = \mathbf{u}_3 - 0 \cdot \mathbf{w}_1 - 0 \cdot \mathbf{w}_2 = \mathbf{u}_3$. $\mathbf{w}_3 = \frac{1}{\sqrt{2}}(0, 1, 0, 1)$.

Hence,

$$Q = [\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3] = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{-1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \text{and} \quad R = \begin{bmatrix} 2 & 1 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{bmatrix}.$$

1. rank $(A) = 3$,
2. $A = QR$ with $Q^t Q = I_3$, and
3. R a 3×4 upper triangular matrix with rank $(R) = 3$.

• GS also works for a countable linearly independent set.

• We also apply GS to a countable linearly dependent set. Here a $\mathbf{v}_i$ contributes a $\mathbf{w}_i$ if and only if $\mathbf{v}_i' \neq \mathbf{0}$. That is, if and only if $\mathbf{v}_i$ is not a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}$.

• GS method is applied to find a maximal lin.ind set from a finite set of vectors.

Cor Every finite dimensional IPS (not $\{\mathbf{0}\}$) has an orthonormal basis. ‼

Cor Any nonempty orthonormal set in a finite dimensional IPS can be extended to an orthonormal basis. ‼

• If we apply GS to $\mathbf{v}_1, \ldots, \mathbf{v}_n$ (here at least one $\mathbf{v}_i$ is nonzero) we obtain an orthonormal basis of $\text{LS}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$.

• If we apply GS to a rearrangement of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ we may obtain another orthonormal basis of $\text{LS}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$. Nevertheless, both the bases will have the same size.

Q We apply GS to $\mathbf{v}_1, \ldots, \mathbf{v}_9$. We get $\mathbf{v}_5' = 0$. What do you conclude?

• Apply GS to $\left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

• We have $\mathbf{w}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and

$$\mathbf{v}_2' = \mathbf{v}_2 - \langle \mathbf{v}_2, \mathbf{w}_1 \rangle \mathbf{w}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{3}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

So $\mathbf{w}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and resulting orthonormal set $\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\}$.

Ex[Bessel's Inequality] Let $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\} \subseteq \mathbb{V} \setminus \{\mathbf{0}\}$ be orthogonal, $\mathbf{u} \in \mathbb{V}$. Then $\sum_{k=1}^{n} \frac{|\langle \mathbf{u}, \mathbf{v}_k \rangle|^2}{\|\mathbf{v}_k\|^2} \leq \|\mathbf{u}\|^2$. Equality holds <u>iff</u> $\mathbf{u} = \sum_{k=1}^{n} \frac{\langle \mathbf{u}, \mathbf{v}_k \rangle}{\|\mathbf{v}_k\|^2} \mathbf{v}_k$.

Ex[Parseval's formula] Let $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ be an orthonormal basis of $\mathbb{V}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{V}$. Then $\langle \mathbf{x}, \mathbf{y}\rangle = \sum_{i=1}^{n} \langle \mathbf{x}, \mathbf{v}_i\rangle \overline{\langle \mathbf{y}, \mathbf{v}_i\rangle}$. When $\mathbf{x} = \mathbf{y}$, we get $\|\mathbf{x}\|^2 = \sum_{i=1}^{n} |\langle \mathbf{x}, \mathbf{v}_i\rangle|^2$, a generalization of Phythagoras Theorem.

Ex Let $A \in \mathcal{M}_{m\times n}(\mathbb{C})$. Then there exists a unique B s.t. $\langle A\mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{x}, B\mathbf{y}\rangle$ for all $\mathbf{x} \in \mathbb{C}^n, \mathbf{y} \in \mathbb{C}^m$ and in fact $B = A^*$.

P. **Inner Product-Range and nullspace**

- Let $A_{m\times n}$ be a matrix. Then,

  $\mathcal{N}(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$

  $\text{col}(A) = \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\}$.

  nullity of $A = \textsc{dim}(\mathcal{N}(A))$.

Ex What is the nullity of $\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$? Give two bases in the null space.

Th Let $A_{m\times n}$ be a matrix. Then $\textsc{dim}(\textsc{col}(A)) + \textsc{dim}(\mathcal{N}(A)) = n$.

Po. Let $\textsc{dim}(\textsc{col}(A)) = r < n$ and $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ an orthonormal basis of $\textsc{null}(A)$. Extend $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ to get $\{\mathbf{u}_1, \ldots, \mathbf{u}_r, \mathbf{u}_{r+1}, \ldots, \mathbf{u}_n\}$ as an orthonormal basis of $\mathbb{C}^n$. Then, $\textsc{col}(A) = LS(A\mathbf{u}_1, \ldots, A\mathbf{u}_n) = L(\mathbf{0}, \ldots, \mathbf{0}, A\mathbf{u}_{r+1}, \ldots, A\mathbf{u}_n) \quad = L(A\mathbf{u}_{r+1}, \ldots, A\mathbf{u}_n)$.

Need to prove $\{A\mathbf{u}_{r+1}, \ldots, A\mathbf{u}_n\}$ is Lin.ind. Consider the system $\alpha_1 A\mathbf{u}_{r+1} + \cdots + \alpha_{n-r} A\mathbf{u}_n = \mathbf{0}$ in the unknowns $\alpha_1, \ldots, \alpha_{n-r}$.

So $A(\alpha_1\mathbf{u}_{r+1} + \cdots + \alpha_{n-r}\mathbf{u}_n) = \mathbf{0}$ or $\alpha_1\mathbf{u}_{r+1} + \cdots + \alpha_{n-r}\mathbf{u}_n \in \textsc{null}(A) = LS(\mathbf{u}_1, \ldots, \mathbf{u}_r)$. Therefore, there exist β_i's s.t. $\alpha_1\mathbf{u}_{r+1} + \cdots + \alpha_{n-r}\mathbf{u}_n = \beta_1\mathbf{u}_1 + \cdots + \beta_r\mathbf{u}_r$.

So $\beta_1\mathbf{u}_1 + \cdots + \beta_r\mathbf{u}_r - \alpha_1\mathbf{u}_{r+1} - \cdots - \alpha_{n-r}\mathbf{u}_n = \mathbf{0}$.

As $\{\mathbf{u}_1, \ldots, \mathbf{u}_r, \mathbf{u}_{r+1}, \ldots, \mathbf{u}_n\}$ is a basis, the only solution is $\alpha_i = 0$ for $1 \le i \le n-r$ and $\beta_j = 0$ for $1 \le j \le r$. Hence, $\{A\mathbf{u}_{r+1}, \ldots, A\mathbf{u}_n\}$ is Lin.ind.

P. **Fundamental Theorem of Linear Algebra**

Th[Fundamental Theorem of Linear Algebra]

Let $A_{m\times n}$ be a complex matrix. Then

a) $\textsc{dim}(\textsc{col}(A)) + \textsc{dim}(\textsc{null}(A)) = n$.

b) $\textsc{null}(A) = [\textsc{col}(A^*)]^{\perp}$ and $\textsc{null}(A^*) = [\textsc{col}(A)]^{\perp}$.

c) $\textsc{dim}(\textsc{col}(A)) = \textsc{dim}(\textsc{col}(A^*))$.

Po. Part a): Done in the previous theorem (see above).

Part b): We first show that $\textsc{null}\,A \subseteq [\textsc{col}\,A^*]^{\perp}$

Let $\mathbf{x} \in \textsc{null}\,A$ then $A\mathbf{x} = \mathbf{0}$, $0 = \langle A\mathbf{x}, \mathbf{u}\rangle = \mathbf{u}^* A\mathbf{x} = (A^*\mathbf{u})^*\mathbf{x} = \langle \mathbf{x}, A^*\mathbf{u}\rangle$ for all $\mathbf{u} \in \mathbb{C}^n$. Thus, $\mathbf{x} \in [\textsc{col}\,A^*]^{\perp}$. So $\textsc{null}\,A \subseteq [\textsc{col}\,A^*]^{\perp}$

We now show that $\text{COL}(A^*)^\perp \subseteq \text{NULL}(A)$.

Let $\mathbf{x} \in \text{COL}(A^*)^\perp$. Then, for every $\mathbf{y} \in \mathbb{C}^n$, $\langle \mathbf{x}, A^*\mathbf{y}\rangle = 0$. So $0 = \langle \mathbf{x}, A^*\mathbf{y}\rangle = (A^*\mathbf{y})^*\mathbf{x} = \mathbf{y}^*(A^*)^*\mathbf{x} = \mathbf{y}^*A\mathbf{x} = \langle A\mathbf{x}, \mathbf{y}\rangle$. Take $\mathbf{y} = A\mathbf{x}$, then $\|A\mathbf{x}\|^2 = 0$. So $A\mathbf{x} = \mathbf{0}$ and $\mathbf{x} \in \text{NULL}(A)$.

So, the proof of Part b) is complete.

c): Left as an exercise. □

P. **Inner Product-Projection Matrix**

Th[decomposition] Let $\mathbf{x} \in \mathbb{V}$ (IPS) and H be a finite dimensional subspace with an orthonormal basis $\{\mathbf{f}_1, \ldots, \mathbf{f}_k\}$. Then $\mathbf{y} = \sum_1^k \langle \mathbf{x}, \mathbf{f}_i\rangle \mathbf{f}_i$ is the only closest point in H from $\mathbf{x}$. Putting $\mathbf{z} = \mathbf{x} - \mathbf{y}$, we have a unique $\mathbf{y} \in H$, a unique $\mathbf{z} \in H^\perp$ s.t. $\mathbf{x} = \mathbf{y} + \mathbf{z}$.

Po. As $\langle \mathbf{x} - \mathbf{y}, \mathbf{f}_i\rangle = 0$, we have $(\mathbf{x} - \mathbf{y}) \perp H$. So for each $\mathbf{w} \in H$, $\|\mathbf{x} - \mathbf{w}\|^2 = \|\mathbf{x} - \mathbf{y}\|^2 + \|\mathbf{y} - \mathbf{w}\|^2 \geq \|\mathbf{x} - \mathbf{y}\|^2$. Uniqueness **!!** □

- In the above, $\mathbf{y}$ is called the projection of $\mathbf{x}$ on H.

Q What is the projection of $\mathbf{e}_1 + \mathbf{e}_2 - 3\mathbf{e}_3$ on $H : \mathbf{x}_1 - \mathbf{x}_2 + \mathbf{x}_4 = 0$? $\mathbf{e}_1 + \mathbf{e}_2 - 3e_3$? Projection of $(1, -1, 0, 0)^t$ is $\frac{1}{3}(1, -1, 0, -2)$. Note that $\frac{1}{3}\begin{bmatrix} 2 & 1 & 0 & -1 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 2 \end{bmatrix}$ is the projection matrix.

- Important: The Projection matrix in the standard basis is given by $A = \sum_1^k \mathbf{f}_i \mathbf{f}_i^t$ as

$$A\mathbf{x} = \left(\sum_1^k \mathbf{f}_i \mathbf{f}_i^t\right)\mathbf{x} = \sum_1^k \mathbf{f}_i \left(\mathbf{f}_i^t \mathbf{x}\right) = \sum_1^k \langle \mathbf{x}, \mathbf{f}_i\rangle \mathbf{f}_i = \mathbf{y}.$$