Summary - Rules of Natural Deduction.

$\wedge$

$$\frac{\phi \quad \psi}{\phi \wedge \psi}\ \wedge i \qquad \frac{\phi \wedge \psi}{\phi}\ \wedge e_1 \qquad \frac{\phi \wedge \psi}{\psi}\ \wedge e_2$$

$\vee$

$$\frac{\phi}{\phi \vee \psi}\ \vee i_1 \qquad \frac{\psi}{\phi \vee \psi}\ \vee i_2 \qquad \frac{\phi \vee \psi \quad \boxed{\begin{array}{c}\phi\\ \vdots\\ \lambda\end{array}} \quad \boxed{\begin{array}{c}\psi\\ \vdots\\ \lambda\end{array}}}{\lambda}$$

$\rightarrow$

$$\frac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \psi\end{array}}}{\phi \rightarrow \psi}\ \rightarrow i \qquad \frac{\phi \quad \phi \rightarrow \psi}{\psi}\ \rightarrow e$$

$\neg$

$$\frac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \bot\end{array}}}{\neg \phi}\ \neg i \qquad \frac{\phi \quad \neg \phi}{\bot}\ \neg e$$

$\bot$

$$\frac{\bot}{\phi}\ \bot e$$

$\neg\neg$

$$\frac{\phi}{\neg\neg \phi}\ \neg\neg i$$ (can be derived)

$$\frac{\neg\neg \phi}{\phi}\ \neg\neg e.$$

Derived Rules.

$$\frac{\phi \rightarrow \psi \quad \neg \psi}{\neg \phi}\ MT \qquad \frac{}{\phi \vee \neg \phi}\ LEM$$

$\phi_1, \phi_2, \ldots \phi_n$ – propositional logic formulas.

$$\underbrace{\phi_1, \phi_2, \ldots \phi_n \models \psi}.$$

Semantic entailment / restricted validity

For all valuations v where $v \models \phi_i \ \forall i \in \{1, \ldots n\}$,
$v \models \psi$. (v satisfies ψ).

$\vdash$ is sound if for all $\phi_1, \ldots \phi_n$ and conclusion ψ, if $\phi_1 \ldots \phi_n \vdash \psi$ then $\phi_1 \ldots \phi_n \models \psi$

$\vdash$ is complete if for all $\phi_1 \ldots \phi_n$ and ψ,

if $\phi_1 \ldots \phi_n \models \psi$ then $\phi_1 \ldots \phi_n \vdash \psi$.

Theorem. $\vdash$ is sound and complete.

- if $\vdash \psi$ then $\models \psi$
- if $\models \psi$ then $\vdash \psi$

Lemma. if $\phi_1 \ldots \phi_k \vdash \psi$ then $\phi_1 \ldots \phi_k \models \psi$.

Proof. $\Rightarrow$ $\exists$ a proof of ψ from premises $\phi_1 \ldots \phi_k$.

Induction on the length of the proof.

Base $\phi \vdash \phi \Rightarrow \phi \models \phi$

Induction step.

For all formulas $\phi_1 \ldots \phi_k, \psi$ for all ND proofs $\phi_1, \ldots \phi_k \vdash \psi$ containing less than n steps, $\phi_1 \ldots \phi_k \models \psi$

To prove: For all $\phi_1, \ldots, \phi_k, \psi$ for all ND proofs $\phi_1 \ldots \phi_k \vdash \psi$ containing n steps, $\phi_1 \ldots \phi_k \models \psi$.

1. ϕ_1 premise ⋮ k. ϕ_k premise. ⋮ n ψ justification (application of some rule)	Rule: $\wedge i$. $\psi = \psi_1 \wedge \psi_2$. By IH $\phi_1 \ldots \phi_n \models \psi_1$ $\phi_1 \ldots \phi_n \models \psi_2$ This implies $\phi_1 \ldots \phi_n \models \psi_1 \wedge \psi_2 = \psi$.

Completeness

- if $\phi_1, \ldots \phi_k \vDash \psi$ then $\phi_1, \ldots \phi_k \vdash \psi$.

Proof.

Step 1. if $\phi_1, \ldots \phi_k \vDash \psi$ then $\vDash \phi_1 \to (\phi_2 \to \cdots (\phi_k \to \psi) \cdots)$

Step 2. if $\vDash \phi_1 \to (\phi_2 \to \cdots (\phi_k \to \psi) \cdots)$ then

$\vdash \phi_1 \to (\phi_2 \to \cdots (\phi_k \to \psi) \cdots)$

Step 3. if $\vdash \phi_1 \to (\phi_2 \to \cdots (\phi_k \to \psi) \cdots)$ then

$\phi_1, \ldots \phi_k \vdash \psi$

Suppose $\vdash \phi_1 \to (\phi_2 \to \cdots (\phi_k \to \psi) \cdots)$: $\exists$ a proof PF

A proof of $\phi_1, \ldots \phi_k \vdash \psi$.

1. ϕ_1 premise.
$\vdots$
k ϕ_k premise.

PF

m. $\phi_1 \to (\phi_2 \to \cdots (\phi_k \to \psi) \cdots)$

m+1 $\phi_2 \to (\cdots (\phi_k \to \psi) \cdots)$ $\to e$ m, 1.

m+2 $\phi_3 \to (\cdots \phi_k \to \psi \cdots)$ $\to e$ m+1, 2

$\vdots$

ψ $\to e$.

Resolution

An inference rule for clauses.

clause: ϕ_1 and ϕ_2 and proposition: p

$$\frac{p \vee \phi_1 \qquad \neg p \vee \phi_2}{\phi_1 \vee \phi_2}\ \text{Res.}$$

A derived rule in ND proof system.

1. $p \vee \phi_1$ premise
2. $\neg p \vee \phi_2$ premise

3. p Assumption

4. $\neg p$ Assumption

5. $\perp$ $\neg e$ 3,4

6. $\phi_1 \vee \phi_2$ $\perp e$ 5

4'. ϕ_2 Assumption

5'. $\phi_1 \vee \phi_2$ $\vee i_2$ 4'

7. $\phi_1 \vee \phi_2$ $\vee e$ 2, 4-6, 4'-5'.

8. ϕ_1 Assumption

9. $\phi_1 \vee \phi_2$ $\vee i_1$ 8

10. $\phi_1 \vee \phi_2$ $\vee e$ 1, 3-7, 8-9

$\neg e$ is a special case of resolution.

$$\frac{p \qquad \neg p}{\perp}\ \neg e$$

Resolution is sound and complete to prove unsatisfiability of clauses - restricted validity -

Lemma 1. $\phi_1, \ldots \phi_n \models \psi$ iff $\phi_1 \wedge \ldots \wedge \phi_n \wedge \neg\psi$ is not satisfiable

Lemma 2. $\phi \vdash_{Res} \perp$ iff ϕ is not satisfiable.

To show $\phi_1, \ldots \phi_n \models \psi$

1. Convert $\phi_1 \wedge \phi_2 \wedge \ldots \wedge \phi_n \wedge \neg\psi$ to CNF.
   (a set of clauses).

2. Apply sequence of resolutions-
   if it is possible to derive $\perp$ then $\phi_1, \ldots \phi_n \models \psi$.
   otherwise $\phi_1, \ldots \phi_n \not\models \psi$.