Advanced

# Image Processing and Stochastic Modeling

## TP4: Wavelets

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## Submission

Please archive your report and codes in "Name_Surname.**zip**" (replace "Name" and "Surname" with your real name), and upload to "`Assignments/TP4: Wavelets`" on `https://chamilo.unige.ch` before Wednesday, May 17 2017, 23:59 PM. Note, **the assessment is mainly based on your report, which should include your answers to all questions and the experimental results**.

In the previous TP, you have seen the Discrete Fourier Transform and some of its enormous range of applications in image processing. The Fourier transform is based on the idea that an image or signal can be decomposed into periodic sine and cosine functions. To obtain a certain periodicity in discrete images, images are wrapped around left and right and top to bottom. Further more, the Fourier Transform gives us the frequency and phase decomposition, and as a direct consequence, sacrifices time. Wavelets were in part designed to overcome these two things.

Wavelets are a mathematical tool for hierarchically decomposing signals. They are an elegant technique to represent different levels of detail present in some signal. They have many applications in many domains, including image compression and denoising.

# 1 Introduction

One can consider a wavelet to be a little part of a wave (Figure 1). It is the graph belonging to $y = \sin(x)\exp^{-x^2}$.

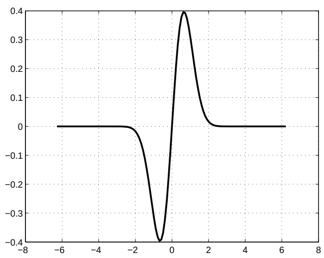


**Figure 1** – A wavelet example. Shown here is $y = \sin(x)\exp^{-x^2}$ on the range $[-2 * \pi, \ldots 2 * \pi]$.

Suppose we are given a generic wavelet:

$$f = w(x)$$

Given your knowledge of the Fourier Transform and its basis functions, it should come as now surprise that we can express any function $f(x)$ as the sum of wavelets in the form:

$$f = aw(bx + c)$$

One could for example:

- Dilate the wavelet by applying a scaling factor to $x$. $f(2x)$ would squash the wavelet, whereas $f(x/2)$ would expand it.
- Translate the wavelet by adding or subtracting a value from $x$. Shifting the wavelet 2 to the right could be done via $f(x-2)$.
- Change the hight of the wavelet my multiplying $f$ with a constant.

It is important to note that there is no standard wavelet. Pending on the specific application different wavelets have been invented that all have slightly different properties or advantages.

## A Simple Wavelet Transform

Let there be a 1D discrete signal $f(x)$ on 8 points:

$$f(x) = [71, 67, 24, 26, 36, 32, 14, 18]$$

We will perform a wavelet decomposition by using only two operations, the average and the difference for pairs of 2 coefficients. The pairwise average of $f(x)$ gives:

$$\begin{aligned} f_{\mu_1}(x) &= \left[\frac{71+67}{2}, \frac{24+26}{2}, \frac{36+32}{2}, \frac{14+18}{2}\right] \\ &= [69, 25, 34, 16] \end{aligned}$$

From the averages we can not reconstruct the original signal $f(x)$. For this we additionally store the differences between the first coefficient of each pair, and the average:

$$\begin{aligned} f_{d_1}(x) &= [71-69, 24-25, 36-34, 14-16] \\ &= [2, -1, 2, -2] \end{aligned}$$

The concatenation of $f_{\mu_1}(x)$ and $f_{d_1}(x)$ is the Discrete Wavelet Transform, at scale 1:

$$d_1 = [\underbrace{69, 25, 34, 16}_{\text{averages}}, \quad \underbrace{2, -1, 2, -2}_{\text{differences}}]$$

We can continue this process, by taking the 4 averages from the previous scale, and repeating all steps:

$$\begin{aligned} f_{\mu_2}(x) &= \left[\frac{69+25}{2}, \frac{34+16}{2}\right] && = [47, 25] \\ f_{d_2}(x) &= [69-47, 34-25] && = [22, 9] \end{aligned}$$

The Discrete Wavelet Transform, at scale 2 becomes:

$$d_2 = [47, 25, 22, 9, 2, -1, 2, -2]$$

The last possible scale 3 Discrete Wavelet Transform then becomes:

$$d_3 = [36, 11, 22, 9, 2, -1, 2, -2]$$

Reconstruction can be achieved straightforward by adding and subtracting the first two elements, then the first four elements, and finally, using all elements:

| | |
|---|---|
| $d_3 = [36, 11, 22, 9, 2, -1, 2, -2]$ | DWT scale 3 |
| $d_2 = [36 + 11, 36 - 11, 22, 9, 2, -1, 2, -2]$ | Add and subtract difference from first element |
| $d_2 = [47, 25, 22, 9, 2, -1, 2, -2]$ | DWT scale 2 reconstruction |
| $d_1 = [47 + 22, 47 - 22, 22 + 9, 22 - 9, 2, -1, 2, -2]$ | Add and subtract difference from first two elements |
| $d_1 = [69, 25, 34, 16, 2, -1, 2, -2]$ | DWT scale 1 reconstruction |
| $d_0 = [69 + 2, 69 - 2, 25 + (-1), 25 - (-1), ...$ | |
| $34 + 2, 34 - 2, 16 + (-2), 16 - (-2)]$ | Add and subtract difference from four elements |
| $d_0 = [71, 67, 24, 26, 36, 32, 14, 18]$ | Reconstructed original signal |

You will notice that obviously the differences are small the moment the difference between original signal values are close together. This can be used for compression, by setting small differences to zero.

### 1.1 Exercise

Give a manual decomposition of $g(x) = [9, 7, 3, 5]$ in 2 scales.

### 1.2 Exercise

Let us take $f(x) = [71, 67, 24, 26, 36, 32, 14, 18]$ again, and suppose we convert the Discrete Wavelet Transform at scale 3 to:

$$d'_3 = [36, 11, 22, 9, 2, 0, 2, 0]$$

setting all negative values of the original $d_3$ to zero.
Manually reconstruct the original signal $f'(x)$. What is the difference between $f(x)$ and $f'(x)$?

## 2 The 1D Haar Wavelet

The Haar wavelet is world's most simple wavelet, which has also been deployed in image processing.

#### Orthonormal basis

Again, we will start by thinking of a signal as a set of coefficients multiplied against some set of basis functions. In particular, we will think of them as piecewise-constant functions on the

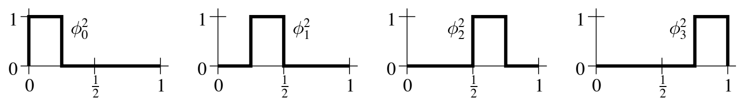


**Figure 2** – The 'box' basis for vector-space $V^2$.

half-open interval $[0, 1)$. A 1D signal with one value or image with a single pixel can be thought of as a continuous function over the entire interval $[0, 1)$. This vector-space will be denoted as $V^0$. An image with two pixels, has thus two constant pieces: one on the interval $[0, 0.5)$, the other on the interval $[0.5, 1)$. This is vector-space $V^1$. This means that vector-space $V^j$, on range $[0, 1)$ has coefficients that each will occupy one of the $2^j$ equally sized sub-intervals. This also means we can think of each 1D signal or image with $2^j$ coefficients as an element in $V^j$. Note that these spaces are nested :

$$V^0 \subset V^1 \subset V^2 \subset \cdots$$

We will now define the *basis vectors* for each vector-space $V^j$. They are known as the *scaling functions*. A very simple basis for $V^j$ is given by a set of scaled "box" functions.:

$$\phi_i^j(x) = \phi(2^j x - i), \tag{1}$$

for $i = 0, \ldots, 2^j - 1$, where

$$\phi(x) = \begin{cases} 1 & \text{for } 0 \le x < 1 \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Figure 2 shows the four box functions that form the basis of $V^2$.

**Example**

For $V^2$, following (1) this gives $j = 2$ and $i = 0, 1, 2, 3$, $(3 = 2^2 - 1)$. The four box-basis functions therefore become:

$$\phi_0^2 = \phi(4x) \qquad \phi_2^2 = \phi(4x - 2)$$
$$\phi_1^2 = \phi(4x - 1) \qquad \phi_3^2 = \phi(4x - 3)$$

For $x$ on the interval $[0, 1)$ and using (2), the results in the box figures in Figure 2.

## 2.1 Exercise

Draw all box functions for vector-space $V^0$ and $V^1$.

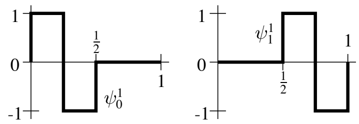


**Figure 3** – The Haar wavelets for $W^1$.

### Inner product

We define the inner product on vector-space $V^j$ to be the normal inner product:

$$\langle f|g\rangle = \int_0^1 f(x)g(x)dx,$$

for elements $f, g \in V^j$.

### Wavelets

Let us now define the vector-space $W^j$ as the space of all functions from $V^{j+1}$ that are *orthogonal* to all functions from vector-space $V^j$ under the defined inner product. Informally, we will use $W^j$ to represent parts of some signal, that can not be represented in $V^j$.

The collection of *linearly independent functions* $\psi_i^j(x)$ spanning $W^j$ are *wavelets.* Wavelets have two important properties:

1. The basis functions of $\psi_i^j$ of $W^j$ together with the basis functions $\phi_i^j$ of $V^j$, form the basis for vector-space $V^{j+1}$.
2. Every basis function $\psi_i^j$ of $W^j$ is orthogonal to every basis function $\phi_i^j$ of $V^j$ under the defined inner product.

## The 1D Haar wavelet

The wavelets corresponding to the "box" like basis vectors are known as the *Haar wavelets*, and are defined as:

$$\psi_i^j(x) = \psi(2^j x - i) \tag{3}$$

for $i = 0, \ldots, 2^j - 1$, where

$$\psi(x) = \begin{cases} 1 & \text{for } 0 \leq x < 0.5 \\ -1 & \text{for } 0.5 \leq x < 1 \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Figure 3 shows the Haar wavelets that span $W^1$.

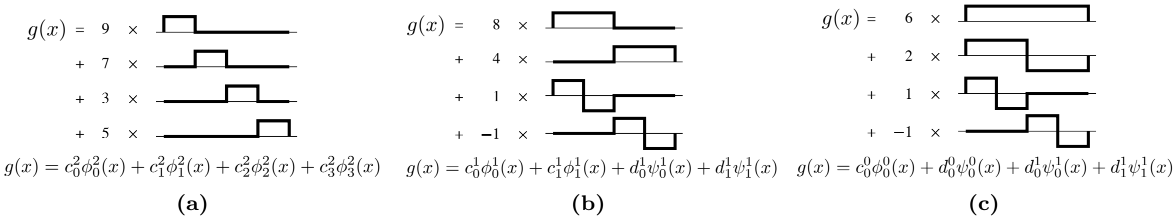


**Figure 4** – The Haar wavelets decomposition for $g(x)$.

### Example

Given our example signal $g(x) = [9, 7, 3, 5]$ we can think of it, as a set of linear combinations in vector-space $V^2$:

$$g(x) = c_0^2\phi_0^2(x) + c_1^2\phi_1^2(x) + c_2^2\phi_2^2(x) + c_3^2\phi_3^2(x)$$

The coefficients $c_0^2, \dots c_3^2$ are thus just the original four signal values $[9, 7, 3, 5]$.

We can rewrite the expression for $g(x)$ in terms of the basis functions $V^1$ and $W^1$. Informally, this amount to the averaging and differencing we saw earlier:

$$g(x) = c_0^1\phi_0^1(x) + c_1^1\phi_1^1(x) + d_0^1\psi_0^1(x) + d_1^1\psi_1^1(x)$$

This gives us four coefficients, $c_0^1, c_1^1, d_0^1, d_1^1$ which equal $[8, 4, 1, -1]$. These should look familiar to you.

Finally, we can rewrite $g(x)$ as the sum of the basis functions in $V^0$, $W^0$ and $W^1$:

$$g(x) = c_0^0\phi_0^0(x) + d_0^0\psi_0^0(x) + d_0^1\psi_0^1(x) + d_1^1\psi_1^1(x)$$

Again, this gives us four coefficients, $c_0^0, d_0^0, d_0^1, d_1^1$ which equal $[6, 2, 1, -1]$. They form the Haar basis for $V^2$.

Graphically, this can be seen in Figure 4.

### Normalization

To ensure that for the chosen inner product and all chosen basis-functions $u(x)$ the following holds:

$$\langle u|u \rangle = 1$$

a normalization step is needed. For the Haar wavelet this van be achieved by slightly modifying the definitions from (1, 3) into:

$$\phi_i^j(x) = 2^{j/2}\phi(2^j x - i) \tag{5}$$

$$\psi_i^j(x) = 2^{j/2}\psi(2^j x - i) \tag{6}$$

### 2.2 Exercise

Give the normalized coefficients of the Haar wavelet decomposition of $f(x)$ into $V^0$, $W^0$ and $W^1$.

## 3 Imaging

For the imaging part of this TP, we will utilize the Matlab Wavelet toolbox.

### 3.1 Exercise

Compute a two-dimensional Haar wavelet decomposition of an image using `dwt2` in Matlab. Visualize it. You may also use the auxiliary function `wcodemat`.

### Compression

The wavelet transform compacts the energy of a signal. The small coefficients, that represent the high detail could be removed to facilitate compression. The actual compression is out of scope of this exercise, but we will have a look at various reconstructions for images from which wavelet coefficients have been removed.

### 3.2 Exercise

- Compute a two-dimensional Haar wavelet decomposition of an image using `wavedec2` in Matlab.
- Sort the coefficients in decreasing magnitude and plot them. Report the results.

### 3.3 Exercise

You will now threshold the coefficients of the Haar wavelet decomposition for 3 scales and reconstruct the image.

- Perform a Haar wavelet decomposition for 3 scales.
- Threshold the coefficients with a suitable value range of your choice. Coefficients that are smaller than the threshold are set to zero.
- Display the original image, the decomposition and the reconstructed image.
- Calculate the PSNR between the image and the reconstruction and the percentage of zero coefficients for different threshold values. Then plot the relative error versus the percentage of zeros. Discuss.