TP5: Information Theory

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1 Quantifiers of information

The first step is to get the probability mass function of each random variable:

U/(V,W)	**00**	**01**	**10**	**11**	***U***
0	$\frac{1}{8}$	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$
1	0	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$
(V,W)	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$	1

Table 1. Probability of $U/(V,W)$

V/W	**0**	**1**	***V***
0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$
1	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{5}{8}$
W	$\frac{1}{4}$	$\frac{3}{4}$	1

Table 2. Probability of V/W

U/V	**0**	**1**	***U***
0	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{1}{2}$
1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
V	$\frac{3}{8}$	$\frac{5}{8}$	1

Table 3. Probability U/V

U/W	**0**	**1**	***U***
0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
1	0	$\frac{1}{2}$	$\frac{1}{2}$
W	$\frac{1}{4}$	$\frac{3}{4}$	1

Table 4. Probability of U/W

The basic formulas of entropy are the following:

$$H(X) = -\sum_{x\in\mathcal{X}} p_X(x)\log_2(p_X(x)) = -E[\log_2(p_X(x))]$$

$$H(X|Y) = -\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p_{X,Y}(x,y)\log_2(p_{X|Y}(x|y)) = \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p_{X,Y}(x,y)\cdot\log_2\left(\frac{p_Y(y)}{p_{X,Y}(x,y)}\right)$$

$$H(X,Y) = H(Y,X) = -\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x,y)\log_2(p(x,y)) - E_{p(x,y)}[\log_2(p(x,y))]$$

$$H(X,Y) = H(X) + H(Y|X) = H(Y) + H(X|Y)$$

$$I(X;Y) = H(X) - H(X|Y) = H(Y) - H(Y|X)$$

where $H(X|Y) \neq H(Y|X)$.

Prociding with the exercise:

a) $H(U), H(V), H(W)$ using data from table 1 and 2

$$H(U) = -\sum_{u\in\mathcal{U}} p(u)\log_2(p(u)) = -\left(2\cdot\frac{1}{2}\cdot\log_2\left(\frac{1}{2}\right)\right) = -\log_2\left(\frac{1}{2}\right) = 1$$

$$H(V) = -\left(\frac{3}{8}\cdot\log_2\left(\frac{3}{8}\right) + \frac{5}{8}\cdot\log_2\left(\frac{5}{8}\right)\right) = -\left(-\frac{3}{8}\cdot 1.42 - \frac{5}{8}\cdot 0.68\right) = 0.95$$

$$H(W) = -\left(\frac{1}{4}\cdot\log_2\left(\frac{1}{4}\right) + \frac{3}{4}\cdot\log_2\left(\frac{3}{4}\right)\right) = -\left(-\frac{2}{4} - \frac{3}{4}\cdot 0.42\right) = 0.81$$

b) $H(U|V), H(V|U)$ and $H(W|U)$ using data from table 2, 3 and 4 and previous results

$$H(U|V) = \sum_{u\in\mathcal{U}}\sum_{v\in\mathcal{V}} p(u,v)\cdot\log_2\left(\frac{p(v)}{p(u,v)}\right) = \frac{1}{8}\cdot\log_2(3) + \frac{3}{8}\cdot\log_2\left(\frac{5}{3}\right) + \frac{1}{4}\cdot\log_2\left(\frac{3}{2}\right) + \frac{1}{4}\cdot\log_2\left(\frac{5}{2}\right) = 0.95$$

$$H(V|U) = H(V) + H(U|V) - H(U) = 0.95 + 0.95 - 1 = 0.9$$

$$H(W|U) = \sum_{w\in\mathcal{W}}\sum_{u\in\mathcal{U}} p(w,u)\cdot\log_2\left(\frac{p(u)}{p(w,u)}\right) = \frac{1}{4}\cdot\log_2(2) + \frac{1}{4}\cdot\log_2(2) + 0 + \frac{1}{2}\cdot\log_2(1) = 0.5$$

c) $I(U;V), I(U;W), I(U;V,W)$ using previous results

$$I(U;V) = H(U) - H(V|U) = 1 - 0.9 = 0.1$$

$$I(U;W) = H(U) - H(W|U) = 1 - 0.5 = 0.5$$

$$I(U;V,W) = H(U) - H(U|V,W)$$

$$H(U|V,W) = \frac{1}{8}\cdot\log_2(1) + 0 + \frac{1}{8}\cdot\log_2(1) + \frac{1}{4}\cdot\log_2(2) + 0 + \frac{1}{4}\cdot\log_2(1) + 0 + \frac{1}{4}\cdot\log_2(2) = 0.5$$

$$I(U;V,W) = 1.0 - 0.5 = 0.5$$

d) H(U, V, W) using previous data

$$H(U,V,W) = H(U|V,W) + H(V|W) + H(W)$$

$$H(V|W) = \frac{1}{8}\cdot\log_2(2) + \frac{1}{4}\cdot\log_2(3) + \frac{1}{8}\cdot\log_2(2) + \frac{1}{2}\cdot\log_2\left(\frac{3}{2}\right) = 0.94$$

$$H(U,V,W) = 0.5 + 0.94 + 0.81 = 2.25$$

2 Communication through noisy channels

First I implemented a simple Binary Symmetric Channel in Octave and two method to code and decode words repeating each bit n times.

Then I analyzed the trend of the *errors* and *rate of communication* using $n = \{1,3,5,11,23\}$. The results can be seen in Figure 1 and Figure 2.

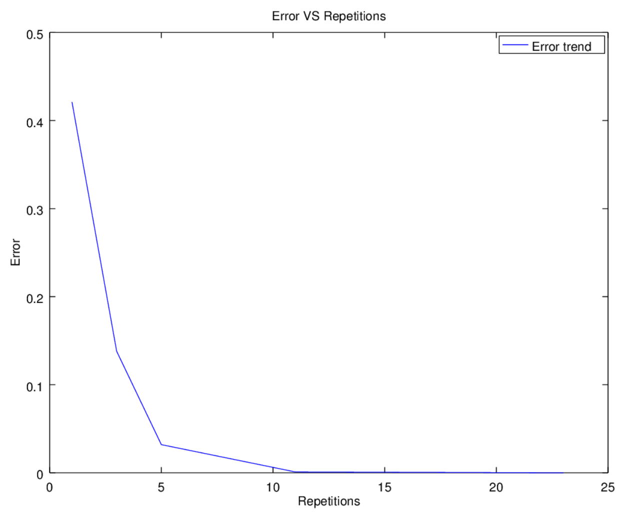

Figure 1. Error VS Repetitions

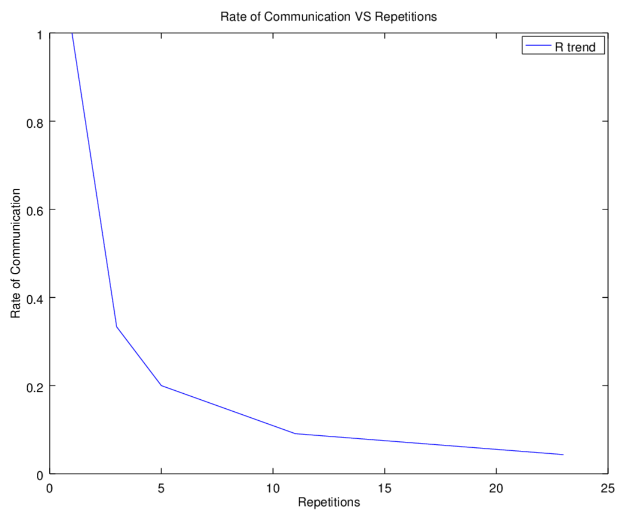

Figure 2. Rate of Communication VS Repetitions

As it possible to see both error and rate of comunication decrease with the increasing of n. In particualr with $n \geqslant \frac{1}{P_{\mathrm{be}}}$ the error become ≈ 0.

In particular with $n = 11$ the ratio of erroroneus sentences is only 0.001 while the rate of communication is 0.1.