

TP5: Information Theory

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1 Quantifiers of information

The first step is to get the probability mass function of each random variable:

$U/(V, W)$	00	01	10	11	U	V/W	0	1	V
0	$\frac{1}{8}$	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$
1	0	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	1	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{5}{8}$
(V, W)	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$	1	W	$\frac{1}{4}$	$\frac{3}{4}$	1

Table 1. Probability of $U/(V, W)$

Table 2. Probability of V/W

U/V	0	1	U	U/W	0	1	U
0	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{1}{2}$
V	$\frac{3}{8}$	$\frac{5}{8}$	1	W	$\frac{1}{4}$	$\frac{3}{4}$	1

Table 3. Probability U/V **Table 4.** Probability of U/W

The basic formulas of entropy are the following:

$$\begin{aligned}
 H(X) &= -\sum_{x \in \mathcal{X}} p_X(x) \log_2(p_X(x)) = -E[\log_2(p_X(x))] \\
 H(X|Y) &= -\sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p_{X,Y}(x, y) \log_2(p_{X|Y}(x|y)) = \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p_{X,Y}(x, y) \cdot \log_2\left(\frac{p_Y(y)}{p_{X,Y}(x, y)}\right) \\
 H(X, Y) &= H(Y, X) = -\sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) \log_2(p(x, y)) - E_{p(x, y)}[\log_2(p(x, y))] \\
 H(X, Y) &= H(X) + H(Y|X) = H(Y) + H(X|Y) \\
 I(X; Y) &= H(X) - H(X|Y) = H(Y) - H(Y|X)
 \end{aligned}$$

where $H(X|Y) \neq H(Y|X)$.

Proceeding with the exercise:

a) $H(U), H(V), H(W)$ using data from table 1 and 2

$$\begin{aligned}
 H(U) &= -\sum_{u \in \mathcal{U}} p(u) \log_2(p(u)) = -\left(2 \cdot \frac{1}{2} \cdot \log_2\left(\frac{1}{2}\right)\right) = -\log_2\left(\frac{1}{2}\right) = 1 \\
 H(V) &= -\left(\frac{3}{8} \cdot \log_2\left(\frac{3}{8}\right) + \frac{5}{8} \cdot \log_2\left(\frac{5}{8}\right)\right) = -\left(-\frac{3}{8} \cdot 1.42 - \frac{5}{8} \cdot 0.68\right) = 0.95 \\
 H(W) &= -\left(\frac{1}{4} \cdot \log_2\left(\frac{1}{4}\right) + \frac{3}{4} \cdot \log_2\left(\frac{3}{4}\right)\right) = -\left(-\frac{2}{4} - \frac{3}{4} \cdot 0.42\right) = 0.81
 \end{aligned}$$

b) $H(U|V), H(V|U)$ and $H(W|U)$ using data from table 2, 3 and 4 and previous results

$$\begin{aligned}
 H(U|V) &= \sum_{u \in \mathcal{U}} \sum_{v \in \mathcal{V}} p(u, v) \cdot \log_2\left(\frac{p(v)}{p(u, v)}\right) = \frac{1}{8} \cdot \log_2(3) + \frac{3}{8} \cdot \log_2\left(\frac{5}{3}\right) + \frac{1}{4} \cdot \log_2\left(\frac{3}{2}\right) + \frac{1}{4} \cdot \log_2\left(\frac{5}{2}\right) = 0.95 \\
 H(V|U) &= H(V) + H(U|V) - H(U) = 0.95 + 0.95 - 1 = 0.9
 \end{aligned}$$

$$H(W|U) = \sum_{w \in \mathcal{W}} \sum_{u \in \mathcal{U}} p(w, u) \cdot \log_2\left(\frac{p(u)}{p(w, u)}\right) = \frac{1}{4} \cdot \log_2(2) + \frac{1}{4} \cdot \log_2(2) + 0 + \frac{1}{2} \cdot \log_2(1) = 0.5$$

c) $I(U; V), I(U; W), I(U; V, W)$ using previous results

$$\begin{aligned}
 I(U; V) &= H(U) - H(V|U) = 1 - 0.9 = 0.1 \\
 I(U; W) &= H(U) - H(W|U) = 1 - 0.5 = 0.5 \\
 I(U; V, W) &= H(U) - H(U|V, W) \\
 H(U|V, W) &= \frac{1}{8} \cdot \log_2(1) + 0 + \frac{1}{8} \cdot \log_2(1) + \frac{1}{4} \cdot \log_2(2) + 0 + \frac{1}{4} \cdot \log_2(1) + 0 + \frac{1}{4} \cdot \log_2(2) = 0.5 \\
 I(U; V, W) &= 1.0 - 0.5 = 0.5
 \end{aligned}$$

d) $H(U, V, W)$ using previous data

$$H(U, V, W) = H(U|V, W) + H(V|W) + H(W)$$

$$H(V|W) = \frac{1}{8} \cdot \log_2(2) + \frac{1}{4} \cdot \log_2(3) + \frac{1}{8} \cdot \log_2(2) + \frac{1}{2} \cdot \log_2\left(\frac{3}{2}\right) = 0.94$$

$$H(U, V, W) = 0.5 + 0.94 + 0.81 = 2.25$$

2 Communication through noisy channels

First I implemented a simple Binary Symmetric Channel in Octave and two method to code and decode words repeating each bit n times.

Then I analyzed the trend of the *errors* and *rate of communication* using $n = \{1, 3, 5, 11, 23\}$. The results can be seen in Figure 1 and Figure 2.

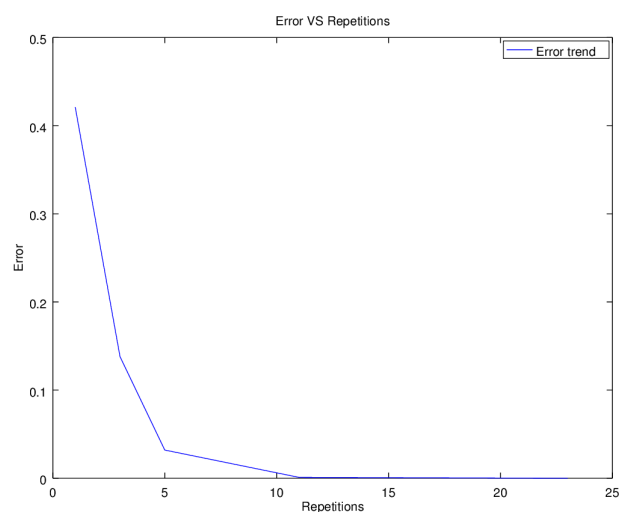


Figure 1. Error VS Repetitions

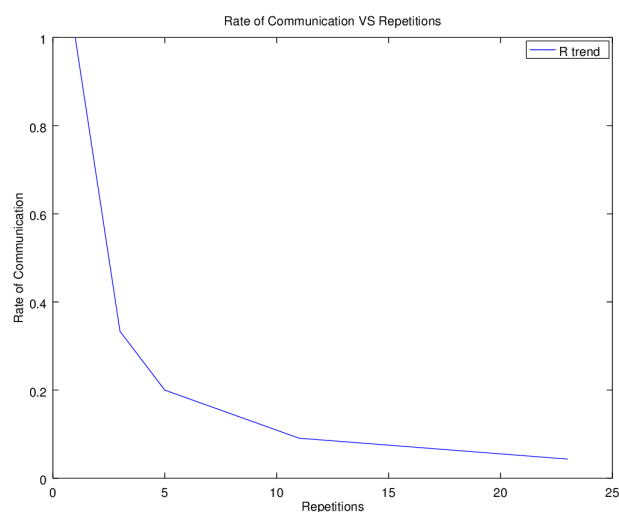


Figure 2. Rate of Communication VS Repetitions

As it possible to see both error and rate of communication decrease with the increasing of n . In particular with $n \geq \frac{1}{P_{be}}$ the error become ≈ 0 .

In particular with $n = 11$ the ratio of errorneous sentences is only 0.001 while the rate of communication is 0.1.

The probabily of change of a bit originally was 0.1 however duplicating this bit n times the data become more robust however each bit carry less information and that impact the rate of communication.