The Quadratic Assignment Problem

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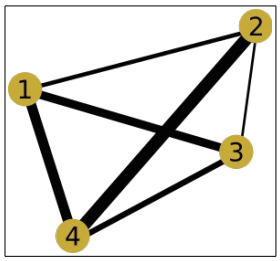
1. Quadratic Assignment Problem

The quadratic assignment problem (QAP) is one of the fundamental combinatorial optimization problems.
It models one common logistic problem, and it is part of the *facility location* category of problems.



The QAP models the following situation:
There are a set of n facilities and a set of n locations. For each pair of locations, a distance is specified and for each pair of facilities a weight is specified (e.g., the amount of supplies transported between the two facilities). The problem is to assign all facilities to different locations with the goal of minimizing the sum of the distances multiplied by the corresponding flows.[1]

We can formalize the problem in this way:

$n \in \mathbb{N}$ the number of facilities

$D \in \mathbb{N}^n$ the distance matrix (symmetric with diagonal 0, as the distances are symmetric)

$W \in \mathbb{N}^n$ the weight matrix (symmetric with diagonal 0, as the flows are symmetric)

The space of solution S is composed of all the possible ($n!$) permutation of the n facilities.
And the fitness is defined as $f(\psi) = \sum_{i,\, j=1..n} W_{[i][j]} \cdot D_{[\psi_i][\psi_j]}$ where ψ_i is the location of the facility i in the solution ψ .

2. Introduction to the Tabu Search metaheuristic

The Tabu Search (TS) metaheurisitic is an evolution of the *local search* heuristic created by Fred W. Glover in 1986.
The local search tries to find an acceptable solution to a problem choosing the best neighbor of the currently selected solution. A limitation of this method is that it tends to get stuck in sub-optimal regions (where there are local minimum or maximum) and so it limits the solutions space S exploration.
To avoid this problem and to explore regions otherwise left unexplored, the tabu search modify the neighborhood space $N(\psi_i)$ during the search.
To do so, it create it defines $N^1(\psi_i)$ as $\forall \psi' \in N^1(\psi_i) \rightarrow \psi' \in N(\psi_i) \wedge \nexists \psi' \in N^1(\psi) \rightarrow \psi' \in tl$.
The tl is the tabu list (the is not necessary implemented as a list), a memory structure where are stored the last l solutions visited or movements done.
In this way the search will walk away from the solutions already visited and it will explore more parts of S , the size of l (tabu tenure) is very important and can vary the behavior of the algorithm, I will explain and demonstrate it later.

1 From QAP wikipedia page

3. Tabu Search and the QAP

To implement any metaheurisitic there is the need to define 5 points:

- The search space S
- The start solution
- The neighborhood of a solution $N(\psi)$
- The exploration operator U
- The stop condition

We already defined $S \in \mathbb{R}^n$ as the set of all possible permutation of n facilities.

The **start solution** will be generated randomly.

The **neighborhood** $N(\psi)$ of a solution ψ is composed by all the solutions ψ' with only two facilities swapped from the original one.

In this way the size of the neighborhood will be $n \cdot (n-1)/2$.

But, as I explained before, the tabu search modify the set of neighbors during the research, and to do so it need a tabu list. In this case the elements of the tabu list of tenure l are represented in the form of (i, r) , with i a facility and r a location.

This means that a neighbor that exchange the facilities i, j in the respective locations s, r is forbidden if both (i, r) and (j, s) are in the tabu list, meaning that the same permutation was done in the last l iterations.

The U operator will choose the best (in terms of fitness, in this case the lower fitness) neighbor in the set $N^1(\psi)$. To improve the performance we implement an **aspiration** and **diversification** process.

With the aspiration process U will choose a neighbor ψ' even if it is not in the $N^1(\psi)$ if its fitness is better of the best found one.

With the diversification process U will choose a neighbor ψ' even if it has not the best fitness if the associated permutation has not been chosen in the last u iterations.

Finally the **stop condition** is defined as a the number of iterations to do.

4. Implementation

The peculiarities of this implementation are 3:

- The use of a matrix $n \times n$ as tabu list
- The use of the same matrix for the diversification process
- The use of the Δfitness to speed up the computation of the neighbors fitness

The tabu matrix (**tm**) represent all the n possible facilities and all the n possible locations.

When leaving a solution to another one, created exchanging the facility i at the location r with the facility j at the location s , at the iteration t the algorithm store t in the matrix at the positions (i, r) and (j, s) . Now checking if a permutation is forbidden is simply looking if $t' \leq (tm[i', s'] + l)$ and $t' \leq (tm[j', r'] + l)$ where l is the tenure of the tabu list.

For the diversification process we know that we have to perform a permutation if it has not been done in the last u iterations. Verify this condition at the iteration t' using the tm matrix is simply checking if $t' \leq (tm[i', s'] + l)$ and $t' > (tm[j', r'] + u)$.

The definition of the fitness of a solution in the QAP is:

$$f(\psi) = \sum_{i,\, j=1..n} W_{[i][j]} \cdot D_{[\psi_i][\psi_j]}$$

This means that the time to compute the fitness for one solution is n^2 , but as the D and W matrices are symmetric and with diagonal 0 (the distance between a and b is the same of the one between b and a and the distance between a and a is 0) the time can be reduced to $n \cdot (n-1)/2$.

The number of neighbors of a solution is again $n \cdot (n-1)/2$, meaning that the complexity to compute all the fitness of the neighborhood would be of type n^4 .

To compute quicker the fitness of a neighbor is possible to use the partial cost or Δfitness, in fact the difference of the fitness between a solution ψ and one of its neighbor depends only from the facilities (i, j) switched and can be computed as:

$$\Delta f(\psi, i, j) = 2 \cdot \sum_{k \neq i,\, j}^{n} (W_{[i][k]} - W_{[j][k]}) \cdot (D_{[\psi_j][\psi_k]} - D_{[\psi_i][\psi_k]})$$

On this way the complexity to compute the fitness of the neighborhood would be of type n^3 .

Finally, I implemented the algorithm both in python and C++ for two reason, first to verify my results rewriting the algorithms from scratch and second to improve the performance (over 50 times better with no optimization in the code).

5. Results

Before starting to analyze the performance and the results of the metaheuristic I wanted to know more about the landscape of the problem. If not specified, the analysis are done using the "1.dat" file representing a QAP of size 12. I choose to explore S by generate 1000 of random solutions and compute its fitness. The result is the follow: $\mu : 813$, $\sigma = 40$

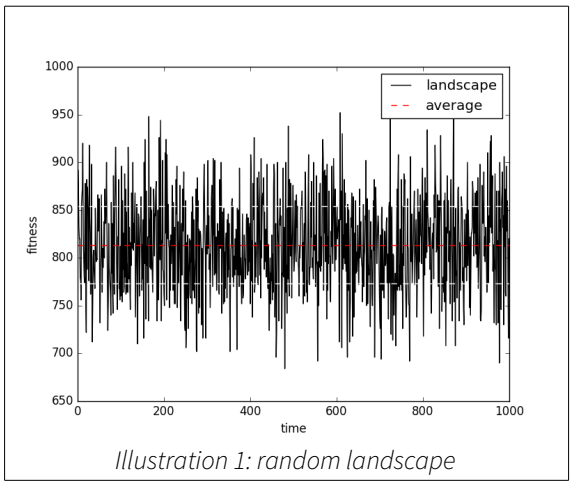


Illustration 1: random landscape

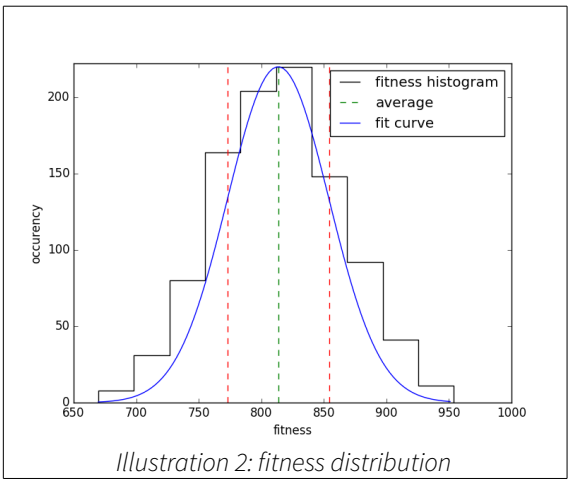


Illustration 2: fitness distribution

Then running the metaheurisitic over the same file multiple times give me a stable optimal solution of **578**, using the parameter found before, the probability of this results it's around 3.2×10^{-8} .
As the size of S is $12!\simeq5^8$ and it's still a reasonable number I choose to perform a brute force research in the solution space to look for any better solution. After few minutes of computation it results that **578** is not only a local minimum but also the **global** one (actually there are around 15 solutions with fitness of 578). The fact that 578 is a global minimum and that the size of S still reasonably small explain the stability of the solution using the metaheuristic.

As introduced before the behavior of the metaheurisitic is influenced by two factors, the size of l and the optionally diversification process.
To understand better how the research is changing I choose to plot the fitness of the solution explored by the time for different configuration (always using the file "1.dat" as source)

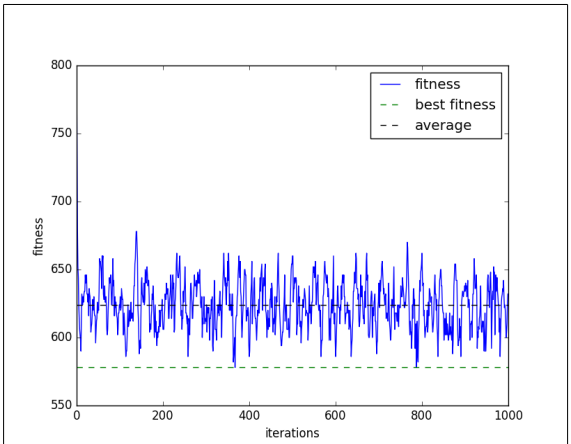


Illustration 3: research with l=1n

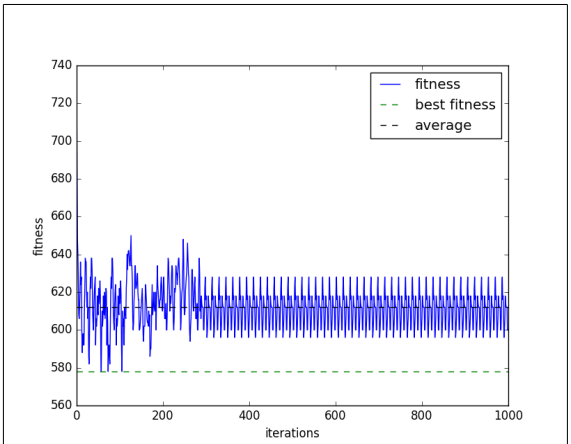


Illustration 4: research with l=0.5n

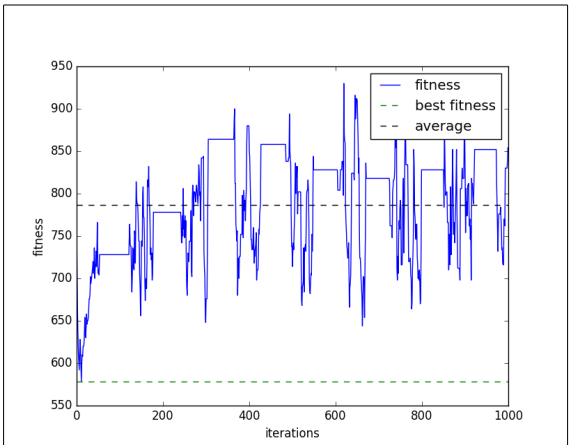


Illustration 5: research with l=10n

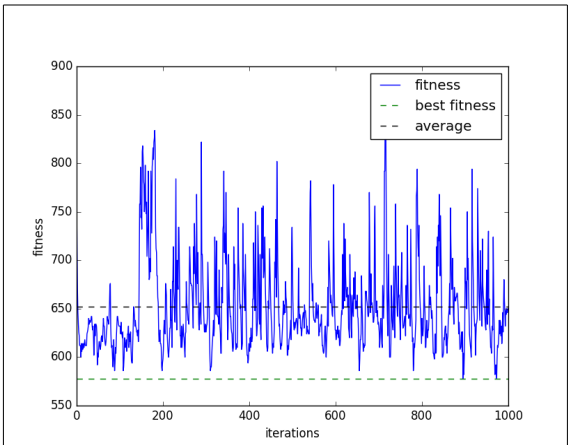


Illustration 6: research with diversification

In the illustration 3 the configuration was $l=n$ and no diversification, it results in a research without any pattern repetition (it means the research is not trapped in any local/global minimum) and it has a mean fitness around 640, that is reached after few iterations.
With a much smaller ratio l/n , like in the illustration 4 where $l=0.5\cdot n$, it's possible to find that the algorithm get stuck in some local minimum, because the tabu list empties itself to quickly and the algorithm fall

back. With lower l the algorithm is privileging the intensification, that's why the average fitness in this case is around 610, instead of 640 before.
When otherwise the ratio l/n is way to big, like in the illustration 5 where $l=10\cdot n$, the algorithm will end up in situations where all the neighbors are forbidden, and (in my case) it will end up skipping many iterations waiting to have at least one free neighbor. More over is possible to see that with higher l the algorithm is diversifying much more, looking in solutions with worst fitness to explore new areas. In fact in this case the average fitness it's around 790 (very close to the landscape average) .
Finally it's possible to have a good mixture of diversification and intensification using the diversification process (in the plot is easy to see that start after n^2 or 144 iterations) , in this way the algorithm will intensify the research in areas never explored (instead of only diversify), this is easily recognizable in the illustration 6, where the average still around 650 but it's possible to see that the algorithms searched in different unexplored areas (where the high fitness peaks are).
In the first experiment I will try to variate l and enable or disable the diversification.

Experiment over file 1.dat			
	L=1n	L=0.9n	L=0.5n
Diversification off	Bf=578, μ=578, σ=0	Bf=578, μ=580, σ=3	Bf=578, μ=584, σ=3.5
Diversification on	Bf=578, μ=578, σ=0	Bf=578, μ=578, σ=0	Bf=578, μ=578, σ=0

The results are, as expected, better for values of l close to n or with diversification on, for the reasons already explicated before.
In the next experiment I will execute the metaheurisitic over QAP with different size but with the same settings, $l=0.5n$ and diversification enabled, to have a good mixture of diversification and intensification .

Experiment over QAP with different sizes (with diversification on)					
N	File	Iterations	Best fitness	Mean	Standard deviation
40	40.dat	20000	491,718	493,374	1,033
50	50.dat	20000	1,259,690	1,264,160	1,988
80	80.dat	20000	8,664,514	8,695,340	12,404
100	100.dat	5000			

7. Project structure

cpp/	It contains the c++ implementation of the metaheuristic used for the experiments
cpp/bin	It contains the compilation script and the *.dat file generated for the experiments
→ analyzer	Perform analysis over one run of n iterations of the metaheurisitic
→ generator	Generate a dat file containing the n,D,W matrix of a random QAP problem of size n
→ statistic	Perform analysis over multiple executions of the metaheurisitic over the same QAP problem
→ landscape	Perform some analysis over the landscape of a QAP problem
→ brute_force	Perform a brute force search of a QAP problem
python/	It contains the python code used for the first implementation of the metaheurisitic