Chapter 1 – Information Coding

Modern computers are implemented using digital electronic technology, with basic building blocks such as transistors and logic gates. In the machine, every piece of information (numbers, text, pictures, programs) is encoded in terms of boolean values i.e. zeroes and ones. In other words, the computer thinks in binary. The goal for today is to get familiar with binary.

1 Counting in base 2

You're used to counting in base 10:

- decimal digits take their value in {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}
- values bigger than 9 are represented by concatenating digits for successive powers of 10:
- $234 = 2 \times 10^2 + 3 \times 10^1 + 4 \times 10^0$
- $45678 = 4 \times 10^4 + 5 \times 10^3 + 6 \times 10^2 + 7 \times 10^1 + 8 \times 10^0$

Binary numbers are the exact same idea but in base 2:

- binary digits (**bits**) take their value in {0, 1}
- values bigger than 1 are represented by concatenating bits for successive powers of 2:
- binary 110 $= 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 =$ decimal 6
- binary 1100011 $= 1 \times 2^6 + 1 \times 2^5 + 0 \times 2^4 + 0 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 =$ decimal 99

When the context is not enough to distinguish "binary 110" from "decimal 110" we will use a "0b" prefix to indicate binary numbers, e.g. 0b1010=10.

Binary-to-Decimal Conversion Let x be a sequence of n bits $x_{n-1}, x_{n-2}, \cdots, x_1, x_0$ (warning: we always write bits x_i with **descending** values of i, down to zero). To interpret the value of x as a natural number, we apply the same formula as in decimal, but with a radix of 2.

$$x = \sum_{i=0}^{n-1} x_i \cdot 2^i$$

Example: 0b110010 reads as 32+16+0+0+2+0 = 50. (Please don't learn the powers of 2 by heart ! We provide all you need on the following page.)

Exercise Convert 0b101010 to decimal. Same question with 0b111110100.

Exercise (10 min max) Show that with n bits, the largest integer that can be represented is $2^n - 1$.

Binary numbers: bit length vs numeric range A sequence of n bits can take any one one of 2^n configurations. The smallest natural number encoded on n bits is 0. The largest natural number encoded on n bits is $2^n - 1$.

Examples:

- $n = 4$: $x \in [0..15] \Longrightarrow$ 16 different values
- $n = 8$: $x \in [0..255] \Longrightarrow$ 256 different values
- $n = 16$: $x \in [0..65535] \Longrightarrow$ 65536 different values
- $n = 32$: $x \in [0..4294967295] \Longrightarrow$ 4294967296 different values (that's about $4x10^9 \sim$ 4 Billion $\sim$ 4 Giga)
- $n = 64$: $x \in [0..18446744073709551615] \Longrightarrow$ 18446744073709551616 different values (that's about $1.8x10^{19} \sim$ 18 Quintillion)

Useful: Powers of 2

2^{0} =	1	2^{16} =	65 536	2^{32} =	4 294 967 296	2^{48} =	281 474 976 710 656
2^{1} =	2	2^{17} =	131 072	2^{33} =	8 589 934 592	2^{49} =	562 949 953 421 312
2^{2} =	4	2^{18} =	262 144	2^{34} =	17 179 869 184	2^{50} =	1 125 899 906 842 624
2^{3} =	8	2^{19} =	524 288	2^{35} =	34 359 738 368	2^{51} =	2 251 799 813 685 248
2^{4} =	16	2^{20} =	1 048 576	2^{36} =	68 719 476 736	2^{52} =	4 503 599 627 370 496
2^{5} =	32	2^{21} =	2 097 152	2^{37} =	137 438 953 472	2^{53} =	9 007 199 254 740 992
2^{6} =	64	2^{22} =	4 194 304	2^{38} =	274 877 906 944	2^{54} =	18 014 398 509 481 984
2^{7} =	128	2^{23} =	8 388 608	2^{39} =	549 755 813 888	2^{55} =	36 028 797 018 963 968
2^{8} =	256	2^{24} =	16 777 216	2^{40} =	1 099 511 627 776	2^{56} =	72 057 594 037 927 936
2^{9} =	512	2^{25} =	33 554 432	2^{41} =	2 199 023 255 552	2^{57} =	144 115 188 075 855 488
2^{10} =	1024	2^{26} =	67 108 864	2^{42} =	4 398 046 511 104	2^{58} =	288 230 376 151 711 744
2^{11} =	2048	2^{27} =	134 217 728	2^{43} =	8 796 093 022 208	2^{59} =	576 460 752 303 423 488
2^{12} =	4 096	2^{28} =	268 435 456	2^{44} =	17 592 186 044 416	2^{60} =	1 152 921 504 606 846 976
2^{13} =	8 192	2^{29} =	536 870 912	2^{45} =	35 184 372 088 832	2^{61} =	2 305 843 009 213 693 952
2^{14} =	16 384	2^{30} =	1 073 741 824	2^{46} =	70 368 744 177 664	2^{62} =	4 611 686 018 427 387 904
2^{15} =	32 768	2^{31} =	2 147 483 648	2^{47} =	140 737 488 355 328	2^{63} =	9 223 372 036 854 775 808
						2^{64} =	18 446 744 073 709 551 616

Exercise (hands-on) Open a terminal window and type `python` (or maybe `python3` ; ask for help until you get to the standard prompt i.e. ">>>"). Type a binary number e.g. `0b110010` then press Return. Observe how python prints numbers in decimal by default (it can be changed, as we'll see later)

Exercise (hands-on) Besides usual arithmetic operations (+, –, $\times$ etc) python also offers a built-in exponentiation operator: writing `x**y` will compute x^y i.e. x to the power of y. Try computing a few examples of powers of two e.g. `2**10`, or `2**32` etc. (btw, this is how we generated the table above !)

2 Decimal-to-Binary Conversion

Method A: divide by two, repeatedly Given a number x, the remainder of the euclidean division of x by 2 gives the right-most digit of its binary representation (in other words $x_0 = x \mod 2$):

$$\begin{aligned} x &= x_{n-1}\cdot 2^{n-1} + x_{n-2}\cdot 2^{n-2} + \cdots + x_2\cdot 2^2 + x_1\cdot 2^1 + x_0 \\ &= \underbrace{\left(x_{n-1}\cdot 2^{n-2} + x_{p-2}\cdot 2^{n-3} + \cdots + x_2\cdot 2^1 + x_1\right)}_{\text{quotient}}\cdot 2 \;+\; \underbrace{x_0}_{\text{remainder}} \end{aligned}$$

If we repeat this procedure to the quotient until we reach 0, then we get all successive digits x_1, x_2 and so on (from right to left). Example with $x = 423$:

$$\begin{array}{rcrcl} 423 &=& 211\times 2 &+& 1 \\ 211 &=& 105\times 2 &+& 1 \\ 105 &=& 52\times 2 &+& 1 \\ 52 &=& 26\times 2 &+& 0 \\ 26 &=& 13\times 2 &+& 0 \\ 13 &=& 6\times 2 &+& 1 \\ 6 &=& 3\times 2 &+& 0 \\ 3 &=& 1\times 2 &+& 1 \\ 1 &=& 0\times 2 &+& 1 \end{array}$$

From this we conclude that: 423 = 0b110100111

Method B: subtract powers of two Given a number x, we search the biggest i such that $2^i < x$. This tells us that bit $x_i = 1$. Then we subtract 2^i from x and repeat until x reaches one or zero, which tells us our final bit x_0.

Example with $x = 423$ again:

- what's the biggest power of 2 smaller than 423 ?
    - that's $256 = 2^8$
    - $423 - 256 = 167$
- what's the biggest power of 2 smaller than 167 ?
    - that's $128 = 2^7$
    - $167 - 128 = 39$
- what's the biggest power of 2 smaller than 39 ?
    - that's $32 = 2^5$
    - $39 - 32 = 7$
- what's the biggest power of 2 smaller than 7 ?
    - that's $4 = 2^2$
    - $7 - 4 = 3$
- what's the biggest power of 2 smaller than 3 ?
    - that's $2 = 2^1$
    - $3 - 2 = 1$

In conclusion, we find that $423 = 2^8 + 2^7 + 2^5 + 2^2 + 2^1 + 1$ = 0b110100111

Exercise (pen & paper) Convert 23 to binary. Same question with 200.

Exercise (hands-on) Play with the `bin()` function of python to convert a few numbers to binary notation. For instance, `bin(423)` returns 0b110100111.

Remark Throughout this course, you should never hesitate to "cheat" (by using a computer) to save time and/or to check your results. But you should still be able to do things by hand when required !

3 Hexadecimal Notation

Significant Digits On paper we usually only write significant digits (i.e. 0011 really is the same as 11) but within the computer, bits are stored in bundles of 8 bits called **bytes**. A byte can take 256 distinct values, from 00000000 to 11111111. Larger numbers are usually represented on 32 bits (4 bytes aka a **word**) or sometimes on 16 bits (2 bytes).

Counting in base 16 is actually easier than it sounds. Indeed $16=2^4$, so one digit in base 16 represents 4 digits in base 2. This means that we can represent each bundle of 4 bits with one **hexadecimal digit**, as illustrated below.

Dec	Hex	Bin
0	0	0
1	1	1
2	2	10
3	3	11
4	4	100
5	5	101
6	6	110
7	7	111
8	8	1000
9	9	1001
10	A	1010
11	B	1011
12	C	1100
13	D	1101
14	E	1110
15	F	1111
16	10	10000
17	11	10001
18	12	10010
19	13	10011

Hexadecimal is very useful as a compact notation for binary numbers, but we will never compute in base 16, or do direct conversions. To avoid confusion, hexadecimal notation uses the "0x" prefix, e.g. 423=0x1A7.

Exercise (pen & paper) Fill in the following table by converting each value to all notations.

Decimal	Binary	Hexadecimal
218		
		AB
40		
		87
	1 0 1 1 1 1 1 0	

Exercise (hands-on) Use python to check your answers to the previous exercise. Play with the `hex()` function to convert a number to hexadecimal: `hex(423)` will return 0x1A7.

4 Representing text with ASCII

Binary encoding is also useful to represent text. There are several standards in use today (e.g. unicode, ISO-latin, windows-125x, etc) most of which are extensions of ASCII, the “American Standard Code for Information Interchange” created in the 60s. Originally based on the English alphabet, ASCII encodes 128 “characters” into seven-bit integers, from 0 to 0x7F. In practice today we store each character on one eight-bit byte.

The majority of ASCII characters are ordinary letters, digits and punctuation (codes 0x20 to 0x7E). Code 0x20 represents the **space** between words. Codes 0x21 to 0x7E are known as the **printable characters**. In addition, the standard defines so-called **control characters** (codes 0 to 0x1F, and 0x7F) which originated with electromechanical teletypewriter machines, but most of these values are obsolete nowadays and you can safely ignore them.

ASCII table (binary) In the chart below, the line position indicates bits 6–5 and the column position indicates bits 4–0. For instance letter “A” is encoded as 0b1000001 (line 0b10, column 0b00001).

	00000	00001	00010	00011	00100	00101	00110	00111	01000	01001	01010	01011	01100	01101	01110	01111	10000	10001	10010	10011	10100	10101	10110	10111	11000	11001	11010	11011	11100	11101	11110	11111
00	NUL	SOH	STX	ETX	EOT	ENQ	ACK	BEL	BS	TAB	LF	VT	FF	CR	SO	SI	DLE	DC1	DC2	DC3	DC4	NAK	SYN	ETB	CAN	EM	SUB	ESC	FS	GS	RS	US
01	␣	!	"	#	$	%	&	’	(	)	*	+	,	-	.	/	0	1	2	3	4	5	6	7	8	9	:	;	<	=	>	?
10	@	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	[	\	]	ˆ	_
11	`	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	{	|	}	~	DEL

ASCII table (hex) The chart below shows the exact same information but with hexadecimal numbers: the (half-)line position indicates the first hex digit, and the column position indicates the second hex digit.

	-0	-1	-2	-3	-4	-5	-6	-7	-8	-9	-A	-B	-C	-D	-E	-F
0-	NUL	SOH	STX	ETX	EOT	ENQ	ACK	BEL	BS	TAB	LF	VT	FF	CR	SO	SI
2-	␣	!	"	#	$	%	&	’	(	)	*	+	,	-	.	/
4-	@	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
6-	`	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o

	-0	-1	-2	-3	-4	-5	-6	-7	-8	-9	-A	-B	-C	-D	-E	-F
1-	DLE	DC1	DC2	DC3	DC4	NAK	SYN	ETB	CAN	EM	SUB	ESC	FS	GS	RS	US
3-	0	1	2	3	4	5	6	7	8	9	:	;	<	=	>	?
5-	P	Q	R	S	T	U	V	W	X	Y	Z	[	\	]	ˆ	_
7-	p	q	r	s	t	u	v	w	x	y	z	{	|	}	~	DEL

Exercise (pen & paper) What is the ASCII code of digit “7” ?

Exercise (pen & paper) Decode the message below:

Byte (hex)	48	65	6c	6c	6f	2c	20	77	6f	72	6c	64	20	21
Text														

Exercise (pen & paper) Encode your first name in ASCII.

Exercise (hands-on) Use python to play with ASCII encoding and decoding. The relevant functions are `chr()` to go from integers to characters, and `ord()` to go the other way.

5 RGB

Binary numbers can also represent colors. In the RGB model, a color is described by indicating how much of each of the red, green, and blue light is included. The color is expressed as an RGB triplet (r,g,b), where each component can vary from zero to a defined maximum value. If all the components are at zero the result is black; if all are at maximum, the result is the brightest representable white.

Exercise (hands-on) Download `rgb.py` from Moodle, and type `python3 rgb.py FFFFFF` to execute it. The last argument on the command-line is a **hex triplet** describing the desired color with three byte values (from 00 to FF in hexadecimal). In other words `000000` means “black” and `FFFFFF` means “white”. Try both. Then, using trial and error, find out the hex triplets corresponding to standard colors like yellow, purple, orange, cyan, etc.