Homework 2 - Berkeley STAT 157

15/18

Handout 1/29/2019, due 2/5/2019 by 4pm in Git by committing to your repository.

```
In [207]: from mxnet import nd, autograd, gluon
```

1. Multinomial Sampling

Implement a sampler from a discrete distribution from scratch, mimicking the function `mxnet.ndarray.random.multinomial`. Its arguments should be a vector of probabilities p. You can assume that the probabilities are normalized, i.e. tha they sum up to 1. Make the call signature as follows:

```
samples = sampler(probs, shape)

probs   : An ndarray vector of size n of nonnegative numbers summing up to 1
shape   : A list of dimensions for the output
samples : Samples from probs with shape matching shape
```

Hints:

1. Use `mxnet.ndarray.random.uniform` to get a sample from $U[0, 1]$.
2. You can simplify things for `probs` by computing the cumulative sum over `probs`.

```
In [208]: def sampler(probs, shape):
              ## Add your codes here
              for i in range(1, len(probs)):
                  probs[i] += probs[i-1]

              tmp = nd.random.uniform(0, 1, shape)
              result = nd.zeros(shape)
              for i in range(len(probs)):
                  result += 1.0 * (tmp > probs[i])
              return result

          # a simple test
          sampler(nd.array([0.1, 0.2, 0.3, 0.2, 0.05, 0.15]), (9,20))
```

```
Out[208]: [[0. 5. 2. 2. 1. 3. 4. 3. 0. 2. 5. 1. 2. 2. 2. 2. 0. 2. 1. 4.]
           [2. 2. 5. 3. 0. 3. 5. 0. 5. 0. 2. 0. 2. 0. 1. 1. 3. 4. 0. 2.]
           [0. 2. 2. 3. 0. 2. 1. 0. 1. 2. 1. 3. 1. 2. 0. 2. 1. 5. 3. 1.]
           [0. 2. 0. 1. 0. 2. 1. 1. 5. 1. 2. 1. 2. 5. 4. 1. 1. 1. 1. 0.]
           [2. 2. 5. 2. 2. 2. 0. 4. 4. 1. 1. 3. 4. 3. 2. 2. 5. 3. 3. 2.]
           [2. 2. 1. 2. 0. 1. 1. 5. 0. 5. 1. 3. 1. 3. 3. 0. 2. 3. 0. 1.]
           [2. 2. 3. 1. 1. 5. 1. 2. 1. 5. 1. 1. 0. 0. 3. 0. 1. 1. 1. 2.]
           [0. 5. 1. 3. 1. 2. 1. 1. 2. 0. 2. 5. 4. 1. 5. 1. 0. 2. 1. 2.]
           [3. 1. 1. 1. 5. 4. 1. 3. 3. 0. 3. 3. 3. 1. 0. 4. 1. 2. 3. 0.]]
          <NDArray 9x20 @cpu(0)>
```

4/4

2. Central Limit Theorem

Let's explore the Central Limit Theorem when applied to text processing.

- Download https://www.gutenberg.org/ebooks/84 (https://www.gutenberg.org/files/84/84-0.txt) from Project Gutenberg
- Remove punctuation, uppercase / lowercase, and split the text up into individual tokens (words).
- For the words `a`, `and`, `the`, `i`, `is` compute their respective counts as the book progresses, i.e.

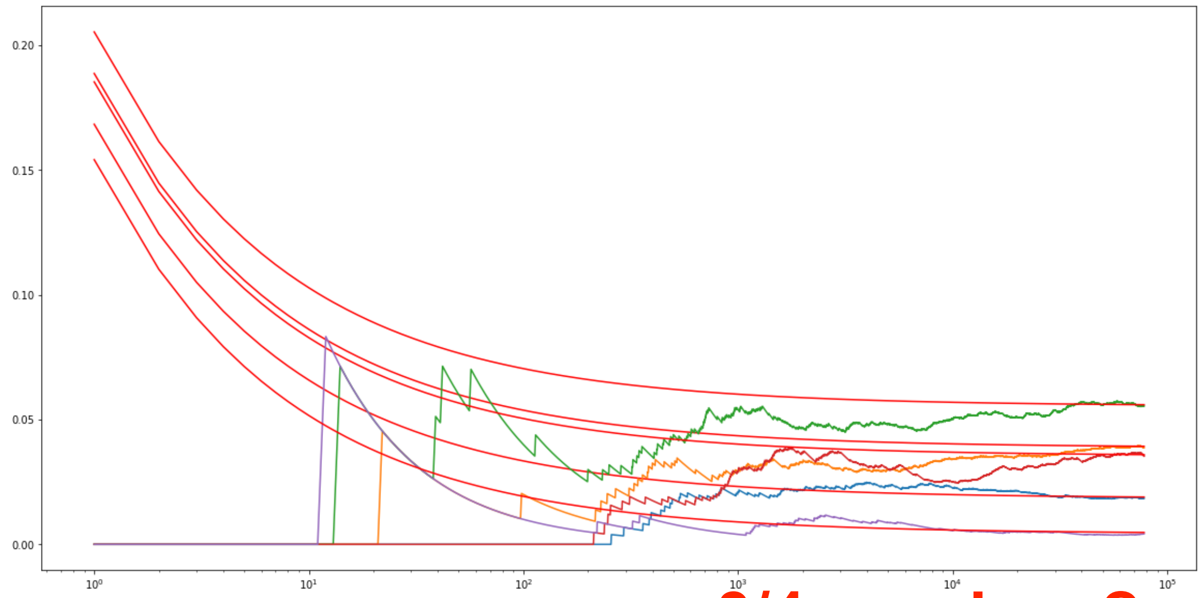
$$n_{\text{the}}[i] = \sum_{j=1}^{i} \{w_j = \text{the}\}$$

- Plot the proportions $n_{\text{word}}[i]/i$ over the document in one plot.
- Find an envelope of the shape $O(1/\sqrt{i})$ for each of these five words.
- Why can we **not** apply the Central Limit Theorem directly?
- How would we have to change the text for it to apply?
- Why does it still work quite well?

In [209]:
```
import string
import numpy as np
from matplotlib import pyplot as plt

filename = gluon.utils.download('https://www.gutenberg.org/files/84/84-
0.txt')
with open(filename) as f:
    book = f.read()
for c in string.punctuation:
    book = book.replace(c,"")
book = book.lower() ##convert all to lowercase
book = book.split()
words = len(book)
counts = [0,0,0,0,0]
for k in range(5):
    counts[k] = nd.zeros(words).asnumpy()
for i in range(words):
    if book[i] == "a":
        counts[0][i] = 1
        continue
    if book[i] == "and":
        counts[1][i] = 1
        continue
    if book[i] == "the":
        counts[2][i] = 1
        continue
    if book[i] == "i":
        counts[3][i] = 1
        continue
    if book[i] == "is":
        counts[4][i] = 1
        continue
var = 0.15
x = np.arange(1, words+1)
for k in range(5):
    counts[k] = np.cumsum(counts[k])/x
    counts[k].reshape(words, 1)
plt.figure(figsize=(20, 10))
for k in range(5):
    plt.semilogx(x, counts[k])
    mean = counts[k][words - 1]
    plt.semilogx(x, var*np.power(x,-0.5) + mean, "r")

plt.show()
```



3/4 envelope?

Q: Why can we not apply the Central Limit Theorem directly?

A: CLT asks for a normalized sum of sufficiently many i.i.d random variables, but in this text, the words don't appear independantly from each other. Take "the" for example: whether "the" appears at one position is not independent from the ones near it (a "the the" phrase might never appear).

Q: How would we have to change the text for it to apply?

A: Following up on the answer above, we can re-shuffle all the words in the text to make each word distribute more evenly, to reach the desired independance.

Q: Why does it still work quite well?

A: Because the proportions are very small in mean and variance, and as the total number of words increse, the correlation between words tend to be trivial.

4/4

3. Denominator-layout notation

We used the numerator-layout notation for matrix calculus in class, now let's examine the denominator-layout notation.

Given $x, y \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$, we have

$$\frac{\partial y}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial y}{\partial x_1} \\ \frac{\partial y}{\partial x_2} \\ \vdots \\ \frac{\partial y}{\partial x_n} \end{bmatrix}, \quad \frac{\partial \mathbf{y}}{\partial x} = \left[\frac{\partial y_1}{\partial x}, \frac{\partial y_2}{\partial x}, \ldots, \frac{\partial y_m}{\partial x} \right]$$

and

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial \mathbf{y}}{\partial x_1} \\ \frac{\partial \mathbf{y}}{\partial x_2} \\ \vdots \\ \frac{\partial \mathbf{y}}{\partial x_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1}, \frac{\partial y_2}{\partial x_1}, \ldots, \frac{\partial y_m}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2}, \frac{\partial y_2}{\partial x_2}, \ldots, \frac{\partial y_m}{\partial x_2} \\ \vdots \\ \frac{\partial y_1}{\partial x_n}, \frac{\partial y_2}{\partial x_n}, \ldots, \frac{\partial y_m}{\partial x_n} \end{bmatrix}$$

Questions:

1. Assume $\mathbf{y} = f(\mathbf{u})$ and $\mathbf{u} = g(\mathbf{x})$, write down the chain rule for $\frac{\partial \mathbf{y}}{\partial \mathbf{x}}$

A: for the jith entry in the resulting matrix:

$$\frac{\partial(y_i)}{\partial(x_j)} = \frac{\partial(y_i)}{\partial(u_i)} * \frac{\partial(g_j)}{\partial(x_j)}$$

1. Given $\mathbf{X} \in \mathbb{R}^{m \times n}$, $\mathbf{w} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^m$, assume $z = \|\mathbf{X}\mathbf{w} - \mathbf{y}\|^2$, compute $\frac{\partial z}{\partial \mathbf{w}}$.

Write out expression for z with entries in X, W, y:

$$z = (\Sigma_{i=1}^n x_{1i} w_i - y_1)^2 + \ldots + (\Sigma_{i=1}^n x_{mi} w_i - y_1)^2$$

$$\frac{\partial z}{\partial w_1} = 2(\Sigma_{i=1}^n x_{1i} w_1 - y_1)(x_{11}) + \ldots + 2(\Sigma_{i=1}^n x_{mi} w_i - y_1)(x_{m1})$$

Thus for $k = 1, \ldots n$

$$\frac{\partial z}{\partial w_k} = 2(\Sigma_{i=1}^n x_{1i} w_1 - y_1)(x_{1k}) + \ldots + 2(\Sigma_{i=1}^n x_{mi} w_i - y_1)(x_{mk})$$

2/3 vectorized form?

4. Numerical Precision

Given scalars `x` and `y`, implement the following `log_exp` function such that it returns a numerically stable version of

$$-\log\left(\frac{e^x}{e^x + e^y}\right)$$

```
In [210]: from math import exp, log
          def log_exp(x, y):
              ## add your solution here
              first = nd.exp(x)+ nd.exp(y)
              first = nd.log(first)
              return first - x
```

Test your codes with normal inputs:

```
In [211]: x, y = nd.array([2]), nd.array([3])
          z = log_exp(x, y)
          z

Out[211]: [1.3132617]
          <NDArray 1 @cpu(0)>
```

Now implement a function to compute $\partial z / \partial x$ and $\partial z / \partial y$ with `autograd`

```
In [212]: def grad(forward_func, x, y):
              ## Add your codes here
              x.attach_grad()
              with autograd.record():
                  z = forward_func(x, y)
              z.backward()
              print('x.grad =', x.grad)
              y.attach_grad()
              with autograd.record():
                  z = forward_func(x, y)
              z.backward()
              print('y.grad =', y.grad)
```

Test your codes, it should print the results nicely.

```
In [213]: grad(log_exp, x, y)
```

```
x.grad =
[-0.7310586]
<NDArray 1 @cpu(0)>
y.grad =
[0.7310586]
<NDArray 1 @cpu(0)>
```

But now let's try some "hard" inputs

```
In [214]: x, y = nd.array([50]), nd.array([100])
          grad(log_exp, x, y)
          print(exp(100))
          print(exp(50))
```

```
x.grad =
[-1.]
<NDArray 1 @cpu(0)>
y.grad =
[nan]
<NDArray 1 @cpu(0)>
2.6881171418161356e+43
5.184705528587072e+21
```

Does your code return correct results? If not, try to understand the reason. (Hint, evaluate `exp(100)`). Now develop a new function `stable_log_exp` that is identical to `log_exp` in math, but returns a more numerical stable result.

```
In [215]: def stable_log_exp(x, y):
              ## Add your codes here
              inside = nd.exp(y-x) + 1
              return nd.log(inside)

          grad(stable_log_exp, x, y)
```

2/3

```
x.grad =
[-1.]
<NDArray 1 @cpu(0)>
y.grad =
[1.]
<NDArray 1 @cpu(0)>
```